Chapter 14

Gamma and Beta Functions

14.1 Introduction

We shall define a function known as *Gamma function*, $\Gamma(x)$ which has many interesting properties. The function is defined in terms of an improper integral depending on a parameter. It has great importance in Analysis and in applications. We shall also introduce a function closely related to Γ is *Beta function*, $B(x, y)$.

14.1.1 The Gamma Function

DEFINITION. The *gamma function* denoted by $\Gamma(x)$ is defined (Refer to Art. 11.4.1.) by

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt \quad \text{for} \quad x > 0. \tag{14.1}$$

We have already seen in Art. 11.4.1 that if $0 < x < 1$, the integrand becomes infinite as $t \to 0+$ but the improper integral corresponding to the interval is absolutely convergent when $x > 0$. Again the integral corresponding to the interval $1 \le x < \infty$ converges absolutely for all values of x. Hence $\Gamma(x)$ is well-defined by (14.1) for $x > 0$.

Theorem 1. *$\Gamma(x)$ is continuous and differentiable in $x > 0$.*

Proof. Let x_0 be an arbitrary point in $0 < x < \infty$. Choose a and b such that $0 < a < x_0 < b$. We next examine $\Gamma(x)$ on the interval $a \le x \le b$.

Let $\Gamma(x) = \Gamma_1(x) + \Gamma_2(x)$, where $\Gamma_1(x)$ and $\Gamma_2(x)$ are defined by

$$\Gamma_1(x) = \int_0^1 e^{-t}\, t^{x-1}\, dt \qquad \text{and} \quad \Gamma_2(x) = \int_1^\infty e^{-t}\, t^{x-1}\, dt.$$

Now $\Gamma_1(x)$ converges uniformly for $a \le x \le b$ and for $0 \le t \le 1$, since

$$\left|e^{-t}\, t^{x-1}\right| \le t^{a-1} = M_1(t) \quad \text{and} \quad \int_0^1 M_1(t)\, dt \quad \text{converges.}$$

Also $\Gamma_2(x)$ converges uniformly for $a \le x \le b$ and for $t \ge 1$, since

$$\left|e^{-t}\, t^{x-1}\right| \le e^{-t}\, t^{b-1} = M_2(t) \quad \text{and} \quad \int_1^\infty M_2(t)\, dt \quad \text{converges.}$$

Thus $\Gamma_1(x)$ and $\Gamma_2(x)$ and consequently their sum $\Gamma(x)$ are all continuous in $a \le x \le b$.

In particular $\Gamma(x)$ is continuous at x_0 and x_0 being an arbitrary positive number, $\Gamma(x)$ is continuous for all $x > 0$.

Similarly it can be shown that differentiation under the sign of integration is permissible for all $x > 0$.

Hence the theorem.

Let us now proceed to establish a few relations concerning gamma functions.

Relation 1. For any $a > 0$,

$$\int_0^\infty e^{-at}\, t^{x-1}\, dt = \frac{\Gamma(x)}{a^x}, \quad x > 0.$$

Proof. Put $at = u$, then

$$\int_\epsilon^B e^{-at}\, t^{x-1}\, dt = \int_{a\epsilon}^{aB} e^{-u} \cdot \frac{u^{x-1}}{a^{x-1}} \cdot \frac{du}{a}.$$

As $\epsilon \to 0+$ and $B \to \infty$,

$$\int_0^\infty e^{-at}\, t^{x-1}\, dt = \frac{1}{a^x} \int_0^\infty e^{-u}\, u^{x-1}\, du = \frac{\Gamma(x)}{a^x}.$$

Relation 2. $\boldsymbol{\Gamma(x+1) = x\,\Gamma(x)}$, $\quad x > 0$. [C.H. 1990]

Proof. An integration by parts gives

$$\int_\epsilon^B \overbrace{e^{-t}}^{u}\, \overbrace{t^{x-1}\, dt}^{dv} = \left[e^{-t} \cdot \frac{t^x}{x}\right]_\epsilon^B + \frac{1}{x}\int_\epsilon^B e^{-t}\, t^x\, dt$$

as $B \to \infty$ and $\epsilon \to 0+$, the integrated part vanishes at both limits and therefore

$$\int_0^\infty e^{-t}\, t^{x-1}\, dt = \frac{1}{x}\int_0^\infty e^{-t}\, t^x\, dt$$

i.e., $\quad \Gamma(x) = \dfrac{1}{x}\,\Gamma(x+1), \quad$ or, $\quad \Gamma(x+1) = x\,\Gamma(x), \; x > 0.$

Corollary 1. For all integers $n \geq 1$ and all $x > 0$,

$$\boldsymbol{\Gamma(x+n) = (x+n-1)(x+n-2)\ldots(x+1)x\,\Gamma(x).}$$

The proof follows from Relation 2.

