

Chapter 2

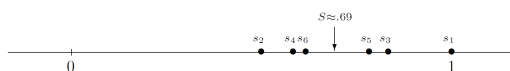
Sequences and Series

2.1 Discussion: Rearrangements of Infinite Series

Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

If we just add from left to right, we get a series of *partial sums*: $s_1 = 1$, $s_2 = 1/2$, $s_3 = 5/6$, and so on. We also see that the sums oscillate such that $s_1 > s_3 > s_5 > \dots$ and $s_2 < s_4 < s_6 < \dots$.



$$s_2 < s_4 < s_6 < \dots < S < \dots < s_5 < s_3 < s_1$$

It is reasonable to say that this series converges to a number $S = 0.69$ (by experimentation with s_N where N is a large number). It is tempting to think that the sum of all those numbers "add" up to S , but for that we must redefine addition for infinite sums. Treating this series algebraically, let's multiply through by $1/2$ and add it back.

$$\begin{aligned} \frac{1}{2}S &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \dots \\ + S &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \dots \\ \hline \frac{3}{2}S &= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} - \dots \end{aligned}$$

The resulting series has the same terms as the original series except in a different order. It has two positive terms and then the negative term instead of switching

each time. But $\frac{3}{2}S \neq S$. This is also seen by experimentation with large N s. Addition, in this infinite setting, is not commutative.

Let us look at another series

$$\sum_{n=0}^{\infty} (-1/2)^n$$

Using $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for geometric series, we get

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots = \frac{1}{1-1/2} = \frac{2}{3}$$

If we rearrange this into two positive and then a negative, you get the same result. Hence addition in an infinite setting is sometimes commutative.

This is applied to the double summation of numbers in a *grid*. For example, $a_{ij} : i, j \in \mathbf{N}$, where $a_{ij} = 1/2^{j-i}$ if $j > i$, $a_{ij} = -1$ if $j = i$, and $a_{ij} = 0$ if $j < i$.

$$\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots \\ 0 & -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ 0 & 0 & -1 & \frac{1}{2} & \frac{1}{4} & \cdots \end{bmatrix}$$

We are trying to give

$$\sum_{i,j=1}^{\infty} a_{ij}$$

mathematical meaning. If we sum over all of the j while holding i for each row we get

$$\sum_{i,j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{ij} \right) = \sum_{i=1}^{\infty} 0 = 0$$

since the sum of each row is zero. If we hold j constant and iterate over i first we get

$$\sum_{i,j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} a_{ij} \right) = \sum_{j=1}^{\infty} \left(\frac{-1}{2^{j-1}} \right) = -2$$

The order in which we add causes us to get different results. This double summation occurs when we are multiplying two series:

$$\sum a_i \sum b_j = \sum_{i,j} a_i b_j$$

Now consider the associative property of addition. Consider $\sum_{n=1}^{\infty} (-1)^n$.

$$\sum_{n=1}^{\infty} (-1)^n = (-1+1) + (-1+1) + (-1+1) + (-1+1) + (-1+1) + \cdots = 0$$

$$\sum_{n=1}^{\infty} (-1)^n = -1 + (1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + \cdots = -1$$

Different associations cause use to get different results. Manipulations that are legitimate in finite settings do not always extend to infinite settings.

2.2 The Limit of a Sequence

Definition 2.2.1. A *sequence* is a function whose domain is \mathbf{N} . Each of the following are common ways to describe a sequence.

1. $(1, \frac{1}{2}, \frac{1}{3}, \dots)$
2. $(\frac{1}{n})_{n=1}^{\infty}$
3. (a_n) , where $a_n = 1/n$ for each $n \in \mathbf{N}$
4. (x_n) , where $x_1 = 2$ and $x_{n+1} = \frac{x_n + 1}{2}$

We do not need to start the sequence at $n = 1$, we can start it at $n = 0$ or $n = n_0$ where $n_0 \in \mathbf{N}$.

