# LA Assignment - 1

**Q1**

$F(\{0,1\}, \wedge, \cdot)$ $\wedge$ = XOR

$\cdot$ = AND

**XOR**

① $\forall a,b \in \{0,1\}$, $a \wedge b \in \{0,1\}$

$\therefore$ $\wedge$ is closed under binary operation

② $\forall a,b,c \in \{0,1\} \Rightarrow (a \wedge b) \wedge c = a \wedge (b \wedge c)$ (associativity)

$\therefore$ $\wedge$ is a semigroup

③ $\forall a \in \{0,1\}$ identity element

$a \wedge e = a$ if $\boxed{e = 1}$

$\therefore$ $\wedge$ is closed under monoid

④ $a \wedge a^{-1} = e$

for $\forall a \in \{0,1\} \Rightarrow$ $a \wedge a = 0 \Rightarrow \boxed{a^{-1} = a}$

$0 \wedge 0 = 0$ so inverse

$1 \wedge 1 = 0$ exists

$\therefore$ It's a group

⑤ $\forall b, a \in \{0,1\}$

$a \wedge b = b \wedge a$ (commutativity)

$\therefore$ XOR is an abelian group

which is required in field

AND $\forall a, b, c \in \{0, 1\}$ and defined everywhere

① $a \cdot b \in \{0,1\}$ $\Rightarrow$ closed under binary operation

② $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ $\Rightarrow$ associativity holds

③ if $e = 1$, $a \cdot e = a$, $\boxed{a \cdot 1 = a}$ $\Rightarrow$ identity element exists

④ $a \cdot b = b \cdot a$ $\Rightarrow$ commutativity holds true

⑤ INVERSE, $\boxed{1^{-1} = 1 \Rightarrow 1 \cdot 1^{-1} = 1 \cdot 1 = 1}$

$\boxed{0^{-1} \text{ does NOT exist}}$

So inverse exists for NON ZERO elements which is the required property in fields.

$\therefore$ F is a field

Q2 (a)

Assumption - For field $(F, +, *)$
Here $+$ and $*$ are normal addition & multiplication of real no.s

(a) $V = \mathbb{R}$, $F = \mathbb{N}$

For $F = \mathbb{N}$, $F = (\mathbb{N}, +, *)$

⇒ for addition operation - It should be an abelian group

⇒ For that it must have identity element

⇒ For addition of numbers, identity element is ZERO, $\boxed{e = 0}$

BUT $0 \notin N$

so here field $(\mathbb{N}, +, *)$ is not closed for '+' as identity does not exist

∴ since F is NOT a field so, it is NOT a valid vector space

(b) $V = \mathbb{Q}$, $F = \mathbb{R}$

for field $F = \mathbb{R}$ $\boxed{F = (\mathbb{R}, +, *)}$

(1) F is a valid abelian group under '+'

i) Associativity holds

$a, b, c \in \mathbb{R}$ $(a+b)+c = a+(b+c)$

ii) Binary operation holds true – its defined everywhere
$a, b \in \mathbb{R} \Rightarrow a+b \in \mathbb{R}$

(iii) Identity exists

$e=0, \quad a+e=a \quad \forall\, a \in \mathbb{R}$
$0 \in \mathbb{R}$

iv) Inverse exist

$\forall a \in \mathbb{R}, \boxed{\bar{a}^1 = -a} \quad \& -a \in \mathbb{R}$

$\boxed{a+\bar{a}^1 = a+(-a) = 0 = e}$

$\cancel{a * \bar{a}^1 = a * \frac{1}{a} = 1 \quad \& \quad 1 \in \mathbb{R}}$

(v) Commutativity holds

$a+b = b+a \quad \forall a, b \in \mathbb{R}$

Similarly above properties are valid under '$*$' also so F is a valid field

But for $V = \mathbb{Q}$

It should satisfy $a\alpha \in V$ where $a \in F$
$\alpha \in V$

But for $a = \pi$ where $\pi \in \mathbb{R}$
$\& \pi$ = irrational

$\pi\alpha$ will be irrational

$\boxed{\pi\alpha \notin \mathbb{Q}}$

So it is NOT a valid field

(c) $V = \mathbb{R}$, $F = \mathbb{Q}$

For field $F = \mathbb{Q}$ $\boxed{F = (\mathbb{Q}, +, *)}$

$\boxed{+, * \text{ are addition over } \mathbb{Q} \text{ \& multiplication}}$

**For Addition**

$\forall a, b, c \in \mathbb{Q}$

- $(a+b)+c = a+(b+c)$
  Associativity holds
- $a+b \in \mathbb{Q}$
  Defined everywhere
- $a+0 = a$
  $e=0$
  Identity exist
- $a+(-a) = 0 = e$
  Inverse exist
- $a+b = b+a$
  Commutativity holds

