Chapter 2

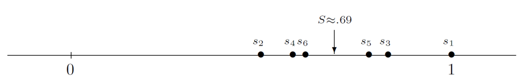
Sequences and Series

2.1 Discussion: Rearrangements of Infinite Series

Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

If we just add from left to right, we get a series of *partial sums*: $s_1 = 1,\ s_2 = 1/2,\ s_3 = 5/6$, and so on. We also see that the sums oscillate such that $s_1 > s_3 > s_5 > \dots$ and $s_2 < s_4 < s_6 < \dots$.



$$s_2 < s_4 < s_6 < \dots < S < \dots < s_5 < s_3 < s_1$$

It is reasonable to say that this series converges to a number $S = 0.69$ (by experimentation with s_N where N is a large number). It is tempting to think that the sum of all those numbers "add" up to S, but for that we must redefine addition for infinite sums. Treating this series algebraically, lets multiply through by 1/2 and add it back.

$$\begin{array}{rl}
\frac{1}{2}S = & \quad \frac{1}{2} \quad - \frac{1}{4} \quad + \frac{1}{6} \quad - \frac{1}{8} \quad + \frac{1}{10} \quad - \frac{1}{12} \quad + \cdots \\
+\ S = & 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \cdots \\
\hline
\frac{3}{2}S = & 1 \quad + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} \quad + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} \quad + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} \quad \cdots
\end{array}$$

The resulting series has the same terms as the original series except in a different order. It has two positive terms and then the negative term instead of switching each time. But $\frac{3}{2}S \neq S$. This is also seen by experimentation with large Ns. Addition, in this infinite setting, is not commutative.

Let us look at another series

$$\sum_{n=0}^{\infty}(-1/2)^n$$

Using $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for geometric series, we get

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots = \frac{1}{1 - 1/2} = \frac{2}{3}$$

If we rearrange this into two positive and then a negative, you get the same result. Hence addition in an infinite setting is sometimes commutative.

This is applied to the double summation of numbers in a *grid*. For example, $a_{ij} : i, j \in \mathbf{N}$, where $a_{ij}1/2^{j-i}$ if $j > i$, $a_{ij} = -1$ if $j = i$, and $a_{ij} = 0$ if $j < i$.

$$\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots \\ 0 & -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ 0 & 0 & -1 & \frac{1}{2} & \frac{1}{4} & \cdots \end{bmatrix}$$

We are trying to give

$$\sum_{i,j=1}^{\infty} a_{ij}$$

mathematical meaning. If we sum over all of the j while holding i for each row we get

$$\sum_{i,j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty}(\sum_{j=1}^{\infty} a_{ij}) = \sum_{i=1}^{\infty} 0 = 0$$

since the sum of each row is zero. If we hold j constant and iterate over i first we get

$$\sum_{i,j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}(\sum_{i=1}^{\infty} a_{ij}) = \sum_{j=1}^{\infty}(\frac{-1}{2^{j-1}}) = -2$$

The order in which we add causes us to get different results. This double summation occurs when we are multiplying two series:

$$\sum a_i \sum b_j = \sum_{i,j} a_i b_j$$

Now consider the associative property of addition. Consider $\sum_{n=1}^{\infty}(-1)^n$.

$$sum_{n=1}^{\infty}(-1)^n = (-1+1) + (-1+1) + (-1+1) + (-1+1) + (-1+1) + \cdots = 0$$

$$sum_{n=1}^{\infty}(-1)^n = -1 + (1-1) + (1-1) + (1-1) + (1-1) + (1-1) + \cdots = -1$$

Different associations cause use to get different results. Manipulations that are legitimate in finite settings do not always extend to infinite settings.

2.2 The Limit of a Sequence

Definition 2.2.1. A *sequence* is a function whose domain is $\mathbf{N}$.
Each of the following are common ways to describe a sequence.

