2.7 Double Summations and Products of Infinite Series

Given a doubly indexed array of real numbers $\{a_{ij} : i, j \in \mathbf{N}\}$, it is not clear how to define $\sum_{i,j=1}^{\infty} a_{ij}$ since

$$\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij} \neq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$$

We can define a partial sum by adding together finite rectangles within the array (order does not matter since it is finite).

$$s_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}$$

Then we can define

$$\sum_{i,j=1}^{\infty} a_{ij} = \lim_{n\to\infty} s_{nn}$$

THEOREM 2.7.1. *Let $\{a_{ij} : i, j \in \boldsymbol{N}\}$ be a doubly indexed array of real numbers. If*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}|$$

converges, the both $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$ and $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$ converge to the same value. Moreover,

$$\lim_{n\to\infty} s_{nn} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$$

where $s_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}$.

Proof. Define

$$t_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|$$

TODO: Finish proof in Excercise 2.8.3 □

Products of Series

$$\Big(\sum_{i=1}^{\infty} a_i\Big)\Big(\sum_{j=1}^{\infty} b_j\Big) = (a_1 + a_2 + \dots)(b_1 + b_2 + \dots)$$

$$= a_1b_1 + (a_1b_2 + a_2b_1) + (a_3b_1 + a_2b_2 + a_1b_3) + \cdots = \sum_{k=2}^{\infty} d_k$$

where

$$d_k = a_1b_{k-1} + a_2b_{k-2} + \cdots + a_{k-1}b_1$$

This is called the *Cauchy product*o f two series.