Chapter 2

First Order Differential Equations

2.1 Linear Equations; Method of Integrating Factors

A **first order linear equation** is an equation that is only linearly dependent on y. Its general form is

$$\frac{dy}{dt} + p(t)y = g(t) \text{ or } P(t)\frac{dy}{dt} + Q(t)y = G(t)$$

Most equations cannot be solved with simple integration, so we can use an **integrating factor** $\mu(t)$. This works for any first order linear equation.

$$\frac{dy}{dt} + ay = g(t)$$

Now, we need to find a a $\mu(t)$ so that

$$\frac{d\mu}{dt} = a\mu$$

which yields $\mu(t) = e^{at}$. Multiplying the original equation by $\mu(t)$

$$e^{at}\frac{dy}{dt} + ae^{at}y = e^{at}g(t)$$

or

$$\frac{d}{dt}(e^{at}y) = e^{at}g(t)$$

By integrating both sides,

$$e^{at}y = \int e^{at}g(t)dt + c$$

$$y = e^{-at} \int_{t_0}^{t} e^{as} g(s) ds + ce^{-at}$$

Now, let's extend this to the general first order linear equation:

$$\frac{dy}{dt} + p(t)y = g(t)$$

Multiply by $\mu(t)$

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t)$$

We know that $\frac{d(\mu(t)y)}{dt} = \mu(t)\frac{dy}{dt} + p(t)\mu(t)y$, if

$$\frac{d\mu(t)}{dt} = p(t)\mu(t)$$

if $p(t) > 0$

$$\frac{d\mu(t)/dt}{\mu(t)} = p(t)$$

$$\ln \mu(t) = \int p(t)dt + k$$

By choosing k to be 0, we obtain the simplest possible $\mu(t)$

$$\mu(t) = \exp \int p(t)dt$$

Now,

$$\frac{d}{dt}[\mu(t)y] = \mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t)$$

By integrating,

$$\mu(t)y = \int \mu(t)g(t)dt + c$$

$$y = \frac{1}{\mu(t)}[\int_{t_0}^{t} \mu(t)g(s)ds + c]$$

2.2 Solutions of Some Differential Equations

If a differential equation $dy/dx = f(x, y)$ can be written as

$$M(x) + N(y)\frac{dy}{dx} = 0$$

then it is said to be **separable** since it can be written as

$$M(x)dx + N(y)dy = 0$$

and integrated.

Let

$$H_1'(x) = M(x) \qquad H_2'(y) = M(y)$$

so then the previous equation becomes

$$H_1'(x) + H_2'(y)\frac{dy}{dx} = 0$$

By the chain rule:

$$H_2'(y)\frac{dy}{dx} = \frac{d}{dy}H_2(y)\frac{dy}{dx} = \frac{d}{dx}H_2(y)$$

which gives us

$$\frac{d}{dx}[H_1(x) + H_2(y)] = 0$$

By integrating, we get

$$H_1(x) + H_2(y) = c$$

Any differentiable function $y = \phi(x)$ that satisfies $H_1(x) + H_2(y) = c$ is a solution of the original differentiable equation. The differential equation and the initial condition $y(x_0) = y_0$ forms an initial value problem. We can use the initial value to find the correct c:

$$c = H_1(x_0) + H_2(y_0)$$

so

$$c = H_1(x_0) + H_2(y_0) = H_1(x) + H_2(y)$$
$$(H_1(x) - H_1(x_0)) + (H_2(y) - H_2(y_0)) = 0$$
$$\int_{x_0}^{x} M(s)ds + \int_{y_0}^{y} N(s)ds = 0$$

since

$$H_1(x) - H_1(x_0) = \int_{x_0}^{x} M(s)ds \qquad H_2(y) - H_2(y_0) = \int_{y_0}^{y} N(s)ds$$

Note 1: Sometimes the solution to

$$\frac{dy}{dx} = f(x, y)$$

has a constant solution $y = y_0$, which occurs when $f(x, y_0) = 0$ for all x and for y_0. For example,

$$\frac{dy}{dx} = \frac{(y-3)\cos x}{1 + 2y^2}$$

has a solution $y = 3$.

Note 2: Sometimes if a function is non-linear it helps to regard both x and y as functions of a third variable t.

$$\frac{dy}{dt} = \frac{dy/dt}{dx/dt}$$

Note 3: Sometimes it is not easy to solve explicitly for y as a function of x. In these cases, it is better to leave the solution in implicit form.

