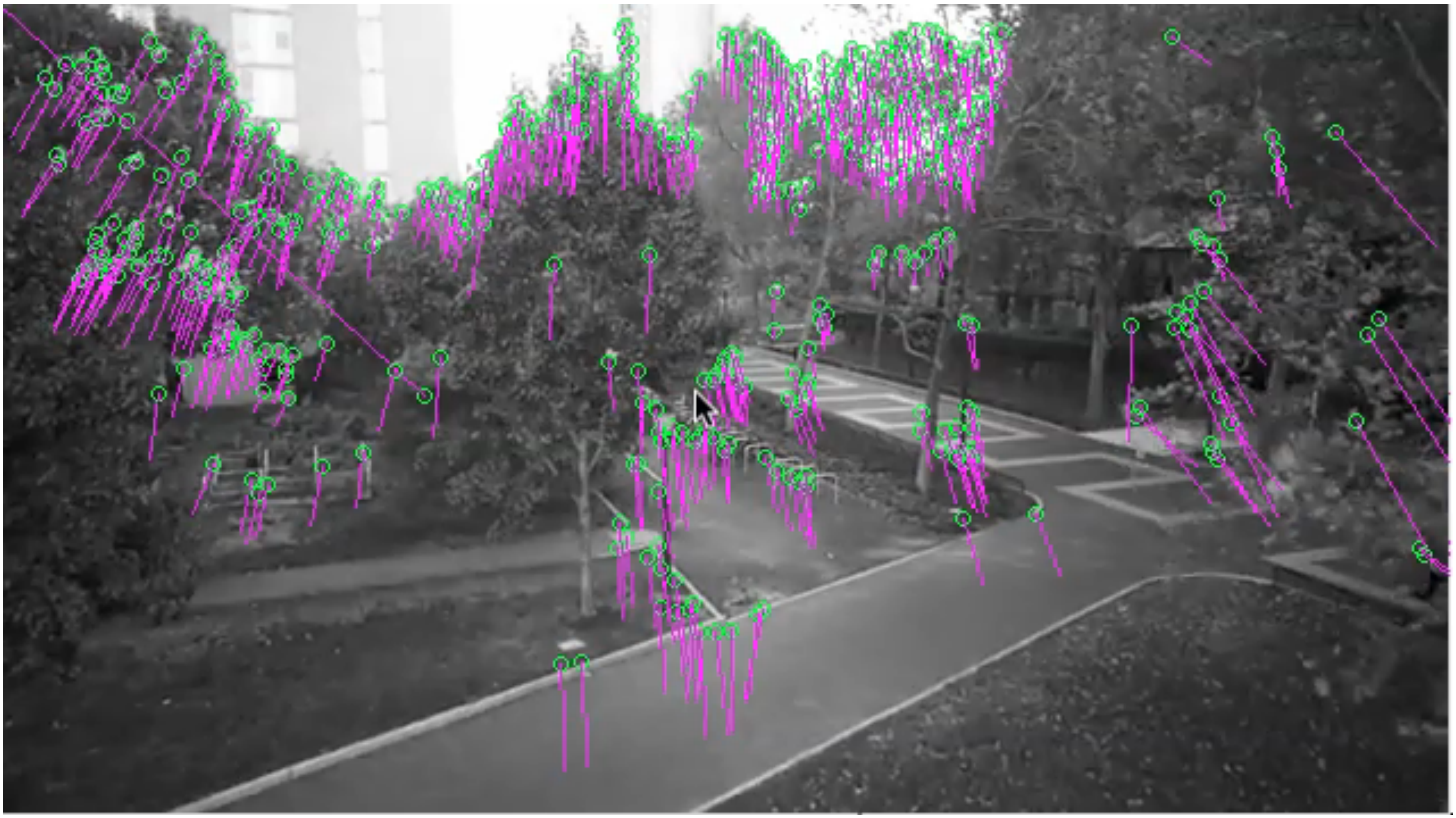
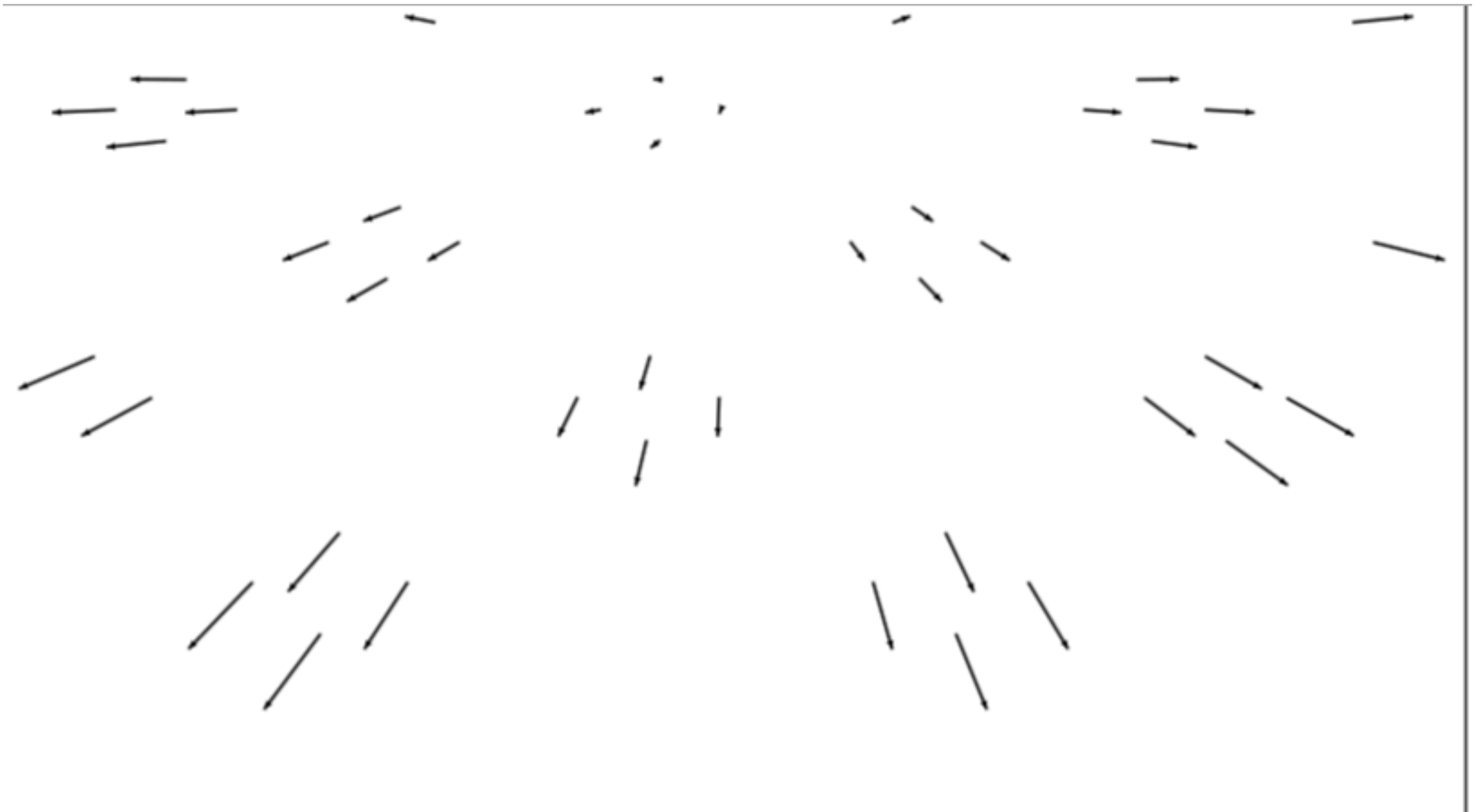
Perception:
3D Velocities from Optical Flow

