Chapter 1

The Real Numbers

1.1 Discussion: the Irrationality of $\sqrt{2}$

THEOREM 1.1.1. *There is no rational number whose square is 2.*

Proof. A rational number can be written in the form $\frac{p}{q}$ where p and q are integers. We will use an indirect proof. First, assume there is a rational so that its square is 2. It can be written that

$$(\frac{p}{q})^2 = 2$$

We can assume p and q have no common factors since they would cancel anyways and give us a new p and q. Now we can written

$$p^2 = 2q^2$$

which implies that p^2 is an even number, which implies p is an even number. So we can let $p = 2r$. Plugging this in

$$2r^2 = q^2$$

With the same logic as for with p, q is also even. So p and q share a common factor of 2 which contradicts the assumption made in the beginning that they share no common factors. $\square$

Important number systems as sets

Natural Numbers

$$\mathbf{N} = \{1, 2, 3, 4, 5, \dots\}$$

Addition works well he, but there is no additive identity or inverse.

Integers

$$\mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

This includes the additive identity (0) and the additive inverses, which define subtraction. The multiplicative identity is 1, but for multiplicative inverses we need to extend to ...

Rational Numbers

$$\mathbf{Q} = \{\text{all fractions } \frac{p}{q} \text{ where } p \text{ and } q \text{ are integers and } q \neq 0\}$$

The multiplicative inverses define division. All of these properties of **Q** make it into a *field*. A field is any set where addition and multiplication are well-defined operations that are commutative, associative, and obey the distributive property: $a(b + c) = ab + bc$. There must be an additive and multiplicative identity, and each element must have an additive and multiplicative inverse.

The set **Q** has a natural *order*. Given two rational numbers r and s, one of the following is true:

$$r < s,\ r = s, \text{ or } r > s$$

This ordering is transitive: if $r < s$ and $s < t$, then $r < t$. Also, between any two rational numbers, r and s, there is a rational number between them: $\frac{r+s}{2}$, which implies that rational numbers are densely packed.

Q is has holes in the spots of irrationals, such as $\sqrt{2}$ and $\sqrt{3}$. To fill these we add ...

Real Numbers

$$\mathbf{R} = \{\text{all real numbers}\}$$

Just like **Q**, **R** is a field. **R** is added as a superset of **Q**. $\mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R}$.

1.2 Some Preliminaries

Sets

A *set* is a collection of object, usually real numbers. The objects that make up the set are *elements*.

Notation

- $x \in A$ means x is in A
- $A \cup B$ (union of A and B) is defined by: if $x \in A \cup B$ then $x \in A$ or $x \in B$ (or both)
- $A \cap B$ (intersection of A and B) is defined by: if $x \in A \cap B$ then $x \in A$ and $x \in B$
- $\emptyset$ is an *empty set*, or a set without any elements in it
- if $A \cap B = \emptyset$, then A and B are *disjoint*
- $A \supseteq B$ or $B \subseteq A$ every element of B is in A so for each $x \in B$, $x \in A$. So B is a *subset* of A, or A *contains* B
- $A = B$ means each element of $A \subseteq B$ and $B \subseteq B$. So the sets are the same.

- $\bigcup_{n=1}^{\infty} A_n$ or $\bigcup_{n \in \mathbf{N}} A$ means $A_1 \cup A_2 \cup \cdots \cup A_\infty$
- $\bigcap_{n=1}^{\infty} A_n$ or $\bigcap_{n \in \mathbf{N}} A$ means $A_1 \cap A_2 \cap \cdots \cap A_\infty$
- $A^c = \{x \in \mathbf{R} : x \notin A\}$

You can define a set by listing items ($N = \{1, 2, 3, \dots\}$), with words (let E be all even natural numbers), or with a rule or algorithm ($S = \{r \in \mathbf{Q} : r^2 < 2\}$).

