2.8 The Existence and Uniqueness Theorem

Consider the initial value problem

$$y' = f(t, y), \qquad y(0) = 0$$

If an initial value problem is not of this form, we can apply a translation of the coordinate axes that will take (t_0, y_0) to the origin.

THEOREM 2.8.1. *If f and $\frac{\partial f}{\partial y}$ are continuous in a rectangle $R : |t| \leq a, |y| \leq b$, then there is some interval $|t| \leq h \leq a$ in which there exists a unique solution $y = \phi(t)$ of the initial value problem.*

If we integrate the initial value problem equation, we get

$$\phi(t) = \int_0^t f[s, \phi(s)]ds$$

This is called the **integral equation**, which is equivalent to the initial value equation.

One method of showing that the integral equation has a unique solution is the **method of successive approximations** or Picard's **iteration method**. In using this method, we start by choosing an initial function ϕ_0, either arbitrarily or to approximate the solution. The simplest choice is

$$\phi_0(t) = 0$$

ϕ_0 satisfies the initial condition, but probably not the differential equation. The next apprximation ϕ_1 is obtained using ϕ_0.

$$\phi_1(t) = \int_0^t f[s, \phi_0(s)]ds$$

and, in general,

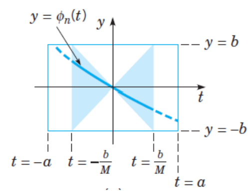
$$\phi_{n+1}(t) = \int_0^t f[s, \phi_n(s)]ds$$

This creates a sequence of functions $\phi_n = \phi_0, \phi_1, \phi_2, \ldots, \phi_n, \ldots$. Each element of the sequence satisfies the initial condition, but in general none satisfies the differential equation. If at some stage $\phi_{k+1}(t) = \phi_k(t)$, then it follows that ϕ_k is a solution and does follow the differential equation.

1. **Do all members of the sequence ϕ_n exist?**
   If f and $\frac{\partial f}{\partial y}$ are continuous in the whole ty-plane, then each ϕ_n is known to exist and can be calculated. But if f and $\frac{\partial f}{\partial y}$ are only assumed continuous in a rectangle $R : |t| \leq a, |y| \leq b$, then some members of the sequence cannot be explicitly determined. If we restrict t to a smaller intercal $|t| \leq a$, we can avoid this danger. Since f must be bounded on R

   $$|f(t, y)| \leq M \qquad (t, y) \text{ in } R$$

   Since $f[t, \phi_k(t)] = \phi'_{k+1}(t)$, the maximum slope of $y = \phi_{k+1}(t)$ is M. So ϕ_{k+1} must lie in R as long as R contains this region:



which is for $|t| \leq b/M$.

2. **Does the sequence $\phi_n(t)$ converge?**
We can identify $\phi_n(t) = \phi_1(t) + [\phi_2(t) - \phi_1(t)] + [\phi_3(t) - \phi_2(t)] + \cdots + [\phi_n(t) - \phi_{n-1}(t)]$ as the nth partial sum of the series

$$\phi_1(t) + \sum_{k+1}^{\infty} [\phi_{k+1}(t) - \phi_k(t)]$$

The convergence of the sequence $\phi_n(t)$ is established by showing that this series converges.
TODO: PROVE THIS USING PROBLEMS 15-18
We denote the limit function by ϕ, so that

$$\phi(t) = \lim_{n\to\infty} \phi_n(t)$$

3. **What are the properties of the limit function ϕ?**
We know that ϕ is continuous since the sequence $\{phi_n$converges in a certain manner, known as uniform convergence. We proof we used proves this as well. Now let us return to

$$\phi_{n+1}(t) = \int_0^t f[s, \phi_n(s)]ds$$

Allowing n to approach ∞ on both sides, we obtain

$$\phi(t) = \lim_{n\to\infty} \int_0^t f[s, \phi_n(s)]ds$$

We can move the limit inside the integral since the sequence converges uniformly.

$$\phi(t) = \int_0^t \lim_{n\to\infty} f[s, \phi_n(s)]ds$$

Then we take it inside the function

$$\phi(t) = \int_0^t f[s, \lim_{n\to\infty} \phi_n(s)]ds$$

so

$$\phi(t) = \int_0^t f[s, \phi(s)]ds$$

Moving the limit inside the function is saying that f is continuous in its second variable, which is known. So this last equation shows that ϕ satisfies the integral equation so it is a solution for the initial value problem.

4. **Are there other solutions of the integral equation besides $y = \phi(t)$?**
Assume another solution $y = \psi(t)$. It can be shown that

$$|\phi(t) - \psi(t)| \leq A \int_0^t |\phi(s) - \psi(s)| ds$$

TODO: Prove this using PROBLEM 19
for $0 \leq t \leq h$ and a suitable positive number A. It is now convenient to introduce U as

$$U(t) = \int_0^t |\phi(s) - \psi(s)| ds$$

$$U(0) = 0$$
$$U(t) \geq 0, \quad \text{for } t \geq 0$$

U is differentiable, and $U'(t) = |\phi(t) - \psi(t)|$.

$$U'(t) - AU(t) \leq 0 \text{ for } 0 \leq t \leq A/2$$

Multiplying by e^{-At} gives

$$[e^{-At}U(t)]' \leq 0 \text{ for } 0 \leq t \leq A/2$$

Then, by integration from 0 to t

$$e^{-At}U(t) \leq 0 \text{ for } 0 \leq t \leq A/2$$

Hence $U(t) \leq 0$ for $0 \leq t \leq A/2$, but since A is arbitrary $U(t) \leq 0$ for all positive t. Since $U(t) \geq 0$, $U(t) = 0$, so $U'(t) = 0$ and $\psi(t) = \phi(t)$. This proves there connot be two different solutions to one initial value problem for $t \geq 0$. A slight modification of this argument shows the same is true for $t \leq 0$.