Chapter 2

First Order Differential Equations

This chapter deals with differential equations of the form

$$\frac{dy}{dt} = f(y, t)$$

2.1 Some Basic Mathematical Models; Directional Fields

Equations containing derivatives are **differential equations**.
A differential equation that describes some physical process is a **mathematical model** of the process.
Direction Fields are valuable tools in studying the solutions of differential equations of the form

$$\frac{dy}{dt} = f(t, y)$$

where f is a given function of the two variables t and y, sometimes called the **rate function**.
To model population growth, an equation in the form

$$\frac{dp}{dt} = rp - k$$

may work, where r is the growth rate and k is the predation rate. The equilibrium solution for this equation is k/r.

2.2 Solutions of Some Differential Equations

Consider a differential equation of the form

$$\frac{dy}{dt} = ay - b$$

Solution:
First, we perform some basic operations:

$$\frac{dy/dt}{ay-b} = 1$$

$$\frac{dy/dt}{y-b/a} = a$$

Then by the chain rule we get:

$$\frac{d}{dt}\ln|y-b/a| = a$$

Then we integrate with respect to t

$$\int \frac{d}{dt}\ln|y-b/a|dt = \int adt$$

$$ln|y-b/a| = at + C$$

Then through some basic manipulation

$$|y-b/a| = e^{at+C} = e^{C}e^{at}$$

$$y-b/a = \pm e^{C}e^{at}$$

Let $c = \pm e^{C}$

$$y-b/a = ce^{at}$$

$$y = b/a + ce^{at}$$

So, $y = b/a + ce^{at}$ is the **general solution** to $\frac{dy}{dt} = ay - b$. If you have an initial condition y_o, that is when $t = 0$, $y = y_o$, we can write c in terms of y_o. If we let $c = y_o - b/a$, then when $t = 0$, $y = y_o$.

$$y = b/a + (y_o - b/a)e^{at}$$

2.3 Classifications of Differential Equations

If a function depends on a single variable, and then you take the derivative with respect to that variable, you are taking an ordinary derivative.

$$f(x) = 2x^2$$

$$\frac{df}{dx} = 4x$$

If a function depends on multiple variables, and you are taking a derivative with respect to one of them, then it is a partial derivative.

$$f(x,y) = 2xy$$

$$\frac{\partial f}{\partial x} = 2y$$

An equation with only ordinary derivatives is an **ordinary differential equation**, while an equation with both ordinary and partial derivatives is a **partial differential equation**.

If there are more than one unknown functions that you are looking for, use a system of differential equations.

The **order** of a differential equation is the order of the highest derivative that appears in the equation. Using $y = u(t)$

$$F(t, u(t), u'(t), u''(t), \dots, u^{(n)}(t)) = F(t, y, y', y'', \dots, y^{(n)}) = 0$$

A differential equation is said to be **linear** if it can be written in the form

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = \sum_{i=0}^{n} a_i(t)y^{(n-i)} = g(t)$$

An equation that cannot be written in that form is called a **nonlinear** equation. For example,

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

Since the methods solving linear differential equations are highly developed, it is sometimes useful to use linear approximations of nonlinear functions. This process of approximating a nonlinear equation by a linear one is called **linearization**. For example, when θ is small, $sin\theta \approx \theta$. So the previous equation becomes

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

A **solution** of an ordinary differential equation on the interval $\alpha < t < \beta$ is a function ϕ such that ϕ', ϕ'', ..., $\phi^{(n)}$ exist and satisy

$$\phi^{(n)}(t) = f[t, \phi(t), \phi'(t), \phi''(t), \dots, \phi^{(n-1)}(t)]$$

for every t in $\alpha < t < \beta$, $t \in \{r \in \mathbf{R} : \alpha < t < \beta\}$, or $t \in (\alpha, \beta)$.