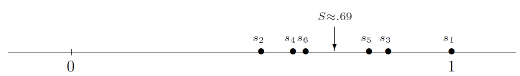
2.1 Discussion: Rearrangements of Infinite Series

Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

If we just add from left to right, we get a series of *partial sums*: $s_1 = 1$, $s_2 = 1/2$, $s_3 = 5/6$, and so on. We also see that the sums oscillate such that $s_1 > s_3 > s_5 > \dots$ and $s_2 < s_4 < s_6 < \dots$.



$$s_2 < s_4 < s_6 < \dots < S < \dots < s_5 < s_3 < s_1$$

It is reasonable to say that this series converges to a number $S = 0.69$ (by experimentation with s_N where N is a large number). It is tempting to think that the sum of all those numbers "add" up to S, but for that we must redefine addition for infinite sums. Treating this series algebraically, lets multiply through by 1/2 and add it back.

$$\begin{array}{rcl}
\frac{1}{2}S = & & \frac{1}{2} \quad - \frac{1}{4} \quad + \frac{1}{6} \quad - \frac{1}{8} \quad + \frac{1}{10} \quad - \frac{1}{12} \quad + \cdots \\
+\ S = & 1 & - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \cdots \\
\hline
\frac{3}{2}S = & 1 & + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} \quad + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} \quad + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} \quad \cdots
\end{array}$$

The resulting series has the same terms as the original series except in a different order. It has two positive terms and then the negative term instead of switching each time. But $\frac{3}{2}S \neq S$. This is also seen by experimentation with large Ns. Addition, in this infinite setting, is not commutative.

Let us look at another series

$$\sum_{n=0}^{\infty} (-1/2)^n$$

Using $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for geometric series, we get

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \dots = \frac{1}{1 - 1/2} = \frac{2}{3}$$

If we rearrange this into two positive and then a negative, you get the same result. Hence addition in an infinite setting is sometimes commutative.

This is applied to the double summation of numbers in a *grid*. For example, $a_{ij} : i, j \in \mathbf{N}$, where $a_{ij} 1/2^{j-i}$ if $j > i$, $a_{ij} = -1$ if $j = i$, and $a_{ij} = 0$ if $j < i$.

$$\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots \\ 0 & -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ 0 & 0 & -1 & \frac{1}{2} & \frac{1}{4} & \cdots \end{bmatrix}$$

We are trying to give

$$\sum_{i,j=1}^{\infty} a_{ij}$$

mathematical meaning. If we sum over all of the j while holding i for each row we get

$$\sum_{i,j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} (\sum_{j=1}^{\infty} a_{ij}) = \sum_{i=1}^{\infty} 0 = 0$$

since the sum of each row is zero. If we hold j constant and iterate over i first we get

$$\sum_{i,j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} (\sum_{i=1}^{\infty} a_{ij}) = \sum_{j=1}^{\infty} (\frac{-1}{2^{j-1}}) = -2$$

The order in which we add causes us to get different results. This double summation occurs when we are multiplying two series:

$$\sum a_i \sum b_j = \sum_{i,j} a_i b_j$$

Now consider the associative property of addition. Consider $\sum_{n=1}^{\infty} (-1)^n$.

$sum_{n=1}^{\infty} (-1)^n = (-1+1) + (-1+1) + (-1+1) + (-1+1) + (-1+1) + \cdots = 0$

$sum_{n=1}^{\infty} (-1)^n = -1 + (1-1) + (1-1) + (1-1) + (1-1) + (1-1) + \cdots = -1$

Different associations cause use to get different results. Manipulations that are legitimate in finite settings do not always extend to infinite settings.