$\underbrace{\qquad\qquad}_{\text{Abelian group}}$

**For multiplication**

$\forall a, b, c \in \mathbb{Q}$

- $(a*b)*c = a*(b*c)$
  Associativity holds
- $a*b \in \mathbb{Q}$
  Defined everywhere
- $a*1 = a$
  $e=1$ Identity exists
- $a * \frac{1}{a} = 1 = e \quad \forall a \neq 0$
  Inverse exist for NON ZERO elements only
- $a*b = b*a$
  Commutativity holds

$\underbrace{\qquad\qquad}_{\text{Abelian group for NONZERO}}$

So F is a valid field

For $V = \mathbb{R}$

for $\forall\ \alpha, \beta, \gamma \in V \in \mathbb{R}$

$\forall\ a, b \in F \in \mathbb{Q}$

- $\alpha + \beta \in \mathbb{R}$
  Adding real no. gives real no.
- $\alpha + \beta = \beta + \alpha$
  Addition is commutative
- $(\alpha+\beta) + \gamma = \alpha + (\beta+\gamma)$
  Associativity holds true

- $\alpha + 0 = \alpha$
  $\phi = 0$
  Unique identity exist
- $\alpha + (-\alpha) = 0 = \phi$
  Inverse exist for real no.
- $a\alpha \in \mathbb{R}$
  $\boxed{\mathbb{R} * \mathbb{Q} = \mathbb{R}}$

- $a(\alpha + \beta) = a\alpha + a\beta$
  $*$ is distributive over plus
- $(a+b)\alpha = a\alpha + b\alpha$
  evaluation gives real no. always so $\in \mathbb{R}$
- $(ab)\alpha = a(b\alpha)$
  . Associativity holds & always gives real no. so $\in \mathbb{R}$

- $1 \cdot \alpha = \alpha$
  Multiplying by 1 gives itself for $\mathbb{R}$

So V is a valid vector space

(d) $V = \mathbb{R}$ , $F = \mathbb{C}$

- For $\alpha \in V$ , $a \in F$

  $a\alpha \in V$ ideally for a vector space

- This property does NOT hold here

Here is how –

let $i \in \mathbb{C} \in F$ $\boxed{i = \sqrt{-1}}$

$i\alpha \notin \mathbb{R} \Rightarrow i\alpha \notin V$

So it is NOT a vector space

Q3 "Direct sum of field F will form a vector space V over F."

Proof – Direct sum means $F+F=F^2$

$F+F+F=F^3$

$$\boxed{F+F+F+\dots + n\text{ times} = F^n}$$

Assumptions :- let $\alpha, \beta \in F^n$

$$\alpha = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \beta = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

1) $\boxed{\alpha+\beta \in V}$

$$\alpha+\beta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix} \left\{ \begin{array}{l} a_i+b_i \in F \\ \forall i \in [1,n] \end{array} \right.$$

$\Rightarrow$ So $\alpha+\beta \in F^n$

2.) $\boxed{\alpha+\beta=\beta+\alpha}$ Commutativity

$$\alpha+\beta=\begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix} = \begin{bmatrix} b_1+a_1 \\ b_2+a_2 \\ \vdots \\ b_n+a_n \end{bmatrix} = \beta+\alpha$$

For a field F, addition is an abelian group which includes commutative property also

3.) $\boxed{(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)}$ ASSOCIATIVITY

$$\gamma=\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$\begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \quad \Bigg\| \quad \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1+c_1 \\ b_2+c_2 \\ \vdots \\ b_n+c_n \end{bmatrix}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\begin{bmatrix} a_1+b_1+c_1 \\ a_2+b_2+c_2 \\ \vdots \\ a_n+b_n+c_n \end{bmatrix} = \begin{bmatrix} a_1+b_1+c_1 \\ a_2+b_2+c_2 \\ \vdots \\ a_n+b_n+c_2 \end{bmatrix}$$

4.) $\boxed{\alpha+\phi=\alpha}$

$$\phi=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Additive identity always exists in a field

5.) $\boxed{\alpha + (-\alpha) = \phi}$

$$-\alpha = \begin{bmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_n \end{bmatrix}, \quad \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \phi$$

Additive inverse exist in field always as 'addition' is an abelian group in it.

