2.4 Differences Between Linear and Nonlinear Equations

THEOREM 2.4.1. *If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation*

$$y' + p(t)y = g(t)$$

for each $t \in I$, and that also satisfies the initial condition

$$y(t_0) = y_0$$

where y_0 is an arbitrary prescribed initial value.

Proof. In section 2.1, we showed that a general solution to an equation of this form is

$$y = \frac{1}{\mu(t)}[\int_{t_0}^{t} \mu(s)g(s)ds + c]$$

where $\mu(t) = \exp \int_{t_0}^{t} p(s)ds$. To satisfy the initial condition, we choose $c = y_0$. So,

$$y = \frac{1}{\mu(t)}[\int_{t_0}^{t} \mu(s)g(s)ds + y_0]$$

y is continuous since $\frac{1}{\mu(t)}$ is continuous ($\mu(t)$ is never 0), and the integral of something is differential and hence continuous. □

THEOREM 2.4.2. *Let the functions f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the intial value problem*

$$y' = f(t, y), \qquad y(t_0) = y_0$$

Theorem 2.4.2 is the same as Theorem 2.4.1 when

$$f(t, y) = -p(t)y + g(t) \text{ and } \partial f(t, y)/\partial y = -p(t)$$

so the continuity of f and $\partial f/\partial y$ is the same as the continuity of p and g.

Both these theorems show the existence and uniqueness of a solution to the initial value problem. Any solution to a first order differential equation cannot intersect another since otherwise the initial value problem with initial value at that point would have multiple solutions.

Interval of Definition. By Theorem 2.4.1, discontinuities in the solution of

$$y' + p(t)y = g(t)$$

with the initial condition $y(t_0) = y_0$ can only exist where there is a discontinuity in either p or q.

For a nonlinear initial value problem, the interval is hard to determine since it must contain $[t, \phi(t)]$ and $\phi(t)$ is not known.

General Solution. First order linear equations have a general solution containing one arbitrary constant. This is not really true for nonlinear differential equations.

Implicit Solution. First order linear equations have an explicit formula for the solution for $y = \phi(t)$. Nonlinear equations do not, and the best you can do is find

$$F(t, y) = 0$$

involving t and y that satisfy $y = \phi(t)$. Sometimes you can explicitly solve for the solution, but sometimes you must use numeric approximations with an implicit solution.

Graphical or Numerical Construction of Integral Curves. Sometimes when you cannot find the solution analytically, you can use a computer or a graph.