Part B : Theory

Ans 1 : Given,

$$p(x|\omega_1) \sim N\left(\begin{bmatrix}0\\0\end{bmatrix}, I_2\right)$$

$$p(x|\omega_2) \sim N\left(\begin{bmatrix}0\\0\end{bmatrix}, I_2\right)$$

$$p(x|\omega_3)\ 0.5N\left(\begin{bmatrix}0.5\\0.5\end{bmatrix}, I_2\right) + 0.5N\left(\begin{bmatrix}-0.5\\-0.5\end{bmatrix}, I_2\right)$$

where I_2 is 2×2 identity matrix : $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}_{2\times2}$

and $p(\omega_1) = p(\omega_2) = p(\omega_3) = \frac{1}{3}$ [∵ equi-probable classes]

New point $x = \begin{bmatrix}0.3\\0.3\end{bmatrix}$

Discriminant Function :

$$g_i(x) = \ln p(\omega_i|x)$$
$$= \ln p(x|\omega_i) + \ln p(\omega_i)$$

Calculating $g_1(x)$, $g_2(x)$ and $g_3(x)$

$$g_1(x) = \ln\left[\frac{1}{(2\pi)^{2/2}} \times \frac{1}{(1)} \times \exp\left[-\frac{1}{2}\left\{\left(\begin{bmatrix}0.3\\0.3\end{bmatrix} - \begin{bmatrix}0\\0\end{bmatrix}\right)^T \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \cdot \times \left(\begin{bmatrix}0.3\\0.3\end{bmatrix} - \begin{bmatrix}0\\0\end{bmatrix}\right)\right\}\right]\right] + \ln(1/3)$$

$$= \ln(1/2\pi) - 0.09 + \ln(1/3)$$

For $g_2(x)$:

$$g_2(x) = \ln[P(x|\omega_2)\,P(\omega_2)]$$
$$= \ln P(x|\omega_2) + \ln P(\omega_2)$$

$$= \ln\left[\frac{1}{(2\pi)^{2\times 2}} \times \frac{1}{(1)} \exp\left\{\frac{-1}{2}\left(\begin{bmatrix}0.3\\0.3\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix}\right)^T \times \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}^{'} \times \left(\begin{bmatrix}0.3\\0.3\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix}\right)\right\}\right] + \ln(1/3)$$

$$= \ln\frac{1}{2\pi} - 0.49 + \ln(1/3)$$

Similarly, $g_3(x) = \ln P(x|\omega_3) + \ln P(\omega_3)$

~~$= \ln\left[0.5 \times \frac{1}{(2\pi)^{2/2}}\right.$~~

$$= \ln\left[0.5 \times \frac{1}{(2\pi)^{2/2}} \cdot \frac{1}{(1)} \cdot \exp\left\{\frac{-1}{2}\left(\begin{bmatrix}0.3\\0.3\end{bmatrix} - \begin{bmatrix}0.5\\0.5\end{bmatrix}\right)^T \times \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \times \left(\begin{bmatrix}0.3\\0.3\end{bmatrix} - \begin{bmatrix}0.5\\0.5\end{bmatrix}\right)\right\}\right] + \ln(1/3)$$

$$= \ln\left[\frac{0.5}{2\pi} \times \exp\left\{\frac{-1}{2} \times (0.08)\right\} + \frac{0.5}{2\pi} \exp\left\{\frac{-1}{2} \times (0.68)\right\}\right] + \ln(1/3)$$

$$= \ln\left(\frac{0.5}{2\pi}\right) + \ln\{\exp(-0.04) + \exp(-0.34)\} + \ln(1/3)$$

$$= \ln(0.5) + \ln\left(\frac{1}{2\pi}\right) + \ln[\exp(-0.04) + \exp(-0.34)] + \ln(1/3)$$

so, $g_3(x) > g_1(x)$ and $g_3(x) > g_2(x)$

Therefore the data point $x = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}$ belongs to 3rd category ω_3.

