2.9 First Order Difference Equations

For some problems, we need to model a discrete process. Differential equations model a continuous process, while **difference equation**s model a discrete process. An example is modelling population growth of a species whose generations do not overlap. The population y_{n+1} of the species in the year $n+1$ is some function of n and the population y_N in the preceding year:

$$y_{n+1} = f(n, y_n) \qquad n \in \mathbf{N}$$

This is a **first order difference equation** since it only depends on the step before it. The equation is **linear** if f is a linear function of y_n; otherwise, it is **nonlinear**. A **solution** of the difference equation is a sequence of numbers $y_0, y_1, y_2, \ldots$ that satisfy the equation for each n. There also must be an **initial condition**.

$$y_0 = \alpha$$

Now for convenience, temporarily let $f(y_n) = f(n, y_n)$.

$$y_1 = f(y_0)$$

$$y_2 = f(y_1) = f[f(y_0)]$$
$$y_1 = f(y_2) = f[f[f(y_0)]] = f^3(y_0)$$
$$y_n = f(y_{n-1}) = f^n(y_0)$$

Solutions for which y_n has the same value for all n are called **equilibrium solutions**. You can find this by setting $y_{n+1} = y_n$, and solve

$$y_n = f(y_n)$$

for y_n.

Linear Equations. These could be of the form

$$y_{n+1} = \rho_n y_n \qquad n \in \mathbf{N}$$

THe reproduction rate rho_n may differ from year to year. This can easily be solved by iteration. We obtain

$$y_1 = \rho_0 y_0$$

$$y_2 = \rho_1 y_1 = \rho_0 \rho_1 y_0$$

and, in general,

$$y_n = (\Pi_{i=0}^{n-1} \rho_i) y_0$$

If the reproduction rate is the same for all n, $\rho_n = \rho$, then

$$y_{n+1} = \rho y_n$$

$$y_n = \rho^n y_0$$

The limiting behavior can easily be determined.

$$\lim_{n\to\infty} y_n = \begin{cases} 0, & \text{if } |\rho| < 1; \\ y_0, & \text{if } |\rho| = 1; \\ \text{does not exist}, & \text{if } |\rho| > 1; \end{cases}$$

The equilibrium solution $y_n = 0$ is asymptotically stable for $|\rho| < 1$ and unstable for $|\rho| > 1$.

If a population has immigration or emigration, we must have a b_n as the net increase in population in year n due to immigration.

$$y_{n+1} = \rho y_n + b_n$$

We can solve for y_n through iteration as well.

$$y_1 = \rho y_0 + b_0$$

$$y_2 = \rho(\rho y_0 + b_0) + b_1 = \rho^2 y_0 + \rho b_0 + b_1$$
$$y_3 = \rho(\rho^2 y_0 + \rho b_0 + b_1) + b_2 = \rho^3 y_0 + \rho^2 b_0 + \rho b_1 + b_2$$

and in general,

$$y_n = \rho^n y_0 + \sum_{j=0}^{n-1} \rho^{n-1-j} b_j$$

The more general solution for linear equations of the form

$$y_{n+1} = \rho_n y_n + b_n$$

then by iterations,

$$y_n = (\Pi_{i=0}^{n-1} \rho_i) y_0 + \sum_{i=0}^{n-1} (\Pi_{j=i+1}^{n-1} \rho_j) b_{n-1}$$

The first term on the right side represents thr descendants of the original population.

If $b_n = b \neq 0$,

$$y_{n+1} = \rho y_n + b$$

and from the previous equation the solution is

$$y_n = \rho^n y_0 + (1 + \rho + \rho^2 + \cdots + \rho^{n-1}) b$$

or for $\rho \neq 1$,

$$y_n = \rho_n y_0 + \frac{1 - \rho^n}{1 - \rho}$$

which is the same as

$$y_n = \rho^n (y_0 - \frac{b}{1 - \rho}) + \frac{b}{1 - \rho}$$

which makes the limit behavior of y_n more clear. For $\rho = 1$,

$$y_n = y_0 + nb$$

The limit behavior is defined by

$$\lim_{n\to\infty} y_n = \begin{cases} b/(1-\rho), & \text{if } |\rho| < 1; \\ \textit{unbounded}, & \text{if } |\rho| = 1; \\ \text{does not exist}, & \text{if } |\rho| > 1; \end{cases}$$

when $|\rho|$ is not less than 1, it converges to y_0, if $y_0 = b/(1-\rho)$ since this is an equilibrium solution.

The model can also be used for interest, where $\rho_n = 1 + r_n$, where r_n is the interest rate, and b_n is the amount deposited or withdrawn.

Nonlinear Equations. Consider the logistic difference equation

$$y_{n+1} = \rho y_n (1 - \frac{y_n}{k})$$

If we scale y_n to $u_n = y_n/k$

$$u_{n+1} = \rho u_n (1 - u_n)$$

$(\rho = k\rho)$, a positive parameter. We can find an equilibrium solution by setting $u_{n+1} = u_n$.

$$u_n = \rho u_n - \rho u_n^2$$

which gives us

$$u_n = 0, \quad u_n = \frac{\rho - 1}{\rho}$$

To check if these equilibrium solutions are asymptotically stable or unstable, we can use linear approximations. Near $u_n = 0$, u_n^2 is small compared to u_n. So,

$$u_{n+1} \approx \rho u_n$$

We already know that this only approaches 0 for $|\rho| < 1$, or $0 < \rho < 1$ since ρ is positive. So $u_n = 0$ is stable for $0 < \rho < 1$.

To test solutions in the neighborhood of $u_n = (\rho - 1)/\rho$, we write

$$u_n = \frac{\rho - 1}{\rho} + v_n$$

where v_n is small. By substituting this into $u_{n+1} = \rho u_n (1 - u_n)$, we get

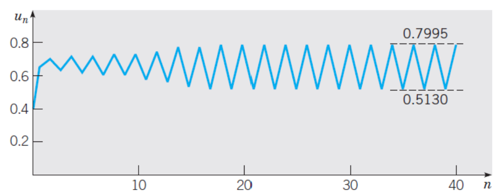
$$v_{n+1} = (2 - \rho) v_n - \rho v_n^2$$

Since v_n is small, $v_n^2 \approx 0$.

$$v_{n+1} = (2 - \rho) v_n$$

In the same manner we know $v_n \to 0$ as $n \to \infty$ for $|2-\rho| < 1$, or $1 < \rho < 3$. Therefore, in this range of values for ρ, $u_n = (\rho - 1)/\rho$ is an asymptotically stable equilibrium solution. $\rho = 1$ is an **exchange of stability** from one equilibrium solution to the other.

If $\rho > 3$, the solution will oscillate between two values; it is period 2. At about $\rho = 3.449$, the solution becomes periodic with period 4. The appearance of a new solution at a certain parameter value is called a **bifurcation**.



The ρ-values at which the successive period doublings occur approach a limit that is approximately 3.57, so for $\rho > 3.57$, the solutions have some regularity. It's fine structure is unpredictable, hence the term **chaotic**. One of the features of chaotic solutions is extreme sensitivity to initial conditions.