18th Oct., 2022

Project - Midsem Report

→ DFT of an odd signal is purely imaginary & hence odd since sin is odd / $X(k) = \overline{X(N-k)}$

23rd Oct., 2022

Odd Signals mod r

Def.

$x_r(n)$ is an odd signal (mod r)

$\iff$

$$x_r(n) = \begin{cases} x_r(\gcd(n,r)), & \text{if } \frac{r}{\gcd(n,r)} \equiv 0 \bmod 4, \\ & \text{and } \frac{n}{\gcd(n,r)} \equiv 1 \bmod 4 \\ -x_r(\gcd(n,r)), & \text{if } \frac{r}{\gcd(n,r)} \equiv 0 \bmod 4, \quad \forall n. \\ & \text{and } \frac{n}{\gcd(n,r)} \equiv 3 \bmod 4 \\ 0, & \text{elsewhere.} \end{cases} \quad (1)$$

Properties:

From def.,

for nonzero $x_r(n)$, $r \equiv 0 \bmod 4$ (ie) $4 \mid r$

$x_r(n)$ is periodic with period r

Pf.

For $x_r(n+kr)$,

$\gcd(n+kr, r) = \gcd(n+kr-kr, r)$

$= \gcd(n, r)$ by euclids algo.

$x_r(n)$ is odd.

(ie) $x_r(-n) = -x_r(n)$

Pf. see paper.

How many distinct values can $x_r(n)$ take?

So there are certain limitations imposed on what values $x_r(n)$ could take cuz of above.

Since it is periodic with period r,
max. no. of distinct values is r.

But turns out, we can come up with a better answer using stuff given in [6]'s paper.

The result is,

For a given r,

$4 | r$ & $\frac{r}{4}$ can be written as a product of primes by uniquely prime factorising.

$$r = 4 \underbrace{P_1^{m_1} P_2^{m_2} \cdots P_{\text{no. of prime factors}}^{m_{\text{no. of prime factors}}}}$$

Then,
there are $Z(r)$ distinct values which $x_r(n)$ can take, where,

$$Z(r) = (m_1 + 1)(m_2 + 1) \cdots (m_{\text{no. of prime factors}} + 1)$$

So by this, if $r = 4$,
$Z(r) = 1$
$r = 8$, $Z(r) = 2$,
etc.

Can we express $x_r(n)$ additively as a sum of simpler signals? – a natural question.

Well, yes. Similar to [6], we have,

$$x_r(n) = \sum_{\substack{d \mid r \\ d \geq 1}} x_r\Big(\overbrace{\tfrac{r}{d}}^{\in \mathbb{N}}\Big) \times h_{r,d}(n) \qquad \text{--- } (4)$$

summing over the divisors of r.

where

$h_{r,d}$ is periodic in r & is defined for $n \in [0, r-1]$ as,

Here we are defining just for the period & then making it repeat periodically for values after that defined period.

$$h_{r,d}(n) = \begin{cases} 1, & \text{if } \frac{r}{\gcd(n,r)} = d \equiv 0 \bmod 4,\ \frac{dn}{r} \equiv 1 \bmod 4 \\ -1, & \text{if } \frac{r}{\gcd(n,r)} = d \equiv 0 \bmod 4,\ \frac{dn}{r} \equiv 3 \bmod 4 \\ 0, & \text{otherwise.} \end{cases} \qquad \text{--- } (1)$$

A simple signal of 1, -1, 0.

So, we have expressed our $x_r(n)$ as a scaled sum of simpler $h_{r,d}(n)$ signals.

How? Well, similar to the approach in [6] ofc. (Go read [6] paper's page 2 for that.)

Clearly, $h_{r,d}(n)$ signals are also odd mod r.

by our defn. earlier.

just cross-examine the two formulae. chumma trivial.

Now, we have ①.

if $\frac{dn}{r} \equiv 1 \bmod 4$,

$\frac{dn}{r}$ is of the form $4k+1$

For that case we want 1.

we can write it as

$i \cdot i^{-dn/r}$ cuz $i \cdot (i^{4k+1})^{-1} = i \times i^{-1} = \frac{i}{i} = 1$

if $\frac{dn}{r} \equiv 3 \bmod 4$, $\frac{dn}{r} = 4k+3$.

We want -1.

we can write it as,

$i \cdot i^{-dn/r}$ again!

cuz $\underbrace{} = i \cdot (i^{4k+3})^{-1} = \frac{i}{-i} = -1$

So,

$$① \Longrightarrow h_{r,d}(n) = \begin{cases} i \cdot i^{-\frac{dn}{r}}, & \text{if } \frac{r}{\gcd(n,r)} = d \equiv 0 \bmod 4 \\ 0, & \text{otherwise.} \end{cases} \quad — ②$$

Our job was to find DFT of $x_r(n)$, which we have now expressed as a linear combination of $h_{r,d}(n)$'s.

