3.1 Homogeneous Equations with Constant Coefficients

A second order ordinary differential equation has the form

$$\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt})$$

This equation is said to be linear if it has the form

$$f(t, y, \frac{dy}{dt}) = g(t) - p(t)\frac{dy}{dt} - q(t)y$$

that is, if f is **linear** in y and dy/dt. It can also be written as

$$y'' + p(t)y' + q(t)y = g(t)$$

or

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

If an equation is not of this form, it is **nonlinear**.

An initial value problem consists of a differential equation together with a pair of initial conditions

$$y(t_0) = y_0 \qquad y'(t_0) = y_0'$$

where y_0 and y_0' are given numbers prescribing values for y and y' at the initial point t_0.

A second order linear equation is said to be **homogeneous** if the term $g(t)$ or $G(t)$, depending on which form its in, is 0 for all t. Otherwise, the equation is **nonhomogeneous**. As a result, $g(t)$ or $G(t)$ is sometimes called the nonhomogeneous term. Homogeneous equations can be written as

$$P(t)y'' + Q(t)y' + R(t)y = 0$$

If the functions P, Q, and R are constants, we have

$$ay'' + by' + cy = 0$$

To solve this, we start by seeking exponential solutions of the form $y = e^{rt}$, where r is a parameter to be determined. Then it follows that $y' = re^{rt}$ and $y'' = r^2e^{rt}$. So,

$$(ar^2 + br + c)e^{rt} = 0$$

Since $e^{rt} \neq 0$

$$ar^2 + br + c = 0$$

This is called the **characteristic equation**, and the roots of this correspond to r in the solution $y = e^{rt}$. We assume two real unique roots r_1 and r_2, so $y = e^{r_1 t}$

and $y = e^{r_2 t}$ are solutions. Not only this, but also any linear combination of the two

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

is also a solution. To verify this, we differentiate:

$$y' = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t}$$

$$y'' = c_1 r_1^2 e^{r_1 t} + c_2 r_2^2 e^{r_2 t}$$

Substituting it back in,

$$ay'' + by' + cy = c_1(ar_1^2 + br_1 + c)e^{r_1 t} + c_2(ar_2^2 + br_2 + c)e^{r_2 t}$$

the terms in the parentheses on the right are 0 since r_1 and r_2 are roots of the characteristic equation.

Using the initial conditions

$$y(t_0) = y_0 \qquad y'(t_0) = y_0'$$

$$c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} = y_0$$

$$c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} = y_0'$$

$$c_1 = \frac{y_0' - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0}, \qquad c_2 = \frac{y_0 r_1 - y_0'}{r_1 - r_2} e^{-r_2 t_0}$$

$r_1 - r_2 \neq 0$ since we established that the roots are different.

It can be shown that the solutions of our homogeneous initial value problem are the same solutions as a nonhomogeneous one. So, these solutions are the general solution for the second order linear equations with constants.