2.5 Autonomous Equations and Population Dynamics

Autonomous equations are those of the form

$$dy/dt = f(y)$$

This form of equation is separable.
Exponential Growth. Exponential growth has an equation of the form

$$dy/dt = ry$$

where the constant of proportionality r is called the **rate of growth or decline**. Solving this with the initial condition

$$y(0) = y_0$$

we obtain

$$y = y_0 e^{rt}$$

For many populations this equation holds true to a certain extent but this is not sustainable since the population would grow rapidly.
Logistic Growth. Since the growth rate actually depending on the population, we replace the constant r with $h(y)$. So,

$$dy/dt = h(y)y$$

The Verhulst equation or the **logistic equation** is of the form

$$dy/dt = (r - ay)y$$

which is the same as the last equation with $h(y) = (r - ay)$ so that $h(y) \approx r$ when y is small and it decreases as y increases. The logistic equation if often written as

$$\frac{dy}{dt} = r(1 - \frac{y}{K})y$$

where $K = r/a$. r is called the **intrinsic growth rate**, the growth rate in the absence of limiting factors.

The constant solutions, or **equilibrium solutions**, occur when

$$\frac{dy}{dt} = r(1 - \frac{y}{K})y = 0$$

or at $y = 0$ and $y = K$. These are also called **critical points**.

Other solutions for this equation always asymptotically approach K as shown.

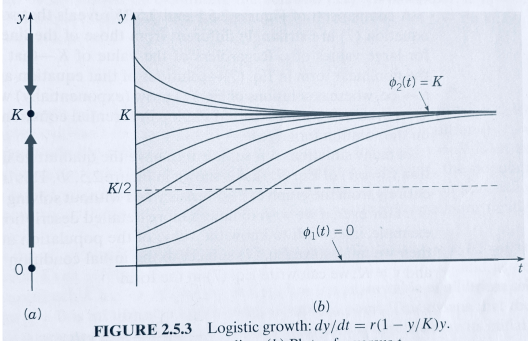

FIGURE 2.5.3 Logistic growth: $dy/dt = r(1 - y/K)y$.
(a) The phase line. (b) Plots of y versus t.