QuickSort

- The quick sort algorithm designed by Hoare is a simple yet highly efficient algorithm.
- It works as follows:
  - Start with the given array A of n elements.
  - Consider a pivot, say A[n].
  - Now, partition the elements of A into two arrays A_L and A_R such that:
    - the elements in A_L are less than A[n]
    - the elements in A_R are greater than A[n].
  - Sort A_L and A_R, recursively.

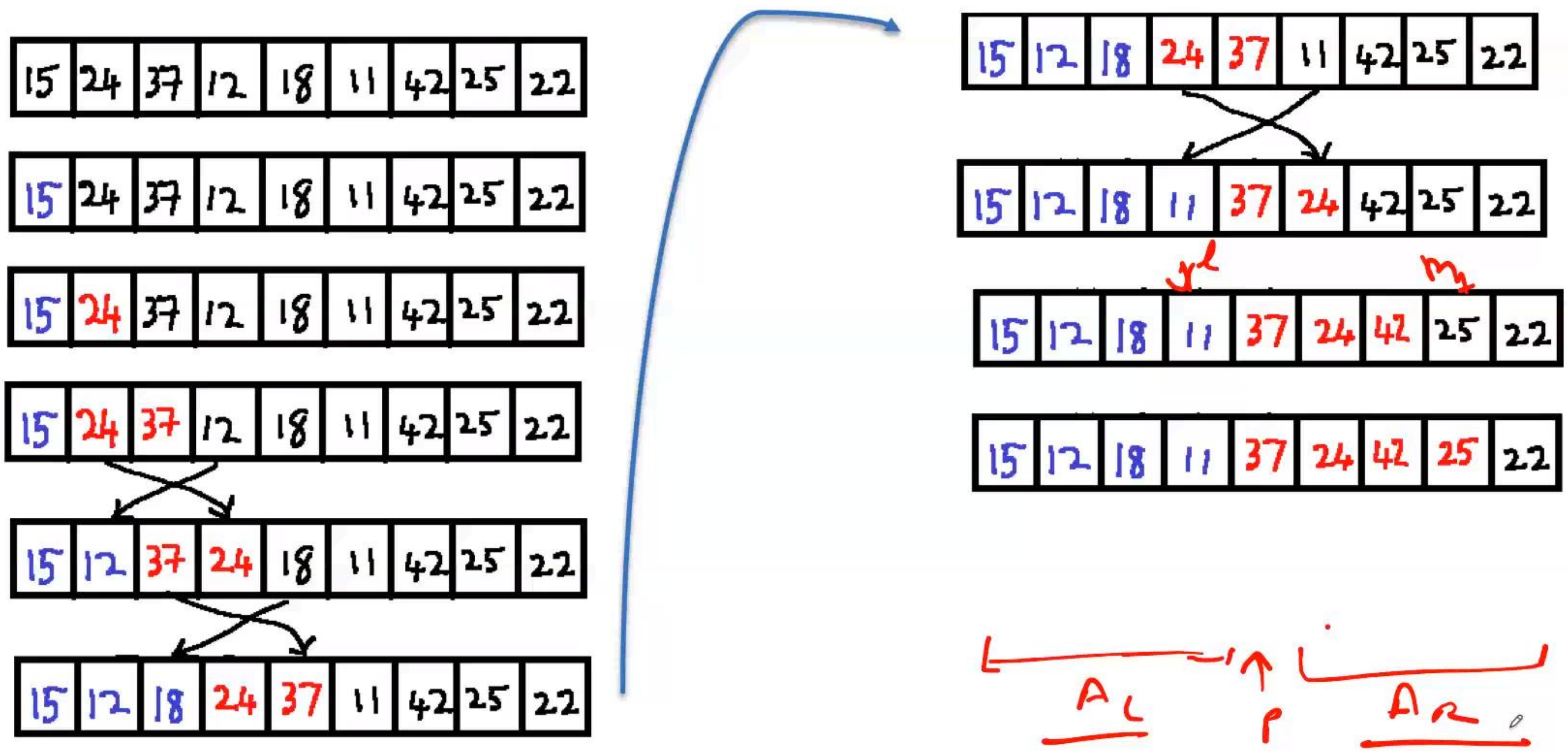
QuickSort

- How to partition ?
  - Suppose we take each element, compare it with A[n] and then move it to A_L or A_R accordingly.
  - Works in O(n) time.
  - Can write the program easily.
  - But, recall that space is also an resource. The above approach requires extra space for the arrays A_L and A_R
  - A better approach exists.

