An Elementary Introduction to Distributions

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Consider the Heaviside step function and the Dirac δ "function"

$$H(x) := \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases} \qquad \delta(x) := \begin{cases} \infty, & x = 0, \\ 0, & x \neq 0, \end{cases}$$

respectively. In practice, one unfortunately tends to witness the mathematically imprecise statement: $\mathrm{d}H/\mathrm{d}x = \delta$. Apart from the Dirac δ function not being well-defined, the Heaviside step function H is quite obviously discontinuous and hence not differentiable.

Can one then make the statement $\mathrm{d}H/\mathrm{d}x = \delta$ *mathematically precise?*

The answer is yes. Both, H and δ, are given a new meaning as distributions, and are related by a "distributional derivative". This ability to differentiate functions that need not be differentiable in the conventional sense turns out to be useful in finding more solutions to partial differential equations.

In Section 1, we present the definition of distributions and illustrate a couple of important examples. In Section 2, we provide the technique of differentiating distributions in a manner that is compatible with the differentiation of functions. In Section 3, we apply the aforementioned concepts in obtaining a larger class of solutions for the homogeneous wave equation.

1 Distributions

In this section and the next, all functions will be considered over $\mathbb{R}$. Consequently, we consider distributions only on $\mathbb{R}$. This is not a restriction as analogous results can be derived on replacing $\mathbb{R}$ with $\mathbb{R}^n$.

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1.1 Definition.
A function $f : \mathbb{R} \to \mathbb{R}$ is said to be
(i) *smooth* if it can be continuously differentiated as many ever times, and
(ii) *compactly supported* if $f(x) = 0$ outside some compact region in $\mathbb{R}$.
The set of compactly supported smooth functions is denoted as $C_c^\infty(\mathbb{R})$. $\diamond$

1.2 Exercise.
Show that $C_c^\infty(\mathbb{R})$ is a vector space.

1.3 Definition.
A *distribution* is a map $u : C_c^\infty(\mathbb{R}) \to \mathbb{R}$ that is
(i) *linear*: if $\lambda \in \mathbb{R}$ and $f, g \in C_c^\infty(\mathbb{R})$, then

$$\text{(a) } u(f+g) = u(f) + u(g), \qquad \text{(b) } u(\lambda f) = \lambda u(f);$$

(ii) *continuous*: if $(f_n)_{n\in\mathbb{N}}$ is a sequence in $C_c^\infty(\mathbb{R})$ that converges to some $f \in C_c^\infty(\mathbb{R})$, then the values $u(f_n)$ converge to $u(f)$ in $\mathbb{R}$. $\diamond$

To simplify the exposition, it is not mentioned *how* the sequence $(f_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R})$ converges to $f \in C_c^\infty(\mathbb{R})$. While we reluctantly sweep these technical arguments under the rug, we want the reader to note that, if $f_n \to f$ in $C_c^\infty(\mathbb{R})$, there exists a compact region $[a, b] \subset \mathbb{R}$ such that
(i) $f_n(x) = 0$ and $f(x) = 0$ for all $x \notin [a, b]$, and
(ii) $\sup_{x\in[a,b]} |f_n(x) - f(x)| \to 0$.

1.4 Exercise.
Show that the set of distributions is a vector space.

To obtain a better understanding of distributions, consider the following examples. Since integration is a nice way to map functions to $\mathbb{R}$, the first example illustrates how distributions can be constructed from functions that satisfy some sort of integrability criterion, thereby justifying the alternative nomenclature of distributions as "generalised functions".

1.5 Example.
Let $\psi : \mathbb{R} \to \mathbb{R}$ be a function that is integrable over any compact region in $\mathbb{R}$. If $f \in C_c^\infty(\mathbb{R})$, then

$$f \mapsto \Lambda_\psi(f) := \int_{\mathbb{R}} \psi(x)\, f(x)\, \mathrm{d}x$$

defines a map $\Lambda_\psi : C_c^\infty(\mathbb{R}) \to \mathbb{R}$. It is obvious that Λ_ψ is linear. To see that Λ_ψ is continuous: if $f_n \to f$ in $C_c^\infty(\mathbb{R})$, then

$$\begin{aligned} |\Lambda_\psi(f_n - f)| = \Big| \int_a^b \psi(x)\, \big(f_n(x) - f(x)\big)\, \mathrm{d}x \Big| &\leq \int_a^b |\psi(x)|\, |f_n(x) - f(x)|\, \mathrm{d}x \\ &\leq \sup_{x \in [a,b]} |f_n(x) - f(x)| \int_a^b |\psi(x)|\, \mathrm{d}x \to 0. \end{aligned}$$

In this calculation, we have reduced the integral over $\mathbb{R}$ to an integral over the compact region $[a, b]$ within which $(f_n)_{n \in \mathbb{N}}$ converges to f. Distributions such as Λ_ψ are known as *regular distributions*. $\diamond$

1.6 Example.
The *Dirac δ-distribution* (not "function"!) is defined as the map

$$f \mapsto \delta(f) := f(0)$$

for all $f \in C_c^\infty(\mathbb{R})$. It is also obviously linear, while its continuity follows from

$$|\delta(f_n - f)| = |f_n(0) - f(0)| \leq \sup_{x \in K} |f_n(x) - f(x)| \to 0$$

when $f_n \to f$ in $C_c^\infty(\mathbb{R})$. The δ-distribution is *not* a regular distribution, *i.e.* it cannot be written as an integral (irrespective of what physicists might say). $\diamond$

1.7 Exercise.
Verify that the distributions Λ_ψ *and* δ *are linear.*

2 Distributional Derivatives

An advantage of working with distributions is that they admit a concept of differentiation regardless of their "differentiability". While the following definition may at first glance appear arbitrary, it is motivated by the subsequent proposition.

