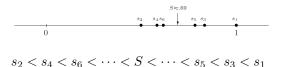
2.1 Discussion: Rearrangements of Infinite Series

Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

If we just add from left to right, we get a series of partial sums: $s_1 = 1$, $s_2 = 1/2$, $s_3 = 5/6$, and so on. We also see that the sums oscillate such that $s_1 > s_3 > s_5 > \ldots$ and $s_2 < s_4 < s_6 < \ldots$



It is reasonable to say that this series converges to a number S=0.69 (by experimentation with s_N where N is a large number). It is tempting to think that the sum of all those numbers "add" up to S, but for that we must redefine addition for infinite sums. That is a large number in the large state of the sum of of the su

fine addition for infinite sums. Treating this series algebraically, lets multiply through by 1/2 and add it back.

$$\frac{1}{2}S = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots$$

$$+ S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \cdots$$

$$\frac{3}{2}S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \cdots$$

The resulting series has the same terms as the original series except in a different order. It has two positive terms and then the negative term instead of switching each time. But $\frac{3}{2}S \neq S$. This is also seen by experimentation with large Ns. Addition, in this infinite setting, is not commutative.

Let us look at another series

$$\sum_{n=0}^{\infty} (-1/2)^n$$

Using $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for geometric series, we get

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \dots = \frac{1}{1 - 1/2} = \frac{2}{3}$$

If we rearrange this into two positive and then a negative, you get the same result. Hence addition in an infinite setting is sometimes commutative.

This is applied to the double summation of numbers in a *grid*. For example, $a_{ij}: i, j \in \mathbb{N}$, where $a_{ij}1/2^{j-i}$ if j > i, $a_{ij} = -1$ if j = i, and $a_{ij} = 0$ if j < i.

$$\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \dots \\ 0 & -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \dots \\ 0 & 0 & -1 & \frac{1}{2} & \frac{1}{4} & \dots \end{bmatrix}$$

We are trying to give

$$\sum_{i,j=1}^{\infty} a_{ij}$$

mathematical meaning. If we sum over all of the j while holding i for each row we get

$$\sum_{i,j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} (\sum_{j=1}^{\infty} a_{ij}) = \sum_{i=1}^{\infty} 0 = 0$$

since the sum of each row is zero. If we hold j constant and iterate over i first we get

$$\sum_{i,j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} (\sum_{i=1}^{\infty} a_{ij}) = \sum_{j=1}^{\infty} (\frac{-1}{2^{j-1}}) = -2$$

The order in which we add causes us to get different results. This double summation occurs when we are multiplying two series:

$$\sum a_i \sum b_j = \sum_{i,j} a_i b_j$$

Now consider the associative property of addition. Consider $\sum_{n=1}^{\infty} (-1)^n$.

$$\sum_{n=1}^{\infty} (-1)^n = (-1+1) + (-1+1) + (-1+1) + (-1+1) + (-1+1) + \cdots = 0$$

$$\sum_{n=1}^{\infty} (-1)^n = -1 + (1-1) + (1-1) + (1-1) + (1-1) + (1-1) + \dots = -1$$

Different associations cause use to get different results. Manipulations that are legitimate in finite settings do not always extend to infinite settings.