Q3. Prove the convolution theorem for 2D Discrete Fourier transforms.

Ans. The convolution theorem is for 2D Fourier transform is:

$$\mathfrak{F}(f(x,y) * h(x,y)) = F(u,v)H(u,v)$$

Fourier transform for a signal z(x,y) of size MxN is:

$$Z(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} z(x,y)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

Convolution of two discrete signals f(x,y) and h(x,y) of size MxN is given by:

$$f(x,y) * h(x,y) = \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} f(i,j)h(x-i,y-j)$$

We have,

$$\begin{aligned}
\mathfrak{F}(f(x,y) * h(x,y)) &= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1}((f(x,y) * h(x,y))e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})} \\
&= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1}(\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} f(i,j)h(x-i,y-j))e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})} \\
&= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} f(i,j)h(x-i,y-j)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})} \\
&= \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} f(i,j)(\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} h(x-i,y-j)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})}) \quad \text{(x,y and i,j are independent)} \\
&= \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} f(i,j)H(u,v)e^{-2j\pi(\frac{ui}{M}+\frac{jv}{N})} \quad \text{(Shifting Property of Fourier Transform for 2D signals)} \\
&= H(u,v)\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} f(i,j)e^{-2j\pi(\frac{ui}{M}+\frac{jv}{N})} \quad \text{(H(u,v) is independent of i,j)} \\
&= H(u,v)F(u,v)
\end{aligned}$$

Hence Proved . If signal sizes are not same, we first zero pad them to ensure they have the same size and then apply the convolution theorem