The Bianchi(-Behr) Classification and Anisotropic Cosmological Spacetimes

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Abstract

After a brief review of the mathematical formulation of symmetries, spacetimes with a three-dimensional isometry (sub)group are considered. For the Bianchi(-Behr) classification of its generating Lie algebra, a partial construction using the derived subalgebra and a complete construction using the structure constants reveal ten distinct types. In the Bianchi I universe, a generalisation of the FLRW model, the shear is seen to be the source of anisotropy. As a consequence, more types of singularities appear. Other examples of spacetimes given are Kantowski-Sachs and Taub-NUT.

1 Introduction

In cosmology, the Friedmann-Lemaître-Robertson-Walker (FLRW) model is generally accepted as a representation of the universe. However, it possesses a high degree of symmetry, in terms of spatial homogeneity and isotropy. Since the universe is not perfectly isotropic, it is the aim of this paper to identify the source and analyse the consequences of anisotropy, while still retaining the notion of homogeneity.

Section 2 recalls the notions of Lie groups and Lie algebras, followed by their application in defining an isometry group from the real Lie algebra of Killing vector fields. Considering spacetimes with a three-dimensional isometry (sub)group, section 3 introduces the classification of three-dimensional real Lie algebras using two different approaches. Section 4 first provides an argument as to why the assumption of spacetime homogeneity is unphysical, and then presents some anisotropic spacetimes - spatially homogeneous Bianchi I, non-spatially homogeneous Kantowski-Sachs, and partially spatially homogeneous Taub-NUT. Lastly, a singularity theorem for spatially homogeneous spacetimes is stated.

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2 Symmetries of a spacetime

2.1 Lie groups and Lie algebras

A *Lie group* G is a group and a smooth manifold, and the maps $m : G \times G \to G$ and $i : G \to G$ given by $m(g_1, g_2) := g_1 g_2$ and $i(g) := g^{-1}$ respectively are both smooth. If M is a smooth manifold, a left-action of G on M is defined as the smooth map

$$\begin{aligned} G \times M &\to M \\ (g, p) &\mapsto g \cdot p \end{aligned} \tag{1}$$

which fulfills $g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p$ and $e \cdot p = p$, where e is the identity of G. With reference to a point $p \in M$, the set of all points under the action of G is called its *orbit*,

$$G \cdot p := \{g \cdot p \mid g \in G\}. \tag{2}$$

If any g leaves p invariant, it belongs to the *stabilizer subgroup*, $\{g \mid g \cdot p = p\}$. The action is said to be *transitive* if the orbit is M, and to be *free* if all stabilizer subgroups are trivial. [9]

A real *Lie algebra* $\mathfrak{g}$ is a real vector space equipped with a bilinear, antisymmetric map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the *Lie bracket*, that satisfies the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \tag{3}$$

for $X, Y, Z \in \mathfrak{g}$. Given a basis $\{E_A\}$ of $\mathfrak{g}$, the Lie bracket introduces coefficients $C^D{}_{AB}$, called the *structure constants*, as $[E_A, E_B] := C^D{}_{AB} E_D$, in terms of which the Jacobi identity becomes

$$C^A{}_{[DF} C^B{}_{G]A} = 0. \tag{4}$$

Lie algebras with identical structure constants are isomorphic. Therefore, the classification of Lie algebras relies on the *derived subalgebra*

$$[\mathfrak{g}, \mathfrak{g}] := \{[X, Y] \mid X, Y \in \mathfrak{g}\} = \mathrm{span}\{C^D{}_{AB} E_D \mid \{E_D\} \text{ basis of } \mathfrak{g}\}. \tag{5}$$

To see the relation between a Lie group and a Lie algebra, consider the left-translation map $L_g : G \to G$ given by $L_g g' = gg'$. Every vector in the tangent space $\mathrm{T}_e G$ generates a unique left-invariant vector field on G as $X(g) = (L_g)_*(e) X(e)$. For any two left-invariant vector fields X, Y and a smooth function f on G, the Lie bracket

$$[X, Y] f := X(Y(f)) - Y(X(f)) \tag{6}$$

is also left-invariant and establishes $\mathrm{T}_e G$ as a Lie algebra. Since the map $\exp : \mathrm{T}_e G \to G$ is known to be a local diffeomorphism, the Lie algebra is said to generate the Lie group.