Kostas Daniilidis

Which direction is the vehicle moving?

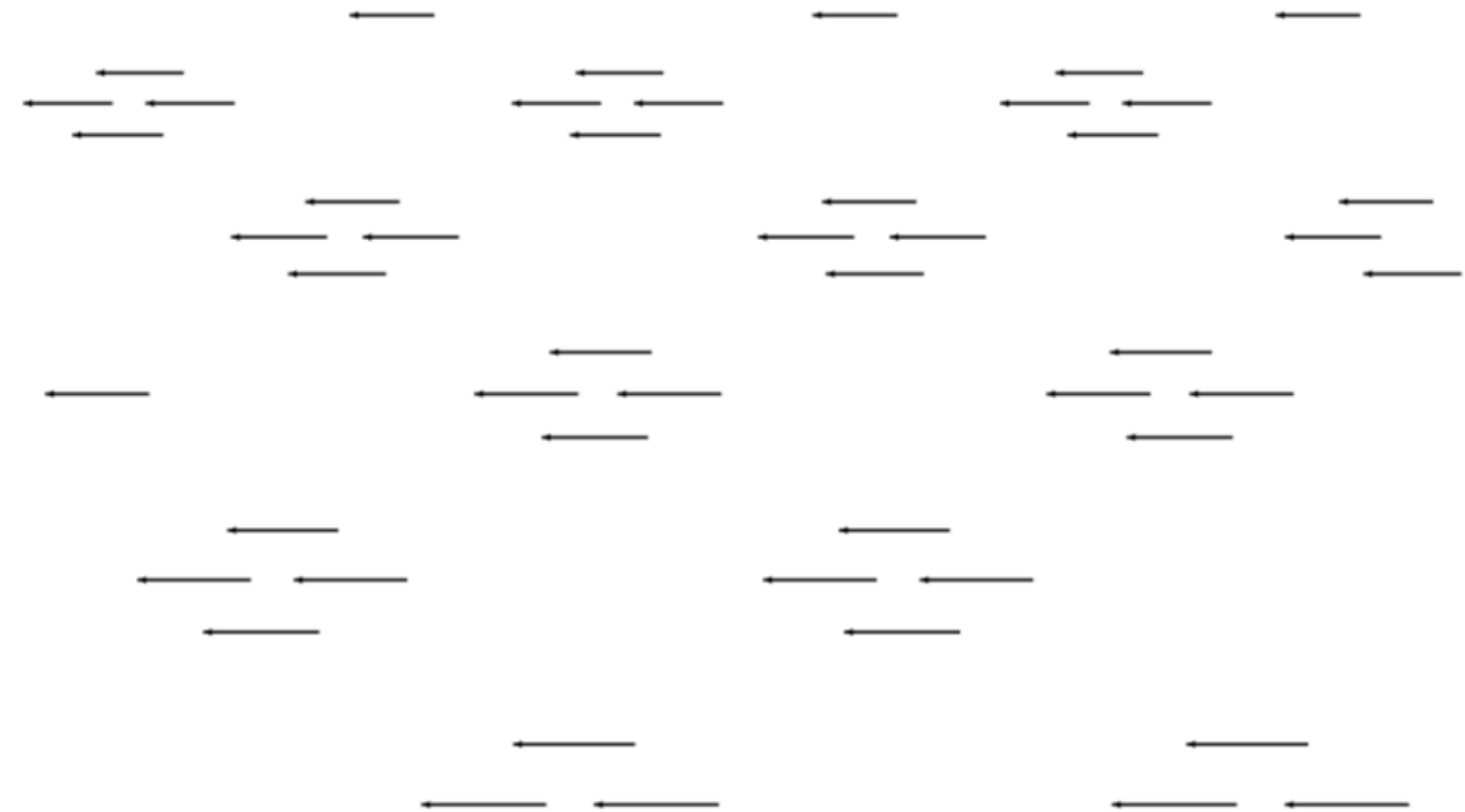


Pure translation

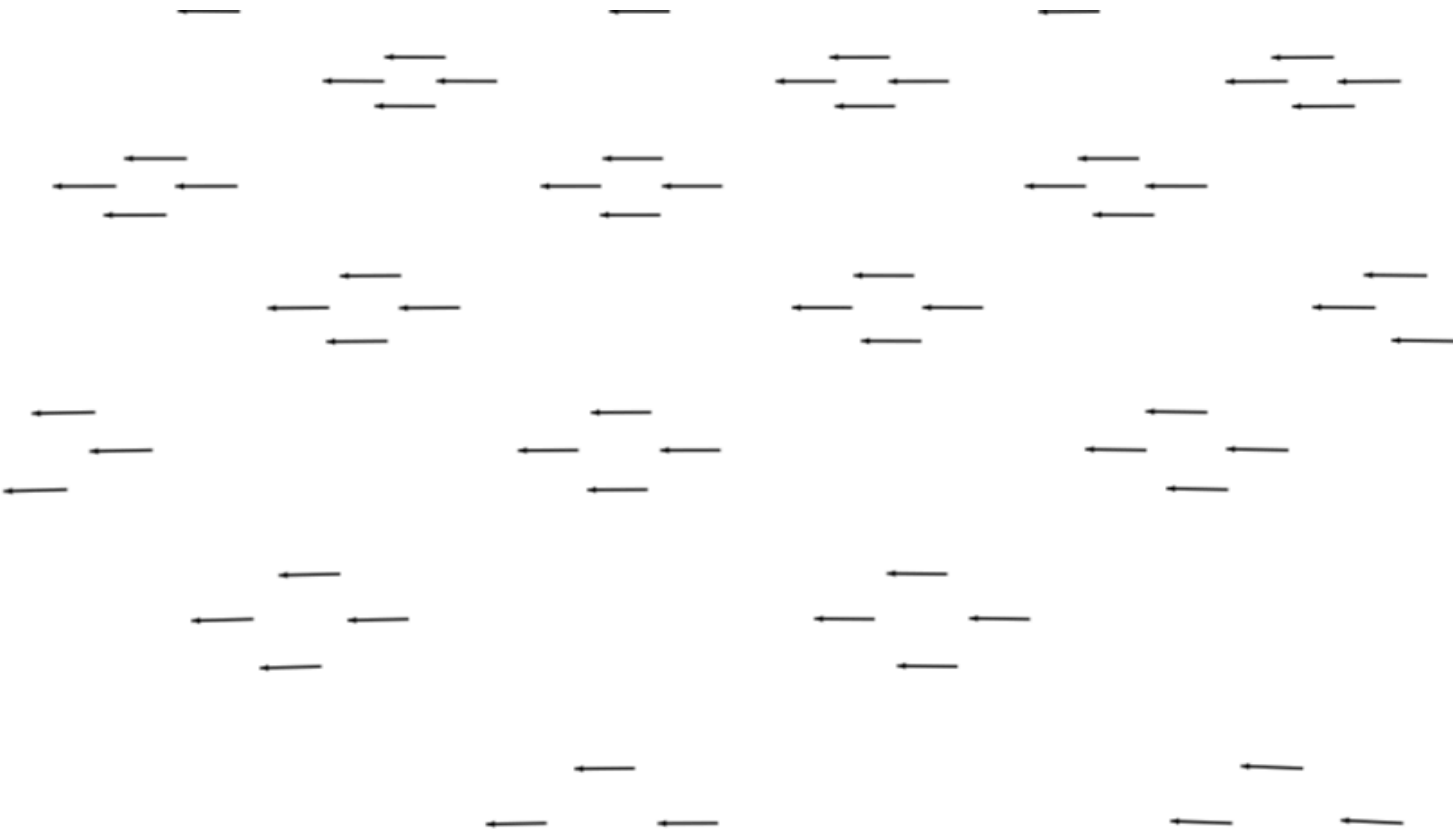


