1.3 The Axiom of Completeness

Axiom of Completeness. *Every nonempty set of real numbers that is bounded above has a least upper bound*

Least Upper Bounds and Greatest Lower Bounds

Definition A set $A \in \mathbf{R}$ is *bounded above* if there exists a number $b \in \mathbf{R}$ such that $a \leq b$ fro all $a \in A$. The number b is an *upper bound* for A.

The set A is *bounded below* if there exists a *lower bound* $l \in \mathbf{R}$ so that $l \leq a$ for all $a \in A$.

Definition A real number s is the *least upper bound* for a set $A \in \mathbf{R}$ if it meets two criteria:

(i) s is an upper bound for A;

(ii) if b is any upper bound for A, then $s \leq b$;

The least upper bound is also called the *supremum* of A. So, $s = \text{lub } A = \sup A$.

The *greatest lower bound* or *infimum* for A is defined similarly and is denoted by $\inf A$.

A set can have many upper bounds, but only one least upper bound. If s_1 and s_2 are both least upper bounds, then by property (ii) we can assert $s_1 \leq s_2$ and $s_2 \leq s_1$, and that $s_1 = s_2$.

A real number a_0 is a *maximum* of set A if a_0 is an element of A and $a_0 \geq a$ for each $a \in A$. Similarly, a number a_1 is a *minimum* of A if $a_1 \in A$ and $a_1 \leq a$ for each $a \in A$.

An upper bounded set is guaranteed to have a least upper bound by *The Axiom of Completeness*, but it is not guaranteed to have a maximum. A supremum can exist and not be a maximum (if the supremum does not exist in the set), but when a maximum exists it is also the supremum. **Lemma** *Assume $s \in \boldsymbol{R}$ is an upper bound for a set $A \in \boldsymbol{R}$. Then, $s = sup\ A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$*

Proof. Given that s is an upper bound, s is the leastupper bound if and only if any number smaller than s is not an upper bound.

$\Rightarrow$ Assume $s = \sup A$ and consider $s - \epsilon$, where $\epsilon > 0$ has been chosen. Since $s - \epsilon < s$, $s - \epsilon$ is not an upper bound for A. So there must be an $a \in A$ such that $s - \epsilon < a$.

$\Leftarrow$ Assume s is an upper bound so that for every $\epsilon > 0$, $s - \epsilon$ is no longer an upper bound for A. $s = \sup A$ since s is an upper bound, and any real number $b < s$ is not an upper bound. This is apparent by setting $\epsilon = s - b$. $\square$