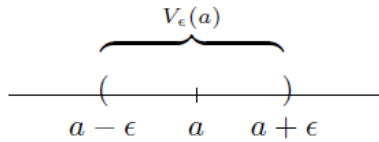
Definition 2.2.3 (Convergence of a Sequence). A sequence (a_n) *converges* to a real number a if, for every positive number ϵ , there exists an $N \in \mathbf{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \epsilon$.

To indicate that (a_n) converges to a , we write either $\lim_{n \rightarrow \infty} a_n = \lim a_n = a$ or $(a_n) \rightarrow a$.

Definition 2.2.4. Given a real number $a \in \mathbf{R}$ and a positive number $\epsilon > 0$, the set

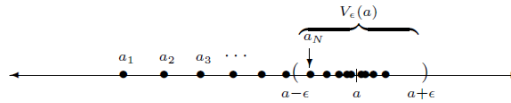
$$V_{\epsilon}(a) = \{x \in \mathbf{R} : |x - a| < \epsilon\}$$

is called the ϵ -neighborhood of a .



Definition 2.2.3B (Convergence of a Sequence: Topological Version).

A sequence (a_n) converges to a if, given any ϵ -neighborhood $V_{\epsilon}(a)$ of a , there exists a point in the sequence after which all of the terms are in $V_{\epsilon}(a)$. In other words, every ϵ -neighborhood contains all but a finite number of the terms of (a_n) .



Quantifiers

TEMPLATE FOR A PROOF THAT $(x_n) \rightarrow x$:

- “Let $\epsilon > 0$ be arbitrary.”
- Demonstrate a choice for $N \in \mathbf{N}$. This step usually requires the most work, almost all of which is done prior to actually writing the formal proof.
- Now, show that N actually works.
- “Assume $n \geq N$.”
- With N well chosen, it should be possible to derive the inequality $|x_n - x| < \epsilon$.

Divergence

Definition 2.2.8. A sequence that does not converge is said to *diverge*.

2.3 The Algebraic and Order Limit Theorems

Definition 2.3.1. A sequence (x_n) is *bounded* if there exists a number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbf{N}$.

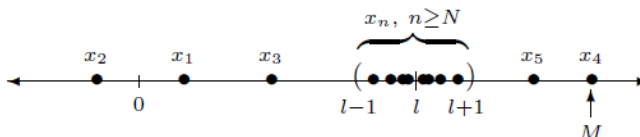
This means $[-M, M]$ contains every term in (x_n)

THEOREM 2.3.2. *Every convergent sequence is bounded.*

Proof. Assume (x_n) converges to a limit l . So for any value of ϵ , there exists an $N \in \mathbf{N}$ such that if $n \geq N$, then x_n is in the interval $(l - \epsilon, l + \epsilon)$, or

$$|x_n| < |l| + \epsilon$$

for all $n \geq N$, for any value of ϵ .



Since there are only a finite number of terms before N , we let

$$M = \max\{|x_1|, |x_2|, |x_3|, \dots, |x_{N-1}|, |l| + \epsilon\}$$

Then it follows that $|x_n| \leq M$ for all $n \in \mathbf{N}$ as desired. \square

THEOREM 2.3.3 (Algebraic Limit Theorem). *Let $\lim a_n = a$, and $\lim b_n = b$. Then,*

- (i) $\lim(ca_n) = ca$, for all $c \in \mathbf{R}$;
- (ii) $\lim(a_n + b_n) = a + b$;
- (iii) $\lim(a_nb_n) = ab$;
- (iv) $\lim(a_n/b_n) = a/b$, provided $b \neq 0$;

Proof. (i) Consider if $c \neq 0$. Let ϵ be some arbitrary positive number. We want to show that after some point in the sequence (ca_n) ,

$$|ca_n - ca| < \epsilon$$

Now,

$$|ca_n - ca| = |c||a_n - a|$$

Since $(a_n) \rightarrow a$, we can make $|a_n - a|$ as small as we want. So we choose an N so

$$|a_n - a| < \frac{\epsilon}{|c|}$$

whenever $n \geq N$. Then,

$$|ca_n - ca| = |c||a_n - a| < |c|\frac{\epsilon}{|c|} = \epsilon$$

The case $c = 0$ reduces to showing the constant sequence $(0, 0, 0, \dots)$ converges to 0. Let $\epsilon > 0$ be arbitrary. Then for any $N \in \mathbf{N}$, $|ca_n - ca| < \epsilon$ for all $n \geq N$ since $|0 - 0| = 0 < \epsilon$.