Relation 3. $\boldsymbol{\Gamma(1) = 1}$.

Proof. By direct computation, since $\Gamma(1)$ converges

$$\Gamma(1) = \int_0^\infty e^{-t} dt = \lim_{B\to\infty}\int_0^B -e^{-t} dt = \lim_{B\to\infty}\left(1 - e^{-B}\right) = 1.$$

Relation 4. $\boldsymbol{\Gamma(n+1) = n!}$, n being a positive integer. [C.H. 1987]

Proof. Combining both the relations 2 and 3, when n is a positive integer,

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n(n-1)(n-2)\ldots 3.2.1.\Gamma(1) = n!$$

Relation 5. Show that $\Gamma(x) > \dfrac{1}{e}\displaystyle\int_0^1 t^{x-1} dt = \dfrac{1}{e^x}$ for $x > 0$.

Hence deduce that $\Gamma(0+) = \lim_{x\to 0+}\Gamma(x) = \infty$. [C.H. 1983, '98]

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Proof. Since the integrand of $\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt$, $x > 0$ is positive, we have

$$\Gamma(x) > e^{-1} \int_0^1 t^{x-1}\, dt = e^{-1}\left[\frac{t^x}{x}\right]_0^1 = \frac{1}{ex}, \quad x > 0$$

and therefore when $x \to 0+$, $\Gamma(x) \to \infty$.

14.2 The Beta Function

DEFINITION. The *beta function* denoted by $B(x, y)$ is defined for positive values of x and y by the integral (referred to Art. 11.4.1)

$$B(x, y) = \int_0^1 t^{x-1}\,(1-t)^{y-1}\, dt; \quad x, y > 0. \tag{14.2}$$

Breaking the integral into two parts

$$B(x, y) = \int_0^{\frac{1}{2}} t^{x-1}\,(1-t)^{y-1}\, dt + \int_{\frac{1}{2}}^1 t^{x-1}\,(1-t)^{y-1}\, dt$$

we have shown in Art. 11.4.1 that the first integral on the right converges for $0 < x < 1$ and is proper for $x \geq 1$, no matter what the value of y may be. Thus it converges when $x > 0$.

Next if we make the change of variable $u = 1 - t$

$$\int_{\frac{1}{2}}^1 t^{x-1}\,(1-t)^{y-1}\, dt = \int_0^{\frac{1}{2}} u^{y-1}\,(1-u)^{x-1}\, du$$

that is the second integral reduces to the first with x and y interchanged. Thus $B(x, y)$ is well defined by (14.2) for $x, y > 0$.

Theorem 2. *$B(x, y)$ is continuous for $x > 0$, $y > 0$.*

This can be proved by applying Weierstrass's M-test on the same lines as in Theorem 1 and is left for the students as an exercise.

Relation 6. $\boldsymbol{B(x, y) = B(y, x)}$**, for** $\boldsymbol{x, y > 0}$. **[C.H. 1997]**

Proof. Put $t = 1 - u$, then

$$\int_\epsilon^1 t^{x-1}\,(1-t)^{y-1}\, dt = \int_\delta^{1-\epsilon} u^{y-1}\,(1-u)^{x-1}\, du$$

and on letting $\epsilon \to 0+$, $\delta \to 0+$, we have

$$B(x, y) = B(y, x).$$

Relation 7. $\displaystyle \boldsymbol{B(x, y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\, dt = \int_0^\infty \frac{t^{y-1}}{(1+t)^{x+y}}\, dt;} \quad \boldsymbol{x, y > 0.}$ **[C.H. 1995, '99]**

Proof. Put $t = \frac{1}{1+u}$, then

$$\int_\epsilon^{1-\delta} t^{x-1}\,(1-t)^{y-1}\, dt = \int_{\frac{1}{\epsilon}-1}^{\frac{\delta}{1-\delta}} \frac{1}{(1+u)^{x-1}} \cdot \frac{u^{y-1}}{(1+u)^{y-1}} \cdot \left\{-\frac{1}{(1+u)^2}\right\} du$$

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and on letting $\epsilon \to 0+$ and $\delta \to 0+$, we have

$$\int_0^1 t^{x-1}(1-t)^{y-1}\,dt = B(x,y) = \int_0^\infty \frac{t^{y-1}}{(1+t)^{x+y}}\,dt.$$

Also since $B(x,y) = B(y,x)$, the result is established.

