Fibonacci Heaps

Fibonacci Heaps

- Binomial heaps support the mergeable heap operations (INSERT, MINIMUM, EXTRACT_MIN, UNION plus, DECREASE_KEY and DELETE) in $O(\lg n)$ worst-case time.
- Fibonacci heaps support the mergeable heap operations that do not involve deleting an element in $O(1)$ amortized time.

Fibonacci Heaps

- Fibonacci heaps are especially desirable when the number of EXTRACT-MIN and DELETE operations is small relative to the number of other operations.
- Fibonacci heaps are loosely based on binomial heaps.
- A collection of trees if neither DECREASE-KEY nor DELETE is ever invoked.
- Each tree is like a binomial tree.

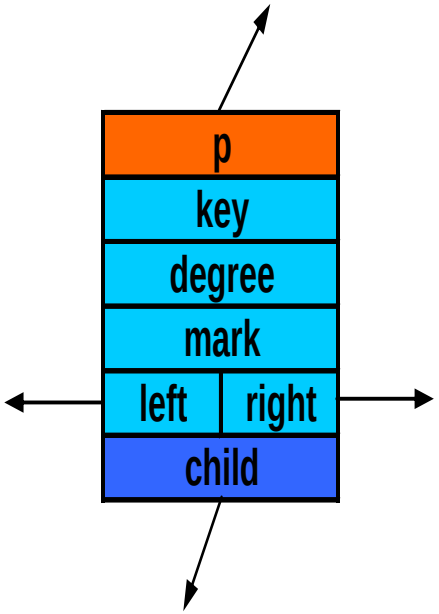
Fibonacci Heaps

- Fibonacci heaps differ from binomial-heaps, however, in that they have more more relaxed structure allowing for improved asymptotic time bounds work that maintains the structure is delayed until it is convenient to perform.
- Like a binomial heap, a fibonacci heap is a collection of heap-ordered trees however, trees are not constrained to be binomial trees.
- Trees within fibonacci heaps are rooted but unordered.

Structure of Fibonacci Heaps

Each node *x* contains:

- A pointer *p[x]* to its parent
- A pointer *child[x]* to one of its children
  - The children of *x* are linked together in a circular, doubly-linked list which is called the child-list.



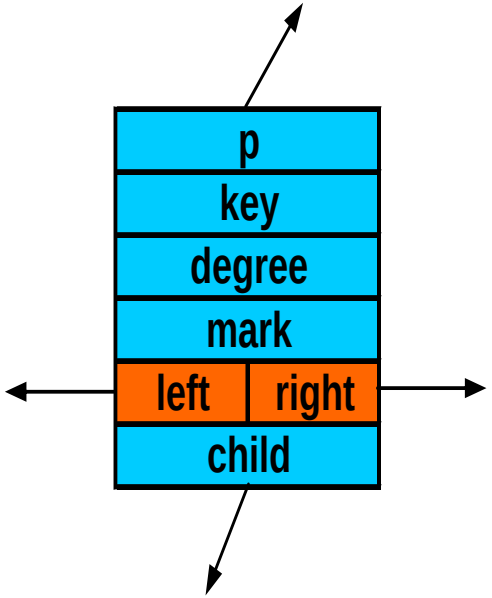
Structure of Fibonacci Heaps

- Each child *y* in a child list has pointers *left[y]* & *right[y]* that point to *y*'s left & right siblings respectively.
- If *y* is an only child, then

$$left[y] = right[y] = y.$$

Structure of Fibonacci Heaps

The roots of all trees are also linked together using their *left & right* pointers into a circular, doubly-linked list which is called the root list.



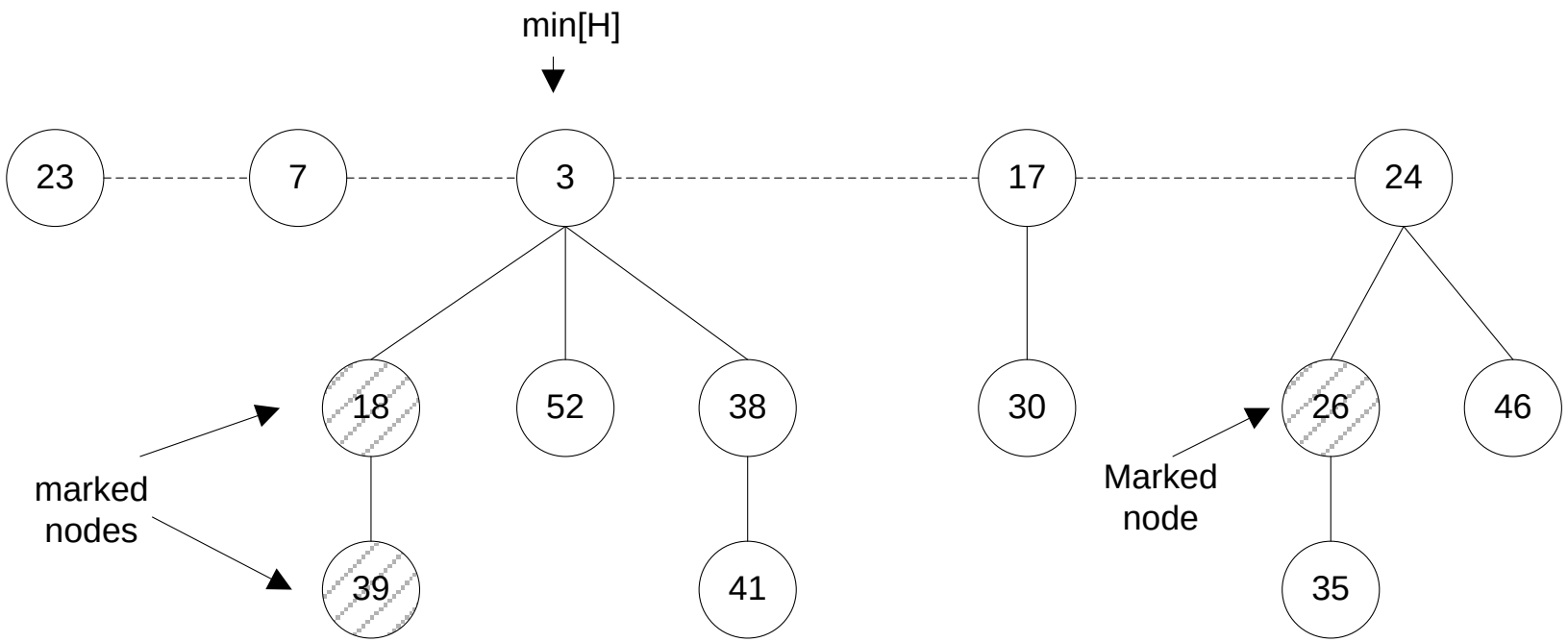
Structure of Fibonacci Heaps

- Circular, doubly-linked lists have two advantages for use in fib-heaps:
  - we can remove a node in *O(1)* time
  - given two such lists, we can concatenate them in *O(1)* time.

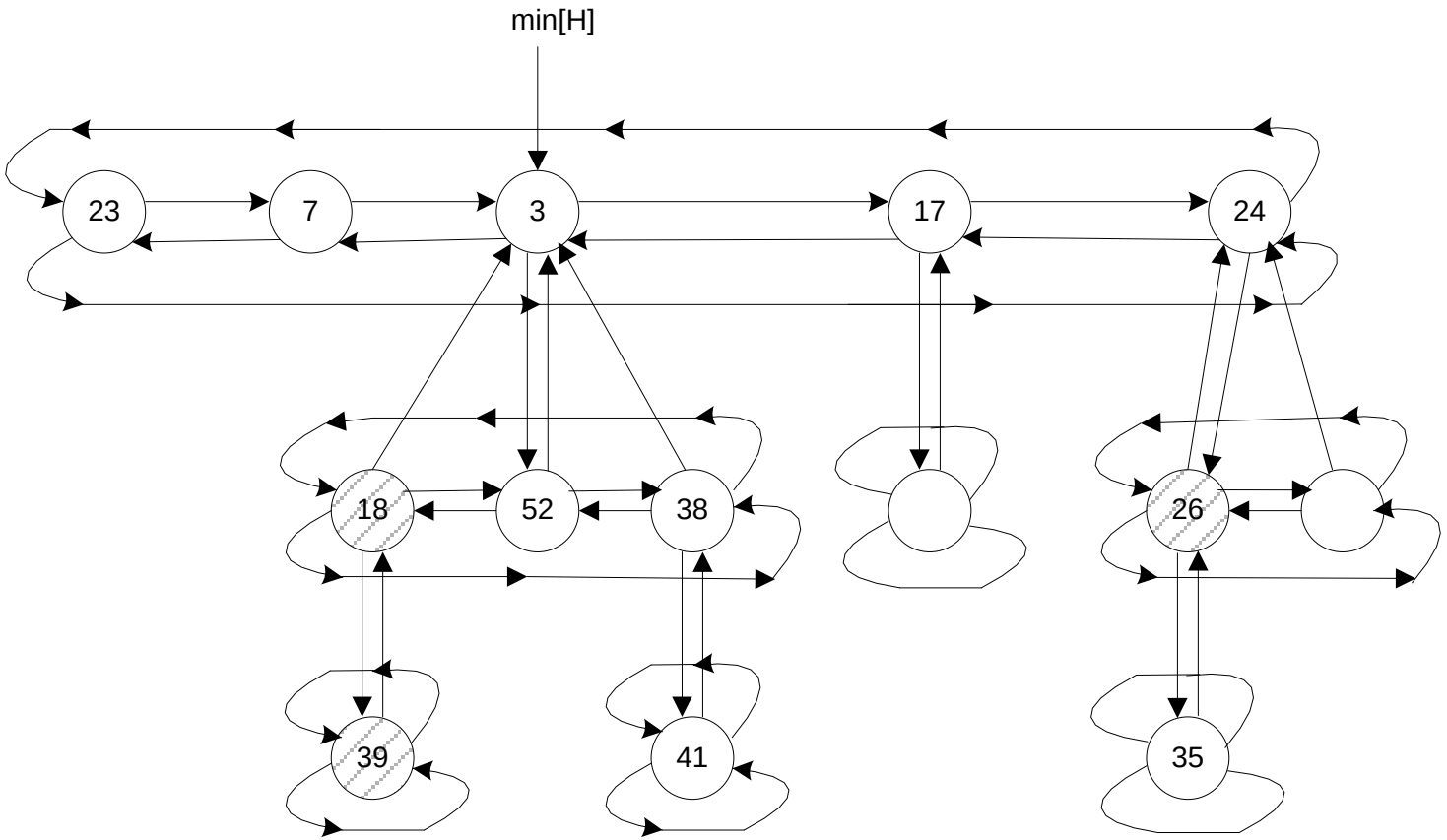
Structure of Fibonacci Heaps

- Two other fields in each node *x*
  - *degreee[x]:* the number of children in the child list of *x*
  - *mark[x]:* a boolean-valued field
    - indicates whether node *x* has lost a child since the last time *x* was made the child of another one
    - newly created nodes are unmarked
    - A node *x* becomes unmarked whenever it is made the child of another node

