2.1 Linear Equations; Method of Integrating Factors

A **first order linear equation** is an equation that is only linearly dependent on y. Its general form is

$$\frac{dy}{dt} + p(t)y = g(t) \text{ or } P(t)\frac{dy}{dt} + Q(t)y = G(t)$$

Most equations cannot be solved with simple integration, so we can use an **integrating factor** $\mu(t)$. This works for any first order linear equation.

$$\frac{dy}{dt} + ay = g(t)$$

Now, we need to find a a $\mu(t)$ so that

$$\frac{d\mu}{dt} = a\mu$$

which yields $\mu(t) = e^{at}$. Multiplying the original equation by $\mu(t)$

$$e^{at}\frac{dy}{dt} + ae^{at}y = e^{at}g(t)$$

or

$$\frac{d}{dt}(e^{at}y) = e^{at}g(t)$$

By integrating both sides,

$$e^{at}y = \int e^{at}g(t)dt + c$$

$$y = e^{-at}\int_{t_0}^{t} e^{as}g(s)ds + ce^{-at}$$

Now, let's extend this to the general first order linear equation:

$$\frac{dy}{dt} + p(t)y = g(t)$$

Multiply by $\mu(t)$

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t)$$

We know that $\frac{d(\mu(t)y)}{dt} = \mu(t)\frac{dy}{dt} + p(t)\mu(t)y$, if

$$\frac{d\mu(t)}{dt} = p(t)\mu(t)$$

if $p(t) > 0$

$$\frac{d\mu(t)/dt}{\mu(t)} = p(t)$$

$$\ln \mu(t) = \int p(t)dt + k$$

By choosing k to be 0, we obtain the simplest possible $\mu(t)$

$$\mu(t) = \exp \int p(t)dt$$

Now,

$$\frac{d}{dt}[\mu(t)y] = \mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t)$$

By integrating,

$$\mu(t)y = \int \mu(t)g(t)dt + c$$

$$y = \frac{1}{\mu(t)}[\int_{t_0}^{t} \mu(t)g(s)ds + c]$$