Asymptotic Distribution of Eigenvalues of the Dirichlet Laplacian

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Contents

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1 Introduction

Let Ω be an open subset of $\mathbb{R}^d$. To construct the Dirichlet and Neumann Laplacians, first define the sesquilinear forms [Sch12, Section 10.6]

$$\mathfrak{q}_{D/N,\Omega}[f,g] = \int_\Omega \overline{(\nabla f(x))} \cdot (\nabla g(x))\, \mathrm{d}x \tag{1.1}$$

on $\mathrm{dom}(\mathfrak{t}_{D,\Omega}) = H^1_0(\Omega)$ and $\mathrm{dom}(\mathfrak{t}_{N,\Omega}) = H^1(\Omega)$ respectively. Here, $H^1(\Omega)$ denotes the Sobolev space $W^{1,2}(\Omega)$ [EE18, Chapter 5], the vector space of functions $f \in L^2(\Omega)$ such that for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq 1$, the weak derivative $\partial^\alpha f$ exists and also belongs to $L^2(\Omega)$. It is a Hilbert space with inner product

$$\langle f, g\rangle_{H^1(\Omega)} = \sum_{|\alpha|\leq 1} \langle \partial^\alpha f, \partial^\alpha g\rangle_{L^2(\Omega)}.$$

The subspace $H^1_0(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$, with respect to the norm induced by the inner product. Therefore the form (1.1) is, in both cases, densely defined

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on the Hilbert space $L^2(\Omega)$. Furthermore, it is positive, so the form norm

$$\|f\|^2_{\mathfrak{q}_{D/N,\Omega}} = \mathfrak{q}_{D/N,\Omega}[f] + \|f\|^2_{L^2(\Omega)}$$

coincides with that of the Sobolev space $H^1(\Omega)$. Since $H^1_0(\Omega)$ and $H^1(\Omega)$ are complete with respect to the Sobolev norm, the forms are closed. By the representation theorem for forms, the *Dirichlet* and *Neumann Laplacians* are the positive self-adjoint operators $-\Delta_{D,\Omega}$ and $-\Delta_{N,\Omega}$ associated with the positive densely defined closed forms $\mathfrak{t}_{D,\Omega}$ and $\mathfrak{t}_{N,\Omega}$ respectively. If Ω is bounded, $-\Delta_{D,\Omega}$ has a purely discrete spectrum and, assuming certain regularity conditions on the boundary $\partial\Omega$, the same is true even for $-\Delta_{N,\Omega}$.

The main result to be derived is Weyl's asymptotic formula [Sch12, Section 12.3]. It addresses the question posed by Kac: *"Can one hear the shape of a drum?"* In other words, to what extent does the spectrum of the operator encode geometric or topological information? The field of spectral geometry generalizes the problem from Laplacians on $\mathbb{R}^d$ to Laplace-Beltrami operators on Riemannian manifolds.

Section 2 establishes order relations between self-adjoint operators and subsequently between eigenvalue distributions. Section 3 presents the spectrum and eigenfunctions of the Laplacians when Ω is a cube. Section 4 includes Weyl's asymptotic formula, the proof of which approximates Ω by cubes and then incorporates the order relations in bounding the eigenvalue distribution from both sides. A certain measurability condition then ensures that the asymptotic limit exists.

2 Order Relations between Laplacians

2.1 Definition. Let T and S be self-adjoint operators on Hilbert spaces $\mathscr{G}$ and $\mathscr{H}$, such that $\mathscr{G}$ is a subspace of $\mathscr{H}$. If $\mathfrak{t}$ and $\mathfrak{s}$ are the corresponding quadratic forms, the ordering $T \geq S$ (or $S \leq T$) is defined by the conditions $\mathrm{dom}(\mathfrak{t}) \subseteq \mathrm{dom}(\mathfrak{s})$ and $\mathfrak{t}[x] \geq \mathfrak{s}[x]$ for all $x \in \mathrm{dom}(\mathfrak{t})$.

2.2 Lemma. *Let Ω and Σ be open subsets of $\mathbb{R}^d$.*

(a) *If $\Omega \subseteq \Sigma$, then $-\Delta_{D,\Sigma} \leq -\Delta_{D,\Omega}$.*

(b) $-\Delta_{N,\Omega} \leq -\Delta_{D,\Omega}$.

