2.3 The Algebraic and Order Limit Theorems

Definition 2.3.1. A sequence (x_n) is *bounded* if there exists a number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbf{N}$.

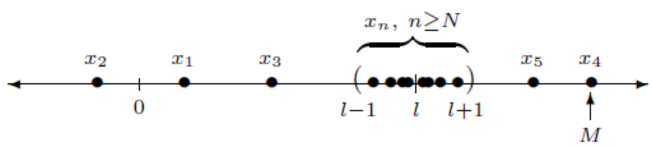
This means $[-M, M]$ contains every term in (x_n)

THEOREM 2.3.2. *Every convergent sequence is bounded.*

Proof. Assume (x_n) converges to a limit l. So for any value of ϵ, there exists an $N \in \mathbf{N}$ such that if $n \geq N$, then x_n is in the interval $(l-\epsilon, l+\epsilon)$, or

$$|x_n| < |l| + \epsilon$$

for all $n \geq N$, for any value of ϵ.



Since there are only a finite number of terms before N, we let

$$M = max\{|x_1|, |x_2|, |x_3|, \dots, |x_{N-1}|, |l| + \epsilon\}$$

Then it follows that $|x_n| \leq M$ for all $n \in \mathbf{N}$ as desired. □

THEOREM 2.3.3 (Algebraic Limit Theorem)**.** *Let* $\lim a_n = a$*, and* $\lim b_n = b$*. Then,*
(i) $\lim(ca_n) = ca$*, for all* $c \in \boldsymbol{R}$*;*
(ii) $\lim(a_n + b_n) = a + b$*;*
(iii) $\lim(a_n b_n) = ab$*;*
(iv) $\lim(a_n/b_n) = a/b$*, provided* $b \neq 0$*;*

Proof. (i) Consider if $c \neq 0$. Let ϵ be some arbitrary positive number. We want to show that after some point in the sequence (ca_n),

$$|ca_n - ca| < \epsilon$$

Now,

$$|ca_n - ca| = |c||a_n - a|$$

Since $(a_n) \to a$, we can make $|a_n - a|$ as small as we want. So we choose an N so

$$|a_n - a| < \frac{\epsilon}{|c|}$$

whenever $n \geq N$. Then,

$$|ca_n - ca| = |c||a_n - a| < |c|\frac{\epsilon}{|c|} = \epsilon$$

The case $c = 0$ reduces to showing the constant sequence $(0, 0, 0, \dots)$ converges to 0. Let $\epsilon > 0$ be arbitrary. Then for any $N \in \mathbf{N}$, $|ca_n - ca| < \epsilon$ for all $n \geq \mathbf{N}$ since $|0 - 0| = 0 < \epsilon$.

(ii) Now, we are proving

$$|(a_n + b_n) - (a + b)|$$

can be made less than an arbitrary ϵ. First, use the triangle inequality to say

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$$

Since $(a_n) \to a$ and $(b_n) \to b$, we know there exists an N_1 and N_2 such that

$$|a_n - a| < \frac{\epsilon}{2} \text{ whenever } n \geq N_1$$

and

$$|b_n - b| < \frac{\epsilon}{2} \text{ whenever } n \geq N_2$$

Now, let $N = max\{N_1, N_2\}$ so that when $n \geq N$, then $n \geq N_1$ and $n \geq N_2$. So,

$$\begin{aligned} |(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| &\leq |a_n - a| + |b_n - b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for all $n \geq N$, as desired.

(iii) To begin,

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |b_n||a_n - a| + |a||b_n - b| \end{aligned}$$

Let $\epsilon > 0$ be arbitrary. For $|a||b_n - b|$, we can choose N_1 so that

$$n \geq N_1 \text{ implies } |b_n - b| < \frac{1}{|a|}\frac{\epsilon}{2}$$

as long as $a \neq 0$. This causes the right side to be less than $\frac{\epsilon}{2}$ Now for $|b_n||a_n - a|$, we know $|b_n| \leq M$ for some M since it is bounded. So,

$$|b_n||a_n - a| \leq M|a_n - a|$$

So we choose an N_2 so that

$$|a_n - a| < \frac{1}{M}\frac{\epsilon}{2} \text{ whenever } n \geq N_2$$

Now, pick $N = max\{N_1, N_2\}$, and observe that if $n \geq N$, then

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |b_n||a_n - a| + |a||b_n - b| \\ &\leq M|a_n - a| + |a||b_n - b| \\ &< M(\frac{\epsilon}{M2}) + |a|(\frac{\epsilon}{|a|2}) = \epsilon \end{aligned}$$

(iv) This is proven by (iii) if we can prove that

$$(b_n) \to b \text{ implies } (\frac{1}{b_n}) \to \frac{1}{b}$$

whenever $b \neq 0$.

$$|\frac{1}{b_n} - \frac{1}{b}| = \frac{|b - b_n|}{|b||b_n|}$$

We can make $|b - b_n|$ as small as we want. To find a worst case estimate of $|b||b_n|$, we must find a lower bound greater than 0. Consider $\epsilon_0 = |b|/2$. Since $(b_n) \to b$, there exists an N_1 such that $|b_n - b| < |b|/2$ for all $n \geq N_1$. This implies $|b_n| > |b|/2 > 0$.

Next, choose N_2 so that $n \geq N$ implies

$$|b_n - b| < \frac{\epsilon|b|^2}{2}$$

Finally, set $N = max\{N_1, N_2\}$, then $n \geq N$ implies

$$|\frac{1}{b_n} - \frac{1}{b}| = |b - b_n|\frac{1}{|b||b_n|} < \frac{\epsilon|b|^2}{2}\frac{1}{|b|\frac{|b|}{2}} = \epsilon$$

□

Limits and Order

THEOREM 2.3.4 (Order Limit Theorem)**.** *Assume* $\lim a_n = a$ *and* $\lim b_n = b$
(i) if $a_n \geq 0$ *for all* $n \in \boldsymbol{N}$*, then* $a \geq 0$*.*
(ii) if $a_n \geq b_n$ *for all* $n \in \boldsymbol{N}$*, then* $a \geq b$*.*
(iii) If there exists $c \in \boldsymbol{R}$ *for which* $c \leq b_n$ *for all* $n \in \boldsymbol{N}$*, then* $c \leq b$*. And same for* a_n *and* a*.*

Proof. (i) We prove this by contradiction. Assume $a < 0$. Then, consider a value of $\epsilon_0 = |a|$. The definition of convergence guarantees that we can find an N such that $|a_n - a| < |a|$ for all $n \geq N$. This means that $|a_N - a| < |a|$, which implies $a_N < 0$, which contradicts that $a_N \geq 0$. We therefore conslude that $a \geq 0$.
(ii) The Algebraic Limit Theorem ensures that the sequence $(b_n - a_n)$ converges to $b - a$. Because $b_n - a_n \geq 0$, we can apply part (i) to get that $b - a \geq 0$.
(iii) Take $a_n = c$ (or $b_n = c$) for all $n \in \mathbf{N}$, and apply (ii). □

In this theorem, we assumed things for all $n \in \mathbf{N}$, but these properties hold true if these assumptions are true for all $n \geq N$, where N is a finite natural number. If a property is of this form it is said to be *eventually* true. Theorem 2.3.4, part (i), could be restated, "Convergent sequences that are eventually nonnegative converge to nonnegative limits."