Differential Equations

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Chapter 1

Introduction

1.1 Some Basic Mathematical Models; Directional Fields

Equations containing derivatives are **differential equations**.
A differential equation that describes some physical process is a **mathematical model** of the process.
Direction Fields are valuable tools in studying the solutions of differential equations of the form

$$\frac{dy}{dt} = f(t, y)$$

where f is a given function of the two variables t and y, sometimes called the **rate function**.
To model population growth, an equation in the form

$$\frac{dp}{dt} = rp - k$$

may work, where r is the growth rate and k is the predation rate. The equilibrium solution for this equation is k/r.

1.2 Solutions of Some Differential Equations

Consider a differential equation of the form

$$\frac{dy}{dt} = ay - b$$

Solution:
First, we perform some basic operations:

$$\frac{dy/dt}{ay - b} = 1$$

$$\frac{dy/dt}{y-b/a} = a$$

Then by the chain rule we get:

$$\frac{d}{dt}\ln|y-b/a| = a$$

Then we integrate with respect to t

$$\int \frac{d}{dt}\ln|y-b/a|dt = \int adt$$

$$ln|y-b/a| = at + C$$

Then through some basic manipulation

$$|y-b/a| = e^{at+C} = e^C e^{at}$$

$$y-b/a = \pm e^C e^{at}$$

Let $c = \pm e^C$

$$y-b/a = ce^{at}$$

$$y = b/a + ce^{at}$$

So, $y = b/a + ce^{at}$ is the **general solution** to $\frac{dy}{dt} = ay - b$. If you have an initial condition y_o, that is when $t = 0$, $y = y_o$, we can write c in terms of y_o. If we let $c = y_o - b/a$, then when $t = 0$, $y = y_o$.

$$y = b/a + (y_o - b/a)e^{at}$$

1.3 Classifications of Differential Equations

If a function depends on a single variable, and then you take the derivative with respect to that variable, you are taking an ordinary derivative.

$$f(x) = 2x^2$$

$$\frac{df}{dx} = 4x$$

If a function depends on multiple variables, and you are taking a derivative with respect to one of them, then it is a partial derivative.

$$f(x,y) = 2xy$$

$$\frac{\partial f}{\partial x} = 2y$$

An equation with only ordinary derivatives is an **ordinary differential equation**, while an equation with both ordinary and partial derivatives is a **partial differential equation**.

If there are more than one unknown functions that you are looking for, use a system of differential equations.

The **order** of a differential equation is the order of the highest derivative that appears in the equation. Using $y = u(t)$

$$F(t, u(t), u'(t), u''(t), \dots, u^{(n)}(t)) = F(t, y, y', y'', \dots, y^{(n)}) = 0$$

A differential equation is said to be **linear** if it can be written in the form

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = \sum_{i=0}^{n} a_i(t)y^{(n-i)} = g(t)$$

An equation that cannot be written in that form is called a **nonlinear** equation. For example,

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

Since the methods solving linear differential equations are highly developed, it is sometimes useful to use linear approximations of nonlinear functions. This process of approximating a nonlinear equation by a linear one is called **linearization**. For example, when θ is small, $sin\theta \approx \theta$. So the previous equation becomes

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

A **solution** of an ordinary differential equation on the interval $\alpha < t < \beta$ is a function ϕ such that ϕ', ϕ'', ..., $\phi^{(n)}$ exist and satisy

$$\phi^{(n)}(t) = f[t, \phi(t), \phi'(t), \phi''(t), \dots, \phi^{(n-1)}(t)]$$

for every t in $\alpha < t < \beta$, $t \in \{r \in \mathbf{R} : \alpha < t < \beta\}$, or $t \in (\alpha, \beta)$.

Chapter 2

First Order Differential Equations

2.1 Linear Equations; Method of Integrating Factors

A **first order linear equation** is an equation that is only linearly dependent on y. Its general form is

$$\frac{dy}{dt} + p(t)y = g(t) \text{ or } P(t)\frac{dy}{dt} + Q(t)y = G(t)$$

Most equations cannot be solved with simple integration, so we can use an **integrating factor** $\mu(t)$. This works for any first order linear equation.

$$\frac{dy}{dt} + ay = g(t)$$

Now, we need to find a a $\mu(t)$ so that

$$\frac{d\mu}{dt} = a\mu$$

which yields $\mu(t) = e^{at}$. Multiplying the original equation by $\mu(t)$

$$e^{at}\frac{dy}{dt} + ae^{at}y = e^{at}g(t)$$

or

$$\frac{d}{dt}(e^{at}y) = e^{at}g(t)$$

By integrating both sides,

$$e^{at}y = \int e^{at}g(t)dt + c$$

$$y = e^{-at}\int_{t_0}^{t} e^{as}g(s)ds + ce^{-at}$$

Now, let's extend this to the general first order linear equation:

$$\frac{dy}{dt} + p(t)y = g(t)$$

Multiply by $\mu(t)$

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t)$$

We know that $\frac{d(\mu(t)y)}{dt} = \mu(t)\frac{dy}{dt} + p(t)\mu(t)y$, if

$$\frac{d\mu(t)}{dt} = p(t)\mu(t)$$

if $p(t) > 0$

$$\frac{d\mu(t)/dt}{\mu(t)} = p(t)$$

$$\ln \mu(t) = \int p(t)dt + k$$

By choosing k to be 0, we obtain the simplest possible $\mu(t)$

$$\mu(t) = \exp\int p(t)dt$$

Now,

$$\frac{d}{dt}[\mu(t)y] = \mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t)$$

By integrating,

$$\mu(t)y = \int \mu(t)g(t)dt + c$$

$$y = \frac{1}{\mu(t)}[\int_{t_0}^{t} \mu(t)g(s)ds + c]$$

2.2 Separable Equations

If a differential equation $dy/dx = f(x,y)$ can be written as

$$M(x) + N(y)\frac{dy}{dx} = 0$$

then it is said to be **separable** since it can be written as

$$M(x)dx + N(y)dy = 0$$

and integrated.

Let

$$H_1'(x) = M(x) \qquad H_2'(y) = M(y)$$

so then the previous equation becomes

$$H_1'(x) + H_2'(y)\frac{dy}{dx} = 0$$

By the chain rule:

$$H_2'(y)\frac{dy}{dx} = \frac{d}{dy}H_2(y)\frac{dy}{dx} = \frac{d}{dx}H_2(y)$$

which gives us

$$\frac{d}{dx}[H_1(x) + H_2(y)] = 0$$

By integrating, we get

$$H_1(x) + H_2(y) = c$$

Any differentiable function $y = \phi(x)$ that satisfies $H_1(x) + H_2(y) = c$ is a solution of the original differentiable equation. The differential equation and the initial condition $y(x_0) = y_0$ forms an initial value problem. We can use the initial value to find the correct c:

$$c = H_1(x_0) + H_2(y_0)$$

so

$$c = H_1(x_0) + H_2(y_0) = H_1(x) + H_2(y)$$
$$(H_1(x) - H_1(x_0)) + (H_2(y) - H_2(y_0)) = 0$$
$$\int_{x_0}^{x} M(s)ds + \int_{y_0}^{y} N(s)ds = 0$$

since

$$H_1(x) - H_1(x_0) = \int_{x_0}^{x} M(s)ds \qquad H_2(y) - H_2(y_0) = \int_{y_0}^{y} N(s)ds$$

Note 1: Sometimes the solution to

$$\frac{dy}{dx} = f(x, y)$$

has a constant solution $y = y_0$, which occurs when $f(x, y_0) = 0$ for all x and for y_0. For example,

$$\frac{dy}{dx} = \frac{(y-3)\cos x}{1+2y^2}$$

has a solution $y = 3$.

Note 2: Sometimes if a function is non-linear it helps to regard both x and y as functions of a third variable t.

$$\frac{dy}{dt} = \frac{dy/dt}{dx/dt}$$

Note 3: Sometimes it is not easy to solve explicitly for y as a function of x. In these cases, it is better to leave the solution in implicit form.

2.3 Modelling with First Order Equations

Construction of the Model. In this step you translate the physical situation into mathematical terms. Mathematical equations are almost always only an approximate description of the actual process. Sometimes you will conceptually replacement of a discrete process by a continuous one.