Relation 8. $B(x,y) = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta\,\cos^{2y-1}\theta\,d\theta; \quad x, y > 0.$

[C.H. 1985, '88, '93]

Proof. Put $t = \sin^2\theta$, then

$$\int_\epsilon^{1-\delta} t^{x-1}(1-t)^{y-1}dt = \int_{\sin^{-1}\sqrt{\epsilon}}^{\sin^{-1}\sqrt{1-\delta}} \sin^{2x-2}\theta\,\cos^{2y-2}\theta\cdot 2\sin\theta\cos\theta\,d\theta$$

and on letting $\epsilon \to 0+$, $\delta \to 0+$, we have

$$\int_0^1 t^{x-1}(1-t)^{y-1}\,dt = B(x,y) = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta\,\cos^{2y-1}\theta\,d\theta; \quad x, y > 0.$$

Relation 9. $B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi.$

Proof. Put $x = \frac{1}{2} = y$ in Relation 8, then

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = 2\int_0^{\frac{\pi}{2}} d\theta = \pi.$$

14.3 Relation Between Beta and Gamma Functions

We state below (the proof is given in Ex. 26 in Art. 15.4.7) an important relation between Beta and Gamma functions.

Relation 10. $B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}; \quad x, y > 0.$ [C.H. 1984, '86; B.H. 2001]

An immediate consequence of this is:

Relation 11. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$

Proof. Relation 9 may now take the form

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \pi.$$

But $\Gamma(1) = 1$. Thus $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Relation 12. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^m\theta\,\cos^n\theta\,d\theta = \frac{1}{2}B\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)}; \quad m, n > -1.$$

[C.H. 1995, '97]

Proof. In Relation 8, put $2x-1=m$, $2y-1=n$, then

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\,d\theta = \frac{1}{2}B\left(\frac{m+1}{2},\frac{n+1}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)};\quad m,n>-1.$$

Relation 13. $\Gamma(x) = 2\displaystyle\int_0^\infty e^{-t^2}t^{2x-1}\,dt,\quad x>0.$

[C.H. 1996]

Proof. The substitution $t=u^2$ turns $\int_\epsilon^B e^{-t}\,t^{x-1}\,dt$ into $\int_{\sqrt{\epsilon}}^{\sqrt{B}} e^{-u^2}u^{2x-2}\cdot 2u\,du$ and on letting $\epsilon\to 0+$ and $B\to\infty$, we have

$$\int_0^\infty e^{-t}\,t^{x-1}\,dt = \Gamma(x) = 2\int_0^\infty e^{-t^2}t^{2x-1}\,dt,\quad x>0.$$

Relation 14. $\displaystyle\int_0^\infty e^{-t^2}dt = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2},\quad \int_{-\infty}^\infty e^{-t^2}dt = \sqrt{\pi}.$

[C.H. 1990, '92]

Proof. Put $x=\frac{1}{2}$ in Relation 13, then

$$\int_0^\infty e^{-t^2}dt = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Next since $e^{-t^2}=e^{-(-t)^2}$, it is an even function of t, whereby

$$\int_{-\infty}^\infty e^{-t^2}dt = 2\times\int_0^\infty e^{-t^2}dt = \sqrt{\pi}.$$

An alternative proof of $\displaystyle\int_0^\infty e^{-t^2}dt$.

In $\int_\epsilon^B e^{-t^2}dt$, put $t^2=u$, then it becomes $\int_{\epsilon^2}^{B^2} e^{-u}\cdot\frac{1}{2\sqrt{u}}du$ and letting $\epsilon\to 0$ and $B\to\infty$,

$$\int_0^\infty e^{-t^2}dt = \frac{1}{2}\int_0^\infty e^{-t}t^{-\frac{1}{2}}dt = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Note. A proof of this relation based on multiple integrals is also given in Ex. 25 in Art. 15.4.7.

Relation 15. $\displaystyle\int_0^{\frac{\pi}{2}}\sin^m x\,dx = \int_0^{\frac{\pi}{2}}\cos^m x\,dx = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)},\quad m>-1.$

Proof. In Relation 12, put $n=0$, then

$$\int_0^{\frac{\pi}{2}}\sin^m\theta\,d\theta = \frac{1}{2}\cdot\frac{\Gamma\left(\frac{m+1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)} = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)},\quad m>-1.$$

Similarly for the other.