6.) Let $x \in F$
$\alpha \in F^n$

$$x\alpha = x \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} xa_1 \\ xa_2 \\ \vdots \\ xa_n \end{bmatrix} \in F^n$$

$xa_i \in F \quad \forall i \in [1, n]$

$\therefore \boxed{x\alpha \in F^n}$

7.) $\boxed{x(\alpha + \beta) = x\alpha + x\beta}$

$$x \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix} = \begin{bmatrix} x(a_1 + b_1) \\ x(a_2 + b_2) \\ \vdots \\ x(a_n + b_n) \end{bmatrix} = \underbrace{\begin{bmatrix} xa_1 \\ xa_2 \\ \vdots \\ xa_n \end{bmatrix}}_{\in F^n} + \underbrace{\begin{bmatrix} xb_1 \\ xb_2 \\ \vdots \\ xb_n \end{bmatrix}}_{\in F^n}$$

[as proved in 6]

so there addition is $\in F^n$
[proved in 1]

8.) $\boxed{(x+y)\alpha = x\alpha + y\alpha}$ $x, y \in F$

$\alpha \in F^n$

$$(x+y)\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = x\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + y\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} xa_1 + ya_1 \\ xa_2 + ya_2 \\ \vdots \\ xa_n + ya_n \end{bmatrix}$$

$$= \begin{bmatrix} xa_1 \\ xa_2 \\ \vdots \\ xa_n \end{bmatrix} + \begin{bmatrix} ya_1 \\ ya_2 \\ \vdots \\ ya_n \end{bmatrix} = \underbrace{\underbrace{x\alpha} + \underbrace{y\alpha}}$$

both these are individually closed & their addition is also closed as proved in previous points

9.) $\boxed{(ab)\alpha = a(b\alpha)}$

$$ab\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} aba_1 \\ aba_2 \\ \vdots \\ aba_n \end{bmatrix} = a\begin{bmatrix} ba_1 \\ ba_2 \\ \vdots \\ ba_n \end{bmatrix} = a(b\alpha)$$

10.) $1 \in F$, $1 \cdot \alpha = 1\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 1 \cdot a_1 \\ 1 \cdot a_2 \\ \vdots \\ 1 \cdot a_n \end{bmatrix} = \alpha$

~~multiplicative identity~~ as $1 \times a_i = a_i \; \forall i \in [1, n]$

So $F^n$ is closed under all 10 properties of vector space & direct sum of $\mathcal{O}_F$ forms vector space

" EVERY FIELD F IS A VECTOR SPACE OVER ITSELF "

field = $(F, +, *)$ properties of field $\boxed{① \to ⑤}$

1.) It is abelian group under addition $(F,+)$ = abelian group

2.) $(F, *)$ is semi abelian group
means its bino closed under binary operation, semi group, monoid

3.) $*$ is distributive over $+$
$a*(b+c) = a*b + a*c \quad \forall\ a, b, c \in F$

4.) Every non-zero element in F has multiplicative inverse

5.) F has 1 (unit element)

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PROOF :- Let $\gamma, \alpha, \beta \in F$

(i) $\alpha + \beta = \beta + \alpha$ ~~as~~ from Pt. ① Commutativity

(ii) $\alpha + \beta \in F$ — from Pt. ① closure

(iii) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ — from Pt. ① Associativity

(iv) $\alpha + \phi = \alpha$ — from Pt. ① — Identity exists

(v) $\alpha + (-\alpha) = \phi$ — Additive inverse exist from Pt. ①

(vi) $a \in F$, $a\alpha \in F$ "

(vii) $a(\alpha+\beta) = a\alpha + a\beta$ — from pt. (3)

(viii) $(a+b)\alpha = a\alpha + b\alpha$ — from pt. (3)

(ix) $(ab)\alpha = a(b\alpha)$ Scalar multiplication is associative
$a, b \in F$

(x) $1 \in F$
$1 \cdot \alpha = \alpha$ from pt. (5)

Q4

Let $\alpha=(x_1,y_1)$, $\beta=(x_2,y_2)$, $\alpha,\beta \in V$

Let $a,b \in F$

- For V to be a vector it should satisfy all 10 properties.

