LA - Assignment - 4

Rishabh Malik
2020201074

Q1

Given - Basis : $\{x^2, x, 1\}$

Given $T(x^2) = x+m$, $T(x) = \cancel{x+m}\ (m-1)x$, $T(1) = x^2+m$

$\Rightarrow T(x^2) = \begin{bmatrix} 0 \\ 1 \\ m \end{bmatrix}$, $T(x) = \begin{bmatrix} 0 \\ m-1 \\ 0 \end{bmatrix}$

$T(1) = \begin{bmatrix} 1 \\ 0 \\ m \end{bmatrix}$

(a)

$$\Rightarrow \boxed{\text{Transformation matrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{bmatrix}}$$

(b) Let $x, y, z \in$ Kernel (T)

From the definition of kernel, we know that

$$\text{Transformation matrix} * \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{z = 0}$

$\Rightarrow x + y(m-1) = 0 \Rightarrow \boxed{x + ym - y = 0}$

$\Rightarrow xm + zm = 0$

$x(m) + 0.m = 0$

$\boxed{mx = 0}$

We have

$z = 0$ — (1)

$mx = 0$ — (2)

$x + ym - y = 0$ — (3)

- Now $\Rightarrow \boxed{\text{if } m = 0}$

eq^n (3) gives $x = y$

So kernel space $K \in \mathbb{R}$ in this case

But

- If $\boxed{m \neq 0}$

$\Rightarrow$ In eq^n $xm + zm = 0$

$z = 0$ from eq^n (1)

so $\boxed{x = 0}$

so eq^n (3) gives $0 + ym - y = 0$

$ym = y$

- For $m = 1$, Kernel space $= \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}$

- When $m \neq 1$ & $m \neq 0$, $y = 0$

Kernel space $= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(c) Let T be $V \to W$

& $\{x, y, z\} \in V$

Image of T in W $= \begin{bmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$= \begin{bmatrix} z \\ x + ym - y \\ xm + zm \end{bmatrix}$

Q2 Given -

$T(x, y, z) = (x + 2yz, 2x + 3y + z, 4x + 7yz)$

For linearity of transformation T,

we already know that, $T(\alpha x) = \alpha T(x)$

α = any scalar

x = vector

We have here,

$T(2x, 2y, 2z) = 2 T(x, y, z)$

$= (2x + 2(2y)(2z), 2(2x) + 3(2y) + (2z),$

$4(2x) + 7(2y)(2z))$

$= 2\,(x + 4yz,\ 2x + 3y + z,\ 4x + 14yz)$

so we get

$T(2x, 2y, 2z) \neq 2T(x, y, z)$

Hence we get that this transformation is non linear

Q3

Given : T is one-one

As T is a linear transformation, it will send zero vector O_v of V to zero vector O_w of W.

$$T(O_v) = T(O_v - O_v) = T(O_v + (-1)O_v)$$
$$= T(O_v) + (-1)T(O_v)$$
$$= T(O_v) - T(O_v)$$
$$T(O_v) = O_w - O_w = O_w$$

$\therefore O_v \in N(T)$

$\Rightarrow$ If a vector $\vec{v} \in N(T)$,

$$T(\vec{v}) = O_w = T(O_v)$$

$\vec{v} = O_v$ (as T is one-one)

Hence $N(T) = \{O_v\}$

$\therefore$ If Kernel ~~is~~ has only null vector than T is one-one

- Let $T(\vec{v}_1) = T(\vec{v}_2)$ for $\forall \vec{v}_1, \vec{v}_2 \in V$

$\Rightarrow$ $O_w = T(\vec{v}_1) - T(\vec{v}_2)$

$= T(\vec{v}_1) + (-1)T(\vec{v}_2)$

$O_w = T(\vec{v}_1 - \vec{v}_2)$

$\therefore (\vec{v}_1 - \vec{v}_2)$ is in null vector space $N(T) = \{O_v\}$

$\vec{v}_1 - \vec{v}_2 = O_v$

$\boxed{\vec{v}_1 = \vec{v}_2}$

$\therefore$ $T: V \to W$ is one-one

Therefore, A linear transformation $T: V \to W$ between 2 vectors space is one-one iff Kernel has only the null vector

