Technical communique

On adaptive sliding mode control without a priori bounded uncertainty☆

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A B S T R A C T

Adaptive Sliding Mode Control (ASMC) aims to adapt the switching gain in such a way to cope with possibly unknown uncertainty. In state-of-the-art ASMC methods, a priori boundedness of the uncertainty is crucial to ensure boundedness for the switching gain and uniformly ultimately boundedness. A priori bounded uncertainty might impose a priori bounds on the system state before obtaining closed-loop stability. A design removing this assumption is still missing in literature. A positive answer to this quest is given by this note where a novel ASMC methodology is proposed which does not require a priori bounded uncertainty. An illustrative example is presented to highlight the main features of the approach, after which a general class of Euler–Lagrange systems is taken as a case study to show the applicability of the proposed design.

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1. Introduction

A design challenge in sliding mode control is to tackle uncertainties in the system to be controlled without prior knowledge about them. The pursuit of this objective has led to several adaptive sliding mode control (ASMC) methods, where the switching gain is adapted in such a way to cope with possibly unknown uncertainty. One possibility for adaptation is to increase monotonically the switching gain (Chang, 2009; Nasiri, Nguang, & Swain, 2014; Negrete-Chávez & Moreno, 2016; Su & Stepanenko, 1994). However, as this approach might lead to high gain (Plestan, Shtessel, Bregeault, & Poznyak, 2010), alternative ASMC methods have been proposed, that can be categorized into two families: (i) increasing–decreasing ASMC (Plestan et al., 2010; Shtessel, Taleb, & Plestan, 2012; Zhang, Obeid, & Laghrouche, 2018); (ii) equivalent control ASMC (Edwards & Shtessel, 2016; Oliveira, Cunha, & Hsu, 2016; Utkin & Poznyak, 2013). Note that most ASMC designs assume either the uncertainty (Chang, 2009; Nasiri

et al., 2014; Negrete-Chávez & Moreno, 2016; Oliveira et al., 2016; Plestan et al., 2010; Su & Stepanenko, 1994; Zhang et al., 2018) or its time-derivative (Edwards & Shtessel, 2016; Shtessel et al., 2012; Utkin & Poznyak, 2013) to be upper bounded by a constant *a priori*. When the uncertainty has explicit dependency on the system states, such prior constant upper bound might be very restrictive because it requires the states to be upper bounded a priori before obtaining system stability. An example illustrating the consequences of such restriction is provided below:

1.1. Illustrative example

Consider the following scalar system

$$\dot{q}(t) = -cq(t) + \tau(t) + d(q, t), \tag{1}$$

where $c > 0$ is possibly unknown, τ is the control input and d denotes combination of state-dependent unmodelled dynamics and time-dependent bounded disturbance. For a choice of sliding variable $s(q, t)$, we have

$$\dot{s} = \frac{\partial s}{\partial t} + \frac{\partial s}{\partial q}\dot{q} = \underbrace{\frac{\partial s}{\partial t} + \frac{\partial s}{\partial q}(-cq + d)}_{a(q,t)} + \underbrace{\frac{\partial s}{\partial q}}_{b(q,t)}\tau, \tag{2}$$

