

2.2 Solutions of Some Differential Equations

If a differential equation $dy/dx = f(x, y)$ can be written as

$$M(x) + N(y) \frac{dy}{dx} = 0$$

then it is said to be **separable** since it can be written as

$$M(x)dx + N(y)dy = 0$$

and integrated.

Let

$$H_1'(x) = M(x) \quad H_2'(y) = N(y)$$

so then the previous equation becomes

$$H_1'(x) + H_2'(y) \frac{dy}{dx} = 0$$

By the chain rule:

$$H_2'(y) \frac{dy}{dx} = \frac{d}{dy} H_2(y) \frac{dy}{dx} = \frac{d}{dx} H_2(y)$$

which gives us

$$\frac{d}{dx} [H_1(x) + H_2(y)] = 0$$

By integrating, we get

$$H_1(x) + H_2(y) = c$$

Any differentiable function $y = \phi(x)$ that satisfies $H_1(x) + H_2(y) = c$ is a solution of the original differential equation. The differential equation and the initial condition $y(x_0) = y_0$ forms an initial value problem. We can use the initial value to find the correct c :

$$c = H_1(x_0) + H_2(y_0)$$

so

$$\begin{aligned} c &= H_1(x_0) + H_2(y_0) = H_1(x) + H_2(y) \\ (H_1(x) - H_1(x_0)) + (H_2(y) - H_2(y_0)) &= 0 \\ \int_{x_0}^x M(s)ds + \int_{y_0}^y N(s)ds &= 0 \end{aligned}$$

since

$$H_1(x) - H_1(x_0) = \int_{x_0}^x M(s)ds \quad H_2(y) - H_2(y_0) = \int_{y_0}^y N(s)ds$$

Note 1: Sometimes the solution to

$$\frac{dy}{dx} = f(x, y)$$

has a constant solution $y = y_0$, which occurs when $f(x, y_0) = 0$ for all x and for y_0 . For example,

$$\frac{dy}{dx} = \frac{(y - 3) \cos x}{1 + 2y^2}$$

has a solution $y = 3$.

Note 2: Sometimes if a function is non-linear it helps to regard both x and y as functions of a third variable t .

$$\frac{dy}{dt} = \frac{dy/dt}{dx/dt}$$

Note 3: Sometimes it is not easy to solve explicitly for y as a function of x . In these cases, it is better to leave the solution in implicit form.