Chapter 1

The Real Numbers

1.1 Discussion: the Irrationality of $\sqrt{2}$

THEOREM 1.1.1. *There is no rational number whose square is 2.*

Proof. A rational number can be written in the form $\frac{p}{q}$ where p and q are integers. We will use an indirect proof. First, assume there is a rational so that its square is 2. It can be written that

$$(\frac{p}{q})^2 = 2$$

We can assume p and q have no common factors since they would cancel anyways and give us a new p and q. Now we can written

$$p^2 = 2q^2$$

which implies that p^2 is an even number, which implies p is an even number. So we can let $p = 2r$. Plugging this in

$$2r^2 = q^2$$

With the same logic as for with p, q is also even. So p and q share a common factor of 2 which contradicts the assumption made in the beginning that they share no common factors. $\square$

Important number systems as sets

Natural Numbers

$$\mathbf{N} = \{1, 2, 3, 4, 5, \dots\}$$

Addition works well he, but there is no additive identity or inverse.

Integers

$$\mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

This includes the additive identity (0) and the additive inverses, which define subtraction. The multiplicative identity is 1, but for multiplicative inverses we need to extend to ...

Rational Numbers

$$\mathbf{Q} = \{\text{all fractions } \frac{p}{q} \text{ where } p \text{ and } q \text{ are integers and } q \neq 0\}$$

The multiplicative inverses define division. All of these properties of $\mathbf{Q}$ make it into a *field*. A field is any set where addition and multiplication are well-defined operations that are commutative, associative, and obey the distributive property: $a(b + c) = ab + bc$. There must be an additive and multiplicative identity, and each element must have an additive and multiplicative inverse.

The set $\mathbf{Q}$ has a natural *order*. Given two rational numbers r and s, one of the following is true:

$$r < s, \; r = s, \text{ or } r > s$$

This ordering is transitive: if $r < s$ and $s < t$, then $r < t$. Also, between any two rational numbers, r and s, there is a rational number between them: $\frac{r+s}{2}$, which implies that rational numbers are densely packed.

$\mathbf{Q}$ is has holes in the spots of irrationals, such as $\sqrt{2}$ and $\sqrt{3}$. To fill these we add ...

Real Numbers

$$\mathbf{R} = \{\text{all real numbers}\}$$

Just like $\mathbf{Q}$, $\mathbf{R}$ is a field. $\mathbf{R}$ is added as a superset of $\mathbf{Q}$. $\mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R}$.

1.2 Some Preliminaries

Sets

A *set* is a collection of object, usually real numbers. The objects that make up the set are *elements*.

Notation

- $x \in A$ means x is in A
- $A \cup B$ (union of A and B) is defined by: if $x \in A \cup B$ then $x \in A$ or $x \in B$ (or both)
- $A \cap B$ (intersection of A and B) is defined by: if $x \in A \cap B$ then $x \in A$ and $x \in B$
- $\emptyset$ is an *empty set*, or a set without any elements in it
- if $A \cap B = \emptyset$, then A and B are *disjoint*
- $A \supseteq B$ or $B \subseteq A$ every element of B is in A so for each $x \in B$, $x \in A$. So B is a *subset* of A, or A *contains* B
- $A = B$ means each element of $A \subseteq B$ and $B \subseteq B$. So the sets are the same.

- $\bigcup_{n=1}^{\infty} A_n$ or $\bigcup_{n\in\mathbf{N}} A$ means $A_1 \cup A_2 \cup \cdots \cup A_\infty$
- $\bigcap_{n=1}^{\infty} A_n$ or $\bigcap_{n\in\mathbf{N}} A$ means $A_1 \cap A_2 \cap \cdots \cap A_\infty$
- $A^c = \{x \in \mathbf{R} : x \notin A\}$

You can define a set by listing items ($N = \{1, 2, 3, \dots\}$), with words (let E be all even natural numbers), or with a rule or algorithm ($S = \{r \in \mathbf{Q} : r^2 < 2\}$).

