

## 4.2 Homogeneous Equations with Constant Coefficients

Consider the  $n$ th order linear homogeneous differential equation

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

$$L[e^{rt}] = e^{rt}(a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n) = e^{rt} Z(r)$$

for all  $r$ , where

$$Z(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n$$

For those values of  $r$  for which  $Z(r) = 0$ , it follows that  $L[e^{rt}] = 0$  and  $y = e^{rt}$  is a solution. The polynomial  $Z(r)$  is called the **characteristic polynomial**, and the equation  $Z(r) = 0$  is the **characteristic equation** of the differential equation. By factoring, we get

$$Z(r) = a_0(r - r_1)(r - r_2) \cdots (r - r_n)$$

where  $r_1, r_2, \dots, r_n$  are the zeroes, some of which may be equal.

**Real and Unreal Roots.** If the roots of the characteristic equation are real and no two are equal, then we have  $n$  distinct solutions  $e^{r_1 t}, e^{r_2 t}, \dots, e^{r_n t}$ . If these functions are linearly independent, then the general solution is

$$y = \sum_{i=1}^n c_i e^{r_i t}$$

If the Wronskian determinant is non zero, then they are linearly independent.

**Complex Roots.** If the characteristic equation has complex roots, they must occur in conjugate pairs,  $\lambda \pm i\mu$ , since the coefficients  $a_0, a_1, \dots, a_n$  are real numbers. The general solution is still of the same form, but we can replace  $e^{(\lambda+i\mu)t}$  and  $e^{(\lambda-i\mu)t}$  by

$$e^{\lambda t} \cos \mu t, \quad e^{\lambda t} \sin \mu t$$

**Repeated Roots.** If the roots of the characteristic equation are not distinct, then the solution is not as clear. If a root of  $Z(r) = 0$ , say  $r = r_1$  has a multiplicity  $s$  (occurs  $s$  times), then

$$e^{r_1 t}, t e^{r_1 t}, t^2 e^{r_1 t}, \dots, t^{s-1} e^{r_1 t}$$

are corresponding solutions. For a complex root, every time  $\lambda + i\mu$  is repeated,  $\lambda - i\mu$  must also repeat.