Q3. Consider a clean image I(x,y) which gets corrupted by additive noise randomly and independently from a zero mean Gaussian distribution with standard deviation σ. Derive an expression for the PDF of the resulting noisy image. Assume continuous-valued intensities.

Ans. Let I(x,y) be the intensity of the image at location (x, y). If a Gaussian noise $\frac{e^{-\frac{x^2+y^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}$ is added to this intensity value, the intensity value is:

$$\begin{aligned} Z(x,y) &= I(x,y) + \eta(0,\sigma) \\ &= I(x,y) + \frac{e^{-\frac{x^2+y^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} \end{aligned}$$

The pdf of this function will be convolution of the given image and the Gaussian noise:

$$\begin{aligned} P(Z=z) &= P(I(x,y) + \eta(0,\sigma) = z) \\ &= \int_{i=0}^{i=z} P_{(I,\eta)}(i, z-i) = \int_{i=0}^{i=z} P(I(x,y) = i, \eta(0,\sigma) = z-i) \\ &= \int_{i=0}^{i=z} P(I(x,y) = i) P(\eta(0,\sigma) = z-i) \qquad \text{As Gaussian noise is independent of the image} \\ &= I(x,y) * \eta(0,\sigma) = \eta(0,\sigma) * I(x,y) \end{aligned}$$