

# Analysis Notes

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# Chapter 1

## The Real Numbers

### 1.1 Discussion: the Irrationality of $\sqrt{2}$

**THEOREM 1.1.1.** *There is no rational number whose square is 2.*

*Proof.* A rational number can be written in the form  $\frac{p}{q}$  where  $p$  and  $q$  are integers. We will use an indirect proof. First, assume there is a rational so that its square is 2. It can be written that

$$\left(\frac{p}{q}\right)^2 = 2$$

We can assume  $p$  and  $q$  have no common factors since they would cancel anyways and give us a new  $p$  and  $q$ . Now we can written

$$p^2 = 2q^2$$

which implies that  $p^2$  is an even number, which implies  $p$  is an even number. So we can let  $p = 2r$ . Plugging this in

$$2r^2 = q^2$$

With the same logic as for with  $p$ ,  $q$  is also even. So  $p$  and  $q$  share a common factor of 2 which contradicts the assumption made in the beginning that they share no common factors.  $\square$

### Important number systems as sets

*Natural Numbers*

$$\mathbf{N} = \{1, 2, 3, 4, 5, \dots\}$$

Addition works well here, but there is no additive identity or inverse.

*Integers*

$$\mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

This includes the additive identity (0) and the additive inverses, which define subtraction. The multiplicative identity is 1, but for multiplicative inverses we need to extend to ...

### Rational Numbers

$$\mathbf{Q} = \{\text{all fractions } \frac{p}{q} \text{ where } p \text{ and } q \text{ are integers and } q \neq 0\}$$

The multiplicative inverses define division. All of these properties of  $\mathbf{Q}$  make it into a *field*. A field is any set where addition and multiplication are well-defined operations that are commutative, associative, and obey the distributive property:  $a(b + c) = ab + bc$ . There must be an additive and multiplicative identity, and each element must have an additive and multiplicative inverse.

The set  $\mathbf{Q}$  has a natural *order*. Given two rational numbers  $r$  and  $s$ , one of the following is true:

$$r < s, r = s, \text{ or } r > s$$

This ordering is transitive: if  $r < s$  and  $s < t$ , then  $r < t$ . Also, between any two rational numbers,  $r$  and  $s$ , there is a rational number between them:  $\frac{r+s}{2}$ , which implies that rational numbers are densely packed.

$\mathbf{Q}$  has holes in the spots of irrationals, such as  $\sqrt{2}$  and  $\sqrt{3}$ . To fill these we add ...

### Real Numbers

$$\mathbf{R} = \{\text{all real numbers}\}$$

Just like  $\mathbf{Q}$ ,  $\mathbf{R}$  is a field.  $\mathbf{R}$  is added as a superset of  $\mathbf{Q}$ .  $\mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R}$ .

## 1.2 Some Preliminaries

### Sets

A *set* is a collection of object, usually real numbers. The objects that make up the set are *elements*.

#### Notation

- $x \in A$  means  $x$  is in  $A$
- $A \cup B$  (union of  $A$  and  $B$ ) is defined by: if  $x \in A \cup B$  then  $x \in A$  or  $x \in B$  (or both)
- $A \cap B$  (intersection of  $A$  and  $B$ ) is defined by: if  $x \in A \cap B$  then  $x \in A$  and  $x \in B$
- $\emptyset$  is an *empty set*, or a set without any elements in it
- if  $A \cap B = \emptyset$ , then  $A$  and  $B$  are *disjoint*
- $A \supseteq B$  or  $B \subseteq A$  every element of  $B$  is in  $A$  so for each  $x \in B$ ,  $x \in A$ . So  $B$  is a *subset* of  $A$ , or  $A$  *contains*  $B$
- $A = B$  means each element of  $A \subseteq B$  and  $B \subseteq A$ . So the sets are the same.

- $\bigcup_{n=1}^{\infty} A_n$  or  $\bigcup_{n \in \mathbf{N}} A$  means  $A_1 \cup A_2 \cup \dots \cup A_{\infty}$
- $\bigcap_{n=1}^{\infty} A_n$  or  $\bigcap_{n \in \mathbf{N}} A$  means  $A_1 \cap A_2 \cap \dots \cap A_{\infty}$
- $A^c = \{x \in \mathbf{R} : x \notin A\}$

You can define a set by listing items ( $N = \{1, 2, 3, \dots\}$ ), with words (let  $E$  be all even natural numbers), or with a rule or algorithm ( $S = \{r \in \mathbf{Q} : r^2 < 2\}$ ).