(ii) Now, we are proving

$$|(a_n + b_n) - (a + b)|$$

can be made less than an arbitrary ϵ . First, use the triangle inequality to say

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$$

Since $(a_n) \rightarrow a$ and $(b_n) \rightarrow b$, we know there exists an N_1 and N_2 such that

$$|a_n - a| < \frac{\epsilon}{2} \text{ whenever } n \geq N_1$$

and

$$|b_n - b| < \frac{\epsilon}{2} \text{ whenever } n \geq N_2$$

Now, let $N = \max\{N_1, N_2\}$ so that when $n \geq N$, then $n \geq N_1$ and $n \geq N_2$. So,

$$\begin{aligned} |(a_n + b_n) - (a + b)| &= |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for all $n \geq N$, as desired.

(iii) To begin,

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |b_n| |a_n - a| + |a| |b_n - b| \end{aligned}$$

Let $\epsilon > 0$ be arbitrary. For $|a| |b_n - b|$, we can choose N_1 so that

$$n \geq N_1 \text{ implies } |b_n - b| < \frac{1}{|a|} \frac{\epsilon}{2}$$

as long as $a \neq 0$. This causes the right side to be less than $\frac{\epsilon}{2}$. Now for $|b_n| |a_n - a|$, we know $|b_n| \leq M$ for some M since it is bounded. So,

$$|b_n| |a_n - a| \leq M |a_n - a|$$

So we choose an N_2 so that

$$|a_n - a| < \frac{1}{M} \frac{\epsilon}{2} \text{ whenever } n \geq N_2$$

Now, pick $N = \max\{N_1, N_2\}$, and observe that if $n \geq N$, then

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |b_n| |a_n - a| + |a| |b_n - b| \\ &\leq M |a_n - a| + |a| |b_n - b| \\ &< M \left(\frac{\epsilon}{M2} \right) + |a| \left(\frac{\epsilon}{|a|2} \right) = \epsilon \end{aligned}$$

(iv) This is proven by (iii) if we can prove that

$$(b_n) \rightarrow b \text{ implies } \left(\frac{1}{b_n} \right) \rightarrow \frac{1}{b}$$

whenever $b \neq 0$.

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| = \frac{|b - b_n|}{|b| |b_n|}$$

We can make $|b - b_n|$ as small as we want. To find a worst case estimate of $|b| |b_n|$, we must find a lower bound greater than 0. Consider $\epsilon_0 = |b|/2$. Since $(b_n) \rightarrow b$, there exists an N_1 such that $|b_n - b| < |b|/2$ for all $n \geq N_1$. This implies $|b_n| > |b|/2 > 0$.

Next, choose N_2 so that $n \geq N$ implies

$$|b_n - b| < \frac{\epsilon |b|^2}{2}$$

Finally, set $N = \max\{N_1, N_2\}$, then $n \geq N$ implies

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| = |b - b_n| \frac{1}{|b| |b_n|} < \frac{\epsilon |b|^2}{2} \frac{1}{|b| \frac{|b|}{2}} = \epsilon$$

□

Limits and Order

THEOREM 2.3.4 (Order Limit Theorem). Assume $\lim a_n = a$ and $\lim b_n = b$

- (i) if $a_n \geq 0$ for all $n \in \mathbf{N}$, then $a \geq 0$.
- (ii) if $a_n \geq b_n$ for all $n \in \mathbf{N}$, then $a \geq b$.
- (iii) If there exists $c \in \mathbf{R}$ for which $c \leq b_n$ for all $n \in \mathbf{N}$, then $c \leq b$. And same for a_n and a .

