5.2 Errors on Integrals

Since our numerical integrals are only approximations, the largest source of error is *approximation error*. Consider the integral $\int_a^b f(x)dx$. Let $x_k = a+kh$. These are called *sample points*. Consider a slice of the integral (for the trapezoidal rule), one that falls between x_{k-1} and x_k, and let us perform a Taylor expansion of $f(x)$ about x_{k-1}:

$$f(x) = f(x_{k-1}) + (x - x_{k-1})f'(x_{k-1}) + \frac{1}{2}(x - x_{k-1})^2 f''(x_{k-1}) + \dots$$

By integration

$$\begin{aligned}\int_{x_{k-1}}^{x_k} f(x)dx = f(x_{k-1}) \int_{x_{k-1}}^{x_k} dx + f'(x_{k-1}) \int_{x_{k-1}}^{x_k} (x - x_{k-1})dx \\ + \frac{1}{2} f''(x_{k-1}) \int_{x_{k-1}}^{x_k} (x - x_{k-1})^2 dx + \dots\end{aligned}$$

By u substitution with $u = x - x_{k-1}$. All integrals integral from 0 to h, which is the width of the slice.

$$\int_{x_{k-1}}^{x_k} f(x)dx = hf(x_{k-1}) + \frac{1}{2}h^2 f'(x_{k-1}) + \frac{1}{6}h^3 f''(x_{k-1}) + O(h^4)$$

where $O(h^4)$ denotes the rest of the terms, which we neglect.