- Proof of preservation of linear independence

Let $S = \{v_1, v_2, v_3, \ldots v_t\}$ is a linearly independent subset of V

$a_1 T(v_1) + a_2 T(v_2) + a_3 T(v_3) + \cdots + a_t T(v_t) = 0$

$T(a_1 v_1 + a_2 v_2 + \cdots + a_t v_t) = 0$ (T being linear)

$\therefore a_1 v_1 + a_2 v_2 \cdots + a_t v_t \in N(T)$

$a_1 v_1 + a_2 v_2 + \dots + a_t v_t = 0 \quad (N(T) = \{0\})$

$\therefore a_1 = a_2 = \dots = a_t = 0$

$\therefore \{v_1, v_2, \dots v_t\}$ is linearly independent

Q4

- Pointwise addition − $(T_1 + T_2)(v) = T_1(v) + T_2(v)$ where $T_1, T_2 \in \Lambda$
  $v \in V$

- Scalar Multiplication − $(\alpha T)(v) = \alpha T(v)$ where $T \in \Lambda$, $v \in V, \alpha \in F$

⇒ Closed under addition − $\boxed{\forall T_1, T_2 \in \Lambda \text{ \& } \forall v \in V}$

↳ will be same everywhere

$(T_1 + T_2) v = T_1(v) + T_2(v)$

where $T_1(v), T_2(v) \in W$

$T_1(v) + T_2(v) \in W$ ∵ W is a vector space

$T_1 + T_2 \in \Lambda$

⇒ Commutative addition

$(T_1 + T_2)(v) = T_1(v) + T_2(v) = T_2(v) + T_1(v)$

$(T_1 + T_2)(v) = (T_2 + T_1)(v)$

$\therefore T_1 + T_2 = T_2 + T_1$

→ Associative addition

$\forall T_1, T_2, T_3 \in \Lambda$

$$((T_1 + T_2) + T_3)(v) = (T_1 + T_2)(v) + T_3(v)$$
$$= T_1(v) + T_2(v) + T_3(v)$$
$$= T_1(v) + (T_2(v) + T_3(v))$$
$$= T_1(v) + (T_2 + T_3)(v)$$

$(T_1 + T_2) + T_3 = T_1 + (T_2 + T_3)$

⇒ Additive Identity

Let $0 : V \to W$ be a mapping where $0(v) = 0$ is the additive identity

$(T + 0)(v) = T(v) + 0(v)$
$= T(v) + 0$
$= T(v)$

$\therefore T + 0 = T$ & hence 0 = Additive identity

⇒ Additive Inverse

$(T + (-T))(v) = T(v) + -T(v) = 0 = 0(v)$

$T + (-T) = 0$ & hence $-T$ is additive inverse

so $\forall T \; \exists (-T)$

⇒ Closed under Multiplication

$\boxed{T \in \Lambda, \; c \in F, \; v \in V}$ → will be same everywhere

$(cT)(v) = cT(v)$

so $cT \in \Lambda$ as $T(v) \in W$
W is vector space

⇒ Associtive multiplication

$\boxed{\text{let } c_1, c_2 \in F}$ → will be same everywhere

$(c(dT))(v) = c(dT(v)) = cdT(v)$
$= ((cd)T)(v)$

⇒ Distributive

$$c(T_1+T_2)(v) = c\,(T_1(v)+T_2(v))$$
$$= cT_1(v) + cT_2(v)$$
$$= (cT_1 + cT_2)v$$

$$c(T_1+T_2) = cT_1 + cT_2$$

⇒ Unit scalar $(1 \in F)$

$$(1.T)(v) = 1.T(v) = T(v)$$
$$1.T = T$$

⇒ $$((C_1+C_2)T_1)(v) = (C_1+C_2)\,T_1(v)$$
$$= C_1T_1(v) + C_2T_2(v)$$
$$= (C_1T_1 + C_2T_2)(v)$$