Proof. (i) We prove this by contradiction. Assume $a < 0$. Then, consider a value of $\epsilon_0 = |a|$. The definition of convergence guarantees that we can find an N such that $|a_n - a| < |a|$ for all $n \geq N$. This means that $|a_N - a| < |a|$, which implies $a_N < 0$, which contradicts that $a_n \geq 0$. We therefore conclude that $a \geq 0$.

(ii) The Algebraic Limit Theorem ensures that the sequence $(b_n - a_n)$ converges to $b - a$. Because $b_n - a_n \geq 0$, we can apply part (i) to get that $b - a \geq 0$.

(iii) Take $a_n = c$ (or $b_n = c$) for all $n \in \mathbf{N}$, and apply (ii). \square

In this theorem, we assumed things for all $n \in \mathbf{N}$, but these properties hold true if these assumptions are true for all $n \geq N$, where N is a finite natural number. If a property is of this form it is said to be *eventually* true. Theorem 2.3.4, part (i), could be restated, "Convergent sequences that are eventually nonnegative converge to nonnegative limits."

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

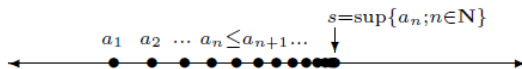
Definition 2.4.1. A sequence (a_n) is *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbf{N}$ and *decreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbf{N}$. A sequence is *monotone* if it is either increasing or decreasing.

THEOREM 2.4.2 (Monotone Convergence Theorem). If a sequence is monotone and bounded, then it converges

Proof. Let (a_n) be monotone and bounded. Let's assume (a_n) is increasing (the decreasing case is handled similarly), and consider the set of points $\{a_n : n \in \mathbf{N}\}$. Since the series is bounded, this set is also bounded, so using the Axiom of Completeness, we can let

$$s = \sup\{a_n : n \in \mathbf{N}\}$$

It seems reasonable for $\lim(a_n) = s$



Let $\epsilon > 0$ be arbitrary. Since s is the least upper bound, $s - \epsilon$ is not an upper bound, so there exists a point in the sequence a_N such that $s - \epsilon < a_N$. Now, since (a_n) is increasing, $a_N \leq a_n$ for all $n \geq N$. Hence,

$$s - \epsilon < a_N \leq a_n \leq s \leq s + \epsilon$$

which implies $|a_n - s| < \epsilon$, as desired. \square

The Monotone Convergence Theorem is useful for infinite series, since it asserts convergence without any mention of the actual limit. **Definition 2.4.3.** Let (b_n) be a sequence. An *infinite series* is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \dots$$

We define the corresponding *sequence of partial sums* (s_m) by

$$s_m = \sum_{n=1}^m b_n = b_1 + b_2 + b_3 + \dots + b_m$$

and say the series $\sum_{n=1}^{\infty} b_n$ converges to B if the sequence (s_m) converges to B . In this case, we write $\sum_{n=1}^{\infty} b_n = B$.

THEOREM 2.4.7 (Cauchy Condensation Test). *Suppose (b_n) is decreasing and satisfies $b_n \geq 0$ for all $n \in \mathbf{N}$. Then, the series $\sum_{n=1}^{\infty} b_n$ converges if and only if the series*

$$\sum_{n=0}^{\infty} 2^n b_{2^n}$$

converges.

Proof. First, assume that $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges. Then the partial sums

$$t_k = b_1 + 2b_2 + 4b_4 + \dots + 2^k b_{2^k}$$

are bounded. There exists an $M > 0$ such that $t_k \leq M$ for all $k \in \mathbf{N}$. Since $b_n \geq 0$, we now that the partial sums are increasing, so we only need to show that

$$s_m = b_1 + b_2 + \dots + b_m$$

is bounded.

