2.] Given :- $r_i = d_{T_i} + n_i$

Here r_i is the range measurement from i^{th} landmark point to the object's true position(d_{T_i}) corrupted with a gaussian noise (n_i) having zero mean and σ_i variance

$$d_{T_i} = \| [x_T, y_T]^T - [x_i, y_i]^T \|$$

$$= \| [x_T - x_i, y_T - y_i]^T \|$$

$$= \sqrt{(x_T - x_i)^2 + (y_T - y_i)^2} \quad \left[\text{Norm of the vector}\right]$$

$$n_i \sim \mathcal{N}(0, \sigma_i^2)$$

$[x_T, y_T] \rightarrow$ Object's true location

$$P\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = (2\pi \sigma_x \sigma_y)^{-1} e^{-\frac{1}{2}[x \ y]\begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}$$

Prior for the location of the object for a given point $[x \ y]$ [candidate]

OBJECTIVE :- Given $K \in \{1, 2, 3, 4\}$, and that position of K can lie anywhere on the unit circle radius, find the optimization problem that needs to be solved to determine the MAP estimate of the object position.

Solution : We have r_i which is the range measurement of Object position from i^{th} Landmark point.

With the linearity property

$$x = Az + b. \quad \text{where } z \sim \mathcal{N}(0, \Sigma)$$
$$\text{and } x \sim \mathcal{N}(b, AA^T)$$

we know, that r_i is also a gaussian distribution

we have, $r_i = d_{Ti} + n_i$

$$\Rightarrow r_i = n_i + d_{Ti}$$

Comparing with the property equation, we ~~know~~ get $r_i \sim \mathcal{N}(d_{Ti}, \sigma_i^2)$

To find the MAP estimate of the object position $[x, y]$, we have to find the maximum ~~likelihood~~ posterior of position $[x, y]$ given r_i.
i.e.

$$\begin{bmatrix} x \\ y \end{bmatrix}_{MAP} = \underset{[x \ y]^T}{\text{argmax}} \ P\left(\begin{bmatrix} x \\ y \end{bmatrix} / r_i\right)$$

Applying Bayes Rule :

$$\begin{bmatrix} x \\ y \end{bmatrix}_{MAP} = \underset{\begin{bmatrix} x \\ y \end{bmatrix}}{\text{argmax}} \ \underbrace{P\left(r_i / \begin{bmatrix} x \\ y \end{bmatrix}\right)}_{\text{(likelihood)}} \times \underbrace{P\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)}_{\text{prior}} \Big/ \underbrace{p(r_i)}_{\text{evidence}}$$

[Irrelevant to Estimation]

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix}_{MAP} = \underset{\begin{bmatrix} x \\ y \end{bmatrix}}{\operatorname{argmax}} \sum_{i=1}^{K} P\left(r_i \middle| \begin{bmatrix} x \\ y \end{bmatrix}\right) \times P\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

Taking log on both sides

$$\ln \begin{bmatrix} x \\ y \end{bmatrix}_{MAP} = \underset{\begin{bmatrix} x \\ y \end{bmatrix}}{\operatorname{argmax}} \sum_{i=1}^{K} \ln\left[P\left(r_i \middle| [x\ y]^T\right)\right] + \ln\left[P([x\ y])\right] \quad \rightarrow (1)$$

Solving each term on R.H.S. individually;

Part 1:

$$\ln\left[P(r_i \mid [x\ y]^T)\right] = \ln \frac{1}{\sqrt{2\pi}} * (\Sigma)^{+1/2} \times e^{-\frac{1}{2} \frac{(r_i - \mu)^2}{\sigma_i^2}}$$

[For Parameter Estimation of a Normal]

$$= -\frac{1}{2} \ln(2\pi) + \ln \Sigma^{1/2} \times e^{-\frac{1}{2} \frac{(r_i - \mu)^2}{\sigma_1^2}}$$

As we know, Here $\mu = d_{Ti}$ & $\Sigma = 1$ for r_i

$$\Rightarrow -\frac{1}{2} \ln(2\pi) + \ln(1)^{1/2} \times e^{-\frac{1}{2} \frac{(r_i - d_{Ti})^2}{\sigma_1^2}}$$

$$\Rightarrow \underbrace{-\frac{1}{2} \ln(2\pi)}_{\text{constant}} + \left(-\frac{1}{2}\right) \frac{(r_i - d_{Ti})^2}{\sigma_i^2}$$

constant $\downarrow$ Does not affect the maximization problem

$$\Rightarrow -\frac{1}{2} \frac{(r_i - d_{Ti})^2}{\sigma_i^2} \quad \rightarrow (2)$$

Part 2 :

$$\ln P\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \ln\left[(2\pi\sigma_x\sigma_y)^{-1} e^{-\frac{1}{2}[x\ y]\begin{bmatrix}\sigma_x^2 & 0 \\ 0 & \sigma_y^2\end{bmatrix}^{-1}\begin{bmatrix}x \\ y\end{bmatrix}}\right]$$

$$= \underbrace{-\ln(2\pi) - \ln\sigma_x - \ln\sigma_y}_{\text{Constant} \to \text{Irrelevant}} - \frac{1}{2}[x\ y]\begin{bmatrix}1/\sigma_x^2 & 0 \\ 0 & 1/\sigma_y^2\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix}$$

$$= -\frac{1}{2}\left[\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right] \to (3)$$

From (1), (2) & (3)

$$\ln\begin{bmatrix}x \\ y\end{bmatrix}_{MAP} = \underset{\begin{bmatrix}x \\ y\end{bmatrix}}{\operatorname{argmax}} \sum_{i=1}^{K} -\frac{1}{2}\left(\frac{r_i - d_{T_i}}{\sigma_i}\right)^2 - \frac{1}{2}\left[\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right]$$

$$\Rightarrow \underset{\begin{bmatrix}x \\ y\end{bmatrix}}{\operatorname{argmax}} -\frac{1}{2}\left[\sum_{i=1}^{K}\frac{(r_i - d_{T_i})^2}{\sigma_i^2} + \left[\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right]\right]$$

For a negative value inside argmax, we can take the argmin and remove the constant, giving us our Objective function as:-

$$\boxed{\ln\begin{bmatrix}x \\ y\end{bmatrix}_{MAP} = \underset{\begin{bmatrix}x \\ y\end{bmatrix}}{\operatorname{argmin}} \underbrace{\sum_{i=1}^{K}\frac{(r_i - d_{T_i})^2}{\sigma_i^2}}_{\text{LIKELIHOOD}} + \underbrace{\left[\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right]}_{\text{PRIOR}}} \to (4)$$