Pure translation to the right

Pure horizontal translation



Pure rotation around vertical axis



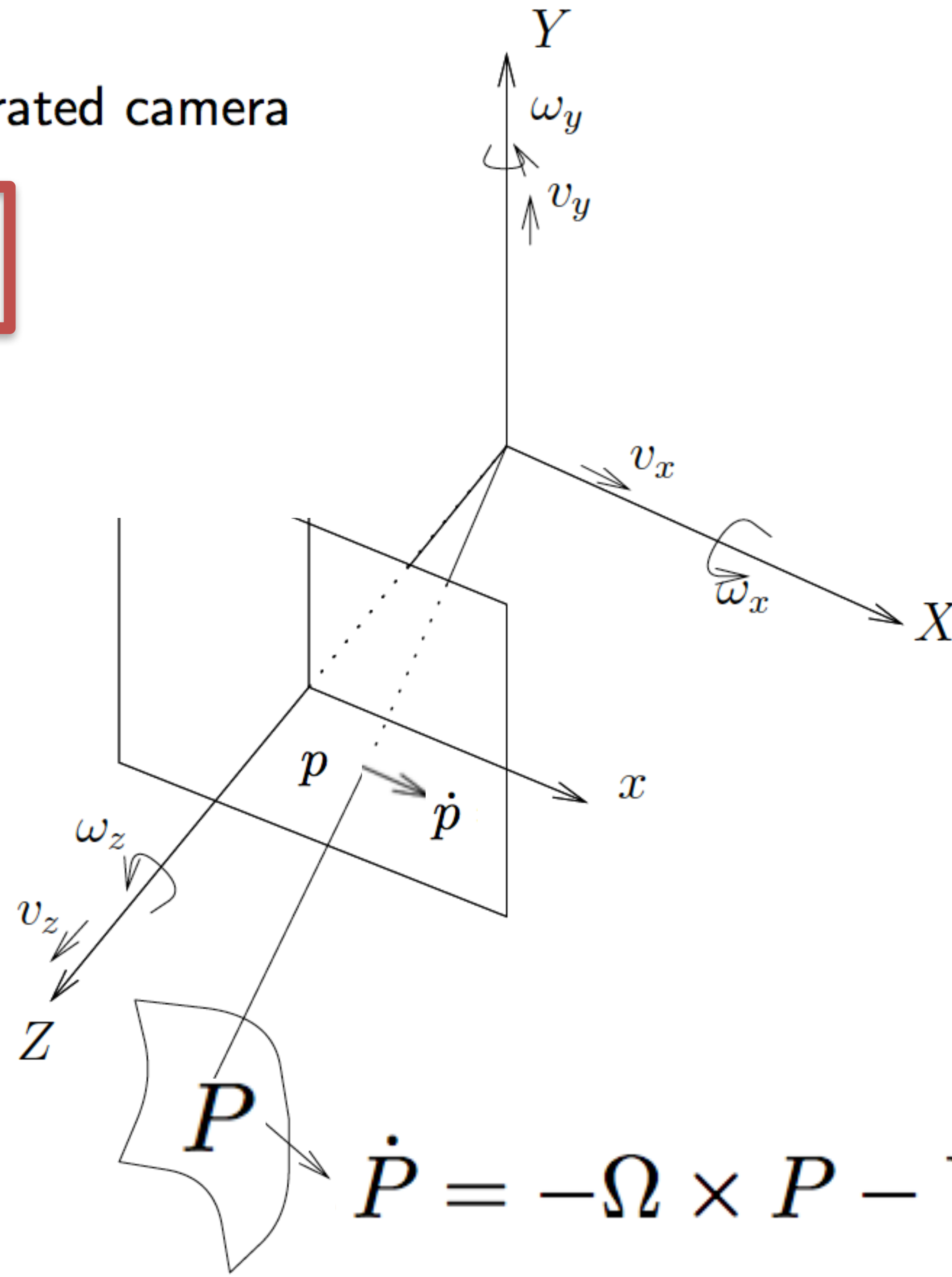
Projection equations for calibrated camera

