3.4 Repeated Roots; Reduction of Order

Consider the differential equation

$$ay'' + by' + cy = 0$$

when the roots of the characteristic equation

$$ar^2 + br + c = 0$$

are equal. So, $b^2 - 4ac = 0$. Then,

$$r_1 = r_2 = -b/2a$$

So one solution for the equation is

$$y_1(t) = e^{-bt/2a}$$

Finding a second solution is not obvious. We start with the assumption that

$$y = v(t)y_1(t) = v(t)e^{-bt/2a}$$

So,

$$y' = v'(t)e^{-bt/2a} - \frac{b}{2a}v(t)e^{-bt/2a}$$

$$y'' = v''(t)e^{-bt/2a} - \frac{b}{a}v'(t)e^{-bt/2a} + \frac{b^2}{4a^2}v(t)e^{-bt/2a}$$

Plugging these into our original differential equation

$$\big\{a\big[v''(t) - \frac{b}{a}v'(t) + \frac{b^2}{4a^2}v(t)\big] + b\big[v'(t) - \frac{b}{2a}v(t)\big] + cv(t)\big\}e^{-bt/2a} = 0$$

$$a\big[v''(t) - \frac{b}{a}v'(t) + \frac{b^2}{4a^2}v(t)\big] + b\big[v'(t) - \frac{b}{2a}v(t)\big] + cv(t) = 0$$

$$av''(t) + (-b + b)v'(t) + \big(\frac{b^2}{4a} - \frac{b^2}{2a} + c\big)v(t) = 0$$

$-b + b$ is obviously 0. The coefficient of the last term (let us call it z) is also 0 since $-4az = b^2 - 4ac = 0$. So,

$$v''(t) = 0$$

$$v'(t) = c_2$$

$$v(t) = c_1 + c_2 t$$

So we choose 0 and 1 and we have

$$y = c_1 e^{-bt/2a} + c_2 t e^{-bt/2a}$$

and y is a linear combination of

$$y_1(t) = e^{-bt/2a}, \qquad y_2(t) = te^{-bt/2a}$$

The Wronskian of these two solutions is

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-bt/2a} & te^{-bt/2a} \\ -\frac{b}{2a}e^{-bt/2a} & (1 - \frac{bt}{2a})e^{-bt/2a} \end{vmatrix} = e^{-bt/2a}$$

which is never zero so these solutions form a fundamental set.
Reduction of Order. If we have an equation

$$y'' + p(t)y' + q(t)y = 0$$

and we have a solution y_1, to find a second solution, let

$$y = v(t)y_1(t)$$

then,

$$y' = v'(t)y_1(t) + v(t)y_1'(t)$$

and

$$y'' = v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t)$$

Substituting for y, y', and y'', and rearranging

$$y_1v'' + (2y_1' + py_1)v' + (y_1''py_1' + qy_1)v = 0$$

Since y_1 is a solution of the equation, the coefficient of v is 0, so

$$y_1v'' + (2y_1' + py_1)v' = 0$$

This is a first order equation of v'. Once v' is found, we can integrate to find v and now we have our second solution $y_2 = vy_1$.