Analysis of the Model. In this step, you are either solving the differential equation or finding out as many properties about it as possible. Sometimes further approximations help to solve this equation. THese approximations should be examined from a physical point of view so that it still reflects the physical features of the process.

Comparison with Experiment or Observation. Now you interpret your solution or information in the context in which the problem arose. It should appear physically reasonable. If possible, check the solution with a known point.

2.4 Differences Between Linear and Nonlinear Equations

THEOREM 2.4.1. *If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation*

$$y' + p(t)y = g(t)$$

for each $t \in I$, and that also satisfies the initial condition

$$y(t_0) = y_0$$

where y_0 is an arbitrary prescribed initial value.

Proof. In section 2.1, we showed that a general solution to an equation of this form is

$$y = \frac{1}{\mu(t)}[\int_{t_0}^{t} \mu(s)g(s)ds + c]$$

where $\mu(t) = \exp \int_{t_0}^{t} p(s)ds$. To satisfy the initial condition, we choose $c = y_0$. So,

$$y = \frac{1}{\mu(t)}[\int_{t_0}^{t} \mu(s)g(s)ds + y_0]$$

y is continuous since $\frac{1}{\mu(t)}$ is continuous ($\mu(t)$ is never 0), and the integral of something is differential and hence continuous. □

THEOREM 2.4.2. *Let the functions f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some* *interval* $t_0 - h < t < t_0 + h$ *contained in* $\alpha < t < \beta$*, there is a unique solution* $y = \phi(t)$ *of the intial value problem*

$$y' = f(t, y), \qquad y(t_0) = y_0$$

Theorem 2.4.2 is the same as Theorem 2.4.1 when

$$f(t, y) = -p(t)y + g(t) \text{ and } \partial f(t, y)/\partial y = -p(t)$$

so the continuity of f and $\partial f/\partial y$ is the same as the continuity of p and g.

Both these theorems show the existence and uniqueness of a solution to the initial value problem. Any solution to a first order differential equation cannot intersect another since otherwise the initial value problem with initial value at that point would have multiple solutions.

Interval of Definition. By Theorem 2.4.1, discontinuities in the solution of

$$y' + p(t)y = g(t)$$

with the initial condition $y(t_0) = y_0$ can only exist where there is a discontinuity in either p or q.

For a nonlinear initial value problem, the interval is hard to determine since it must contain $[t, \phi(t)]$ and $\phi(t)$ is not known.

General Solution. First order linear equations have a general solution containing one arbitrary constant. This is not really true for nonlinear differential equations.

Implicit Solution. First order linear equations have an explicit formula for the solution for $y = \phi(t)$. Nonlinear equations do not, and the best you can do is find

$$F(t, y) = 0$$

involving t and y that satisfy $y = \phi(t)$. Sometimes you can explicitly solve for the solution, but sometimes you must use numeric approximations with an implicit solution.

Graphical or Numerical Construction of Integral Curves. Sometimes when you cannot find the solution analytically, you can use a computer or a graph.

2.5 Autonomous Equations and Population Dynamics

Autonomous equations are those of the form

$$dy/dt = f(y)$$

This form of equation is separable.

Exponential Growth. Exponential growth has an equation of the form

$$dy/dt = ry$$

where the constant of proportionality r is called the **rate of growth or decline**. Solving this with the initial condition

$$y(0) = y_0$$

we obtain

$$y = y_0 e^{rt}$$

For many populations this equation holds true to a certain extent but this is not sustainable since the population would grow rapidly.

Logistic Growth. Since the growth rate actually depending on the population, we replace the constant r with $h(y)$. So,

$$dy/dt = h(y)y$$

The Verhulst equation or the **logistic equation** is of the form

$$dy/dt = (r - ay)y$$

which is the same as the last equation with $h(y) = (r - ay)$ so that $h(y) \approx r$ when y is small and it decreases as y increases. The logistic equation if often written as

$$\frac{dy}{dt} = r(1 - \frac{y}{K})y$$

where $K = r/a$. r is called the **intrinsic growth rate**, the growth rate in the absence of limiting factors.

The constant solutions, or **equilibrium solutions**, occur when

$$\frac{dy}{dt} = r(1 - \frac{y}{K})y = 0$$

or at $y = 0$ and $y = K$. These are also called **critical points**.

Other solutions for this equation always asymptotically approach K as shown.

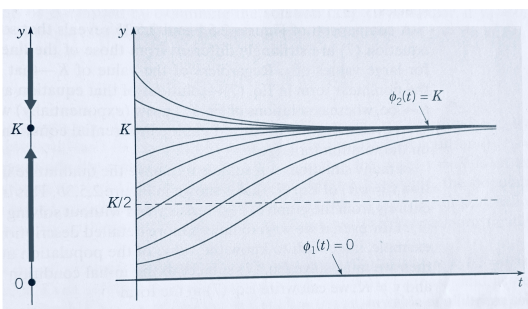


FIGURE 2.5.3 Logistic growth: $dy/dt = r(1 - y/K)y$.
(*a*) The phase line. (*b*) Plots of y versus t.

K is often referred to as the **saturation level**, or the **environmental carrying capacity**, for a given species.

Sometimes this qualitative knowledge of the solution is enough, but to solve it, we can rewrite the equation as

$$\frac{dy}{(1-y/K)y} = r\ dt$$

where $y \neq 0$ and $y \neq K$. Using a partial fraction expansion, we have

$$(\frac{1}{y} + \frac{1/K}{1-y/K})dy = r\ dt$$

By integration,

$$\ln|y| - \ln|1 - \frac{y}{K}| = rt + c$$

since if $0 < y_0 < K$ then y will remain in this interval, we can remove the absolute values. Then by taking the exponential of both sides,

$$\frac{y}{1-(y/K)} = Ce^{rt}$$

where $C = e^c$. $C = y_0/[1-(y_0/K)]$ to satisfy $y(0) = y_0$.

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

We notice that

$$\lim_{t\to\infty} y(t) = K$$

hence $y = K$ is an **asymptotically stable solution**, and $y = 0$ is an **unstable equilibrium solution**.
A Critical Threshold. Consider the equation

$$\frac{dy}{dt} = -r(1 - \frac{y}{T})y$$

The function $dy/dt = f(y)$ is a parabola with zeros at $y = 0$ and $y = K$ opening up. For $0 < y < T$, $f(y) < 0$. Hence T is the **threshold level** since no growth occurs below it. Above T, y grows indefinitely. It becomes unbounded in a finite amount of time $t*$.

$$t* = \frac{1}{r}\ln\frac{y_0}{y_0 - T}$$

which we find by setting the denominator of the solution to 0.
Logistic Growth with a Threshold. This equation can be modified so that unbounded growth does not occur when y is above T. We need another factor that makes dy/dt negative when y is large.

$$\frac{dy}{dt} = -r(1 - \frac{y}{T})(1 - \frac{y}{K})y$$

This pulls dy/dt back down after a certain point. $f(y)$ is a cubic function now. The solutions of the equation are similar to the equations with unbounded growth except when $y > T$, y will approach K.

2.6 Exact Equations and Integrating Factors

If we have a differential equation:

$$M(x,y) + N(x,y)y' = 0$$

find a function $\psi(x,y)$ such that

$$\frac{\partial \psi}{\partial x}(x,y) = M(x,y), \qquad \frac{\partial \psi}{\partial y}(x,y) = N(x,y)$$

Then,

$$M(x,y) + N(x,y)y' = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\psi[x,\phi(x)] = 0$$

where $y = \phi(x)$ is the solution. By integration we get

$$\psi(x,y) = c$$

which implicitly defines the solutions for the original differential equation. In this case the original differential equation is an **exact** differential equation.

THEOREM 2.6.1. *Let the functions M, N, M_y, N_x, where the subscripts denote partial derivatives be continuous in the rectangular region $R : \alpha < x < \beta, \gamma < y < \delta$. The differential equation*

$$M(x,y) + N(x,y)y' = 0$$

is an exact differential equation in R if and only if

$$M_y(x,y) = N_x(x,y)$$

for each point in R. That is, there exists a function ψ such that

$$\frac{\partial \psi}{\partial x}(x,y) = M(x,y), \qquad \frac{\partial \psi}{\partial y}(x,y) = N(x,y)$$

if and only if M and N satisfy that constraint.