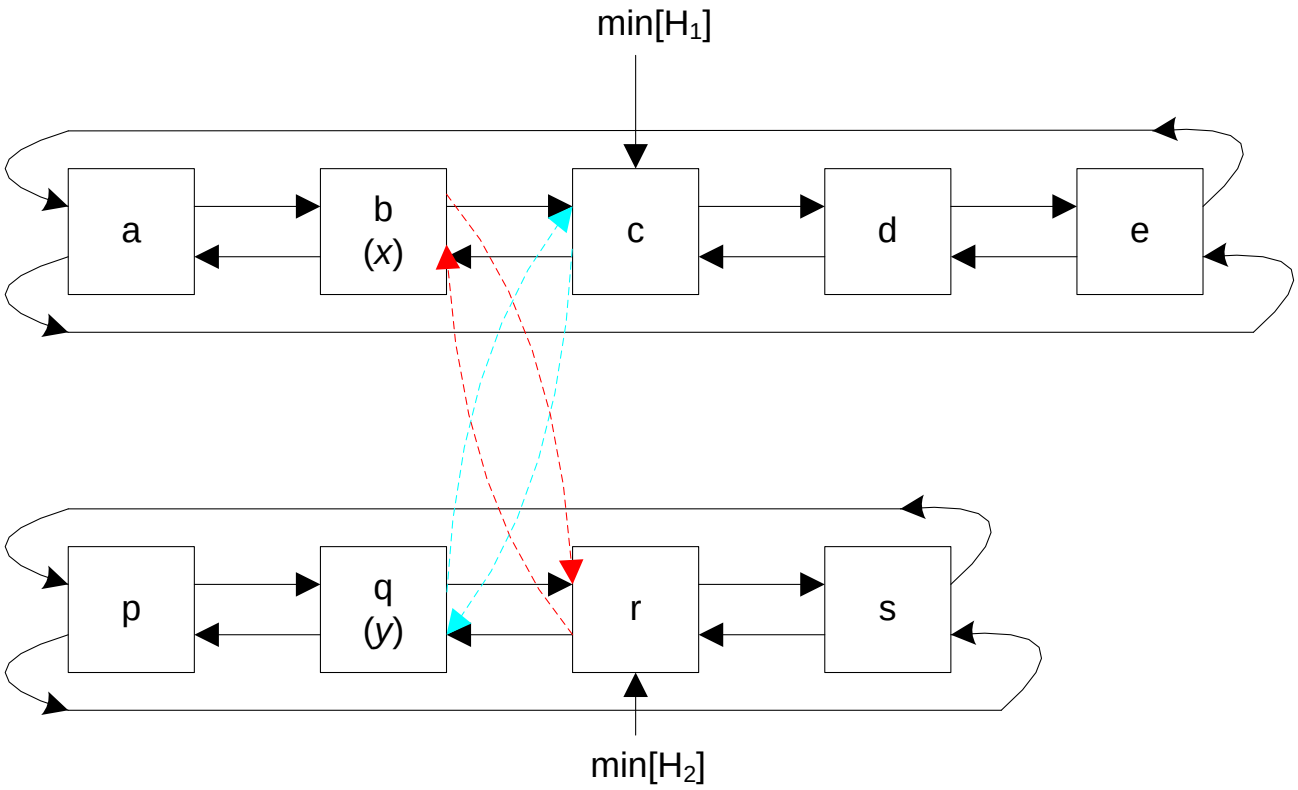
Structure of Fibonacci Heaps



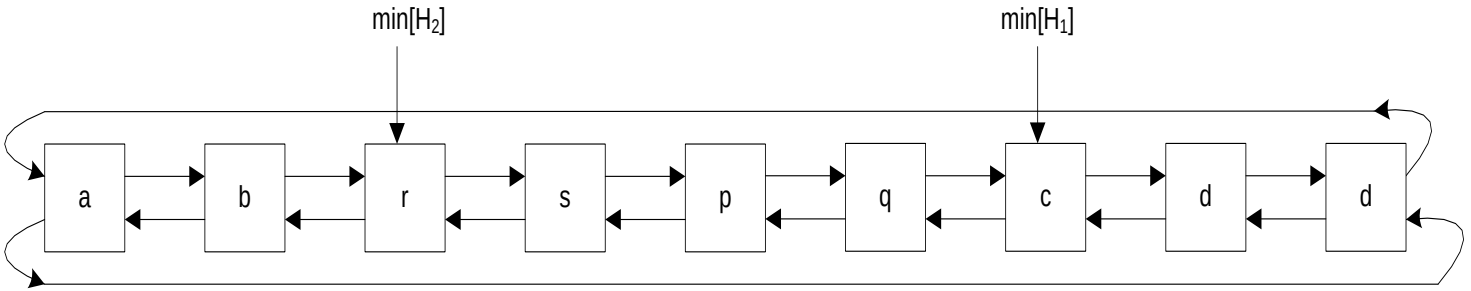
Structure of Fibonacci Heaps



Concetenation of Two Circular, Doubly – Linked Lists



Concetenation of Two Circular, Doubly – Linked Lists



Concetenation of Two Circular, Doubly – Linked Lists

CONCATENATE (H_1, H_2)

$x \leftarrow$ left[min[H_1]]

$y \leftarrow$ left[min[H_2]]

right[x] $\leftarrow$ min[H_2]

left[min[H_2]] $\leftarrow x$

right[y] $\leftarrow$ min[H_1]

left[min[H_1]] $\leftarrow y$

end

Running time is O(*1*)

Potential Function

- A given fibonacci heap H
  - $t(H)$: the number of trees in root list of H
  - $m(H)$: the number of marked nodes in H
- The potential of fibonacci heap H is:

$$\Phi(H) = t(H) + 2\, m(H)$$

- A fibonacci heap application begins with an empty heap:

the initial potential = 0

- The potential is non-negative at all subsequent times.

Maximum Degree

- We will assume that there is aknown upper bound $D(n)$ on the maximum degree of any node in an n node heap
- If only mergeable-heap operations are supported

$$D(n) = \lfloor \lg n \rfloor$$

- If decrease key & delete operations are supported

  $D(n) = O(\lg n)$

Mergeable Heap Operations

MAKE-HEAP, INSERT, MINIMUM, EXTRACT-MIN, UNION

If only these operations are to be supported, each fibonacci-heap is a collection of unordered binomial trees.

Mergeable Heap Operations

- An unordered binomial tree U_k
  - is like a binomial tree
  - defined recursively:
    - U_0 consists of a single node
    - U_k consists of two U_{k-1}'s for which the root of one is made into any child of the root of the other

Mergeable Heap Operations

- Lemma which gives properties of binomial trees holds for unordered binomial trees as well but with the following variation on property 4
- Property 4': For the unordered binomial tree U_k:
  - The root has degree k > the degree of any other node
  - The children of the root are the roots of subtrees

    $U_0, U_1, \ldots\ldots\ldots, U_{k-1}$ in some order

Mergeable Heap Operations

- The key idea in the mergeable heap operations on fibonacci heaps is to delay work as long as possible.
- Performance trade-off among implementations of the various operations:
  - If the number of trees is small we can quickly determine the new min node during EXTRACT-MIN
  - However we pay a price for ensuring that the number of trees is small

Mergeable Heap Operations

- However we pay a price for ensuring that the number of trees is small
- However it can take up to $\Omega(\lg n)$ time
  - to insert a node into a binomial heap
  - or to unite two binomial heaps
- We do not consolidate trees in a fibonacci heap when we insert a new node or unite two heaps
- We delay the consolidation for the EXTRACT-MIN operation when we really need to find the new minimum node.

Mergeable Heap Operations

Creating a new fibonacci heap:

MAKE-FIB-HEAP procedure

- allocates and returns the fibonacci heap object *H*
- Where *n[H]* = 0 and min[H] = NIL
- There are no trees in the heap

because *t(H)* = 0 and *m(H)* = 0 => *Φ(H)* = 0

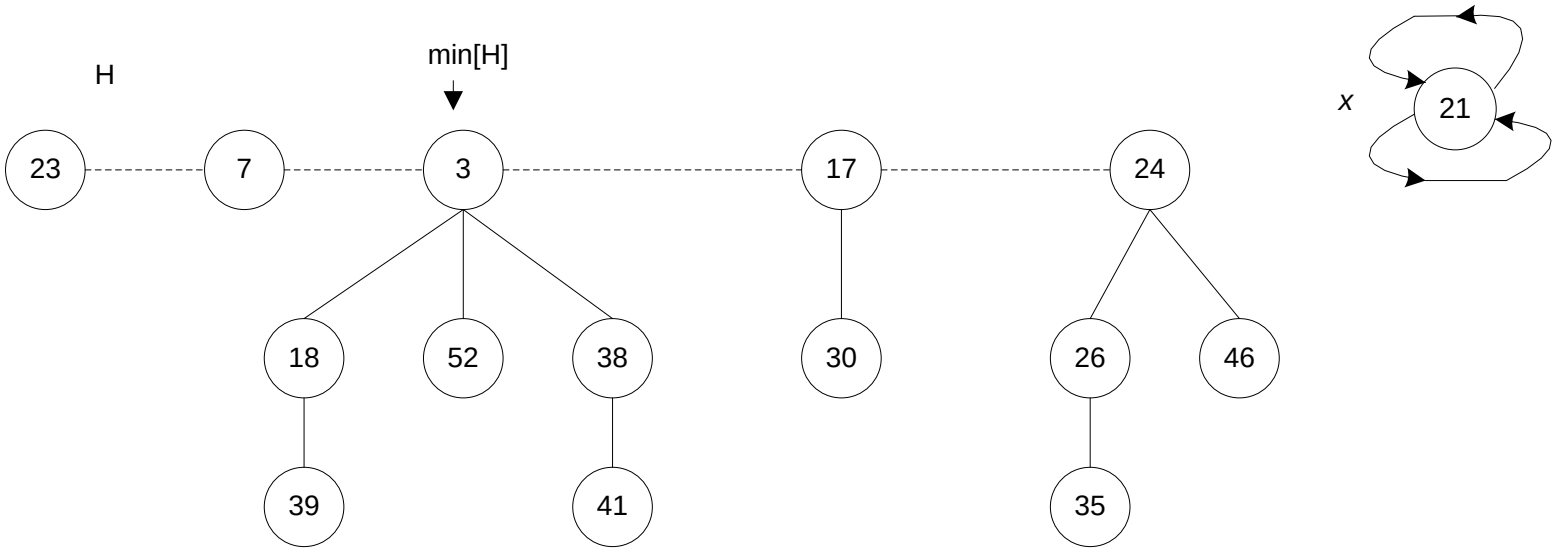
the amortized cost = *O(1)* = the actual cost