2.3 Modelling with First Order Equations

Construction of the Model. In this step you translate the physical situation into mathematical terms. Mathematical equations are almost always only an approximate description of the actual process. Sometimes you will conceptually replacement of a discrete process by a continuous one.

Analysis of the Model. In this step, you are either solving the differential equation or finding out as many properties about it as possible. Sometimes further approximations help to solve this equation. THese approximations should be examined from a physical point of view so that it still reflects the physical features of the process.

Comparison with Experiment or Observation. Now you interpret your solution or information in the context in which the problem arose. It should appear physically reasonable. If possible, check the solution with a known point.

2.4 Differences Between Linear and Nonlinear Equations

THEOREM 2.4.1. *If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation*

$$y' + p(t)y = g(t)$$

for each $t \in I$, and that also satisfies the initial condition

$$y(t_0) = y_0$$

where y_0 is an arbitrary prescribed initial value.

Proof. In section 2.1, we showed that a general solution to an equation of this form is

$$y = \frac{1}{\mu(t)}[\int_{t_0}^{t} \mu(s)g(s)ds + c]$$

where $\mu(t) = \exp \int_{t_0}^{t} p(s)ds$. To satisfy the initial condition, we choose $c = y_0$. So,

$$y = \frac{1}{\mu(t)}[\int_{t_0}^{t} \mu(s)g(s)ds + y_0]$$

y is continuous since $\frac{1}{\mu(t)}$ is continuous ($\mu(t)$ is never 0), and the integral of something is differential and hence continuous. □

THEOREM 2.4.2. *Let the functions f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some* *interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the intial value problem*

$$y' = f(t, y), \qquad y(t_0) = y_0$$

Theorem 2.4.2 is the same as Theorem 2.4.1 when

$$f(t, y) = -p(t)y + g(t) \text{ and } \partial f(t, y)/\partial y = -p(t)$$

so the continuity of f and $\partial f/\partial y$ is the same as the continuity of p and g.

Both these theorems show the existence and uniqueness of a solution to the initial value problem. Any solution to a first order differential equation cannot intersect another since otherwise the initial value problem with initial value at that point would have multiple solutions.

Interval of Definition. By Theorem 2.4.1, discontinuities in the solution of

$$y' + p(t)y = g(t)$$

with the initial condition $y(t_0) = y_0$ can only exist where there is a discontinuity in either p or q.

For a nonlinear initial value problem, the interval is hard to determine since it must contain $[t, \phi(t)]$ and $\phi(t)$ is not known.

General Solution. First order linear equations have a general solution containing one arbitrary constant. This is not really true for nonlinear differential equations.

Implicit Solution. First order linear equations have an explicit formula for the solution for $y = \phi(t)$. Nonlinear equations do not, and the best you can do is find

$$F(t, y) = 0$$

involving t and y that satisfy $y = \phi(t)$. Sometimes you can explicitly solve for the solution, but sometimes you must use numeric approximations with an implicit solution.

Graphical or Numerical Construction of Integral Curves. Sometimes when you cannot find the solution analytically, you can use a computer or a graph.

2.5 Autonomous Equations and Population Dynamics

Autonomous equations are those of the form

$$dy/dt = f(y)$$

This form of equation is separable.

Exponential Growth. Exponential growth has an equation of the form

$$dy/dt = ry$$

where the constant of proportionality r is called the **rate of growth or decline**. Solving this with the initial condition

$$y(0) = y_0$$

we obtain

$$y = y_0 e^{rt}$$

For many populations this equation holds true to a certain extent but this is not sustainable since the population would grow rapidly.

Logistic Growth. Since the growth rate actually depending on the population, we replace the constant r with $h(y)$. So,

$$dy/dt = h(y)y$$

The Verhulst equation or the **logistic equation** is of the form

$$dy/dt = (r - ay)y$$

which is the same as the last equation with $h(y) = (r - ay)$ so that $h(y) \approx r$ when y is small and it decreases as y increases. The logistic equation if often written as

$$\frac{dy}{dt} = r(1 - \frac{y}{K})y$$

where $K = r/a$. r is called the **intrinsic growth rate**, the growth rate in the absence of limiting factors.

The constant solutions, or **equilibrium solutions**, occur when

$$\frac{dy}{dt} = r(1 - \frac{y}{K})y = 0$$

or at $y = 0$ and $y = K$. These are also called **critical points**.

