Q3. Consider a matrix A of size m x n. Define $P = A^T A$ and $Q = AA^T$. (Note: all matrices, vectors and scalars involved in this question are real-valued).

(a) Prove that for any vector y with appropriate number of elements, we have $y^T Py \geq 0$. Similarly show that $z^T Qz$ 0 for a vector z with appropriate number of elements. Why are the eigenvalues of P and Q non-negative?.

Ans. We have $y^T Py = y^T A^T Ay$ and want to prove that it is ≥ 0. We define $Ay = X$ and consequently $X^T = (Ay)^T = y^T A^T$. The size of X will be m x n . n x $1 = m$ x 1 or $X = [x_1 \ x_2 \ x_3 \ ... \ x_m]^T$. Hence, $y^T A^T Ay = X^T X$. As X is a 1D vector, $X^T X = \sum_{i=1}^{i=m} x_i^2$ which is ≥ 0. Similarly, $z^T Qz = z^T AA^T z$ and we take $X = A^T z$ with size n x 1. Hence, $z^T AA^T z = X^T X = \sum_{i=1}^{i=n} x_i^2$ which is also ≥ 0. Now, if λ is the eigenvalue of P, we have $Pu = \lambda u \implies A^T Au = \lambda u$. Pre-multiplying with u^T, we get $u^T A^T Au = u^T \lambda u \implies (Au)^T Au = \lambda u^T u$ or $||Au||^2 = \lambda ||u||^2$. As both LHS and RHS are non-negative, λ must also be non-negative to satisfy the equation. Similiary for Q, we have $AA^T v = \mu v$. Pre-multiplying with v^t, we get $||A^T v||^2 = \mu ||v||^2$ and hence μ must also be non-negative. therefore, the eigenvalues of P and Q are non-negative.

(b) If u is an eigenvector of P with eigenvalue λ, show that Au is an eigenvector of Q with eigenvalue λ. If v is an eigenvector of Q with eigenvalue μ, show that $A^T v$ is an eigenvector of P with eigenvalue μ. What will be the number of elements in u and v?

Ans. If u is an eigenvector of P with eigenvalue λ, we have by definition $Pu = \lambda u \implies A^T Au = \lambda u$. Now, if Au were the eigenvector of Q, we would have $Q(Au) = \lambda' Au \implies AA^T Au = \lambda' Au$. But we know $A^T Au = \lambda u$, hence $AA^T Au = A\lambda u = \lambda Au$. Hence Au is an eigenvector of Q with eigenvalue $\lambda' = \lambda$.Similarly, if $A^T v$ is an eigenvector of P with v as the eigenvector of Q, $A^T AA^T v = \mu' A^T v$ but $AA^T = \mu v$. Hence $A^T AA^T v = \mu'^{A^T} v \implies A^T(\mu v) = \mu' A^T v$. Hence $A^T v$ is an eigenvector of P with eigenvalue $\mu' = \mu$.

(c) If v_i is an eigenvector of Q and we define $u_i = \frac{A^T v_i}{||A^T v_i||_2}$. Then prove that there will exist some real, non-negative γ_i such that $Au_i = \gamma_i v_i$.

Ans. We want to prove that $Au_i = \gamma_i v_i$ for $u_i = \frac{A^T v_i}{||A^T v_i||_2}$. Substituting, we have $\frac{AA^T v_i}{||A^T v_i||_2} = \gamma_i v_i$. But, $AA^T = Q$ and hence $\frac{AA^T v_i}{||A^T v_i||_2} = \gamma_i v_i \implies \frac{Qv_i}{||A^T v_i||_2} = \gamma_i v_i$. Also, it is mentioned that v_i is the eigenvector of Q, that is $Qv_i = \lambda v_i$. Now replacing this, we have satisfied the condition as $\frac{Qv_i}{||A^T v_i||_2} = \gamma_i v_i \implies (\frac{\lambda}{||A^T v_i||_2})v_i = \gamma v_i$ and $\gamma = \frac{\lambda}{||A^T v_i||_2}$.

(d) It can be shown that $u_i^T u_j = 0$ for $i \neq j$ and likewise $v_i^T v_j = 0$ for $i \neq j$ for correspondingly distinct eigen-values. Now, define $U = [v_1|v_2|v_3| \ ::: \ |v_m]$ and $V = [u_1|u_2|u_3| \ ::: \ |u_m]$. Now show that $A = U\Gamma V^T$ where Γ is a diagonal matrix containing the non-negative values $\gamma_1, \gamma_2, ..., \gamma_m$.

Ans. We have $U\Gamma V^T \implies ([v_1|v_2|v_3| \ ::: \ |v_m]\Gamma)V^T \implies [\gamma_1 v_1|\gamma_2 v_2|\gamma_3 v_3| \ ::: \ |\gamma_m v_m]([u_1|u_2|u_3| \ ::: \ |u_m])^T = [\gamma_1 v_1|\gamma_2 v_2|\gamma_3 v_3| \ ::: \ |\gamma_m v_m]([\frac{A^T v_1}{||A^T v_1||_2}|\frac{A^T v_2}{||A^T v_2||_2}|\frac{A^T v_3}{||A^T v_3||_2}| \ ::: \ |\frac{A^T v_m}{||A^T v_m||_2}])^T$, where the last step follows from Q3 definition of u_i. Then, we have $\begin{bmatrix}\gamma_1 v_1 & \gamma_2 v_2 & \gamma_3 v_3 & ... & \gamma_m v_m\end{bmatrix} \begin{bmatrix} \frac{(v_1)^T A}{||A^T v_1||_2} \\ \frac{(v_2)^T A}{||A^T v_2||_2} \\ \frac{(v_3)^T A}{||A^T v_3||_2} \\ ... \\ \frac{(v_m)^T A}{||A^T v_m||_2} \end{bmatrix} = (\sum_{i=1}^{i=m} \frac{\gamma_i ||v_i||_2}{||A^T v_i||_2})A$, where the last term has norm in it has the orthogonal v_i yield zero. Now, if the γ_i are chosen properly to make the sum as unity, we will have $U\Gamma V^T = A$.