Ans 4: Given, R.V X with mean μ and std deviation σ and scalars a, b.

a) $E[ax+b] = E[ax] + E[b]$

$= aE[x] + E[b]$

$= aE[x] + b$ $\quad [\because E[x] = \mu,\ E[\text{scalar}] = \text{scalar}]$

$= a\mu + b$

b) $\text{Variance}(aX+b) = \text{var}(ax) + \text{var}(b)$

$= a^2\text{var}(x) + 0$ $\quad [\because \text{var}[\text{constant}] = 0]$

$= a^2\sigma^2$

Ans 2: Given,

$$p(x|\omega_i) = \frac{1}{2b} \cdot \exp\left(\frac{-|x-a_i|}{b}\right),$$ for $i = 1, 2$ & $a_1 = 0,\ a_2 = 1,\ b = 1$

$$P(x|\omega_1) = \frac{1}{2(1)} \cdot \exp\left(\frac{-|x-a_1|}{(1)}\right)$$

$$= \frac{e^{-|x|}}{2}$$

$$p(x|\omega_2) = \frac{1}{2b} \exp\left(\frac{-|x-a_2|}{b}\right)$$

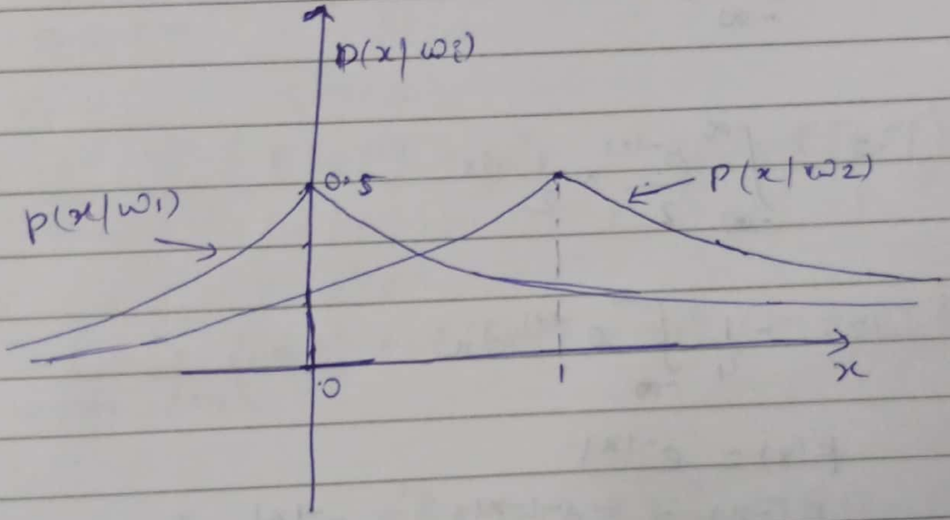
$$= \frac{1}{2(1)} * \exp\left(\frac{-|x-1|}{(1)}\right) = \frac{e^{-|x-1|}}{2}$$

e) And two priors with equally probable

$$p(\omega_1) = p(\omega_2) = \frac{1}{2}$$

a) Likelihood · Ratio :

$$\frac{p(-x|\omega_1)}{p(x|\omega_2)} = \frac{e^{-|x|}}{e^{-|x-1|}} = e^{|x-1|-|x|}$$



b) For Decision Boundary :

$$g_1(x) = g_2(x)$$

$$\Rightarrow \ln p(\omega_1|x) = \ln p(\omega_2|x)$$

$$\Rightarrow p(\omega_1|x) = p(\omega_2|x)$$

$$\Rightarrow p(x|\omega_1) \times \cancel{p(\omega_1)} = p(x|\omega_2) \cdot \cancel{p(\omega_2)}$$

$$\Rightarrow p(x|\omega_1) = p(x|\omega_2)$$

$$\Rightarrow \cancel{\frac{1}{2}} e^{-|x|} = \cancel{\frac{1}{2}} e^{-|x-1|}$$

$$\Rightarrow |x| = |x-1|$$

$$\Rightarrow \left[\,|x| - |x-1| = 0\,\right] \Rightarrow \text{No solution}$$

Then, There will be no decision boundary exist.