Other solutions for this equation always asymptotically approach K as shown.

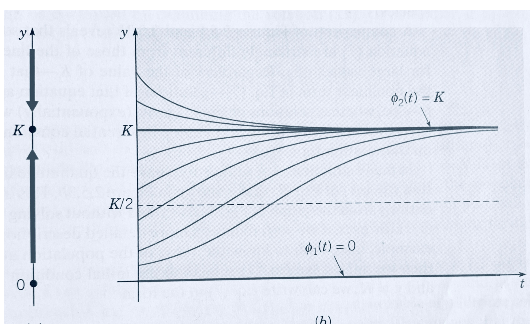


FIGURE 2.5.3 Logistic growth: $dy/dt = r(1 - y/K)y$. (a) The phase line. (b) Plots of y versus t.

K is often referred to as the **saturation level**, or the **environmental carrying capacity**, for a given species.

Sometimes this qualitative knowledge of the solution is enough, but to solve it, we can rewrite the equation as

$$\frac{dy}{(1-y/K)y} = r\ dt$$

where $y \neq 0$ and $y \neq K$. Using a partial fraction expansion, we have

$$(\frac{1}{y} + \frac{1/K}{1-y/K})dy = r\ dt$$

By integration,

$$\ln|y| - \ln|1-\frac{y}{K}| = rt + c$$

since if $0 < y_0 < K$ then y will remain in this interval, we can remove the absolute values. Then by taking the exponential of both sides,

$$\frac{y}{1-(y/K)} = Ce^{rt}$$

where $C = e^c$. $C = y_0/[1-(y_0/K)]$ to satisfy $y(0) = y_0$.

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

We notice that

$$\lim_{t\to\infty} y(t) = K$$

hence $y = K$ is an **asymptotically stable solution**, and $y = 0$ is an **unstable equilibrium solution**.

A Critical Threshold. Consider the equation

$$\frac{dy}{dt} = -r(1-\frac{y}{T})y$$

The function $dy/dt = f(y)$ is a parabola with zeros at $y = 0$ and $y = K$ opening up. For $0 < y < T$, $f(y) < 0$. Hence T is the **threshold level** since no growth occurs below it. Above T, y grows indefinitely. It becomes unbounded in a finite amount of time $t*$.

$$t* = \frac{1}{r}\ln\frac{y_0}{y_0 - T}$$

which we find by setting the denominator of the solution to 0.

Logistic Growth with a Threshold. This equation can be modified so that unbounded growth does not occur when y is above T. We need another factor that makes dy/dt negative when y is large.

$$\frac{dy}{dt} = -r(1-\frac{y}{T})(1-\frac{y}{K})y$$

This pulls dy/dt back down after a certain point. $f(y)$ is a cubic function now. The solutions of the equation are similar to the equations with unbounded growth except when $y > T$, y will approach K.

2.6 Exact Equations and Integrating Factors

If we have a differential equation:

$$M(x,y) + N(x,y)y' = 0$$

find a function $\psi(x,y)$ such that

$$\frac{\partial\psi}{\partial x}(x,y) = M(x,y), \qquad \frac{\partial\psi}{\partial y}(x,y) = N(x,y)$$

Then,

$$M(x,y) + N(x,y)y' = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\psi[x,\phi(x)] = 0$$

where $y = \phi(x)$ is the solution. By integration we get

$$\psi(x,y) = c$$

which implicitly defines the solutions for the original differential equation. In this case the original differential equation is an **exact** differential equation.

THEOREM 2.6.1. *Let the functions M, N, M_y, N_x, where the subscripts denote partial derivatives be continuous in the rectangular region $R : \alpha < x < \beta, \gamma < y < \delta$. The differential equation*

$$M(x,y) + N(x,y)y' = 0$$

is an exact differential equation in R if and only if

$$M_y(x,y) = N_x(x,y)$$

for each point in R. That is, there exists a function ψ such that

$$\frac{\partial\psi}{\partial x}(x,y) = M(x,y), \qquad \frac{\partial\psi}{\partial y}(x,y) = N(x,y)$$

if and only if M and N satisfy that constraint.