De Morgan's Laws
$(A \cap B)^c = A^c \cup B^c$ and $(A \cup B)^c = A^c \cap B^c$

Functions

Given two sets A and B, a *function* from A to B is a rule or mapping that takes each element $x \in A$ to a single element in B. We can write $f\colon A \to B$. Given $x \in A$, $f(x)$ represents an element of B associated with x by f. A is the domain of f. The range is a subset of B.
Triangle Inequality
Absolute Value Function:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The Absolute Value Function satisfies:

$$|ab| = |a||b|$$

$$|a + b| \leq |a| + |b|$$

Logic and Proofs

A type of indirect proof previously used is *proof by contradiction*,which starts by negating what we are proving and then finding a contradiction. Most proofs are direct, which means it starts from a true statement and then gets to the theorems conclusion.

THEOREM 1.2.1. *Two real numbers a and b are equal if and only if for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$*

Proof. Must prove both:
$\Rightarrow$ If $a = b$, then for every real number ϵ it follows that $|a - b| < \epsilon$.
If $a = b$, then $|a - b| = 0$, and $|a - b| < \epsilon$ for any $\epsilon > 0$.

$\Leftarrow$ If for every real number $\epsilon > 0$ if follows that $|a - b| < \epsilon$, then we must have $a = b$.
Assume $a \neq b$,

let $\epsilon_0 = |a - b| > 0$ since $a \neq b$
But $|a - b| = \epsilon_0$ contradicts $|a - b| < \epsilon_0$, which was given. So $a \neq b$ is unacceptable, and a must equal b. □

Induction

The fundamental principle behind induction is that if S is a subset of **N** so that S contains 1 and if S contains n, then S contains $n + 1$, then by induction $S = \mathbf{N}$.

1.3 The Axiom of Completeness

Axiom of Completeness. *Every nonempty set of real numbers that is bounded above has a least upper bound*

Least Upper Bounds and Greatest Lower Bounds

Definition A set $A \in \mathbf{R}$ is *bounded above* if there exists a number $b \in \mathbf{R}$ such that $a \leq b$ fro all $a \in A$. The number b is an *upper bound* for A.

The set A is *bounded below* if there exists a *lower bound* $l \in \mathbf{R}$ so that $l \leq a$ for all $a \in A$.

Definition A real number s is the *least upper bound* for a set $A \in \mathbf{R}$ if it meets two criteria:

(i) s is an upper bound for A;

(ii) if b is any upper bound for A, then $s \leq b$;

The least upper bound is also called the *supremum* of A. So, $s = \text{lub } A = \sup A$.

The *greatest lower bound* or *infimum* for A is defined similarly and is denoted by $\inf A$.

A set can have many upper bounds, but only one least upper bound. If s_1 and s_2 are both least upper bounds, then by property (ii) we can assert $s_1 \leq s_2$ and $s_2 \leq s_1$, and that $s_1 = s_2$.

A real number a_0 is a *maximum* of set A if a_0 is an element of A and $a_0 \geq a$ for each $a \in A$. Similarly, a number a_1 is a *minimum* of A if $a_1 \in A$ and $a_1 \leq a$ for each $a \in A$.

An upper bounded set is guaranteed to have a least upper bound by *The Axiom of Completeness*, but it is not guaranteed to have a maximum. A supremum can exist and not be a maximum (if the supremum does not exist in the set), but when a maximum exists it is also the supremum. **Lemma** *Assume $s \in \boldsymbol{R}$ is an upper bound for a set $A \in \boldsymbol{R}$. Then, $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$*

Proof. Given that s is an upper bound, s is the leastupper bound if and only if any number smaller than s is not an upper bound.

$\Rightarrow$ Assume $s = \sup A$ and consider $s - \epsilon$, where $\epsilon > 0$ has been chosen. Since $s - \epsilon < s$, $s - \epsilon$ is not an upper bound for A. So there must be an $a \in A$ such that $s - \epsilon < a$.

$\Leftarrow$ Assume s is an upper bound so that for every $\epsilon > 0$, $s - \epsilon$ is no longer an upper bound for A. $s = \sup A$ since s is an upper bound, and any real number $b < s$ is not an upper bound. This is apparent by setting $\epsilon = s - b$. $\square$