Proof. Computing M_y and N_x, we get,

$$M_y(x,y) = \psi_{xy}(x,y), \qquad M_y(x,y) = \psi_{yx}(x,y)$$

and $\psi_{xy}(x,y) = \psi_{yx}(x,y)$ so if the equation is exact, then $M_y(x,y) = N_x(x,y)$. Now, we must prove the other way around.

We need to find a ψ so that

$$\psi_x\,(x,y) = M(x,y), \qquad \psi_y\,(x,y) = N(x,y)$$

By integrating the first half the the equation above we get

$$\psi(x,y) = Q(x,y) + h(y)$$

where

$$Q(x,y) = \int_{x_0}^{x} M(s,y)ds$$

and $h(y)$ acts as a constant (with respect to x). Now we choose h to satisfy

$$\psi_y(x,y) = \frac{\partial Q}{\partial y}(x,y) + h'(y) = N(x,y)$$

So we have

$$h'(y) = N(x,y) - \frac{\partial Q}{\partial y}(x,y)$$

For this equation to be true the right side of the equation must be only a function of y, so the partial derivative with respect to x should be 0.

$$\frac{\partial N}{\partial x}(x,y) - \frac{\partial}{\partial x}\frac{\partial Q}{\partial y}(x,y) = 0$$

$$\frac{\partial N}{\partial x}(x,y) - \frac{\partial}{\partial y}\frac{\partial Q}{\partial x}(x,y) = 0$$

And we know $\frac{\partial Q}{\partial x}(x,y) = \frac{\partial \psi}{\partial x}(x,y) = M(x,y)$. So,

$$\frac{\partial N}{\partial x}(x,y) - \frac{\partial M}{\partial y}(x,y) = 0$$

and

$$M_y(x,y) = N_x(x,y)$$

$h(y)$ can be found be integrating $N(x,y) - \frac{\partial Q}{\partial y}(x,y)$. □

Integrating Factors. Sometimes if an equation is not exact, then it is possible to use an integrating factor $\mu(x,y)$ to make it exact. If we have,

$$M(x,y) + N(x,y)y' = 0$$

and we multiply it by $\mu(x,y)$,

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)y' = 0$$

By Theorem 2.6.1, this equation is exact if and only if

$$(\mu M)_y = (\mu N)_x$$

By the product rule, we get another differential equation

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0$$

Solving this gets you $\mu(x,y)$, which will make the original equation exact, so you could solve that too.

If $\mu(x,y) = \mu(x)$ is only a function of x, we can set $\mu_x = \frac{d\mu}{dx}$ and $\mu_y = 0$. So we get

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$$

which is both linear and separable.

2.7 Numerical Approximations: Euler's Method

Sometimes analytical solutions are not possible so we can use numerical approximations to get close to the actual solution. One of these approximations is called **Euler's Method**. The basis of Euler's Method is using the differential equation to create a tangent line at the initial value (t_0, y_0), and use it to create an approximation for another point (t, y).

$$y = y_0 + f(t_0, y_0)(t - t_0)$$

where $\frac{dy}{dt} = f(t_0, y_0)$. For a better approximation we can repeat this n times instead of just doing it once. So, if we needed to find the solution at t, we could use Euler's Method with n steps by

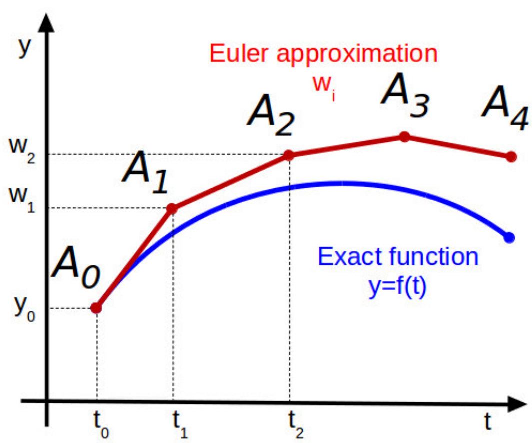
$$y_{i+1} = y_i + f(t_i, y_i)(t_{i+1} - t_i)$$

now if we use equal intervals

$$y_{i+1} = y_i + f(t_i, y_i)h$$

where h is $(t - t_0)/n$. Or,

$$y = y_0 + \sum_{i=1}^{n} f(t_i, y_i)h$$



As you increase n, the approximation gets better. The Euler method uses the solution that passes through step to approximate the solution that we are looking for. For converging solutions, this works well since all the solutions converge to similar values, while this causes large errors for diverging situations since each step takes you further from the solution.

2.8 The Existence and Uniqueness Theorem

Consider the initial value problem

$$y' = f(t, y), \qquad y(0) = 0$$

If an initial value problem is not of this form, we can apply a translation of the coordinate axes that will take (t_0, y_0) to the origin.

THEOREM 2.8.1. *If f and $\frac{\partial f}{\partial y}$ are continuous in a rectangle $R : |t| \leq a, |y| \leq b$, then there is some interval $|t| \leq h \leq a$ in which there exists a unique solution $y = \phi(t)$ of the initial value problem.*

If we integrate the initial value problem equation, we get

$$\phi(t) = \int_0^t f[s, \phi(s)]ds$$

This is called the **integral equation**, which is equivalent to the initial value equation.

One method of showing that the integral equation has a unique solution is the **method of successive approximations** or Picard's **iteration method**. In using this method, we start by choosing an initial function ϕ_0, either arbitrarily or to approximate the solution. The simplest choice is

$$\phi_0(t) = 0$$

ϕ_0 satisfies the initial condition, but probably not the differential equation. The next apprximation ϕ_1 is obtained using ϕ_0.

$$\phi_1(t) = \int_0^t f[s, \phi_0(s)]ds$$

and, in general,

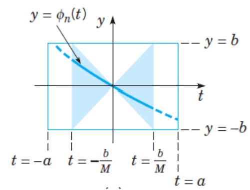
$$\phi_{n+1}(t) = \int_0^t f[s, \phi_n(s)]ds$$

This creates a sequence of functions $\phi_n = \phi_0, \phi_1, \phi_2, \ldots, \phi_n, \ldots$. Each element of the sequence satisfies the initial condition, but in general none satisfies the differential equation. If at some stage $\phi_{k+1}(t) = \phi_k(t)$, then it follows that ϕ_k is a solution and does follow the differential equation.

1. **Do all members of the sequence ϕ_n exist?**
   If f and $\frac{\partial f}{\partial y}$ are continuous in the whole ty-plane, then each ϕ_n is known to exist and can be calculated. But if f and $\frac{\partial f}{\partial y}$ are only assumed continuous in a rectangle $R : |t| \leq a, |y| \leq b$, then some members of the sequence cannot be explicitly determined. If we restrict t to a smaller intercal $|t| \leq a$, we can avoid this danger. Since f must be bounded on R

   $$|f(t, y)| \leq M \qquad (t, y) \text{ in } R$$

   Since $f[t, \phi_k(t)] = \phi'_{k+1}(t)$, the maximum slope of $y = \phi_{k+1}(t)$ is M. So ϕ_{k+1} must lie in R as long as R contains this region:



which is for $|t| \leq b/M$.

2. **Does the sequence $\phi_n(t)$ converge?**
We can identify $\phi_n(t) = \phi_1(t) + [\phi_2(t) - \phi_1(t)] + [\phi_3(t) - \phi_2(t)] + \cdots + [\phi_n(t) - \phi_{n-1}(t)]$ as the nth partial sum of the series

$$\phi_1(t) + \sum_{k+1}^{\infty} [\phi_{k+1}(t) - \phi_k(t)]$$

The convergence of the sequence $\phi_n(t)$ is established by showing that this series converges.
TODO: PROVE THIS USING PROBLEMS 15-18
We denote the limit function by ϕ, so that

$$\phi(t) = \lim_{n \to \infty} \phi_n(t)$$

3. **What are the properties of the limit function ϕ?**
We know that ϕ is continuous since the sequence $\{phi_n$converges in a certain manner, known as uniform convergence. We proof we used proves this as well. Now let us return to

$$\phi_{n+1}(t) = \int_0^t f[s, \phi_n(s)]ds$$

Allowing n to approach ∞ on both sides, we obtain

$$\phi(t) = \lim_{n \to \infty} \int_0^t f[s, \phi_n(s)]ds$$

We can move the limit inside the integral since the sequence converges uniformly.

$$\phi(t) = \int_0^t \lim_{n \to \infty} f[s, \phi_n(s)]ds$$

Then we take it inside the function

$$\phi(t) = \int_0^t f[s, \lim_{n \to \infty} \phi_n(s)]ds$$

so

$$\phi(t) = \int_0^t f[s, \phi(s)]ds$$

Moving the limit inside the function is saying that f is continuous in its second variable, which is known. So this last equation shows that ϕ satisfies the integral equation so it is a solution for the initial value problem.

