END TERM PROJECT REPORT

DIGITAL SIGNAL PROCESSING

Odd Ramanujan sums of Complex Roots of Unity

(By Soo-Chang Pei, and Kuo-Wei Chang, IEEE Signal Processing Letters, 2007)

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Abstract

This project shows how odd Ramanujan sums can be used as weighting coefficients to compute pure imaginary discrete fourier transform integer-valued coefficients of odd symmetric length 4-N periodic signals. In an earlier paper, an approach was proposed where the discrete fourier transform (DFT) coefficients of even symmetric signals could be computed by forming a weighted average of the signals using integer valued coefficients. These integer valued coefficients are Ramanujan Sums, i.e. special sums of complex roots of unity which are computed using closed form formulae.

Recently, a recursive method was implemented to compute the DFT through an infinite impulse response (IIR) filter. This project proposes a similar approach for odd symmetric signals.

Contents

1. Introduction

a. Ramanujan Sums

Ramanujan sums are special sums of the complex roots of unity. It is represented as a function of two positive integer variables *q* and *n* given as follows

$$c_q(n) = \sum_{\substack{1 \le a \le q \\ (a,q)=1}} e^{2\pi i \frac{a}{q} n},$$

where *(a,q)* denotes the greatest common divisor of *a* and *q*.

This is the expression for an even function which is then transformed into an odd function by simple circular shifts. This transformation is done to build up the imaginary discrete fourier transform integer value coefficients to introduce an IIR system. This is done by replacing n with n+q/4 which gives an odd function. The proof for the same is given as:

$$
\begin{aligned}
c_d\left(-n + \frac{d}{4}\right) &= c_d\left(n - \frac{d}{4}\right) = c_d\left(y - \frac{d}{2}\right) \\
&= \sum_{\substack{0 \le U \le d-1 \\ \gcd(U,d)=1}} W_d^{-(y-\frac{d}{2})U} \\
&= \sum_{\substack{0 \le U \le d-1 \\ \gcd(U,d)=1}} \exp\left(\frac{-i2\pi \cdot \left(y - \frac{d}{2}\right) U}{d}\right) \\
&= \sum_{\substack{0 \le U \le d-1 \\ \gcd(U,d)=1}} \exp\left(\frac{-i2\pi y}{d} + i\pi U\right) \\
&= \sum_{\substack{0 \le U \le d-1 \\ \gcd(U,d)=1}} \exp\left(\frac{-i2\pi y}{d}\right) \exp(i\pi U) \\
&\because \gcd(U,d) = 1 \quad \text{and} \quad d \equiv 0 \bmod 4 \\
&\Rightarrow U \equiv 1 \bmod 2 \Rightarrow \exp(i\pi U) = -1 \\
&\therefore c_d\left(-n + \frac{d}{4}\right) \\
&= \sum_{\substack{0 \le U \le d-1 \\ \gcd(U,d)=1}} \exp\left(\frac{-i2\pi y}{d}\right) \exp(i\pi U) \\
&= - \sum_{\substack{0 \le U \le d-1 \\ \gcd(U,d)=1}} \exp\left(\frac{-i2\pi y}{d}\right) \\
&= -c_d(y) = -c_d\left(n + \frac{d}{4}\right).
\end{aligned}
$$

Note: in the proof a=U and q=d.

b. Discrete Fourier Transform

The discrete Fourier transform transforms a sequence of N complex numbers into another sequence of complex numbers, which is defined by

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn}$$

The DFT can be evaluated outside the domain of K $\in$ [0, N-1].

c. Odd Signals

These signals are defined with respect to a fixed positive integer r for all values of the time index n. A signal $x_r(n)$ is called an odd signal (mod r) if

$$x_r(n) = \begin{cases} x_r(\gcd(n,r)), & \text{if } \frac{r}{\gcd(n,r)} \equiv 0 \bmod 4, \\ & \text{and } \frac{n}{\gcd(n,r)} \equiv 1 \bmod 4 \\ -x_r(\gcd(n,r)), & \text{if } \frac{r}{\gcd(n,r)} \equiv 0 \bmod 4, \quad \forall n. \\ & \text{and } \frac{n}{\gcd(n,r)} \equiv 3 \bmod 4 \\ 0, & \text{elsewhere.} \end{cases} \tag{1}$$

