LA - ASSIGNMENT-2

Q1

- Let $S = \{v_1, v_2, v_3, -- v_K\}$
- S is linearly independent set of vectors in vector space $V(F)$.
- β is vector such that $\beta \in V$ and $\beta \notin S$

$\Rightarrow$ PROOF - $\boxed{S_1 = S + \{\beta\}}$

- Let S_1 be linearly independent.
- Let $\beta \in S$. So, there exist scalar $\{a_1, a_2, -- a_K\} \in F$ such that $\beta = \sum_{i=1}^{K} a_i v_i$

So $\Rightarrow$ $S_1 = S \cup \{\beta\}$ is linearly dependent.

$\Rightarrow$ This is a contradiction to hypothesis that S_1 is linearly independent.

Hence $\beta \notin S$.

- Conversely, if $\beta \notin S$. Then we have to prove that S_1 is linearly independent. If possible, let us assume S_1 to be linearly dependent.

Now, there exists a finite non-empty subset 'X' of S_1 that is linearly dependent.

$\Rightarrow$ $X \subset S_1$; $S_1 = S \cup \{\beta\}$, S is linearly independent

- X has β & some vectors in $S = \{v_1, v_2, -- v_K\}$
- β is a linear combination of some vectors in S

So,

$\beta \in S$

Which contradicts to our assumption

So, S_1 is linearly independent

Q2

Let V be a vector space

$S \subset V$, where S is a subspace in V

For proving, it suffices to show that span(S) is closed under linear combination.

Suppose $a, b \in$ span(S)

α, β be constants

By definition of span, we have constants c_i & d_i such that

$$a = c_1 s_1 + c_2 s_2 + \cdots$$

$$b = d_1 s_1 + d_2 s_2 + \cdots$$

$$\alpha a + \beta b = \alpha(c_1 s_1 + c_2 s_2 + \cdots) + \beta(d_1 s_1 + d_2 s_2 + \cdots)$$

$$= (\alpha_1 c_1 + \beta d_1) s_1 + (\alpha c_2 + \beta d_2) s_2 + \cdots$$

This sum is a linear combination of elements of S & is thus in span(S). So span(S) is closed under linear combination & is the subspace in V as well

Q3 (a) $\begin{bmatrix} 12 & 4 & 4 \\ 5 & 4 & 5 \\ 2 & 3 & 4 \end{bmatrix}$

$\left[\begin{array}{l} R_2 = R_2 - \frac{5}{12} R_1 \\ R_3 = R_3 - \frac{1}{6} R_1 \end{array} \right] \Rightarrow$ we get $\begin{bmatrix} 12 & 4 & 4 \\ 0 & \frac{7}{3} & \frac{10}{3} \\ 0 & \frac{7}{3} & \frac{10}{3} \end{bmatrix}$

$R_3 = R_3 - R_2$

$\begin{bmatrix} 12 & 4 & 4 \\ 0 & 7/3 & 10/3 \\ 0 & 0 & 0 \end{bmatrix} \Leftarrow$ Echleon form

$$\begin{bmatrix} 12 & 4 & 4 \\ 0 & \frac{7}{3} & \frac{10}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\begin{array}{c} 12x_1 + 4x_2 + 4x_3 = 0 \\ \frac{7}{3}x_2 + \frac{10}{3}x_3 = 0 \end{array}}$$

Now let $\boxed{x_3 = y}$

$\frac{7}{3}x_2 + \frac{10}{3}y = 0$

$\boxed{x_2 = -\frac{10}{7}y}$

$12x_1 = -4(x_2 + x_3)$

$12x_1 = -4y\left(-\frac{10}{7} + 1\right)$

$\boxed{x_1 = \frac{y}{7}}$

$$\boxed{X = \begin{bmatrix} y/7 \\ -10/7y \\ y \end{bmatrix} = \begin{bmatrix} 1/7 \\ -10/7 \\ 1 \end{bmatrix}}$$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

$\left.\begin{array}{l} R_2 = R_2 - 6R_1 \\ R_3 = R_3 - 3R_1 \end{array}\right\} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -14 \\ 0 & -2 & -5 \end{bmatrix}$

$\left. R_3 = R_3 - \left(\frac{2}{7}\right)R_2 \right\} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -14 \\ 0 & 0 & -1 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -14 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 + 2x_2 + 3x_3 = 0$