Proof.

(a) With the identification of $C_0^\infty(\Omega)$ and $\{f \in C_0^\infty(\Sigma) : f \restriction \Omega \in C_0^\infty(\Omega), f(\Sigma\backslash\Omega) = 0\}$, the space $C_0^\infty(\Omega)$ may be interpreted as a subspace of $C_0^\infty(\Sigma)$ when considered as the form domain. Since the form norm is also preserved, their closures yield $\mathrm{dom}(\mathfrak{q}_{D,\Omega}) \subseteq \mathrm{dom}(\mathfrak{q}_{D,\Sigma})$. On $\mathrm{dom}(\mathfrak{q}_{D,\Omega})$, the quadratic forms are identical and so $-\Delta_{D,\Sigma} \leq -\Delta_{D,\Omega}$.

(b) The inclusion $\mathrm{dom}(\mathfrak{q}_{D,\Omega}) = H_0^1(\Omega) \subseteq H^1(\Omega) = \mathrm{dom}(\mathfrak{q}_{N,\Omega})$ and the quadratic form being identical on $\mathrm{dom}(\mathfrak{q}_{D,\Omega})$ together imply $-\Delta_{N,\Omega} \leq -\Delta_{D,\Omega}$. $\square$

There is no preferred extension of $H^1(\Omega)$ to Σ, which is why there is no order relation between Neumann Laplacians on different domains. However, that is not the case for direct sums of domains.

2.3 Lemma. *Let $\Omega_1, \dots, \Omega_n$ be pairwise disjoint open subsets of $\mathbb{R}^d$.*

(a) *If $\Omega = \bigcup_{i=1}^n \Omega_i$, then $-\Delta_{D,\Omega} = \bigoplus_{i=1}^n -\Delta_{D,\Omega_i}$ on $L^2(\Omega) = \bigoplus_{i=1}^n L^2(\Omega_i)$.*

(b) *Let Ω be an open subset of $\mathbb{R}^d$ such that $\bigcup_{i=1}^n \Omega_i \subseteq \Omega$ and $\Omega \setminus \bigcup_{i=1}^n \Omega_i$ is a Lebesgue null set. Then $-\Delta_{N,\Omega} \geq \bigoplus_{i=1}^n -\Delta_{N,\Omega_i}$ on $L^2(\Omega) = \bigoplus_{i=1}^n L^2(\Omega_i)$.*

Proof. Let f_i denote the restriction of $f \in L^2(\Omega)$ to Ω_i. By linearity of the integral, it is clear that the Hilbert space can be written as $L^2(\Omega) = \bigoplus_{i=1}^n L^2(\Omega_i)$.

(a) For $f, g \in C_0^\infty(\Omega)$, it follows from Ω being the union of disjoint sets Ω_i that

$$\mathfrak{q}_{D,\Omega}[f,g] = \int_\Omega \overline{(\nabla f)(x)} \cdot (\nabla g)(x)\, \mathrm{d}x = \sum_{i=1}^n \int_{\Omega_i} \overline{(\nabla f_i)(x)} \cdot (\nabla g_i)(x)\, \mathrm{d}x = \sum_{i=1}^n \mathfrak{q}_{D,\Omega_i}[f_i, g_i],$$

and $C_0^\infty(\Omega) = \bigoplus_{i=1}^n C_0^\infty(\Omega_i)$; by continuity, the former extends to the closure of the latter, that is, $\mathrm{dom}(\mathfrak{q}_{D,\Omega}) = \bigoplus_{i=1}^n \mathrm{dom}(\mathfrak{q}_{D,\Omega_i})$. The corresponding Dirichlet Laplacians therefore fulfill $-\Delta_{D,\Omega} = \bigoplus_{i=1}^n -\Delta_{D,\Omega_i}$.