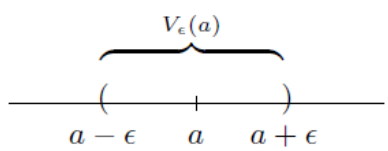
1. $(1, \frac{1}{2}, \frac{1}{3}, \dots)$
2. $(\frac{1}{n})_{n=1}^{\infty}$
3. (a_n), where $a_n = 1/n$ for each $n \in \mathbf{N}$
4. (x_n), where $x_1 = 2$ and $x_{n+1} = \frac{x_n+1}{2}$

We do not need to start the sequence at $n = 1$, we can start it at $n = 0$ or $n = n_0$ where $n_0 \in \mathbf{N}$.
Definition 2.2.3 (Convergence of a Sequence). A sequence (a_n) *converges* to a real number a if, for every positive number ϵ, there exists an $N \in \mathbf{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \epsilon$.

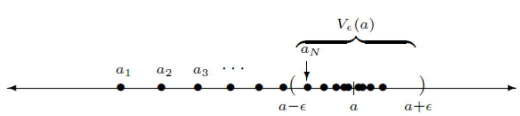
To indicate that (a_n) converges to a, we write either $\lim_{n\to\infty} a_n = \lim a_n = a$ or $(a_n) \to a$.
Definition 2.24. Given a real number $a \in \mathbf{R}$ and a positive number $\epsilon > 0$, the set

$$V_\epsilon(a) = x \in \mathbf{R} : |x - a| < \epsilon$$

is called the *ϵ-neighborhood of a.*



Definition 2.2.3B (Convergence of a Sequence: Topological Version). A sequence (a_n) converges to a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, there exists a point in the sequence after which all of the terms are in $V_\epsilon(a)$. In other words, every ϵ-neighborhood contains all but a finite number of the terms of (a_n).



Quantifiers

TEMPLATE FOR A PROOF THAT $(x_n) \to x$:

- "Let $\epsilon > 0$ be arbitrary."
- Demonstrate a choice for $N \in \mathbf{N}$. This step usually requires the most work, almost all of which is done prior to actually writing the formal proof.
- Now, show that N actually works.
- "Assume $n \geq N$."
- With N well chosen, it should be possible to derive the inequality $|x_n - x| < \epsilon$.

Divergence

Definition 2.2.8. A sequence that does not converge is said to *diverge*.

2.3 The Algebraic and Order Limit Theorems

Definition 2.3.1. A sequence (x_n) is *bounded* if there exists a number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbf{N}$.

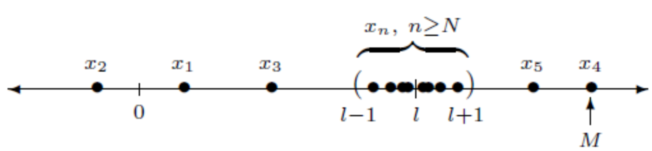
This means $[-M, M]$ contains every term in (x_n)

THEOREM 2.3.2. *Every convergent sequence is bounded.*

Proof. Assume (x_n) converges to a limit l. So for any value of ϵ, there exists an $N \in \mathbf{N}$ such that if $n \geq N$, then x_n is in the interval $(l - \epsilon, l + \epsilon)$, or

$$|x_n| < |l| + \epsilon$$

for all $n \geq N$, for any value of ϵ.



Since there are only a finite number of terms before N, we let

$$M = max\{|x_1|, |x_2|, |x_3|, \ldots, |x_{N-1}|, |l| + \epsilon\}$$

Then it follows that $|x_n| \leq M$ for all $n \in \mathbf{N}$ as desired. $\square$

THEOREM 2.3.3 (Algebraic Limit Theorem)**.** *Let* $\lim a_n = a$*, and* $\lim b_n = b$*. Then,*
(i) $\lim(ca_n) = ca$*, for all* $c \in \boldsymbol{R}$*;*
(ii) $\lim(a_n + b_n) = a + b$*;*
(iii) $\lim(a_n b_n) = ab$*;*
(iv) $\lim(a_n/b_n) = a/b$*, provided* $b \neq 0$*;*

