LA Assignment - 3

Q1

(a) $x + 3y + 5z = 14$
$2x - y - 3z = 3$
$4x + 5y - z = 7$

Augmented Matrix $\begin{bmatrix} 1 & 3 & 5 & 14 \\ 2 & -1 & -3 & 3 \\ 4 & 5 & -1 & 7 \end{bmatrix}$

$$\left.\begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 4R_1 \end{matrix}\right\} \Rightarrow \begin{bmatrix} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 0 & -7 & -21 & -49 \end{bmatrix}$$

$$\left. R_3 = R_3 - R_2 \right\} \Rightarrow \begin{bmatrix} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 0 & 0 & -8 & -24 \end{bmatrix}$$

$x + 3y + 5z = 14$
$-7y - 13z = -25$
$-8z = -24$

$\boxed{z = 3}$

$-7y - 13(3) = -25$

$\boxed{y = -2}$

$x + 3(3) + 5(-2) = 14$

$\boxed{x = 5}$

(b) $y + z = 4$

$3x + 6y - 3z = 3$

$-2x - 3y + 7z = 10$

$$\text{Augmented Matrix} = \begin{bmatrix} 0 & 1 & 1 & 4 \\ 3 & 6 & -3 & 3 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

$$\left.\begin{matrix} R_1 \leftrightarrow R_2 \\ (\text{swap}) \end{matrix}\right\} \Rightarrow \begin{bmatrix} 3 & 6 & -3 & 3 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

$$R_1 = R_1/3 \Big\} \Rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

$$R_3 = R_3 + 2R_1 \Big\} \Rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{bmatrix}$$

$$R_3 = R_3 - R_2 \Big\} \Rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{bmatrix} \Rightarrow R_3 = R_3/8 \Rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$R_1 = R_1 - 2R_2 \Big\} \Rightarrow \begin{bmatrix} 1 & 0 & -3 & -7 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow R_1 = R_1 + 3R_3 \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$R_2 = R_2 - R_3 \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow \boxed{\begin{matrix} x = -1 \\ y = 2 \\ z = 2 \end{matrix}}$$

Q2

For the following to be a subspace, they should satisfy the following properties -

