Q4. Prove or disprove:
(a) The Laplacian mask with a -4 in the center (see class slides) is a separable filter.
(b) The Laplacian mask with a -4 in the center (see class slides) can be implemented entirely using 1D convolutions

Ans.
(a) For a 2D 3x3 filter to be separable, it must be a product of two 1D 1x3 and 3x1 filter. Let the two filters be $[a_1\ a_2\ a_3]^T$ and $[b_1\ b_2\ b_3]$, then we have:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

This is not possible as from the first entry, either a_1 or b_1 should be 0. If a_1 is 0, then the first row should be 0 and if b_1 is 0, then the first column should be 0. Hence Laplacian filter is not separable.

(b) The Laplacian filter for a given image f is given by $\nabla^2 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = f(x+1, y) + f(x-1, y) - 2f(x, y) + f(x, y+1) + f(x, y-1) - 2f(x, y)$. The first three terms have y constant and can be implemented as 1D convolution with $[1\ -2\ 1]^T$ filter. The last three terms have x constant can be implemented using $[1\ -2\ 1]$ filter. The sum of these two will give us Laplacian filter. If the image f is impulse matrix, then we should get the Laplacian filter as impulse response and then any image can be convolved to get the Laplacian operator, which is shown below:

$$\begin{aligned}
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} * \left(\begin{bmatrix} 1 & -2 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}
\end{aligned}$$

Hence, the Laplacian mask can be implemented directly using sum of two 1D convolutions only.