4.1 General Theory of nth Order Linear Equations

An nth order linear differential equation is an equation of the form

$$P_0(t)\frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + P_{n-1}(t)\frac{dy}{dt} + P_n(t)y = G(t)$$

Dividing by $P_0(t)$,

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t)$$

The linear differential operator L of order n is defined above.

For this, we have n initial conditions,

$$y(t_0) = y_0 \; y'(t_0) = y_0' \; \ldots \; y^(n-1)(t_0) = y_0^(n-1)$$

THEOREM 4.1.1. *If the functions $p_1, p_2, \ldots, p_n$, and g are continuous on the open interval I, then there exists exactly one solution $y = \phi(t)$ of the differential equation that also satisfies the initial conditions, where t_0 is any point in I. This solution exists throughout the interval I.*

The Homogeneous Equation.

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0$$

If the functions $y_1, y_2, \ldots, y_n$ are solutions of the previous equation, then it follows by direct computation that the linear combination

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

where $c_1, \ldots, c_n$ are arbitrary constants, is also a solution. This family of solutions encompasses all the solutions for all initial conditions. For this the Wronskian

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

must be non zero at $t = t_0$.

THEOREM 4.1.2. *If the functions $p_1, p_2, \ldots, p_n$ are continuous on the open interval I, if the functions $y_1, y_2, \ldots, y_n$ are solutions, and if $W \neq 0$ for at least one point in I, then every solution can be expressed as a linear combination of the solutions $y_1, y_2, \ldots, y_n$.*

The set is called a **fundamental set of solutions** if the Wronskian is 0. The **general solution** is a linear combination of these with arbitrary constants.

Linear Dependence and Independence. $f_1, f_2, \ldots, f_n$ are said to be **linearly dependent** if for a set of constants $k_1, k_2, \ldots, k_n$, not all zero,

$$\sum_{i=0}^{n} k_i f_i(t) = 0$$

These functions are **linearly independent** if they are not linearly dependent.

THEOREM 4.1.3. *If $y_1(t), y_2(t), \ldots, y_n(t)$ is a fundamental set of solutions, then it is linearly independent. If a set is linearly independent, they form a fundamental set*

Proof. Since its a fundamental set the Wronskian is nonzero, so the only solution of the linear dependence condition is if all the ks are zero, so it is linearly independent. □

The Nonhomogeneous Equation.

$$L[y] = g(t)$$

If Y_1 and Y_2 are two solutions of the nonhomogeneous equation, then

$$L[Y_1 - Y_2] = L[Y_1](t) - L[Y_2](t) = g(t) - g(t) = 0$$

hence the difference is a solution to the homogeneous equation.

So the general solution to a nonhomogeneous equation is

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + Y(t)$$