3.3 Complex Roots of the Characteristic Equation

We continue our discussion of the equation

$$ay'' + by' + cy = 0$$

where a, b, and c are given real numbers. If we seek solutions of the form $y = e^{rt}$, the r must be a root of the characteristic equation

$$ar^2 + br + c = 0$$

We previously showed that if r_1 and r_2 are real and different, which occurs if $b^2 - 4ac > 0$, then the general solution of our differential equation is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Now if $b^2 - 4ac < 0$, then the roots of the characteristic equation are conjugate complex numbers,

$$r_1 = \lambda + i\mu, \qquad r_2 = \lambda - i\mu$$

where λ and μ are real. The corresponding expressions for y are

$$y_1(t) = \exp[(\lambda + i\mu)t], \qquad y_2(t) = \exp[(\lambda - i\mu)t]$$

What does it mean to raise the number e to a complex power. **Euler's Formula.** A Taylor series for e^t about $t = 0$ is

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

If we replace t with it,

$$e^{it} = \sum_{n=0}^{\infty} \frac{i^n t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!}$$

The real portion of the expression is the Taylor expansion for $\cos t$, and the imaginary part is the Taylor expansion for $\sin t$, so

$$e^{it} = \cos t + i \sin t$$

which is known as Euler's formula.

$$e^{-it} = \cos t - i \sin t$$

since $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$. In general if t is replaced with μt

$$e^{i\mu t} = \cos \mu t + i \sin \mu t$$

Now if we have a complex number $\lambda + i\mu$

$$e^{(\lambda+i\mu)t} = e^{\lambda t}e^{i\mu t} = e^{\lambda t}(\cos\mu t + i\sin\mu) = e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t$$

$$\frac{d}{dt}(e^{rt}) = re^{rt}$$

still holds true with complex rs and we can verify it using the definition above. **Complex Roots; The General Case.** If we have two solutions for y, $e^{\lambda t}\cos\mu t \pm ie^{\lambda t}\sin\mu t$, we can use the real and imaginary parts as a fundamental set of solutions since

$$W(u,v)(t) = \mu e^{2\lambda t}$$

by direct computation. $W \neq 0$ as long as $\mu \neq 0$, so u and v form a fundamental set. If μ is zero then the roots are real and we already talked about that. So the general solution of the original equation is

$$y = c_1 e^{\lambda t}\cos\mu t + c_2 e^{\lambda t}\sin\mu t$$