2.2 Killing vector fields and the isometry group

On an n-dimensional (pseudo-)Riemannian manifold (M, g), a *Killing vector field* ξ is the generator of a one-parameter group of isometries; the metric is preserved under its flow, $\mathscr{L}_\xi g = 0$. With the Lie derivative being $\mathbb{R}$-linear and fulfilling $\mathscr{L}_{[X,Y]} = [\mathscr{L}_X, \mathscr{L}_Y]$, the set of Killing vector fields $\{\xi_A\}$ is closed under the bracket, forming a real Lie algebra $\mathfrak{g}_r$ of dimension r. That this Lie algebra is finite-dimensional follows from the fact that M admits no more than $\frac{1}{2}n(n+1)$ linearly independent Killing vector fields. Consequently, $\mathfrak{g}_r$ generates an *isometry group* G_r.

Let (M, g) now represent a spacetime[1]. For reasons outlined in section 4, it will be assumed that the spacetime is only *spatially homogeneous*, i.e. there is an isometry (sub)group G_3 whose action is free and transitive on spacelike hypersurfaces. Naturally then, this implies the bounds $3 \leq r \leq 6$. The difference $s = r - 3 \leq 3$ determines the dimension of the stabilizer subgroups, also called the *isotropy groups*. Of interest here are the cases $s = 1$, *local rotational symmetry* (LRS), and $s = 0$, *anisotropy*. The former includes the Kantowski-Sachs universe and the LRS Bianchi universes, while the latter includes the remaining Bianchi universes [2].

3 The Bianchi(-Behr) classification

Bianchi introduced a classification of three-dimensional real Lie algebras [7], which will prove beneficial in identifying the types of $\mathfrak{g}_3$ that generate the transitive isometry subgroup G_3 of the spacelike hypersurfaces. Reconstructed here is the initial part of the approach [1] which makes apparent the role of the derived subalgebra.

Consider a three-dimensional real Lie algebra $\mathfrak{g}$. If the derived subalgebra $[\mathfrak{g}, \mathfrak{g}]$ has dimension zero, $\mathfrak{g}$ is abelian and represents Bianchi Type I. Let the derived subalgebra have dimension one, with a basis $\{E_1\}$. Its extension to a basis of $\mathfrak{g}$ as $\{E_1, E_2, E_3\}$ has the general form

$$[E_1, E_2] = \alpha E_1, \qquad [E_1, E_3] = \beta E_1, \qquad [E_2, E_3] = \gamma E_1, \tag{7}$$

with α, β, $\gamma \in \mathbb{R}$. For the first case, assume $\beta \neq 0$. Introduce the new basis $\{X_1, X_2, X_3\}$ as

$$X_1 = E_1, \qquad X_2 = \gamma E_1 - \beta E_2 + \alpha E_3, \qquad X_3 = (1/\beta) E_3, \tag{8}$$

so that the structure constants are simplified

$$\begin{aligned}
[X_1, X_2] &= -\beta[E_1, E_2] + \alpha[E_1, E_3] = -\beta\alpha E_1 + \alpha\beta E_1 = 0, && (9)\\
[X_1, X_3] &= \frac{1}{\beta}[E_1, E_3] = \frac{1}{\beta}\beta E_1 = X_1, && (10)\\
[X_2, X_3] &= \frac{\gamma}{\beta}[E_1, E_3] - \frac{\beta}{\beta}[E_2, E_3] = \frac{\gamma}{\beta}\beta E_1 - \gamma E_1 = 0, && (11)
\end{aligned}$$

representing Bianchi Type III. The case $\alpha \neq 0$ is of the same type under the exchange $X_2 \leftrightarrow X_3$.

[1] A four-dimensional connected Lorentzian manifold with a time orientation.

Since either α or β being nonzero results in Bianchi Type III, the last case to consider is $\gamma \neq 0$ with $\alpha = \beta = 0$. Rescale the basis as $X_1 = E_1$, $X_2 = E_2$, $X_3 = (1/\gamma)E_3$, so that

$$[X_1, X_2] = 0, \qquad [X_1, X_3] = 0, \qquad [X_2, X_3] = X_1, \tag{12}$$

representing Bianchi Type II. Further cases of higher-dimensional derived subalgebras are more complicated and can be referred in [1]. The final result is a classification of the three-dimensional real Lie algebras into nine distinct types, as shown in table 1. Note that types VI and VII are one-parameter families of Lie algebras; different values of the parameter q correspond to non-isomorphic Lie algebras.