Fix m and let k be large enough to ensure $m \leq 2^{k+1} - 1$. Then, $s_m \leq s_{2^{k+1}-1}$ and

$$\begin{aligned} s_{2^{k+1}-1} &= b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + \dots + (b_{2^k} + \dots + b_{2^{k+1}-1}) \\ &\leq b_1 + (b_2 + b_2) + (b_4 + b_4 + b_4 + b_4) + \dots + (b_{2^k} + \dots + b_{2^k}) = b_1 + 2b_2 + 4b_4 + \dots + 2^k b_{2^k} = t_k \end{aligned}$$

Thus, $s_m \leq t_k \leq M$, and the sequence (s_m) is bounded. By the Monotone Convergence Theorem, we can conclude that $\sum_{n=1}^{\infty} b_n$ converges.

Now, if $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges. Fix m and let k be big enough to ensure $m \leq 2^k$. Then,

$$\begin{aligned} s_{2^k} &= b_1 + b_2 + (b_3 + b_4) + (b_5 + b_6 + b_7 + b_8) + \cdots + (b_{2^{k-1}+1} + \cdots + b_{2^k}) \\ &\geq b_1 + b_2 + (b_4 + b_4) + (b_8 + b_8 + b_8 + b_8) + \cdots + (b_{2^k} + b_{2^k} + \cdots + b_{2^k}) \\ &= b_1 + b_2 + 2b_4 + 4b_8 + \cdots + k(b_k) \\ &= b_1 + (t_k - b_1)/2 = (b_1 + t_k)/2 \end{aligned}$$

So, $s_m > (b_1 + t_k)/2$, which diverges since t_k diverges so s_m diverges. \square

Corollary 2.4.7 *The series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$*

2.5 Subsequence and the Bolzano-Weierstrass Theorem

Definition 2.5.1. Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < n_4 < \cdots$ be an increasing sequence of natural numbers. Then the sequence

$$a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, a_{n_6}, \dots$$

is called a *subsequence* of (a_n) and is denoted by (a_{n_j}) , where $j \in \mathbf{N}$ indexes the subsequence.

The terms in a subsequence are in the same order as the original sequence, and repetitions are not allowed.

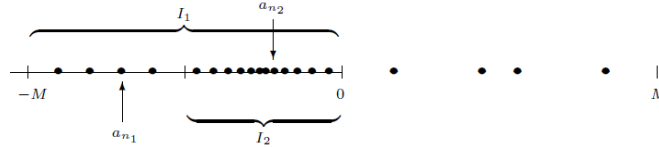
THEOREM 2.5.2. *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

Proof. TODO: Exercise 2.5.1 \square

The Bolzano-Weierstrass Theorem

THEOREM 2.5.5 (The Bolzano-Weierstrass Theorem). *Every bounded sequence contains a convergent subsequence.*

Proof. Let (a_n) be a bounded sequence so that there exists $M > 0$ satisfying $|a_n| \leq M$ for all $n \in \mathbf{N}$. Split $[-M, M]$ into $[-M, 0]$ and $[0, M]$. At least one of these intervals contain an infinite number of the points in the sequence (a_n) . Select a half for which this is the case and label that interval as I_1 . Then, let a_{n_1} be some point in the sequence (a_n) satisfying $a_{n_1} \in I_1$.



Next, we bisect I_1 into closed intervals of equal length, and let I_2 be a half that again contains an infinite number of points of the original sequence. Then choose an a_{n_2} such that $n_2 > n_1$ and $a_{n_2} \in I_2$. In general, we construct the closed interval I_k by taking a half of I_{k-1} containing an infinite number of points of (a_n) and then select $n_k > n_{k-1} > \cdots > n_2 > n_1$ so that $a_{n_k} \in I_k$. The sets

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

form a nested sequence of closed intervals, so by the Nested Interval Property there exists at least one point $x \in \mathbf{R}$ contained in every I_k . Now, we will show that $(a_{n_k} \rightarrow x)$.

Let $\epsilon > 0$. By construct, the length of I_k is $M(1/2)^{k-1}$ which converges to zero. Choose N so that $k \geq N$ implies that the length of I_k is less than ϵ . Since x and a_{n_k} are both in I_k , it follows that $|a_{n_k} - x| < \epsilon$. \square