Chapter 2

First Order Differential Equations

2.1 Linear Equations; Method of Integrating Factors

A **first order linear equation** is an equation that is only linearly dependent on y. Its general form is

$$\frac{dy}{dt} + p(t)y = g(t) \text{ or } P(t)\frac{dy}{dt} + Q(t)y = G(t)$$

Most equations cannot be solved with simple integration, so we can use an **integrating factor** $\mu(t)$. This works for any first order linear equation.

$$\frac{dy}{dt} + ay = g(t)$$

Now, we need to find a a $\mu(t)$ so that

$$\frac{d\mu}{dt} = a\mu$$

which yields $\mu(t) = e^{at}$. Multiplying the original equation by $\mu(t)$

$$e^{at}\frac{dy}{dt} + ae^{at}y = e^{at}g(t)$$

or

$$\frac{d}{dt}(e^{at}y) = e^{at}g(t)$$

By integrating both sides,

$$e^{at}y = \int e^{at}g(t)dt + c$$

$$y = e^{-at} \int_{t_0}^{t} e^{as} g(s) ds + ce^{-at}$$

Now, let's extend this to the general first order linear equation:

$$\frac{dy}{dt} + p(t)y = g(t)$$

Multiply by $\mu(t)$

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t)$$

We know that $\frac{d(\mu(t)y)}{dt} = \mu(t)\frac{dy}{dt} + p(t)\mu(t)y$, if

$$\frac{d\mu(t)}{dt} = p(t)\mu(t)$$

if $p(t) > 0$

$$\frac{d\mu(t)/dt}{\mu(t)} = p(t)$$

$$\ln \mu(t) = \int p(t)dt + k$$

By choosing k to be 0, we obtain the simplest possible $\mu(t)$

$$\mu(t) = \exp \int p(t)dt$$

Now,

$$\frac{d}{dt}[\mu(t)y] = \mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t)$$

By integrating,

$$\mu(t)y = \int \mu(t)g(t)dt + c$$

$$y = \frac{1}{\mu(t)}[\int_{t_0}^{t} \mu(t)g(s)ds + c]$$

2.2 Solutions of Some Differential Equations

If a differential equation $dy/dx = f(x, y)$ can be written as

$$M(x) + N(y)\frac{dy}{dx} = 0$$

then it is said to be **separable** since it can be written as

$$M(x)dx + N(y)dy = 0$$

and integrated.

Let

$$H_1'(x) = M(x) \qquad H_2'(y) = M(y)$$

so then the previous equation becomes

$$H_1'(x) + H_2'(y)\frac{dy}{dx} = 0$$

By the chain rule:

$$H_2'(y)\frac{dy}{dx} = \frac{d}{dy}H_2(y)\frac{dy}{dx} = \frac{d}{dx}H_2(y)$$

which gives us

$$\frac{d}{dx}[H_1(x) + H_2(y)] = 0$$

By integrating, we get

$$H_1(x) + H_2(y) = c$$

Any differentiable function $y = \phi(x)$ that satisfies $H_1(x) + H_2(y) = c$ is a solution of the original differentiable equation. The differential equation and the initial condition $y(x_0) = y_0$ forms an initial value problem. We can use the initial value to find the correct c:

$$c = H_1(x_0) + H_2(y_0)$$

so

$$c = H_1(x_0) + H_2(y_0) = H_1(x) + H_2(y)$$
$$(H_1(x) - H_1(x_0)) + (H_2(y) - H_2(y_0)) = 0$$
$$\int_{x_0}^{x} M(s)ds + \int_{y_0}^{y} N(s)ds = 0$$

since

$$H_1(x) - H_1(x_0) = \int_{x_0}^{x} M(s)ds \qquad H_2(y) - H_2(y_0) = \int_{y_0}^{y} N(s)ds$$

Note 1: Sometimes the solution to

$$\frac{dy}{dx} = f(x, y)$$

has a constant solution $y = y_0$, which occurs when $f(x, y_0) = 0$ for all x and for y_0. For example,

$$\frac{dy}{dx} = \frac{(y-3)\cos x}{1 + 2y^2}$$

has a solution $y = 3$.

Note 2: Sometimes if a function is non-linear it helps to regard both x and y as functions of a third variable t.

$$\frac{dy}{dt} = \frac{dy/dt}{dx/dt}$$

Note 3: Sometimes it is not easy to solve explicitly for y as a function of x. In these cases, it is better to leave the solution in implicit form.

2.3 Modelling with First Order Equations

Construction of the Model. In this step you translate the physical situation into mathematical terms. Mathematical equations are almost always only an approximate description of the actual process. Sometimes you will conceptually replacement of a discrete process by a continuous one.

Analysis of the Model. In this step, you are either solving the differential equation or finding out as many properties about it as possible. Sometimes further approximations help to solve this equation. THese approximations should be examined from a physical point of view so that it still reflects the physical features of the process.

Comparison with Experiment or Observation. Now you interpret your solution or information in the context in which the problem arose. It should appear physically reasonable. If possible, check the solution with a known point.

2.4 Differences Between Linear and Nonlinear Equations

THEOREM 2.4.1. *If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation*

$$y' + p(t)y = g(t)$$

for each $t \in I$, and that also satisfies the initial condition

$$y(t_0) = y_0$$

where y_0 is an arbitrary prescribed initial value.

Proof. In section 2.1, we showed that a general solution to an equation of this form is

$$y = \frac{1}{\mu(t)}[\int_{t_0}^{t} \mu(s)g(s)ds + c]$$

where $\mu(t) = \exp \int_{t_0}^{t} p(s)ds$. To satisfy the initial condition, we choose $c = y_0$. So,

$$y = \frac{1}{\mu(t)}[\int_{t_0}^{t} \mu(s)g(s)ds + y_0]$$

y is continuous since $\frac{1}{\mu(t)}$ is continuous ($\mu(t)$ is never 0), and the integral of something is differential and hence continuous. $\square$

THEOREM 2.4.2. *Let the functions f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some* *interval* $t_0 - h < t < t_0 + h$ *contained in* $\alpha < t < \beta$, *there is a unique solution* $y = \phi(t)$ *of the intial value problem*

$$y' = f(t, y), \qquad y(t_0) = y_0$$

Theorem 2.4.2 is the same as Theorem 2.4.1 when

$$f(t, y) = -p(t)y + g(t) \text{ and } \partial f(t, y)/\partial y = -p(t)$$

so the continuity of f and $\partial f/\partial y$ is the same as the continuity of p and g.

Both these theorems show the existence and uniqueness of a solution to the initial value problem. Any solution to a first order differential equation cannot intersect another since otherwise the initial value problem with initial value at that point would have multiple solutions.

Interval of Definition. By Theorem 2.4.1, discontinuities in the solution of

$$y' + p(t)y = g(t)$$

with the initial condition $y(t_0) = y_0$ can only exist where there is a discontinuity in either p or q.

For a nonlinear initial value problem, the interval is hard to determine since it must contain $[t, \phi(t)]$ and $\phi(t)$ is not known.

General Solution. First order linear equations have a general solution containing one arbitrary constant. This is not really true for nonlinear differential equations.

Implicit Solution. First order linear equations have an explicit formula for the solution for $y = \phi(t)$. Nonlinear equations do not, and the best you can do is find

$$F(t, y) = 0$$

involving t and y that satisfy $y = \phi(t)$. Sometimes you can explicitly solve for the solution, but sometimes you must use numeric approximations with an implicit solution.

Graphical or Numerical Construction of Integral Curves. Sometimes when you cannot find the solution analytically, you can use a computer or a graph.