## 3.5 Variation of Parameters

Consider

$$y'' + p(t)y' + q(t)y = g(t)$$

where $p$, $q$, and $g$ are given continuous functions. To start we know the general solution of the corresponding homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

is

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t)$$

The crucial idea of this method is to replace the constants $c_1$ and $c_2$ with functions $u_1(t)$ and $u_2(t)$; thus we have

$$y = u_1(t)y_1(t) + u_2(t)y_2(t)$$

and

$$y' = u_1'(t)y_1(t) + u_1(t)y_1'(t) + u_2'(t)y_2(t) + u_2(t)y_2'(t)$$

We now set the terms involving $u_1'(t)$ and $u_2'(t)$ to zero:

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$$

so

$$y' = u_1(t)y_1'(t) + u_2(t)y_2'(t)$$

and

$$y'' = u_1'(t)y_1'(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t)$$

Now we substitute $y$, $y'$ and $y''$ and rearrange

$$\begin{aligned} u_1(t)[y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)] \\ +u_2(t)[y_2''(t) + p(t)y_2'(t) + q(t)y_2(t)] \\ +u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t) \end{aligned}$$

The expressions in the square brackets are zero since both $y_1$ and $y_2$ are solutions to the homogeneous equation.

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t)$$

and

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$$

form a system of two linear algebraic equations for the derivatives $u_1'(t)$ and $u_2'(t)$ of the unknown functions. By solving we get

$$u_1'(t) = -\frac{y_2(t)g(t)}{W(y_1, y_2)(t)}, \qquad u_2'(t) = -\frac{y_1(t)g(t)}{W(y_1, y_2)(t)}$$

By integrating we get

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)} ds, \qquad u_2(t) = -\int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} ds$$

**THEOREM 3.5.1.** *If the functions $p$, $q$, and $g$ are continuous on an open interval $I$, and if the functions $y_1$ and $y_2$ are a fundamental set of solutions of the homogeneous equation corresponding to the nonhomogeneous equation*

$$y'' + p(t)y' + q(t)y = g(t)$$

*then a particular solution is*

$$Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}ds + y_2(t)\int_{t_0}^{t} \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}ds$$

*where $t_0$ is any conveniently chosen point in $I$. The general solution is*

$$y = c_1y_1(t) + c_2y_2(t) + Y(t)$$

The difficulties of using this is evaluating the integral and finding the fundamental set of the corresponding homogeneous equation.