4. **Are there other solutions of the integral equation besides** $y = \phi(t)$**?**
Assume another solution $y = \psi(t)$. It can be shown that

$$|\phi(t) - \psi(t)| \leq A \int_0^t |\phi(s) - \psi(s)| ds$$

TODO: Prove this using PROBLEM 19
for $0 \leq t \leq h$ and a suitable positive number A. It is now convenient to introduce U as

$$U(t) = \int_0^t |\phi(s) - \psi(s)| ds$$

$$U(0) = 0$$
$$U(t) \geq 0, \quad \text{for } t \geq 0$$

U is differentiable, and $U'(t) = |\phi(t) - \psi(t)|$.

$$U'(t) - AU(t) \leq 0 \text{ for } 0 \leq t \leq A/2$$

Multiplying by e^{-At} gives

$$[e^{-At}U(t)]' \leq 0 \text{ for } 0 \leq t \leq A/2$$

Then, by integration from 0 to t

$$e^{-At}U(t) \leq 0 \text{ for } 0 \leq t \leq A/2$$

Hence $U(t) \leq 0$ for $0 \leq t \leq A/2$, but since A is arbitrary $U(t) \leq 0$ for all positive t. Since $U(t) \geq 0$, $U(t) = 0$, so $U'(t) = 0$ and $\psi(t) = \phi(t)$. This proves there connot be two different solutions to one initial value problem for $t \geq 0$. A slight modification of this argument shows the same is true for $t \leq 0$.

2.9 First Order Difference Equations

For some problems, we need to model a discrete process. Differential equations model a continuous process, while **difference equation**s model a discrete process. An example is modelling population growth of a species whose generations do not overlap. The population y_{n+1} of the species in the year $n + 1$ is some function of n and the population y_N in the preceding year:

$$y_{n+1} = f(n, y_n) \qquad n \in \mathbf{N}$$

This is a **first order difference equation** since it only depends on the step before it. The equation is **linear** if f is a linear function of y_n; otherwise, it is **nonlinear**. A **solution** of the difference equation is a sequence of numbers $y_0, y_1, y_2, \ldots$ that satisfy the equation for each n. There also must be an **initial condition**.

$$y_0 = \alpha$$

Now for convenience, temporarily let $f(y_n) = f(n, y_n)$.

$$y_1 = f(y_0)$$

$$y_2 = f(y_1) = f[f(y_0)]$$

$$y_1 = f(y_2) = f[f[f(y_0)]] = f^3(y_0)$$

$$y_n = f(y_{n-1}) = f^n(y_0)$$

Solutions for which y_n has the same value for all n are called **equilibrium solutions**. You can find this by setting $y_{n+1} = y_n$, and solve

$$y_n = f(y_n)$$

for y_n.

Linear Equations. These could be of the form

$$y_{n+1} = \rho_n y_n \qquad n \in \mathbf{N}$$

THe reproduction rate rho_n may differ from year to year. This can easily be solved by iteration. We obtain

$$y_1 = \rho_0 y_0$$

$$y_2 = \rho_1 y_1 = \rho_0 \rho_1 y_0$$

and, in general,

$$y_n = (\Pi_{i=0}^{n-1} \rho_i) y_0$$

If the reproduction rate is the same for all n, $\rho_n = \rho$, then

$$y_{n+1} = \rho y_n$$

$$y_n = \rho^n y_0$$

The limiting behavior can easily be determined.

$$\lim_{n\to\infty} y_n = \begin{cases} 0, & \text{if } |\rho| < 1; \\ y_0, & \text{if } |\rho| = 1; \\ \text{does not exist}, & \text{if } |\rho| > 1; \end{cases}$$

The equilibrium solution $y_n = 0$ is asymptotically stable for $|\rho| < 1$ and unstable for $|\rho| > 1$.

If a population has immigration or emigration, we must have a b_n as the net increase in population in year n due to immigration.

$$y_{n+1} = \rho y_n + b_n$$

We can solve for y_n through iteration as well.

$$y_1 = \rho y_0 + b_0$$

$$y_2 = \rho(\rho y_0 + b_0) + b_1 = \rho^2 y_0 + \rho b_0 + b_1$$
$$y_3 = \rho(\rho^2 y_0 + \rho b_0 + b_1) + b_2 = \rho^3 y_0 + \rho^2 b_0 + \rho b_1 + b_2$$

and in general,

$$y_n = \rho^n y_0 + \sum_{j=0}^{n-1} \rho^{n-1-j} b_j$$

The more general solution for linear equations of the form

$$y_{n+1} = \rho_n y_n + b_n$$

then by iterations,

$$y_n = (\Pi_{i=0}^{n-1} \rho_i) y_0 + \sum_{i=0}^{n-1} (\Pi_{j=i+1}^{n-1} \rho_j) b_{n-1}$$

The first term on the right side represents thr descendants of the original population.

If $b_n = b \neq 0$,

$$y_{n+1} = \rho y_n + b$$

and from the previous equation the solution is

$$y_n = \rho^n y_0 + (1 + \rho + \rho^2 + \cdots + \rho^{n-1}) b$$

or for $\rho \neq 1$,

$$y_n = \rho_n y_0 + \frac{1 - \rho^n}{1 - \rho}$$

which is the same as

$$y_n = \rho^n (y_0 - \frac{b}{1 - \rho}) + \frac{b}{1 - \rho}$$

which makes the limit behavior of y_n more clear. For $\rho = 1$,

$$y_n = y_0 + nb$$

The limit behavior is defined by

$$\lim_{n \to \infty} y_n = \begin{cases} b/(1-\rho), & \text{if } |\rho| < 1; \\ \textit{unbounded}, & \text{if } |\rho| = 1; \\ \text{does not exist}, & \text{if } |\rho| > 1; \end{cases}$$

when $|\rho|$ is not less than 1, it converges to y_0, if $y_0 = b/(1-\rho)$ since this is an equilibrium solution.

The model can also be used for interest, where $\rho_n = 1 + r_n$, where r_n is the interest rate, and b_n is the amount deposited or withdrawn.

Nonlinear Equations. Consider the logistic difference equation

$$y_{n+1} = \rho y_n (1 - \frac{y_n}{k})$$

If we scale y_n to $u_n = y_n/k$

$$u_{n+1} = \rho u_n (1 - u_n)$$

$(\rho = k\rho)$, a positive parameter. We can find an equilibrium solution by setting $u_{n+1} = u_n$.

$$u_n = \rho u_n - \rho u_n^2$$

which gives us

$$u_n = 0, \quad u_n = \frac{\rho - 1}{\rho}$$

To check if these equilibrium solutions are asymptotically stable or unstable, we can use linear approximations. Near $u_n = 0$, u_n^2 is small compared to u_n. So,

$$u_{n+1} \approx \rho u_n$$

We already know that this only approaches 0 for $|\rho| < 1$, or $0 < \rho < 1$ since ρ is positive. So $u_n = 0$ is stable for $0 < \rho < 1$.

