2.2 Separable Equations

If a differential equation $dy/dx = f(x, y)$ can be written as

$$M(x) + N(y)\frac{dy}{dx} = 0$$

then it is said to be **separable** since it can be written as

$$M(x)dx + N(y)dy = 0$$

and integrated.

Let

$$H_1'(x) = M(x) \qquad H_2'(y) = M(y)$$

so then the previous equation becomes

$$H_1'(x) + H_2'(y)\frac{dy}{dx} = 0$$

By the chain rule:

$$H_2'(y)\frac{dy}{dx} = \frac{d}{dy}H_2(y)\frac{dy}{dx} = \frac{d}{dx}H_2(y)$$

which gives us

$$\frac{d}{dx}[H_1(x) + H_2(y)] = 0$$

By integrating, we get

$$H_1(x) + H_2(y) = c$$

Any differentiable function $y = \phi(x)$ that satisfies $H_1(x) + H_2(y) = c$ is a solution of the original differentiable equation. The differential equation and the initial condition $y(x_0) = y_0$ forms an initial value problem. We can use the initial value to find the correct c:

$$c = H_1(x_0) + H_2(y_0)$$

so

$$c = H_1(x_0) + H_2(y_0) = H_1(x) + H_2(y)$$
$$(H_1(x) - H_1(x_0)) + (H_2(y) - H_2(y_0)) = 0$$
$$\int_{x_0}^{x} M(s)ds + \int_{y_0}^{y} N(s)ds = 0$$

since

$$H_1(x) - H_1(x_0) = \int_{x_0}^{x} M(s)ds \qquad H_2(y) - H_2(y_0) = \int_{y_0}^{y} N(s)ds$$

Note 1: Sometimes the solution to

$$\frac{dy}{dx} = f(x, y)$$

has a constant solution $y = y_0$, which occurs when $f(x, y_0) = 0$ for all x and for y_0. For example,

$$\frac{dy}{dx} = \frac{(y - 3)\cos x}{1 + 2y^2}$$

has a solution $y = 3$.

Note 2: Sometimes if a function is non-linear it helps to regard both x and y as functions of a third variable t.

$$\frac{dy}{dt} = \frac{dy/dt}{dx/dt}$$

Note 3: Sometimes it is not easy to solve explicitly for y as a function of x. In these cases, it is better to leave the solution in implicit form.