(b) Let $f, g \in H^1(\Omega)$. The condition $\bigcup_{i=1}^n \Omega_i \subseteq \Omega$ obviously implies $f_i, g_i \in H^1(\Omega_i)$. Since $\Omega \setminus \bigcup_{i=1}^n \Omega_i$ is a Lebesgue null set,

$$\mathfrak{q}_{N,\Omega}[f,g] = \int_\Omega \overline{(\nabla f)(x)} \cdot (\nabla g(x))\, \mathrm{d}x = \sum_{i=1}^n \int_{\Omega_i} \overline{(\nabla f_i)(x)} \cdot (\nabla g_i)(x)\, \mathrm{d}x = \sum_{i=1}^n \mathfrak{q}_{N,\Omega_i}[f_i, g_i].$$

Due to the inclusion $\mathrm{dom}(q_{N,\Omega}) = H^1(\Omega) \subseteq \bigoplus_{i=1}^n H^1(\Omega_i) = \bigoplus_{i=1}^n \mathrm{dom}(q_{N,\Omega_i})$, the corresponding Neumann Laplacians fulfill $-\Delta_{N,\Omega} \geq \bigoplus_{i=1}^n -\Delta_{N,\Omega_i}$. $\square$

Let T be a self-adjoint operator on $\mathscr{H}$, with spectral measure E_T. The *eigenvalue distribution* of T is defined as $N_T(\lambda) = \dim E_T((-\infty, \lambda))\mathscr{H}$ for $\lambda \in \mathbb{R}$. The sequence $\lambda_n(T) = \inf\{\lambda : N_T(\lambda) \geq n\}$ then enumerates the eigenvalues of T in increasing order, counted with multiplicity, and converges from below to $\inf \sigma_{\mathrm{ess}}(T)$.

2.4 Lemma. *Let T and S be self-adjoint operators on $\mathscr{H}$.*

(a) *If $T \geq S$, then $N_T(\lambda) \leq N_S(\lambda)$.*

(b) *If $T = \bigoplus_{i=1}^n T_i$ is an orthogonal sum of self-adjoint operators T_i on $\mathscr{H} = \bigoplus_{i=1}^n \mathscr{H}_i$, then $N_T(\lambda) = \sum_{i=1}^n N_{T_i}(\lambda)$.*

Proof.

(a) Let $\mathfrak{t}$ and $\mathfrak{s}$ denote the quadratic forms associated with T and S respectively. By definition, $T \geq S$ implies $\mathfrak{t}[x] \geq \mathfrak{s}[x]$ for all $x \in \operatorname{dom}(\mathfrak{t})$. If $F_n(\mathscr{H})$ denotes the set of linear subspaces having at most n dimensions, the min-max principle [Sch12, Theorem 12.1] shows that

$$\begin{aligned}
\lambda_n(T) &= \sup_{D \in F_{n-1}(\mathscr{H})} \inf\{\mathfrak{t}[x] : x \in \operatorname{dom}(\mathfrak{t}), \|x\| = 1, x \perp D\} \\
&\geq \sup_{D \in F_{n-1}(\mathscr{H})} \inf\{\mathfrak{s}[x] : x \in \operatorname{dom}(\mathfrak{t}), \|x\| = 1, x \perp D\} \\
&\geq \sup_{D \in F_{n-1}(\mathscr{H})} \inf\{\mathfrak{s}[x] : x \in \operatorname{dom}(\mathfrak{s}), \|x\| = 1, x \perp D\} = \lambda_n(S),
\end{aligned}$$

and consequently $N_T(\lambda) \leq N_S(\lambda)$.

(b) Since $T = \bigoplus_{i=1}^n T_i$ is an orthogonal sum of self-adjoint operators, the corresponding spectral measures are given by $E_T = \bigoplus_{i=1}^n E_{T_i}$. Then

$$N_T(\lambda) = \dim \bigoplus_{i=1}^n E_{T_i}((-\infty, \lambda))\mathscr{H} = \sum_{i=1}^n \dim E_{T_i}((-\infty, \lambda))\mathscr{H} = \sum_{i=1}^n N_{T_i}(\lambda)$$

by the definition of direct sums of Hilbert spaces [Con07, Section 1.6]. □

Therefore, the min-max principle converts the order relations of self-adjoint operators into an ordering of their eigenvalue distributions. For the Laplacians, the eigenvalue distributions $N_{-\Delta_{D/N,\Omega}}$ will henceforth be abbreviated as $N_{D/N,\Omega}$.

