2.5 Autonomous Equations and Population Dynamics

Autonomous equations are those of the form

$$dy/dt = f(y)$$

This form of equation is separable.
Exponential Growth. Exponential growth has an equation of the form

$$dy/dt = ry$$

where the constant of proportionality r is called the **rate of growth or decline**. Solving this with the initial condition

$$y(0) = y_0$$

we obtain

$$y = y_0 e^{rt}$$

For many populations this equation holds true to a certain extent but this is not sustainable since the population would grow rapidly.
Logistic Growth. Since the growth rate actually depending on the population, we replace the constant r with $h(y)$. So,

$$dy/dt = h(y)y$$

The Verhulst equation or the **logistic equation** is of the form

$$dy/dt = (r - ay)y$$

which is the same as the last equation with $h(y) = (r - ay)$ so that $h(y) \approx r$ when y is small and it decreases as y increases. The logistic equation if often written as

$$\frac{dy}{dt} = r(1 - \frac{y}{K})y$$

where $K = r/a$. r is called the **intrinsic growth rate**, the growth rate in the absence of limiting factors.

The constant solutions, or **equilibrium solutions**, occur when

$$\frac{dy}{dt} = r(1 - \frac{y}{K})y = 0$$

or at $y = 0$ and $y = K$. These are also called **critical points**.

Other solutions for this equation always asymptotically approach K as shown.

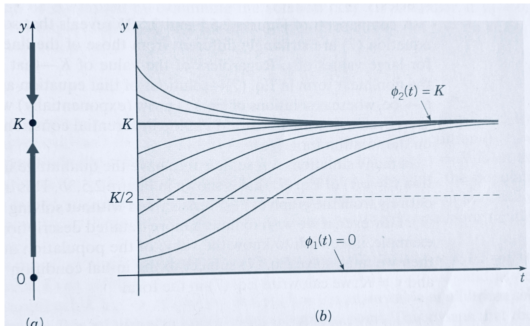

FIGURE 2.5.3 Logistic growth: $dy/dt = r(1 - y/K)y$. (a) The phase line. (b) Plots of y versus t.

K is often referred to as the **saturation level**, or the **environmental carrying capacity**, for a given species.

Sometimes this qualitative knowledge of the solution is enough, but to solve it, we can rewrite the equation as

$$\frac{dy}{(1 - y/K)y} = r\ dt$$

where $y \neq 0$ and $y \neq K$. Using a partial fraction expansion, we have

$$(\frac{1}{y} + \frac{1/K}{1 - y/K})dy = r\ dt$$

By integration,

$$\ln|y| - \ln|1 - \frac{y}{K}| = rt + c$$

since if $0 < y_0 < K$ then y will remain in this interval, we can remove the absolute values. Then by taking the exponential of both sides,

$$\frac{y}{1 - (y/K)} = Ce^{rt}$$

where $C = e^c$. $C = y_0/[1 - (y_0/K)]$ to satisfy $y(0) = y_0$.

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

We notice that

$$\lim_{t \to \infty} y(t) = K$$

hence $y = K$ is an **asymptotically stable solution**, and $y = 0$ is an **unstable equilibrium solution**.

A Critical Threshold. Consider the equation

$$\frac{dy}{dt} = -r(1 - \frac{y}{T})y$$

The function $dy/dt = f(y)$ is a parabola with zeros at $y = 0$ and $y = K$ opening up. For $0 < y < T$, $f(y) < 0$. Hence T is the **threshold level** since no growth occurs below it. Above T, y grows indefinitely. It becomes unbounded in a finite amount of time $t*$.

$$t* = \frac{1}{r} \ln \frac{y_0}{y_0 - T}$$

which we find by setting the denominator of the solution to 0.

Logistic Growth with a Threshold. This equation can be modified so that unbounded growth does not occur when y is above T. We need another factor that makes dy/dt negative when y is large.

$$\frac{dy}{dt} = -r(1 - \frac{y}{T})(1 - \frac{y}{K})y$$

This pulls dy/dt back down after a certain point. $f(y)$ is a cubic function now. The solutions of the equation are similar to the equations with unbounded growth except when $y > T$, y will approach K.