Chapter 4

Higher Order Linear Equations

4.1 General Theory of nth Order Linear Equations

An nth order linear differential equation is an equation of the form

$$P_0(t)\frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + P_{n-1}(t)\frac{dy}{dt} + P_n(t)y = G(t)$$

Dividing by $P_0(t)$,

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t)$$

The linear differential operator L of order n is defined above.

For this, we have n initial conditions,

$$y(t_0) = y_0 \; y'(t_0) = y_0' \; \ldots \; y^(n-1)(t_0) = y_0^(n-1)$$

THEOREM 4.1.1. *If the functions $p_1, p_2, \ldots, p_n$, and g are continuous on the open interval I, then there exists exactly one solution $y = \phi(t)$ of the differential equation that also satisfies the initial conditions, where t_0 is any point in I. This solution exists throughout the interval I.*

The Homogeneous Equation.

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0$$

If the functions $y_1, y_2, \ldots, y_n$ are solutions of the previous equation, then it follows by direct computation that the linear combination

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

where $c_1, \dots, c_n$ are arbitrary constants, is also a solution. This family of solutions encompasses all the solutions for all initial conditions. For this the Wronskian

$$W(y_1, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1'' & y_2'' & \dots & y_n'' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

must be non zero at $t = t_0$.

THEOREM 4.1.2. *If the functions $p_1, p_2, \dots, p_n$ are continuous on the open interval I, if the functions $y_1, y_2, \dots, y_n$ are solutions, and if $W \neq 0$ for at least one point in I, then every solution can be expressed as a linear combination of the solutions $y_1, y_2, \dots, y_n$.*

The set is called a **fundamental set of solutions** if the Wronskian is 0. The **general solution** is a linear combination of these with arbitrary constants.
Linear Dependence and Independence. $f_1, f_2, \dots, f_n$ are said to be **linearly dependent** if for a set of constants $k_1, k_2, \dots, k_n$, not all zero,

$$\sum_{i=0}^{n} k_i f_i(t) = 0$$

These functions are **linearly independent** if they are not linearly dependent.

THEOREM 4.1.3. *If $y_1(t), y_2(t), \dots, y_n(t)$ is a fundamental set of solutions, then it is linearly independent. If a set is linearly independent, they form a fundamental set*

Proof. Since its a fundamental set the Wronskian is nonzero, so the only solution of the linear dependence condition is if all the ks are zero, so it is linearly independent. ☐

The Nonhomogeneous Equation.

$$L[y] = g(t)$$

If Y_1 and Y_2 are two solutions of the nonhomogeneous equation, then

$$L[Y_1 - Y_2] = L[Y_1](t) - L[Y_2](t) = g(t) - g(t) = 0$$

hence the difference is a solution to the homogeneous equation.

So the general solution to a nonhomogeneous equation is

$$y = c_1 y_1(t) + c_2 y_2(t) + \dots + c_n y_n(t) + Y(t)$$

4.2 Homogeneous Equations with Constant Coefficients

Consider the nth order linear homogeneous differential equation

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$
$$L[e^{rt}] = e^{rt}(a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n) = e^{rt} Z(r)$$

for all r, where

$$Z(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n$$

For those values of r for which $Z(r) = 0$, it follows that $L[e^{rt}] = 0$ and $y = e^{rt}$ is a solution. The polynomial $Z(r)$ is called the **characteristic polynomial**, and the equation $Z(r) = 0$ is the **characteristic equation** of the differential equation. By factoring, we get

$$Z(r) = a_0(r - r_1)(r - r_2) \ldots (r - r_n)$$

where $r_1, r_2, \ldots, r_n$ are the zeroes, some of which may be equal.
Real and Unreal Roots. If the roots of the characteristic equation are real and no two are equation, then we have n distinct solutions $e^{r_1 t}, e^{r_2 t}, \ldots, e^{r_n t}$. If these functions are linearly independent, then the general solution is

$$y = \sum_{i=0}^{n} c_i e^{r_i t}$$

If the Wronskian determinant is non zero, then they are linearly independent.
Complex Roots. If the characteristic equation has complex roots, they must occur in conjugate pairs, $\lambda \pm i\mu$, since the coefficient $a_0, a_1, \ldots, a_n$ are real numbers. The general solution is still of the same form, but we can replace $e^{(\lambda + i\mu)t}$ and $e^{(\lambda - i\mu)t}$ by

$$e^{\lambda t} \cos \mu t, \quad e^{\lambda t} \sin \mu t$$

Repeated Roots. If the roots of the characteristic equation are not distinct, then the solution is not as clear. If a root of $Z(r) = 0$, say $r = r_1$ has a multiplicity s (occurs s times), then

$$e^{r_1 t}, te^{r_1 t}, t^2 e^{r_1 t}, \ldots, t^{s-1} e^{r_1 t}$$

are corresponding solutions. For a complex root, every time $\lambda + i\mu$ is repeated, $\lambda - i\mu$ must also repeat.