$x = \frac{X}{Z}$, $y = \frac{Y}{Z}$

or in vector notation $\boxed{p = \frac{1}{Z}P}$

Differentiating w.r.t. time yields:

$$\dot{p} = \frac{\dot{P}}{Z} - \frac{\dot{Z}}{Z}p$$



$$\dot{P} = -\Omega \times P - V$$

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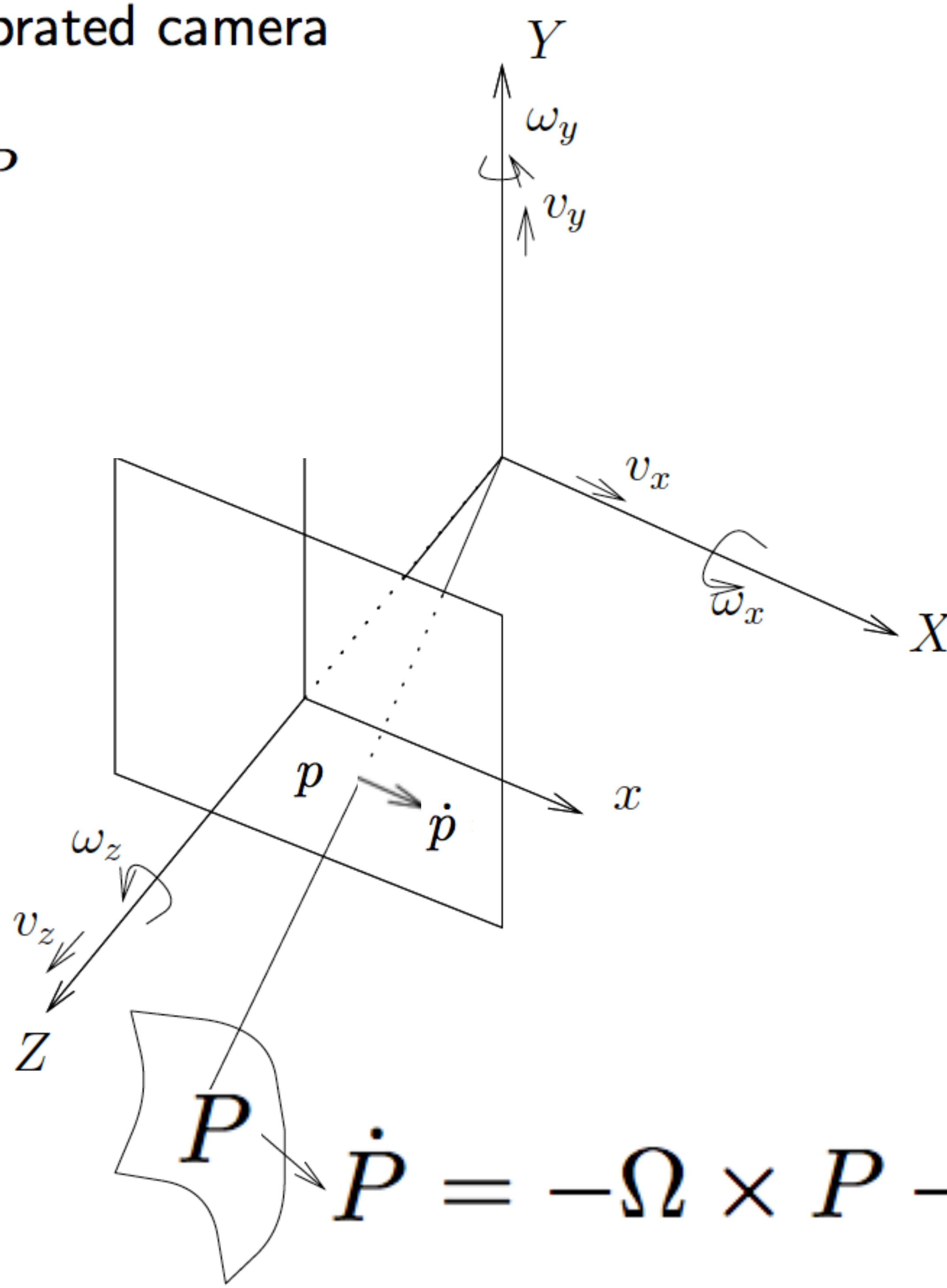