To test solutions in the neighborhood of $u_n = (\rho - 1)/\rho$, we write

$$u_n = \frac{\rho - 1}{\rho} + v_n$$

where v_n is small. By substituting this into $u_{n+1} = \rho u_n(1 - u_n)$, we get

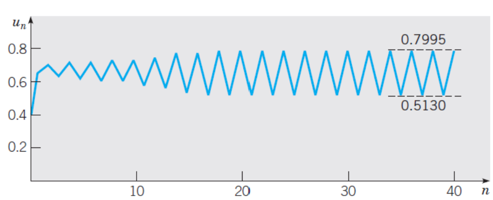
$$v_{n+1} = (2 - \rho)v_n - \rho v_n^2$$

Since v_n is small, $v_n^2 \approx 0$.

$$v_{n+1} = (2 - \rho)v_n$$

In the same manner we know $v_n \to 0$ as $n \to \infty$ for $|2 - \rho| < 1$, or $1 < \rho < 3$. Therefore, in this range of values for ρ, $u_n = (\rho - 1)/\rho$ is an asymptotically stable equilibrium solution. $\rho = 1$ is an **exchange of stability** from one equilibrium solution to the other.

If $\rho > 3$, the solution will oscillate between two values; it is period 2. At about $\rho = 3.449$, the solution becomes periodic with period 4. The appearance of a new solution at a certain parameter value is called a **bifurcation**.



The ρ-values at which the successive period doublings occur approach a limit that is approximately 3.57, so for $\rho > 3.57$, the solutions have some regularity. It's fine structure is unpredictable, hence the term **chaotic**. One of the features of chaotic solutions is extreme sensitivity to initial conditions.

Chapter 3

Second Order Linear Equations

3.1 Homogeneous Equations with Constant Coefficients

A second order ordinary differential equation has the form

$$\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt})$$

This equation is said to be linear if it has the form

$$f(t, y, \frac{dy}{dt}) = g(t) - p(t)\frac{dy}{dt} - q(t)y$$

that is, if f is **linear** in y and dy/dt. It can also be written as

$$y'' + p(t)y' + q(t)y = g(t)$$

or

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

If an equation is not of this form, it is **nonlinear**.

An initial value problem consists of a differential equation together with a pair of initial conditions

$$y(t_0) = y_0 \qquad y'(t_0) = y'_0$$

where y_0 and y'_0 are given numbers prescribing values for y and y' at the initial point t_0.

A second order linear equation is said to be **homogeneous** if the term $g(t)$ or $G(t)$, depending on which form its in, is 0 for all t. Otherwise, the equation is **nonhomogeneous**. As a result, $g(t)$ or $G(t)$ is sometimes called the nonhomogeneous term. Homogeneous equations can be written as

$$P(t)y'' + Q(t)y' + R(t)y = 0$$

If the functions P, Q, and R are constants, we have

$$ay'' + by' + cy = 0$$

To solve this, we start by seeking exponential solutions of the form $y = e^{rt}$, where r is a parameter to be determined. Then it follows that $y' = re^{rt}$ and $y'' = r^2e^{rt}$. So,

$$(ar^2 + br + c)e^{rt} = 0$$

Since $e^{rt} \neq 0$

$$ar^2 + br + c = 0$$

This is called the **characteristic equation**, and the roots of this correspond to r in the solution $y = e^{rt}$. We assume two real unique roots r_1 and r_2, so $y = e^{r_1 t}$ and $y = e^{r_2 t}$ are solutions. Not only this, but also any linear combination of the two

$$y = c_1e^{r_1 t} + c_2e^{r_2 t}$$

is also a solution. To verify this, we differentiate:

$$y' = c_1r_1e^{r_1 t} + c_2r_2e^{r_2 t}$$

$$y'' = c_1r_1^2e^{r_1 t} + c_2r_2^2e^{r_2 t}$$

Substituting it back in,

$$ay'' + by' + cy = c_1(ar_1^2 + br_1 + c)e^{r_1 t} + c_2(ar_2^2 + br_2 + c)e^{r_2 t}$$

the terms in the parentheses on the right are 0 since r_1 and r_2 are roots of the characteristic equation.

Using the initial conditions

$$y(t_0) = y_0 \qquad y'(t_0) = y_0'$$

$$c_1e^{r_1 t_0} + c_2e^{r_2 t_0} = y_0$$

$$c_1r_1e^{r_1 t_0} + c_2r_2e^{r_2 t_0} = y_0'$$

$$c_1 = \frac{y_0' - y_0r_2}{r_1 - r_2}e^{-r_1 t_0}, \qquad c_2 = \frac{y_0r_1 - y_0'}{r_1 - r_2}e^{-r_2 t_0}$$

$r_1 - r_2 \neq 0$ since we established that the roots are different.

It can be shown that the solutions of our homogeneous initial value problem are the same solutions as a nonhomogeneous one. So, these solutions are the general solution for the second order linear equations with constants.

3.2 Solutions of Linear Homogeneous Equations; the Wronskian

Let p and q be continuous functions on an open interval $I = (\alpha, \beta)$ where α and β can be anything including ∞. Then, for any function ϕ that is twice differentiable on I,

$$L[\phi] = \phi'' + p\phi' + q\phi$$

So,

$$L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t)$$

The operator L is often written as $L = D^2 + pD + q$, where D is the derivative operator.

So the initial value problem is

$$L[y] = y'' + p(t)y' + q(t)y$$

$$y(t_0) = y_0 \qquad y'(t_0) = y_0'$$

THEOREM 3.2.1. *Consider the initial value problem*

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \qquad y'(t_0) = y_0'$$

where p, q, and g are continuous on an open interval I that contains the point t_0. Then there is exactly one solution $y = \phi(t)$ of this problem, and the solution exists throughout the interval I.

This theorem says three things:

1. A solution *exists*
2. The solution is *unique*
3. The solution ϕ is defined *throughout the interval* I where the coefficients are continuous and is at least twice differentiable there.

For most second order problems, we cannot write a useful expression for the solution. This is a major difference between first order and second order linear equations.

THEOREM 3.2.2 (Principle of Superposition)**.** *If y_1 and y_2 are two solutions fo the differential equation,*

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constants c_1 and c_2

Proof. To prove this, we substitute

$$y = c_1 y_1(t) + c_2 y_2(t)$$

for y.

$$\begin{aligned} L[c_1y_1 + c_2y_2] &= [c_1y_1 + c_2y_2]'' + p[c_1y_1 + c_2y_2]' + q[c_1y_1 + c_2y_2] \\ &= c_1y_1'' + c_2y_2'' + c_1py_1' + c_2py_2' + c_1qy_1 + c_2qy_2 \\ &= c_1[y_1'' + py_1' + qy_1] + c_2[y_2'' + py_2' + qy_2] \\ &= c_1L[y_1] + c_2L[y_2] = c_1(0) + c_2(0) = 0 \end{aligned}$$

since $L[y_1] = L[y_2] = 0$ because they are both solutions. □

This theorem essentially states that beginning with two solutions, we can construct an infinite family of solutions. Now, to address if all solutions of the equation are included in this family. First, we find constants to match our intial values.

$$c_1y_1(t_0) + c_2y_2(t_0) = y_0$$
$$c_1y_1'(t_0) + c_2y_2'(t_0) = y_0'$$

which can be written as

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

The determinant of the coefficients of the system

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)$$

If $W \neq 0$,

$$c_1 = \frac{y_0y_2'(t_0) - y_0'y_2(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}, \qquad c_2 = \frac{-y_0y_1'(t_0) + y_0'y_1(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}$$

or

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}, \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

These values of c_1 and c_2 satisfy the initial conditions and differential equation. But this is only if $W \neq 0$. If $W = 0$ then there are no solutions unless the numerators are equal to 0. Since otherwise the initial cannot be satisfied no matter what constants are chosen.

The determinant W is called the **Wronskian determinant**, or simply the **Wronskian**, of the solutions y_1 and y_2. Sometimes we use $W(y_1, y_2)(t_0)$.

THEOREM 3.2.3. *Suppose that y_1 and y_2 are two solutions of*

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

and that the initial conditions

$$y(t_0) = y_0 \qquad y'(t_0) = y'_0$$

are assigned. Then it is always possible to choose the constants c_1, c_2 so that

$$y = c_1 y_1(t) + c_2 y_2(t)$$

satisfies the differential equation and the initial conditions if and only if the Wronskian

$$W = y_1 y'_2 - y'_1 y_2$$

is not zero at t_0.