Proof. (i) Consider if $c \neq 0$. Let ϵ be some arbitrary positive number. We want to show that after some point in the sequence (ca_n),

$$|ca_n - ca| < \epsilon$$

Now,

$$|ca_n - ca| = |c||a_n - a|$$

Since $(a_n) \to a$, we can make $|a_n - a|$ as small as we want. So we choose an N so

$$|a_n - a| < \frac{\epsilon}{|c|}$$

whenever $n \geq N$. Then,

$$|ca_n - ca| = |c||a_n - a| < |c|\frac{\epsilon}{|c|} = \epsilon$$

The case $c = 0$ reduces to showing the constant sequence $(0, 0, 0, \dots)$ converges to 0. Let $\epsilon > 0$ be arbitrary. Then for any $N \in \mathbf{N}$, $|ca_n - ca| < \epsilon$ for all $n \geq \mathbf{N}$ since $|0 - 0| = 0 < \epsilon$.

(ii) Now, we are proving

$$|(a_n + b_n) - (a + b)|$$

can be made less than an arbitrary ϵ. First, use the triangle inequality to say

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$$

Since $(a_n) \to a$ and $(b_n) \to b$, we know there exists an N_1 and N_2 such that

$$|a_n - a| < \frac{\epsilon}{2} \text{ whenever } n \geq N_1$$

and

$$|b_n - b| < \frac{\epsilon}{2} \text{ whenever } n \geq N_2$$

Now, let $N = max\{N_1, N_2\}$ so that when $n \geq N$, then $n \geq N_1$ and $n \geq N_2$. So,

$$\begin{aligned}|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| &\leq |a_n - a| + |b_n - b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

for all $n \geq N$, as desired.

(iii) To begin,

$$\begin{aligned}|a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |b_n||a_n - a| + |a||b_n - b|\end{aligned}$$

Let $\epsilon > 0$ be arbitrary. For $|a||b_n - b|$, we can choose N_1 so that

$$n \geq N_1 \text{ implies } |b_n - b| < \frac{1}{|a|}\frac{\epsilon}{2}$$

as long as $a \neq 0$. This causes the right side to be less than $\frac{\epsilon}{2}$ Now for $|b_n||a_n - a|$, we know $|b_n| \leq M$ for some M since it is bounded. So,

$$|b_n||a_n - a| \leq M|a_n - a|$$

So we choose an N_2 so that

$$|a_n - a| < \frac{1}{M}\frac{\epsilon}{2} \text{ whenever } n \geq N_2$$

Now, pick $N = max\{N_1, N_2\}$, and observe that if $n \geq N$, then

$$\begin{aligned}|a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |b_n||a_n - a| + |a||b_n - b| \\ &\leq M|a_n - a| + |a||b_n - b| \\ &< M(\frac{\epsilon}{M2}) + |a|(\frac{\epsilon}{|a|2}) = \epsilon\end{aligned}$$

(iv) This is proven by (iii) if we can prove that

$$(b_n) \to b \text{ implies } (\frac{1}{b_n}) \to \frac{1}{b}$$

whenever $b \neq 0$.

$$|\frac{1}{b_n} - \frac{1}{b}| = \frac{|b - b_n|}{|b||b_n|}$$

We can make $|b - b_n|$ as small as we want. To find a worst case estimate of $|b||b_n|$, we must find a lower bound greater than 0. Consider $\epsilon_0 = |b|/2$. Since $(b_n) \to b$, there exists an N_1 such that $|b_n - b| < |b|/2$ for all $n \geq N_1$. This implies $|b_n| > |b|/2 > 0$.