De Morgan's Laws
$(A \cap B)^c = A^c \cup B^c$ and $(A \cup B)^c = A^c \cap B^c$

Functions

Given two sets A and B, a *function* from A to B is a rule or mapping that takes each element $x \in A$ to a single element in B. We can write $f\colon A \to B$. Given $x \in A$, $f(x)$ represents an element of B associated with x by f. A is the domain of f. The range is a subset of B.
Triangle Inequality
Absolute Value Function:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The Absolute Value Function satisfies:

$$|ab| = |a||b|$$

$$|a + b| \leq |a| + |b|$$

Logic and Proofs

A type of indirect proof previously used is *proof by contradiction*,which starts by negating what we are proving and then finding a contradiction. Most proofs are direct, which means it starts from a true statement and then gets to the theorems conclusion.

THEOREM 1.2.1. *Two real numbers a and b are equal if and only if for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$*

Proof. Must prove both:
$\Rightarrow$ If $a = b$, then for every real number ϵ it follows that $|a - b| < \epsilon$.
If $a = b$, then $|a - b| = 0$, and $|a - b| < \epsilon$ for any $\epsilon > 0$.

$\Leftarrow$ If for every real number $\epsilon > 0$ if follows that $|a - b| < \epsilon$, then we must have $a = b$.
Assume $a \neq b$,

let $\epsilon_0 = |a - b| > 0$ since $a \neq b$
But $|a - b| = \epsilon_0$ contradicts $|a - b| < \epsilon_0$, which was given. So $a \neq b$ is unacceptable, and a must equal b. □

Induction

The fundamental principle behind induction is that if S is a subset of **N** so that S contains 1 and if S contains n, then S contains $n + 1$, then by induction $S = \mathbf{N}$.

1.3 The Axiom of Completeness

Axiom of Completeness. *Every nonempty set of real numbers that is bounded above has a least upper bound*

Least Upper Bounds and Greatest Lower Bounds

Definition A set $A \in \mathbf{R}$ is *bounded above* if there exists a number $b \in \mathbf{R}$ such that $a \leq b$ fro all $a \in A$. The number b is an *upper bound* for A.

The set A is *bounded below* if there exists a *lower bound* $l \in \mathbf{R}$ so that $l \leq a$ for all $a \in A$.

Definition A real number s is the *least upper bound* for a set $A \in \mathbf{R}$ if it meets two criteria:

(i) s is an upper bound for A;

(ii) if b is any upper bound for A, then $s \leq b$;

The least upper bound is also called the *supremum* of A. So, $s = \text{lub } A = \sup A$.

The *greatest lower bound* or *infimum* for A is defined similarly and is denoted by $\inf A$.

A set can have many upper bounds, but only one least upper bound. If s_1 and s_2 are both least upper bounds, then by property (ii) we can assert $s_1 \leq s_2$ and $s_2 \leq s_1$, and that $s_1 = s_2$.

A real number a_0 is a *maximum* of set A if a_0 is an element of A and $a_0 \geq a$ for each $a \in A$. Similarly, a number a_1 is a *minimum* of A if $a_1 \in A$ and $a_1 \leq a$ for each $a \in A$.

An upper bounded set is guaranteed to have a least upper bound by *The Axiom of Completeness*, but it is not guaranteed to have a maximum. A supremum can exist and not be a maximum (if the supremum does not exist in the set), but when a maximum exists it is also the supremum. **Lemma** *Assume $s \in \boldsymbol{R}$ is an upper bound for a set $A \in \boldsymbol{R}$. Then, $s = sup\ A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$*

Proof. Given that s is an upper bound, s is the leastupper bound if and only if any number smaller than s is not an upper bound.

$\Rightarrow$ Assume $s = \sup A$ and consider $s - \epsilon$, where $\epsilon > 0$ has been chosen. Since $s - \epsilon < s$, $s - \epsilon$ is not an upper bound for A. So there must be an $a \in A$ such that $s - \epsilon < a$.