where $a(q, t)$ can be referred to as "uncertainty", as it contains unknown dynamics and disturbances stemming from c and d. Let
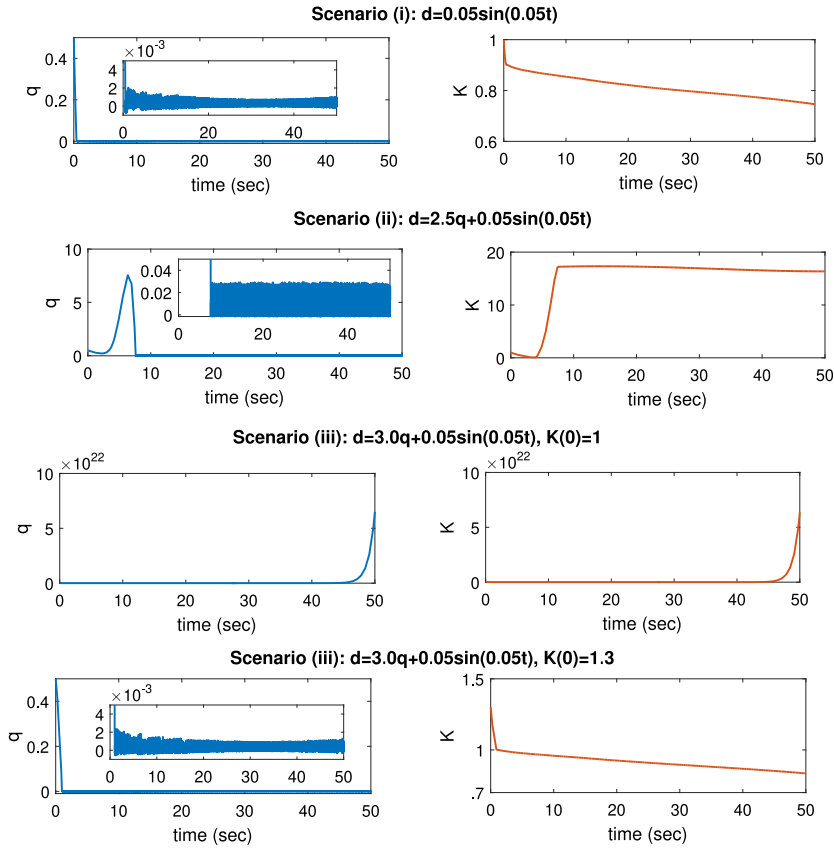
Fig. 1. Performance and gain evolution of control law (3) for various scenarios..

us now consider an increasing–decreasing ASMC law as proposed in Plestan et al. (2010):

$$\tau(t) = -K \operatorname{sgn}(s(t)), \tag{3a}$$

$$\dot{K}(t) = \begin{cases} \bar{K}|s(t)| \operatorname{sgn}(|s(t)| - \epsilon) & \text{if } K(t) \geq \mu \\ \mu & \text{if } K(t) < \mu, \end{cases} \tag{3b}$$

where K is the switching gain and ϵ, μ and $\bar{K}$ are positive user-defined gains. The following stability result can be established for (1) and (3):

Lemma 1 (*Rephrased from Plestan et al. (2010)*)*. Under the assumptions that $\exists \bar{a}, \bar{b} \in \mathbb{R}^+$ such that $a(q, t) \leq \bar{a}$, $b(q, t) \leq \bar{b}$, $\exists K^* = \bar{a}/\bar{b}$ such that $K(t) \leq K^* \ \forall t > 0$ and there exists finite time t_f such that $|s(t)| \leq \delta_1$ for $t \geq t_f$ with*

$$\delta_1 = \sqrt{\epsilon^2 + \frac{\bar{a}^2}{\bar{K}\bar{b}}}. \tag{4}$$

Remark 1 (*A Priori Boundedness*)*. Lemma 1 relies on a priori boundedness of the uncertainty, which allows to ensure boundedness of the switching gain and of the sliding variable. Similar a priori assumptions on the uncertainty (Chang, 2009; Nasiri et al., 2014; Negrete-Chávez & Moreno, 2016; Oliveira et al., 2016; Plestan et al., 2010; Su & Stepanenko, 1994; Zhang et al., 2018) or its time-derivative (Edwards & Shtessel, 2016; Shtessel et al., 2012; Utkin & Poznyak, 2013) appear in other ASMC methods. When the assumption of a priori bound is violated, unboundedness of the switching gains cannot be excluded, as clarified below.

Let us simulate the closed loop (1) and (3) with $c = 1$, $\epsilon = 1$, $\mu = 0.1$, $\bar{K} = 1$, $q(0) = 0.5$ and $s(t) = q(t)$ giving

$$\dot{s}(t) = \underbrace{(-cq(t) + d(q, t))}_{a(q,t)} + \tau(t), \tag{5}$$

and $b(q, t) = 1$ from (2). We consider different scenarios, with state-independent and state-dependent d

(i) $d(t) = 0.05 \sin(0.05t)$, $K(0) = 1$,
(ii) $d(q, t) = 2.5q + 0.05 \sin(0.05t)$, $K(0) = 1$,
(iii) $d(q, t) = 3q + 0.05 \sin(0.05t)$, $K(0) = 1$ & $K(0) = 1.3$.

