3D Composite Transformation

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Composite Transformation: A number of **transformations** or sequence of **transformations** can be combined into single one called as composition. The resulting matrix is called as **composite** matrix. The process of combining is called as concatenation

3D Point Homogenous Coordinate

- We don't lose anything
- The main advantage: it is easier to compose translation and rotation
- Everything is matrix multiplication

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Scaling About Fix point

Scaling with respect to an arbitrary fixed point is not as simple as scaling with respect to the origin .

The procedure of scaling with respect to an arbitrary fixed point is:

- Translate the object so that the fixed point coincides with the origin.
- Scale the object with respect to the origin.
- Use the inverse translation of step 1 to return the objects to its original position.

3D scaling

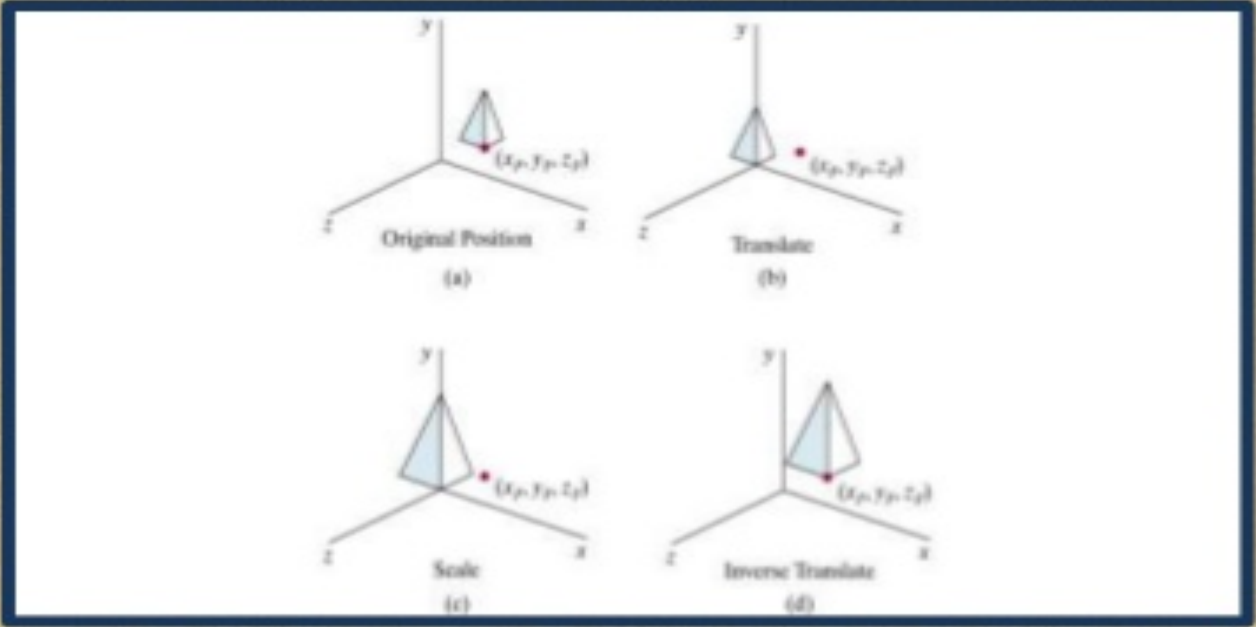
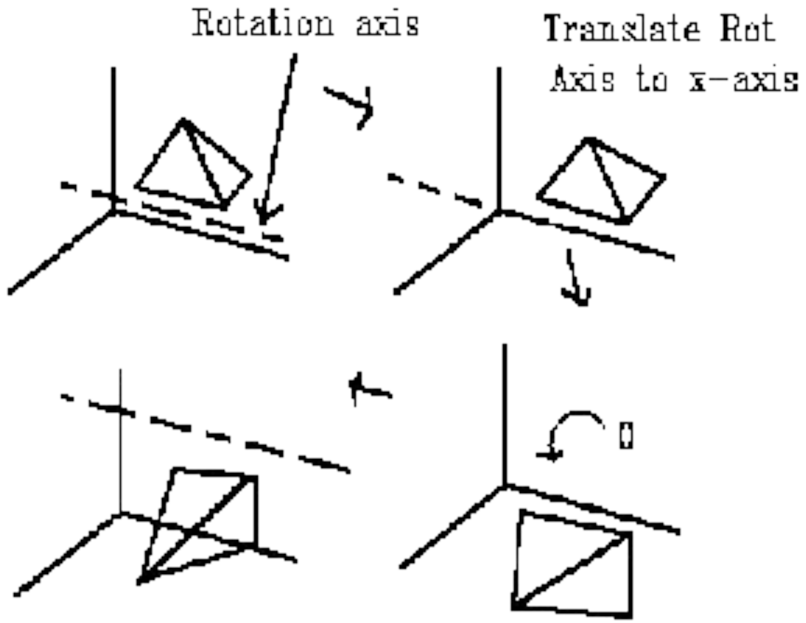


