Q5. Suppose I convolve an image f with a mean-filter of size $(2a+1) \times (2a+1)$ where $a > 0$ is an integer to produce a result f_1. Suppose I convolve the resultant image f_1 with the same mean-filter once again to produce an image f_2, and so on until you get image f_k in the K^{th} iteration. Can you express f_k as a convolution of f with some kernel. If not, why not? If yes, with what kernel? Justify.

Ans. Mean-filter of size $(2a+1) \times (2a+1)$ is given by:

$$M = \frac{1}{(2a+1)^2} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{(2a+1)\times(2a+1)}$$

One round of filtering on an image f will give us $M * f$. Let us assume we have the K^{th} kernel M_k. Then another round of filtering on image f will give us:

$$\begin{aligned} M * (M_k * f) &= (M * M_k) * f \qquad \text{As convolution is an associative operator} \\ &= M_{k+1} * f \end{aligned}$$

Hence, by induction, the kernel operator for K^{th} operator is:

$$M_k = M * M * M * .... * M \qquad \text{K times}$$

It is possible to convolve the image f with this filter to get the same result as convolving the image K times with the mean-filter. It is very difficult to get a closed matrix form of M_k in terms of a and k but as k becomes very large, we can approximate the overall filter to be a Gaussian filter of size $(2a+1) \times (2a+1)$ according to the central limit theorem.