Fig. 1 shows that as the state-dependent term of d gets larger, the switching gain gets higher and higher to stabilize the system (cf. scenarios (i)–(ii)). Eventually, if the initial condition $K(0)$ is not high enough, instability might arise (cf. scenario (iii)). The value $K(0)$ for which the system can be stabilized depends on $d(q, t)$, and it is thus unknown. These problems happen because the assumptions of Lemma 1 are violated, which provides us with the motivation to look for alternative approaches.

2. A candidate solution

For a state-dependent structure of d, let us consider the state-dependent upper bound

$$|a(q, t)| \leq K_0^* + K_1^*|q|, \ \forall q, \forall t \geq 0 \tag{6}$$

with unknown $K_0^*, K_1^* \in \mathbb{R}^+$ (note that (6) does not impose a priori bounds on the uncertainty and it covers all scenarios of Section 1.1). Based on the structure of (6), the following control law is proposed in place of (3)

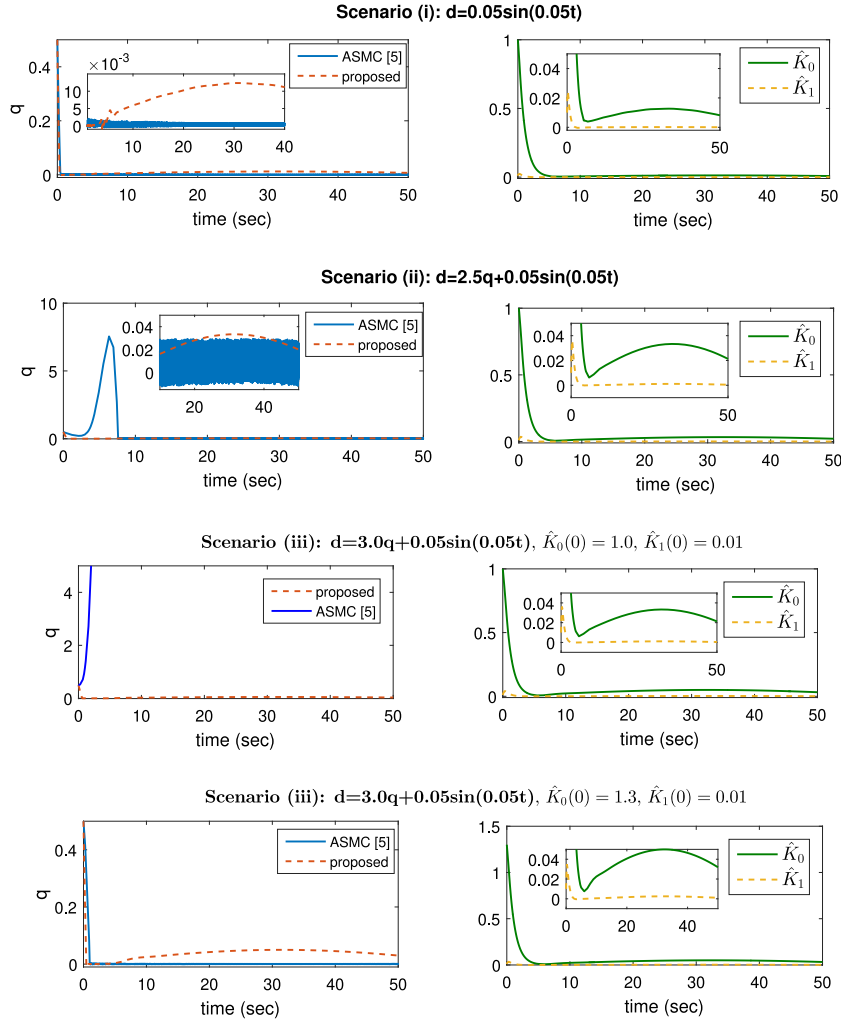$$\tau(t) = -\Lambda s(t) - \rho(t) \operatorname{sgn}(s(t)) \tag{7}$$

Fig. 2. Performance and gain evolution of the proposed control law for various scenarios.