fig : fixed point scaling

The corresponding composite transformation matrix is:

$$\mathbf{T}(x_f, y_f, z_f) \cdot \mathbf{S}(s_x, s_y, s_z) \cdot \mathbf{T}(-x_f, -y_f, -z_f) = \begin{bmatrix} s_x & 0 & 0 & (1-s_x)x_f \\ 0 & s_y & 0 & (1-s_y)y_f \\ 0 & 0 & s_z & (1-s_z)z_f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

General Three-Dimensional Rotations



- **<u>Case1 : When an object is to be rotated about an axis that is parallel to one of the coordinate axes</u>**

We can attain this desired rotation with the following transformation sequence.

1. Translate the object so that the rotation axis coincides with the parallel coordinate axis.

2. Perform the specified rotation about that axis.

3. Translate the object so that the rotation axis is moved back to its original position.

Any coordinate position P on the object is transformed with the sequence shown as

$$\mathbf{P' = T^{-1}\, R(\theta\,)T\ (P)}$$

General Three-Dimensional Rotations

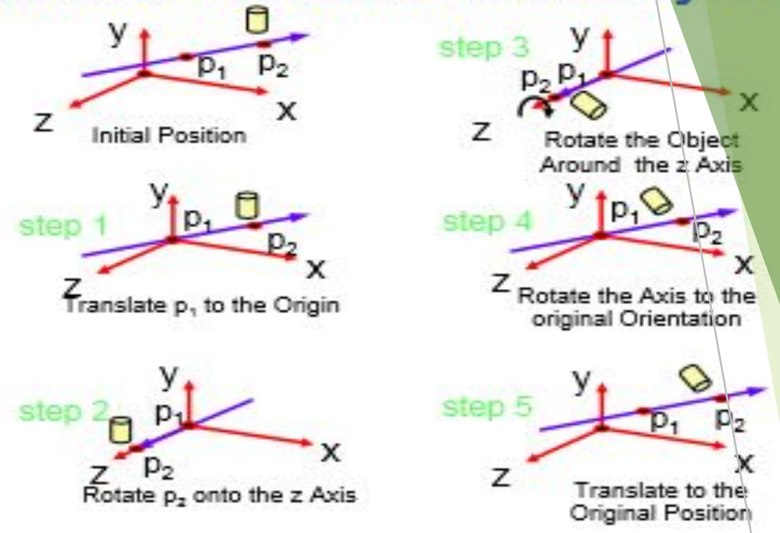
► Case 2: Rotation about Arbitrary Axis

When the object is rotated about an axis that is not parallel to any one of co-ordinate axis, i.e., x, y, z. Then additional transformations are required. First of all, alignment is needed, and then the object is being back to the original position. Following steps are required

1. Translate the object to the origin
2. Rotate object so that axis of object coincide with any of coordinate axis.
3. Perform rotation about co-ordinate axis with whom coinciding is done.
4. Apply inverse rotation to bring rotation back to the original position

$$P' = T^{-1} R_x^{-1} R_y^{-1} R_z R_y R_x T (P)$$

Rotation About an Arbitrary Axis



Rotation About Arbitrary Axis

- Constructing an orthonormal system along the rotation axis:
  - A vector *W* parallel to the rotation axis:

$$s = p_2 - p_1; \qquad w = \frac{s}{|s|}$$

  - A vector *V* perpendicular to *W*:

$$a = w \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \; ; \quad v = \frac{a}{|a|}$$

  - A vector *U* forming a right-handed orthogonal system with *W* and *V*:

$$u = v \times w$$

Let an axis of rotation L be specified by a direction vector $\mathbf{V}$ and a location point P. Find the transformation for a rotation of $\theta°$ about L. Refer to Fig. 6-5.