Type	$[X_1, X_2]$	$[X_1, X_3]$	$[X_2, X_3]$	
I	0	0	0	
II	0	0	X_1	
III	0	X_1	0	
IV	0	X_1	$X_1 + X_2$	
V	0	X_1	X_2	
VI	0	X_1	qX_2	$q \neq 0, 1$
VII	0	X_2	$-X_1 + qX_2$	$q^2 < 4$
VIII	X_1	$2X_2$	X_3	
IX	X_3	$-X_2$	X_1	

Table 1: Bianchi classification of three-dimensional real Lie algebras [7].

Behr introduced a different technique in deriving a stronger classification [4]. Let $C^E{}_{BD}$ denote the structure constants of $\mathfrak{g}$. Using the antisymmetric Levi-Civita tensor density, ε^{FBD} with $\varepsilon^{123} = 1$, the nine independent structure constants can be extracted[2] into a rank two tensor which is further subdivided into a symmetric and antisymmetric part [11] as

$$C^E{}_{BD}\varepsilon^{FBD} =: C^{EF} = C^{(EF)} + C^{[EF]} =: N^{EF} + \varepsilon^{EFG}A_G. \tag{13}$$

In the last equality, the three independent components of $C^{[EF]}$ are written as a one-form A_G. Contracting equation (4) in a similar manner, and repeatedly using equation (13),

$$\begin{aligned} 0 = \varepsilon^{DFG}C^E{}_{[DF}C^B{}_{G]E} = C^{ED}C^B{}_{ED} &= \varepsilon^{EDF}A_F C^B{}_{ED} \\ &= A_F C^{BF} \\ &= A_F(N^{BF} + \varepsilon^{BFD}A_D), \end{aligned} \tag{14}$$

where the constant that arises in the second equality has been ignored. Due to the antisymmetry

[2] In a sense, similar to the action of the Hodge dual.

of ε^{BFD}, the second term vanishes, resulting in a simpler form of the Jacobi identity,

$$N^{BF}A_F = 0. \tag{15}$$

Since N^{BF} is symmetric, it can be diagonalized as $N^{BF} = \text{diag}(N_1, N_2, N_3)$. The above equation also shows that A_F is an eigenvector of N^{BF}, and so it can be expressed as $A_F = (A, 0, 0)$. The Jacobi identity becomes $N_1 A = 0$, introducing the first classification: $A = 0$ is called class A, while $A \neq 0$ is called class B. Within each class, the rank and signature of N^{BF} determine the types, as can be seen in table 2.

Class	Type	A	N_1	N_2	N_3	
A	I	0	0	0	0	
	II	0	+	0	0	
	VI_0	0	+	−	0	
	VII_0	0	+	+	0	
	VIII	0	+	+	−	
	IX	0	+	+	+	
B	V	+	0	0	0	
	IV	+	0	0	+	
	VI_h	+	0	+	−	$h < 0$
	VII_h	+	0	+	+	$h > 0$

Table 2: Behr's version of the Bianchi classification [3].

For type VI_h, the parameter h is related to q by $h = -(1+q)^2/(1-q)^2$, and for type VII_h, by $h = q^2/(4-q^2)$. Whenever $AN_2N_3 \neq 0$, the parameter h is absorbed into A as $A^2 = hN_2N_3$. Also, Bianchi Type III, which seems to be missing, corresponds to Type VI_{-1}.

4 Anisotropic cosmological spacetimes

Homogeneous spacetimes are not physical because there cannot be any expansion. To see this, consider the stress-energy tensor of a perfect fluid [3],

$$T_{ab} = \rho u_a u_b + p(g_{ab} + u_a u_b), \tag{16}$$

with normalised velocity vector field u^a, energy density ρ, and pressure p. If the spacetime is homogeneous, the energy density ρ is constant everywhere, and $\nabla_a \rho = 0$. Conservation of the stress-energy tensor, $\nabla^b T_{ab} = 0$, then establishes the constraint

$$u^a \nabla_a \rho = -(\rho + p)\nabla_a u^a \quad \implies \quad \nabla_a u^a = 0. \tag{17}$$

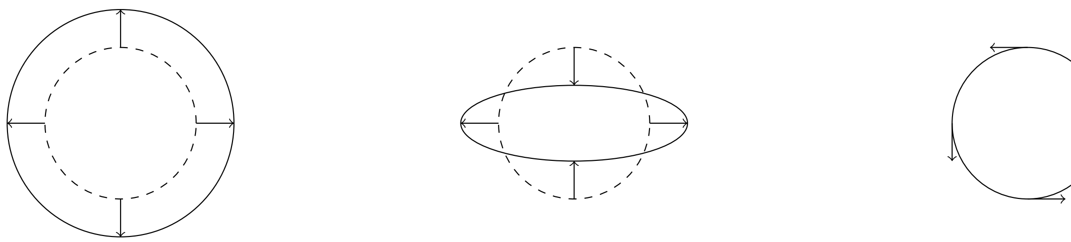

Figure 1: Expansion θ, shear σ_{ab}, and rotation ω_{ab}.