From the above definition, the following properties can be obtained,

1. $x_r(n)$ is a periodic signal with the period r

   i.e, $x_r(n+kr) = x_r(n)$

2. $x_r(n)$ has odd symmetry

   i.e, $x_r(r-n) = - x_r(n)$

Since the signal is periodic with r it may have a maximum of r distinct values. The value of r can be expressed as a product of primes and therefore, the maximum values that x_r (n) can now assume is given by $\tau(r)$,

$$\tau(r) = \prod_{p^{\text{prime}}} (\mathrm{m_p} + 1)$$

2. Project Implementation

a. Expressing a signal as sum of smaller signals

$$x_r(n) = \sum_{\substack{d|r \\ d \geq 1}} x_r\left(\frac{r}{d}\right) h_{r,d}(n) \tag{3}$$

where the signal $h_{r,d}(n)$ is periodic with period r, which is defined for $n \in [0, r-1]$ as

$$h_{r,d}(n) = \begin{cases} 1, & \text{if } \frac{r}{\gcd(n,r)} = d \equiv 0 \bmod 4, \frac{dn}{r} \equiv 1 \bmod 4 \\ -1, & \text{if } \frac{r}{\gcd(n,r)} = d \equiv 0 \bmod 4, \frac{dn}{r} \equiv 3 \bmod 4 \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

It is clear that the signals $h_{r,d}(n)$ are odd $(\bmod\ r)$.

We then proceed to simplify the definition for the odd signal $h_{r,d}(n)$.

It is clear that the signals $h_{r,d}(n)$ are odd $(\bmod\ r)$.
Let $i = \sqrt{-1}$ and use $1 = i^{4k}, -1 = i^{4k+2}, \forall k \in N$, so

$$h_{r,d}(n) = \begin{cases} i \cdot i^{-\frac{dn}{r}}, & \text{if } \frac{r}{\gcd(n,r)} = d \equiv 0 \bmod 4 \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

In this section we have redefined $x_r(n)$ as a linear combination of $h_{r,d}(n)$. Thus finding the DFT of $h_{r,d}(n)$ brings us closer to finding the DFT of $x_r(n)$.

b. Writing the DFT in terms of Ramanujan sums

DFT of $h_{r,d}(n)$ can be written as,

$$H_{r,d}(n) = \sum_{0 \leq k \leq r-1} h_{r,d}(k) W_r^{-nk}$$

Here, $W_r = \exp(2\pi i/r)$.

Using that we rewrite the DFT of $h_{r,d}(n)$, $H_{r,d}(n)$ as,

$$
\begin{aligned}
H_{r,d}(n) &= \sum_{0 \le k \le r-1} h_{r,d}(k) W_r^{-nk} \\
&= i \cdot \sum_{0 \le k \le r-1} W_r^{-nk} \cdot i^{-\frac{dk}{r}} \\
&= i \cdot \sum_{\substack{0 \le k' \le d-1 \\ \gcd(d,k')=1 \\ d \equiv 0 \bmod 4}} W_d^{-nk'} \cdot i^{-k'}
\end{aligned}
$$

where $k' = dk/r$, and then using $i = W_d^{d/4}$, so

$$
\begin{aligned}
H_{r,d}(n) &= i \cdot \sum_{\substack{0 \le k' \le d-1 \\ \gcd(d,k')=1 \\ d \equiv 0 \bmod 4}} W_d^{-nk'} \cdot i^{-k'} \\
&= i \cdot \sum_{\substack{0 \le k' \le d-1 \\ \gcd(d,k')=1 \\ d \equiv 0 \bmod 4}} W_d^{-k'(n+\frac{d}{4})}.
\end{aligned}
$$

The last complex sum is another odd type of the Ramanujan sum. We use the symbol $c_d(n)$ to denote the value of the even Ramanujan sum and rewrite the equation is,

$$
H_{r,d}(n) = i \cdot c_d\left(n + \frac{d}{4}\right), \quad d \geqslant 1, \quad d \equiv 0 \mod 4.
$$