$$\rho(t) = \hat{K}_0(t) + \hat{K}_1(t)|q(t)|, \tag{8}$$

with $\Lambda > 0$ and where $\hat{K}_i(t)$ can be regarded as the counterpart of $K(t)$ in (3b), adapted by

$$\dot{\hat{K}}_0(t) = |s(t)| - \alpha_0 \hat{K}_0(t),$$

$$\dot{\hat{K}}_1(t) = |s(t)||q(t)| - \alpha_1 \hat{K}_1(t), \tag{9a}$$

$$\text{with } \hat{K}_i(0) > 0, \ \alpha_i > 0, \ i = 0, 1. \tag{9b}$$

We simulate system (1) and the controller (7)–(9) with $\Lambda = 2$, $\alpha_i = 1.1$ and with same system parameters and disturbances as in Section 1.1. To make the initial conditions of (9a) consistent with that of (3b), we initialize $\hat{K}_0(0) = K(0)$ and select a small $\hat{K}_1(0) = 0.01$. Fig. 2 shows that the performance of (3) is better in scenario (i), but the proposed design provides better performance as the state-dependency in d gets larger (cf. scenarios (ii) and (iii)). In addition, no sufficiently high initial gain is required in the proposed controller to achieve stability.

Remark 2 (*Monotonically Increasing Gains*). It is to be observed that when $\epsilon, \mu \to 0$ in (3b) or $\alpha_i \to 0$ in (9a) both adaptive laws yield monotonically increasing gains (Chang, 2009; Nasiri et al., 2014; Negrete-Chávez & Moreno, 2016; Su & Stepanenko, 1994). The crucial difference is that the law (3b) departs from such high-gain adaptive laws in a way to handle only a priori bounded uncertainties, whereas the proposed law in (9a) can handle state-dependent uncertainties without a priori bound.

In the following, the closed-loop stability is analysed via the notion of Globally Uniformly Ultimately Bounded (GUUB) solutions (Khalil, 2002, Definition 4.6 (Sect. 4.8))).

3. Stability analysis of the candidate solution

Theorem 1. *The closed-loop trajectories in* (5) *employing the control laws* (7) *and* (8) *with adaptive law* (9), *are GUUB where an ultimate bound $\overline{\omega}$ on s is given by*

$$\overline{\omega} = \sqrt{\frac{\sum_{i=0}^{1} \alpha_i K_i^{*2}}{(\overline{\varrho} - \bar{\kappa})}}, \tag{10}$$

with $\overline{\varrho} \triangleq 2\min_i\{\Lambda, \ \alpha_i/2\}$, $0 < \bar{\kappa} < \overline{\varrho}$.

Proof. The laws (9a) with initial condition in (9b) give

$$\hat{K}_i(t) = \underbrace{\exp(-\alpha_i t)\hat{K}_i(0)}_{\geq 0}$$

$$+ \underbrace{\int_0^t \exp(-\alpha_i(t - \tau))(|s(\tau)||q(\tau)|^i)\mathrm{d}\tau}_{\geq 0} \tag{11}$$

$$\Rightarrow \hat{K}_i(t) \geq 0, \ i = 0, 1, \ \forall t \geq 0. \tag{12}$$

Stability is analysed via the Lyapunov function (for compactness, we omit time dependency when unambiguous)

$$V = \frac{1}{2}s^2 + \sum_{i=0}^{1} \frac{1}{2}(\hat{K}_i - K_i^*)^2. \tag{13}$$

Utilizing (5), the upper bound (6) and the control laws (7)–(8), the time derivative of (13) yields

$$\dot{V} = s(-\Lambda s - \rho\,\mathrm{sgn}(s) + a) + \sum_{i=0}^{1}(\hat{K}_i - K_i^*)\dot{\hat{K}}_i$$

$$\leq -\Lambda s^2 - \sum_{i=0}^{1}\{(\hat{K}_i - K_i^*)(|q|^i|s| - \dot{\hat{K}}_i)\}, \tag{14}$$

where we have used the fact that $\hat{K}_i \geq 0$ from (12) and thus $\rho \geq 0$. The following equality holds

$$(\hat{K}_i - K_i^*)\dot{\hat{K}}_i = (\hat{K}_i - K_i^*)(|s||q|^i - \alpha_i\hat{K}_i)$$

$$= (\hat{K}_i - K_i^*)|q|^i|s| + \alpha_i\hat{K}_iK_i^* - \alpha_i\hat{K}_i^2. \tag{15}$$

