Mini-project 2: Traffic Jams

CX 4230, Spring 2023[1]

Spring 2023: Due Apr 9 at 11:59 pm

[1] Last updated: `Sun Mar 11 18:28:28 EST 2023`.

In this mini-project, you will implement several models of one-dimensional traffic flow on a freeway, based on models described in the reading,[2]

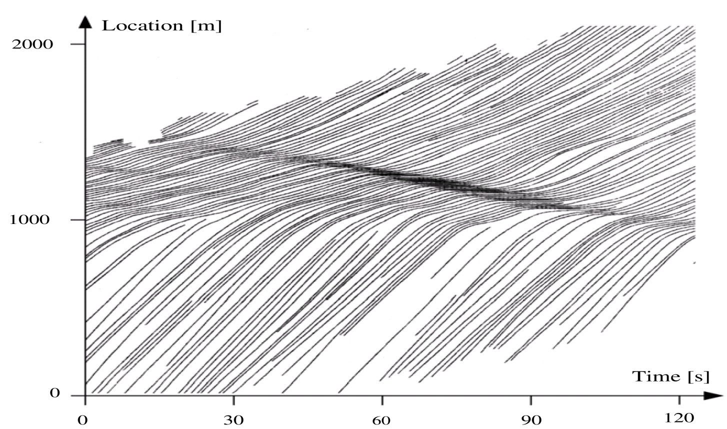
> H-J. Bungartz et al. (2014). *Modeling and simulation: an application-oriented approach*. Springer. [Online text via GT Library Proxy]

[2] Hans-Joachim Bungartz, Stefan Zimmer, Martin Buchholz, and Dirk Pflüger. *Modeling and simulation: an application-oriented introduction*. Springer, 2014

Here are the basic ground rules:

- You may work individually or in pairs. See section 3.
- You can implement your solution in any programming language.
- You will submit a written write-up (PDF file) *and* your source code. See section 4.
- The assignment is due on Sunday, April 9, at 11:59 pm. See section 4 for the late policy.

Context. The goal of this mini-project is to model the behavior of freeway traffic. Such behavior was measured for traffic flowing in one direction on a British highway and is depicted in fig. 1.



The original: trajectories showing a stop-and-go wave on a British motorway segment. [Adapted from: J. Treiterer et al. (1970)]

Figure 1: Traffic trajectories: Each line traces the location of a vehicle (y-axis) over time (x-axis). Regions of slower-moving traffic and jamming correspond with areas where there are higher densities of lines having nearly zero slopes.

Each line of fig. 1 represents the trajectory of a car, that is, its position (y-axis) over time (x-axis). The slope of a line at a given point in time represents a vehicle's instantaneous speed, with lower slopes meaning lower speeds. Additionally, higher densities of lines represent closely spaced cars; the combination of high density and small slopes correspond to traffic jams. You can see such a jam forming in the region between the 1 km and 1.4 km, and it sharpens around time 30 s and persists, appearing to "move backwards." This type of behavior in the study of fluids is referred to as a *shock wave*.

1 Part 1: A PDE-based model (50 points)

Recall the Lighthill-Whitham-Richards (LWR) model presented in class.[3,4] Let the system be described by the state variables,

[3] Lighthill and Whitham 1955

[4] Richards 1956

- $\rho = \rho(x,t)$ be the density of cars at position x and time t, having units of, say, number of vehicles per km;
- $v = v(x,t)$ be the velocity of a car at position x and time t, in units of km/h; and
- $f = f(x,t)$ be the flow of cars at position x and time t, in units of number of vehicles per hour.