Next, choose N_2 so that $n \geq N$ implies

$$|b_n - b| < \frac{\epsilon|b|^2}{2}$$

Finally, set $N = max\{N_1, N_2\}$, then $n \geq N$ implies

$$|\frac{1}{b_n} - \frac{1}{b}| = |b - b_n|\frac{1}{|b||b_n|} < \frac{\epsilon|b|^2}{2}\frac{1}{|b|\frac{|b|}{2}} = \epsilon$$

□

Limits and Order

THEOREM 2.3.4 (Order Limit Theorem)**.** *Assume* $\lim a_n = a$ *and* $\lim b_n = b$
(i) if $a_n \geq 0$ *for all* $n \in \boldsymbol{N}$*, then* $a \geq 0$*.*
(ii) if $a_n \geq b_n$ *for all* $n \in \boldsymbol{N}$*, then* $a \geq b$*.*
(iii) If there exists $c \in \boldsymbol{R}$ *for which* $c \leq b_n$ *for all* $n \in \boldsymbol{N}$*, then* $c \leq b$*. And same for* a_n *and* a*.*

Proof. (i) We prove this by contradiction. Assume $a < 0$. Then, consider a value of $\epsilon_0 = |a|$. The definition of convergence guarantees that we can find an N such that $|a_n - a| < |a|$ for all $n \geq N$. This means that $|a_N - a| < |a|$, which implies $a_N < 0$, which contradicts that $a_N \geq 0$. We therefore conslude that $a \geq 0$.
(ii) The Algebraic Limit Theorem ensures that the sequence $(b_n - a_n)$ converges to $b - a$. Because $b_n - a_n \geq 0$, we can apply part (i) to get that $b - a \geq 0$.
(iii) Take $a_n = c$ (or $b_n = c$) for all $n \in \mathbf{N}$, and apply (ii). □

In this theorem, we assumed things for all $n \in \mathbf{N}$, but these properties hold true if these assumptions are true for all $n \geq N$, where N is a finite natural number. If a property is of this form it is said to be *eventually* true. Theorem 2.3.4, part (i), could be restated, "Convergent sequences that are eventually nonnegative converge to nonnegative limits."

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

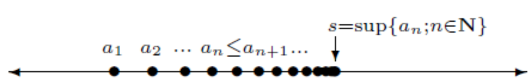
Definition 2.4.1. A sequence (a_n) is *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbf{N}$ and *decreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbf{N}$. A sequence is *monotone* if it is either increasing or decreasing.

THEOREM 2.4.2 (Monotone Convergence Theorem)**.** *If a sequence is monotone and bounded, then it convergences*

Proof. Let (a_n) be monotone and bounded. Let's assume (a_n) is increasing (the decreasing case is handled similarly), and consider the *set* of points $\{a_n : n \in \mathbf{N}\}$. Since the series is bounded, this set it also bounded, so using the Axiom of Completeness, we can let

$$s = \sup\{a_n : n \in \mathbf{N}\}$$

It seems reasonable for $\lim(a_n) = s$



Let $\epsilon > 0$ be arbitrary. Since s is the least upper bound, $s - \epsilon$ is not an upper bound, so there exists a point in the sequence a_N such that $s - \epsilon < a_N$. Now, since (a_n) is increasing, $a_N \leq a_n$ for all $n \geq N$. Hence,

$$s - \epsilon < a_N \leq a_n \leq s \leq s + \epsilon$$

which implies $|a_n - s| < \epsilon$, as desired. □

The Monotone Convergence Theorem is useful for infinite series, since it asserts convergences without any mention of the actual limit. **Definition 2.4.3.** Let (b_n) be a sequence. An *infinite series* is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \dots$$

We define the corresponding *sequence of partial sums* (s_m) by

$$s_m = \sum_{n=1}^{m} b_n = b_1 + b_2 + b_3 + \cdots + b_m$$

and say the the series $\sum_{n=1}^{\infty} b_n$ *converges to* B if the sequence (s_m) converges to B. In this case, we write $\sum_{n=1}^{\infty} b_n = B$.

THEOREM 2.4.7 (Cauchy Condensation Test)**.** *Suppose* (b_n) *is decreasing and satisfies* $b_n \geq 0$ *for all* $n \in \boldsymbol{N}$. *Then, the series* $\sum_{n=1}^{\infty} b_n$ *converges if and only if the series*

$$\sum_{n=0}^{\infty} 2^n b_{2^n}$$

converges.

