FINDING AND PROVING AN IDENTITY FOR K^(C)[p,m] where p=13 and m=2 (quadratic residue case)

This worksheet has **Startup Code**

```
> myseeds:=[[15, -1, -3, -2, -2, -2, -3]];
```

$$myseeds := [[15, -1, -3, -2, -2, -2, -3]] \quad (1)$$

NOTE 1 : myseeds generates 1*6 = 6 functions. For m=2 we need these 6 functions. Also we need to multiply by **(eta(13*tau)/eta(tau))^(2*k)**, k=-1. Thus the list [-2] in the plantseeds function.

```
> BIGBAS:=plantseeds(myseeds,[-2],13):
```

NOTE 2 : Now to finish getting the basis for m=2 we need to multiply all the functions by f[13,1]/f[13,6]. This is achieved using the **mult_nv_by_fp_quot** function.

```
> nvLA:=BIGBAS:
> BIGBAS;
```

$$[[15, -1, -3, -2, -2, -2, -3], [15, -3, -1, -2, -3, -2, -2], [15, -2, -2, -1, -2, -3, -3], [15, -2, -3, -2, -1, -3, -2], [15, -2, -2, -3, -3, -1, -2], [15, -3, -2, -3, -2, -2, -1], [3, 1, -1, 0, 0, 0, -1], [3, -1, 1, 0, -1, 0, 0], [3, 0, 0, 1, 0, -1, -1], [3, 0, -1, 0, 1, -1, 0], [3, 0, 0, -1, -1, 1, 0], [3, -1, 0, -1, 0, 0, 1]] \quad (2)$$

```
> nvL:=map(nv->mult_nv_by_fp_quot(nv,13,1,6),nvLA):
> nops(nvL);
```

$$12 \quad (3)$$

We now have a list of 12 functions in our basis list and we are ready to find an prove the identity for m=2.

```
> do_alg_steps(13,2,nvL);
--------------------------------------------------
STEP 1: check modularity
        modularity checks
--------------------------------------------------
STEP 2: find k0 and divide by j0
        k0 = 12
--------------------------------------------------
STEP 3: Compute table of ORDS at all cusps for each func
```

"CUSPS: ", [[1, 0], [0, 1], [1, 2], [1, 3], [1, 4], [1, 5], [1, 6], [2, 13], [3, 13], [4, 13], [5, 13], [6, 13]]

"TABLE of ords"

3, −1, −1, −1, −1, −1, −1, 0, 2, 1, 1, −1

2, −1, −1, −1, −1, −1, −1, 0, 0, 1, 2, 1

3, −1, −1, −1, −1, −1, −1, −1, 1, 2, 1, 0

2, −1, −1, −1, −1, −1, −1, 0, 2, 2, 0, 0

2, −1, −1, −1, −1, −1, −1, 1, 1, 0, 2, 0

1, −1, −1, −1, −1, −1, −1, 1, 1, 1, 1, 1

2, 0, 0, 0, 0, 0, 0, −1, 1, 0, 0, −2

1, 0, 0, 0, 0, 0, 0, −1, −1, 0, 1, 0

2, 0, 0, 0, 0, 0, 0, −2, 0, 1, 0, −1

$$1, 0, 0, 0, 0, 0, 0, -1, 1, 1, -1, -1$$

$$1, 0, 0, 0, 0, 0, 0, 0, 0, -1, 1, -1$$

$$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0$$

```
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STEP 4: Compute LOWER BOUND for ORD of _Kpm/_j0 at each cusp
```

"TABLE :"

$$\_cusp, \_LOWER\_BOUND\_of\_ORD, \frac{\_Kpm}{\_j0}, \_at\_cusp$$

$$\_cusp = 0, \_LOWER\_BOUND = -1$$

$$\_cusp = \frac{1}{2}, \_LOWER\_BOUND = 0$$

$$\_cusp = \frac{1}{3}, \_LOWER\_BOUND = 0$$

$$\_cusp = \frac{1}{4}, \_LOWER\_BOUND = 0$$

$$\_cusp = \frac{1}{5}, \_LOWER\_BOUND = 0$$

$$\_cusp = \frac{1}{6}, \_LOWER\_BOUND = 0$$

$$\_cusp = \frac{2}{13}, \_LOWER\_BOUND = -\frac{14}{13}$$

$$\_cusp = \frac{3}{13}, \_LOWER\_BOUND = -\frac{11}{13}$$

$$\_cusp = \frac{4}{13}, \_LOWER\_BOUND = \frac{1}{13}$$

$$\_cusp = \frac{5}{13}, \_LOWER\_BOUND = -\frac{4}{13}$$

$$\_cusp = \frac{6}{13}, \_LOWER\_BOUND = -1$$

```
---------------------------------------------------
STEP 5: Compile LHS vs RHS ORD table at cusps and find constant B
```

"TABLE ORD lower bounds"

$$\_cusp, \_width, \_ORD\_LHS, \_ORD\_RHS, \_ORD\_LHS\_minus\_RHS$$

$$0, 13, -1, -1, -1$$

$$\frac{1}{2}, 13, 0, -1, -1$$

$$\frac{1}{3}, 13, 0, -1, -1$$

$$\frac{1}{4}, 13, 0, -1, -1$$

$$\frac{1}{5}, 13, 0, -1, -1$$

$$\frac{1}{6}, 13, 0, -1, -1$$

$$\frac{2}{13}, 1, -1, -2, -2$$

$$\frac{3}{13}, 1, 0, -1, -1$$

$$\frac{4}{13}, 1, 1, -1, -1$$

$$\frac{5}{13}, 1, 0, -1, -1$$

$$\frac{6}{13}, 1, -1, -2, -2$$

```
        This implies that B = -13
-------------------------------------------------
STEP 6: Prove and check identity
```

"Coefficients in CKpm identity"

$$\_k = 1, -\zeta^{11} - \zeta^{9} - \zeta^{7} - \zeta^{6} - \zeta^{4} - \zeta^{2} - 1$$

$$\_k = 2, \zeta^{11} - \zeta^{9} - \zeta^{8} - \zeta^{5} - \zeta^{4} + \zeta^{2} - 1$$

$$\_k = 3, -\zeta^{9} - \zeta^{4}$$

$$\_k = 4, \zeta^{11} + \zeta^{10} + \zeta^{8} + \zeta^{5} + \zeta^{3} + \zeta^{2} + 2$$

$$\_k = 5, -\zeta^{10} - \zeta^{7} - \zeta^{6} - \zeta^{3}$$

$$\_k = 6, -\zeta^{11} - \zeta^{10} - \zeta^{9} - 2\,\zeta^{8} - 2\,\zeta^{7} - 2\,\zeta^{6} - 2\,\zeta^{5} - \zeta^{4} - \zeta^{3} - \zeta^{2}$$

$$\_k = 7, 0$$

$$\_k = 8, 0$$

$$\_k = 9, 0$$

$$\_k = 10, 0$$

$$\_k = 11, 0$$

$$\_k = 12, -\zeta^{10} - \zeta^{7} - \zeta^{6} - \zeta^{3}$$

"Proving and checking identity"

"IDENTITY CHECKED AND PROVEN"

"IDENTITY checked for ", $\_O\left(q^{\_topq + 1}\right) = \_O\left(q^{154}\right)$

```
and _topq + 1 > -_B = 13
-------------------------------------------------
```