Differential Equations and Complex Analysis Notes

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Chapter 1

Introduction

1.1 Newton's Laws

Isaac Newton introduced the three basic laws of mechanics that are known as Newton's Laws.

Newton's Second Law

$$\vec{\mathbf{F}}_{net} = m\vec{\mathbf{a}}$$

Sometimes this equation is written as

$$\vec{\mathbf{F}}_{net} = \frac{d\vec{\mathbf{p}}}{dt}$$

to account for a changing mass, where $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$.

Newton's First Law

Newton's first law is a special case of the second law:

$$\text{If } \vec{\mathbf{F}}_{net} = \vec{\mathbf{0}}, \text{ the motion is uniform}$$

which means that velocity is constant and accusation is 0.

Gravity

Acceleration caused by gravity is the same on every object. This seems contradictory to what we have seen from Newton's second law, $\vec{\mathbf{a}} = \vec{\mathbf{F}}_{new}/m$, but since the force of gravity is proportional to the mass, the acceleration is independent of the mass.

Hooke's Law

The equation for spring force is $F = -kx$. So,

$$-kx = m\frac{d^2x}{dt}$$

The solution of this differential equation is

$$x(t) = A\sin(\omega t + \varphi)$$

where A is the *amplitude* of the motion, and φ is the textitphase. Both of these quantities are determined by the initial conditions. ω is determined by the spring and the mass:

$$\omega = \sqrt{k/m}$$

where ω is the *angular frequency*, which is related to the *period* T and *frequency* f.

$$T = \frac{1}{f}$$

so $f = \omega/2\pi$.

Newton's Third Law

Every force has an equal and opposite reaction force.

$$\vec{\mathbf{F}}_12 = -\vec{\mathbf{F}}_21$$

This can be restated in terms of momentum.

$$\frac{d\vec{\mathbf{p_1}}}{dt} = -\frac{d\vec{\mathbf{p_2}}}{dt} \text{ or } \frac{d}{dt}(\vec{\mathbf{p_1}} + \vec{\mathbf{p_2}}) = 0$$

So,

$$\vec{\mathbf{P}}_{tot} = \text{a constant vector}$$

1.2 Work, Energy, and the Conservation of Energy

For an object traveling in one dimension with a constant net force F and with velocity changing linearly with time ($v \propto t$), we have

$$F \times (x_f - x_i) = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

The quantity $mv^2/2$ is called the **kinetic energy** K, while the left side is referred to as the **work** W done by the force. For forces that are not constant we have

$$W = \int_{x_i}^{x_f} F(x)dx$$

To extend to multiple dimensions

$$W = \int_{\vec{\mathbf{r}}_i}^{\vec{\mathbf{r}}_f} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

With these new definitions, the first equation of this section can be written as

$$W_{net} = \Delta K = K_f - K_i$$

aka the **work-energy theorem**. A **conservative force** is a force which the work done by the force is path independent (it only depends on the start and end state).

$$\int_{\vec{\mathbf{r}}_i}^{\vec{\mathbf{r}}_f} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}_f) - f(\vec{\mathbf{r}}_i)$$

We define potential energy for these forces as

$$U(\vec{\mathbf{r}}) = -\int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + U(\vec{\mathbf{r}}_0)$$

So $U(\vec{\mathbf{r}}_f) - U(\vec{\mathbf{r}}_i) = -\int_{\vec{\mathbf{r}}_i}^{\vec{\mathbf{r}}_f} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. We also set **total energy** to

$$E \equiv U(\vec{\mathbf{r}}) + K$$

Important potential energies

$$\text{Local Gravity: } U(h) = mgh$$

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$$\text{Gravitation: } U(r) = -\frac{Gm_1m_2}{r}$$

Given the potential energy, we can find the force using the gradient

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}) = -\vec{\boldsymbol{\nabla}}U$$

$$F(x) = -\frac{dU}{dx} \text{ (in one dimension)}$$

1.3 Rotations and the Center of Mass

The **angular velocity** $\vec{\omega}$ is the rate at which the angle θ changes. Its direction is determined by the right hand rule (Fingers in direction of motion, thumb is direction of $\vec{\omega}$). The rate at which ω changes is the **angular acceleration** vector $\vec{\alpha}$. The rotation eq uivalents for mass and force, are **rotational inertia** I and **torque** $\vec{\tau}$. So Newton's second law for rotation motion is

$$\vec{\tau} = I\vec{\alpha} = \frac{d\vec{\mathbf{L}}}{dt}$$

where $\vec{\mathbf{L}}$ is **angular momentum**, which is defined as

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$$

Torque is similarly defined as

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$$

The rotational inertia I is dependent on the way an objects mass is distributed. By breaking up the object of total mass

$$M = \sum_i \Delta m_i$$

into many discrete portions of mass, we can set

$$I \equiv \sum_i (\Delta m_i) r_i^2 = \int r^2 dm$$

Work one in a rotation rigid body is defined as

$$W = \int_{\theta_i}^{\theta_f} \tau d\theta$$

Kinetic energy of a rotating object is

$$K = \frac{1}{2} I \omega^2$$

The **center of mass** is defined as

$$\vec{\mathbf{R}} = \frac{\sum_i m_i \vec{\mathbf{r}}_i}{M}$$

For an external force moving the object, the mass will act as a point mass at $\vec{\mathbf{R}}$, so

$$\vec{\mathbf{F}}_{\text{net, external}} = M\vec{\mathbf{A}} = M\frac{d^2\vec{\mathbf{R}}}{dt^2}$$

1.4 Elastic Media and Waves

Internal forces that resemble the force in Hooke's law lead to wave motion. This means that if $h(x, t)$ is some variation from equilibrium (one dimension but could be generalized to $\vec{\mathbf{r}}$) then the variable h must satisfy the wave equation.

