1.5 Cantor's Theorem

Cantor's Diagonalization Method

Cantor initially published his discovery that **R** is uncountable in 1874, but in 1891 he offered another simpler proof that relies on decimal representations for real numbers.

THEOREM 1.5.1. *The open interval* $(0, 1) = \{x \in \boldsymbol{R} : 0 < x < 1\}$ *is uncountable.*

Power Sets and Cantor's Theorem

Given a set A, the *power set* $P(A)$ refers to the collection of all subsets of A.
Example:

$$P(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

THEOREM 1.5.2 (Cantor's Theorem)**.** *Given any set A, there does not exist a function $f : A \to P(A)$ that is onto.*

Proof. For contradiction, assume that $f : A \to P(A)$ is onto. So for each element $a \in A$, $f(a)$ is a particular subset of A. Since f is onto, eery subset of A appears as $f(a)$ for some $a \in A$. Now, let B be a subset of A ($B \subseteq A$) following

$$B = \{a \in A : a \notin f(a)\}$$

Since f is onto $B = f(a')$ for some $a' \in A$.

If a' is in B ($a' \in B$), $a' \notin f(a')$ since this is a requirement to be in B. Since $a' \notin f(a')$ and $f(a') = B$ implies $a' \notin B$ and we assumed that $a' \in B$, we have a contradiction.

If a' is not in B ($a' \notin B$), $a' \in f(a')$ since it would otherwise be in B. Since $a' \in f(a')$ and $f(a') = B$ implies $a' \in B$ and we assumed that $a' \notin B$, we have a contradiction. $\square$