Proof. Computing M_y and N_x, we get,

$$M_y(x,y) = \psi_{xy}(x,y), \qquad M_y(x,y) = \psi_{yx}(x,y)$$

and $\psi_{xy}(x,y) = \psi_{yx}(x,y)$ so if the equation is exact, then $M_y(x,y) = N_x(x,y)$. Now, we must prove the other way around.

We need to find a ψ so that

$$\psi_x\,(x,y) = M(x,y), \qquad \psi_y\,(x,y) = N(x,y)$$

By integrating the first half the the equation above we get

$$\psi(x,y) = Q(x,y) + h(y)$$

where

$$Q(x,y) = \int_{x_0}^{x} M(s,y)ds$$

and $h(y)$ acts as a constant (with respect to x). Now we choose h to satisfy

$$\psi_y(x,y) = \frac{\partial Q}{\partial y}(x,y) + h'(y) = N(x,y)$$

So we have

$$h'(y) = N(x,y) - \frac{\partial Q}{\partial y}(x,y)$$

For this equation to be true the right side of the equation must be only a function of y, so the partial derivative with respect to x should be 0.

$$\frac{\partial N}{\partial x}(x,y) - \frac{\partial}{\partial x}\frac{\partial Q}{\partial y}(x,y) = 0$$

$$\frac{\partial N}{\partial x}(x,y) - \frac{\partial}{\partial y}\frac{\partial Q}{\partial x}(x,y) = 0$$

And we know $\frac{\partial Q}{\partial x}(x,y) = \frac{\partial \psi}{\partial x}(x,y) = M(x,y)$. So,

$$\frac{\partial N}{\partial x}(x,y) - \frac{\partial M}{\partial y}(x,y) = 0$$

and

$$M_y(x,y) = N_x(x,y)$$

$h(y)$ can be found be integrating $N(x,y) - \frac{\partial Q}{\partial y}(x,y)$. □

Integrating Factors. Sometimes if an equation is not exact, then it is possible to use an integrating factor $\mu(x,y)$ to make it exact. If we have,

$$M(x,y) + N(x,y)y' = 0$$

and we multiply it by $\mu(x,y)$,

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)y' = 0$$

By Theorem 2.6.1, this equation is exact if and only if

$$(\mu M)_y = (\mu N)_x$$

By the product rule, we get another differential equation

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0$$

Solving this gets you $\mu(x,y)$, which will make the original equation exact, so you could solve that too.

If $\mu(x,y) = \mu(x)$ is only a function of x, we can set $\mu_x = \frac{d\mu}{dx}$ and $\mu_y = 0$. So we get

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$$

which is both linear and separable.

2.7 Numerical Approximations: Euler's Method

Sometimes analytical solutions are not possible so we can use numerical approximations to get close to the actual solution. One of these approximations is called **Euler's Method**. The basis of Euler's Method is using the differential equation to create a tangent line at the initial value (t_0, y_0), and use it to create an approximation for another point (t, y).

$$y = y_0 + f(t_0, y_0)(t - t_0)$$

where $\frac{dy}{dt} = f(t_0, y_0)$. For a better approximation we can repeat this n times instead of just doing it once. So, if we needed to find the solution at t, we could use Euler's Method with n steps by

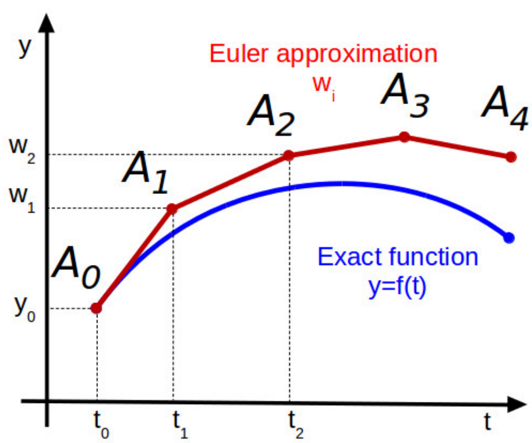
$$y_{i+1} = y_i + f(t_i, y_i)(t_{i+1} - t_i)$$

now if we use equal intervals

$$y_{i+1} = y_i + f(t_i, y_i)h$$

where h is $(t - t_0)/n$. Or,

$$y = y_0 + \sum_{i=1}^{n} f(t_i, y_i)h$$



As you increase n, the approximation gets better. The Euler method uses the solution that passes through step to approximate the solution that we are looking for. For converging solutions, this works well since all the solutions converge to similar values, while this causes large errors for diverging situations since each step takes you further from the solution.