These are interrelated by $f = \rho v$, which is consistent with the units. Suppose the initial conditions $\rho(x,0) \equiv \rho_0(x)$, $v(x,0) \equiv v_0(x)$ are given. Furthermore, we assume that vehicles cannot travel faster than $v_{\max}$, and that maximum density of vehicles at any point on the road is $\rho_{\max}$.[5] The general LWR model says the state of the system evolves in time and space according to the partial differential equation (PDE),

[5] The maximum density would be related to, say, the typical length of a vehicle, which determines the largest number of vehicles that can occupy a region of space.

$$\frac{\partial \rho}{\partial t} = -\frac{\partial f}{\partial x}. \tag{1}$$

We make two additional simplifying assumptions. First, we assume that the flow f is an explicit function of ρ only, i.e., $f = f(\rho)$.[6] Second, we assume f has the form of a logistic function,

[6] When imposing this assumption on eq. (1), the resulting model is sometimes called the *shallow-water equation* or transport equation, and it is an instance of a *nonlinear hyperbolic PDE*.

$$f(\rho) = v_{\max}\rho \left(1 - \frac{\rho}{\rho_{\max}}\right). \tag{2}$$

Substituting eq. (2) into eq. (1) yields

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\frac{\partial f}{\partial x} && (3)\\
&= -\left(\frac{df}{d\rho}\right)\frac{\partial \rho}{\partial x} && (4)\\
&= -v_{\max}\left(1 - 2\frac{\rho}{\rho_{\max}}\right)\frac{\partial \rho}{\partial x}. && (5)
\end{aligned}$$

This model is the one we wish to simulate numerically. You will implement and compare two methods in sections 1.1 and 1.2.

1.1 *Task 1.1: Implement an upwind scheme for an (artificial) shock*

Suppose we wish to understand what section 1 predicts for the initial condition shown in fig. 2. This figure depicts a "shock," which occurs when the density of vehicles is at the maximum $\rho_{\max}$ over some region. A question might be how long does this shock last? That is, how long does it take for traffic to return to an equilibrium distribution?

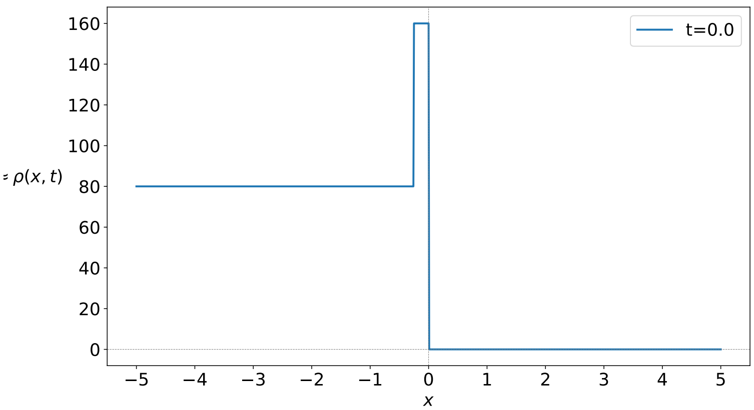


Figure 2: An initial shock. Suppose the maximum density $\rho_{\max}$ is 160 cars per km. Traffic enters from the left (x=−5 km) at 80 cars per km, half the maximum density. This density is fixed until x=−0.25 km, where a "shock" occurs, meaning traffic suddenly hits the maximum density. This shock extends until x=0 km. To the right of that point, the density is zero.

Chapter 7.4.1 of Bungartz et al. 2014 describes a numerical solver for section 1. This solver is sometimes referred to as an *upwind scheme*. The solver is essentially formula (7.12) of Bungartz et al. 2014 with periodic (circular) boundary conditions. Implement this solver, but with the following modifications:

- Consider the spatial domain's coordinates to extend from −5 km to 5 km, rather than 0 to 10 km. (This change is a cosmetic one to match fig. 2 as the model has no explicit dependence on the physical coordinates.)
- Simulate 2 min of logical time with a time step of 0.1 s (one-tenth of a second). Therefore, your simulation loop will evaluate 1,200 time steps in total.
- Use a spatial step size of 0.01 km (10 m). Therefore, your grid will have about 1,000 points.
- To prevent negative densities, after computing the update step (Bungartz equation 7.12) over the entire domain, force any negative densities to zero.

After implementing it, do the following.

1. Validate your implementation, that is, argue that it is correct.
2. Plot your results for several time steps to show the overall pattern. Such plots might include a contour or heat plot of density, flow, or velocity over space and time.
3. Critique these results; that is, how reasonable does they seem compared to what you might have expected?