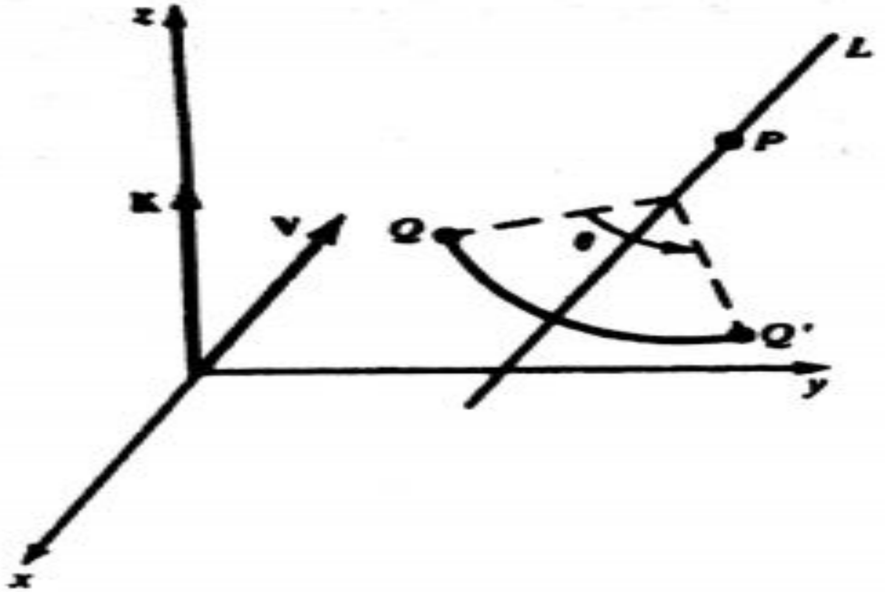


Fig. 6-5

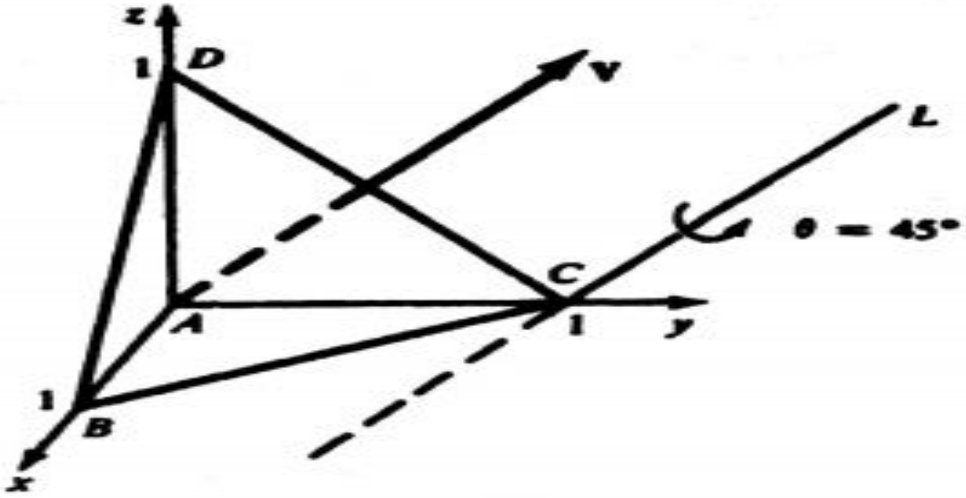


Fig. 6-6

SOLUTION

We can find the required transformation by the following steps:

1. Translate P to the origin.
2. Align $\mathbf{V}$ with the vector $\mathbf{K}$
3. Rotate by $\theta°$ about $\mathbf{K}$.
4. Reverse steps 2 and 1.

So

$$\boxed{R_{\theta,L} = T_{-P}^{-1} \cdot A_{\mathbf{V}}^{-1} \cdot R_{\theta,\mathbf{K}} \cdot A_{\mathbf{V}} \cdot T_{-P}}$$

Here, $A_{\mathbf{V}}$ is the transformation described in Prob. 6.2.

Alignment vector Av calculation

Find a transformation A_V which aligns a given vector **V** with the vector **K** along the positive z axis.

SOLUTION

See Fig. 6-4(a). Let $\mathbf{V} = a\mathbf{I} + b\mathbf{J} + c\mathbf{K}$. We perform the alignment through the following sequence of transformations [Figs. 6-4(b) and 6-4(c)]:

1. Rotate about the x axis by an angle θ_1 so that V rotates into the upper half of the xz plane (as the vector $\mathbf{V}_1$).
2. Rotate the vector $\mathbf{V}_1$ about the y axis by an angle $-\theta_2$ so that $\mathbf{V}_1$ rotates to the positive z axis (as the vector $\mathbf{V}_2$).