Here, $c_d(n+d/4)$ is an odd function. The same was proved in the introduction where we define a Ramanujan sum.

c. DFT of main signal as a linear combination of DFT's of smaller signals

The DFT of the main signal is therefore a linear summation of the DFT's of the smaller signal. Which is given as,

$$
X_r\left(\frac{r}{D}\right) = i \sum_{\substack{d \mid r \\ d > 1 \\ d \equiv 0 \bmod 4}} x_r\left(\frac{r}{d}\right) c_d\left(\frac{r}{D} + \frac{d}{4}\right), \qquad D > 1, \quad D \mid r
$$

As an example we compute $\tau(12) = 2$ distinct Fourier coefficients for $x_{12}(n)$

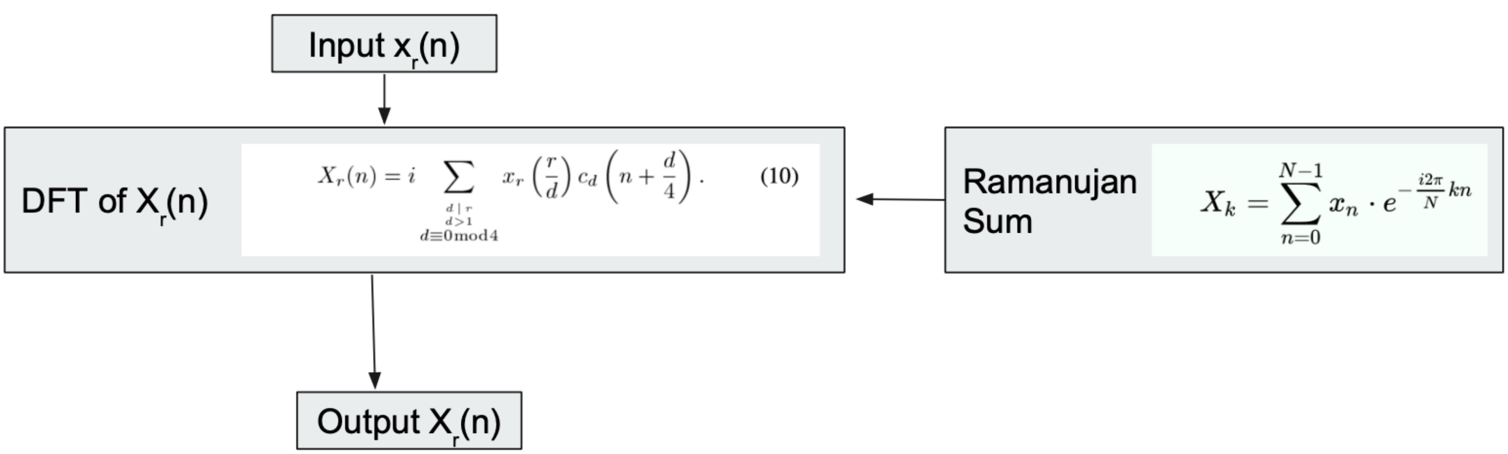
$$\begin{aligned}
X_{12}(0) &= 0 = X_{12}(2) = X_{12}(4) = X_{12}(6) \\
&= X_{12}(8) = X_{12}(10) \\
X_{12}(1) &= X_{12}(5) = -X_{12}(7) = -X_{12}(11) \\
&= -2ix_{12}(1) - 2ix_{12}(3) \\
X_{12}(3) &= -X_{12}(9) = -4ix_{12}(1) + 2ix_{12}(3).
\end{aligned}$$

d. Inverse DFT of the signal

To inversely compute the signal values from its Fourier coefficients, we can use a similar decomposition approach as we use in part (a) of this report. The complex conjugate of the sum is given as $-ic_d(n+d/4)$. This gives us the inverse DFT as,