The decomposition into the trace, symmetric trace-free, and antisymmetric parts, of

$$\nabla_a u_b = \frac{1}{3}\theta(g_{ab} + u_a u_b) + \sigma_{ab} + \omega_{ab} \tag{18}$$

has the physical interpretation of expansion, shear, and twist, respectively. From equation (17), the expansion $\theta = \nabla_a u^a$ vanishes, thus contradicting the observed redshift.

In a spatially homogeneous spacetime, given a Lie algebra $\mathfrak{g}_3$ of Killing vector fields $\{\xi_A\}$ that generates the free and transitive isometry (sub)group G_3, a triad $\{\eta_A\}$ can be constructed such that it is preserved under the Lie derivative along the Killing vector fields, $\mathscr{L}_{\xi_A}\eta_B = [\xi_A, \eta_B] = 0$. In the vacuum case[3], the normal to the spacelike hypersurfaces completes the tetrad as $\eta_0 = \partial_t$. With the duals $\{\omega^A\}$ of this triad, the general form of the metric [10] is

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \gamma_{AB}(t)\omega^A\omega^B, \tag{19}$$

where γ_{AB} is constant on each spacelike hypersurface due to the Killing equation. A complete list of coordinate representations of Killing vector fields and their corresponding triads and duals for each Bianchi type was given by Taub [12].

4.1 Bianchi I universe

As a generalisation of the FLRW spacetime, the metric takes the form

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a_1^2(t)\,\mathrm{d}x^2 + a_2^2(t)\,\mathrm{d}y^2 + a_3^2(t)\,\mathrm{d}z^2\,, \tag{20}$$

allowing for anisotropic expansion, with an averaged scale factor $a^3(t) = a_1a_2a_3$. It is evident that the coordinate vector fields $\xi_A = \partial_A$ are Killing vector fields since the metric has no spatial dependence. Furthermore, these vector fields commute, generating Bianchi Type I.

Considering the dust model [6], the energy density ρ determines a constant M such that $\frac{4}{3}\pi a^3 = M\rho^{-1}$. On solving the field equations, the averaged scale factor is

$$a^3(t) = \frac{9}{2}Mt(t + \Sigma) \tag{21}$$

[3] For a perfect fluid, there are two subcases: if the timelike velocity vector field u^a is normal to the spacelike hypersurfaces, the spacetime is said to be *orthogonal*; if not, it is *tilted* and there are two choices for η_0.

where Σ measures the magnitude of the shear σ_{ab},

$$\Sigma^2 = \frac{1}{2}\Sigma_{ab}\Sigma^{ab} = \frac{1}{2}(a^3\sigma_{ab})(a^3\sigma^{ab}). \tag{22}$$

It is precisely the shear which encodes the anisotropy, without which this spacetime would just be that of FLRW. Meanwhile, the scale factors are

$$a_i(t) = a\left(\frac{t^2}{a^3}\right)^{\frac{2}{3}\sin\alpha_i} \quad \text{where} \quad \alpha_i = \alpha + (i-1)\frac{2\pi}{3} \tag{23}$$

for some constant $\alpha \in (-\frac{\pi}{6}, \frac{\pi}{2}]$. This spacetime is not always anisotropic, as $\alpha = \pi/6$ or $\pi/2$ possess local rotational symmetry since two of the scale factors are equal.

To analyse the singularity structure, consider the expansion rate equation

$$\frac{\dot{a}_i}{a_i} = \frac{2}{3t}\frac{t + \Sigma(1 + 2\sin\alpha_i)}{t + \Sigma}. \tag{24}$$

At late times, the spacetime tends towards isotropy. However, at early times, especially when $t \to 0$, the expansion is dominated by the shear. For the case $\alpha \neq \frac{\pi}{2}$, one of the shear coefficients $(1 + 2\sin\alpha_i)$ is negative while the other two are positive. When visualized under time reversal[4], equation (24) shows that the former will expand divergently while the latter will contract to zero, known as a *cigar* singularity. As a special case, if $\alpha = \frac{\pi}{2}$, one of the shear coefficients is positive while the other two are zero. Again viewed under time reversal, equation (24) shows that the former will contract to zero while the latter contract to finite nonzero values, also called a *pancake* singularity.

