

3.2 Solutions of Linear Homogeneous Equations; the Wronskian

Let p and q be continuous functions on an open interval $I = (\alpha, \beta)$ where α and β can be anything including ∞ . Then, for any function ϕ that is twice differentiable on I ,

$$L[\phi] = \phi'' + p\phi' + q\phi$$

So,

$$L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t)$$

The operator L is often written as $L = D^2 + pD + q$, where D is the derivative operator.

So the initial value problem is

$$L[y] = y'' + p(t)y' + q(t)y$$

$$y(t_0) = y_0 \quad y'(t_0) = y'_0$$

THEOREM 3.2.1. *Consider the initial value problem*

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0$$

where p , q , and g are continuous on an open interval I that contains the point t_0 . Then there is exactly one solution $y = \phi(t)$ of this problem, and the solution exists throughout the interval I .

This theorem says three things:

1. A solution *exists*
2. The solution is *unique*
3. The solution ϕ is defined *throughout the interval I* where the coefficients are continuous and is at least twice differentiable there.

For most second order problems, we cannot write a useful expression for the solution. This is a major difference between first order and second order linear equations.

THEOREM 3.2.2 (Principle of Superposition). *If y_1 and y_2 are two solutions fo the differential equation,*

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constants c_1 and c_2

Proof. To prove this, we substitute

$$y = c_1 y_1(t) + c_2 y_2(t)$$

for y .

$$\begin{aligned} L[c_1 y_1 + c_2 y_2] &= [c_1 y_1 + c_2 y_2]'' + p[c_1 y_1 + c_2 y_2]' + q[c_1 y_1 + c_2 y_2] \\ &= c_1 y_1'' + c_2 y_2'' + c_1 p y_1' + c_2 p y_2' + c_1 q y_1 + c_2 q y_2 \\ &= c_1 [y_1'' + p y_1' + q y_1] + c_2 [y_2'' + p y_2' + q y_2] \\ &= c_1 L[y_1] + c_2 L[y_2] = c_1(0) + c_2(0) = 0 \end{aligned}$$

since $L[y_1] = L[y_2] = 0$ because they are both solutions. \square

This theorem essentially states that beginning with two solutions, we can construct an infinite family of solutions. Now, to address if all solutions of the equation are included in this family. First, we find constants to match our initial values.

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= y_0' \end{aligned}$$

which can be written as

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

The determinant of the coefficients of the system

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)$$

If $W \neq 0$,

$$c_1 = \frac{y_0 y_2'(t_0) - y_0' y_2(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}, \quad c_2 = \frac{-y_0 y_1'(t_0) + y_0' y_1(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}$$

or

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}, \quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

These values of c_1 and c_2 satisfy the initial conditions and differential equation. But this is only if $W \neq 0$. If $W = 0$ then there are no solutions unless the numerators are equal to 0. Since otherwise the initial cannot be satisfied no matter what constants are chosen.

The determinant W is called the **Wronskian determinant**, or simply the **Wronskian**, of the solutions y_1 and y_2 . Sometimes we use $W(y_1, y_2)(t_0)$.

THEOREM 3.2.3. Suppose that y_1 and y_2 are two solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

and that the initial conditions

$$y(t_0) = y_0 \quad y'(t_0) = y'_0$$

are assigned. Then it is always possible to choose the constants c_1, c_2 so that

$$y = c_1 y_1(t) + c_2 y_2(t)$$

satisfies the differential equation and the initial conditions if and only if the Wronskian

$$W = y_1 y'_2 - y'_1 y_2$$

is not zero at t_0 .

THEOREM 3.2.4. Suppose that y_1 and y_2 are two solutions of the differential equation,

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

Then the family of solutions

$$y = c_1 y_1(t) + c_2 y_2(t)$$

with arbitrary coefficients c_1 and c_2 includes every solution of the equation if and only if there is a point t_0 where the Wronskian of y_1 and y_2 is not zero.

Proof. Let ϕ be any arbitrary solution of the equation. To prove this, we show that ϕ as a part of the family $c_1 y_1(t) + c_2 y_2(t)$. Let t_0 be a point where the Wronskian is nonzero. So,

$$y_0 = \phi(t_0), \quad y'_0 = \phi'(t_0)$$

Now, we consider the initial value problem

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y'_0$$

Since the Wronskian is nonzero, it is possible to choose a c_1 and c_2 , so that

$$\phi(t) = c_1 y_1(t) + c_2 y_2(t)$$

so ϕ is included in the family. If there is no point t_0 where the Wronskian is nonzero, there are values y_0 and y'_0 that the system has no solutions for c_1 and c_2 . Select a pair of such values and choose the solution $\phi(t)$ that satisfies the initial condition. So, if the $W = 0$, then there exists a solution that does not fit in the family. \square

$$y = c_1 y_1(t) + c_2 y_2(t)$$

with arbitrary constant coefficients is the **general solution**. And the solutions y_1 and y_2 are said to form a **fundamental set of solutions** if and only if their Wronskian is nonzero.