$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 h}{\partial t^2}$$

This equation arises from applying Newton's second law to the elastic medium. This equation is linear so any linear combination of two known solutions is also a solution (**superposition principle**). v is the **wave speed** and is

$$v = \sqrt{\frac{\text{restoring force factor}}{\text{inertial factor}}}$$

The solution to the wave equation is $h(x,t) = g(x - vt)$ for any function g. h is a **traveling wave**, and it represents some shape described by g moving in the x-direction with velocity v. If g takes a sinusoidal form, h is a **harmonic wave**.

$$h(x,t) = H\sin(k(x - vt)) = H\sin(kx - \omega t)$$

H is the **amplitude**, the maximum departure from equilibrium. k and ω the repeat length and time. The **wavelength** λ (repeat length), **period** T, and frequency $f = 1/T$ are specified by the **wave number** k and **angular frequency** ω.

$$\omega = 2\pi f = \frac{2\pi}{T} \text{ and } k = \frac{2\pi}{\lambda}$$

and wave speed links wavelength and frequency.

$$v = \lambda f$$

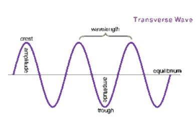
Fourier's theorem states that any wave can be written as a linear combination of harmonic waves.

Another solution for the wave equation is for **standing waves**, with an equation

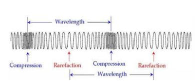
$$h(x,t) = H\sin(kx + \varphi)\cos(\omega t + \delta)$$

These waves oscillate in place rather than traveling. Frequency is still $f = \omega/2\pi$. The wavelength is given by $\lambda = 2L/n$ where n is the number of nodes minus 1.

Waves can either be **transverse** (the displacement is perpendicular to the motion of the wave),



or **longitudinal** (the displacement is parallel to the motion of the wave).



Power and Energy in Waves

The **power** of a wave is the rate at which energy is delivered through a unit area perpendicular to the waves motion. For a harmonic wave propagating on a string of mass density μ,

$$P = \mu v \omega^2 H^2 \cos^2(kx - \omega t)$$

The energy density in the harmonic wave is P/v. Both power and energy density are both traveling waves as well and are proportional to the amplitude squared and the frequency squared.

Reflection and Refraction

Waves that reach boundaries between different media reflect back into the original medium and refract into the new one. The refracted wave is called the **transmitted** wave. The amplitudes of reflected and refracted waves are constrained by the requirement that energy must be conserved.

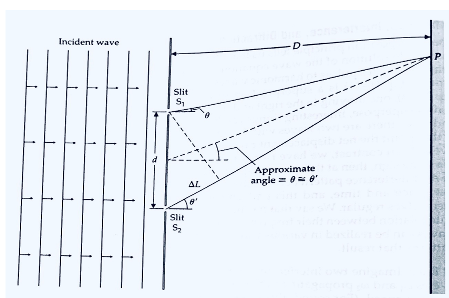
Coherence, Interference, and Diffraction

Interference is essential the superposition of waves. If you have two waves with equal h-values and opposite signs, the superposition of the two waves is destructive interference. If two waves have equal h-values and the same sign, it will be constructive interference. Harmonic waves are **coherent** if there is a definite relation between their frequencies and phases.

Two waves with slightly different angular frequencies ω_1 and ω_2 are interfering in the same medium (with the same speed and amplitude:

$$H\sin(k_1x - \omega_1 t) + H\sin(k_2x - \omega_2 t) = 2H\sin(Kx - \Omega t)\cos\left(\frac{\delta k}{2}x - \frac{\delta\omega}{2}t\right)$$

where $K = (k_1 + k_2)/2$, $\Omega = (\omega_1 + \omega_2)/2$, $\delta k = k_1 - k_2$, and $\delta\omega = \omega_1 - \omega_2$. This results in a product of two waves, one of which has a small angular frequency and wave number $\delta k/2$. The part with the small frequency is the **beat**.



With the setup shown in the figure above, one wave is being split by two slits. ΔL is the difference in the distance required to travel by each "source". The condition for constructive interference is $\Delta L = n\lambda$; the condition for destructive interference is $\Delta L = (n+1/2)\lambda$. n is any integer. Using the approximation ($\theta' = \theta$), $\Delta L = d\sin\theta$, so $d\sin\theta = n\lambda$ shows a pattern of constructive interference.

Gratings are similar to the setup shown in the figure except instead of two sources there are N sources to sharpen the interference pattern.

A wave front can also interfere with itself. Wave front propagation can be thought of as a continual regeneration of "waveletes" that interfere with each other to create the straight-line propagation. When there is a barrier, the constructive interference is no longer present so the wave front will bend. This is called **diffraction**.

The Doppler Shift

This is how the movement of the receiver or emitter of the wave or medium affects the frequency or wavelength of the wave.

Chapter 2

First Order Differential Equations