Digital Signal Processing Project

Odd Ramanujan Sums of Complex Roots of Unity

Objective

This project uses odd Ramanujan sums as weighting coefficients to compute pure imaginary discrete fourier transform integer value coefficients of odd symmetric length 4-N periodic signals.

Originally the discrete fourier transform coefficients of even symmetric signals was computed by forming a weighted average of the signals using integer value coefficients. These integer valued coefficients are Ramanujan Sums.

Recently, a recursive method was developed to compute the DFT of even symmetric signals through an infinite impulse response filter (IIR).

This project uses the same IIR filter but for odd symmetric signals.

Some important definitions

- Ramanujan Sums
- Discrete fourier transform
- Odd Symmetric Signals

Ramanujan Sums

Ramanujan sums are basically special sums of the complex roots of unity. It is represented as a function of two positive integer variables *a* and *q* defined by the formula,

$$c_q(n) = \sum_{\substack{1 \le a \le q \\ (a,q)=1}} e^{2\pi i \frac{a}{q} n},$$

where (*a*,*q*) means that *a* and *q* are co-primes.

Note: this is an even function which can be transformed into an odd function by simple circular shifts to build up the imaginary discrete fourier transform integer value coefficients to introduce an IIR system

Discrete Fourier Transform

The discrete Fourier transform transforms a sequence of N complex numbers into another sequence of complex numbers, which is defined by

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn}$$

Odd Symmetric Signals

These signals are defined with respect to a fixed positive integer r for all values of the time index n. A signal $x_r(n)$ is called an odd signal (mod r) if

$$x_r(n) = \begin{cases} x_r(\gcd(n,r)), & \text{if } \frac{r}{\gcd(n,r)} \equiv 0 \bmod 4, \\ & \text{and } \frac{n}{\gcd(n,r)} \equiv 1 \bmod 4 \\ -x_r(\gcd(n,r)), & \text{if } \frac{r}{\gcd(n,r)} \equiv 0 \bmod 4, \\ & \text{and } \frac{n}{\gcd(n,r)} \equiv 3 \bmod 4 \\ 0, & \text{elsewhere.} \end{cases} \quad \forall n. \tag{1}$$

What do we infer from the definition?

- $x_r(n)$ is a periodic signal with the period r

  i.e, $x_r(n+kr) = x_r(n)$

- $x_r(n)$ has odd symmetry

  i.e, $x_r(r-n) = -x_r(n)$

Example

Let's take a signal satisfying the definition where, $r = 12$ which satisfies $r \equiv 0 \mod 4$. The values of the signal are then represented by a sequence of numbers of the form

$$\begin{aligned}\langle x_{12}(n), n = 0 \ldots 11\rangle \\ = \langle 0, p, 0, q, 0, p, 0, -p, 0, -q, 0, -p\rangle. \quad (2)\end{aligned}$$

This leads us to ask what are the maximum possible number of distinct values that $x_r(n)$ can assume for any given r.

Number of maximum values of $x_r(n)$ is represented as a function of τ where,

$$\tau (r) = \Pi_{pprime}(m_p + 1)$$

Expressing $x_r(n)$ as a sum of simpler signals

$$x_r(n) = \sum_{\substack{d|r \\ d\geq 1}} x_r\left(\frac{r}{d}\right) h_{r,d}(n) \tag{3}$$

where the signal $h_{r,d}(n)$ is periodic with period r, which is defined for $n \in [0, r-1]$ as

$$h_{r,d}(n) = \begin{cases} 1, & \text{if } \frac{r}{\gcd(n,r)} = d \equiv 0 \bmod 4,\ \frac{dn}{r} \equiv 1 \bmod 4 \\ -1, & \text{if } \frac{r}{\gcd(n,r)} = d \equiv 0 \bmod 4,\ \frac{dn}{r} \equiv 3 \bmod 4 \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

It is clear that the signals $h_{r,d}(n)$ are odd $(\bmod\ r)$.

Simplifying $h_{r,d}(n)$

It is clear that the signals $h_{r,d}(n)$ are odd $\pmod r$.

Let $i = \sqrt{-1}$ and use $1 = i^{4k}, -1 = i^{4k+2}, \forall k \in N$, so

$$h_{r,d}(n) = \begin{cases} i \cdot i^{-\frac{dn}{r}}, & \text{if } \frac{r}{\gcd(n,r)} = d \equiv 0 \bmod 4 \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

This simplifies the definition for the odd signal $h_{r,d}(n)$.

Since, we have redefined $x_r(n)$ as a linear combination of $h_{r,d}(n)$ finding the DFT of $h_{r,d}(n)$ brings us closer to finding the DFT of $x_r(n)$.

DFT of $h_{r,d}(n)$

$$H_{r,d}(n) = \sum_{0 \le k \le r-1} h_{r,d}(k) W_r^{-nk} \qquad (6)$$

where $W_r = \exp(2\pi i/r)$. Using (5), we can rewrite (6) as

$$\begin{aligned} H_{r,d}(n) &= \sum_{0 \le k \le r-1} h_{r,d}(k) W_r^{-nk} \\ &= i \cdot \sum_{0 \le k \le r-1} W_r^{-nk} \cdot i^{-\frac{dk}{r}} \\ &= i \cdot \sum_{\substack{0 \le k' \le d-1 \\ \gcd(d,k')=1 \\ d \equiv 0 \bmod 4}} W_d^{-nk'} \cdot i^{-k'} \end{aligned} \qquad (7$$

where $k' = dk/r$, and then using $i = W_d^{d/4}$, so

$$\begin{aligned} H_{r,d}(n) &= i \cdot \sum_{\substack{0 \le k' \le d-1 \\ \gcd(d,k')=1 \\ d \equiv 0 \bmod 4}} W_d^{-nk'} \cdot i^{-k'} \\ &= i \cdot \sum_{\substack{0 \le k' \le d-1 \\ \gcd(d,k')=1 \\ d \equiv 0 \bmod 4}} W_d^{-k'(n+\frac{d}{4})}. \end{aligned} \qquad (8)$$

From the last equation we observe that the complex sum is similar to the ramanujan sum defined earlier.