2.8 The Existence and Uniqueness Theorem

Consider the initial value problem

$$y' = f(t, y), \qquad y(0) = 0$$

If an initial value problem is not of this form, we can apply a translation of the coordinate axes that will take (t_0, y_0) to the origin.

THEOREM 2.8.1. *If f and $\frac{\partial f}{\partial y}$ are continuous in a rectangle $R : |t| \leq a, |y| \leq b$, then there is some interval $|t| \leq h \leq a$ in which there exists a unique solution $y = \phi(t)$ of the initial value problem.*

If we integrate the initial value problem equation, we get

$$\phi(t) = \int_0^t f[s, \phi(s)]ds$$

This is called the **integral equation**, which is equivalent to the initial value equation.

One method of showing that the integral equation has a unique solution is the **method of successive approximations** or Picard's **iteration method**. In using this method, we start by choosing an initial function ϕ_0, either arbitrarily or to approximate the solution. The simplest choice is

$$\phi_0(t) = 0$$

ϕ_0 satisfies the initial condition, but probably not the differential equation. The next apprximation ϕ_1 is obtained using ϕ_0.

$$\phi_1(t) = \int_0^t f[s, \phi_0(s)]ds$$

and, in general,

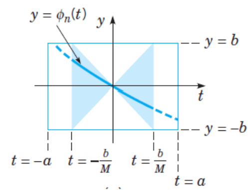
$$\phi_{n+1}(t) = \int_0^t f[s, \phi_n(s)]ds$$

This creates a sequence of functions $\phi_n = \phi_0, \phi_1, \phi_2, \ldots, \phi_n, \ldots$. Each element of the sequence satisfies the initial condition, but in general none satisfies the differential equation. If at some stage $\phi_{k+1}(t) = \phi_k(t)$, then it follows that ϕ_k is a solution and does follow the differential equation.

1. **Do all members of the sequence ϕ_n exist?**
   If f and $\frac{\partial f}{\partial y}$ are continuous in the whole ty-plane, then each ϕ_n is known to exist and can be calculated. But if f and $\frac{\partial f}{\partial y}$ are only assumed continuous in a rectangle $R : |t| \leq a, |y| \leq b$, then some members of the sequence cannot be explicitly determined. If we restrict t to a smaller intercal $|t| \leq a$, we can avoid this danger. Since f must be bounded on R

   $$|f(t, y)| \leq M \qquad (t, y) \text{ in } R$$

   Since $f[t, \phi_k(t)] = \phi'_{k+1}(t)$, the maximum slope of $y = \phi_{k+1}(t)$ is M. So ϕ_{k+1} must lie in R as long as R contains this region:



which is for $|t| \leq b/M$.

2. **Does the sequence $\phi_n(t)$ converge?**
We can identify $\phi_n(t) = \phi_1(t) + [\phi_2(t) - \phi_1(t)] + [\phi_3(t) - \phi_2(t)] + \cdots + [\phi_n(t) - \phi_{n-1}(t)]$ as the nth partial sum of the series

$$\phi_1(t) + \sum_{k+1}^{\infty} [\phi_{k+1}(t) - \phi_k(t)]$$

The convergence of the sequence $\phi_n(t)$ is established by showing that this series converges.
TODO: PROVE THIS USING PROBLEMS 15-18
We denote the limit function by ϕ, so that

$$\phi(t) = \lim_{n \to \infty} \phi_n(t)$$

3. **What are the properties of the limit function ϕ?**
We know that ϕ is continuous since the sequence $\{phi_n$converges in a certain manner, known as uniform convergence. We proof we used proves this as well. Now let us return to

$$\phi_{n+1}(t) = \int_0^t f[s, \phi_n(s)] ds$$

Allowing n to approach ∞ on both sides, we obtain

$$\phi(t) = \lim_{n \to \infty} \int_0^t f[s, \phi_n(s)] ds$$

We can move the limit inside the integral since the sequence converges uniformly.

$$\phi(t) = \int_0^t \lim_{n \to \infty} f[s, \phi_n(s)] ds$$

Then we take it inside the function

$$\phi(t) = \int_0^t f[s, \lim_{n \to \infty} \phi_n(s)] ds$$

so

$$\phi(t) = \int_0^t f[s, \phi(s)] ds$$

Moving the limit inside the function is saying that f is continuous in its second variable, which is known. So this last equation shows that ϕ satisfies the integral equation so it is a solution for the initial value problem.