Mergeable Heap Operations

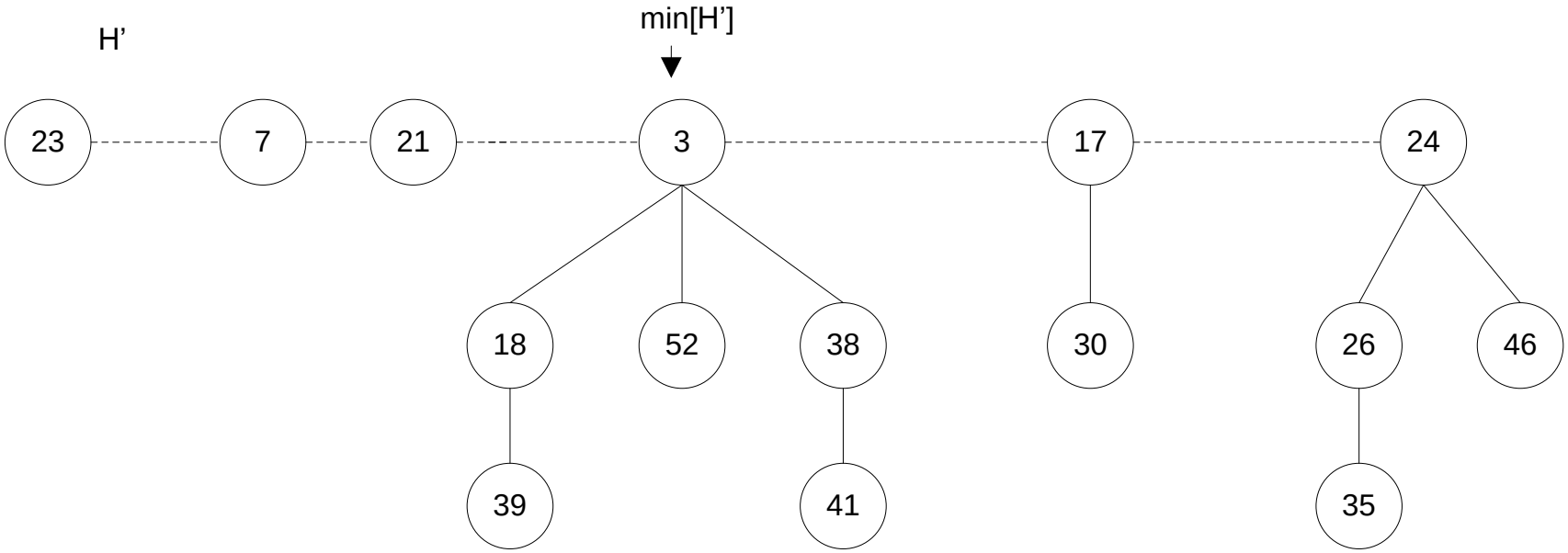
Inserting a node

```
FIB-HEAP-INSERT(H, x)
    degree[x] ← 0
    p[x] ← NIL
    child[x] ← NIL
    left[x] ← x
    right[x] ← x
    mark[x] ← FALSE
    concatenate the root list containing x with root list H
    if key[x] < key[min[H]] then
            min[H] ← x
    endif
    n[H] ← n[H] + 1
end
```

Mergeable Heap Operations



Mergeable Heap Operations



Mergeable Heap Operations

$t(H') = t(H) + 1$

- Increase in potential:

$$\Phi(H') - \Phi(H) = [t(H) + 1 + 2m(H)] - [t(H) + 2m(H)]$$

$$= 1$$

The actual cost = $O(1)$

The amortized cost = $O(1) + 1 = O(1)$

Mergeable Heap Operations

Finding the minimum node:

Given by pointer min[*H*]

actual cost = *O(1)*

amortized cost = actual cost = *O(1)*

since the potential of *H* does not change

Uniting Two Fibonacci Heaps

FIB-HEAP-UNION(H_1, H_2)
    H = MAKE-FIB-HEAP()
    if key[min[H_1]] ≤ key[min[H_2]] then
        min[H] ← min[H_1]
    else
        min[H] ← min[H_2]
    endif
    concatenate the root lists of H_1 and H_2
    n[H] ← n[H_1] + n[H_2]
    Free the objects H_1 and H_2
    return H
end

Uniting Two Fibonacci Heaps

- No consolidation of trees
- Actual cost = $O(1)$
- Change in potential

$$\Phi(H) - (\Phi(H_1) + \Phi(H_2)) =$$
$$= (t(H) + 2m(H)) - ((t(H_1) + 2m(H_1)) + (t(H_2) + 2m(H_2)))$$
$$= 0 \text{ since } t(H) = t(H_1) + t(H_2)$$
$$m(H) = m(H_1) + m(H_2)$$

Therefore amortized cost = actual cost = $O(1)$

Extracting the Minimum Node

The most complicated operation the delayed work of consolidating the trees in the root list occurs

```
FIB-HEAP-EXTRACT-MIN(H)
    z = min[H]
    for each child x of z
            add x to the root list of H
            p[x] ← NIL
    endfor
    remove z from the root list of H
    min[H] ← right[z]
    CONSOLIDATE(H)
end
```

Extracting the Minimum Node

- Repeatedly execute the following steps until every root in the root list has a distinct degree value

(1) Find two roots x and y in the root list with the same degree

where key[x] ≤ key[y]

(2) Link y to x : Remove y from the root list and make y a child of x

This operation is performed by procedure FIB-HEAP-LINK

Procedure CONSOLIDATE uses an auxiliary pointer array A[0.....*D(n)*]

A[i] = y : y is currently a root with degree[y] = i

Extracting the Minimum Node

```
CONSOLIDATE(H)
    for i← 0 to D(n) do
            A[i] ← N IL
    endfor
    for each node w in the root list of H do
            x ← w
            d ← degree[x]
            while A[d] ≠ NIL do
                        y ← A[d]
                        if key[x] > key[y] then
                                    exchange x ↔ y
                        endif
                        FIB-HEAP-LINK(H,y,x)
                        A[d] ← NIL
                        d ← d + 1
            endwhile
            A[d] ← x
    endfor
```

```
min[H] ← +∞
for i ← 0 to D(n[H]) do
if A[i] ≠ NIL then
            add A[i] to the root list of H
            if key[A[i]] < key[min[H]] then
                        min[H] ← A[i]
            endif
            endif
endfor
end
```

Extracting the Minimum Node

FIB-HEAP-LINK(H,*y*,*x*)

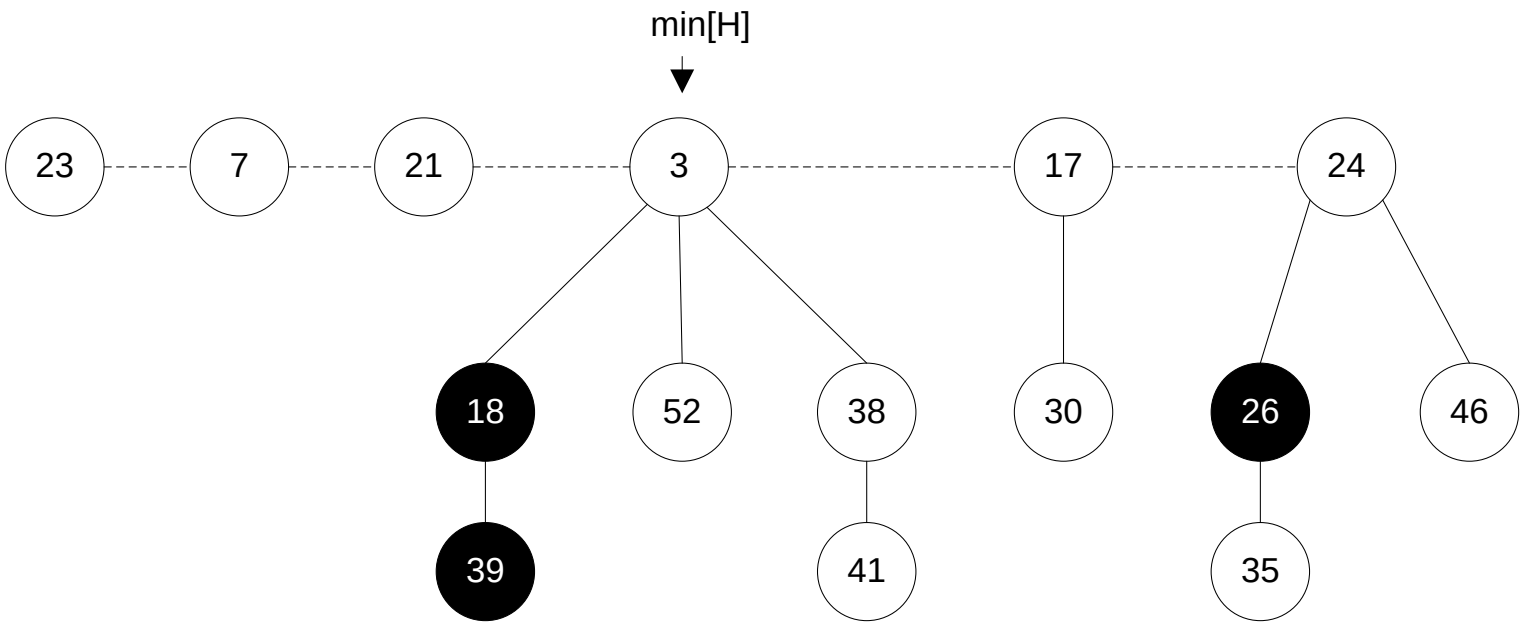
remove *y* from the root list of H

make *y* a child of *x*, incrementing degree[*x*]

