1.2 Some Preliminaries

Sets

A *set* is a collection of object, usually real numbers. The objects that make up the set are *elements*.

Notation

- $x \in A$ means x is in A
- $A \cup B$ (union of A and B) is defined by: if $x \in A \cup B$ then $x \in A$ or $x \in B$ (or both)
- $A \cap B$ (intersection of A and B) is defined by: if $x \in A \cap B$ then $x \in A$ and $x \in B$
- $\emptyset$ is an *empty set*, or a set without any elements in it
- if $A \cap B = \emptyset$, then A and B are *disjoint*
- $A \supseteq B$ or $B \subseteq A$ every element of B is in A so for each $x \in B$, $x \in A$. So B is a *subset* of A, or A *contains* B
- $A = B$ means each element of $A \subseteq B$ and $B \subseteq B$. So the sets are the same.
- $\bigcup_{n=1}^{\infty} A_n$ or $\bigcup_{n \in \mathbf{N}} A$ means $A_1 \cup A_2 \cup \cdots \cup A_\infty$
- $\bigcap_{n=1}^{\infty} A_n$ or $\bigcap_{n \in \mathbf{N}} A$ means $A_1 \cap A_2 \cap \cdots \cap A_\infty$
- $A^c = \{x \in \mathbf{R} : x \notin A\}$

You can define a set by listing items ($N = \{1, 2, 3, \dots\}$), with words (let E be all even natural numbers), or with a rule or algorithm ($S = \{r \in \mathbf{Q} : r^2 < 2\}$).

De Morgan's Laws
$(A \cap B)^c = A^c \cup B^c$ and $(A \cup B)^c = A^c \cap B^c$

Functions

Given two sets A and B, a *function* from A to B is a rule or mapping that takes each element $x \in A$ to a single element in B. We can write $f\colon A \to B$. Given $x \in A$, $f(x)$ represents an element of B associated with x by f. A is the domain of f. The range is a subset of B.

Triangle Inequality

Absolute Value Function:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The Absolute Value Function satisfies:

$$|ab| = |a||b|$$

$$|a + b| \leq |a| + |b|$$

Logic and Proofs

A type of indirect proof previously used is *proof by contradiction*,which starts by negating what we are proving and then finding a contradiction. Most proofs are direct, which means it starts from a true statement and then gets to the theorems conclusion.

THEOREM 1.2.1. *Two real numbers a and b are equal if and only if for every real number* $\epsilon > 0$ *it follows that* $|a - b| < \epsilon$

Proof. Must prove both:
$\Rightarrow$ If $a = b$, then for every real number ϵ it follows that $|a - b| < \epsilon$.
If $a = b$, then $|a - b| = 0$, and $|a - b| < \epsilon$ for any $\epsilon > 0$.

$\Leftarrow$ If for every real number $\epsilon > 0$ if follows that $|a - b| < \epsilon$, then we must have $a = b$.
Assume $a \neq b$,
let $\epsilon_0 = |a - b| > 0$ since $a \neq b$
But $|a - b| = \epsilon_0$ contradicts $|a - b| < \epsilon_0$, which was given. So $a \neq b$ is unacceptable, and a must equal b. $\square$

Induction

The fundamental principle behind induction is that if S is a subset of $\mathbf{N}$ so that S contains 1 and if S contains n, then S contains $n + 1$, then by induction $S = \mathbf{N}$.