2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

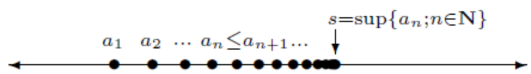
Definition 2.4.1. A sequence (a_n) is *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbf{N}$ and *decreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbf{N}$. A sequence is *monotone* if it is either increasing or decreasing.

THEOREM 2.4.2 (Monotone Convergence Theorem)**.** *If a sequence is monotone and bounded, then it convergences*

Proof. Let (a_n) be monotone and bounded. Let's assume (a_n) is increasing (the decreasing case is handled similarly), and consider the *set* of points $\{a_n : n \in \mathbf{N}\}$. Since the series is bounded, this set it also bounded, so using the Axiom of Completeness, we can let

$$s = \sup\{a_n : n \in \mathbf{N}\}$$

It seems reasonable for $\lim(a_n) = s$



Let $\epsilon > 0$ be arbitrary. Since s is the least upper bound, $s - \epsilon$ is not an upper bound, so there exists a point in the sequence a_N such that $s - \epsilon < a_N$. Now, since (a_n) is increasing, $a_N \leq a_n$ for all $n \geq N$. Hence,

$$s - \epsilon < a_N \leq a_n \leq s \leq s + \epsilon$$

which implies $|a_n - s| < \epsilon$, as desired. □

The Monotone Convergence Theorem is useful for infinite series, since it asserts convergealignnces without any mention of the actual limit. **Definition 2.4.3.** Let (b_n) be a sequence. An *infinite series* is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \ldots$$

We define the corresponding *sequence of partial sums* (s_m) by

$$s_m = \sum_{n=1}^{m} b_n = b_1 + b_2 + b_3 + \cdots + b_m$$

and say the the series $\sum_{n=1}^{\infty} b_n$ *converges to* B if the sequence (s_m) converges to B. In this case, we write $\sum_{n=1}^{\infty} b_n = B$.

THEOREM 2.4.7 (Cauchy Condensation Test)**.** *Suppose* (b_n) *is decreasing and satisfies* $b_n \geq 0$ *for all* $n \in \boldsymbol{N}$*. Then, the series* $\sum_{n=1}^{\infty} b_n$ *converges if and only if the series*

$$\sum_{n=0}^{\infty} 2^n b_{2^n}$$

converges.

Proof. First, assume that $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges. Then the partial sums

$$t_k = b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k}$$

are bounded. There exists an $M > 0$ such that $t_k \leq M$ for all $k \in \mathbf{N}$. Since $b_n \geq -$, we now that the partial sums are increasing, so we only need to show that

$$s_m = b_1 + b_2 + \cdots + b_m$$

is bounded.

Fix m and let k be large enough to ensure $m \leq 2^{k+1} - 1$. Then, $s_m \leq s_{2^{k+1}-1}$ and

$$s_{2^{k+1}-1} = b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + \cdots + (b_{2^k} + \cdots + b_{2^{k+1}-1})$$
$$\leq b_1 + (b_2 + b_2) + (b_4 + b_4 + b_4 + b_4) + \cdots + (b_{2^k} + \cdots + b_{2^k}) = b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k} = t_k$$

Thus, $s_m \leq t_k \leq M$, and the sequence (s_m) is bounded. By the Monotone Convergence Theorem, we can conclude that $\sum_{n=1}^{\infty} b_n$ converges.

Now, if $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges. Fix m and let k be big enough to ensure $m \leq 2^k$. Then,

$$\begin{aligned} s_{2^k} &= b_1 + b_2 + (b_3 + b_4) + (b_5 + b_6 + b_7 + b_8) + \cdots + (b_{2^{k-1}+1} + \dots b_{2^k}) \\ &\geq b_1 + b_2 + (b_4 + b_4) + (b_8 + b_8 + b_8 + b_8) + \cdots + (b_{2^k} + b_{2^k} + \cdots + b_{2^k}) \\ &= b_1 + b_2 + 2b_4 + 4b_8 + \cdots + k(b_k) \\ &= b_1 + (t_k - b_1)/2 = (b_1 + t_k)/2 \end{aligned}$$

So, $s_m > (b_1 + t_k)/2$, which diverges since t_k diverges so s_m diverges. $\square$

Corollary 2.4.7 *The series* $\sum_{n=1}^{\infty} 1/n^p$ *converges if and only if* $p > 1$