$$\dot{P} = -\Omega \times P -$$

If we combine the equations for $\dot{P}$ and the optical flow $\dot{p}$ we obtain

$$\dot{p} = \frac{1}{Z} e_3 \times (p \times V) + e_3 \times (p \times (p \times \Omega))$$

where $e_3 = (0, 0, 1)^T$.

Written out in coordinates

$$\dot{p} = \underbrace{\frac{1}{Z} \begin{bmatrix} xV_z - V_x \\ yV_z - V_y \end{bmatrix}}_{\text{translational flow}} + \underbrace{\begin{bmatrix} xy & -(1+x^2) & y \\ (1+y)^2 & -xy & -x \end{bmatrix} \Omega}_{\text{rotational flow independent of depth}}$$

Optical flow has two additive components: translational and rotational.

$$\dot{p} = \underbrace{\frac{1}{Z}\begin{bmatrix} xV_z - V_x \\ yV_z - V_y \end{bmatrix}}_{\text{translational flow}} + \underbrace{\begin{bmatrix} xy & -(1+x^2) & y \\ (1+y)^2 & -xy & -x \end{bmatrix}}_{\text{rotational flow independent of depth}} \Omega$$

If Z is known, $\dot{p}$ is linear in V and Ω.

Having at least 3 optical flow vectors not on collinear points and corresponding depths we can solve for the 3D velocities from 6 equations.

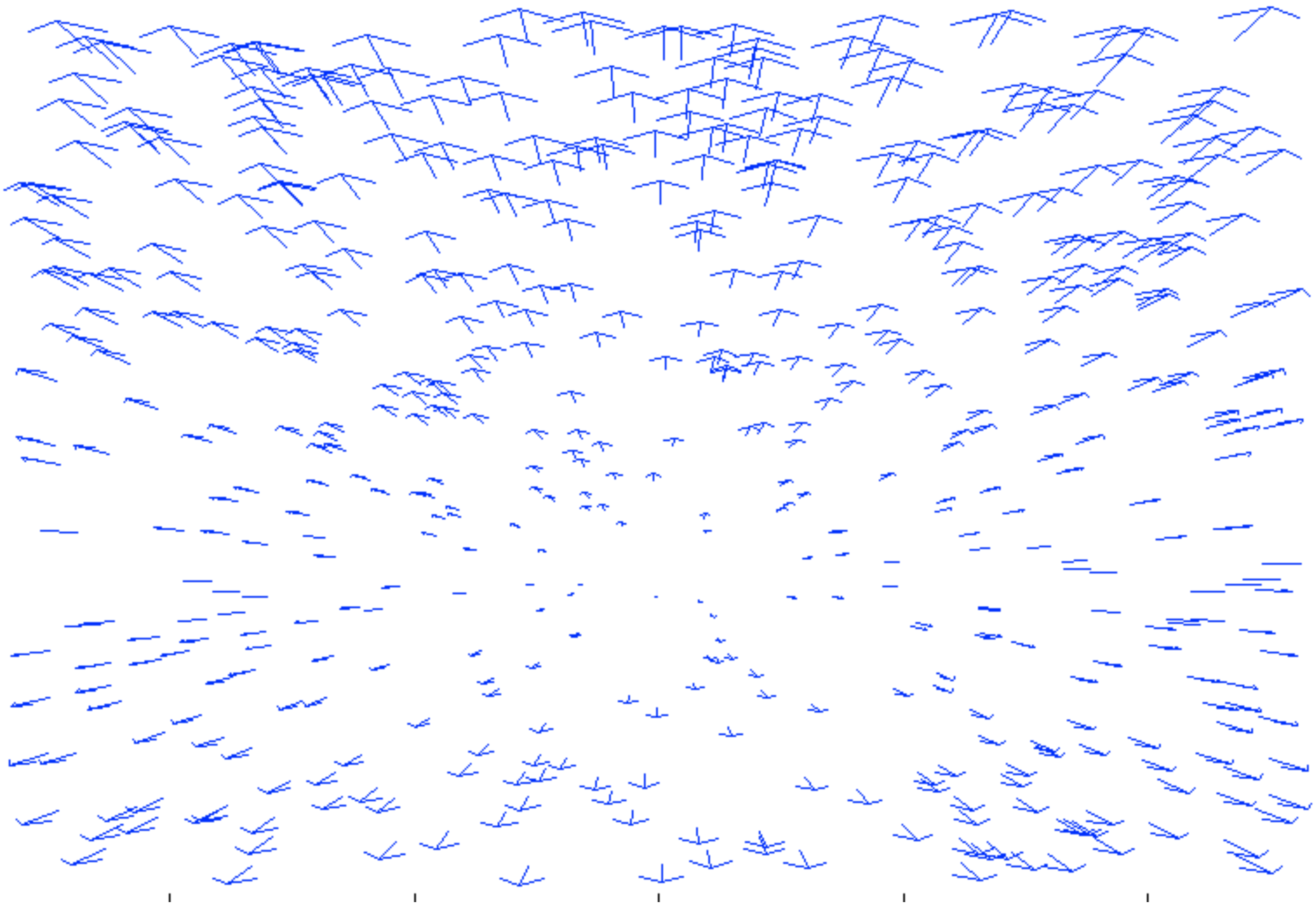
If the field is purely rotational then we have no information about depth.