3 Examples of the Discrete Spectrum

3.1 Example. Let $\Omega = (a, a+l) \subset \mathbb{R}$ where $l > 0$. The Dirichlet Laplacian is the differential operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with boundary conditions $f(a) = f(a+l) = 0$. On solving the eigenvalue equation, the eigenfunctions are

$$\begin{aligned}
\phi_{2k}(x) &= \sin 2k\pi l^{-1}(x - a - l/2), \\
\phi_{2k-1}(x) &= \cos(2k-1)\pi l^{-1}(x - a - l/2), \quad k \in \mathbb{N},
\end{aligned}$$

from which the spectrum can easily be seen as

$$\sigma(-\Delta_{D,\Omega}) = \{n^2\pi^2 l^{-2} : n \in \mathbb{N}\}.$$

To ensure there are no more eigenvalues, it must be proven that the eigenfunctions form a basis of $L^2(\Omega)$. Due to the self-adjointness of $-\Delta_{D,\Omega}$, the eigenfunctions are pairwise orthogonal. It follows from Parseval's identity that $\{e^{-\imath nx} : n \in \mathbb{Z}\}$ is a basis of $L^2((-\pi,\pi))$ [Rud76, Corollary to Theorem 11.40]. On translating the system from $(-\pi,\pi)$ to Ω, the same applies to the eigenfunctions ϕ_n, since all separable infinite-dimensional Hilbert spaces are isomorphic [Con07, Corollary 1.5.5].

Similarly, the Neumann Laplacian is the differential operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with boundary conditions $f'(a) = f'(a+l) = 0$. The eigenfunctions are

$$\begin{aligned} \psi_{2k}(x) &= \cos 2k\pi l^{-1}(x-a-l/2), \\ \psi_{2k+1}(x) &= \sin(2k+1)\pi l^{-1}(x-a-l/2), \quad k \in \mathbb{N}_0, \end{aligned}$$

and the spectrum is

$$\sigma(-\Delta_{N,\Omega}) = \{n^2\pi^2 l^{-2} : n \in \mathbb{N}_0\}.$$

That the eigenfunctions form a basis of $L^2(\Omega)$ is proven as in the Dirichlet case.

3.2 Example. Let $\Omega = (a_1, a_1+l) \times \ldots \times (a_d, a_d+l) \subseteq \mathbb{R}^d$ where $l > 0$. For $n \in \mathbb{N}^d$, the method of separation of variables yields the eigenfunctions $\phi_n(x) = \phi_{n_1}(x_1) \cdots \phi_{n_d}(x_d)$ for $-\Delta_{D,\Omega}$, with spectrum

$$\sigma(-\Delta_{D,\Omega}) = \{(n_1^2 + \ldots + n_d^2)\pi^2 l^{-2} : n_1, \ldots, n_d \in \mathbb{N}\}.$$

Similarly, for $n \in \mathbb{N}_0^d$, the eigenfunctions are $\psi_n(x) = \psi_{n_1}(x_1) \cdots \psi_{n_d}(x_d)$ for $-\Delta_{N,\Omega}$, with spectrum

$$\sigma(-\Delta_{N,\Omega}) = \{(n_1^2 + \ldots + n_d^2)\pi^2 l^{-2} : n_1, \ldots, n_d \in \mathbb{N}_0\}.$$

For both, the Dirichlet and Neumann Laplacians, the eigenfunctions are defined on the tensor product space $L^2((a_1, a_1+l)) \otimes \cdots \otimes L^2((a_d, a_d+l))$, and it is a standard result that the tensor product of their bases is a basis for the tensor product space. Hence, there are no more eigenvalues in the spectrum than those mentioned above.

The existence and completeness of eigenfunctions may also be proven formally, using the weak formulation of the eigenvalue equation and some variational principles [Nic11].

4 Asymptotic Distribution of Eigenvalues

4.1 Lemma. *Let $\Sigma = (a_1, a_1 + l) \times \ldots \times (a_d, a_d + l) \subseteq \mathbb{R}^d$, where $l > 0$. For $\lambda > 0$, the following inequalities hold for the eigenvalue distributions of the Laplacians:*

$$\left| N_{D,\Sigma}(\lambda) - (2\pi)^{-d}\lambda^{d/2}\omega_d l^d \right| \le \sum_{k=1}^{d-1} (2\pi)^{-k}\lambda^{k/2}\omega_k l^k, \tag{4.1}$$

$$\left| N_{N,\Sigma}(\lambda) - (2\pi)^{-d}\lambda^{d/2}\omega_d l^d \right| \le \sum_{k=1}^{d-1} (2\pi)^{-k}\lambda^{k/2}\omega_k l^k, \tag{4.2}$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$.