THEOREM 3.2.4. *Suppose that y_1 and y_2 are two solutions of the differential equation,*

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

Then the family of solutions

$$y = c_1 y_1(t) + c_2 y_2(t)$$

with arbitrary coefficients c_1 and c_2 includes every solution of the equation if and only if there is a point t_0 where the Wronskian of y_1 and y_2 is not zero.

Proof. Let ϕ be any arbitrary solution of the equation. To prove this, we show that ϕ as a part of the family $c_1 y_1(t) + c_2 y_2(t)$. Let t_0 be a point be point where the Wronskian is nonzero. So,

$$y_0 = \phi(t_0), \qquad y'_0 = \phi'(t_0)$$

Now, we consider the initial value problem

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y'_0$$

Since the Wronskian is nonzero, it is possible to choose a c_1 and c_2, so that

$$\phi(t) = c_1 y_1(t) + c_2 y_2(t)$$

so ϕ is included in the family. If there is no point t_0 where the Wronskian is nonzero, there are values y_0 and y'_0 that the system has no solutions for c_1 and c_2. Select a pair of such values and choose the solution $\phi(t)$ that satisfies the initial condition. So, if the $W = 0$, then there exists a solution that does not fit in the family. $\square$

$$y = c_1 y_1(t) + c_2 y_2(t)$$

with arbitrary constant coefficients is the **general solution**. And the solutions y_1 and y_2 are said to form a **fundamental set of solutions** if and only if their Wronskian is nonzero.

THEOREM 3.2.5. *Consider the differential equation*

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

whose coefficients p and q are continuous on some open interval I. Choose some point t_0 in I. Let y_1 be the solution of the equation that also satisfies the initial conditions

$$y(t_0) = 1, \qquad y'(t_0) = 0$$

and let y_2 be the solutions of the equation that satisfies the initial conditions

$$y(t_0) = 0, \qquad y'(t_0) = 1$$

Then y_1 and y_2 form a fundamental set of solutions.

Proof. The *existence* of y_1 and y_2 is ensured by the existence part of theorem 3.2.1, and they form a fundamental set of solutions since

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

□

THEOREM 3.2.6. *Consider the differential equation*

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where p and q are continuous real-valued functions. If $y = u(t) + iv(t)$ is a complex-valued solution, then its real part u and imaginary part v are both also solutions.

Proof. To prove this we substitute $y = u(t) + iv(t)$ in $L[y]$,

$$\begin{aligned} L[y] &= u''(t) + iv''(t) + p(t)[u'(t) + iv'(t)] + q(t)[u(t) + iv(t)] \\ &= u''(t) + p(t)u'(t) + q(t)u(t) + i[v''(t) + p(t)v'(t) + q(t)v(t)] \\ &= L[u(t)] + iL[v(t)] = 0 \end{aligned}$$

For a complex-number to be zero both the real and imaginary parts have to be zero, so both $L[u(t)] = 0$ and $L[v(t)] = 0$, and are both solutions. □

Also, the complex conjugate $\bar{y}$ of a solution y is also a solution since it is a linear combination of $u(t)$ and $v(t)$.

THEOREM 3.2.7 (Abel's Theorem)**.** *If y_1 and y_2 are solutions to the differential equation*

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where p and q are continuous on an open interval I, then the Wronskian W is given by

$$W(y_1, y_2)(t) = c \exp\left[-\int p(t)dt\right]$$

where c is a constant dependent on y_1 and y_2 but not t. So, W is zero for all $t \in I$ if $c = 0$, else it is nonzero for all $t \in I$.

Proof. We know

$$y_1'' + p(t)y_1' + q(t)y_1 = 0$$
$$y_2'' + p(t)y_2' + q(t)y_2 = 0$$

if we multiply the first equation by $-y_2$ and the second with y_1,a dn add the resulting equations, we get

$$(y_1y_2'' - y_1''y_2) + p(t)(y_1y_2' - y_1'y_2) = 0$$

We see that

$$W' = y_1y_2'' - y_1''y_2$$

so

$$W' + p(t)W = 0$$

which we can solve to get

$$W(y_1, y_2)(t) = c \exp\left[-\int p(t)dt\right]$$

and since the exponential function cannot be 0, $W = 0$ if and only if $c = 0$. □

3.3 Complex Roots of the Characteristic Equation

We continue our discussion of the equation

$$ay'' + by' + cy = 0$$

where a, b, and c are given real numbers. If we seek solutions of the form $y = e^{rt}$, the r must be a root of the characteristic equation

$$ar^2 + br + c = 0$$

We previously showed that if r_1 and r_2 are real and different, which occurs if $b^2 - 4ac > 0$, then the general solution of our differential equation is

$$y = c_1e^{r_1t} + c_2e^{r_2t}$$

Now if $b^2 - 4ac < 0$, then the roots of the characteristic equation are conjugate complex numbers,

$$r_1 = \lambda + i\mu, \qquad r_2 = \lambda - i\mu$$

where λ and μ are real. The corresponding expressions for y are

$$y_1(t) = \exp[(\lambda + i\mu)t], \qquad y_2(t) = \exp[(\lambda - i\mu)t]$$

What does it mean to raise the number e to a complex power. **Euler's Formula.** A Taylor series for e^t about $t = 0$ is

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

If we replace t with it,

$$e^{it} = \sum_{n=0}^{\infty} \frac{i^n t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!}$$

The real portion of the expression is the Taylor expansion for $\cos t$, and the imaginary part is the Taylor expansion for $\sin t$, so

$$e^{it} = \cos t + i \sin t$$

which is known as Euler's formula.

$$e^{-it} = \cos t - i \sin t$$

since $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$. In general if t is replaced with μt

$$e^{i\mu t} = \cos \mu t + i \sin \mu t$$

Now if we have a complex number $\lambda + i\mu$

$$e^{(\lambda + i\mu)t} = e^{\lambda t} e^{i\mu t} = e^{\lambda t}(\cos \mu t + i \sin \mu) = e^{\lambda t} \cos \mu t + i e^{\lambda t} \sin \mu t$$

$$\frac{d}{dt}(e^{rt}) = r e^{rt}$$

still holds true with complex rs and we can verify it using the definition above. **Complex Roots; The General Case.** If we have two solutions for y, $e^{\lambda t} \cos \mu t \pm i e^{\lambda t} \sin \mu t$, we can use the real and imaginary parts as a fundamental set of solutions since

$$W(u, v)(t) = \mu e^{2\lambda t}$$

by direct computation. $W \neq 0$ as long as $\mu \neq 0$, so u and v form a fundamental set. If μ is zero then the roots are real and we already talked about that. So the general solution of the original equation is

$$y = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t$$

3.4 Repeated Roots; Reduction of Order

Consider the differential equation

$$ay'' + by' + cy = 0$$

when the roots of the characteristic equation

$$ar^2 + br + c = 0$$

are equal. So, $b^2 - 4ac = 0$. Then,

$$r_1 = r_2 = -b/2a$$

So one solution for the equation is

$$y_1(t) = e^{-bt/2a}$$

Finding a second solution is not obvious. We start with the assumption that

$$y = v(t)y_1(t) = v(t)e^{-bt/2a}$$

So,

$$y' = v'(t)e^{-bt/2a} - \frac{b}{2a}v(t)e^{-bt/2a}$$

$$y'' = v''(t)e^{-bt/2a} - \frac{b}{a}v'(t)e^{-bt/2a} + \frac{b^2}{4a^2}v(t)e^{-bt/2a}$$

Plugging these into our original differential equation

$$\big\{a\big[v''(t) - \frac{b}{a}v'(t) + \frac{b^2}{4a^2}v(t)\big] + b\big[v'(t) - \frac{b}{2a}v(t)\big] + cv(t)\big\}e^{-bt/2a} = 0$$

$$a\big[v''(t) - \frac{b}{a}v'(t) + \frac{b^2}{4a^2}v(t)\big] + b\big[v'(t) - \frac{b}{2a}v(t)\big] + cv(t) = 0$$

$$av''(t) + (-b + b)v'(t) + \big(\frac{b^2}{4a} - \frac{b^2}{2a} + c\big)v(t) = 0$$