So, since DFT is linear, if we can just find a neat way to get DFT of $h_{r,d}(n)$, we would be one step closer to being done. So,

DFT of $h_{r,d}(n)$

From our typical DFT,

$$H_{r,d}(n) = \sum_{0 \le k \le r-1} h_{r,d}(k) W_r^{-nk}$$

where $W_r = e^{2\pi i/r}$

This's notation is different from what we are used to especially with the opposite k & n but this is the one in the paper so yeah just wrap your head around it. p ez.

Using ②,

$$H_{r,d}(n) = i \cdot \sum_{0 \le k \le r-1} W_r^{-nk} \cdot i^{-\frac{dk}{r}}$$

Let $k' = \frac{dk}{r}$

cuz

$$= i \cdot \sum_{0 \le k' \le d-1} W_d^{-nk'} \cdot i^{-k'}$$

from ① → $d \equiv 0 \bmod 4$

$gcd(d, k') = 1$ ** How??

$$W_d = e^{2\pi i/d}$$

$$W_d^{d/4} = e^{\frac{2\pi i}{d} \cdot \frac{d}{4}} = e^{(\pi/2)i} = \cos\frac{\pi}{2} + i \sin\frac{\pi}{2} = i$$

So, $W_d^{d/4} = i$

$$\Rightarrow \quad H_{r,d}(n) = i \cdot \sum_{\substack{\text{same} \\ \text{blah} \\ \text{blah}}} W_d^{-nk'} \cdot \left(W_d^{d/4}\right)^{-k'}$$

$$H_{r,d}(n) = i \cdot \underbrace{\sum_{\substack{0 \le k' \le d-1 \\ d \equiv 0 \bmod 4 \\ \gcd(d,k')=1}} W_d^{-k'(n+\frac{d}{4})}}$$

$\rightarrow \le \phi(n)$ elements cuz additional condition of $d \equiv 0 \bmod 4$.

This is basically an odd type of Ramanujan sum.

basically some sums which are important in number theory. Check wikipedia.

$$\boxed{c_d(n) = \sum_{\substack{0 \le U \le d-1 \\ \gcd(U,d)=1}} W_d^{-nU}}$$

eulers totient function ← $\phi(d)$ elements

This is called the even ramanujan sum.

So, notice similarity of $c_d(n)$ & $H_{r,d}(n)$.

We can write,

$$\boxed{H_{r,d}(n) = i \cdot c_d\left(n + \frac{d}{4}\right) \quad , d \ge 1 \, , d \equiv 0 \bmod 4} \quad \text{— (3)}$$

Although $c_d(n)$ is an even function,

$c_d\left(n + \frac{d}{4}\right)$ when $d \equiv 0 \bmod 4$ is an odd function.

Pf:

tion, $c_d(n+d/4)$ is an odd function of n if $d \equiv 0 \bmod 4$. To show this, let $y = n + d/4$, so $n - d/4 = y - d/2$, and

$$
\begin{aligned}
c_d\left(-n+\frac{d}{4}\right) &= c_d\left(n-\frac{d}{4}\right) = c_d\left(y-\frac{d}{2}\right) \\
&= \sum_{\substack{0\le U\le d-1 \\ \gcd(U,d)=1}} W_d^{-(y-\frac{d}{2})U} \\
&= \sum_{\substack{0\le U\le d-1 \\ \gcd(U,d)=1}} \exp\left(\frac{-i2\pi\cdot\left(y-\frac{d}{2}\right)U}{d}\right) \\
&= \sum_{\substack{0\le U\le d-1 \\ \gcd(U,d)=1}} \exp\left(\frac{-i2\pi y}{d}+i\pi U\right) \\
&= \sum_{\substack{0\le U\le d-1 \\ \gcd(U,d)=1}} \exp\left(\frac{-i2\pi y}{d}\right)\exp(i\pi U)
\end{aligned}
$$

$\because \gcd(U,d)=1$ and $d \equiv 0 \bmod 4$
$\Rightarrow U \equiv 1 \bmod 2 \Rightarrow \exp(i\pi U) = -1$

$$
\begin{aligned}
\therefore c_d\left(-n+\frac{d}{4}\right) &= \sum_{\substack{0\le U\le d-1 \\ \gcd(U,d)=1}} \exp\left(\frac{-i2\pi y}{d}\right)\exp(i\pi U) \\
&= -\sum_{\substack{0\le U\le d-1 \\ \gcd(U,d)=1}} \exp\left(\frac{-i2\pi y}{d}\right) \\
&= -c_d(y) = -c_d\left(n+\frac{d}{4}\right).
\end{aligned}
$$

The proof is complete.

Simple proof.

notice how this is true only because $d \equiv 0 \bmod 4$. Otherwise, this wont be an odd function which would mean that we run into some unneatness / trouble.

Very fascinating. This is partly why in our initial defn., we had to make sure we define it like that so that $4 \mid r$.

DFT of $x_r(n)$

Now, using linearity of DFT & ③, ④,

$$\boxed{X_r(n) = i \cdot \sum_{\substack{d \mid r \\ d>1 \\ d \equiv 0 \bmod 4}} x_r\left(\frac{r}{d}\right)\cdot c_d\left(n+\frac{d}{4}\right)}$$

DFT

$\rightarrow X_r(n)$ is odd.

