

# Chapter 1

## The Real Numbers

### 1.1 Discussion: the Irrationality of $\sqrt{2}$

**THEOREM 1.1.1.** *There is no rational number whose square is 2.*

*Proof.* A rational number can be written in the form  $\frac{p}{q}$  where  $p$  and  $q$  are integers. We will use an indirect proof. First, assume there is a rational so that its square is 2. It can be written that

$$\left(\frac{p}{q}\right)^2 = 2$$

We can assume  $p$  and  $q$  have no common factors since they would cancel anyways and give us a new  $p$  and  $q$ . Now we can written

$$p^2 = 2q^2$$

which implies that  $p^2$  is an even number, which implies  $p$  is an even number. So we can let  $p = 2r$ . Plugging this in

$$2r^2 = q^2$$

With the same logic as for with  $p$ ,  $q$  is also even. So  $p$  and  $q$  share a common factor of 2 which contradicts the assumption made in the beginning that they share no common factors.  $\square$

### Important number systems as sets

*Natural Numbers*

$$\mathbf{N} = \{1, 2, 3, 4, 5, \dots\}$$

Addition works well here, but there is no additive identity or inverse.

*Integers*

$$\mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

This includes the additive identity (0) and the additive inverses, which define subtraction. The multiplicative identity is 1, but for multiplicative inverses we need to extend to ...

### Rational Numbers

$$\mathbf{Q} = \{\text{all fractions } \frac{p}{q} \text{ where } p \text{ and } q \text{ are integers and } q \neq 0\}$$

The multiplicative inverses define division. All of these properties of  $\mathbf{Q}$  make it into a *field*. A field is any set where addition and multiplication are well-defined operations that are commutative, associative, and obey the distributive property:  $a(b + c) = ab + bc$ . There must be an additive and multiplicative identity, and each element must have an additive and multiplicative inverse.

The set  $\mathbf{Q}$  has a natural *order*. Given two rational numbers  $r$  and  $s$ , one of the following is true:

$$r < s, r = s, \text{ or } r > s$$

This ordering is transitive: if  $r < s$  and  $s < t$ , then  $r < t$ . Also, between any two rational numbers,  $r$  and  $s$ , there is a rational number between them:  $\frac{r+s}{2}$ , which implies that rational numbers are densely packed.

$\mathbf{Q}$  has holes in the spots of irrationals, such as  $\sqrt{2}$  and  $\sqrt{3}$ . To fill these we add ...

### Real Numbers

$$\mathbf{R} = \{\text{all real numbers}\}$$

Just like  $\mathbf{Q}$ ,  $\mathbf{R}$  is a field.  $\mathbf{R}$  is added as a superset of  $\mathbf{Q}$ .  $\mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R}$ .

## 1.2 Some Preliminaries

### Sets

A *set* is a collection of object, usually real numbers. The objects that make up the set are *elements*.

#### Notation

- $x \in A$  means  $x$  is in  $A$
- $A \cup B$  (union of  $A$  and  $B$ ) is defined by: if  $x \in A \cup B$  then  $x \in A$  or  $x \in B$  (or both)
- $A \cap B$  (intersection of  $A$  and  $B$ ) is defined by: if  $x \in A \cap B$  then  $x \in A$  and  $x \in B$
- $\emptyset$  is an *empty set*, or a set without any elements in it
- if  $A \cap B = \emptyset$ , then  $A$  and  $B$  are *disjoint*
- $A \supseteq B$  or  $B \subseteq A$  every element of  $B$  is in  $A$  so for each  $x \in B$ ,  $x \in A$ . So  $B$  is a *subset* of  $A$ , or  $A$  *contains*  $B$
- $A = B$  means each element of  $A \subseteq B$  and  $B \subseteq A$ . So the sets are the same.

- $\bigcup_{n=1}^{\infty} A_n$  or  $\bigcup_{n \in \mathbf{N}} A$  means  $A_1 \cup A_2 \cup \dots \cup A_{\infty}$
- $\bigcap_{n=1}^{\infty} A_n$  or  $\bigcap_{n \in \mathbf{N}} A$  means  $A_1 \cap A_2 \cap \dots \cap A_{\infty}$
- $A^c = \{x \in \mathbf{R} : x \notin A\}$

You can define a set by listing items ( $N = \{1, 2, 3, \dots\}$ ), with words (let E be all even natural numbers), or with a rule or algorithm ( $S = \{r \in \mathbf{Q} : r^2 < 2\}$ ).

### De Morgan's Laws

$(A \cap B)^c = A^c \cup B^c$  and  $(A \cup B)^c = A^c \cap B^c$

### Functions

Given two sets A and B, a *function* from A to B is a rule or mapping that takes each element  $x \in A$  to a single element in B. We can write  $f: A \rightarrow B$ . Given  $x \in A$ ,  $f(x)$  represents an element of B associated with x by f. A is the domain of  $f$ . The range is a subset of B.

### Triangle Inequality

*Absolute Value Function:*

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The Absolute Value Function satisfies:

$$|ab| = |a||b|$$

$$|a + b| \leq |a| + |b|$$

### Logic and Proofs

A type of indirect proof previously used is *proof by contradiction*, which starts by negating what we are proving and then finding a contradiction. Most proofs are direct, which means it starts from a true statement and then gets to the theorems conclusion.

**THEOREM 1.2.1.** *Two real numbers a and b are equal if and only if for every real number  $\epsilon > 0$  it follows that  $|a - b| < \epsilon$*

*Proof.* Must prove both:

$\Rightarrow$  If  $a = b$ , then for every real number  $\epsilon$  it follows that  $|a - b| < \epsilon$ .

If  $a = b$ , then  $|a - b| = 0$ , and  $|a - b| < \epsilon$  for any  $\epsilon > 0$ .

$\Leftarrow$  If for every real number  $\epsilon > 0$  it follows that  $|a - b| < \epsilon$ , then we must have  $a = b$ .

Assume  $a \neq b$ ,

let  $\epsilon_0 = |a - b| > 0$  since  $a \neq b$

But  $|a - b| = \epsilon_0$  contradicts  $|a - b| < \epsilon_0$ , which was given. So  $a \neq b$  is unacceptable, and  $a$  must equal  $b$ .  $\square$

## Induction

The fundamental principle behind induction is that if  $S$  is a subset of  $\mathbf{N}$  so that  $S$  contains 1 and if  $S$  contains  $n$ , then  $S$  contains  $n + 1$ , then by induction  $S = \mathbf{N}$ .

## 1.3 The Axiom of Completeness