mark[*y*] ← FALSE

end

Extracting the Minimum Node



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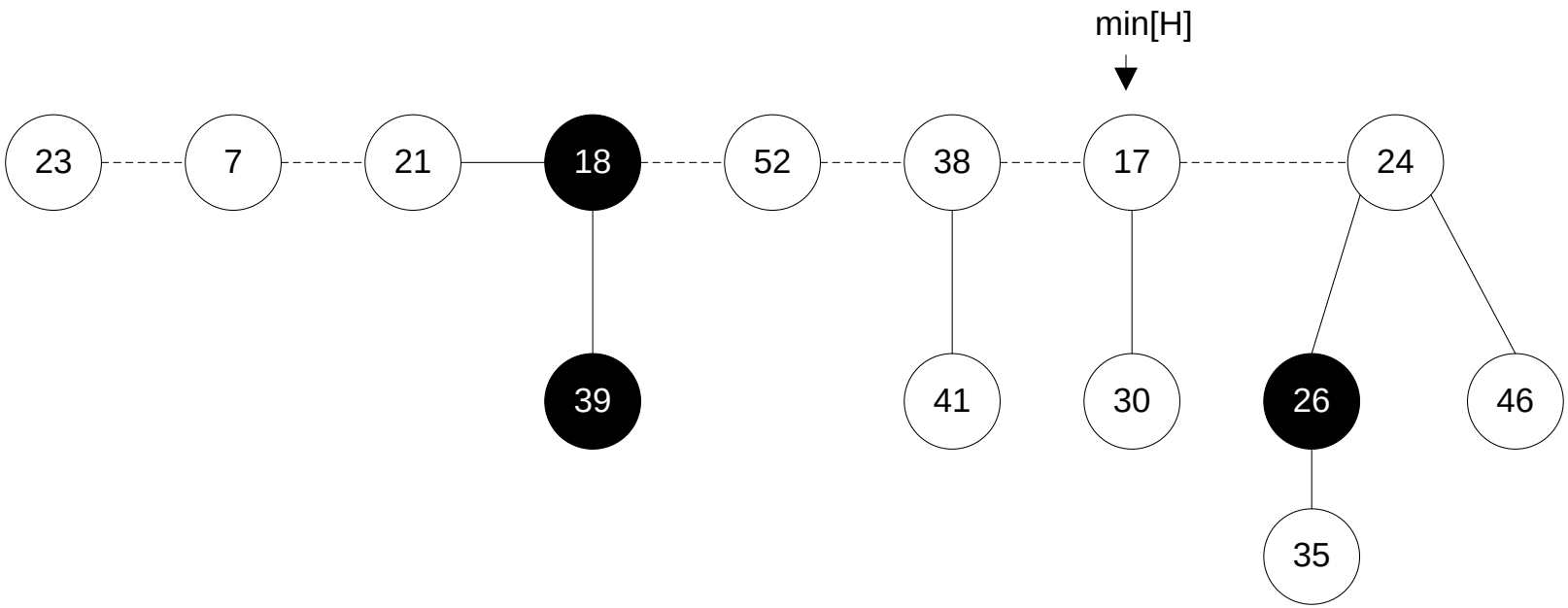

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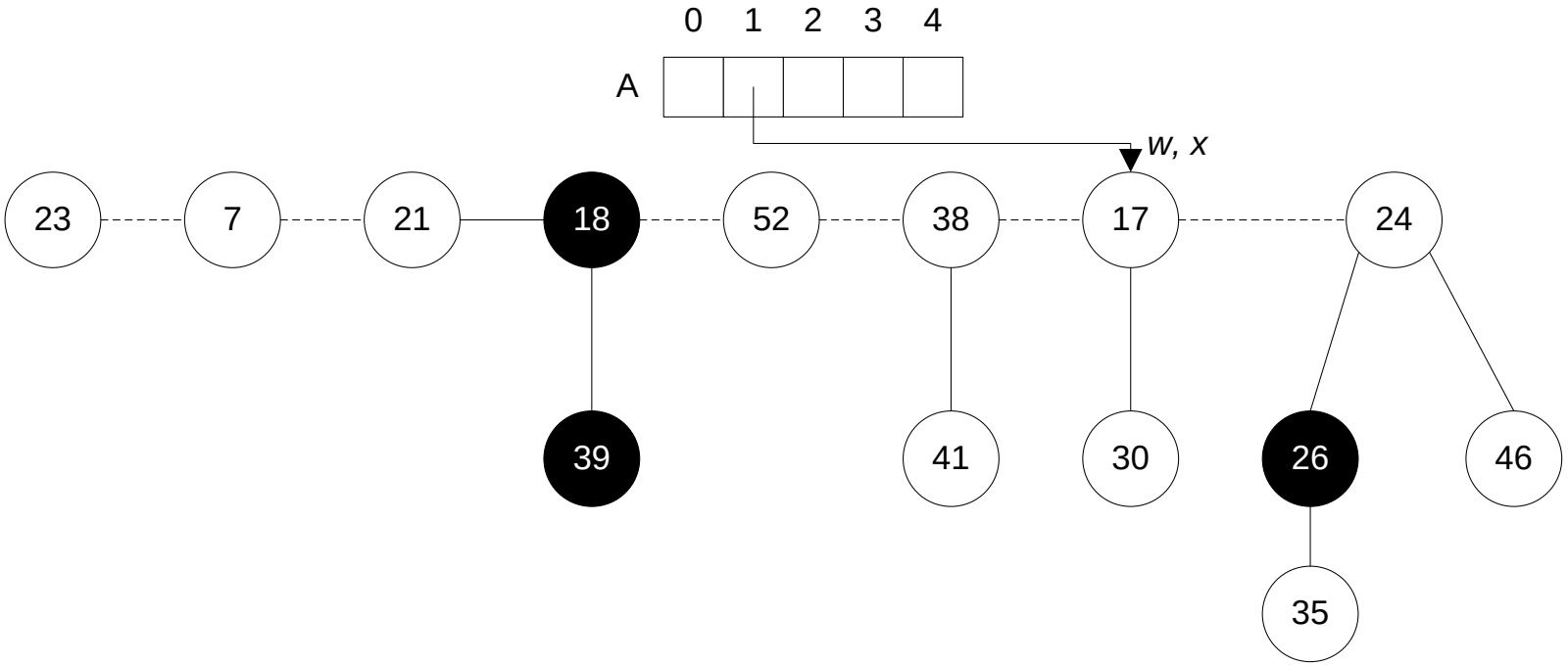

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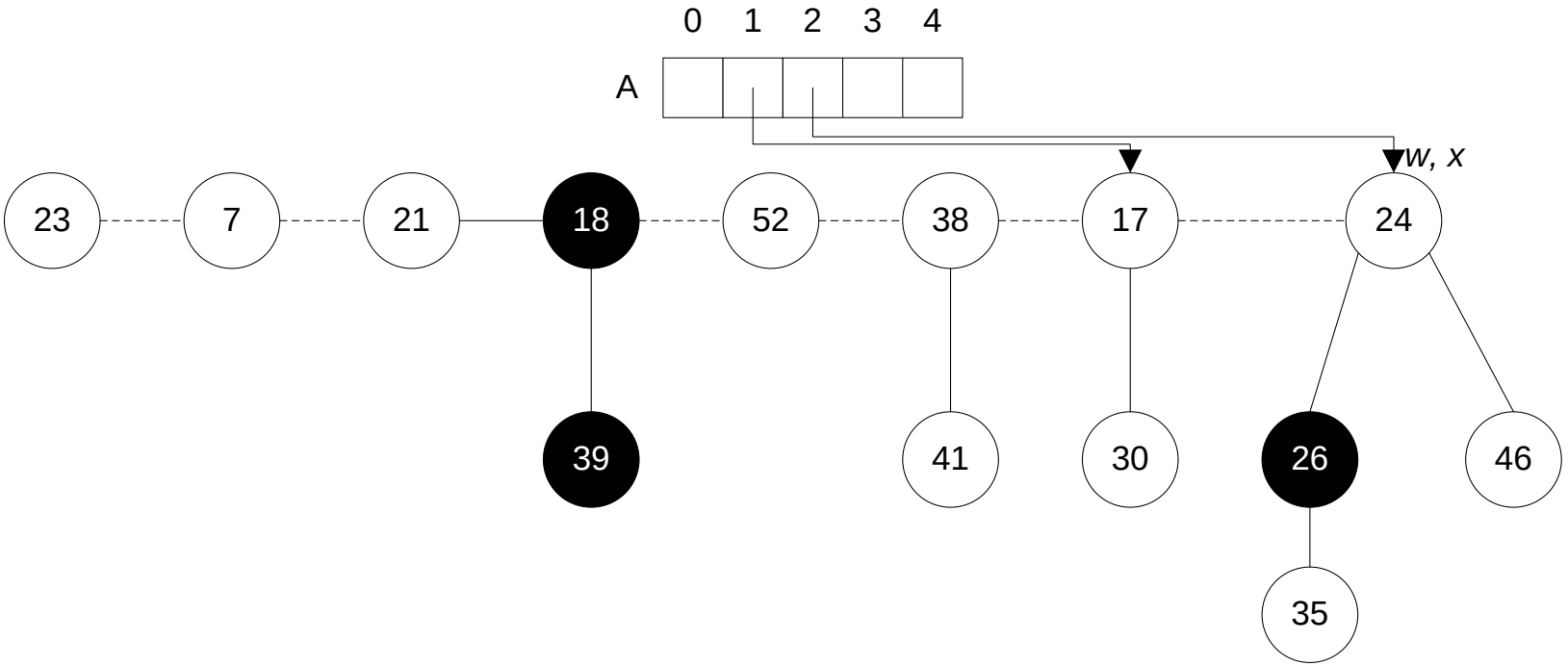