QuickSort

```
Procedure Partition(A,n)
begin
  pivot = A[n];
  less = 0; more = 1;
  for more = 1 to n-1 do
    if A[more] < pivot then
      less++;
      swap(A[more], A[less]);
    end
  end
  swap(A[less+1], A[n]);
end
```

QuickSort


15 24 37 12 18 11 42 25 22
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A L
p
A R

Analyzing Quick Sort

- We know that $|A_L| + |A_R| = n-1$.
- But, if the pivot is such that all elements are smaller (or larger) than the pivot, then $|A_L|$ (or $|A_R|$) $= n-1$.
- The recurrence relation in that case is

$$T(n) = T(n-1) + O(n).$$

- Suppose the same situation happens over every recursive call. So, the above recurrence relation holds during every recursive call.
- When will this happen ?

Analyzing Quick Sort

$n_1 + n_2 = n-1$

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Analyzing Quick Sort

- In general, if the sizes of $|A_L|$ and $|A_R|$ are such that they are a constant away from each other, then the recurrence relation is:

  $$T(n) = T(an) + T((1-a)n) + O(n)$$

  where a is a constant < 1.

- In practice, it turns out that most often the partitions are not too skewed.
- So, quick sort runs in O(n log n) time almost always.

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Divide and Conquer

- Divide the problem P into $k \geq 2$ sub-problems $P_1, P_2, \ldots, P_k$.
- Solve the sub-problems $P_1, P_2, \ldots, P_k$.
- Combine the solutions of the sub-problems to arrive at a solution to P.

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Request control

Divide and Conquer

- A useful paradigm with several applications.
- Examples include merge sort, convex hull, median finding, matrix multiplication, and others.
- Typically, the sub-problems are solved recursively.
  - Recurrence relation

$$T(n) = D(n) + \sum_i T(n_i) + C(n)$$

Divide time

Recursive cost

Combine time

Algorithm Merge

```
Algorithm Merge(L, R)
// L and R are two sorted arrays of size n each.
// The output is written to an array A of size 2n.
int i=1, j=1;
L[n+1] = R[n+1] = MAXINT; // so that index does not
                          // fall over
for k = 1 to 2n do
        if L[i] < R[j] then
                A[k] = L[i]; i++;
        else
                A[k] = R[j]; j++;
end-for
```

Time complexity is O(n).

From Merging to Sorting

- How to use merging to finally sort?
- Using the divide and conquer principle
  - Divide the input array into two halves.
  - Sort each of them.
  - Merge the two sub-arrays. This is indeed procedure Merge.
- The algorithm can now be given as follows.

Analyzing Merge Sort

- Recurrence relation for merge sort as:

$$T(n) = 2T(n/2)+O(n).$$

  - This can be explained by the O(n) time for merge and
  - The two subproblems obtained during the divide step each take T(n/2) time.
  - Now use the general format for divide and conquer based algorithms.
- Solving this recurrence relation is done using say the substitution method giving us $T(n) = O(n \log n)$.
  - Look at previous examples.

Algorithm Merge Sort

```
Algorithm MergeSort(A)
begin
        mid = n/2; //divide step
        L = MergeSort(A[1..mid]);
        R = MergeSort(A[mid+1..n]);
        Merge(L, R); //combine step
end-Algorithm
```

Maxima of n Numbers

- A sequential program resembles the code below.

```
Program Maxima(A)
Begin
        int max = A[1];
        for i = 1 to n do
                if max < A[i]
                        max = A[i]
                End-if
        End-for
End
```

- How do we run this program in parallel?

Parallel Merge Sort

- Need to rethink on a parallel merge algorithm
- Start from the beginning.
  - We have two sorted arrays L and R.
  - Need to merge them into a single sorted array A.
- Define the rank of an element x in a sorted array A as the number of elements of A that are smaller than x.
- To merge L and R, need to know the rank of every element from L and R in the merged array L U R.

Parallel Merge Sort

- Now, consider an element x in L at index k.
- How many elements of L are smaller than x?
  - k-1.
- How many elements of R are smaller than x?
  - No easy answer, but
  - can do binary search for x in R and get the answer.
  - Say k' elements in R are smaller than x.

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Parallel Merge Sort

L = [8 10 12 27]   R = [15 17 24 32]

Element	8	10	12	27	15	17	24	32
Rank in L	0	1	2	3	3	3	3	4
Rank in R	0	0	0	3	0	1	2	3
Rank in L U R	0	1	2	6	3	4	5	7

L U R = [8 10 12 15 17 24 27 32]

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AS

Parallel Merge Sort

```
Algorithm ParallelMergeSort(A)
Begin
        mid = n/2; //divide step
        L = MergeSort(A[1..mid]);
        R = MergeSort(A[mid+1..n]);
        ParallelMerge(L, R); //combine step
 end-Algorithm
```

Parallel Merge Sort

- So, we just have to figure out a way to merge in parallel.

- Recall the merge algorithm as we developed it earlier.
  - Too many dependent tasks.
  - Not feasible in a parallel model.

```
for k = 1 to 2n do
        if L[i] < R[j] then
                A[k] = L[i]; i++;
        else A[k] = R[j]; j++;
end-for
```