Proof. The domain Σ was considered in Example 3.2 and so the spectrum is identical. Let $B(r)$ denote the open ball centred at the origin and of radius r under the Euclidean norm in $\mathbb{R}^d$. For the Dirichlet Laplacian, the eigenvalue distribution $N_{D,\Sigma}(\lambda)$ counts the elements $n \in \mathbb{N}^d$ that fulfill $(n_1^2 + \ldots + n_d^2)\pi^2 l^{-2} < \lambda$, or equivalently, counts the elements in $\mathbb{N}^d \cap B(R)$, where $R = \lambda^{1/2} l \pi^{-1}$. Similarly, for the Neumann Laplacian, the eigenvalue distribution $N_{N,\Sigma}(\lambda)$ counts the elements in $\mathbb{N}_0^d \cap B(R)$.

Associated with every $m \in \mathbb{N}^d \cap B(R)$ and $n \in \mathbb{N}_0^d \cap B(R)$ are disjoint unit cubes

$$P_m = \{x \in \mathbb{R}^d : m_i - 1 \le x_i < m_i,\ i = 1, \ldots, d\},$$
$$Q_n = \{x \in \mathbb{R}^d : n_i \le x_i < n_i + 1,\ i = 1, \ldots, d\},$$

respectively. If $B_+(r) = \{x \in B(r) : x_i \ge 0,\ i = 1, \ldots, d\}$ is the intersection of the non-negative orthant with $B(r)$, then the union of all cubes P_m is contained in $B_+(R)$, while the union of all Q_n contains $B_+(R)$. Since the eigenvalue distribution $N_{D,\Sigma}(\lambda)$ is just the volume of $\bigcup_{m \in \mathbb{N}^d \cap B(R)} P_m$ and $N_{N,\Sigma}(\lambda)$ is that of $\bigcup_{n \in \mathbb{N}_0^d \cap B(R)} Q_n$,

$$N_{D,\Sigma}(\lambda) \le |B_+(R)| = 2^{-d}\omega_d R^d = (2\pi)^{-d}\lambda^{d/2}\omega_d l^d \le N_{N,\Sigma}(\lambda). \tag{4.3}$$

To estimate the difference $N_{N,\Sigma}(\lambda) - N_{D,\Sigma}(\lambda)$, note that these are precisely the elements in $(\mathbb{N}_0^d \setminus \mathbb{N}^d) \cap B(R)$. Equivalently, this is the union $\bigcup_{k=0}^{d-1} Z_k$ where Z_k includes the elements in $\mathbb{N}_0^d \cap B(R)$ that have exactly k nonzero components. Repeating the construction of P_m above for the elements in Z_k shows that the number of these elements is less than the volume of the k-dimensional open ball $B_+^k(R)$, yielding the inequality

$$0 \le N_{N,\Sigma}(\lambda) - N_{D,\Sigma}(\lambda) \le \sum_{k=0}^{d-1} 2^{-k}\omega_k R^k = \sum_{k=0}^{d-1} (2\pi)^{-k}\lambda^{k/2}\omega_k l^k. \tag{4.4}$$

Together, (4.3) and (4.4) imply (4.1) and (4.2). $\square$

Let Ω be a bounded subset of $\mathbb{R}^d$. Then there exists a closed cube $\mathcal{O} = [a,b]^d \subset \mathbb{R}^d$ that contains Ω, and a partition of $\mathcal{O}$ by s^d cubes Σ_k of length $l = (b-a)/s$. If the inner Jordan measure $\underline{J}(\Omega) = \sup_s\{|\bigcup_k \Sigma_k| : \Sigma_k \subset \Omega\}$ equals the outer Jordan measure $\overline{J}(\Omega) = \inf_s\{|\bigcup_k \Sigma_k| : \Sigma_k \cap \Omega \neq \varnothing\}$, the subset Ω is said to be *Jordan measurable* and the common value is called the *Jordan content* $J(\Omega)$.