4. **Are there other solutions of the integral equation besides** $y = \phi(t)$**?**
   Assume another solution $y = \psi(t)$. It can be shown that

$$|\phi(t) - \psi(t)| \leq A \int_0^t |\phi(s) - \psi(s)| ds$$

TODO: Prove this using PROBLEM 19
for $0 \leq t \leq h$ and a suitable positive number A. It is now convenient to introduce U as

$$U(t) = \int_0^t |\phi(s) - \psi(s)| ds$$

$$U(0) = 0$$
$$U(t) \geq 0, \quad \text{for } t \geq 0$$

U is differentiable, and $U'(t) = |\phi(t) - \psi(t)|$.

$$U'(t) - AU(t) \leq 0 \text{ for } 0 \leq t \leq A/2$$

Multiplying by e^{-At} gives

$$[e^{-At}U(t)]' \leq 0 \text{ for } 0 \leq t \leq A/2$$

Then, by integration from 0 to t

$$e^{-At}U(t) \leq 0 \text{ for } 0 \leq t \leq A/2$$

Hence $U(t) \leq 0$ for $0 \leq t \leq A/2$, but since A is arbitrary $U(t) \leq 0$ for all positive t. Since $U(t) \geq 0$, $U(t) = 0$, so $U'(t) = 0$ and $\psi(t) = \phi(t)$. This proves there connot be two different solutions to one initial value problem for $t \geq 0$. A slight modification of this argument shows the same is true for $t \leq 0$.

2.9 First Order Difference Equations

For some problems, we need to model a discrete process. Differential equations model a continuous process, while **difference equation**s model a discrete process. An example is modelling population growth of a species whose generations do not overlap. The population y_{n+1} of the species in the year $n + 1$ is some function of n and the population y_N in the preceding year:

$$y_{n+1} = f(n, y_n) \qquad n \in \mathbf{N}$$

This is a **first order difference equation** since it only depends on the step before it. The equation is **linear** if f is a linear function of y_n; otherwise, it is **nonlinear**. A **solution** of the difference equation is a sequence of numbers $y_0, y_1, y_2, \ldots$ that satisfy the equation for each n. There also must be an **initial condition**.

$$y_0 = \alpha$$

Now for convenience, temporarily let $f(y_n) = f(n, y_n)$.

$$y_1 = f(y_0)$$

$$y_2 = f(y_1) = f[f(y_0)]$$
$$y_1 = f(y_2) = f[f[f(y_0)]] = f^3(y_0)$$
$$y_n = f(y_{n-1}) = f^n(y_0)$$

Solutions for which y_n has the same value for all n are called **equilibrium solutions**. You can find this by setting $y_{n+1} = y_n$, and solve

$$y_n = f(y_n)$$

for y_n.

Linear Equations. These could be of the form

$$y_{n+1} = \rho_n y_n \qquad n \in \mathbf{N}$$

THe reproduction rate rho_n may differ from year to year. This can easily be solved by iteration. We obtain

$$y_1 = \rho_0 y_0$$

$$y_2 = \rho_1 y_1 = \rho_0 \rho_1 y_0$$

and, in general,

$$y_n = (\Pi_{i=0}^{n-1} \rho_i) y_0$$

If the reproduction rate is the same for all n, $\rho_n = \rho$, then

$$y_{n+1} = \rho y_n$$

$$y_n = \rho^n y_0$$

The limiting behavior can easily be determined.

$$\lim_{n \to \infty} y_n = \begin{cases} 0, & \text{if } |\rho| < 1; \\ y_0, & \text{if } |\rho| = 1; \\ \text{does not exist}, & \text{if } |\rho| > 1; \end{cases}$$

The equilibrium solution $y_n = 0$ is asymptotically stable for $|\rho| < 1$ and unstable for $|\rho| > 1$.