$\Leftarrow$ Assume s is an upper bound so that for every $\epsilon > 0$, $s - \epsilon$ is no longer an upper bound for A. $s = \sup A$ since s is an upper bound, and any real number $b < s$ is not an upper bound. This is apparent by setting $\epsilon = s - b$. $\square$

1.4 Consequences of Completeness

THEOREM 1.4.1 (Nested Interval Property)**.** *For each $n \in \boldsymbol{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \boldsymbol{R} : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \dots$$

has a nonempty intersection; that is, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. In order to show $\bigcap_{n=1}^{\infty} I_n$ is not empty, we are going to use the Axiom of Completeness to produce a single real number x satisfying $x \in I_n$ for every $n \in \mathbf{N}$. Consider the set

$$A = \{a_n : n \in \mathbf{N}\}$$

of left-hand endpoints of the intervals. Since the intervals are nested, every b_n is an upper bound for A. let $x = \sup A$. Consider a particular $I_n = [a_n, b_n]$. Since x is an upper bound for A, $a_n \leq x$. Since x is the least upper bound and each b_n are upper bounds, $x \leq b_n$. So $a_n \leq x \leq b_n$ for any n. So $x \in I_n$ for any $n \in \mathbf{N}$. Hence, $x \in \bigcap_{n=1}^{\infty} I_n \neq \emptyset$. $\square$

The Density of Q in R

THEOREM 1.4.2 (Archimedean Property)**.** *(i) Given any number $x \in \boldsymbol{R}$, there exists an $n \in \boldsymbol{N}$ satisfying $n > x$.*

(ii) Given any real number $y > 0$, there exists an $n \in \boldsymbol{N}$ satisfying $1/n < y$.

Proof. Part (i) states that $\mathbf{N}$ is not bounded above. Assume, for contradiction, that $\mathbf{N}$ is bounded above. By AoC, $\mathbf{N}$ has a least upper bound. Let $\alpha = \sup N$. $\alpha - 1$ is not an upper bound, so there is an $n \in \mathbf{N}$, such that $\alpha - 1 < n$, which is the same as saying $\alpha < n + 1$. $n + 1 \in \mathbf{N}$, we have a contradiction to the fact α is an upper bound.

Part (ii) follows from (i) by letting $x = 1/y$. $\square$

THEOREM 1.4.3 (Density of Q in R)**.** *For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.*

Proof. To simplify matters, let's assume $0 \leq a < b$. A rational number is a quotient of integers, so we must product m, $n \in \mathbf{N}$ so that

$$a < \frac{m}{n} < b$$

First, we must choose a large enough n so that an increment of size $1/n$ is small enough so it doesn't step over the interval (a, b). Basically, we need an $n \in \mathbf{N}$ such that

$$\frac{1}{n} < b - a$$

By the first inequality, we can get $na < m < nb$. With n chosen, we need to choose an m to be the smallest natural number greater than na. So,

$$m - 1 \leq na < m$$

which yields $a < m/n$. And $a < b - 1/n$ from the second inequality. So

$$m \leq na + 1 < n(b - \frac{1}{n}) + 1 = nb$$

Because $m < nb$ so $m/n < b$. Now we have $a < m/n < b$. □

Collary *Given any two real numbers $a < b$, there exists an irrational number t satisfying $a < t < b$*

The Existence of Square Roots

THEOREM 1.4.4. *There exists a real numbers $\alpha \in \boldsymbol{R}$ satisfying $\alpha^2 = 2$.*

Proof. Consider the set

$$T = \{t \in \mathbf{R} : t^2 < 2\}$$

and set $\alpha = \sup T$. If $\alpha^2 < 2$. NEED TO FINISH THIS PROOF. □

Countable and Uncountable Sets

Cardinality
Cardinality refers to the size of a set. The cardinalities of finite sets can be compared by attaching a natural number to each set. By using comparisons rather than just length, this idea extends to infinite sets.