4.3 The Method of Undetermined Coefficients

A particular solution Y of the nonhomogeneous nth order linear equation with constant coefficients

$$L[y] = \sum_{i=0}^{n} a_i y^{(n-i)} = g(t)$$

can be obtained by the method of undetermined coefficients, provided that $g(t)$ is of an appropriate form.

When the differential operator L is applied to a polynomial $A_0t^m + A_1t^{m-1} + \cdots + A_m$, an exponential function $e^{\alpha t}$, a sine function $\sin\beta t$, or a cosine function $\cos\beta t$, the result is a polynomial, an exponential function, or a linear combination of sine and cosine functions, respectively. If $g(t)$ is a sum of these functions, we can find a $Y(t)$ by choosing a suitable combination of these functions. The constants are determined by plugging in the assumed expression into the differential equation.

If $g(t)$ is a sum of several terms, it is sometimes easier to split it up and then compute the solutions separately and then add them together for the particular solution.

This method is easy, but it only works in specific cases (with constant coefficients).

4.4 The Method of Undetermined Coefficients

This for equations of the form

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t)$$

The first step is to solve the corresponding homogeneous function. After this it is a general method that works for any function g. If we know a fundamental set of solutions, y_1, y_2, $\ldots$, y_n, of the homogeneous equation, the general solution for the homogeneous equation is

$$y_c(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t)$$

The method of variation of parameters for determining a particular solution rests on the possibility of determining n functions u_1, u_2, $\ldots$, u_n such that $Y(t)$ is of the form

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + \cdots + u_n(t)y_n(t)$$

Since we have n functions to determine, we specify n conditions. By differentiating,

$$Y' = (u_1y_1' + u_2y_2' + \cdots + u_ny_n') + (u_1'y_1 + u_2'y_2 + \cdots + u_n'y_n)$$

The first condition we impose is that

$$u_1'y_1 + u_2'y_2 + \cdots + u_n'y_n = 0$$

so

$$Y' = u_1y_1' + u_2y_2' + \cdots + u_ny_n'$$

We repeat this for more derivatives Y'', $\ldots$, $Y^{(n-1)}$. After each differentiation we set equal to zero the sum of the terms involving derivatives of $u_1, \ldots, u_n$. In this way we obtain $n-2$ further conditions.

$$u_1'y_1^{(m)} + u_2'y_2^{(m)} + \cdots + u_n'y_n^{(m)} = 0, \quad m = 1, 2, \ldots, n-2$$

and using this we get

$$Y^{(m)} = u_1 y_1^{(m)} + u_2 y_2^{(m)} + \cdots + u_n y_n^{(m)}, \quad m = 2, 3, \ldots, n-1$$

Then we differentiate one more time to get

$$Y^{(n)} = (u_1 y_1^{(n)} + \cdots + u_n y_n^{(n)}) + (u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)})$$

To satisfy the original differential equation, substitute for Y and its derivatives, then group the terms involving each of the functions $y_1, \ldots, y_n$ and their derivatives. It then follows that most of the terms in the equation drop out because each of $y_1, \ldots, y_n$ is a solution and therefore $L[y_i] = 0, i = 1, 2, \ldots, n$. The remaining terms yield the relation

$$u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = g$$

These provide n simultaneous linear nonhomogeneous algebraic equations for $u_1', u_2', \ldots, u_n'$:

$$\begin{aligned}
y_1 u_1' + y_2 u_2' + \cdots + y_n u_n' &= 0, \\
y_1' u_1' + y_2' u_2' + \cdots + y_n' u_n' &= 0, \\
y_1'' u_1' + y_2'' u_2' + \cdots + y_n'' u_n' &= 0, \\
&\vdots, \\
y_1^{(n-1)} u_1' + y_2^{(n-1)} u_2' + \cdots + y_n^{(n-1)} u_n' &= 0
\end{aligned}$$

This system is a linear algebraic system for the unknown quantities $u_1', \ldots, u_n'$. Using Cramer's rule, can write the solution of the system of equation in the form

$$u_m'(t) = \frac{g(t) W_m(t)}{W(t)}, \quad m = 1, 2, \ldots, n$$

Here $W(t) = W(y_1, y_2, \ldots, y_n)(t)$, and W_m is the determinant obtained from W by replacing the mth column by the column $(0, 0, \ldots, 1)$. With this notation a particular solution is given by

$$Y(t) = \sum_{m=1}^{n} y_m(t) \int_{t_0}^{t} \frac{g(s) W_m(s)}{W(s)} ds$$

where t_0 is arbitrary. Sometimes it can be simplified using Abel's identity

$$W(t) = W(y_1, \ldots, y_n)(t) = c \exp\left[-\int p_1(t) dt \right]$$

The constant c can be determined evaluating by W at some convenient point.