$-b + b$ is obviously 0. The coefficient of the last term (let us call it z) is also 0 since $-4az = b^2 - 4ac = 0$. So,

$$v''(t) = 0$$

$$v'(t) = c_2$$

$$v(t) = c_1 + c_2t$$

So we choose 0 and 1 and we have

$$y = c_1e^{-bt/2a} + c_2te^{-bt/2a}$$

and y is a linear combination of

$$y_1(t) = e^{-bt/2a}, \qquad y_2(t) = te^{-bt/2a}$$

The Wronskian of these two solutions is

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-bt/2a} & te^{-bt/2a} \\ -\frac{b}{2a}e^{-bt/2a} & (1 - \frac{bt}{2a})e^{-bt/2a} \end{vmatrix} = e^{-bt/2a}$$

which is never zero so these solutions form a fundamental set.
Reduction of Order. If we have an equation

$$y'' + p(t)y' + q(t)y = 0$$

and we have a solution y_1, to find a second solution, let

$$y = v(t)y_1(t)$$

then,

$$y' = v'(t)y_1(t) + v(t)y_1'(t)$$

and

$$y'' = v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t)$$

Substituting for y, y', and y'', and rearranging

$$y_1v'' + (2y_1' + py_1)v' + (y_1''py_1' + qy_1)v = 0$$

Since y_1 is a solution of the equation, the coefficient of v is 0, so

$$y_1v'' + (2y_1' + py_1)v' = 0$$

This is a first order equation of v'. Once v' is found, we can integrate to find v and now we have our second solution $y_2 = vy_1$.

3.5 Nonhomogeneous Equations; Method of Undetermined Coefficients

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

is the homogeneous equation corresponding to the nonhomogeneous equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t)$$

THEOREM 3.5.1. *If Y_1 and Y_2 are two solutions of the nonhomogeneous equation, then their difference $Y_1 - Y_2$ is a solution of the corresponding homogeneous equation. If, in addition, y_1 and y_2 are a fundamental set of solutions of the homogeneous equation, then*

$$Y_1(t) - Y_2(t) = c_1y_1(t) + c_2y_2(t)$$

Proof.

$$L[Y_1](t) = L[Y_2](t) = g(t)$$

$$L[Y_1](t) - L[Y_2](t) = g(t) - g(t) = 0$$

$$L[Y_1](t) - L[Y_2](t) = L[Y_1 - Y_2](t) = 0$$

So, $Y_1 - Y_2$ is a solution to $L[y] = 0$ □

THEOREM 3.5.2. *The general solution of a nonhomogeneous equation can be written in the form*

$$y = \phi(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

where y_1 and y_2 are a fundamental set of solutions of the corresponding homogeneous equation,c_1 and c_2 are arbitrary constants, and Y is some specific solution.

Proof. let $Y_1 = \phi$ and $Y_2 = Y$, so

$$\phi(t) - Y(t) = c_1 y_1(t) + c_2 y_2(t)$$

Then add $Y(t)$ to the other side. Now, ϕ includes all solutions. □

The tough part of this is finding the specific solution Y.

Method of Undetermined Coefficients. This method has us make an initial assumption of the form of the specific solution and then plug it into the differential equation to find the constants. If $g(t)$ is an exponential function $e^{\alpha t}$ then assume Y is proportional to this function. If $g(t)$ is $sin\beta t$ or $cos\beta t$, then assume $Y(t)$ is a linear combination of the two. If $g(t)$ is a polynomial, assume Y is a polynomial of the same degree.

If $g(t)$ is a product of multiple of these situations, Ex: $-8e^t \cos 2t$, then we can say the same, Ex: $Y(t) = Ae^t \cos 2t + Be^t \sin 2t$.

If $g(t)$ is a sum of multiple of these situations $g(t) = g_1(t) + g_2(t)$, then suppose Y_1 and Y_2 are solutions to

$$ay'' + by' + cy = g_1(t)$$

and

$$ay'' + by' + cy = g_2(t)$$

respectively. Then $Y_1 + Y_2$ is a solution for the original equation since $g(t) = g_1(t) + g_2(t) = L[Y_1](t) + L[Y_2](t) = L[Y_1 + Y_2](t)$.

If the form you assume it is in is within the general solution of the corresponding homogeneous equation, then just multiply by t. If that still doesn't work then do it again. For second order equations, you will never have to do it more than twice.

TABLE 3.5.1 The Particular Solution of $ay'' + by' + cy = g_i(t)$

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)$
$P_n(t)e^{\alpha t}$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}$
$P_n(t)e^{\alpha t}\begin{cases}\sin \beta t\\ \cos \beta t\end{cases}$	$t^s[(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}\cos \beta t$ $+ (B_0t^n + B_1t^{n-1} + \cdots + B_n)e^{\alpha t}\sin \beta t]$

3.6 Variation of Parameters

Consider

$$y'' + p(t)y' + q(t)y = g(t)$$

where p, q, and g are given continuous functions. To start we know the general solution of the corresponding homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

is

$$y_c(t) = c_1y_1(t) + c_2y_2(t)$$

The crucial idea of this method is to replace the constants c_1 and c_2 with functions $u_1(t)$ and $u_2(t)$; thus we have

$$y = u_1(t)y_1(t) + u_2(t)y_2(t)$$

and

$$y' = u_1'(t)y_1(t) + u_1(t)y_1'(t) + u_2'(t)y_2(t) + u_2(t)y_2'(t)$$

We now set the terms involving $u_1'(t)$ and $u_2'(t)$ to zero:

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$$

so

$$y' = u_1(t)y_1'(t) + u_2(t)y_2'(t)$$

and

$$y'' = u_1'(t)y_1'(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t)$$

Now we substitute y, y' and y'' and rearrange

$$\begin{aligned} u_1(t)[y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)]\\ +u_2(t)[y_2''(t) + p(t)y_2'(t) + q(t)y_2(t)]\\ +u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t) \end{aligned}$$

The expressions in the square brackets are zero since both y_1 and y_2 are solutions to the homogeneous equation.

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t)$$

and

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$$

form a system of two linear algebraic equations for the derivatives $u_1'(t)$ and $u_2'(t)$ of the unknown functions. By solving we get

$$u_1'(t) = -\frac{y_2(t)g(t)}{W(y_1,y_2)(t)}, \qquad u_2'(t) = -\frac{y_1(t)g(t)}{W(y_1,y_2)(t)}$$

By integrating we get

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}ds, \qquad u_2(t) = -\int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}ds$$

THEOREM 3.6.1. *If the functions p, q, and g are continuous on an open interval I, and if the functions y_1 and y_2 are a fundamental set of solutions of the homogeneous equation corresponding to the nonhomogeneous equation*

$$y'' + p(t)y' + q(t)y = g(t)$$

then a particular solution is

$$Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}ds + y_2(t)\int_{t_0}^{t} \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}ds$$

where t_0 is any conveniently chosen point in I. The general solution is

$$y = c_1y_1(t) + c_2y_2(t) + Y(t)$$

The difficulties of using this is evaluating the integral and finding the fundamental set of the corresponding homogeneous equation.

Chapter 4

Higher Order Linear Equations

4.1 General Theory of nth Order Linear Equations

An nth order linear differential equation is an equation of the form

$$P_0(t)\frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + P_{n-1}(t)\frac{dy}{dt} + P_n(t)y = G(t)$$

Dividing by $P_0(t)$,

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t)$$

The linear differential operator L of order n is defined above.

For this, we have n initial conditions,

$$y(t_0) = y_0 \; y'(t_0) = y'_0 \; \ldots \; y^{(}n-1)(t_0) = y_0^{(}n-1)$$

THEOREM 4.1.1. *If the functions $p_1, p_2, \ldots, p_n$, and g are continuous on the open interval I, then there exists exactly one solution $y = \phi(t)$ of the differential equation that also satisfies the initial conditions, where t_0 is any point in I. This solution exists throughout the interval I.*

The Homogeneous Equation.

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0$$

If the functions $y_1, y_2, \ldots, y_n$ are solutions of the previous equation, then it follows by direct computation that the linear combination

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

where $c_1, \ldots, c_n$ are arbitrary constants, is also a solution. This family of solutions encompasses all the solutions for all initial conditions. For this the Wronskian

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ y_1'' & y_2'' & \ldots & y_n'' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix}$$

must be non zero at $t = t_0$.