Definition A function $f : A \to B$ is one-to-one (1-1) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if given any $b \in B$, it is possible to find the element $a \in A$ such that $f(a) = b$. **Definition** Two sets A and B havethe same cardinality if there exists $f : A \to B$ that is 1-1 and onto. In this case, we write $A \sim B$.
Countable Sets
Definition A set A is *countable* if $N \sim A$. AN infinite set that is countable is called an *uncountable* set.

THEOREM 1.4.5. *(i) The set $\boldsymbol{Q}$ is countable*
(ii) The set $\boldsymbol{R}$ is uncountable

Proof. (i) For each $n \in \mathbf{N}$, let

$$A_n = \{\pm\frac{p}{q} : \text{where } p, q \in \mathbf{N} \text{ are in lowest terms with } p + q = n\}$$

so

$$A_1 = \{\frac{0}{1}\}, \quad A_2 = \{\frac{1}{1}, \frac{-1}{1}\}, \quad A_3 = \{\frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1}\}$$

Our one to one correspondence from $\mathbf{N}$ to $\mathbf{Q}$ is by listing the elements from $\bigcup_{n=1}^{\infty} A_n$. So, $f(n) = (\bigcup_{n=1}^{\infty} A_n)[n]$. For any fraction, like 22/7, it will be in $\bigcup_{n=1}^{\infty} A_n$ exactly once ($22/7 \in A_2 9$). This makes $\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_n \cap A_m = \emptyset$. So, $\mathbf{N} \sim \mathbf{Q}$ and $\mathbf{Q}$ is countable.

(ii) Proof by contradiction. Assume there exists a 1-1 from $\mathbf{N}$ to $\mathbf{R}$. If we let $x_n = f(n)$ for each $n \in \mathbf{N}$, we can write

$$\mathbf{R} = \{x_1, x_2, x_3, \dots\}$$

Let I_1 be a closed interval that does not contain x_1. Then create infinite intervals based on the following rules. Given an I_n, construct I_{n+1} to satisfy

$$\text{(i) } I_{n+1} \subseteq I_n \text{ and}$$

$$\text{(ii) } x_{n+1} \notin I_{n+1}.$$

Given I_n, it is clear that I_{n+1} exists since I_n certainly contains two smaller disjoint closed intervals and x_{n+1} can only be in one of them. Since $x_{n_0} \notin I_{n_0}$,

$$x_{n_0} \notin \bigcap_{n=1}^{\infty} I_n$$

This is true for every natural number n_0, and hence every real number x_{n_0}, so

$$\bigcap_{n=1}^{\infty} I_n = \emptyset$$

which contradicts the Nested Interval Property, which asserts that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Due to this contradiction, $\mathbf{R}$ cannot be countable, and is uncountable. □

Since $\mathbf{R} = \mathbf{Q} \cup \mathbf{I}$, where $\mathbf{I}$ is all irrational numbers, $\mathbf{I}$ cannot be countable because otherwise $\mathbf{R}$ would be.

THEOREM 1.4.6. *If $A \subseteq B$ and B is countable, then A is either countable, finite, or empty.*

THEOREM 1.4.7. *(i) If A_1, A_2, $\dots A_m$ are each countable sets, then the union $\bigcup_{n=1}^{m} A_n$ is countable.*
(ii) If A_n is a countable set for each $n \in \boldsymbol{N}$, then $\bigcup_{n=1}^{\infty} A_n$ is countable.

1.5 Cantor's Theorem

Cantor's Diagonalization Method

Cantor initially published his discovery that **R** is uncountable in 1874, but in 1891 he offered another simpler proof that relies on decimal representations for real numbers.

THEOREM 1.5.1. *The open interval* $(0,1) = \{x \in \boldsymbol{R} : 0 < x < 1\}$ *is uncountable.*

Power Sets and Cantor's Theorem

Given a set A, the *power set* $P(A)$ refers to the collection of all subsets of A.
Example:

$$P(\{a,b\}) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$$

THEOREM 1.5.2 (Cantor's Theorem)**.** *Given any set A, there does not exist a function $f : A \to P(A)$ that is onto.*

Proof. For contradiction, assume that $f : A \to P(A)$ is onto. FINISH THIS PROOF □