Substituting (15) in (14) yields

$$\dot{V} \leq -\Lambda s^2 + \sum_{i=0}^{1}\left(\alpha_i\hat{K}_iK_i^* - \alpha_i\hat{K}_i^2\right)$$

$$\leq -\Lambda s^2 - \sum_{i=0}^{1}\left(\frac{\alpha_i}{2}(\hat{K}_i - K_i^*)^2 - \frac{\alpha_iK_i^{*2}}{2}\right) \tag{16}$$

in view of the fact that

$$\hat{K}_iK_i^* - \hat{K}_i^2 = -\left(\frac{\hat{K}_i}{\sqrt{2}} - \frac{K_i^*}{\sqrt{2}}\right)^2 - \frac{\hat{K}_i^2}{2} + \frac{K_i^{*2}}{2}$$

$$\leq -\left(\frac{\hat{K}_i}{\sqrt{2}} - \frac{K_i^*}{\sqrt{2}}\right)^2 + \frac{K_i^{*2}}{2}. \tag{17}$$

Using the definition of Lyapunov function (13), the condition (16) is further simplified to

$$\dot{V} \leq -\overline{\varrho}V + \frac{1}{2}\sum_{i=0}^{1}\alpha_iK_i^{*2}, \tag{18}$$

where $\overline{\varrho} \triangleq 2\min_i\{\Lambda,\ \alpha_i/2\} > 0$ by design. Using $0 < \bar{\kappa} < \overline{\varrho}$, $\dot{V}$ in (18) simplifies to

$$\dot{V} \leq -\bar{\kappa}V - (\overline{\varrho} - \bar{\kappa})V + \bar{\delta}, \tag{19}$$

where $\bar{\delta} \triangleq (1/2)\sum_{i=0}^{1}\alpha_iK_i^{*2}$. Further, define a scalar $\bar{\mathcal{B}} \triangleq \bar{\delta}/(\overline{\varrho} - \bar{\kappa})$. It can be seen that $\dot{V}(t) \leq -\bar{\kappa}V(t)$ when $V(t) \geq \bar{\mathcal{B}}$ so that

$$V \leq \max\left\{V(0), \bar{\mathcal{B}}\right\}, \quad \forall t \geq 0, \tag{20}$$

and the Lyapunov function enters in finite time inside the ball defined by $\bar{\mathcal{B}}$. The definition of the Lyapunov function (13) yields $V \geq (1/2)|s|^2$ from which we have the ultimate bound (10) which is global and uniform as it is independent of initial conditions. $\square$

Remark 3 (*A Posteriori Boundedness*). Compared to Chang (2009), Edwards and Shtessel (2016), Nasiri et al. (2014), Negrete-Chávez and Moreno (2016), Oliveira et al. (2016), Plestan et al. (2010), Shtessel et al. (2012), Su and Stepanenko (1994), Utkin and Poznyak (2013) and Zhang et al. (2018), no a priori bounded assumption is imposed on the system dynamics or states: state boundedness is obtained a posteriori as a result of stability analysis.

The control laws (7)–(9), proposed for an academic example, are now shown to find application in a wide range of real-life systems modelled by fully-actuated Euler–Lagrange (EL) dynamics (Spong, Hutchinson, & Vidyasagar, 2008). The following notations will be used: $\mathrm{sgn}(\mathbf{s}) = [\mathrm{sgn}(s_1), \ldots, \mathrm{sgn}(s_n)]$ for $\mathbf{s} = [s_1, \ldots, s_n]$; $\|(\cdot)\|$ and $\lambda_{\min}(\cdot)$ denote 1-norm and minimum eigenvalue of $(\cdot)$, respectively.