Theorem 3. Duplication Formula. *Prove that*

$$\sqrt{\pi}\Gamma(2x) = 2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right),\quad \text{for}\quad x>0.$$

[C.H. 1982, '85, '88, '91, '97, '98; B.H. 2000]

Proof. We have from Relations 8 and 10, for $x, y > 0$;

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta \cos^{2y-1}\theta\, d\theta. \tag{14.3}$$

Put $y = x$, then

$$B(x,x) = \frac{\Gamma(x)\Gamma(x)}{\Gamma(2x)} = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta \cos^{2x-1}\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^{2x-1}\, d\theta = \frac{2}{2^{2x-1}}\int_0^{\frac{\pi}{2}} \sin^{2x-1} 2\theta\, d\theta$$

$$= \frac{1}{2^{2x-1}}\int_0^{\pi} \sin^{2x-1}\phi\, d\phi, \quad \text{if we put} \quad 2\theta = \phi$$

$$= \frac{2}{2^{2x-1}}\int_0^{\frac{\pi}{2}} \sin^{2x-1}\phi\, d\phi. \tag{14.4}$$

[since $\sin^{2x-1}(\pi-\phi) = \sin^{2x-1}\phi$]

Next put $y = \frac{1}{2}$ in (14.3),

$$\frac{\Gamma(x)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(x+\frac{1}{2}\right)} = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta\, d\theta \tag{14.5}$$

from (14.4) and (14.5), for $x > 0$

$$\frac{\Gamma(x)\Gamma(x)}{\Gamma(2x)} = \frac{1}{2^{2x-1}} \cdot \frac{\Gamma(x)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(x+\frac{1}{2}\right)} = \frac{\sqrt{\pi}}{2^{2x-1}} \cdot \frac{\Gamma(x)}{\Gamma\left(x+\frac{1}{2}\right)}$$

and the theorem is established.

This formula is known as *Duplication formula*.

Next we are going to prove a theorem with m and n changed from x and y as it will be easier to apply it in various problems.

Theorem 4. *Prove that* $\Gamma(m)\Gamma(1-m) = \pi \operatorname{cosec} m\pi, \quad 0 < m < 1.$

Proof. We have from Relations 7 and 10, for $m, n > 0$

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx.$$

Put $n = 1 - m$. Then $0 < m < 1$,

$$B(m, 1-m) = \frac{\Gamma(m)\Gamma(1-m)}{\Gamma(1)} = \Gamma(m)\Gamma(1-m)$$

$$= \int_0^{\infty} \frac{x^{m-1}}{1+x}\, dx \quad [\text{since } \Gamma(1) = 1]$$

$$= \int_0^{1} \frac{x^{m-1}}{1+x}\, dx + \int_1^{\infty} \frac{x^{m-1}}{1+x}\, dx \tag{14.6}$$

since both these integrals converge for $0 < m < 1$ [vide worked out examples 22 and 23, Art. 11.4.2].

Putting $x = 1/y$ we have

$$\int_1^{B} \frac{x^{m-1}}{1+x}\, dx = \int_{1/B}^{1} \frac{y^{-m}}{1+y}\, dy$$

whereby as $B \to \infty$, the second integral of (14.6) becomes

$$\int_1^\infty \frac{x^{m-1}}{1+x}\,dx = \int_0^1 \frac{y^{-m}}{1+y}\,dy = \int_0^1 \frac{x^{-m}}{1+x}\,dx.$$

Thus from (14.6), for $0 < m < 1$

$$\begin{aligned}\Gamma(m)\Gamma(1-m) &= \int_0^1 \frac{x^{m-1}+x^{-m}}{1+x}\,dx = \int_0^1 \left(x^{m-1}+x^{-m}\right)\left(1-\frac{x}{1+x}\right)dx \\ &= \int_0^1 \left(x^{m-1}+x^{-m}\right)dx - \int_0^1 \frac{x^m+x^{1-m}}{1+x}\,dx, \qquad (14.7)\end{aligned}$$

provided any two of the integrals be convergent [vide 11.1.2 (2)]. Now, since

$$\begin{aligned}\int_\epsilon^1 \left(x^{m-1}+x^{-m}\right)dx &= \left[\frac{x^m}{m}+\frac{x^{1-m}}{1-m}\right]_\epsilon^1 \\ &= \left(\frac{1}{m}+\frac{1}{1-m}\right) - \left(\frac{\epsilon^m}{m}+\frac{\epsilon^{1-m}}{1-m}\right) \\ &\to \frac{1}{m}+\frac{1}{1-m} \quad \text{as} \quad \epsilon \to 0+0;\end{aligned}$$

$$\int_0^1 \left(x^{m-1}+x^{-m}\right)dx \quad \text{converges to} \quad \frac{1}{m}+\frac{1}{1-m}.$$