4.2 Kantowski-Sachs universe

Possessing local rotational symmetry, the isometry group $G_4 \cong \mathbb{R} \times SO(3)$ has a subgroup $G_3 \cong SO(3)$ of Bianchi Type IX. Unlike the spatially homogeneous spacetimes, this subgroup has a transitive action on only two-dimensional surfaces, which have to be of constant curvature K since the maximal number of symmetries of a two-dimensional surface is three. The Kantowski-Sachs spacetime corresponds to $K = 1$, hence the metric is [8]

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + A^2(t)\,\mathrm{d}\chi^2 + B^2(t)\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \tag{25}$$

which easily reflects the topology $\mathbb{R} \times \mathbb{R} \times S^2$.

For a perfect fluid matter model, the spacetime is geodesically incomplete towards the past and the future. A nonzero cosmological constant admits cigar, pancake, and point singularities, while a positive cosmological constant allows two more types of singularities, namely infinite and infinite barrel [13].

[4]Purely as a mathematical tool to analyse the expansion rate equation.

	A	B	Singularity type
$\Lambda \neq 0,\, t \to 0$	$t^{-1/3}$	$t^{2/3}$	Cigar
	t	const	Pancake
	$t^{2/3\gamma}$	$t^{2/3\gamma}$	Point
$\Lambda > 0,\, t \to \pm\infty$	$\mathrm{e}^{\pm\sqrt{\Lambda/3}t}$	$\mathrm{e}^{\pm\sqrt{\Lambda/3}t}$	Infinite
	$\mathrm{e}^{\pm\sqrt{\Lambda}t}$	const	Infinite barrel

Table 3: Asymptotic behaviour of A and B and the associated types of singularities [13]. In the third row, γ is determined by the fluid's equation of state, $p = (\gamma - 1)\rho$.

4.3 Taub-NUT spacetime

In a different form [5] from that given by Misner, the metric

$$\mathrm{d}s^2 = -f(r)\,(\mathrm{d}t - 2l\cos\theta\,\mathrm{d}\phi)^2 + f(r)^{-1}\,\mathrm{d}r^2 + (r^2 + l^2)\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \tag{26}$$

where

$$f(r) = \frac{r^2 - 2mr - l^2}{r^2 + l^2}, \tag{27}$$

includes the NUT parameter l and a constant $m > 0$. If $l = 0$, the metric is seen to be that of Schwarzschild, with m being the mass. Henceforth, l will be assumed nonzero.

At $r = 0$, neither the metric nor the scalar curvature are singular. Thus r can be extended to the full range $(-\infty, \infty)$. However, the metric does become singular when $f(r) = 0$, with the roots $r_\pm = m \pm \sqrt{m^2 + l^2}$ separating the two spacetimes.

The Taub region: $r_- < r < r_+$ such that $f(r) < 0$.
Since t is spacelike and r is timelike, it exists for only finite time, and is a spatially homogeneous spacetime of Bianchi Type IX. At $\theta = 0$ or π, there is an ambiguity in the direction of time, which is fixed by the transformations (Misner's original construction)

$$t \to t + 2l\phi \quad \text{for} \quad 0 < \theta < \pi/2, \tag{28}$$

$$t \to t - 2l\phi \quad \text{for} \quad \pi/2 < \theta < \pi. \tag{29}$$

However, ϕ then imposes a periodicity in t, and the spacetime topology is seen to be $\mathbb{R} \times S^3$.

The NUT regions: $r < r_-$ and $r > r_+$ such that $f(r) > 0$.
Since t is timelike and r is spacelike, the metric is stationary and the roots $r_\pm$ correspond to distinct Killing horizons. When t, r, and θ are constant, the metric

$$\mathrm{d}s^2 = -\left(4l^2 f(r)(1 - \cos\theta)^2 - (r^2 + l^2)\sin^2\theta\right)\mathrm{d}\phi^2 \tag{30}$$

admits, in certain sections, closed timelike curves.