4. Application to Euler–Lagrange systems

Consider the Euler–Lagrange (EL) dynamics

$$\mathbf{M}(\mathbf{q}(t))\ddot{\mathbf{q}}(t) + \mathbf{C}(\mathbf{q}(t), \dot{\mathbf{q}}(t))\dot{\mathbf{q}}(t) +$$

$$\mathbf{G}(\mathbf{q}(t)) + \mathbf{F}(\dot{\mathbf{q}}(t)) + \mathbf{d}(t) = \boldsymbol{\tau}(t), \tag{21}$$

where $\mathbf{q}, \dot{\mathbf{q}} \in \mathbb{R}^n$ are the system states; $\mathbf{M}(\mathbf{q}) \in \mathbb{R}^{n \times n}$ is the mass/inertia matrix; $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^{n \times n}$ denotes the Coriolis, centripetal terms; $\mathbf{G}(\mathbf{q}) \in \mathbb{R}^n$ denotes the gravity vector; $\mathbf{F}(\dot{\mathbf{q}}) \in \mathbb{R}^n$ represents the vector of damping and friction forces; $\mathbf{d}(t) \in \mathbb{R}^n$ denotes an external disturbance and $\boldsymbol{\tau} \in \mathbb{R}^n$ is the generalized control input.

For most EL systems of practical interest, (21) presents a few interesting properties (cf. Spong et al. (2008, Sect. 9.5)), which are later exploited for control design and stability analysis:

Property 1. $\exists \overline{c}, \overline{g}, \overline{f}, \overline{d} \in \mathbb{R}^+$ *such that* $\|\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\| \leq \overline{c}\|\dot{\mathbf{q}}\|$, $\|\mathbf{G}(\mathbf{q})\| \leq \overline{g}$, $\|\mathbf{F}(\dot{\mathbf{q}})\| \leq \overline{f}\|\dot{\mathbf{q}}\|$ *and* $\|\mathbf{d}(t)\| \leq \overline{d}$, $\forall \mathbf{q}, \dot{\mathbf{q}}, \forall t \geq 0$.

Property 2. *The matrix* $\mathbf{M}(\mathbf{q})$ *is symmetric and uniformly positive definite in* $\mathbf{q}$*, i.e.* $\exists \underline{m}, \overline{m} \in \mathbb{R}^+$ *such that*

$$0 < \underline{m}\mathbf{I} \leq \mathbf{M}(\mathbf{q}) \leq \overline{m}\mathbf{I}. \tag{22}$$

Property 3. *The matrix* $(\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}))$ *is skew symmetric, i.e., for any non-zero vector* $\mathbf{z}$*, we have* $\mathbf{z}^T(\mathbf{M}(\mathbf{q})2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}))\mathbf{z} = 0$.

Remark 4 (*State-dependent Uncertainty*). It is considered that $\mathbf{M}, \mathbf{C}, \mathbf{F}, \mathbf{G}, \mathbf{d}$ and their corresponding bounds $\underline{m}, \overline{m}, \overline{c}, \overline{g}, \overline{f}, \overline{d}$ are *unknown*, thus resulting in state-dependent uncertainty in line with Section 1.1.

Let us consider the tracking problem for desired trajectories satisfying $\mathbf{q}^d, \dot{\mathbf{q}}^d, \ddot{\mathbf{q}}^d \in \mathcal{L}_\infty$. Let $\mathbf{e}(t) \triangleq \mathbf{q}(t) - \mathbf{q}^d(t)$ be the tracking error. We define a sliding variable $\mathbf{s}$ as

$$\mathbf{s}(t) \triangleq \dot{\mathbf{e}}(t) + \boldsymbol{\Phi}\mathbf{e}(t), \tag{23}$$

where $\boldsymbol{\Phi} \in \mathbb{R}^{n \times n}$ is positive definite. In the following, let us omit variable dependency for compactness. Multiplying the derivative of (23) by $\mathbf{M}$ and using (21) yields

$$\mathbf{M}\dot{\mathbf{s}} = \mathbf{M}(\ddot{\mathbf{q}} - \ddot{\mathbf{q}}^d + \boldsymbol{\Phi}\dot{\mathbf{e}}) = \boldsymbol{\tau} - \mathbf{C}\mathbf{s} + \boldsymbol{\varphi}, \tag{24}$$

where $\boldsymbol{\varphi} \triangleq -(\mathbf{C}\dot{\mathbf{q}} + \mathbf{G} + \mathbf{F} + \mathbf{d} + \mathbf{M}\ddot{\mathbf{q}}^d - \mathbf{M}\boldsymbol{\Phi}\dot{\mathbf{e}} - \mathbf{C}\mathbf{s})$ represents the overall uncertainty. Using (23) and Properties 1 and 2 we have