Next to calculate the remaining convergent integral

$$\int_0^1 \frac{x^m+x^{1-m}}{1+x}\,dx, \qquad 0 < m < 1$$

we see that

$$\frac{x^m+x^{1-m}}{1+x} = \left(x^m+x^{1-m}\right)\left(1-x+x^2-x^3+\cdots\right)$$

is uniformly convergent in $0 \le x < 1$. Hence integrating term by term within the interval,

$$\begin{aligned}\int_0^x \frac{x^m+x^{1-m}}{1+x}\,dx &= \int_0^x \left(x^m+x^{1-m}\right)\left(1-x+x^2-x^3+\cdots\right)dx \\ &= \left[\frac{x^{m+1}}{m+1}+\frac{x^{2-m}}{2-m}-\frac{x^{m+2}}{m+2}-\frac{x^{3-m}}{3-m}+\cdots\right]_0^x \\ &= \frac{x^{m+1}}{m+1}+\frac{x^{2-m}}{2-m}-\frac{x^{m+2}}{m+2}-\frac{x^{3-m}}{3-m}+\cdots \quad \text{for} \quad 0 \le x < 1.\end{aligned}$$

Also at $x = 1$, the integrated series becomes

$$\frac{1}{m+1}+\frac{1}{2-m}-\frac{1}{m+2}-\frac{1}{3-m}+\cdots$$

which is convergent.

Hence the series represents a continuous function throughout $0 \le x \le 1$ and as $x \to 1-0$, we obtain

$$\int_0^1 \frac{x^m+x^{1-m}}{1+x}\,dx = \frac{1}{m+1}+\frac{1}{2-m}-\frac{1}{m+2}-\frac{1}{3-m}+\cdots$$

Thus

$$\Gamma(m)\Gamma(1-m) = \frac{1}{m} + \frac{1}{1-m} - \frac{1}{m+1} - \frac{1}{2-m} + \frac{1}{m+2} + \frac{1}{3-m} - \cdots$$

$$= \sum_{k=0}^{\infty}(-1)^k \left(\frac{1}{k+m} + \frac{1}{k+1-m}\right) = \pi \operatorname{cosec} m\pi,$$

since we have the relation

$$\operatorname{cosec} x = \sum_{k=0}^{\infty}(-1)^k \left\{\frac{1}{k\pi + x} + \frac{1}{(k+1)\pi - x}\right\}.$$

Theorem 5. *Discuss the convergence of* $\int_0^1 \log\Gamma(x)\,dx$ *and show that its value is* $\frac{1}{2}\log(2\pi)$. **[C.H. 1986, '88]**

Proof. The only singularity is at $x = 0$. Now

$$\Gamma(x+1) = x\Gamma(x), \qquad \text{or,} \quad \Gamma(x) = \frac{\Gamma(x+1)}{x} \quad \text{for} \quad x > 0.$$

$$\therefore \qquad \log\Gamma(x) = \log\Gamma(x+1) - \log x, \quad x > 0.$$

We can also write

$$\int_0^1 \log\Gamma(x)\,dx = \int_0^1 \{\log\Gamma(x+1) - \log x\}\,dx,$$

since $\int_0^1 \log\Gamma(x+1)\,dx$ is proper and $\int_0^1 \log x\,dx$ converges at $x = 0$, the reason being $\lim_{x\to 0+} x^{1/2}\log x = 0$, for $\mu = \frac{1}{2} < 1$.

Therefore, $\int_0^1 \log\Gamma(x)\,dx$ converges. Next

$$\int_0^1 \log\Gamma(x)\,dx = \int_1^0 -\log\Gamma(1-z)\,dz, \quad \text{if} \quad x = 1-z.$$

$$= \int_0^1 \log\Gamma(1-x)\,dx = \frac{1}{2}\int_0^1 \{\log\Gamma(x) + \log\Gamma(1-x)\}\,dx$$

$$= \frac{1}{2}\int_0^1 \log\Gamma(x)\Gamma(1-x)\,dx = \frac{1}{2}\int_0^1 \log\frac{\pi}{\sin\pi x}\,dx$$