1.2 *Task 1.2: Implement the Lax-Friedrichs scheme*

Modify your previous simulation to implement the *Lax-Friedrichs scheme*. Assume the following notation.

- Space is discretized at the points x_1, x_2, x_3, ..., x_i, ..., with step size $x_{i+1} - x_i \equiv s$.
- Time is discreteized at the points t_1, t_2, t_3, ..., t_j, ..., with step size $t_{j+1} - t_j \equiv h$.
- The approximate density at point x_i and time t_j is $\rho_{i,j}$.
- The approximate flow at x_i and t_j is $f_{i,j} = f(\rho_{i,j})$.

Then the Lax-Friedrichs update formula is given by

$$\rho_{i,j+1} = \frac{\rho_{i+1,j} + \rho_{i-1,j}}{2} - \frac{h}{s}\frac{f_{i+1,j} - f_{i-1,j}}{2}. \tag{6}$$

This scheme uses second-order accurate derivative approximations.

After implementing it, do the following.

1. Validate your implementation, that is, argue that it is correct.
2. Plot your results for several time steps to show the overall pattern. Such plots might include a contour or heat plot of density, flow, or velocity over space and time.
3. Critique these results; that is, how reasonable does they seem compared to what you might have expected? How do these results compare to the upwind scheme?

2 *Part 2: A CA-based model (50 points)*

Your next task is to implement cellular automata (CA) models of traffic flow and compare them to your results from section 1.

2.1 Task 2.1: A first CA model

Implement the CA model described in Chapter 8.2 of Bungartz et al. 2014.[7] However, try to choose parameters that more closely match what is simulated in section 1, to facilitate comparison.

When your implementation is complete, do the following.

[7] You may need to read the background material in 8.1, too.

1. Validate your implementation, that is, argue that it is correct.

2. Plot your results for several time steps to show the overall pattern. Such plots might include a contour or heat plot of density, flow, or velocity over space and time.

3. Critique these results; that is, how reasonable does they seem compared to what you might have expected? How does it compare to what you obtained in section 1?

3 Teaming

You may work individually or in pairs. Teams of two have additional requirements for the assignment, as noted above.

To "declare" your team, do the following:

- Visit the Canvas "People" page for Mini-project 2 ("MP2").[8]

- If you are working individually: Pick an unused team number and add yourself.

- If you are working in a team of two: Both of you should pick an unused team number and add yourself to the same unused team number.

[8] Link: `https://gatech.instructure.com/courses/310452/groups#tab-34531`

4 How to Submit

Create a *private* git code repository at `github.gatech.edu` (*not* `github.com`!) For teams, only one person needs to create the repository. Place both your code and a PDF report summarizing your results in this repo. You'll submit the URL to this repository on the page for this assignment in Canvas. In the repository itself, create a `README.md` file that tells us a) your individual or team number and member(s) of the team; b) which file is your PDF report; and c) what the organization of your code is (so we can inspect and evaluate it).

Important! Since your repository will be private, please be sure to add *everyone* from the teaching staff to your repo: the two TAs, Albert Chen (`achen355`) and Varun Lakshmanan (`vlakshmanan30`), and Prof. Vuduc (`rvuduc3`). Otherwise, we will not be able to see and grade your submission and you will get a zero.

Late submissions. There is a 10% penalty if you submit up to one day late (Monday, April 10), a 25% penalty if you submit two days late (Tuesday, April 11), and 50% penalty if you submit three days late (Wednesday, April 12). No assignments will be accepted after that time.

References

Hans-Joachim Bungartz, Stefan Zimmer, Martin Buchholz, and Dirk Pflüger. *Modeling and simulation: an application-oriented introduction*. Springer, 2014.

Michael J Lighthill and Gerald Beresford Whitham. On kinematic waves. ii. a theory of traffic flow on long crowded roads. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 229(1178):317–345, 1955. DOI: 10.1098/rspa.1955.0089. URL https://royalsocietypublishing.org/doi/10.1098/rspa.1955.0089.

Paul I Richards. Shock waves on the highway. *Operations research*, 4 (1):42–51, 1956. URL https://www.jstor.org/stable/167515.