2.6 Exact Equations and Integrating Factors

If we have a differential equation:

$$M(x,y) + N(x,y)y' = 0$$

find a function $\psi(x,y)$ such that

$$\frac{\partial \psi}{\partial x}(x,y) = M(x,y), \qquad \frac{\partial \psi}{\partial y}(x,y) = N(x,y)$$

Then,

$$M(x,y) + N(x,y)y' = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\psi[x,\phi(x)] = 0$$

where $y = \phi(x)$ is the solution. By integration we get

$$\psi(x,y) = c$$

which implicitly defines the solutions for the original differential equation. In this case the original differential equation is an **exact** differential equation.

THEOREM 2.6.1. *Let the functions M, N, M_y, N_x, where the subscripts denote partial derivatives be continuous in the rectangular region $R : \alpha < x < \beta, \gamma < y < \delta$. The differential equation*

$$M(x,y) + N(x,y)y' = 0$$

is an exact differential equation in R if and only if

$$M_y(x,y) = N_x(x,y)$$

for each point in R. That is, there exists a function ψ such that

$$\frac{\partial \psi}{\partial x}(x,y) = M(x,y), \qquad \frac{\partial \psi}{\partial y}(x,y) = N(x,y)$$

if and only if M and N satisfy that constraint.

Proof. Computing M_y and N_x, we get,

$$M_y(x,y) = \psi_{xy}(x,y), \qquad M_y(x,y) = \psi_{yx}(x,y)$$

and $\psi_{xy}(x,y) = \psi_{yx}(x,y)$ so if the equation is exact, then $M_y(x,y) = N_x(x,y)$. Now, we must prove the other way around.

We need to find a ψ so that

$$\psi_x\,(x,y) = M(x,y), \qquad \psi_y\,(x,y) = N(x,y)$$

By integrating the first half the the equation above we get

$$\psi(x,y) = Q(x,y) + h(y)$$

where

$$Q(x,y) = \int_{x_0}^{x} M(s,y)ds$$

and $h(y)$ acts as a constant (with respect to x). Now we choose h to satisfy

$$\psi_y(x,y) = \frac{\partial Q}{\partial y}(x,y) + h'(y) = N(x,y)$$

So we have

$$h'(y) = N(x,y) - \frac{\partial Q}{\partial y}(x,y)$$

For this equation to be true the right side of the equation must be only a function of y, so the partial derivative with respect to x should be 0.

$$\frac{\partial N}{\partial x}(x,y) - \frac{\partial}{\partial x}\frac{\partial Q}{\partial y}(x,y) = 0$$

$$\frac{\partial N}{\partial x}(x,y) - \frac{\partial}{\partial y}\frac{\partial Q}{\partial x}(x,y) = 0$$

And we know $\frac{\partial Q}{\partial x}(x,y) = \frac{\partial \psi}{\partial x}(x,y) = M(x,y)$. So,

$$\frac{\partial N}{\partial x}(x,y) - \frac{\partial M}{\partial y}(x,y) = 0$$

and

$$M_y(x,y) = N_x(x,y)$$

$h(y)$ can be found be integrating $N(x,y) - \frac{\partial Q}{\partial y}(x,y)$. □

Integrating Factors. Sometimes if an equation is not exact, then it is possible to use an integrating factor $\mu(x,y)$ to make it exact. If we have,

$$M(x,y) + N(x,y)y' = 0$$

and we multiply it by $\mu(x,y)$,

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)y' = 0$$

By Theorem 2.6.1, this equation is exact if and only if

$$(\mu M)_y = (\mu N)_x$$

By the product rule, we get another differential equation

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0$$

Solving this gets you $\mu(x,y)$, which will make the original equation exact, so you could solve that too.

If $\mu(x,y) = \mu(x)$ is only a function of x, we can set $\mu_x = \frac{d\mu}{dx}$ and $\mu_y = 0$. So we get

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$$

which is both linear and separable.