Writing $H_{r,d}(n)$ with the odd Ramanujan sum

$$H_{r,d}(n) = i \cdot c_d\left(n + \frac{d}{4}\right), \quad d \geqslant 1, \quad d \equiv 0 \mod 4. \quad (9)$$

Note: although $c_d(n)$ is an even function, $c_d(n+d/4)$ is an odd function of n if $d \equiv 0 \mod 4$.

DFT and IDFT of $x_r(n)$

To compute the DFT coefficients $X_r(n)$ of the odd signal $x_r(n)$, we invoke the linearity of the DFT operation. Using (3),(4), and (9), we can write,

$$X_r(n) = i \sum_{\substack{d \mid r \\ d>1 \\ d\equiv 0 \bmod 4}} x_r\left(\frac{r}{d}\right) c_d\left(n + \frac{d}{4}\right). \qquad (10)$$

Similarly, to compute the inverse DFT we use the same decomposition approach where complex conjugate of the sum is given as $-ic_d(n+d/4)$.

$$x_r(n) = \frac{-i}{r} \sum_{\substack{d \mid r \\ d>1 \\ d\equiv 0 \bmod 4}} X_r\left(\frac{r}{d}\right) c_d\left(n + \frac{d}{4}\right). \qquad (14)$$

Block diagram

Input $x_r(n)$

DFT of $X_r(n)$

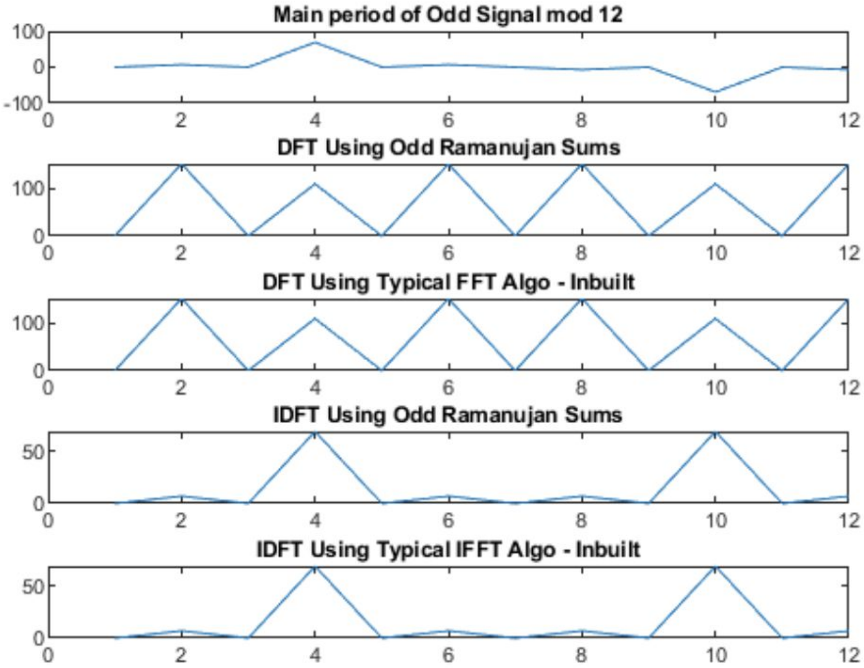
$$X_r(n) = i \sum_{\substack{d \mid r \\ d>1 \\ d \equiv 0 \bmod 4}} x_r\left(\frac{r}{d}\right) c_d\left(n + \frac{d}{4}\right). \qquad (10)$$

Ramanujan Sum

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn}$$

Output $X_r(n)$

Result


Main period of Odd Signal mod 12
DFT Using Odd Ramanujan Sums
DFT Using Typical FFT Algo - Inbuilt
IDFT Using Odd Ramanujan Sums
IDFT Using Typical IFFT Algo - Inbuilt

Further questions to ask: what if we put together even and odd signals?

- If we take the example case of r=12 we observe that there is an increase in the number of distinct values from 6 to 8. As the value of the period increases so does the number of distinct values also known as dimensions.
- The circular shift property of DFT is also used to increase dimensions.
- However, the even-odd method to determine dimensions gives greater number of dimensions than the circular shift method since the circular shift method is just a special case of the even-odd method where the gcd (r , k1)=gcd (r , k2) where r is the period and k1 and k2 are time indices of the even and odd signals respectively.

Z Domain Characterisation

$$\sum_{n=0}^{\infty} x_r(n) z^{-n} = \frac{-i}{r} \sum_{\substack{d \mid r \\ d>1 \\ d \equiv 0 \bmod 4}} X_r\left(\frac{r}{d}\right) C'_d(z)$$

$$C'_d(z) = z^{\frac{d}{4}} C_d(z) - \sum_{i=1}^{\frac{d}{4}} z^i c_d\left(\frac{d}{4} - i\right)$$

Thank You