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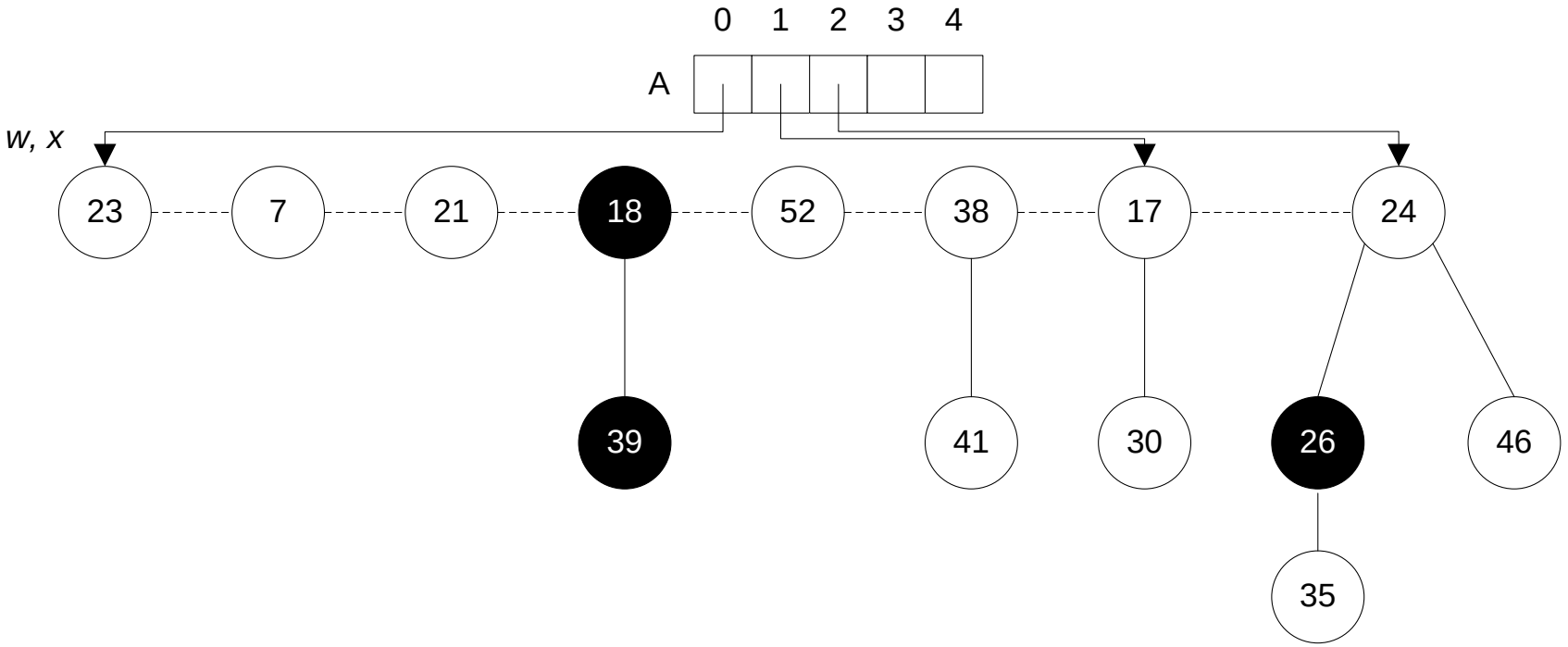

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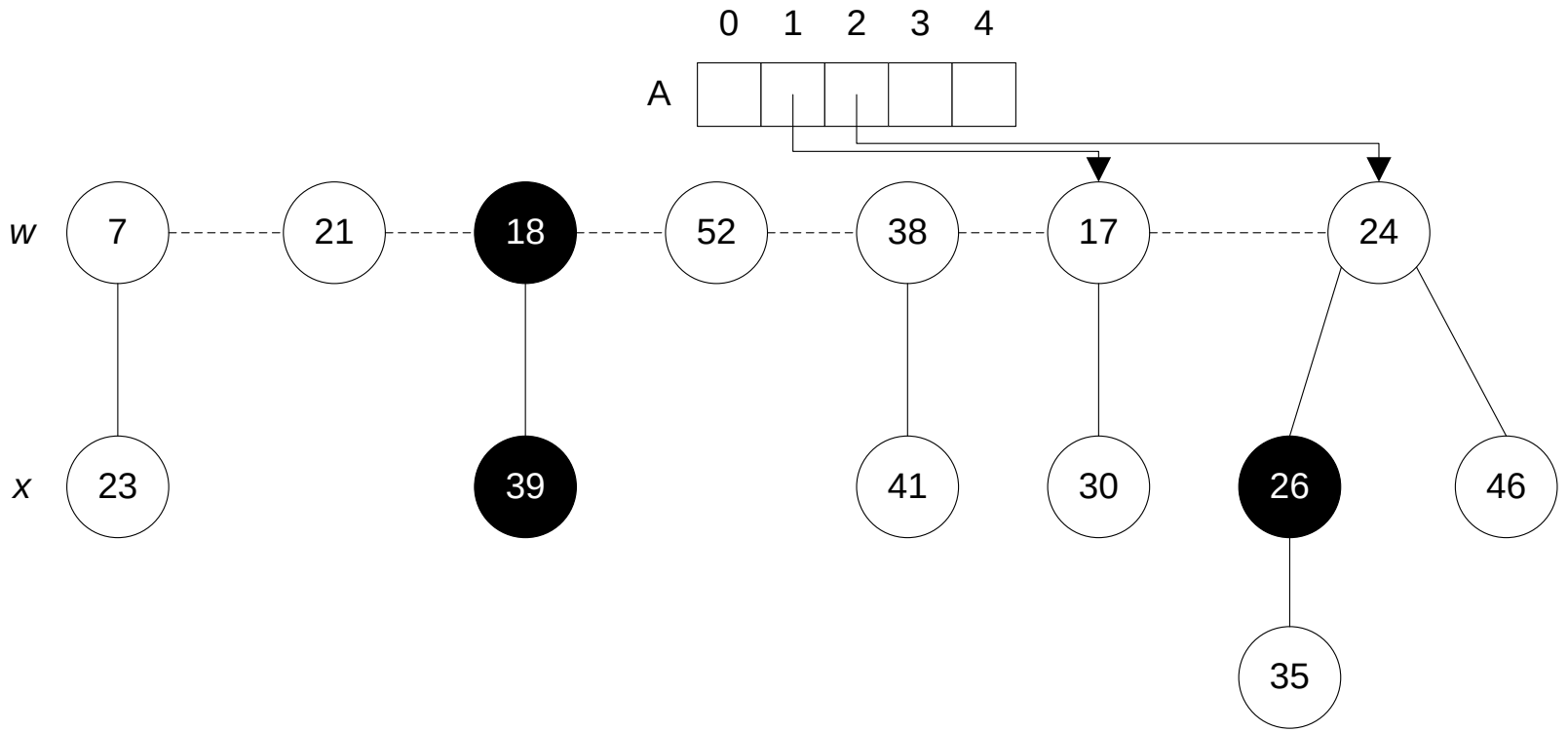

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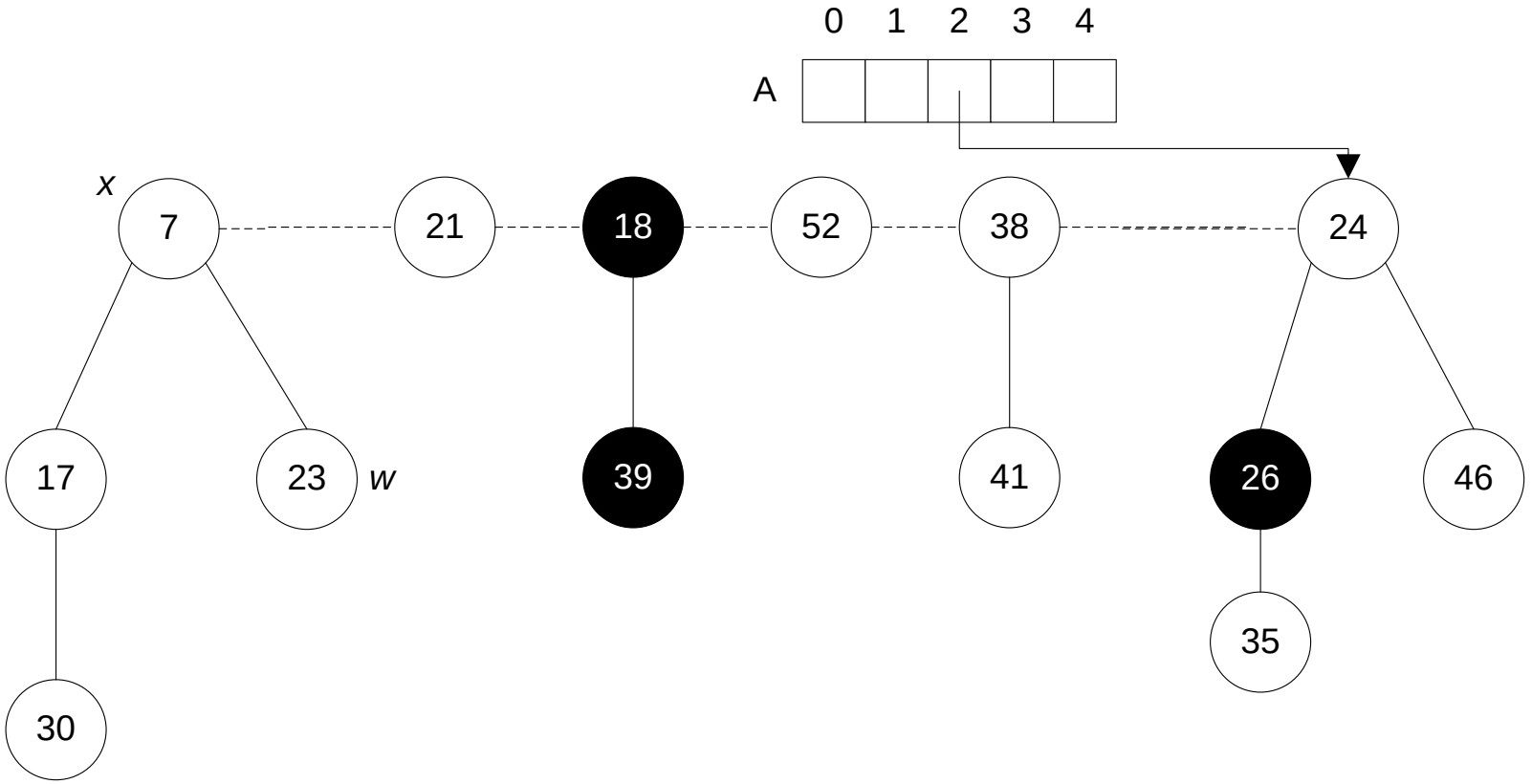

