1.1 Discussion: the Irrationality of $\sqrt{2}$

THEOREM 1.1.1. *There is no rational number whose square is 2.*

Proof. A rational number can be written in the form $\frac{p}{q}$ where p and q are integers. We will use an indirect proof. First, assume there is a rational so that its square is 2. It can be written that

$$(\frac{p}{q})^2 = 2$$

We can assume p and q have no common factors since they would cancel anyways and give us a new p and q. Now we can written

$$p^2 = 2q^2$$

which implies that p^2 is an even number, which implies p is an even number. So we can let $p = 2r$. Plugging this in

$$2r^2 = q^2$$

With the same logic as for with p, q is also even. So p and q share a common factor of 2 which contradicts the assumption made in the beginning that they share no common factors. $\square$

Important number systems as sets

Natural Numbers

$$\mathbf{N} = \{1, 2, 3, 4, 5, \dots\}$$

Addition works well he, but there is no additive identity or inverse.

Integers

$$\mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

This includes the additive identity (0) and the additive inverses, which define subtraction. The multiplicative identity is 1, but for multiplicative inverses we need to extend to ...

Rational Numbers

$$\mathbf{Q} = \{\text{all fractions } \frac{p}{q} \text{ where } p \text{ and } q \text{ are integers and } q \neq 0\}$$

The multiplicative inverses define division. All of these properties of **Q** make it into a *field*. A field is any set where addition and multiplication are well-defined operations that are commutative, associative, and obey the distributive property: $a(b + c) = ab + bc$. There must be an additive and multiplicative identity, and each element must have an additive and multiplicative inverse.

The set $\mathbf{Q}$ has a natural *order*. Given two rational numbers r and s, one of the following is true:

$$r < s,\ r = s, \text{ or } r > s$$

This ordering is transitive: if $r < s$ and $s < t$, then $r < t$. Also, between any two rational numbers, r and s, there is a rational number between them: $\frac{r+s}{2}$, which implies that rational numbers are densely packed.

$\mathbf{Q}$ is has holes in the spots of irrationals, such as $\sqrt{2}$ and $\sqrt{3}$. To fill these we add ...

Real Numbers

$$\mathbf{R} = \{\text{all real numbers}\}$$

Just like $\mathbf{Q}$, $\mathbf{R}$ is a field. $\mathbf{R}$ is added as a superset of $\mathbf{Q}$. $\mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R}$.