The LNM Institute of Information Technology
Jaipur, Rajsthan

MATH-I ■ Assignment #8

(Continuity, Differentiability, Directional derivatives)

Q1. Examine the following functions for continuity at the point $(0,0)$ where $f(0,0)=0$ and $f(x,y)$ for $(x,y)\neq(0,0)$ is given by
(a) $|x|+|y|$, (b) $\frac{-x}{\sqrt{x^2+y^2}}$, (c) $\frac{2x}{x^2+x+y^2}$, (d) $\frac{x^4-y^2}{x^4+y^2}$, (e) $\frac{x^4}{x^4+y^2}$.

Q2. Consider the function $f:\mathbb{R}^2\to\mathbb{R}$ defined by

$$f(x,y)=\begin{cases}1, & \text{if } x=0 \text{ or if } y=0\\ 0, & \text{otherwise.}\end{cases}$$

Show that the function satisfy the following:
(a) The iterated limits $\lim\limits_{x\to0}\left[\lim\limits_{y\to0}f(x,y)\right]$ and $\lim\limits_{y\to0}\left[\lim\limits_{x\to0}f(x,y)\right]$ exist and equals 0,
(b) $\lim\limits_{(x,y)\to(0,0)}f(x,y)$ does not exist,
(c) $f(x,y)$ is not continuous at $(0,0)$,
(d) the partial derivatives exist at $(0,0)$.

Q3. Let

$$f(x,y)=\begin{cases}xy\left(\frac{x^2-y^2}{x^2+y^2}\right), & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0).\end{cases}$$

Prove that
(a) $f_x(0,y)=-y$ and $f_y(x,0)=x$ for all x and y,
(b) $f_{xy}(0,0)=-1$ and $f_{yx}(0,0)=1$ and
(c) $f(x,y)$ is differentiable at $(0,0)$.

Q4. Let

$$f(x,y)=\begin{cases}xy\left(\frac{2x^2-3y^2}{x^2+y^2}\right), & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0).\end{cases}$$

Prove that $f_{xy}(0,0)\neq f_{yx}(0,0)$.

Q5. Suppose f is a function with $f_x(x,y)=f_y(x,y)=0$ for all (x,y). Then show that f is constant.

Q6. Consider the function $f:\mathbb{R}^2\to\mathbb{R}$ defined by

$$f(x,y)=\begin{cases}\frac{y^3}{x^2+y^2}, & \text{if } (x,y)\neq0\\ 0, & \text{otherwise.}\end{cases}$$

Show that f is continuous at $(0,0)$, it has all directional derivatives at $(0,0)$ but not differentiable at $(0,0)$.

Q7. Consider the function $f:\mathbb{R}^2\to\mathbb{R}$ defined by

$$f(x,y)=\begin{cases}\frac{x}{y}, & \text{if } y\neq0\\ 0, & \text{if } y=0.\end{cases}$$

Discuss the continuity, differentiability at $(0,0)$. Determine the directional derivative at $(0,0)$ in all possible direction.