### De Morgan's Laws

$(A \cap B)^c = A^c \cup B^c$  and  $(A \cup B)^c = A^c \cap B^c$

### Functions

Given two sets  $A$  and  $B$ , a *function* from  $A$  to  $B$  is a rule or mapping that takes each element  $x \in A$  to a single element in  $B$ . We can write  $f: A \rightarrow B$ . Given  $x \in A$ ,  $f(x)$  represents an element of  $B$  associated with  $x$  by  $f$ .  $A$  is the domain of  $f$ . The range is a subset of  $B$ .

### Triangle Inequality

*Absolute Value Function:*

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The Absolute Value Function satisfies:

$$|ab| = |a||b|$$

$$|a + b| \leq |a| + |b|$$

### Logic and Proofs

A type of indirect proof previously used is *proof by contradiction*, which starts by negating what we are proving and then finding a contradiction. Most proofs are direct, which means it starts from a true statement and then gets to the theorems conclusion.

**THEOREM 1.2.1.** *Two real numbers  $a$  and  $b$  are equal if and only if for every real number  $\epsilon > 0$  it follows that  $|a - b| < \epsilon$*

*Proof.* Must prove both:

$\Rightarrow$  If  $a = b$ , then for every real number  $\epsilon$  it follows that  $|a - b| < \epsilon$ .

If  $a = b$ , then  $|a - b| = 0$ , and  $|a - b| < \epsilon$  for any  $\epsilon > 0$ .

$\Leftarrow$  If for every real number  $\epsilon > 0$  it follows that  $|a - b| < \epsilon$ , then we must have  $a = b$ .

Assume  $a \neq b$ ,

let  $\epsilon_0 = |a - b| > 0$  since  $a \neq b$

But  $|a - b| = \epsilon_0$  contradicts  $|a - b| < \epsilon_0$ , which was given. So  $a \neq b$  is unacceptable, and  $a$  must equal  $b$ .  $\square$

## Induction

The fundamental principle behind induction is that if  $S$  is a subset of  $\mathbf{N}$  so that  $S$  contains 1 and if  $S$  contains  $n$ , then  $S$  contains  $n + 1$ , then by induction  $S = \mathbf{N}$ .

## 1.3 The Axiom of Completeness

**Axiom of Completeness.** *Every nonempty set of real numbers that is bounded above has a least upper bound*

### Least Upper Bounds and Greatest Lower Bounds

**Definition** A set  $A \in \mathbf{R}$  is *bounded above* if there exists a number  $b \in \mathbf{R}$  such that  $a \leq b$  for all  $a \in A$ . The number  $b$  is an *upper bound* for  $A$ .

The set  $A$  is *bounded below* if there exists a *lower bound*  $l \in \mathbf{R}$  so that  $l \leq a$  for all  $a \in A$ .

**Definition** A real number  $s$  is the *least upper bound* for a set  $A \in \mathbf{R}$  if it meets two criteria:

- (i)  $s$  is an upper bound for  $A$ ;
- (ii) if  $b$  is any upper bound for  $A$ , then  $s \leq b$ ;

The least upper bound is also called the *supremum* of  $A$ . So,  $s = \text{lub } A = \sup A$ .

The *greatest lower bound* or *infimum* for  $A$  is defined similarly and is denoted by  $\inf A$ .

A set can have many upper bounds, but only one least upper bound. If  $s_1$  and  $s_2$  are both least upper bounds, then by property (ii) we can assert  $s_1 \leq s_2$  and  $s_2 \leq s_1$ , and that  $s_1 = s_2$ .

A real number  $a_0$  is a *maximum* of set  $A$  if  $a_0$  is an element of  $A$  and  $a_0 \geq a$  for each  $a \in A$ . Similarly, a number  $a_1$  is a *minimum* of  $A$  if  $a_1 \in A$  and  $a_1 \leq a$  for each  $a \in A$ .

An upper bounded set is guaranteed to have a least upper bound by *The Axiom of Completeness*, but it is not guaranteed to have a maximum. A supremum can exist and not be a maximum (if the supremum does not exist in the set), but when a maximum exists it is also the supremum. **Lemma** *Assume  $s \in \mathbf{R}$  is an upper bound for a set  $A \in \mathbf{R}$ . Then,  $s = \sup A$  if and only if, for every choice of  $\epsilon > 0$ , there exists an element  $a \in A$  satisfying  $s - \epsilon < a$*

*Proof.* Given that  $s$  is an upper bound,  $s$  is the least upper bound if and only if any number smaller than  $s$  is not an upper bound.

$\Rightarrow$  Assume  $s = \sup A$  and consider  $s - \epsilon$ , where  $\epsilon > 0$  has been chosen. Since  $s - \epsilon < s$ ,  $s - \epsilon$  is not an upper bound for  $A$ . So there must be an  $a \in A$  such that  $s - \epsilon < a$ .