$$\|\boldsymbol{\varphi}\| \leq \overline{c}\|\dot{\mathbf{q}}\|^2 + \overline{g} + \overline{f}\|\dot{\mathbf{q}}\| + \overline{d} + \overline{m}(\|\ddot{\mathbf{q}}^d\| + \|\boldsymbol{\Phi}\|\|\dot{\mathbf{e}}\|)$$

$$+ \overline{c}\|\dot{\mathbf{q}}\|(\|\dot{\mathbf{e}}\| + \|\boldsymbol{\Phi}\|\|\mathbf{q}\|). \tag{25}$$

Further, let us define $\boldsymbol{\xi} = [\mathbf{e}^T\ \dot{\mathbf{e}}^T]^T$. Then, using inequalities $\|\boldsymbol{\xi}\| \geq \|\mathbf{e}\|$, $\|\boldsymbol{\xi}\| \geq \|\dot{\mathbf{e}}\|$, boundedness of the desired trajectories, and substituting $\dot{\mathbf{q}} = \dot{\mathbf{e}} + \dot{\mathbf{q}}^d$ into (25) yields

$$\|\boldsymbol{\varphi}\| \leq K_0^* + K_1^*\|\boldsymbol{\xi}\| + K_2^*\|\boldsymbol{\xi}\|^2, \tag{26}$$

where $K_0^* \triangleq \{\overline{c}\|\dot{\mathbf{q}}^d\|^2 + \overline{g} + \overline{f}\|\dot{\mathbf{q}}^d\| + \overline{d} + \overline{m}\|\ddot{\mathbf{q}}^d\|\}$, $K_1^* \triangleq \{\overline{c}\|\dot{\mathbf{q}}^d\|(3 + \|\boldsymbol{\Phi}\|) + \overline{f} + \overline{m}\|\boldsymbol{\Phi}\|\}$, $K_2^* \triangleq \overline{c}\|\dot{\mathbf{q}}^d\|(2 + \|\boldsymbol{\Phi}\|)$ are *unknown* finite scalars. Hence, a state-dependent upper bound structure naturally occurs for EL systems.

Problem: Design an adaptive sliding mode control framework for EL system (21) requiring (i) *no knowledge of the system dynamics terms* in line with Remark 4; (ii) no a priori constant upper bound on the states.

4.1. Controller design

An answer to the problem is constructed. Based on the upper bound structure (26), we propose the control law

$$\boldsymbol{\tau}(t) = -\boldsymbol{\Lambda}\mathbf{s}(t) - \rho(t)\,\mathrm{sgn}(\mathbf{s}(t)), \tag{27}$$

$$\rho(t) = \hat{K}_0(t) + \hat{K}_1(t)\|\boldsymbol{\xi}(t)\| + \hat{K}_2(t)\|\boldsymbol{\xi}(t)\|^2, \tag{28}$$

where $\boldsymbol{\Lambda}$ is a positive definite user-defined matrix. The gains $\hat{K}_i$ are adapted via

$$\dot{\hat{K}}_i(t) = \|\mathbf{s}(t)\|\|\boldsymbol{\xi}(t)\|^i - \alpha_i\hat{K}_i(t), \tag{29a}$$

$$\text{with } \hat{K}_i(0) > 0, \ \alpha_i > 0 \tag{29b}$$

where $\alpha_i \in \mathbb{R}^+$, $i = 0, 1, 2$ are design scalars.

4.2. Stability analysis

Theorem 2. *Under Properties 1–3, the closed-loop trajectories in (24) employing the control laws (27) and (28) with adaptive law (29), are GUUB with an ultimate bound ω on $\mathbf{s}$ given by*[1]

$$\omega = \sqrt{\frac{n\sum_{i=0}^{2}\alpha_i K_i^{*2}}{\underline{m}(\varrho - \kappa)}}, \tag{30}$$

where $\varrho \triangleq \dfrac{\min_i\{\lambda_{\min}(\boldsymbol{\Lambda})/n, \ \alpha_i/2\}}{\max\{\overline{m}/2, \ 1/2\}}$ and $0 < \kappa < \varrho$.