[since $\Gamma(x)\Gamma(1-x) = \pi \operatorname{cosec} \pi x$]

$$= \frac{1}{2}\int_0^1 \log\pi\,dx - \frac{1}{2}\int_0^1 \log\sin\pi x\,dx$$

$$= \frac{1}{2}\log\pi - \frac{1}{2}\int_0^\pi \log\sin t \cdot \frac{1}{\pi}\,dt \qquad (\text{Put } \pi x = t.)$$

$$= \frac{1}{2}\log\pi - \frac{1}{2\pi}\cdot 2\int_0^{\frac{\pi}{2}} \log\sin t\,dt$$

$$= \frac{1}{2}\log\pi - \frac{1}{\pi}\cdot\frac{\pi}{2}\log\frac{1}{2} = \frac{1}{2}\{\log\pi + \log 2\} = \frac{1}{2}\log(2\pi).$$

Hence the theorem.

14.4 Illustrative Examples

Ex. 1. $\Gamma(4) = 3\cdot 2\cdot 1\Gamma(1) = 3\cdot 2\cdot 1\cdot 1 = 6,\quad \Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{15}{8}\sqrt{\pi}.$

$\Gamma\left(\frac{5}{4}\right) = \frac{1}{4}\Gamma\left(\frac{1}{4}\right),\quad \Gamma\left(\frac{8}{3}\right) = \frac{5}{3}\cdot\frac{2}{3}\Gamma\left(\frac{2}{3}\right).$

Ex. 2. (i) $\int_0^\infty e^{-x}x^{\frac{3}{2}}\,dx = \Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}.$

(ii) Evaluate $\int_0^\infty \sqrt{x}e^{-x^3}dx$. [C.H. 1994, '98]

In $\int_\epsilon^B \sqrt{x}e^{-x^3}dx$, put $x^3 = z$, then

$$\int_\epsilon^B \sqrt{x}e^{-x^3}dx = \int_{\epsilon^3}^{B^3} e^{-z}\cdot\frac{1}{3}z^{-\frac{1}{2}}\,dz$$

and on letting $\epsilon\to 0+$ and $B\to\infty$,

$$\int_0^\infty \sqrt{x}e^{-x^3}dx = \frac{1}{3}\int_0^\infty e^{-x}x^{-\frac{1}{2}}\,dx = \frac{1}{3}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{3}.$$

(iii) $\int_0^\infty x^m e^{-ax^n}\,dx$ where m, n and a are positive integers. [C.H. 1999]

Put $ax^n = z$, then

$$\int_\epsilon^B x^m e^{-ax^n}\,dx = \frac{1}{na}\int_{a\epsilon^n}^{aB^n}\left(\frac{z}{a}\right)^{\frac{m+1}{n}-1}\cdot e^{-z}\,dz$$

and on letting $\epsilon\to 0+$, $B\to\infty$,

$$\int_0^\infty x^m e^{-ax^n}\,dx = \frac{1}{na^{\frac{m+1}{n}}}\int_0^\infty e^{-z}z^{\frac{m+1}{n}-1}\,dz = \frac{1}{na^{\frac{m+1}{n}}}\Gamma\left(\frac{m+1}{n}\right).$$

Ex. 3. $\int_0^{\frac{\pi}{2}} \sin^4 x\cos^4 x\,dx = \frac{1}{2}\cdot\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(5)} = \frac{1}{2}\cdot\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{4\cdot 3\cdot 2\cdot 1\Gamma(1)} = \frac{3\pi}{256}.$

Ex. 4. $\int_0^1 x^3\left(1-x^2\right)^{\frac{5}{2}}\,dx$

Solution. Put $x = \sin^2\theta$, then

$$I = \int_0^{\frac{\pi}{2}} \sin^3\theta\cos^6\theta\,d\theta = \frac{1}{2}\cdot\frac{\Gamma(2)\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{11}{2}\right)} = \frac{1}{2}\cdot\frac{1\Gamma(1)\Gamma\left(\frac{7}{2}\right)}{\frac{9}{2}\cdot\frac{7}{2}\cdot\Gamma\left(\frac{7}{2}\right)} = \frac{2}{63}.$$

Ex. 5. $\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx = (b-a)^{m+n-1}B(m,n);\quad m,n>0.$

Solution. Put $x = a\cos^2\theta + b\sin^2\theta$, then

$$I = \int_0^{\frac{\pi}{2}} 2(b-a)^{m+n-1}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta = (b-a)^{m+n-1}\cdot B(m,n).$$