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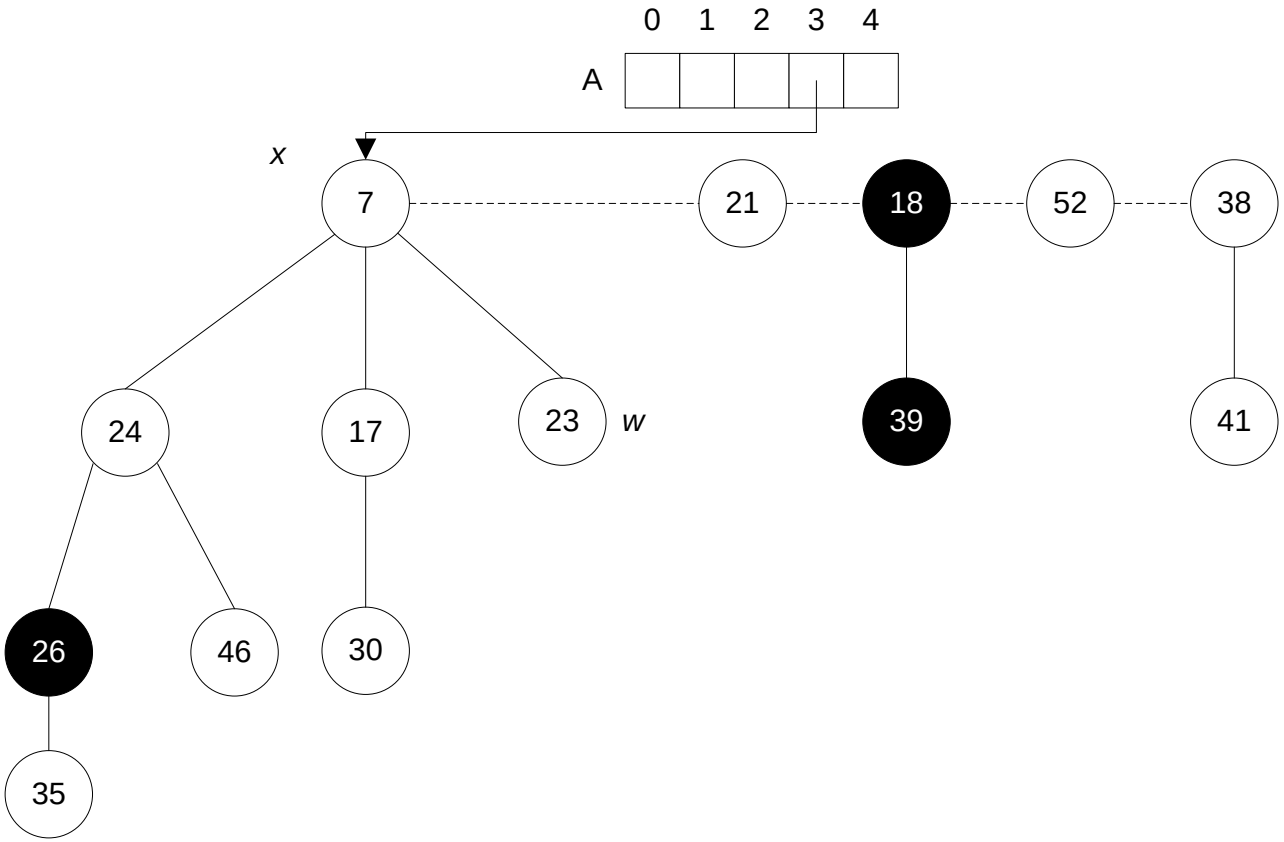

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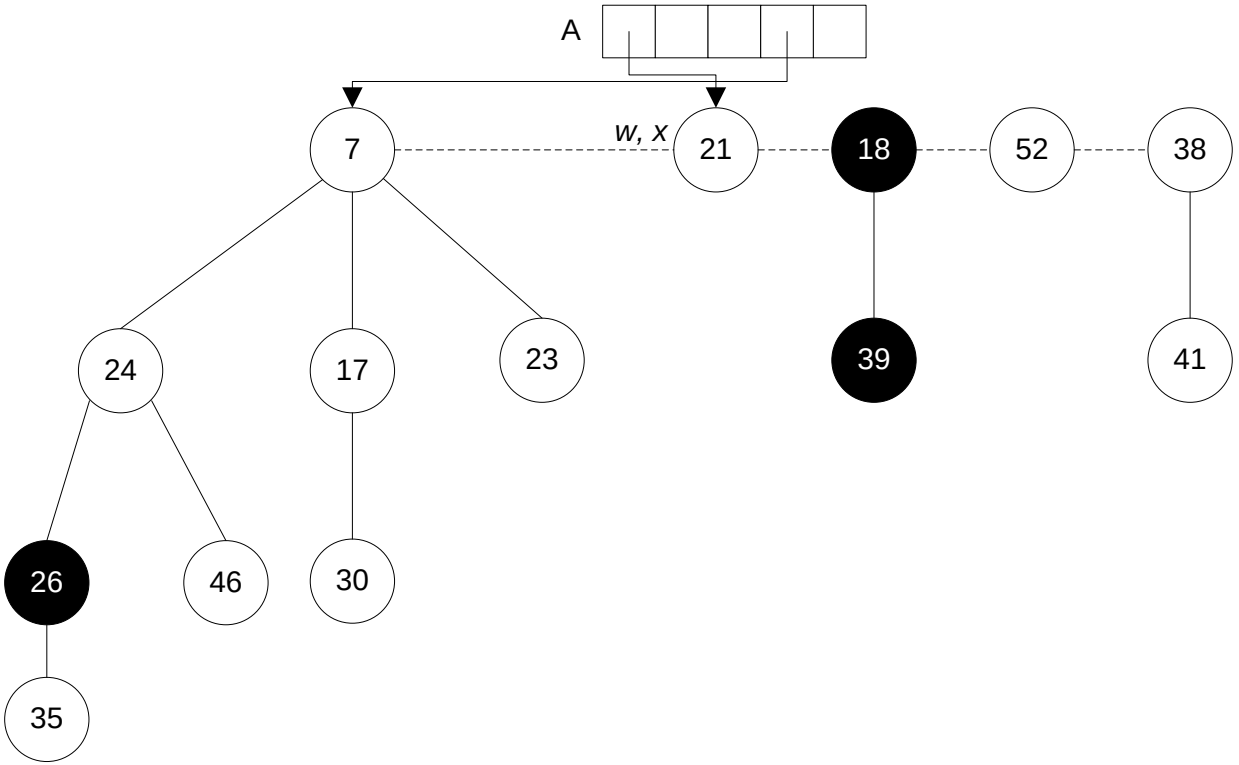

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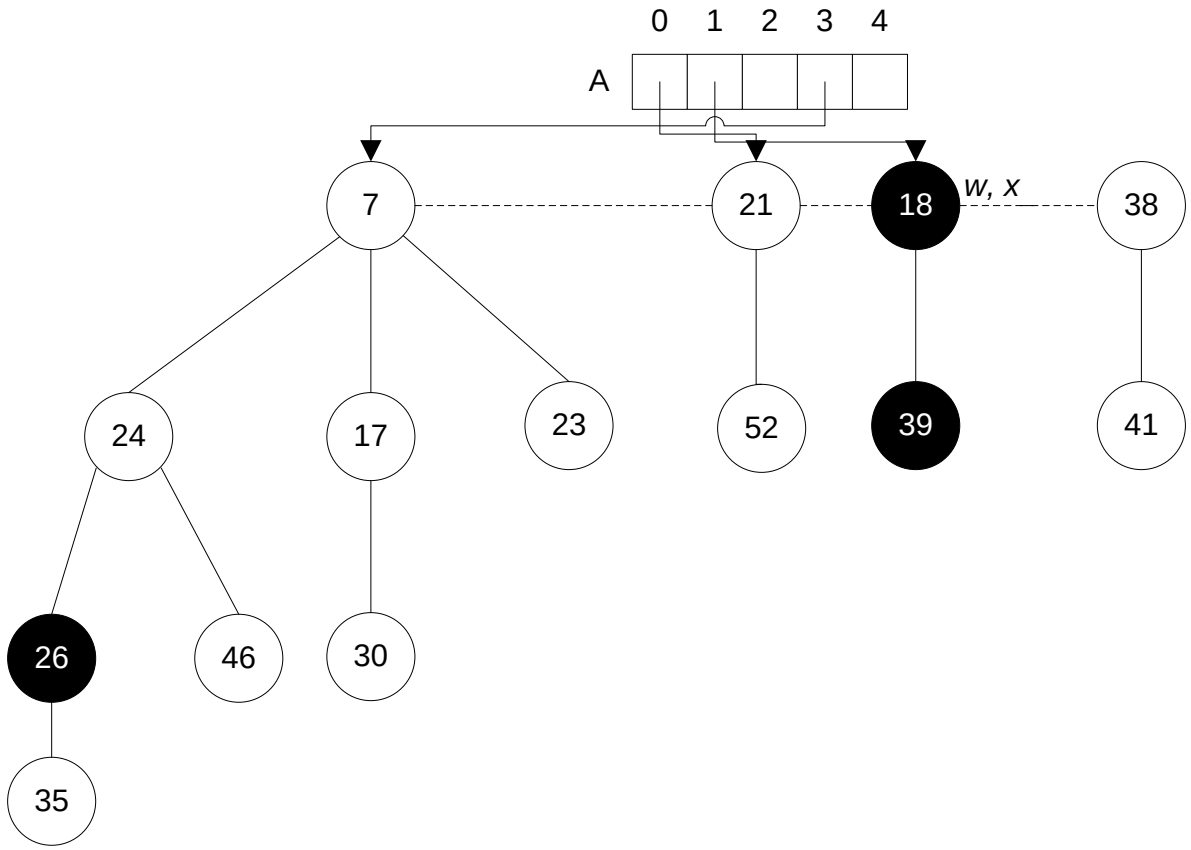