2.1 Definition.
The *distributional derivative* $\mathrm{d}u/\mathrm{d}x$ of a distribution u is the distribution

$$f \mapsto \frac{\mathrm{d}u}{\mathrm{d}x}(f) := -u\Big(\frac{\mathrm{d}f}{\mathrm{d}x}\Big)$$

for all $f \in C_c^\infty(\mathbb{R})$. $\diamond$

2.2 Proposition.
Let $\psi : \mathbb{R} \to \mathbb{R}$ *be a continuously differentiable function. It determines a regular distribution* Λ_ψ *whose distributional derivative* $\mathrm{d}\Lambda_\psi/\mathrm{d}x$ *coincides with the regular distribution determined by the continuous function* $\mathrm{d}\psi/\mathrm{d}x$*:*

$$\frac{\mathrm{d}\Lambda_\psi}{\mathrm{d}x}(f) = \int_{\mathbb{R}} \frac{\mathrm{d}\psi}{\mathrm{d}x}(x)\, f(x)\,\mathrm{d}x$$

for all $f \in C_c^\infty(\mathbb{R})$*.*

Proof. As every continuous function is integrable over every compact region in $\mathbb{R}$, the function ψ determines a regular distribution $\Lambda_\psi(f) = \int_{\mathbb{R}} \psi(x)\, f(x)\,\mathrm{d}x$. If $[a,b] \subset \mathbb{R}$ is a compact region outside which $f(x) = 0$, then

$$\begin{aligned}\frac{\mathrm{d}\Lambda_\psi}{\mathrm{d}x}(f) = -\Lambda_\psi\Big(\frac{\mathrm{d}f}{\mathrm{d}x}\Big) &= -\int_a^b \psi(x)\,\frac{\mathrm{d}f}{\mathrm{d}x}(x)\,\mathrm{d}x \\ &= -\big(\psi(x)\, f(x)\big)\Big|_a^b + \int_a^b \frac{\mathrm{d}\psi}{\mathrm{d}x}(x)\, f(x)\,\mathrm{d}x\end{aligned}$$

$$= \int_{\mathbb{R}} \frac{\mathrm{d}\psi}{\mathrm{d}x}(x)\, f(x)\, \mathrm{d}x.$$

The first equality uses the definition of the distributional derivative, the second rewrites the regular distribution in its integral form, and the third follows from partial integration. In the fourth equality, the boundary term vanishes since $f(a)$ and $f(b)$ are both zero[2]. □

2.3 Example.
The Heaviside step function $H : \mathbb{R} \to \mathbb{R}$ is given as

$$H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

This function is integrable over every compact region in $\mathbb{R}$, and thus determines a regular distribution

$$f \mapsto \Lambda_H(f) = \int_{\mathbb{R}} H(x)\, f(x)\, \mathrm{d}x = \int_0^\infty f(x)\, \mathrm{d}x$$

for all $f \in C_c^\infty(\mathbb{R})$. Now, H is not continuous and therefore not differentiable. In its stead, one can consider Λ_H and take its distributional derivative:

$$\frac{\mathrm{d}\Lambda_H}{\mathrm{d}x}(f) = -\Lambda_H\Big(\frac{\mathrm{d}f}{\mathrm{d}x}\Big) = -\int_0^\infty \frac{\mathrm{d}f}{\mathrm{d}x}(x)\, \mathrm{d}x = -f(x)\Big|_0^\infty = f(0) = \delta(f).$$

The first equality uses the definition of the distributional derivative, the second rewrites the regular distribution in its integral form, the third follows from the fundamental theorem of calculus, the fourth from the fact that $f(x) = 0$ outside a compact region, and the fifth from the definition of the Dirac δ-distribution. In other words, $\mathrm{d}\Lambda_H/\mathrm{d}x = \delta$. ◇

[2]Without loss of generality, assume that $f(b) \neq 0$. By the smoothness (or just continuity) of f, one can find an $\epsilon > 0$ such that $0 < f(b + \epsilon) < f(b)$, thereby contradicting the assumption that $f(x) = 0$ outside $[a, b]$.