4.2 Theorem. (Weyl's asymptotic formula) *Let Ω be a bounded open Jordan measurable subset of $\mathbb{R}^d$. If ω_d is the volume of the unit ball in $\mathbb{R}^d$, the asymptotic distribution of eigenvalues of the Dirichlet Laplacian is given by*

$$\lim_{\lambda\to+\infty} N_{D,\Omega}(\lambda)\lambda^{-d/2} = (2\pi)^{-d}\omega_d J(\Omega).$$

Proof. Consider a cube $\mathcal{O} = [a,b]^d$ that contains Ω and construct a partition Σ_k of $\mathcal{O}$ with length $l = (b-a)/s$.

First, let $P_1, \ldots, P_m$ denote the interior of the cubes Σ_k that lie in the interior of Ω. By construction, $P := \bigcup_{i=1}^m P_i \subseteq O$, and so Lemmas 2.2(a) and 2.3(a) imply

$$-\Delta_{D,\Omega} \leq -\Delta_{D,P} = \bigoplus_{i=1}^m -\Delta_{D,P_i}.$$

Every cube P_i fulfills the criteria of Lemma 4.1, allowing the application of (4.1) after Lemma 2.4 to obtain a lower bound for the eigenvalue distribution

$$N_{D,\Omega}(\lambda) \geq \sum_{i=1}^m N_{D,P_i}(\lambda) \geq (2\pi)^{-d}\lambda^{d/2}\omega_d|P| - m\sum_{i=1}^{d-1}(2\pi)^{-i}\lambda^{i/2}\omega_i l^i, \tag{4.5}$$

where $|P| = ml^d$ is the volume of P.

Second, let $Q_1, \ldots, Q_n$ denote the interior of the cubes Σ_k that intersect Ω and let Q be the interior of $\bigcup_{i=1}^n \overline{Q_i}$. Then $\Omega \subseteq Q$ and $Q \setminus \bigcup_{i=1}^n Q_i$ is a Lebesgue null set, so Lemmas 2.2(a), 2.2(b) and 2.3(b) imply

$$-\Delta_{D,\Omega} \geq -\Delta_{D,Q} \geq -\Delta_{N,Q} \geq \bigoplus_{i=1}^n -\Delta_{N,Q_i}.$$

Every cube Q_i fulfills the criteria of Lemma 4.1, again allowing the application of (4.2) after Lemma 2.4 to now obtain an upper bound for the eigenvalue distribution

$$N_{D,\Omega}(\lambda) \leq \sum_{i=1}^n N_{D,Q_i}(\lambda) \leq (2\pi)^{-d}\lambda^{d/2}\omega_d|Q| + m\sum_{i=1}^{d-1}(2\pi)^{-i}\lambda^{i/2}\omega_i l^i, \tag{4.6}$$

where $|Q| = nl^d$ is the volume of Q.

Noting that the order of λ does not exceed $d/2$ in (4.5) and (4.6), multiplying by $\lambda^{-d/2}$ and taking the limit $\lambda \to +\infty$ yields

$$(2\pi)^{-d}\omega_d|P| \leq \liminf_{\lambda\to+\infty} N_{D,\Omega}(\lambda)\lambda^{-d/2} \leq \limsup_{\lambda\to+\infty} N_{D,\Omega}(\lambda)\lambda^{-d/2} \leq (2\pi)^{-d}\omega_d|Q|.$$

The Jordan measurability of Ω implies that $\sup_s |P| = \inf_s |Q| = J(\Omega)$, and so the limit $\lambda \to +\infty$ exists. Therefore, $\lim_{\lambda\to+\infty} N_{D,\Omega}(\lambda)\lambda^{-d/2} = (2\pi)^{-d}\omega_d J(\Omega)$. □

Asymptotic analysis of the spectrum of the Dirichlet Laplacian therefore reveals the dimension d and Jordan content $J(\Omega)$ of Ω. There exist extensions of Weyl's asymptotic formula to Schrödinger operators $H = -\Delta + V$ [Pan20, Section 8.2], but that will not be considered here.

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