Q6. If F is the continuous Fourier operator, prove that $F(F(F(F(f(t))))) = f(t)$. Hint: Prove that $F(F(f(t))) = f(-t)$ and proceed further from there.

Ans. First we prove $F(F(f(t))) = f(-t)$:

$$
\begin{aligned}
G(f) = F(f(t)) &= \int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}dt \\
H(\tau) = F(G(f)) = F(F(f(t))) &= \int_{-\infty}^{\infty} G(f)e^{-j2\pi f\tau}df \\
&= \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}dt)e^{-j2\pi f\tau}df \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}e^{-j2\pi f\tau}df\,dt \\
&= \int_{-\infty}^{\infty} f(t)(\int_{-\infty}^{\infty} e^{-j2\pi f(t+\tau)}df)dt \quad \text{(f and t are independent)} \\
&= \int_{-\infty}^{\infty} f(t)\delta(t+\tau)dt \quad \text{(Shifting property of Fourier Transform)} \\
&= f(-\tau) \quad \text{(Sifting property of Fourier Transform)} \\
\therefore H(\tau) = f(-\tau) &\implies H(t) = f(-t) \implies F(F(t)) = f(-t)
\end{aligned}
$$

Now, if $F(F(f(t))) = f(-t),\ F(F(F(F(f(t))))) = F(F(f(-t))) = f(-(-t)) = f(t)$. Hence Proved.