

# Differential Equations

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# Chapter 1

## Introduction

### 1.1 Some Basic Mathematical Models; Directional Fields

Equations containing derivatives are **differential equations**.

A differential equation that describes some physical process is a **mathematical model** of the process.

**Direction Fields** are valuable tools in studying the solutions of differential equations of the form

$$\frac{dy}{dt} = f(t, y)$$

where  $f$  is a given function of the two variables  $t$  and  $y$ , sometimes called the **rate function**.

To model population growth, an equation in the form

$$\frac{dp}{dt} = rp - k$$

may work, where  $r$  is the growth rate and  $k$  is the predation rate. The equilibrium solution for this equation is  $k/r$ .

### 1.2 Solutions of Some Differential Equations

Consider a differential equation of the form

$$\frac{dy}{dt} = ay - b$$

Solution:

First, we perform some basic operations:

$$\frac{dy/dt}{ay - b} = 1$$

$$\frac{dy/dt}{y - b/a} = a$$

Then by the chain rule we get:

$$\frac{d}{dt} \ln |y - b/a| = a$$

Then we integrate with respect to  $t$

$$\int \frac{d}{dt} \ln |y - b/a| dt = \int a dt$$

$$\ln |y - b/a| = at + C$$

Then through some basic manipulation

$$|y - b/a| = e^{at+C} = e^C e^{at}$$

$$y - b/a = \pm e^C e^{at}$$

Let  $c = \pm e^C$

$$y - b/a = ce^{at}$$

$$y = b/a + ce^{at}$$

So,  $y = b/a + ce^{at}$  is the **general solution** to  $\frac{dy}{dt} = ay - b$ . If you have an initial condition  $y_o$ , that is when  $t = 0$ ,  $y = y_o$ , we can write  $c$  in terms of  $y_o$ . If we let  $c = y_o - b/a$ , then when  $t = 0$ ,  $y = y_o$ .

$$y = b/a + (y_o - b/a)e^{at}$$

### 1.3 Classifications of Differential Equations

If a function depends on a single variable, and then you take the derivative with respect to that variable, you are taking an ordinary derivative.

$$f(x) = 2x^2$$

$$\frac{df}{dx} = 4x$$

If a function depends on multiple variables, and you are taking a derivative with respect to one of them, then it is a partial derivative.

$$f(x, y) = 2xy$$

$$\frac{\partial f}{\partial x} = 2y$$

An equation with only ordinary derivatives is an **ordinary differential equation**, while an equation with both ordinary and partial derivatives is a **partial differential equation**.

If there are more than one unknown functions that you are looking for, use a system of differential equations.

The **order** of a differential equation is the order of the highest derivative that appears in the equation. Using  $y = u(t)$

$$F(t, u(t), u'(t), u''(t), \dots, u^{(n)}(t)) = F(t, y, y', y'', \dots, y^{(n)}) = 0$$

A differential equation is said to be **linear** if it can be written in the form

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \dots + a_n(t)y = \sum_{i=0}^n a_i(t)y^{(n-i)} = g(t)$$

An equation that cannot be written in that form is called a **nonlinear** equation. For example,

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} \sin \theta = 0$$

Since the methods solving linear differential equations are highly developed, it is sometimes useful to use linear approximations of nonlinear functions. This process of approximating a nonlinear equation by a linear one is called **linearization**. For example, when  $\theta$  is small,  $\sin \theta \approx \theta$ . So the previous equation becomes

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} \theta = 0$$

A **solution** of an ordinary differential equation on the interval  $\alpha < t < \beta$  is a function  $\phi$  such that  $\phi', \phi'', \dots, \phi^{(n)}$  exist and satisfy

$$\phi^{(n)}(t) = f[t, \phi(t), \phi'(t), \phi''(t), \dots, \phi^{(n-1)}(t)]$$

for every  $t$  in  $\alpha < t < \beta$ ,  $t \in \{r \in \mathbf{R} : \alpha < r < \beta\}$ , or  $t \in (\alpha, \beta)$ .

## Chapter 2

# First Order Differential Equations