4.3 Homogeneous Equations with Constant Coefficients

Consider the nth order linear homogeneous differential equation

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

$$L[e^{rt}] = e^{rt}(a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n) = e^{rt} Z(r)$$

for all r, where

$$Z(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n$$

For those values of r for which $Z(r) = 0$, it follows that $L[e^{rt}] = 0$ and $y = e^{rt}$ is a solution. The polynomial $Z(r)$ is called the **characteristic polynomial**, and the equation $Z(r) = 0$ is the **characteristic equation** of the differential equation. By factoring, we get

$$Z(r) = a_0(r - r_1)(r - r_2)\ldots(r - r_n)$$

where $r_1, r_2, \ldots, r_n$ are the zeroes, some of which may be equal.
Real and Unreal Roots. If the roots of the characteristic equation are real and no two are equation, then we have n distinct solutions $e^{r_1 t}, e^{r_2 t}, \ldots, e^{r_n t}$. If these functions are linearly independent, then the general solution is

$$y = \sum_{i=0}^{n} c_i e^{r_i t}$$

If the Wronskian determinant is non zero, then they are linearly independent.
Complex Roots. If the characteristic equation has complex roots, they must occur in conjugate pairs, $\lambda \pm i\mu$, since the coefficient $a_0, a_1, \ldots, a_n$ are real numbers. The general solution is still of the same form, but we can replace $e^{(\lambda + i\mu)t}$ and $e^{(\lambda - i\mu)t}$ by

$$e^{\lambda t} \cos \mu t, \quad e^{\lambda t} \sin \mu t$$

Repeated Roots. If the roots of the characteristic equation are not distinct, then the solution is not as clear. If a root of $Z(r) = 0$, say $r = r_1$ has a multiplicity s (occurs s times), then

$$e^{r_1 t}, te^{r_1 t}, t^2 e^{r_1 t}, \ldots, t^{s-1} e^{r_1 t}$$

are corresponding solutions. For a complex root, every time $\lambda + i\mu$ is repeated, $\lambda - i\mu$ must also repeat.