$$\dot{p} = \underbrace{\begin{bmatrix} xy & -(1+x^2) & y \\ (1+y)^2 & -xy & -x \end{bmatrix} \Omega}_{\text{rotational flow independent of depth}}$$

This also means that if we know Ω from other sources we can *derogate* he flow field without knowing the depth.

Translational Flow:

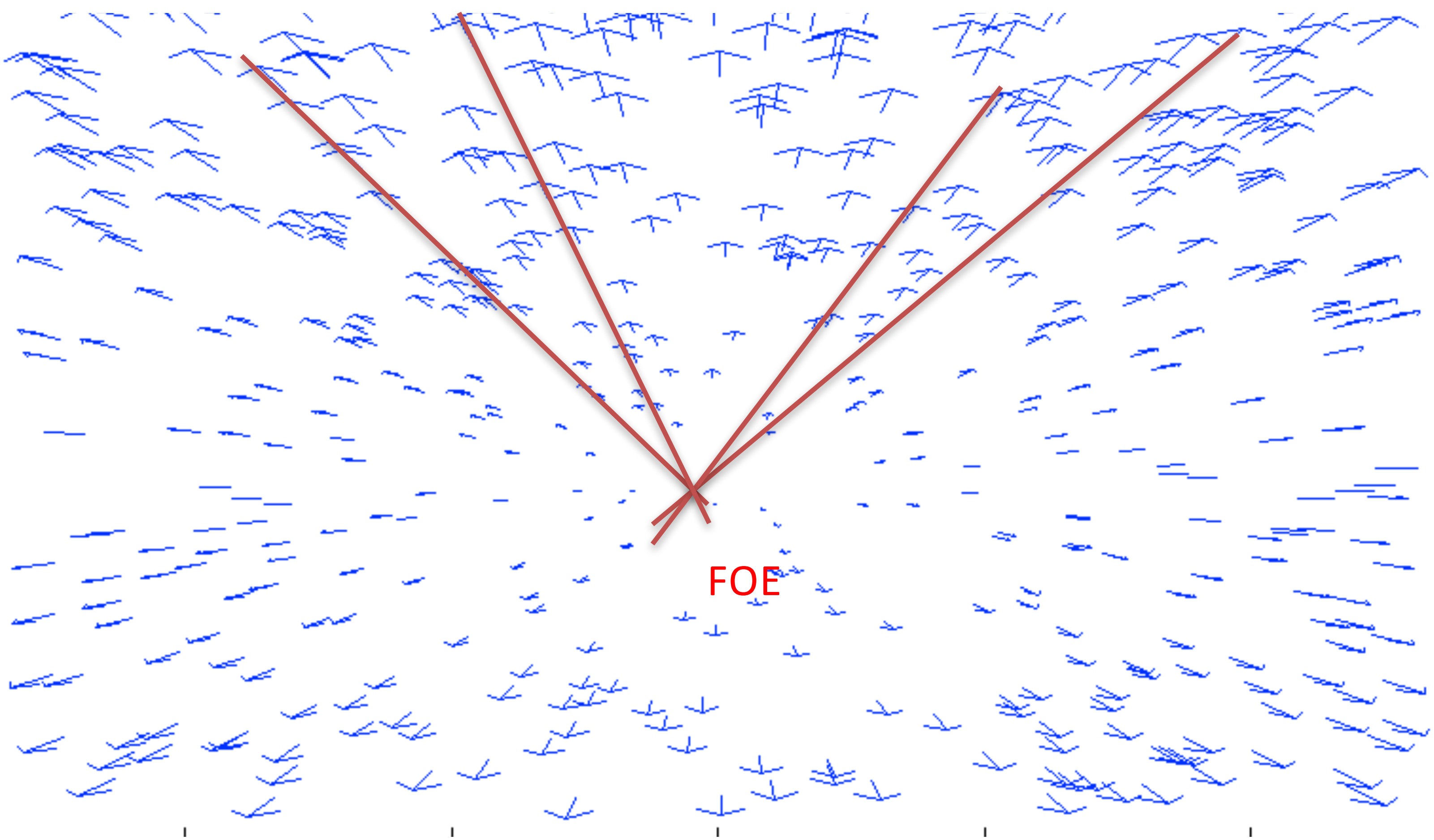
$$\dot{p}_{\text{trans}} = \frac{V_z}{Z} \begin{bmatrix} x - \frac{V_x}{V_z} \\ y - \frac{V_y}{V_z} \end{bmatrix}$$