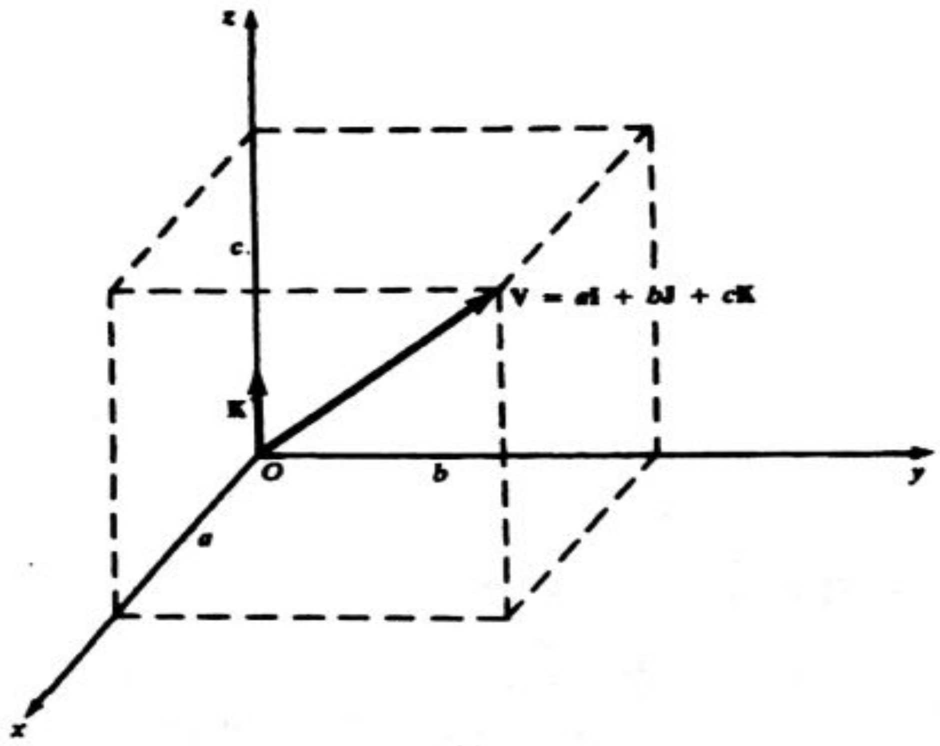
Implementing step 1 from Fig. 6-4(b), we observe that the required angle of rotation θ_1 can be found by looking at the projection of **V** onto the yz plane. (We assume that b and c are not both zero.) From triangle $OP'B$:

$$\sin\theta_1 = \frac{b}{\sqrt{b^2+c^2}} \qquad \cos\theta_1 = \frac{c}{\sqrt{b^2+c^2}}$$