$$(C_1+C_2)T_1 = C_1T_1 + C_2T_2$$

$\therefore$ $\wedge$ is a vector space

Q5

$P: V \to V$ is projection of V along M onto N
as $P(v) = n$, $v \in V$

Since P is along M onto N, basis of N gives the n value

$P(v) = y_1 n_1 + y_2 n_2 + \dots + y_K n_K$ where

$\{n_1, n_2, \dots n_K\}$ is basis of N

so $P(v) \in N$

Let $A = [n_1 \; n_2 \dots n_K]$ be matrix of basis vectors of N

so for any $n \in N$

$$n = [n_1 \; n_2 \dots n_K] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_K \end{bmatrix} = Ay$$

As $\because P(v) \in N$ we get $P(v) = Ay$

$\Rightarrow$ $v - P(v) \in N^{\perp}$

$v - P(v) \in C A^{\perp}$

we already know that left null space is orthogonal to column space

$\Rightarrow$ $v - P(v) \in N(A^T)$

$$A^T(v - P(v)) = 0$$

$$A^T v - P(v) A^T = 0$$

$$A^T v - A^T A y = 0$$

$$A^T A y = A^T v \quad \text{--- (1)}$$

In eqn (1),
As A is matrix consisting of basis vectors, so $A^T A$ is invertible

eqn * $(A^T A)^{-1}$

$$(A^T A)^{-1} A^T A y = (A^T A)^{-1} A^T v$$

$$y = (A^T A)^{-1} A^T v$$

We know,

$$P(v) = Ay = A\left((A^T A)^{-1} A^T v\right)$$

$$P(v) = \underbrace{A\left((A^T A)^{-1} A^T\right)}_{\text{so this is a matrix \& is linearly transformation}} v$$

$\Rightarrow$ so P is linear

- <u>P is idempotent</u>

Take any value & apply P to it 2 times.
It always gives same answer. as in case it has been applied only once. If such a case occurs, then projection is said to be idempotent.

let $v \in V$

we already have, $P(v) = (A(A^T A)^{-1} A^T) v$

$$P^2(v) = P(P(v))$$
$$= P((A(A^T A)^{-1} A^T) v)$$
$$= (A A(^T A)^{-1} A^T)(A(A^T A)^{-1} A^T) v$$

$$= (A(A^T A)^{-1} A^T) v$$
$$= P(v)$$

$\Rightarrow$ so $P^2(v) = P(v)$

$\therefore P$ is idempotent

Q6

Alice - determinant is non-zero
Bob - determinant is zero

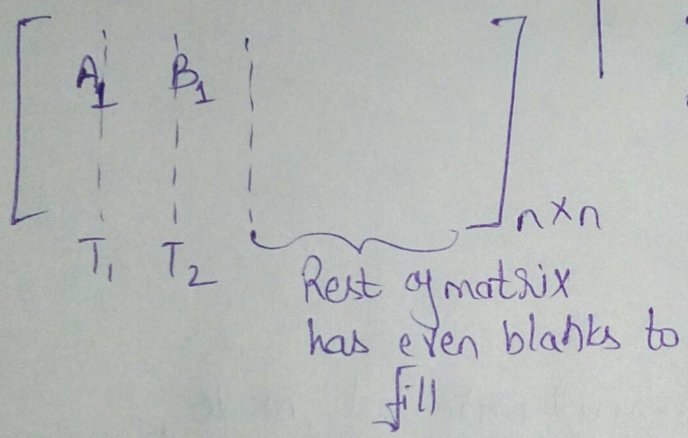
n

even

- If bob starts 2nd, plays optimally & wins
- Alice starts 2nd & wins

odd

- Bob starts 1st & wins
- Alice starts 1st & wins



- A_1 - Alice moves 1st
- B_1 - Bob moves 1st
- T_1 - column 1
- T_2 - column 2
- Total n colums

Strategy

(i) Alice starts at A_1, Bob replies by putting number in same row.

we have to ensure that T_1 & T_2 are linearly dependent as it would make sure that the determinant has value zero