- But it does NOT have a unique identity vector $\phi$ due to which it fails to be a vector space

- Here is how,

$\boxed{\text{ideally, } \alpha+\phi=\alpha}$ — ① Let $\phi=(e_1,e_2)$

Here, $(x_1,y_1)+(e_1,e_2)$

$=(x_1+e_1, 0)$ {according to definition of addition provided in question}

* But $\alpha \neq 0$ & $\beta \neq 0$
(we have taken it to be non zero)

So, $(x_1,y_1)+(e_1+e_2)$ should be equal to $(x_1,y_1)$
(according to eq^n ①)

But this is not the case here

$(x_1+e_1, 0) \neq (x_1,y_1)$

Since, identity element does NOT exist

$\therefore$ It is NOT a vector space

Q5

ⓐ $\boxed{a_1 \geq 0}$

<u>example</u>:- $\alpha_1 = \{2, 3, 4, -1, 0\}$

~~$\alpha_2 = \{$~~

- Scaler multiplication should be closed
- But here for any -ve scalar

<u>let $a = -1$</u>

$a\alpha_1 = -1 * \{2, 3, 4, -1, 0\}$

$= \{-2, -3, -4, 1, 0\}$

$\hookrightarrow a_1 = -2 < 0$ which is $\not\geq 0$

So it is <u>NOT</u> a subspace

ⓒ $\boxed{a_2 = a_2^2}$

<u>example</u> - $\alpha = \{1, 2, 3, 4\}$

let a be a scalar with $a = -1$

- $a\alpha$ (product of a vector space/subspace with a scalar)

should be closed

$\Rightarrow$ But here $a\alpha = -1 * \{1, 2, 3, 4\}$

$= \{-1, -2, -3, 4\}$

$a_2$ now is $-2$

- Definition in question says $a_2 = a_2^2$
  so $a_2$ can never be $-ve$
- But in this case, it is not so

$\therefore$ It is <u>NOT</u> a vector subspace

(d) $\boxed{a_1 a_2 = 0}$

Let us take 2 sample example vector subspaces

- $\alpha_1 = \{1, 0, 2, 3, -1, 0\}$
  here $a_1 = 1$, $a_2 = 0$, $a_1 a_2 = 0$
  so $\alpha_1$ satisfies property

- $\alpha_2 = \{0, 1, 4, 5, 6, 7\}$
  here $a_1 = 0$, $a_2 = 1$, $a_1 a_2 = 0$
  so $\alpha_2$ also satisfies

- <u>Ideally</u> <u>$\alpha_1 + \alpha_2$</u> should <u>also satisfy</u>

BUT

$$\begin{aligned}\alpha_1 + \alpha_2 &= \{1, 0, 2, 3, -1, 0\} \\ &\quad + \{0, 1, 4, 5, 6, 7\} \\ \hline &\quad \{1, 1, 6, 8, 5, 7\}\end{aligned}$$

here $a_1 = a_2 = 1$ & $a_1 a_2 = 1 \neq 0$

So it is NOT a vector subspace

(e) $\boxed{a_2 = \text{Rational}}$

Let $\alpha$ be a vector subspace

example - $\alpha = \{1, -1, 2, 0, 1.5\}$

$\Rightarrow$ for 'a' being a scaler, $a\alpha$ should also be a vector subspace

$\Rightarrow$ $a \in \mathbb{R}$

If $a = \pi$ & $\pi$ is we know is irrational

$\Rightarrow$ so $\pi\alpha$ will give a vector subspace with all elements being irrational

(multiplication / addition with irrational no. gives irrational no.)

$\Rightarrow$ $a_2$ will also be irrational

$\therefore$ It is NOT a vector subspace

(b) $\boxed{a_1 + 3a_2 = a_3}$

Let us take 2 valid examples

$\alpha_1 = \{1, 2, 7\}$ $[1 + 3(2) = 7]$

$\alpha_2 = \{-1, 3, 8\}$ $[-1 + 3(3) = 8]$

Now as per rule $\alpha_1 + \alpha_2$ should be closed under abover property

$$\alpha_1 + \alpha_2 = \{1, 2, 7\} + \{-1, 3, 8\} = (0, 5, 15)$$

$0 + 3(5) = 15$ , holds true

Taking a generic example

$\alpha = \{a_1, a_2, a_3, \ldots a_n\} \Rightarrow a_1 + 3a_2 = a_3$

$\beta = \{b_1, b_2, b_3, \ldots b_n\} \Rightarrow b_1 + 3b_2 = b_3$

* $\alpha + \beta = \{a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots, a_n + b_n\}$

$(a_1 + b_1) + 3(a_2 + b_2) = (a_1 + 3a_2) + (b_1 + 3b_2)$

$= a_3 + b_3$

so $\alpha + \beta$ holds true , $\boxed{\text{It is closed under addition}}$

* for $\boxed{c \in F}$

$c(a_1) + c(3a_2) = c(a_1 + 3a_2) = c \cdot a_3$

So, scalar multiplication is also closed

Hence its a <u>valid</u> vector subspace