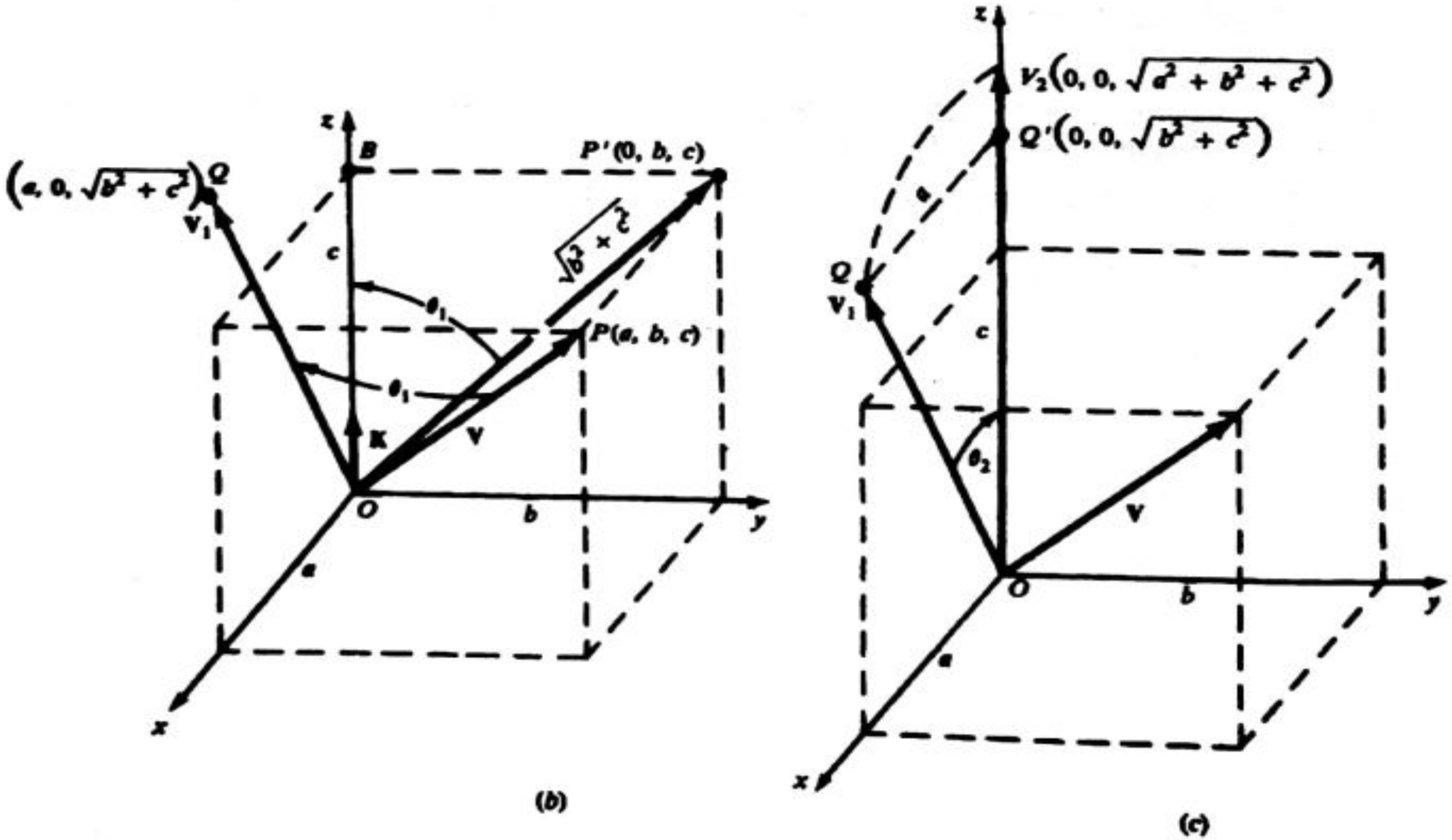
The required rotation is

$$R_{\theta_1,\mathbf{I}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{c}{\sqrt{b^2+c^2}} & -\dfrac{b}{\sqrt{b^2+c^2}} & 0 \\ 0 & \dfrac{b}{\sqrt{b^2+c^2}} & \dfrac{c}{\sqrt{b^2+c^2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Applying this rotation to the vector **V** produces the vector $\mathbf{V}_1$ with the components $(a, 0, \sqrt{b^2+c^2})$.



(a)



z
B
P′(0, b, c)
(a, 0, √(b² + c²)) Q
V₁
c
√(b² + c²)
θ₁
P(a, b, c)
θ₁
K
V
O
b
y
a
x
(b)
z
V₂(0, 0, √(a² + b² + c²))
Q′(0, 0, √(b² + c²))
a
Q
V₁
c
θ₂
V
O
b
y
a
x
(c)

Fig. 6-4

Implementing step 2 from Fig. 6-4(c), we see that a rotation of $-\theta_2$ degrees is required, and so from triangle OQQ':

$$\sin(-\theta_2) = -\sin\theta_2 = -\frac{a}{\sqrt{a^2+b^2+c^2}} \qquad \text{and} \qquad \cos(-\theta_2) = \cos\theta_2 = \frac{\sqrt{b^2+c^2}}{\sqrt{a^2+b^2+c^2}}$$

Then

$$R_{-\theta_2,\mathbf{J}} = \begin{pmatrix} \dfrac{\sqrt{b^2+c^2}}{\sqrt{a^2+b^2+c^2}} & 0 & \dfrac{-a}{\sqrt{a^2+b^2+c^2}} & 0 \\ 0 & 1 & 0 & 0 \\ \dfrac{a}{\sqrt{a^2+b^2+c^2}} & 0 & \dfrac{\sqrt{b^2+c^2}}{\sqrt{a^2+b^2+c^2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since $|\mathbf{V}| = \sqrt{a^2+b^2+c^2}$, and introducing the notation $\lambda = \sqrt{b^2+c^2}$, we find

$$\boxed{\begin{aligned} A_{\mathbf{V}} &= R_{-\theta_2,\mathbf{J}} \cdot R_{\theta_1,\mathbf{I}} \\ &= \begin{pmatrix} \dfrac{\lambda}{|\mathbf{V}|} & \dfrac{-ab}{\lambda|\mathbf{V}|} & \dfrac{-ac}{\lambda|\mathbf{V}|} & 0 \\ 0 & \dfrac{c}{\lambda} & \dfrac{-b}{\lambda} & 0 \\ \dfrac{a}{|\mathbf{V}|} & \dfrac{b}{|\mathbf{V}|} & \dfrac{c}{|\mathbf{V}|} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}}$$