(ii) If Alice puts number in either T_1 or T_2, Bob responds by putting number in another column such that T_1 & T_2 remains dependent

To do this, T_1 & T_2 should be scaler multiples of each other.

eg-

$Alice_1 = 2$	$Bob_1 - 4$
$Alice_2 = 7$	$Bob_2 = 14$
$Bob_3 = \frac{3}{2}$	$Alice_3 = 3$
$\vdots$	$\vdots$
T_1	T_2

Keep doing like this

Since n is even, bob will make last move.
& will ensure that T_1 & T_2 remain dependent

Q7

Let X, U be vector space over field F, $T: X \to U$

(a)

An operator such that $S: U \to X$ where $S(T(x)) = x$ $\forall x \in X$, so S is left inverse of T

To prove - T is injective $\Leftrightarrow$ left inverse (S) of T exists

- S exists if T is injective

Let $x, y \in X$ such that $T(x) = T(y)$

$\Rightarrow x = S(T(x)) = S(T(y)) = y$

$\therefore$ for $T(x) = T(y) \Rightarrow x = y$

$\therefore$ T is injective

- T is injective if S exists

Let $A = \{x_i : i \in I\}$ be basis of X

Then $\{T(x_i) : i \in I\}$ is linearly independent subset of U. So $\exists$ basis B such that $\{T(x_i) : i \in I\} \subset B$

Let $S: U \to X$ be a linear map defined on B by

$$S(u) = \begin{cases} v & \text{if } u \in A \text{ with } u = T(v) \\ 0 & \text{if } u \notin A \end{cases}$$

Now for $x = \sum \lambda_i x_i \in X$ we get

$$
\begin{aligned}
S(T(x)) &= S(T(\sum \lambda_i x_i)) \\
&= \sum \lambda_i S(T(x_i)) \\
&= \sum \lambda_i x_i \\
&= x
\end{aligned}
$$

$\therefore$ S is left inverse of T

(b)

If ~~there~~ $\exists$ a S , $S : U \to X$ such that $T(S(u)) = u \quad \forall u \in U$

an operator

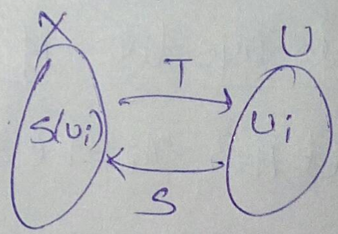
then S is right inverse of T.

- Right inverse exists if T is surjective

As S exists, $\forall u_i \in U$, we have $S(u_i) \in X$

such that $T(S(u_i)) = u_i$

$\therefore$ T is surjective



- T is surjective if S exists

Let $\beta = \{u_i : i \in I\}$ be basis for U

As T is surjective $\exists \{x_i : i \in I\}$ $x_i \in X$

such that $T(x_i) = u_i$

$\{x_i : i \in I\}$

Let $\alpha_1, \alpha_2, \alpha_3, \dots \alpha_n$ be scalars

$$\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = 0$$

$$T(\alpha_1 x_1 + \alpha_2 x_2 + \dots \alpha_n x_n) = 0$$

$$\alpha_1 T_1(x) + \alpha_2 T(x_2) + \dots + \alpha_n T(x_n) = 0$$

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n = 0$$

As u_i is basis for U, $\alpha_1 = \alpha_2 \dots \alpha_n = 0$

$\therefore$ $\{x_i : i \in I\}$ are independent

Now let $S : U \to X$ be a linear map defined on U as follows

$$S(u) = \begin{cases} v & \text{if } u \in B \text{ with } u = T(v) \\ 0 & \text{if } u \notin B \end{cases}$$

Then for $u = \Sigma \lambda_i u_i$ we get

$$T(S(u)) = T(S(\Sigma \lambda_i u_i)) = \Sigma \lambda_i T(S(u_i))$$
$$= \Sigma \lambda_i u_i$$
$$= u$$

$\therefore$ $T(S(u)) = u$

$\therefore$ It's ~~is~~ right inverse exists