Pf: $$X_r(-n) = i \sum_{\substack{d\mid r \\ d>1 \\ d\equiv 0 \bmod 4}} x_r\left(\frac{r}{d}\right) \cdot \underbrace{c_d\left(-n+\frac{d}{4}\right)}_{= -c_d\left(n+\frac{d}{4}\right)}$$

from before.

$$= -X_r(n) \quad \blacksquare$$

So now, we only need to compute $\tau(r)$ unique values of $X_r(n)$. These values are,

$$X_r\left(\frac{r}{D}\right) = i \sum_{\substack{d\mid r \\ d>1 \\ d\equiv 0 \bmod 4}} x_r\left(\frac{r}{d}\right) \cdot c_d\left(\frac{r}{D}+\frac{d}{4}\right), \quad D>1,\ D\mid r$$

and, $X_r(0) = 0$ $\because X_r(n)$ is odd $\Rightarrow X_r(-0) = -X_r(0)$
$\Rightarrow X_r(0) = 0$

Ex:

As a concrete example, let us compute the $\tau(12) = 2$ distinct Fourier coefficients for $x_{12}(n)$. Using (11), we have

$$\begin{aligned}
X_{12}(0) &= 0 = X_{12}(2) = X_{12}(4) = X_{12}(6) \\
&= X_{12}(8) = X_{12}(10) \\
X_{12}(1) &= X_{12}(5) = -X_{12}(7) = -X_{12}(11) \\
&= -2ix_{12}(1) - 2ix_{12}(3) \\
X_{12}(3) &= -X_{12}(9) = -4ix_{12}(1) + 2ix_{12}(3). \qquad (12)
\end{aligned}$$

see paper.

Now, we know how to calculate DFT of odd signals mod r (ie) $x_r(n)$.

What about the 'inverse?

IDFT of $X_r(n)$

We can use the same $H_{d,r}$ linear combination decomposition approach as above & get a very similar expression but with a -ve sign & scaling,

Our regular IDFT,

$$x_r(n) = \frac{1}{r} \sum_{k=0}^{r-1} X_r(n) W_r^{nk}$$

for scaling cuz in DFT we didn't do '$\frac{1}{\sqrt{N}}$' so now we do '$\frac{1}{N}$'.

After going through a process similar to what we did to get our DFT, we get,

$$\boxed{x_r(n) = \frac{-i}{r} \sum_{\substack{d \mid r \\ d>1 \\ d \equiv 0 \bmod 4}} x_r\left(\frac{r}{d}\right) \cdot c_d\left(n + \frac{d}{4}\right)}$$

IDFT

this is different from our DFT eqn.

Now, we have DFT & IDFT for our odd signals mod r. The algo stuff ends here

In [6], we have same thing for even signals.

What if we have a signal which is a

Sum of an odd mod r & an even mod r signal?

Sum of Even & Odd

Our next step is to study when we put together the even signals and odd signals (mod r). Assume $r = 12$; then from [6], the values of the even signal in its main period may be represented by a sequence of numbers of this form

$$\langle y_{12}(n), n = 0 \ldots 11 \rangle = \langle a, b, c, d, e, b, f, b, e, d, c, b \rangle \quad (15\text{-}1)$$

and from (2)

$$\langle x_{12}(n), n = 0 \ldots 11 \rangle = \langle 0, p, 0, q, 0, p, 0, -p, 0, -q, 0, -p \rangle . \quad (15\text{-}2)$$

Combining (15-1) and (15-2), we can get

$$\langle z_{12}(n), n = 0 \ldots 11 \rangle = \langle a, p+b, c, q+d, e, p+b, f, b - p, e, d-q, c, b-p \rangle . \quad (15\text{-}3)$$

So, we see an increase in the no. of distinct values which makes sense.

As we can see, the values we can use are: $a, p+b, c, q+d, e, f, b-p, d-q$, eight distinct values in total. Compared with (15-1), which can only use six values, we find that (15-3) has 33% increase. As a matter of fact, when r is big, the values, or what we called *dimensions*, have nearly 100% increase from that in [6]. We use an example to demonstrate that. If $r = 2^n$, then, from [6], the dimensions are $n+1$, but the odd part provides another $n-1$ dimensions. So overall, it has $2n$ dimensions, which have increased by 100% when $n \gg 2$.

no. of distinct values aka "dimensions"

Similarly,
if you take an even signal & add to it a circular shifted version of itself, its DFT $X_r(n)$ values will have more distinct values (ie) "dimensions".

Also in the paper we have,

dimensions of even/odd method $\geq$ dimensions of circular shift method

Z-domain representation of odd signals mod r is also given in the last section of the paper. But this is not of our concern as Z-transforms are not there in our course & our main

<u>Aim</u>:
To implement the DFT & IDFT algos given in the paper for odd signals mod r which are based on odd ramanujan sums & to discuss how it works & why it is better than just doing the normal DFT directly.

We have discussed how it works but we are yet to see why it is better than doing the normal DFT directly.