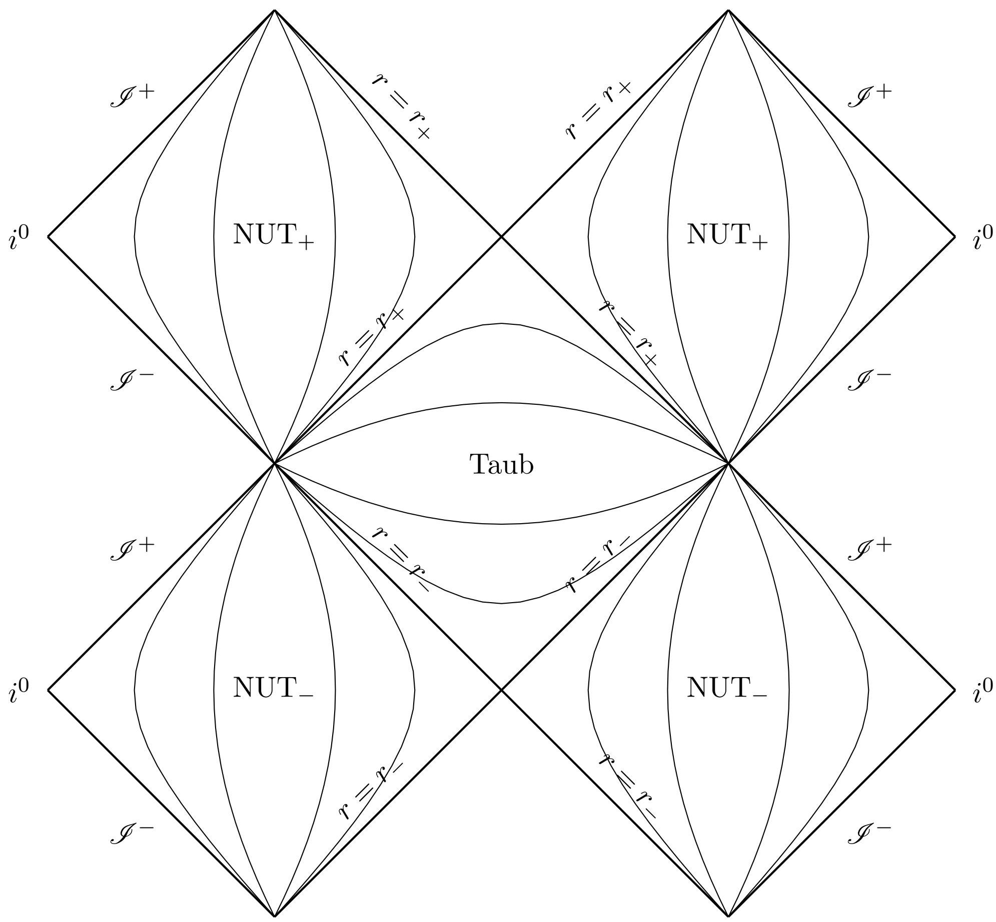

Figure 2: Penrose diagram of Taub-NUT spacetime [5]. Hyperbolic lines correspond to curves of constant r, and show how the Taub region is enclosed within the NUT regions.

4.4 Singularity theorem for spatially homogeneous spacetimes

Hawking and Ellis [6] extended the singularity theorem to spatially homogeneous spacetimes.

Theorem. *The spacetime (M, g) cannot be timelike geodesically complete if*

1. *$Ric(K, K) > 0$ for all timelike and null vectors K (strong energy condition),*
2. *there exist equations of motion for the matter fields such that the Cauchy problem has a unique solution,*
3. *the Cauchy data on some spacelike hypersurface H is invariant under a group of diffeomorphisms of H which is transitive on H.*

5 Conclusion

The symmetries of a spacetime are given by the action of an isometry group, which is generated by a real Lie algebra of Killing vector fields. For spatially homogeneous spacetimes, this Lie algebra is three-dimensional. An efficient construction implementing the antisymmetry of the structure constants gives the ten distinct types listed by the Bianchi(-Behr) classification.

Spacetimes considered here are the spatially homogeneous Bianchi I universe with dust and the non-spatially homogeneous Kantowski-Sachs universe with a perfect fluid. It is seen that the anisotropy is manifest in the shear, which admits up to three and five different types of singularities respectively. Also presented is the Taub-NUT spacetime, in which the enclosed Taub region is seen to be spatially homogeneous albeit of finite time.

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