Proof. First, assume that $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges. Then the partial sums

$$t_k = b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k}$$

are bounded. There exists an $M > 0$ such that $t_k \leq M$ for all $k \in \mathbf{N}$. Since $b_n \geq -$, we now that the partial sums are increasing, so we only need to show that

$$s_m = b_1 + b_2 + \cdots + b_m$$

is bounded.

Fix m and let k be large enough to ensure $m \leq 2^{k+1} - 1$. Then, $s_m \leq s_{2^{k+1}-1}$ and

$$s_{2^{k+1}-1} = b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + \cdots + (b_{2^k} + \cdots + b_{2^{k+1}-1})$$

$$\leq b_1 + (b_2 + b_2) + (b_4 + b_4 + b_4 + b_4) + \cdots + (b_{2^k} + \cdots + b_{2^k}) = b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k} = t_k$$

Thus, $s_m \leq t_k \leq M$, and the sequence (s_m) is bounded. By the Monotone Convergence Theorem, we can conclude that $\sum_{n=1}^{\infty} b_n$ converges.

Now, if $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges. Fix m and let k be big enough to ensure $m \leq 2^k$. Then,

$$s_{2^k} = b_1+b_2+(b_3+b_4)+(b_5+b_6+b_7+b_8)+\cdots+(b_{2^{k-1}+1}+\dots b_{2^k}) \geq b_1+b_2+(b_4+b_4)+(b_8+b_8+b_8+b_8)+\cdots+(b_{2^k}+b$$

So, $s_m > (b_1 + t_k)/2$, which diverges since t_k diverges so s_m diverges. $\square$

Corollary 2.4.7 *The series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$*

2.5 Subsequence and the Bolzano-Weierstrass Theorem

Definition 2.5.1. Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < n_4 < \dots$ be an increasing sequence of natural numbers. Then the sequence

$$a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, a_{n_6}, \dots$$

is called a *subsequence* of (a_n) and is denoted by (a_{n_j}), where $j \in \mathbf{N}$ indexes the subsequence.

The terms in a subsequence are in the same order as the original sequence, and repetitions are not allowed.

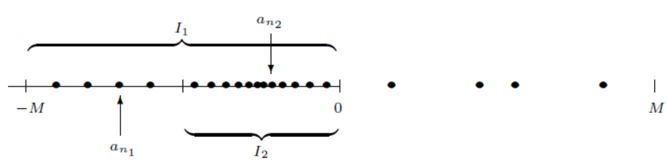
THEOREM 2.5.2. *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

Proof. TODO: Exercise 2.5.1 $\square$

The Bolzano–Weierstrass Theorem

THEOREM 2.5.5 (The Bolzano–Weierstrass Theorem)**.** *Every bounded sequence contains a convergent subsequence.*

Proof. Let (a_n) be a bounded sequence so that there exists $M > 0$ satisfying $|a_n| \leq M$ for all $n \in \mathbf{N}$. Split $[-M, M]$ into $[-M, 0]$ and $[0, M]$. At least one of these intervals contain an infinite number of the points in the sequence (a_n). Select a half for which this is the case and label that interval as I_1. Then, let a_{n_1} be some point in the sequence (a_n) satisfying $a_{n_1} \in I_1$.



Next, we bisect I_1 into closed intervals of equal length, and let I_2 be a half that again contains an infinite number of points of the original sequence. Then choose an a_{n_2} such that $n_2 > n_1$ and $a_{n_2} \in I_2$. In general, we construct the closed interval I_k by taking a half of I_{k-1} containing an infinite number of points of (a_n) and then select $n_k > n_{k-1} > \cdots > n_2 > n_1$ so that $a_{n_k + \in I_k}$. The sets

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$$

form a nested sequence of closed intervals, so by the Nested Interval Property there exists at least one point $x \in \mathbf{R}$ contained in every I_k. Now, we will show that $(a_{n_k} \to x)$.

Let $\epsilon > 0$. By construct, the length of I_k is $M(1/2)^{k-1}$ which converges to zero. Choose N so that $k \geq N$ implies that the length of I_k is less than ϵ. Since x and a_{n_k} are both in I_k, it follows that $|a_{n_k} - x| < \epsilon$. $\square$