THEOREM 4.1.2. *If the functions $p_1, p_2, \ldots, p_n$ are continuous on the open interval I, if the functions $y_1, y_2, \ldots, y_n$ are solutions, and if $W \neq 0$ for at least one point in I, then every solution can be expressed as a linear combination of the solutions $y_1, y_2, \ldots, y_n$.*

The set is called a **fundamental set of solutions** if the Wronskian is 0. The **general solution** is a linear combination of these with arbitrary constants.
Linear Dependence and Independence. $f_1, f_2, \ldots, f_n$ are said to be **linearly dependent** if for a set of constants $k_1, k_2, \ldots, k_n$, not all zero,

$$\sum_{i=0}^{n} k_i f_i(t) = 0$$

These functions are **linearly independent** if they are not linearly dependent.

THEOREM 4.1.3. *If $y_1(t), y_2(t), \ldots, y_n(t)$ is a fundamental set of solutions, then it is linearly independent. If a set is linearly independent, they form a fundamental set*

Proof. Since its a fundamental set the Wronskian is nonzero, so the only solution of the linear dependence condition is if all the ks are zero, so it is linearly independent. □

The Nonhomogeneous Equation.

$$L[y] = g(t)$$

If Y_1 and Y_2 are two solutions of the nonhomogeneous equation, then

$$L[Y_1 - Y_2] = L[Y_1](t) - L[Y_2](t) = g(t) - g(t) = 0$$

hence the difference is a solution to the homogeneous equation.

So the general solution to a nonhomogeneous equation is

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + Y(t)$$

4.2 Homogeneous Equations with Constant Coefficients

Consider the nth order linear homogeneous differential equation

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

$$L[e^{rt}] = e^{rt}(a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n) = e^{rt} Z(r)$$

for all r, where

$$Z(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n$$

For those values of r for which $Z(r) = 0$, it follows that $L[e^{rt}] = 0$ and $y = e^{rt}$ is a solution. The polynomial $Z(r)$ is called the **characteristic polynomial**, and the equation $Z(r) = 0$ is the **characteristic equation** of the differential equation. By factoring, we get

$$Z(r) = a_0 (r - r_1)(r - r_2) \dots (r - r_n)$$

where $r_1, r_2, \dots, r_n$ are the zeroes, some of which may be equal.

Real and Unreal Roots. If the roots of the characteristic equation are real and no two are equation, then we have n distinct solutions $e^{r_1 t}, e^{r_2 t}, \dots, e^{r_n t}$. If these functions are linearly independent, then the general solution is

$$y = \sum_{i=0}^{n} c_i e^{r_i t}$$

If the Wronskian determinant is non zero, then they are linearly independent.

Complex Roots. If the characteristic equation has complex roots, they must occur in conjugate pairs, $\lambda \pm i\mu$, since the coefficient $a_0, a_1, \dots, a_n$ are real numbers. The general solution is still of the same form, but we can replace $e^{(\lambda + i\mu)t}$ and $e^{(\lambda - i\mu)t}$ by

$$e^{\lambda t} \cos \mu t, \quad e^{\lambda t} \sin \mu t$$

Repeated Roots. If the roots of the characteristic equation are not distinct, then the solution is not as clear. If a root of $Z(r) = 0$, say $r = r_1$ has a multiplicity s (occurs s times), then

$$e^{r_1 t}, t e^{r_1 t}, t^2 e^{r_1 t}, \dots, t^{s-1} e^{r_1 t}$$

are corresponding solutions. For a complex root, every time $\lambda + i\mu$ is repeated, $\lambda - i\mu$ must also repeat.

4.3 The Method of Undetermined Coefficients

A particular solution Y of the nonhomogeneous nth order linear equation with constant coefficients

$$L[y] = \sum_{i=0}^{n} a_i y^{(n-i)} = g(t)$$

can be obtained by the method of undetermined coefficients, provided that $g(t)$ is of an appropriate form.
When the differential operator L is applied to a polynomial $A_0t^m + A_1t^{m-1} + \cdots + A_m$, an exponential function $e^{\alpha t}$, a sine function $\sin \beta t$, or a cosine function $\cos \beta t$, the result is a polynomial, an exponential function, or a linear combination of sine and cosine functions, respectively. If $g(t)$ is a sum of these functions, we can find a $Y(t)$ by choosing a suitable combination of these functions. The constants are determined by plugging in the assumed expression into the differential equation.
If $g(t)$ is a sum of several terms, it is sometimes easier to split it up and then compute the solutions separately and then add them together for the particular solution.
This method is easy, but it only works in specific cases (with constant coefficients).

4.4 The Method of Undetermined Coefficients

This for equations of the form

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t)$$

The first step is to solve the corresponding homogeneous function. After this it is a general method that works for any function g. If we know a fundamental set of solutions, $y_1, y_2, \ldots, y_n$, of the homogeneous equation, the general solution for the homogeneous equation is

$$y_c(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t)$$

The method of variation of parameters for determining a particular solution rests on the possibility of determining n functions $u_1, u_2, \ldots, u_n$ such that $Y(t)$ is of the form

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + \cdots + u_n(t)y_n(t)$$

Since we have n functions to determine, we specify n conditions. By differentiating,

$$Y' = (u_1y_1' + u_2y_2' + \cdots + u_ny_n') + (u_1'y_1 + u_2'y_2 + \cdots + u_n'y_n)$$

The first condition we impose is that

$$u_1'y_1 + u_2'y_2 + \cdots + u_n'y_n = 0$$

so

$$Y' = u_1y_1' + u_2y_2' + \cdots + u_ny_n'$$

We repeat this for more derivatives $Y'', \ldots, Y^{(n-1)}$. After each differentiation we set equal to zero the sum of the terms involving derivatives of $u_1, \ldots, u_n$. In this way we obtain $n-2$ further conditions.

$$u_1'y_1^{(m)} + u_2'y_2^{(m)} + \cdots + u_n'y_n^{(m)} = 0, \quad m = 1, 2, \ldots, n-2$$

and using this we get

$$Y^{(m)} = u_1 y_1^{(m)} + u_2 y_2^{(m)} + \cdots + u_n y_n^{(m)}, \quad m = 2, 3, \ldots, n-1$$

Then we differentiate one more time to get

$$Y^{(n)} = (u_1 y_1^{(n)} + \cdots + u_n y_n^{(n)}) + (u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)})$$

To satisfy the original differential equation, substitute for Y and its derivatives, then group the terms involving each of the functions $y_1, \ldots, y_n$ and their derivatives. It then follows that most of the terms in the equation drop out because each of $y_1, \ldots, y_n$ is a solution and therefore $L[y_i] = 0, i = 1, 2, \ldots, n$. The remaining terms yield the relation

$$u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = g$$

These provide n simultaneous linear nonhomogeneous algebraic equations for $u_1', u_2', \ldots, u_n'$:

$$\begin{aligned} y_1 u_1' + y_2 u_2' + \cdots + y_n u_n' &= 0, \\ y_1' u_1' + y_2' u_2' + \cdots + y_n' u_n' &= 0, \\ y_1'' u_1' + y_2'' u_2' + \cdots + y_n'' u_n' &= 0, \\ &\vdots, \\ y_1^{(n-1)} u_1' + y_2^{(n-1)} u_2' + \cdots + y_n^{(n-1)} u_n' &= 0 \end{aligned}$$

This system is a linear algebraic system for the unknown quantities $u_1', \ldots, u_n'$. Using Cramer's rule, can write the solution of the system of equation in the form

$$u_m'(t) = \frac{g(t) W_m(t)}{W(t)}, \quad m = 1, 2, \ldots, n$$

Here $W(t) = W(y_1, y_2, \ldots, y_n)(t)$, and W_m is the determinant obtained from W by replacing the mth column by the column $(0, 0, \ldots, 1)$. With this notation a particular solution is given by

$$Y(t) = \sum_{m=1}^{n} y_m(t) \int_{t_0}^{t} \frac{g(s) W_m(s)}{W(s)} ds$$

where t_0 is arbitrary. Sometimes it can be simplified using Abel's identity

$$W(t) = W(y_1, \ldots, y_n)(t) = c \exp\left[-\int p_1(t) dt \right]$$

The constant c can be determined evaluating by W at some convenient point.