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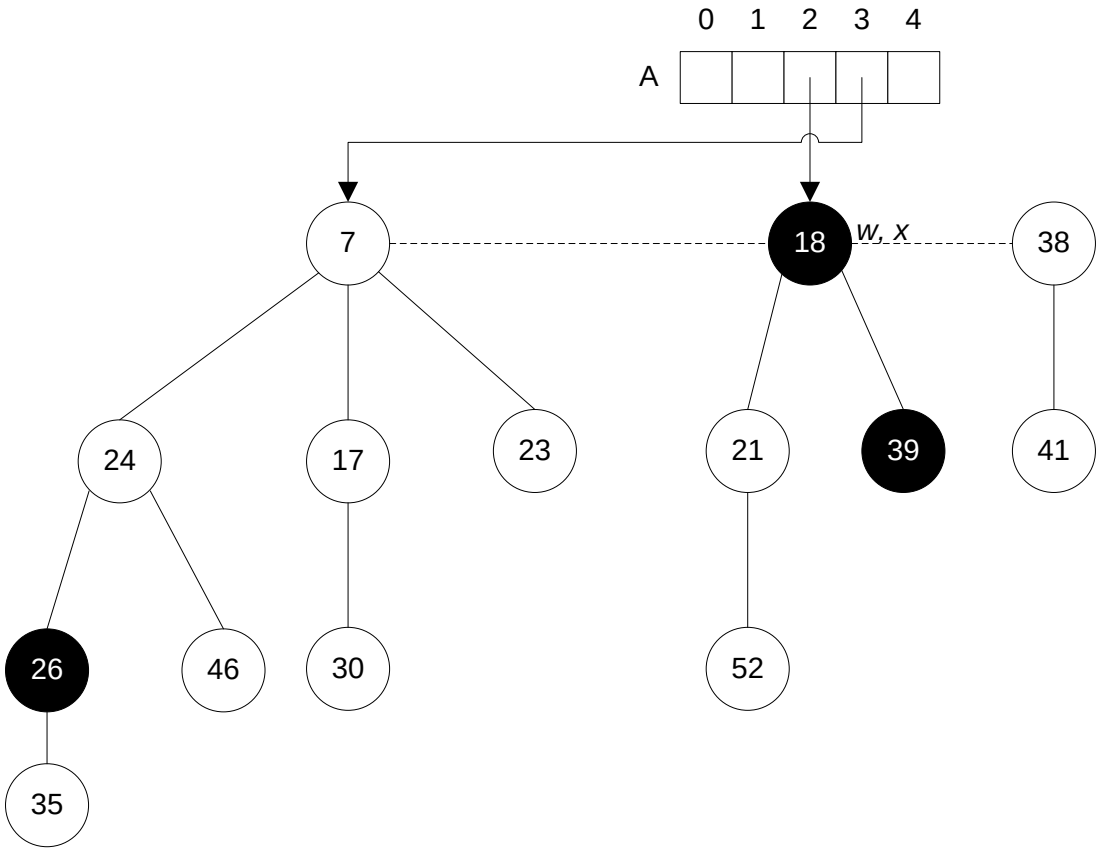

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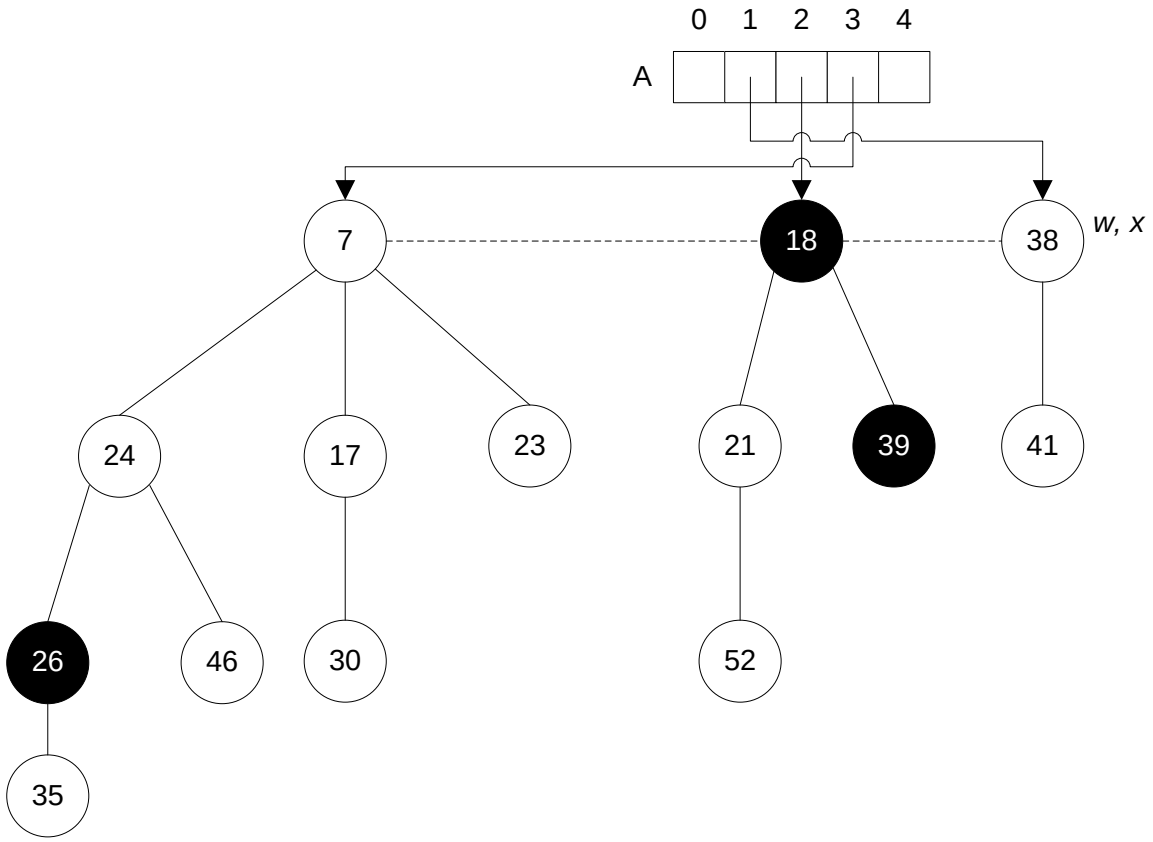

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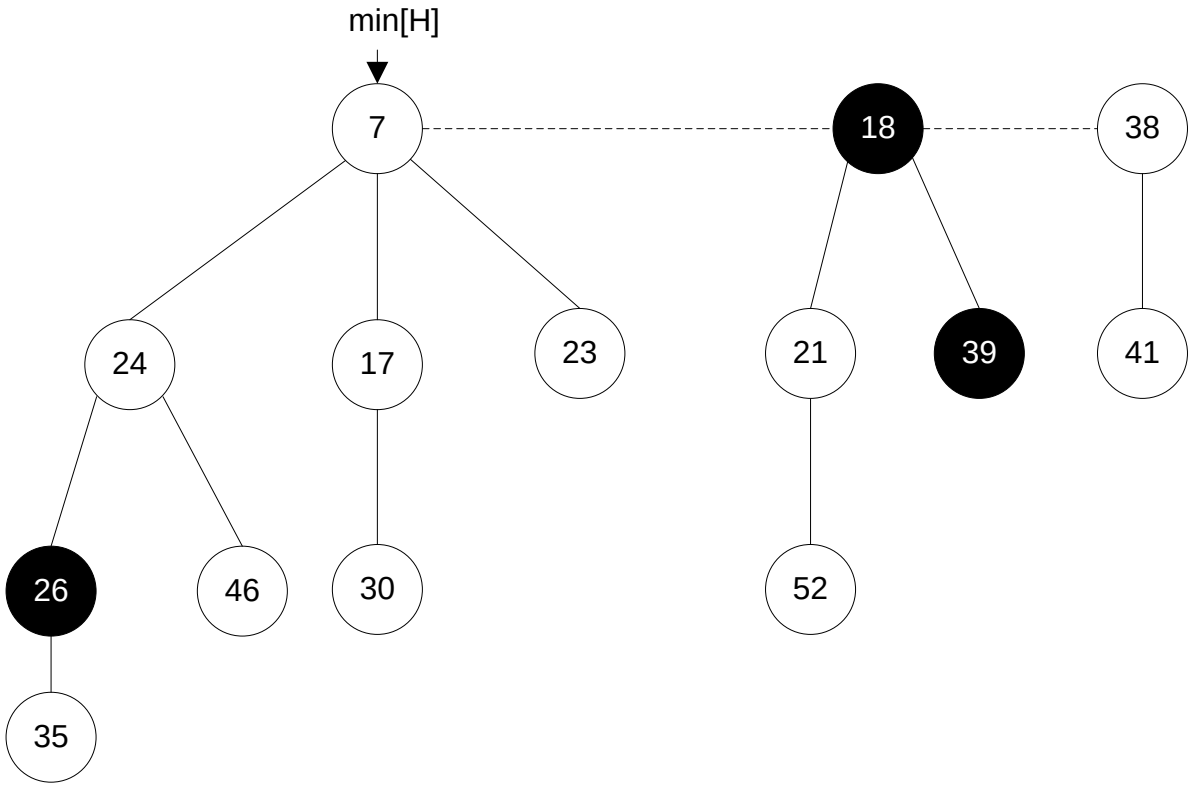


Analysis of the FIB-HEAP-EXTRACT-MIN Procedure

- If all trees in the fib-heap are unordered binomial trees before the execution of the EXTRACT-MIN operation then they are all unordered binomial trees afterward.
- There are two ways in which trees are changed:

  (1) each child of the extracted root node becomes a child, each new tree is itself an unordered binomial tree.

