2.2 The Limit of a Sequence

Definition 2.2.1. A *sequence* is a function whose domain is $\mathbf{N}$.
Each of the following are common ways to describe a sequence.

1. $(1, \frac{1}{2}, \frac{1}{3}, \dots)$
2. $(\frac{1}{n})_{n=1}^{\infty}$
3. (a_n), where $a_n = 1/n$ for each $n \in \mathbf{N}$
4. (x_n), where $x_1 = 2$ and $x_{n+1} = \frac{x_n+1}{2}$

We do not need to start the sequence at $n = 1$, we can start it at $n = 0$ or $n = n_0$ where $n_0 \in \mathbf{N}$.

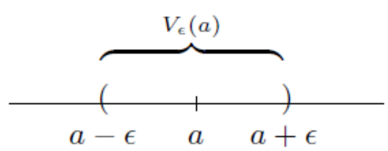
Definition 2.2.3 (Convergence of a Sequence). A sequence (a_n) *converges* to a real number a if, for every positive number ϵ, there exists an $N \in \mathbf{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \epsilon$.

To indicate that (a_n) converges to a, we write either $\lim_{n\to\infty} a_n = \lim a_n = a$ or $(a_n) \to a$.

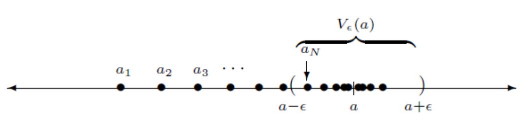
Definition 2.24. Given a real number $a \in \mathbf{R}$ and a positive number $\epsilon > 0$, the set

$$V_\epsilon(a) = x \in \mathbf{R} : |x - a| < \epsilon$$

is called the ϵ*-neighborhood of* a.



Definition 2.2.3B (Convergence of a Sequence: Topological Version). A sequence (a_n) converges to a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, there exists a point in the sequence after which all of the terms are in $V_\epsilon(a)$. In other words, every ϵ-neighborhood contains all but a finite number of the terms of (a_n).



Quantifiers

Template for a proof that $(x_n) \to x$:

- "Let $\epsilon > 0$ be arbitrary."
- Demonstrate a choice for $N \in \mathbf{N}$. This step usually requires the most work, almost all of which is done prior to actually writing the formal proof.
- Now, show that N actually works.
- "Assume $n \geq N$."
- With N well chosen, it should be possible to derive the inequality $|x_n - x| < \epsilon$.

Divergence

Definition 2.2.8. A sequence that does not converge is said to *diverge*.