$\Leftarrow$  Assume  $s$  is an upper bound so that for every  $\epsilon > 0$ ,  $s - \epsilon$  is no longer an upper bound for  $A$ .  $s = \sup A$  since  $s$  is an upper bound, and any real number  $b < s$  is not an upper bound. This is apparent by setting  $\epsilon = s - b$ .  $\square$

## 1.4 Consequences of Completeness

**THEOREM 1.4.1** (Nested Interval Property). *For each  $n \in \mathbf{N}$ , assume we are given a closed interval  $I_n = [a_n, b_n] = \{x \in \mathbf{R} : a_n \leq x \leq b_n\}$ . Assume also that each  $I_n$  contains  $I_{n+1}$ . Then, the resulting nested sequence of closed intervals*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \dots$$

*has a nonempty intersection; that is,  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ .*

*Proof.* In order to show  $\bigcap_{n=1}^{\infty} I_n$  is not empty, we are going to use the Axiom of Completeness to produce a single real number  $x$  satisfying  $x \in I_n$  for every  $n \in \mathbf{N}$ . Consider the set

$$A = \{a_n : n \in \mathbf{N}\}$$

of left-hand endpoints of the intervals. Since the intervals are nested, every  $b_n$  is an upper bound for  $A$ . let  $x = \sup A$ . Consider a particular  $I_n = [a_n, b_n]$ . Since  $x$  is an upper bound for  $A$ ,  $a_n \leq x$ . Since  $x$  is the least upper bound and each  $b_n$  are upper bounds,  $x \leq b_n$ . So  $a_n \leq x \leq b_n$  for any  $n$ . So  $x \in I_n$  for any  $n \in \mathbf{N}$ . Hence,  $x \in \bigcap_{n=1}^{\infty} I_n \neq \emptyset$ .  $\square$

### The Density of $\mathbf{Q}$ in $\mathbf{R}$

**THEOREM 1.4.2** (Archimedean Property). *(i) Given any number  $x \in \mathbf{R}$ , there exists an  $n \in \mathbf{N}$  satisfying  $n > x$ .*

*(ii) Given any real number  $y > 0$ , there exists an  $n \in \mathbf{N}$  satisfying  $1/n < y$ .*

*Proof.* Part (i) states that  $\mathbf{N}$  is not bounded above. Assume, for contradiction, that  $\mathbf{N}$  is bounded above. By AoC,  $\mathbf{N}$  has a least upper bound. Let  $\alpha = \sup \mathbf{N}$ .  $\alpha - 1$  is not an upper bound, so there is an  $n \in \mathbf{N}$ , such that  $\alpha - 1 < n$ , which is the same as saying  $\alpha < n + 1$ .  $n + 1 \in \mathbf{N}$ , we have a contradiction to the fact  $\alpha$  is an upper bound.

Part (ii) follows from (i) by letting  $x = 1/y$ .  $\square$

**THEOREM 1.4.3** (Density of  $\mathbf{Q}$  in  $\mathbf{R}$ ). *For every two real numbers  $a$  and  $b$  with  $a < b$ , there exists a rational number  $r$  satisfying  $a < r < b$ .*

*Proof.* To simplify matters, let's assume  $0 \leq a < b$ . A rational number is a quotient of integers, so we must product  $m, n \in \mathbf{N}$  so that

$$a < \frac{m}{n} < b$$

First, we must choose a large enough  $n$  so that an increment of size  $1/n$  is small enough so it doesn't step over the interval  $(a, b)$ . Basically, we need an  $n \in \mathbf{N}$  such that

$$\frac{1}{n} < b - a$$

By the first inequality, we can get  $na < m < nb$ . With  $n$  chosen, we need to choose an  $m$  to be the smallest natural number greater than  $na$ . So,

$$m - 1 \leq na < m$$

which yields  $a < m/n$ . And  $a < b - 1/n$  from the second inequality. So

$$m \leq na + 1 < n(b - \frac{1}{n}) + 1 = nb$$

Because  $m < nb$  so  $m/n < b$ . Now we have  $a < m/n < b$ . □

**Collary** Given any two real numbers  $a < b$ , there exists an irrational number  $t$  satisfying  $a < t < b$

## The Existence of Square Roots

**THEOREM 1.4.4.** There exists a real numbers  $\alpha \in \mathbf{R}$  satisfying  $\alpha^2 = 2$ .

*Proof.* Consider the set

$$T = \{t \in \mathbf{R} : t^2 < 2\}$$

and set  $\alpha = \sup T$ . If  $\alpha^2 < 2$ . NEED TO FINISH THIS PROOF. □

## Countable and Uncountable Sets

### Cardinality

*Cardinality* refers to the size of a set. The cardinalities of finite sets can be compared by attaching a natural number to each set. By using comparisons rather than just length, this idea extends to infinite sets.