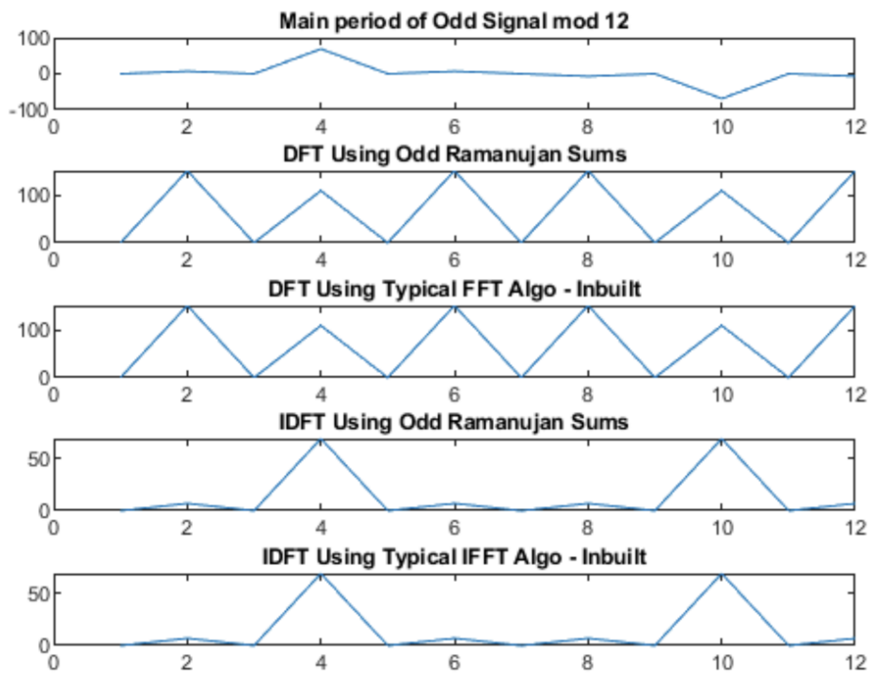
$$x_r(n) = \frac{-i}{r} \sum_{\substack{d \mid r \\ d>1 \\ d \equiv 0 \bmod 4}} X_r\left(\frac{r}{d}\right) c_d\left(n + \frac{d}{4}\right).$$

e. Block Diagram



3. Results and Discussion

a. Waveforms of DFT and IDFT of a Sample Input



Thus, we can see that our algorithm gives the same result as that of the typical fast fourier transform algorithm for our class of odd signals.

4. Future Scope

a. Odd + Even signals using the circular shift property

When we put together odd and even signals and take the example case of r=12 we observe that there is an increase in the number of distinct values from 6 to 8. As the value of the period increases so does the number of distinct values also known as dimensions. This circular shift property of DFT is used to increase dimensions. However, the even-odd method to determine dimensions gives greater number of dimensions than the circular shift method since the circular shift method is just a special case of the even-odd method where the gcd (r , k1)=gcd (r , k2) where r is the period and k1 and k2 are time indices of the even and odd signals respectively.

b. Z Transform

A z-domain representation of the odd signals mod(r) is obtained and is of the form,

$$\sum_{n=0}^{\infty} x_r(n) z^{-n} = \frac{-i}{r} \sum_{\substack{d \mid r \\ d>1 \\ d \equiv 0 \bmod 4}} X_r\left(\frac{r}{d}\right) C'_d(z)$$

Here $C'_d(z)$ is the one sided z-transform of the odd Ramanujan sums $c_d(n+d/4)$. Using the concept of shift property of one sided z-transform, we can also obtain

$$C'_d(z) = z^{\frac{d}{4}} C_d(z) - \sum_{i=1}^{\frac{d}{4}} z^i c_d\left(\frac{d}{4} - i\right)$$

5. Conclusion

Through this project we have shown that a special class of odd periodic signals possess a DFT representation, with the imaginary integer-valued odd Ramanujan sums playing the role of the complex-valued roots of unity. In addition, combining this odd part with the even part, one can get more dimensions. The project also derives an IIR system suitable for these odd signals.

6. References

S. -C. Pei and K. -W. Chang, "Odd Ramanujan Sums of Complex Roots of Unity," in IEEE Signal Processing Letters, vol. 14, no. 1, pp. 20-23, Jan. 2007, doi: 10.1109/LSP.2006.881527.

7. Acknowledgements

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