DECA Chapter 5 Master Test

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1. Solve Airy's equation in powers of x

$$y'' - xy = 0, \qquad -\infty < x < \infty.$$

2. Obtain the recurrence relation for

$$L[y] = x^2y'' + x\big[xp(x)\big] + \big[x^2q(x)\big] = 0,$$

where

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \quad x^2q(x) = \sum_{n=0}^{\infty} q_n x^n,$$

and y is of the form

$$y = \phi(r, x) = \sum_{n=0}^{\infty} a_n x^{r+n}.$$

3. The equation

$$y'' - 2xy' + \lambda y = 0, \quad -\infty < x < \infty$$

where λ is a constant is known as the **Hermite equation**.

   (a) Find the first four terms in each of the two solutions about $x = 0$ and show that they form a fundamental set of solutions.

   (b) Observe that if λ is a nonnegative even integer, then one or the other of the series terminates and becomes a polynomial. Find the polynomial solutions for $\lambda = 0, 2, 4, 6, 8$, and 10.

   (c) The Hermite polynomial $H_n(x)$ is defined as the polynomial of the Hermite equation with solution of the Hermite equation with $\lambda = 2n$ for which the coefficient of x^n is 2^n. Find $H_0(x), ..., H_5(x)$.

4. The Legendre equation is

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0.$$

Show that if α is zero or a positive even integer $2n$, the series solution y_1 reduces to a polynomial of degree $2n$ containing only even powers of x.

Find the polynomials corresponding to $\alpha = 0, 2$, and 4.
Show that if α is a positive odd integer $2n + 1$, the series solution y_2 reduces to a polynomial of degree $2n + 1$ containing only odd powers of x. Find the polynomials corresponding to $\alpha = 1, 3$, and 5.

5. Show that for $n = 0, 1, 2, 3$, the corresponding Legendre polynomial is given by
$$P_n(x) = \frac{1}{2^n n!} \frac{dn}{dx^n}(x^2 - 1)^n.$$

6. Solve the Euler equation
$$L[y] = x^2 y'' + dxy' + \beta y = 0$$
about $x_0 = 0$. Consider $x > 0$ and $x < 0$.

7. Using the method of reduction of order, show that if r_1 is a repeated root of
$$r(r-1) + \alpha r + \beta = 0,$$
then x^{r_1} and $x^{r_1} \ln x$ are solutions of $x^2 y'' + \alpha x y' + \beta y = 0$ for $x > 0$.

8. The Laguerre differential equation is
$$xy'' + (1-x)y' + \lambda y = 0.$$
   (a) Show that $x = 0$ is a regular singular point.
   (b) Determine the indicial equation, its roots, and the recurrence relation.
   (c) Find one solution ($x > 0$). Show that if $\lambda = m$, a positive integer, this solution reduces to a polynomial. When properly normalized, this polynomial is known as the Laguerre polynomial, $L_m(x)$.

9. Consider the differential equation
$$x^3 y'' + \alpha x y' + \beta y' = 0,$$
where α and β are real constants and $\alpha \neq 0$.
   (a) Show that $x = 0$ is an irregular singular point.
   (b) By attempting to determine a solution of the form $\sum_{n=0}^{\infty} a_n x^{r+n}$, show that the indicial equation for r is linear and that, consequently, there is only one formal solution of the assumed form.

10. Show that the Bessel equation of order one-half
$$x^2 y'' + xy' + (x^2 - \frac{1}{4})y = 0, \quad x > 0$$
can be reduced to the equation
$$v'' + v = 0$$

by the change of dependent variable $y = x^{-1/2} v(x)$. From this, conclude that $y_1(x) = x^{-1/2} \cos x$ and $y_2(x) = x^{-1/2} \sin x$ are solutions of the Bessel equation of order one-half.