If a population has immigration or emigration, we must have a b_n as the net increase in population in year n due to immigration.

$$y_{n+1} = \rho y_n + b_n$$

We can solve for y_n through iteration as well.

$$y_1 = \rho y_0 + b_0$$

$$y_2 = \rho(\rho y_0 + b_0) + b_1 = \rho^2 y_0 + \rho b_0 + b_1$$

$$y_3 = \rho(\rho^2 y_0 + \rho b_0 + b_1) + b_2 = \rho^3 y_0 + \rho^2 b_0 + \rho b_1 + b_2$$

and in general,

$$y_n = \rho^n y_0 + \sum_{j=0}^{n-1} \rho^{n-1-j} b_j$$

The more general solution for linear equations of the form

$$y_{n+1} = \rho_n y_n + b_n$$

then by iterations,

$$y_n = (\Pi_{i=0}^{n-1} \rho_i) y_0 + \sum_{i=0}^{n-1} (\Pi_{j=i+1}^{n-1} \rho_j) b_{n-1}$$

The first term on the right side represents thr descendants of the original population.

If $b_n = b \neq 0$,

$$y_{n+1} = \rho y_n + b$$

and from the previous equation the solution is

$$y_n = \rho^n y_0 + (1 + \rho + \rho^2 + \cdots + \rho^{n-1}) b$$

or for $\rho \neq 1$,

$$y_n = \rho_n y_0 + \frac{1 - \rho^n}{1 - \rho}$$

which is the same as

$$y_n = \rho^n (y_0 - \frac{b}{1 - \rho}) + \frac{b}{1 - \rho}$$

which makes the limit behavior of y_n more clear. For $\rho = 1$,

$$y_n = y_0 + nb$$

The limit behavior is defined by

$$\lim_{n \to \infty} y_n = \begin{cases} b/(1 - \rho), & \text{if } |\rho| < 1; \\ \textit{unbounded}, & \text{if } |\rho| = 1; \\ \text{does not exist}, & \text{if } |\rho| > 1; \end{cases}$$

when $|\rho|$ is not less than 1, it converges to y_0, if $y_0 = b/(1-\rho)$ since this is an equilibrium solution.

The model can also be used for interest, where $\rho_n = 1 + r_n$, where r_n is the interest rate, and b_n is the amount deposited or withdrawn.

Nonlinear Equations. Consider the logistic difference equation

$$y_{n+1} = \rho y_n (1 - \frac{y_n}{k})$$

If we scale y_n to $u_n = y_n/k$

$$u_{n+1} = \rho u_n (1 - u_n)$$

$(\rho = k\rho)$, a positive parameter. We can find an equilibrium solution by setting $u_{n+1} = u_n$.

$$u_n = \rho u_n - \rho u_n^2$$

which gives us

$$u_n = 0, \quad u_n = \frac{\rho - 1}{\rho}$$

To check if these equilibrium solutions are asymptotically stable or unstable, we can use linear approximations. Near $u_n = 0$, u_n^2 is small compared to u_n. So,

$$u_{n+1} \approx \rho u_n$$

We already know that this only approaches 0 for $|\rho| < 1$, or $0 < \rho < 1$ since ρ is positive. So $u_n = 0$ is stable for $0 < \rho < 1$.

To test solutions in the neighborhood of $u_n = (\rho - 1)/\rho$, we write

$$u_n = \frac{\rho - 1}{\rho} + v_n$$

where v_n is small. By substituting this into $u_{n+1} = \rho u_n (1 - u_n)$, we get

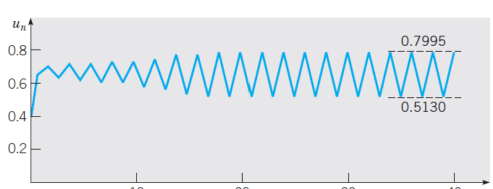
$$v_{n+1} = (2 - \rho) v_n - \rho v_n^2$$

Since v_n is small, $v_n^2 \approx 0$.

$$v_{n+1} = (2 - \rho) v_n$$

In the same manner we know $v_n \to 0$ as $n \to \infty$ for $|2 - \rho| < 1$, or $1 < \rho < 3$. Therefore, in this range of values for ρ, $u_n = (\rho - 1)/\rho$ is an asymptotically stable equilibrium solution. $\rho = 1$ is an **exchange of stability** from one equilibrium solution to the other.

If $\rho > 3$, the solution will oscillate between two values; it is period 2. At about $\rho = 3.449$, the solution becomes periodic with period 4. The appearance of a new solution at a certain parameter value is called a **bifurcation**.



The ρ-values at which the successive period doublings occur approach a limit that is approximately 3.57, so for $\rho > 3.57$, the solutions have some regularity. It's fine structure is unpredictable, hence the term **chaotic**. One of the features of chaotic solutions is extreme sensitivity to initial conditions.