**Definition** A function  $f : A \rightarrow B$  is one-to-one (1-1) if  $a_1 \neq a_2$  in  $A$  implies that  $f(a_1) \neq f(a_2)$  in  $B$ . The function  $f$  is *onto* if given any  $b \in B$ , it is possible to find the element  $a \in A$  such that  $f(a) = b$ . **Definition** Two sets  $A$  and  $B$  have the same cardinality if there exists  $f : A \rightarrow B$  that is 1-1 and onto. In this case, we write  $A \sim B$ .

### Countable Sets

**Definition** A set  $A$  is *countable* if  $N \sim A$ . AN infinite set that is countable is called an *uncountable* set.

**THEOREM 1.4.5.** (i) The set  $\mathbf{Q}$  is countable  
(ii) The set  $\mathbf{R}$  is uncountable

*Proof.* (i) For each  $n \in \mathbf{N}$ , let

$$A_n = \left\{ \pm \frac{p}{q} : \text{where } p, q \in \mathbf{N} \text{ are in lowest terms with } p + q = n \right\}$$

so

$$A_1 = \left\{ \frac{0}{1} \right\}, \quad A_2 = \left\{ \frac{1}{1}, \frac{-1}{1} \right\}, \quad A_3 = \left\{ \frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1} \right\}$$

Our one to one correspondence from  $\mathbf{N}$  to  $\mathbf{Q}$  is by listing the elements from  $\bigcup_{n=1}^{\infty} A_n$ . So,  $f(n) = (\bigcup_{n=1}^{\infty} A_n)[n]$ . For any fraction, like  $22/7$ , it will be in  $\bigcup_{n=1}^{\infty} A_n$  exactly once ( $22/7 \in A_29$ ). This makes  $\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_n \cap A_m = \emptyset$ . So,  $\mathbf{N} \sim \mathbf{Q}$  and  $\mathbf{Q}$  is countable.

(ii) Proof by contradiction. Assume there exists a 1-1 from  $\mathbf{N}$  to  $\mathbf{R}$ . If we let  $x_n = f(n)$  for each  $n \in \mathbf{N}$ , we can write

$$\mathbf{R} = \{x_1, x_2, x_3, \dots\}$$

Let  $I_1$  be a closed interval that does not contain  $x_1$ . Then create infinite intervals based on the following rules. Given an  $I_n$ , construct  $I_{n+1}$  to satisfy

$$(i) \ I_{n+1} \subseteq I_n \text{ and}$$

$$(ii) \ x_{n+1} \notin I_{n+1}.$$

Given  $I_n$ , it is clear that  $I_{n+1}$  exists since  $I_n$  certainly contains two smaller disjoint closed intervals and  $x_{n+1}$  can only be in one of them. Since  $x_{n_0} \notin I_{n_0}$ ,

$$x_{n_0} \notin \bigcap_{n=1}^{\infty} I_n$$

This is true for every natural number  $n_0$ , and hence every real number  $x_{n_0}$ , so

$$\bigcap_{n=1}^{\infty} I_n = \emptyset$$

which contradicts the Nested Interval Property, which asserts that  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ . Due to this contradiction,  $\mathbf{R}$  cannot be countable, and is uncountable.  $\square$

Since  $\mathbf{R} = \mathbf{Q} \cup \mathbf{I}$ , where  $\mathbf{I}$  is all irrational numbers,  $\mathbf{I}$  cannot be countable because otherwise  $\mathbf{R}$  would be.

**THEOREM 1.4.6.** *If  $A \subseteq B$  and  $B$  is countable, then  $A$  is either countable, finite, or empty.*

**THEOREM 1.4.7.** (i) *If  $A_1, A_2, \dots, A_m$  are each countable sets, then the union  $\bigcup_{n=1}^m A_n$  is countable.*

(ii) *If  $A_n$  is a countable set for each  $n \in \mathbf{N}$ , then  $\bigcup_{n=1}^{\infty} A_n$  is countable.*



## 1.5 Cantor's Theorem

### Cantor's Diagonalization Method

Cantor initially published his discovery that  $\mathbf{R}$  is uncountable in 1874, but in 1891 he offered another simpler proof that relies on decimal representations for real numbers.

**THEOREM 1.5.1.** *The open interval  $(0,1) = \{x \in \mathbf{R} : 0 < x < 1\}$  is uncountable.*

### Power Sets and Cantor's Theorem

Given a set  $A$ , the *power set*  $P(A)$  refers to the collection of all subsets of  $A$ .

**Example:**

$$P(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

**THEOREM 1.5.2** (Cantor's Theorem). *Given any set  $A$ , there does not exist a function  $f : A \rightarrow P(A)$  that is onto.*

*Proof.* For contradiction, assume that  $f : A \rightarrow P(A)$  is onto. FINISH THIS PROOF □

## Chapter 2

# Sequences and Series