

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

A horizontal number line segment from 0 to 1. Points are marked with dots and labeled from left to right: s_2 , s_4 , s_6 , s_5 , s_3 , and s_1 . An arrow points to the interval between s_4 and s_5 , with the label $S \approx .69$ above it.

It is reasonable to say that this series converges to a number $S = 0.69$ (by experimentation with s_N where N is a large number). It is tempting to think that the sum of all those numbers "add" up to S , but for that we must redefine addition for infinite sums. Treating this series algebraically, lets multiply through by $1/2$ and add it back.

The resulting series has the same terms as the original series except in a different order. It has two positive terms and then the negative term instead of switching each time. But $\frac{3}{2}S \neq S$. This is also seen by experimentation with large N s. Addition, in this infinite setting, is not commutative.

$$\sum_{n=0}^{\infty} (-1/2)^n$$
$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots = \frac{1}{1 - 1/2} = \frac{2}{3}$$

If we rearrange this into two positive and then a negative, you get the same result. Hence addition in an infinite setting is sometimes commutative.

This is applied to the double summation of numbers in a *grid*. For example, $a_{ij} : i, j \in \mathbf{N}$, where $a_{ij} = 1/2^{j-i}$ if $j > i$, $a_{ij} = -1$ if $j = i$, and $a_{ij} = 0$ if $j < i$.

$$\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \dots \\ 0 & -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \dots \\ 0 & 0 & -1 & \frac{1}{2} & \frac{1}{4} & \dots \end{bmatrix}$$

We are trying to give

$$\sum_{i,j=1}^{\infty} a_{ij}$$

mathematical meaning. If we sum over all of the j while holding i for each row we get

$$\sum_{i,j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{ij} \right) = \sum_{i=1}^{\infty} 0 = 0$$

since the sum of each row is zero. If we hold j constant and iterate over i first we get

$$\sum_{i,j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} a_{ij} \right) = \sum_{j=1}^{\infty} \left(\frac{-1}{2^{j-1}} \right) = -2$$

The order in which we add causes us to get different results. This double summation occurs when we are multiplying two series:

$$\sum a_i \sum b_j = \sum_{i,j} a_i b_j$$

Now consider the associative property of addition. Consider $\sum_{n=1}^{\infty} (-1)^n$.

$$\sum_{n=1}^{\infty} (-1)^n = (-1 + 1) + (-1 + 1) + (-1 + 1) + (-1 + 1) + (-1 + 1) + \dots = 0$$

$$\sum_{n=1}^{\infty} (-1)^n = -1 + (1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + \dots = -1$$

Different associations cause use to get different results. Manipulations that are legitimate in finite settings do not always extend to infinite settings.