4.4 The Method of Undetermined Coefficients

This for equations of the form

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t)$$

The first step is to solve the corresponding homogeneous function. After this it is a general method that works for any function g. If we know a fundamental set of solutions, y_1, y_2, …, y_n, of the homogeneous equation, the general solution for the homogeneous equation is

$$y_c(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t)$$

The method of variation of parameters for determining a particular solution rests on the possibility of determining n functions u_1, u_2, …, u_n such that $Y(t)$ is of the form

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + \cdots + u_n(t)y_n(t)$$

Since we have n functions to determine, we specify n conditions. By differentiating,

$$Y' = (u_1y_1' + u_2y_2' + \cdots + u_ny_n') + (u_1'y_1 + u_2'y_2 + \cdots + u_n'y_n)$$

The first condition we impose is that

$$u_1'y_1 + u_2'y_2 + \cdots + u_n'y_n = 0$$

so

$$Y' = u_1y_1' + u_2y_2' + \cdots + u_ny_n'$$

We repeat this for more derivatives Y'', …, $Y^{(n-1)}$. After each differentiation we set equal to zero the sum of the terms involving derivatives of $u_1, \ldots, u_n$. In this way we obtain $n-2$ further conditions.

$$u_1'y_1^{(m)} + u_2'y_2^{(m)} + \cdots + u_n'y_n^{(m)} = 0, \quad m = 1, 2, \ldots, n-2$$

and using this we get

$$Y^{(m)} = u_1y_1^{(m)} + u_2y_2^{(m)} + \cdots + u_ny_n^{(m)}, \quad m = 2, 3, \ldots, n-1$$

Then we differentiate one more time to get

$$Y^{(n)} = (u_1y_1^{(n)} + \cdots + u_ny_n^{(n)}) + (u_1'y_1^{(n-1)} + \cdots + u_n'y_n^{(n-1)})$$

To satisfy the original differential equation, substitute for Y and its derivatives, then group the terms involving each of the functions $y_1, \ldots, y_n$ and their derivatives. It then follows that most of the terms in the equation drop out because each of $y_1, \ldots, y_n$ is a solution and therefore $L[y_i] = 0, i = 1, 2, \ldots, n$. The remaining terms yield the relation

$$u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = g$$

These provide n simultaneous linear nonhomogeneous algebraic equations for $u_1', u_2', \ldots, u_n'$:

$$\begin{aligned} y_1 u_1' + y_2 u_2' + \cdots + y_n u_n' &= 0, \\ y_1' u_1' + y_2' u_2' + \cdots + y_n' u_n' &= 0, \\ y_1'' u_1' + y_2'' u_2' + \cdots + y_n'' u_n' &= 0, \\ &\vdots, \\ y_1^{(n-1)} u_1' + y_2^{(n-1)} u_2' + \cdots + y_n^{(n-1)} u_n' &= 0 \end{aligned}$$

This system is a linear algebraic system for the unknown quantities $u_1', \ldots, u_n'$. Using Cramer's rule, can write the solution of the system of equation in the form

$$u_m'(t) = \frac{g(t) W_m(t)}{W(t)}, \quad m = 1, 2, \ldots, n$$

Here $W(t) = W(y_1, y_2, \ldots, y_n)(t)$, and W_m is the determinant obtained from W by replacing the mth column by the column $(0, 0, \ldots, 1)$. With this notation a particular solution is given by

$$Y(t) = \sum_{m=1}^{n} y_m(t) \int_{t_0}^{t} \frac{g(s) W_m(s)}{W(s)} ds$$

where t_0 is arbitrary. Sometimes it can be simplified using Abel's identity

$$W(t) = W(y_1, \ldots, y_n)(t) = c \exp\left[- \int p_1(t) dt \right]$$

The constant c can be determined evaluating by W at some convenient point.