1. Zero Vector belongs to it
2. S is closed under vector addition
3. S is closed under scalar multiplication

① Absolutely summable sequences

x_i such that $\sum_{i=1}^{\infty} |x_i| < \infty$

- zero vector

$\{0, 0, 0, ... 0\}$ as $\sum_{i=1}^{\infty} (0) = 0$

- Addition

Let $A \in S$ & $\sum_{i=1}^{\infty} |(a_i)| < k_A$ where $a_i \in A$

$B \in S$ & $\sum_{i=1}^{\infty} |b_i| < k_B$ where $b_i \in B$

Now $C = A + B$

$C = \{a_i, a_2, a_3, ... \}$
$+ \{b_1, \cancel{+}b_2, b_3, ... \}$

$C = \{a_1 + b_1, a_2 + b_2, ... \}$

$|a+b| \leq |a| + |b|$, $a, b \in \mathbb{R}$

$\sum_{i=1}^{\infty} |c_i| = |a_1 + b_1| + |a_2 + b_2| + ...$

$\leq |a_1| + |a_2| + \dots + |b_1| + |b_2| + \dots$

$\leq K_A + K_B$

$\therefore$ C is closed under addition

- Let $A \in S$, $\sum_{i=1}^{\infty} |a_i| \leq K_A$, $a_i \in A$

$KA = \{ka_1, ka_2, ka_3, \dots\}$ $K \in \mathbb{R}$

we already know that $|a.b| \leq |a|.|b|$

$KA = |Ka_1| + |Ka_2| + |Ka_3| + \dots$

$\leq |K||a_1| + |K||a_2| + |K||a_3| + \dots$

$\leq |K| (|a_1| + |a_2| + |a_3| + \dots)$

$\leq |K| K_A$

So its a subspace

② Bounded Sequences

- Zero Vector

For $A = \{0, 0, \dots\}$

$a_i \leq 1 \ \forall a_i \in A$ is bounded

- Addition

let $A \in S$, $|a_i| \in M_A \; \forall a_i \in A$

$B \in S$, $|b_i| \in M_B \; \forall b_i \in B$

$C = A + B$

$= \{(a_1+b_1), (a_2+b_2), (a_3+b_3), \ldots\}$

$C_i = a_i + b_i$

$|C_i| = |a_i + b_i|$

$|C_i| \leq |a_i| + |b_i|$

$|C_i| \leq M_A + M_B$, So addition is closed

- Scalar Multiplication

let $A \in S$, $|a_i| \leq M \; \forall a_i \in A$

Now, $K \cdot A = \{Ka_1, Ka_2, \ldots\}$ K = any scalar

$|Ka_i| \leq |K||a_i| \; \forall a_i \in A$

$\Rightarrow$ KA is bounded & closed under scalar multiplication

$\therefore$ Its a subspace

(c) Arithmetic Sequences

$x_i = a + di$, a, d = parameters of arithmetic sequence

- Zero vector – it is an arithmetic sequence

$a = 0, d = 0$ $\{0, 0, \ldots 0\}$; $a = d = 0$

- Addition

let $A, B \in S$

$a_i = a + di$ $a, d \rightarrow$ parameters of arithmetic

$b_i = b + e_i$ b, e sequence formula

$$A + B = \{(a + d_1 + b + e_1), (a + 2d + b + 2e), \ldots\}$$

$$= \{(a+b) + (d+e), (a+b) + 2(d+e), \ldots\}$$

So, $A + B$ is an arithmetic sequence.

- Scalar Multiplication

$$KA = \{Ka + Kd, Ka + 2Kd, Ka + 3Kd, \ldots\}$$

we can see KA is arithmetic sequence with first value $= Ka$ & common difference $= Kd$

So its a subspace

(d) Geometric Sequences

(x_i such that $x_i = a r^i$, a, r are fixed)

It is not a subspace can be shown by following exam

$A = \{2, 4, 8, \dots\}$ $a = 2, r = 2$

$B = \{1, 1, 1 \dots\}$ $a = 1, r = 1$

$A + B = \{3, 5, 9, \dots\}$

In $A + B$ $\frac{5}{3} \neq \frac{9}{5}$ so its geometric sequence.

So its now a subspace

Q3

Let $f, g, h \in V$

We already know, • sum of 2 continous functions is continous

• Scalar multiplication of a continous functions is continous.

$\Rightarrow$ Addition & scalar multiplication is closed.

Now for all x such that $a \leq x \leq b$, we have from definition & commutative law of real number :-

• $$\left.\begin{aligned} (f+g)x &= f(x)+g(x) \\ &= g(x)+f(x) \\ &= (g+f)x \end{aligned}\right\}$$ As this holds true for all x, Hence conclude –

$f+g = g+f$

which is commutative law of vector addition

•

$$((f+g)+h)(x) = (f+g)x + h(x)$$
$$= [f(x)+g(x)] + h(x)$$
$$= f(x) + g(x) + h(x)$$
$$= (f+(g+h))(x)$$

$\Rightarrow$ So, we conclude that $(f+g)+h = f+(g+h)$ which is associative law

• Let 0 be a constant function with value 0, then for any $f \in V$ we have

$$(f+0)(x) = f(x) + 0 = f(x)$$

$$f+0=f$$

which is the additive identity law.

- Also $(-f)(x) = -(f(x))$

$$(f+(-f))(x) = f(x) - f(x) = 0$$

from which we get that $f+(-f)=0$

which is Additive inverse law.

- $c, d \in \mathbb{R}$

$$c(f+g)(x) = c(f(x)+g(x)) = cf(x)+cg(x)$$
$$= (cf+cg)(x)$$

which is distribute law

- $$((cd)f)(x) = (cd)f(x)$$
$$= c(df(x)) = (c(df))(x)$$

So, this is scalar associative law

Q4

$$f(x)=x, \quad g(x)=e^{x}, \quad h(x)=e^{-x}, \quad x \in [0,1]$$

Let f, g, h be linearly dependent, so there exist a_i $(i=\{1,2,3\})$ not all zero such that

$$a_1 x + a_2 e^{x} + a_3 e^{-x} = 0 \quad \text{for all } x \in [0,1]$$

for $x = 0$, we have $a_2 = -a_3$ — ①

for $x = 1$, we have, $a_1 + a_2 e + \frac{a_3}{e} = 0$ — ②

for $x = 0.5$, we have, $0.5a_1 + \sqrt{e}\, a_2 + \frac{a_3}{e} = 0$ — ③

Put $a_2 = -a_3$ in ② & ③

- $a_1 + a_2\left(e - \frac{1}{e}\right) = 0$

  $a_1 = -a_2\left(e - \frac{1}{e}\right)$

- $0.5a_1 + a_2\left(\sqrt{e} - \frac{1}{\sqrt{e}}\right) = 0$

  $-\frac{a_2}{2}\left(e - \frac{1}{e}\right) + a_2\left(\sqrt{e} - \frac{1}{\sqrt{e}}\right) = 0$

  $a_2 = 0, \; a_3 = 0, \; a_1 = 0$

So, f, g, h are linearly independent