2.7 Properties of Infinite Series

The convergence of a series $\sum_{k=1}^{\infty} a_k$ is defined by the terms of sequence (s_n)

$$\sum_{k=1}^{\infty} a_k = A \text{ means that } \lim s_n = A$$

THEOREM 2.7.1 (Algebraic Limit Theorem for Series)**.** *If* $\sum_{k=1}^{\infty} a_k = A$ *and* $\sum_{k=1}^{\infty} b_k = B$*, then*

1. $\sum_{k=1}^{\infty} ca_k = cA$ *for all* $c \in \boldsymbol{R}$ *and*
2. $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$

Proof. 1. The sequence of partial sums for ca_n must converge to cA if and only if $\sum_{k=1}^{\infty} ca_k$, so

$$t_m = ca_1 + ca_2 + ca_3 + \cdots + ca_m$$

converges to cA. But we are given that $\sum_{k=1}^{\infty} a_k$ converges to A, so

$$s_m = a_1 + a_2 + a_3 + \cdots + a_m$$

converges to A. Since $t_m = cs_m$, $(t_m) \to cA$.

2. TODO: Exercise 2.7.8

□

THEOREM 2.7.2 (Cauchy Criterion for Series)**.** *The series* $\sum_{k=1}^{\infty} a_k$ *converges if and only if, given* $\epsilon > 0$*, there exists an* $N \in \boldsymbol{N}$ *such that whenever* $n > m \geq N$ *it follows that*

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon$$

Proof. Observe that

$$|s_n - s_m| = |a_{m+1} + a_{m+2} + \cdots + a_n|$$

and apply the Cauchy Criterion for sequences. □

THEOREM 2.7.3. *If the series* $\sum_{k=1}^{\infty} a_k$ *converges, then* $(a_k) \to 0$.

Proof. Consider the special case $n = m + 1$ in the Cauchy Criterion for Convergent Series. □

The converse of this statement is not true. Ex: Harmonic Series.

THEOREM 2.7.4 (Comparison Test)**.** *Assume* (a_k) *and* (b_k) *are sequences satisfying* $0 \leq a_k \leq b_k$ *for all* $k \in \boldsymbol{N}$

1. *If* $\sum_{k=1}^{\infty} b_k$ *converges, then* $\sum_{k=1}^{\infty} a_k$ *converges.*

2. *If* $\sum_{k=1}^{\infty} a_k$ *diverges, then* $\sum_{k=1}^{\infty} b_k$ *diverges.*

Proof. Both statements follow immediately from the Cauchy Criterion for Series and the observation that

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |b_{m+1} + b_{m+2} + \cdots + b_n|$$

□

Just like before $a_k \leq b_k$ just has to be *eventually* true.

THEOREM 2.7.5 (Absolute Convergence Test)**.** *If the series* $\sum_{k=1}^{\infty} |a_k|$*, then* $\sum_{k=1}^{\infty} a_k$ *converges as well.*

Proof. Since $\sum_{k=1}^{\infty} |a_k|$ converges, we know that, given an $\epsilon > 0$, there is an $N \in \mathbf{N}$ such that

$$|a_{m+1}| + |a_{m+2}| + \cdots + |a_n| < \epsilon$$

for all $n > m \geq N$. By the triangle inequality,

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |a_{m+1}| + |a_{m+2}| + \cdots + |a_n| < \epsilon$$

so the sufficiency of the Cauchy Criterion guarantees that $\sum_{k=1}^{\infty} a_k$ also converges. □

The converse is not always true. Consider an alternating harmonic series, which converges.

THEOREM 2.7.6 (Alternating Series Test)**.** *Let* (a_n) *be a sequence satisfying*

1. $a_{n+1} > a_n$ *for all* $n \in \boldsymbol{N}$ *and*
2. $(a_n) \to 0$

Then, the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ *converges.*

Proof. TODO: Exercise 2.7.1 □

Definition 2.7.8. If the series $\sum_{k=1}^{\infty} |a_k|$, then $\sum_{k=1}^{\infty} a_k$ *converges absolutely.* If $\sum_{k=1}^{\infty} a_k$ converges, but $\sum_{k=1}^{\infty} |a_k|$ diverges, then $\sum_{k=1}^{\infty} a_k$ *converges conditionally.*

Rearrangements

Rearrangements are just different orders, or you are just permuting the terms in the sum into some other order. **Definition 2.7.9.** Let $\sum_{k=1}^{\infty} a_k$ be a series. A series $\sum_{k=1}^{\infty} b_k$ is called a *rearrangement* of $\sum_{k=1}^{\infty} a_k$ if there exists a one-to-one, onto function $f\colon \mathbf{N} \to \mathbf{N}$ such that $b_{f(k)} = a_k$ for all $k \in \mathbf{N}$.

THEOREM 2.7.10. *If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then any rearrangement of this series converges to the same limit.*

Proof. Assume $\sum_{k=1}^{\infty} a_k$ converges absolutely to A, and let $\sum_{k=1}^{\infty} b_k$ be a rearrangement of $\sum_{k=1}^{\infty} a_k$. Let's use

$$s_n = \sum_{k=1}^{n} a_k \qquad t_m = \sum_{k=1}^{m} b_k$$

We want to show $(t_m) \to A$.

Let $\epsilon > 0$. By hypothesis, $(s_m) \to A$ so choose N_1 such that

$$|s_n - A| < \frac{\epsilon}{2}$$

for all $n \geq N_1$. Since the convergence is absolute, we can choose N_2 such that

$$\sum_{m+1}^{n} |a_k| < \frac{\epsilon}{2}$$

for all $n > m \geq N_2$. Now, take $N = max\{N_1, N_2\}$. We know that the terms $a_1, a_2, a_3, \ldots, a_N$ must all appear in the rearrangement so choose an M so they are all apparent within the partial sum.

$$M = max f(k) : 1 \leq k \leq M$$

Now for $m \geq M$, $(t_m - s_N)$ consists of a finite set of terms, the absolute values of which appear in the tail $\sum_{N+1}^{\infty} |a_k|$. Our choice of N_2 earlier then guarantees $|t_m - s_N| < \frac{\epsilon}{2}$, so

$$\begin{aligned} |t_m - A| &= |t_m - s_N + s_N - A| \\ &\leq |t_m - s_N| + |s_N - A| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

□