By intersecting the lines spanned by $\dot{p}_{\text{trans}}$, we can obtain the Focus of Expansion (FOE) also called Epipole

$$FOE = (V_x/V_z, V_y/V_z)$$

FOE can also be at infinity if $V_z = 0$.



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$$\dot{p}_{\text{trans}} = \frac{V_z}{Z} \begin{bmatrix} x - \frac{V_x}{V_z} \\ y - \frac{V_y}{V_z} \end{bmatrix}$$

The time to collision (which birds and insects estimate) is

$$\frac{Z}{V_z}$$

$$\frac{V_z}{Z} = \frac{||\dot{p}_{\text{trans}}||}{||p - F\vec{O}E||}$$

Points at the same radial distance from FOE have flow vector lengths proportional to inverse depth (or inverse time to collision).

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From

$$\dot{p}_{trans}^T (p \times V) = 0$$

we obtain the following coplanarity condition

$$V^T (p \times \dot{p}_{trans}) = 0$$

which says that image point, flow, and linear velocity lie on the same plane.

We can obtain V from two points

$$V \sim (p_1 \times \dot{p}_1) \times (p_2 \times \dot{p}_2)$$

and from n points we obtain a homogeneous system

$$\underbrace{\begin{pmatrix} (p_1 \times \dot{p}_1)^T \\ p_2 \times \dot{p}_2)^T \\ \dots \\ p_n \times \dot{p}_n)^T \end{pmatrix}}_{A} V = 0 \tag{1}$$

Then V is the nullspace of A which can be obtained from SVD.

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Both V and Ω unknown

Recall that

$$\dot{p} = \frac{1}{Z}F(x,y)V + G(x,y)\Omega$$

This is can be written linearly in inverse depths and Ω:

$$\dot{p} = \left[F(x,y)V \;\; G(x,y)\right] \begin{bmatrix} \frac{1}{Z} \\ \Omega \end{bmatrix}$$

For n points we can write out a system of equations:

$$\begin{pmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \cdots \\ \dot{p}_n \end{pmatrix} = \Phi(V) \begin{pmatrix} \frac{1}{Z_1} \\ \frac{1}{Z_1} \\ \cdots \\ \frac{1}{Z_N} \\ \Omega \end{pmatrix}$$

The Φ matrix is a 2N by (N+3) matrix and is a function of V

$$\dot{d} = \Phi(V) \begin{pmatrix} \frac{1}{Z_1} \\ \frac{1}{Z_1} \\ \dots \\ \frac{1}{Z_N} \\ \Omega \end{pmatrix}$$

If we solve for the unknown vector of inverse depths and Ω we obtain

$$\Phi^+(V)\dot{d}$$

which we can insert back in the objective function.

A search on the sphere yields then V:

$$\arg\min_{V \in S^2} \|\dot{d} - \Phi(V)\Phi(V)^+\dot{d}\|^2$$

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