If both b and c are zero, then $\mathbf{V} = a\mathbf{I}$, and so $\lambda = 0$. In this case, only a $\pm 90°$ rotation about the y axis is required. So if $\lambda = 0$, it follows that

$$A_{\mathbf{v}} = R_{-\theta_2,\mathbf{J}} = \begin{pmatrix} 0 & 0 & \frac{-a}{|a|} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{a}{|a|} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In the same manner we calculate the inverse transformation that aligns the vector $\mathbf{K}$ with the vector $\mathbf{V}$:

$$A_{\mathbf{v}}^{-1} = (R_{-\theta_2,\mathbf{J}} \cdot R_{\theta_1,\mathbf{I}})^{-1} = R_{\theta_1,\mathbf{I}}^{-1} \cdot R_{-\theta_2,\mathbf{J}}^{-1} = R_{-\theta_1,\mathbf{I}} \cdot R_{\theta_2,\mathbf{J}}$$

$$= \begin{pmatrix} \frac{\lambda}{|\mathbf{V}|} & 0 & \frac{a}{|\mathbf{V}|} & 0 \\ \frac{-ab}{\lambda|\mathbf{V}|} & \frac{c}{\lambda} & \frac{b}{|\mathbf{V}|} & 0 \\ \frac{-ac}{\lambda|\mathbf{V}|} & -\frac{b}{\lambda} & \frac{c}{|\mathbf{V}|} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The pyramid defined by the coordinates $A(0,0,0)$, $B(1,0,0)$, $C(0,1,0)$, and $D(0,0,1)$ is rotated $45°$ about the line L that has the direction $\mathbf{V} = \mathbf{J} + \mathbf{K}$ and passing through point $C(0,1,0)$ (Fig. 6-6). Find the coordinates of the rotated figure.

SOLUTION

From Prob. 6.3, the rotation matrix $R_{\theta,L}$ can be found by concatenating the matrices

$$R_{\theta,L} = T_{-P}^{-1} \cdot A_{\mathbf{V}}^{-1} \cdot R_{\theta,\mathbf{K}} \cdot A_{\mathbf{V}} \cdot T_{-P}$$

With $P = (0,1,0)$, then

$$T_{-P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now $\mathbf{V} = \mathbf{J} + \mathbf{K}$. So from Prob. 6.2, with $a = 0$, $b = 1$, $c = 1$, we find $\lambda = \sqrt{2}$, $|\mathbf{V}| = \sqrt{2}$, and

$$A_{\mathbf{V}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{\mathbf{V}}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Also

$$R_{45^\circ,\mathbf{K}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad T_{-P}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

en

$$R_{\theta,L} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{2+\sqrt{2}}{4} & \frac{2-\sqrt{2}}{4} & \frac{2-\sqrt{2}}{4} \\ -\frac{1}{2} & \frac{2-\sqrt{2}}{4} & \frac{2+\sqrt{2}}{4} & \frac{\sqrt{2}-2}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

find the coordinates of the rotated figure, we apply the rotation matrix $R_{\theta,L}$ to the matrix of homog rdinates of the vertices A, B, C, and D:

$$C = (ABCD) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$R_{\theta,L}\cdot C = \begin{pmatrix} \frac{1}{2} & \frac{1+\sqrt{2}}{2} & 0 & 1 \\ \frac{2-\sqrt{2}}{4} & \frac{4-\sqrt{2}}{4} & 1 & \frac{2-\sqrt{2}}{2} \\ \frac{\sqrt{2}-2}{4} & \frac{\sqrt{2}-4}{4} & 0 & \frac{\sqrt{2}}{2} \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

The rotated coordinates are (Fig. 6-7)

$$A' = \left(\frac{1}{2}, \frac{2-\sqrt{2}}{4}, \frac{\sqrt{2}-2}{4}\right) \qquad C' = (0, 1, 0)$$

$$B' = \left(\frac{1+\sqrt{2}}{2}, \frac{4-\sqrt{2}}{4}, \frac{\sqrt{2}-4}{4}\right) \qquad D' = \left(1, \frac{2-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$