1.4 Consequences of Completeness

THEOREM 1.4.1 (Nested Interval Property)**.** *For each $n \in \boldsymbol{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \boldsymbol{R} : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \dots$$

has a nonempty intersection; that is, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. In order to show $\bigcap_{n=1}^{\infty} I_n$ is not empty, we are going to use the Axiom of Completeness to produce a single real number x satisfying $x \in I_n$ for every $n \in \mathbf{N}$. Consider the set

$$A = \{a_n : n \in \mathbf{N}\}$$

of left-hand endpoints of the intervals. Since the intervals are nested, every b_n is an upper bound for A. let $x = \sup A$. Consider a particular $I_n = [a_n, b_n]$. Since x is an upper bound for A, $a_n \leq x$. Since x is the least upper bound and each b_n are upper bounds, $x \leq b_n$. So $a_n \leq x \leq b_n$ for any n. So $x \in I_n$ for any $n \in \mathbf{N}$. Hence, $x \in \bigcap_{n=1}^{\infty} I_n \neq \emptyset$. □

The Density of Q in R

THEOREM 1.4.2 (Archimedean Property)**.** *(i) Given any number $x \in \boldsymbol{R}$, there exists an $n \in \boldsymbol{N}$ satisfying $n > x$.*

(ii) Given any real number $y > 0$, there exists an $n \in \boldsymbol{N}$ satisfying $1/n < y$.

Proof. Part (i) states that $\mathbf{N}$ is not bounded above. Assume, for contradiction, that $\mathbf{N}$ is bounded above. By AoC, $\mathbf{N}$ has a least upper bound. Let $\alpha = \sup N$. $\alpha - 1$ is not an upper bound, so there is an $n \in \mathbf{N}$, such that $\alpha - 1 < n$, which is the same as saying $\alpha < n + 1$. $n + 1 \in \mathbf{N}$, we have a contradiction to the fact α is an upper bound.

Part (ii) follows from (i) by letting $x = 1/y$. □

THEOREM 1.4.3 (Density of Q in R)**.** *For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.*

Proof. To simplify matters, let's assume $0 \leq a < b$. A rational number is a quotient of integers, so we must product m, $n \in \mathbf{N}$ so that

$$a < \frac{m}{n} < b$$

First, we must choose a large enough n so that an increment of size $1/n$ is small enough so it doesn't step over the interval (a, b). Basically, we need an $n \in \mathbf{N}$ such that

$$\frac{1}{n} < b - a$$

By the first inequality, we can get $na < m < nb$. With n chosen, we need to choose an m to be the smallest natural number greater than na. So,

$$m - 1 \leq na < m$$

which yields $a < m/n$. And $a < b - 1/n$ from the second inequality. So

$$m \leq na + 1 < n(b - \frac{1}{n}) + 1 = nb$$

Because $m < nb$ so $m/n < b$. Now we have $a < m/n < b$. □

Collary *Given any two real numbers $a < b$, there exists an irrational number t satisfying $a < t < b$*

The Existence of Square Roots

THEOREM 1.4.4. *There exists a real numbers $\alpha \in \boldsymbol{R}$ satisfying $\alpha^2 = 2$.*

Proof. Consider the set

$$T = \{t \in \mathbf{R} : t^2 < 2\}$$

and set $\alpha = \sup T$. If $\alpha^2 < 2$. NEED TO FINISH THIS PROOF. □

Countable and Uncountable Sets

Cardinality
Cardinality refers to the size of a set. The cardinalities of finite sets can be compared by attaching a natural number to each set. By using comparisons rather than just length, this idea extends to infinite sets.

Definition A function $f : A \to B$ is one-to-one (1-1) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if given any $b \in B$, it is possible to find the element $a \in A$ such that $f(a) = b$. **Definition** Two sets A and B havethe same cardinality if there exists $f : A \to B$ that is 1-1 and onto. In this case, we write $A \sim B$.
Countable Sets
Definition A set A is *countable* if $N \sim A$. AN infinite set that is countable is called an *uncountable* set.

THEOREM 1.4.5. *(i) The set $\boldsymbol{Q}$ is countable*
(ii) The set $\boldsymbol{R}$ is uncountable

Proof. (i) For each $n \in \mathbf{N}$, let

$$A_n = \{\pm\frac{p}{q} : \text{where } p, q \in \mathbf{N} \text{ are in lowest terms with } p + q = n\}$$

so

$$A_1 = \{\frac{0}{1}\}, \quad A_2 = \{\frac{1}{1}, \frac{-1}{1}\}, \quad A_3 = \{\frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1}\}$$

Our one to one correspondence from $\mathbf{N}$ to $\mathbf{Q}$ is by listing the elements from $\bigcup_{n=1}^{\infty} A_n$. So, $f(n) = (\bigcup_{n=1}^{\infty} A_n)[n]$. For any fraction, like 22/7, it will be in $\bigcup_{n=1}^{\infty} A_n$ exactly once ($22/7 \in A_29$). This makes $\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_n \cap A_m = \emptyset$. So, $\mathbf{N} \sim \mathbf{Q}$ and $\mathbf{Q}$ is countable.

(ii) Proof by contradiction. Assume there exists a 1-1 from $\mathbf{N}$ to $\mathbf{R}$. If we let $x_n = f(n)$ for each $n \in \mathbf{N}$, we can write

$$\mathbf{R} = \{x_1, x_2, x_3, \dots\}$$

Let I_1 be a closed interval that does not contain x_1. Then create infinite intervals based on the following rules. Given an I_n, construct I_{n+1} to satisfy

$$\text{(i) } I_{n+1} \subseteq I_n \text{ and}$$

$$\text{(ii) } x_{n+1} \notin I_{n+1}.$$

Given I_n, it is clear that I_{n+1} exists since I_n certainly contains two smaller disjoint closed intervals and x_{n+1} can only be in one of them. Since $x_{n_0} \notin I_{n_0}$,

$$x_{n_0} \notin \bigcap_{n=1}^{\infty} I_n$$

This is true for every natural number n_0, and hence every real number x_{n_0}, so

$$\bigcap_{n=1}^{\infty} I_n = \emptyset$$

which contradicts the Nested Interval Property, which asserts that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Due to this contradiction, $\mathbf{R}$ cannot be countable, and is uncountable. $\square$

Since $\mathbf{R} = \mathbf{Q} \cup \mathbf{I}$, where $\mathbf{I}$ is all irrational numbers, $\mathbf{I}$ cannot be countable because otherwise $\mathbf{R}$ would be.

THEOREM 1.4.6. *If $A \subseteq B$ and B is countable, then A is either countable, finite, or empty.*

THEOREM 1.4.7. *(i) If A_1, A_2, $\dots A_m$ are each countable sets, then the union $\bigcup_{n=1}^{m} A_n$ is countable.*
(ii) If A_n is a countable set for each $n \in \boldsymbol{N}$, then $\bigcup_{n=1}^{\infty} A_n$ is countable.