2.5 Subsequence and the Bolzano-Weierstrass Theorem

Definition 2.5.1. Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < n_4 < \dots$ be an increasing sequence of natural numbers. Then the sequence

$$a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, a_{n_6}, \dots$$

is called a *subsequence* of (a_n) and is denoted by (a_{n_j}), where $j \in \mathbf{N}$ indexes the subsequence.

The terms in a subsequence are in the same order as the original sequence, and repetitions are not allowed.

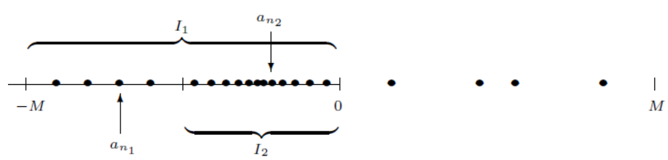
THEOREM 2.5.2. *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

Proof. TODO: Exercise 2.5.1 □

The Bolzano–Weierstrass Theorem

THEOREM 2.5.5 (The Bolzano–Weierstrass Theorem)**.** *Every bounded sequence contains a convergent subsequence.*

Proof. Let (a_n) be a bounded sequence so that there exists $M > 0$ satisfying $|a_n| \leq M$ for all $n \in \mathbf{N}$. Split $[-M, M]$ into $[-M, 0]$ and $[0, M]$. At least one of these intervals contain an infinite number of the points in the sequence (a_n). Select a half for which this is the case and label that interval as I_1. Then, let a_{n_1} be some point in the sequence (a_n) satisfying $a_{n_1} \in I_1$.



Next, we bisect I_1 into closed intervals of equal length, and let I_2 be a half that again contains an infinite number of points of the original sequence. Then choose an a_{n_2} such that $n_2 > n_1$ and $a_{n_2} \in I_2$. In general, we construct the closed interval I_k by taking a half of I_{k-1} containing an infinite number of points of (a_n) and then select $n_k > n_{k-1} > \dots > n_2 > n_1$ so that $a_{n_k + \in I_k}$. The sets

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$$

form a nested sequence of closed intervals, so by the Nested Interval Property there exists at least one point $x \in \mathbf{R}$ contained in every I_k. Now, we will show that $(a_{n_k} \to x)$.

Let $\epsilon > 0$. By construct, the length of I_k is $M(1/2)^{k-1}$ which converges to zero. Choose N so that $k \geq N$ implies that the length of I_k is less than ϵ. Since x and a_{n_k} are both in I_k, it follows that $|a_{n_k} - x| < \epsilon$. □