Chapter 2

First Order Differential Equations

2.1 Linear Equations; Method of Integrating Factors

A **first order linear equation** is an equation that is only linearly dependent on y. Its general form is

$$\frac{dy}{dt} + p(t)y = g(t) \text{ or } P(t)\frac{dy}{dt} + Q(t)y = G(t)$$

Most equations cannot be solved with simple integration, so we can use an **integrating factor** $\mu(t)$. This works for any first order linear equation.

$$\frac{dy}{dt} + ay = g(t)$$

Now, we need to find a a $\mu(t)$ so that

$$\frac{d\mu}{dt} = a\mu$$

which yields $\mu(t) = e^{at}$. Multiplying the original equation by $\mu(t)$

$$e^{at}\frac{dy}{dt} + ae^{at}y = e^{at}g(t)$$

or

$$\frac{d}{dt}(e^{at}y) = e^{at}g(t)$$

By integrating both sides,

$$e^{at}y = \int e^{at}g(t)dt + c$$

$$y = e^{-at} \int_{t_0}^{t} e^{as} g(s) ds + ce^{-at}$$

Now, let's extend this to the general first order linear equation:

$$\frac{dy}{dt} + p(t)y = g(t)$$

Multiply by $\mu(t)$

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t)$$

We know that $\frac{d(\mu(t)y)}{dt} = \mu(t)\frac{dy}{dt} + p(t)\mu(t)y$, if

$$\frac{d\mu(t)}{dt} = p(t)\mu(t)$$

if $p(t) > 0$

$$\frac{d\mu(t)/dt}{\mu(t)} = p(t)$$

$$\ln \mu(t) = \int p(t)dt + k$$

By choosing k to be 0, we obtain the simplest possible $\mu(t)$

$$\mu(t) = \exp \int p(t)dt$$

Now,

$$\frac{d}{dt}[\mu(t)y] = \mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t)$$

By integrating,

$$\mu(t)y = \int \mu(t)g(t)dt + c$$

$$y = \frac{1}{\mu(t)}[\int_{t_0}^{t} \mu(t)g(s)ds + c]$$

2.2 Solutions of Some Differential Equations

If a differential equation $dy/dx = f(x, y)$ can be written as

$$M(x) + N(y)\frac{dy}{dx} = 0$$

then it is said to be **separable** since it can be written as

$$M(x)dx + N(y)dy = 0$$

and integrated.

Let

$$H_1'(x) = M(x) \qquad H_2'(y) = M(y)$$

so then the previous equation becomes

$$H_1'(x) + H_2'(y)\frac{dy}{dx} = 0$$

By the chain rule:

$$H_2'(y)\frac{dy}{dx} = \frac{d}{dy}H_2(y)\frac{dy}{dx} = \frac{d}{dx}H_2(y)$$

which gives us

$$\frac{d}{dx}[H_1(x) + H_2(y)] = 0$$

By integrating, we get

$$H_1(x) + H_2(y) = c$$

Any differentiable function $y = \phi(x)$ that satisfies $H_1(x) + H_2(y) = c$ is a solution of the original differentiable equation. The differential equation and the initial condition $y(x_0) = y_0$ forms an initial value problem. We can use the initial value to find the correct c:

$$c = H_1(x_0) + H_2(y_0)$$

so

$$c = H_1(x_0) + H_2(y_0) = H_1(x) + H_2(y)$$
$$(H_1(x) - H_1(x_0)) + (H_2(y) - H_2(y_0)) = 0$$
$$\int_{x_0}^{x} M(s)ds + \int_{y_0}^{y} N(s)ds = 0$$

since

$$H_1(x) - H_1(x_0) = \int_{x_0}^{x} M(s)ds \qquad H_2(y) - H_2(y_0) = \int_{y_0}^{y} N(s)ds$$

Note 1: Sometimes the solution to

$$\frac{dy}{dx} = f(x,y)$$

has a constant solution $y = y_0$, which occurs when $f(x, y_0) = 0$ for all x and for y_0. For example,

$$\frac{dy}{dx} = \frac{(y-3)\cos x}{1+2y^2}$$

has a solution $y = 3$.

Note 2: Sometimes if a function is non-linear it helps to regard both x and y as functions of a third variable t.

$$\frac{dy}{dt} = \frac{dy/dt}{dx/dt}$$

Note 3: Sometimes it is not easy to solve explicitly for y as a function of x. In these cases, it is better to leave the solution in implicit form.

2.3 Modelling with First Order Equations

Construction of the Model. In this step you translate the physical situation into mathematical terms. Mathematical equations are almost always only an approximate description of the actual process. Sometimes you will conceptually replacement of a discrete process by a continuous one.

Analysis of the Model. In this step, you are either solving the differential equation or finding out as many properties about it as possible. Sometimes further approximations help to solve this equation. THese approximations should be examined from a physical point of view so that it still reflects the physical features of the process.

Comparison with Experiment or Observation. Now you interpret your solution or information in the context in which the problem arose. It should appear physically reasonable. If possible, check the solution with a known point.

2.4 Differences Between Linear and Non-Linear Equations

THEOREM 2.4.1. *There is no rational number whose square is 2.*