$-7x_2 - 14x_3 = 0$

$1 \cdot x_3 = 0$

$\boxed{x_3 = 0}$

$-7x_2 + (-14)0 = 0$

$\boxed{x_2 = 0}$

$x_1 + 2(0) + 3(0) = 0$

$\boxed{x_1 = 0}$

so,

$X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 12 & 3 & 9 \\ 20 & 5 & 15 \\ 16 & 4 & 12 \end{bmatrix}$

$$\left.\begin{array}{l} R_2 = R_2 - \frac{5}{3} R_1 \\ R_3 = R_3 - \frac{4}{3} R_1 \end{array}\right\} \Rightarrow \begin{bmatrix} 12 & 3 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 12 & 3 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$12x_1 + 3x_2 + 9x_3 = 0$

let $x_2 = y$ & $x_3 = z$

$12x_1 + 3y + 9z = 0$

$x_1 = -\frac{1}{12}(3y + 9z)$

$x_1 = -\frac{y}{4} - \frac{3}{4}z$

$$X = \begin{bmatrix} -\frac{3z}{4} - \frac{y}{4} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 0 \\ 0 \end{bmatrix} y + \begin{bmatrix} -3/4 \\ 0 \\ 0 \end{bmatrix} z$$

This is the null space

Q4 Let V be vector space in field F

Given $\text{Basis}(W) = \{\alpha_i \,;\, i \in W\}$

where W is a subspace of V

$\text{Span}(W) = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_m\alpha_m$ (By definition of span)

where $\{c_i \,;\, i \in [1,m]\} \in F$

$\{\alpha_i \,;\, i \in [1,m]\} \in$ Basis of W

$\Rightarrow$ Now we have set $\{\alpha_i + \beta \,;\, i \in [m]\}$, $\beta = V/W$

We will calculate span(S) over it. $(V/W = V - W)$

$$S = c_1(\alpha_1+\beta) + c_2(\alpha_2+\beta) + \cdots + c_m(\alpha_m+\beta)$$

$$S = c_1\alpha_1 + \cancel{c_2\alpha_2}\, c_1\beta + c_2\alpha_2 + c_2\beta + \cdots + c_m\alpha_m + c_m\beta$$

$$S = (c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 \cdots + c_m\alpha_m) + (c_1\beta + c_2\beta + \cdots + c_m\beta)$$

$$S = (c_1\alpha_1 + c_2\alpha_2 + \cdots + c_m\alpha_m) + (c_1 + c_2 + \cdots + c_m)\beta$$

$\Rightarrow c_1 + c_2 + \cdots + c_m = \gamma$

where γ is a scalar, $\gamma \in F$

$$S = c_1\alpha_1 + c_2\alpha_2 + \dots + c_m\alpha_m + \gamma\beta$$

But know

$$\sum_{i=1}^{m} c_i\alpha_i = \text{Span}(\alpha)$$

$\gamma\beta$ is a vector in $V(F)$; $\gamma\beta = \alpha Z$ (αZ is a vector in $V(F)$)

So,

$$S = c_1\alpha_1 + c_2\alpha_2 + \dots + c_m\alpha_m + Z$$

- On Adding vector in span, dimensionality of span remains unchanged, it only shifts the span

- Therefore the dimensionality of span over $\{\alpha_i + \beta ; i \in [m]\}$ is same as that of span over $\{\alpha_i ; i \in [m]\}$

Q5 (a)

Linear transformations are represented in the form of matrices & the elements in this matrix belong to the field (F).

Number of elements in $F = p^n$ (given)

Since Vector space V is K dimensional
So matrix of linear transformation is of order $K \times K$ $\qquad T : V \to V$

$$K \text{ rows} \left\{ \underbrace{\begin{bmatrix} \vdots & & \\ \vdots & & \\ - & - & - \end{bmatrix}}_{K \text{ columns}} \right._{K \times K}$$

No. of linear transformations of $T : V \to V$ can be found by the fact that each element of matrix is uniquely represented

$$= (p^n)^{K^2}$$

- K^2 is no. of elements in matrix
- p^n is the no. of possibility for each element

(b). Now we already know about $(K \times K)$ matrix of linear transformation

- Here, for $(K \times K)$ matrix, we have to find no. of ways of forming it, such that there is NO dependent vectors

No. ways of selecting 1st Row $= {}^{p^n}C_K \cdot K!$ — ①

Let $y = {}^{p^n}C_K$

- Selecting 2nd row of matrix

$$= {}^{p^n}C_K K! - p^n \text{ --- } (1)$$

↳ excluding combinations of 1st row

$$= y - p^n \text{ --- } (2)$$

- Selecting 3rd Row

$$= {}^{p^n}C_K K! - p^n p^n \text{ --- } (3)$$

↳ Excludes linear combinations of row 1 & 2.

$$= y - p^{2n} \text{ --- } (3)$$

Selecting K^{th} Row

$$= {}^{p^n}C_K K! - p^{(K-1)n}$$

$$= y - p^{(K-1)n} \quad \text{—— (K)}$$

Multiply eq^n (1), (2), -- (K)
we get

$$= y(y-p^n)(y-p^{2n})(y-p^{3n}) \ldots (y-p^{(K-1)n})$$

On generalizing above eq^n, we get

$$\boxed{= y \prod_{i=1}^{K-1} (y - p^{in}) \quad \text{where } y = {}^{p^n}C_K K!}$$