  (2) trees are linked by FIB-HEAP-LINK procedure only if they have the same degree hence U_k is linked to U_k to form a U_{k+1}

Complexity Analysis of the FIB-HEAP-EXTRACT-MIN Procedure

Actual Cost

1-st for loop: Contributes O(D(n))

3-rd for loop: Contributes O(D(n))

2-nd for loop:

Size of the root-list upon calling CONSOLIDATE is at most:

$$D(n) + t(H) - 1$$

D(n): upper bound on the number of children of the extracted node

t(H) – 1: original *t(H)* root list nodes – the extracted node

Complexity Analysis of the FIB-HEAP-EXTRACT-MIN Procedure

Each iteration of the inner while-loop links one root to another thus reducing the size of the root list by 1

Therefore the total amount work performed in the 2-nd for loop is at most proportional to $D(n) + t(H)$

Thus, the total actual cost is $O(D(n) + t(H))$

Complexity Analysis of the FIB-HEAP-EXTRACT-MIN Procedure

Amortized Cost

Potential before: $t(H) + 2m(H)$

Potential after: at most $(D(n) + 1)+2m(H)$ since at most $D(n)+1$ roots remain & no nodes marked

Amortized cost = $O(D(n) + t(H)) +$

$[(D(n) + 1) + 2m(H)] –$

$[t(H) + 2m(H)]$

$= O(D(n)) + O(t(H)) – D(n) – t(H)$

$= \boldsymbol{O(D(n))}$

Complexity Analysis of the FIB-HEAP-EXTRACT-MIN Procedure

The cost of performing each link is paid for by the reduction in potential due to the link reducing the number of roots by one

EXTRACT-MIN Procedure for Fibonacci Heaps

```
FIB-HEAP-EXTRACT-MIN (H)
z ← min[ H ]
if  z ≠ NIL then
        for each child x of z do
                add x to the root list of H
                p [ x ] ← NIL
        endfor
        remove z from the root list of H
        if right [ z ] = z then
                min [ H ] ← NIL
        else
                min [ H ] ← right [ z ]
                CONSOLIDATE (H)
        endif
        n [ H ] ← n [ H ] – 1
endif
return z
end
```

EXTRACT-MIN Procedure for Fibonacci Heaps

FIB-HEAP-LINK (H, *y*, *x*)

remove *y* from the root list of H

make *y* a child of *x*, incrementing degree [x]

mark [*y*] ← FALSE

end

EXTRACT-MIN Procedure for Fibonacci Heaps

```
CONSOLIDATE ( H )
    for i ← 0 to D ( n ( H ) )
            A[ i ] ← NIL
    endfor
    for each node w in the root list of H do
            x ← w
            d ← degree [ x ]
            while A [ d ] ≠ NIL do
                    y ← A [ d ]
                    if key [ x ] > key [ y ] then
                                exchange x ↔ y
                    endif
                    FIB-HEAP-LINK ( H , y, x )
                    A [ d ] ← NIL
                    d ← d + 1
            endwhile
            A [ d ] ← x
    endfor
```

```
min [ H ] ← NIL
   for i ← 0 to D ( n [ H ] ) do
      if A [ i ] ≠ NIL then
         Add A [ i ] to the root list of H
         if min [ H ] = NIL or key [ A [ i ] ] <
                              key [ min [ H ] ] then
            min [ H ] ← A [ i ]
         endif
      endif
   endfor
end
```

Bounding the Maximum Degree

For each node x within a fibonacci heap, define

size(x): the number of nodes, including itself, in the subtree rooted at x

NOTE: x need not to be in the root list, it can be any node at all.

We shall show that size(x) is exponential in degree[x]

Bounding the Maximum Degree

Lemma 1: Let x be a node with degree[x]=k

Let $y_1, y_2, ..., y_k$ denote the children of x in the order in which they are linked to x, from earliest to the latest, then

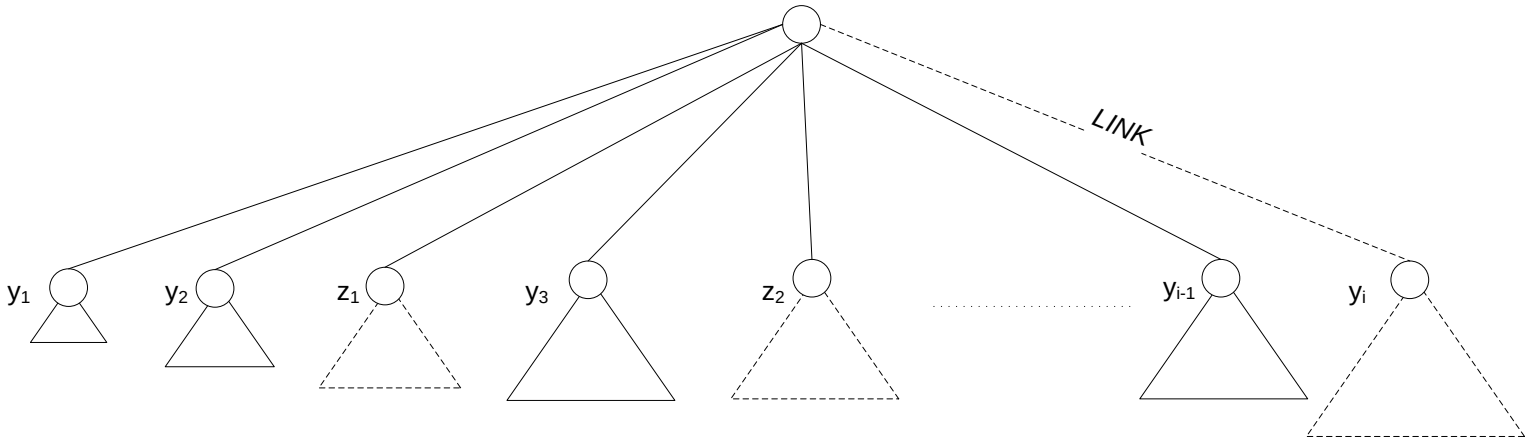
$$\text{degree}[y_1] \geq 0 \text{ and } \text{degree}[y_i] \geq i-2$$

$$\text{for } i = 2,3,...,k$$

Bounding the Maximum Degree

Proof: degree[y_1] ≥ 0 =>
obvious

For $i \geq 2$:



Bounding the Maximum Degree

- When y_i is linked to x: at least $y_1, y_2, \ldots, y_{i-1}$ were all children of x so we must have had degree[x] $\geq i - 1$
- NOTE: z node(s) denotes the node(s)

  that were children of x just before the link of y_i that are lost after the link of y_i
- When y_i is linked to x:

  $$\text{degree}[y_i] = \text{degree}[x] \geq i - 1$$

  since then, node y_i has lost at most one child, we conclude that degree[y_i] $\geq$ i-2

Bounding the Maximum Degree

As in Lemma-1, let $y_1, y_2, ..., y_k$ denote the children of node *x* in the order in which they were linked to *x*

$$size(x) \geq S_k \geq 1 + 1 + \sum_{i=2}^{k} S_{i-2} = 2 + \sum_{i=2}^{k} S_{i-2}$$

for x itself

S_1 for y_1

for $y_2, ..., y_k$ due to Lemma-1

Bounding the Maximum Degree

Corollary: Max. degree *D(n)* in an n node fib-heap is *O(lg n)*

Proof: Let *x* be any node with degree[*x*] = k in an n-node fib-heap by Lemma-3 we have

$$n \geq \text{size}(x) \geq \Phi^k$$

taking base-Φ log => $k \leq \log_\Phi n$

therefore *D(n) = O(lg n)*

Decreasing a Key

```
FIB-HEAP-DECREASE-KEY(H, x, k)
    key[x] ← k
    y ← p[x]
    if y ≠ NIL and key[x] < key[y] then
            CUT(H, x, y)
            CASCADING-CUT(H,y)
    endif   /* else no structural change */
    if key[x] < key[min[H]] then
            min[H] ← x
    endif
end
```

Decreasing a Key

CUT(H, x, y)

remove x from the child list of y, decrementing degree y

add x to the root list of H

p[x] ← NIL

mark[x] ← FALSE

end

Decreasing a Key

```
CASCADING-CUT(H, y)
        z ← p[y]
        if z ≠ NIL then
                if mark[y] = FALSE then
                        mark[y] = TRUE
                else
                        CUT(H, y, z)
                        CASCADING-CUT(H, z)
                endif
        endif
end
```

Amortized Cost of FIB-HEAP-DECREASE-KEY Procedure

Actual Cost

O(1) time + the time required to perform the cascading cuts, suppose that CASCADING-CUT is recursively called *c* times each call takes *O(1)* time exclusive of recursive calls therefore,

the actual cost = $O(1) + O(c) = \boldsymbol{O(c)}$

Amortized Cost of FIB-HEAP-DECREASE-KEY Procedure

Amortized Cost

Let H denote the fib-heap prior to the DECREASE-KEY operation.

Each recursive call of CASCADING-CUT, except for the last one, cuts a marked node and last call of cascading cut may mark a node.

Amortized Cost of FIB-HEAP-DECREASE-KEY Procedure

Hence, after the DECREASE-KEY operation

$$number\,of\,trees = t(H) + 1 + (c-1) = t(H) + C$$

tree rooted at x

trees produced by cascading cuts

$$number\,of\,marked\,nodes = m(H) - (c-1) + 1 = m(H) - c + 2$$

unmarked during the first c-1 CASCADING-CUTS

marked during the last CASCADING-CUT

Amortized Cost of FIB-HEAP-DECREASE-KEY Procedure

Potential Difference

$= [(t(H) + c) + 2(m(H)-c+2)]-[t(H)+2m(H)]$

$= 4 - c$

Amortized Cost $= O(c) + 4 - c = \mathbf{O(1)}$

Deleting a Node

FIB-HEAP-DELETE(H, x)

FIB-HEAP-DECREASE-KEY(H, x, -∞) => *O(1)*

FIB-HEAP-EXTRACT-MIN(H) => *O(D(n))*

end

Amortized Cost = *O(1) + O(D(n)) = O(D(n))*

Bounding the Maximum Degree

- Why do we apply CASCADING-CUT during DECREASE-KEY operation?
- To maintain the size of any tree/subtree exponential in the degree of its root node.

  e.g.: to prevent cases where

  $$size[x] = degree[x] + 1$$