c) $p(\text{error}) = \min\{p(\omega_1|x), p(\omega_2|x)\}$

$$= \int_{-\infty}^{\infty} p(\omega_1|x)\cdot p(x)\,dx$$

$$= \int_{-\infty}^{\infty} p(x|\omega_1)\cdot P(\omega_1)\cdot p(x)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{-|x|}}{2} \times \frac{1}{2}\,dx$$

$$= -\frac{1}{4}\int_{-\infty}^{\infty} e^{-|x|}\,dx$$

$F(x) = e^{-|x|}$

$F(-x) = e^{-|-x|} = e^{-|x|}$.

$F(-x) = F(x)$

$$= -\frac{1}{4} \times \int_{0}^{\infty} 2\, e^{-|x|}\,dx$$

$$= -\frac{1}{2} \times \left[\frac{e^{-x}}{-1}\right]_0^{\infty} = \frac{1}{2}\left[e^{-x}\right]_0^{\infty}$$

$$= \frac{1}{2}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{1}{2}\left[0 - 1\right] = -0.5$$

Ans 3: To prove: $\left[\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2\right]$ is unbiased.

$$E\left[\frac{1}{n-1}\cdot\sum_{i=1}^{n}(x_i-\bar{x})^2\right]$$

$$= \frac{1}{n-1}\cdot\sum_{i=1}^{n} E(x_i-\bar{x})^2$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}\left(E[x_i^2] + E[\mu_{ML}^2] - 2E[x_i\mu_{ML}]\right)$$

$$= \frac{1}{n-1}\cdot\sum_{i=1}^{n}\left(E[x_i^2] + \mu_{ML}^2 - 2\mu_{ML}\cdot E[x_i]\right)$$

$$= \frac{1}{n-1}\cdot\sum_{i=1}^{n}\left(\sigma_{ML}^2 + E^2[x_i] + \mu_{ML}^2 - 2\mu_{ML}E[x_i]\right)$$

$$= \frac{n\sigma_{ML}^2}{n-1} + \frac{1}{n-1}\sum_{i=1}^{n}\left(E^2[x_i] + \mu_{ML}^2 - 2\mu_{ML}E[x_i]\right)$$

$$= \left(\frac{n}{n-1}\right)\sigma_{ML}^2 + \frac{1}{(n-1)}\sum_{i=1}^{n}\left(\cancel{\mu_{ML}^2} + \cancel{\mu_{ML}^2} - \cancel{2\mu_{ML}\cdot\mu_{ML}}\right)$$

$$= \left(\frac{n}{n-1}\right)\sigma_{ML}^2 \quad \text{[Unbiased]}$$

$$= \frac{1}{n-1}\cdot n\sigma_{ML}^2$$

$$= \frac{1}{n-1}\cdot\sum_{i=1}^{n}(x_i-\mu_{ML})^2 = RHS$$

Hence, proved.

Ans 5 : we have Cauchy's Distribution :

$$p(x) = \frac{1}{\pi b} \times \frac{1}{1+\left(\frac{x-a}{b}\right)^2}$$

Consider two categories are $\underset{\substack{\downarrow \\ c}}{\text{"correct"}}$ and $\underset{\substack{\downarrow \\ \bar{c}}}{\text{"incorre}}$

$$p(x|c) = \frac{1}{\pi b} \times \frac{1}{1+\left(\frac{x-a}{b}\right)^2}$$

Now,

$$p(c|x) = 1 - p(error|x)$$

$$\frac{p(x|c)\cdot p(c)}{p(x)} = 1 - p(error|x)$$

$$p(c) = \frac{[1 - p(error|x)] \times p(x)}{p(x|c)}$$

$$\left[\quad p(c) = \frac{[1 - \min\{p(c|x), p(\bar{c}|x)\}]\cdot p(x)}{p(x|c)} \quad \right]$$