3 Weak Solutions of the Wave Equation

Having presented the basic theory of distributions, as well as some examples, we now depict their application to (partial) differential equations. In particular, the wave equation serves as a good example. For this, we must replace $C_c^\infty(\mathbb{R})$ with $C_c^\infty(\mathbb{R}^2)$, where the latter consists of functions $f : \mathbb{R}^2 \to \mathbb{R}$ that are smooth and compactly supported. As a result, a distribution is a map $u : C_c^\infty(\mathbb{R}^2) \to \mathbb{R}$ that is linear and continuous. With this adjustment, the definition of a distributional derivative carries forward on replacing the derivative by a partial derivative.

3.1 Definition.

A *classical solution* of the homogeneous wave equation is a function $\Psi : \mathbb{R}^2 \to \mathbb{R}$ that is twice continuously differentiable and fulfills the equation

$$\frac{\partial^2 \Psi}{\partial t^2}(t,x) - \frac{\partial^2 \Psi}{\partial x^2}(t,x) = 0.$$

A *weak solution* of the homogeneous wave equation is a function $\Psi : \mathbb{R}^2 \to \mathbb{R}$ that is integrable over compact regions in $\mathbb{R}^2$ and fulfills the equation

$$\frac{\partial^2 \Lambda_\Psi}{\partial t^2}(f) - \frac{\partial^2 \Lambda_\Psi}{\partial x^2}(f) = 0$$

for all $f \in C_c^\infty(\mathbb{R}^2)$. Here, Λ_Ψ is the regular distribution determined by Ψ. $\diamond$

3.2 Theorem.

Every classical solution of the homogeneous wave equation is also a weak solution.

Proof. Let $\Psi : \mathbb{R}^2 \to \mathbb{R}$ be a classical solution of the homogeneous wave equation. Since Ψ is twice continuously differentiable, Proposition 2.2 (adapted for partial derivatives) applies twice to give

$$\frac{\partial^2 \Lambda_\Psi}{\partial t^2}(f) - \frac{\partial^2 \Lambda_\Psi}{\partial x^2}(f)$$

$$
\begin{aligned}
&= \iint_{\mathbb{R}^2} \frac{\partial^2 \Psi}{\partial t^2}(t,x)\, f(t,x)\,\mathrm{d}t\,\mathrm{d}x - \iint_{\mathbb{R}^2} \frac{\partial^2 \Psi}{\partial x^2}(t,x)\, f(t,x)\,\mathrm{d}t\,\mathrm{d}x \\
&= \iint_{\mathbb{R}^2} \Big(\frac{\partial^2 \Psi}{\partial t^2}(t,x) - \frac{\partial^2 \Psi}{\partial x^2}(t,x)\Big)\, f(t,x)\,\mathrm{d}t\,\mathrm{d}x \\
&= 0
\end{aligned}
$$

for all $f \in C_c^\infty(\mathbb{R}^2)$. Therefore, Ψ is also a weak solution. $\square$

3.3 Example.
Recall the Heaviside step function $H : \mathbb{R} \to \mathbb{R}$ from Example 2.3. The function $\Psi(t,x) := H(t-x)$ is a weak solution of the homogeneous wave equation. To see this, observe that

$$
\frac{\partial^2 \Lambda_\Psi}{\partial t^2}(f) - \frac{\partial^2 \Lambda_\Psi}{\partial x^2}(f) = \Lambda_\Psi\Big(\frac{\partial^2 f}{\partial t^2}\Big) - \Lambda_\Psi\Big(\frac{\partial^2 f}{\partial x^2}\Big) = \Lambda_\Psi\Big(\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2}\Big)
$$

by definition of the distributional derivative and by linearity of the distribution. On rewriting the regular distribution in its integral form, the change of variables $(t,x) \mapsto (u,v) := (t+x, t-x)$ gives

$$
\begin{aligned}
\Lambda_\Psi\Big(\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2}\Big) &= \iint_{\mathbb{R}^2} H(t-x)\,\Big(\frac{\partial^2 f}{\partial t^2}(t,x) - \frac{\partial^2 f}{\partial x^2}(t,x)\Big)\,\mathrm{d}t\,\mathrm{d}x \\
&= 4 \iint_{\mathbb{R}^2} H(v)\,\frac{\partial^2 f}{\partial u \partial v}(u,v)\,\mathrm{d}u\,\mathrm{d}v \\
&= 4 \int_0^\infty \int_{\mathbb{R}} \frac{\partial^2 f}{\partial u \partial v}(u,v)\,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}
$$

Since $f \in C_c^\infty(\mathbb{R}^2)$ is compactly supported, so is the function $\partial f/\partial v$. This results in the inner integral always evaluating to zero:

$$
\int_{\mathbb{R}} \frac{\partial}{\partial u}\frac{\partial f}{\partial v}(u,v)\,\mathrm{d}u = \frac{\partial f}{\partial v}(-,v)\Big|_{-\infty}^{\infty} = 0
$$

for all $f \in C_c^\infty(\mathbb{R}^2)$. Hence, $\Psi(t,x) = H(t-x)$ is a weak solution of the homogeneous wave equation. $\diamond$

3.4 Exercise.
Verify the change of variables $(t,x) \mapsto (u,v) := (t+x, t-x)$ *within the integral of the previous example. (Hint: factorise the second order differential operator as a product of first order differential operators to simplify the calculation.)*