Proof. Note that the laws (29a) with initial condition in (29b) give

$$\hat{K}_i(t) = \underbrace{\exp(-\alpha_i t)\hat{K}_i(0)}_{\geq 0}$$

$$+ \underbrace{\int_0^t \exp(-\alpha_i(t-\tau))(\|\mathbf{s}(\tau)\|\|\boldsymbol{\xi}(\tau)\|^i)\mathrm{d}\tau}_{\geq 0} \tag{31}$$

$$\Rightarrow \hat{K}_i(t) \geq 0 \ \forall t \geq 0. \tag{32}$$

Closed-loop stability is analysed using the Lyapunov function candidate:

$$V = \frac{1}{2}\mathbf{s}^T\mathbf{M}\mathbf{s} + \sum_{i=0}^{2}\frac{1}{2}(\hat{K}_i - K_i^*)^2. \tag{33}$$

Using (24) and (27), the time derivative of (33) yields

$$\dot{V} = \mathbf{s}^T\mathbf{M}\dot{\mathbf{s}} + \frac{1}{2}\mathbf{s}^T\dot{\mathbf{M}}\mathbf{s} + \sum_{i=0}^{2}(\hat{K}_i - K_i^*)\dot{\hat{K}}_i$$

$$= \mathbf{s}^T(\boldsymbol{\tau} - \mathbf{C}\mathbf{s} + \boldsymbol{\varphi}) + \frac{1}{2}\mathbf{s}^T\dot{\mathbf{M}}\mathbf{s} + \sum_{i=0}^{2}(\hat{K}_i - K_i^*)\dot{\hat{K}}_i$$

$$= \mathbf{s}^T(-\boldsymbol{\Lambda}\mathbf{s} - \rho\,\mathrm{sgn}(\mathbf{s}) + \boldsymbol{\varphi})$$

$$+ \frac{1}{2}\mathbf{s}^T(\dot{\mathbf{M}} - 2\mathbf{C})\mathbf{s} + \sum_{i=0}^{2}(\hat{K}_i - K_i^*)\dot{\hat{K}}_i. \tag{34}$$

Property 3 implies $\mathbf{s}^T(\dot{\mathbf{M}} - 2\mathbf{C})\mathbf{s} = 0$. Then, utilizing the upper bound structure (26) and the fact that $\rho \geq 0$ from (32), $\dot{V}$ gets

simplified to

$$\dot{V} = \mathbf{s}^T(-\boldsymbol{\Lambda}\mathbf{s} - \rho\,\mathrm{sgn}(\mathbf{s}) + \boldsymbol{\varphi}) + \sum_{i=0}^{2}(\hat{K}_i - K_i^*)\dot{\hat{K}}_i$$

$$\leq -\mathbf{s}^T\boldsymbol{\Lambda}\mathbf{s} - \sum_{i=0}^{2}\{(\hat{K}_i - K_i^*)(\|\boldsymbol{\xi}\|^i\|\mathbf{s}\| - \dot{\hat{K}}_i)\}. \tag{35}$$

Using (29a) we have

$$(\hat{K}_i - K_i^*)\dot{\hat{K}}_i = \|\mathbf{s}\|(\hat{K}_i - K_i^*)\|\boldsymbol{\xi}\|^i + \alpha_i\hat{K}_i K_i^* - \alpha_i\hat{K}_i^2. \tag{36}$$

Substituting (36) in (35) yields

$$\dot{V} \leq -\frac{\lambda_{\min}(\boldsymbol{\Lambda})\|\mathbf{s}\|^2}{n} + \sum_{i=0}^{2}\left(\alpha_i\hat{K}_i K_i^* - \alpha_i\hat{K}_i^2\right)$$

$$\leq -\frac{\lambda_{\min}(\boldsymbol{\Lambda})\|\mathbf{s}\|^2}{n} - \sum_{i=0}^{2}\left(\frac{\alpha_i(\hat{K}_i - K_i^*)^2}{2} - \frac{\alpha_i K_i^{*2}}{2}\right) \tag{37}$$

where the last inequality arises via using (17). Further, the definition of Lyapunov function (33) yields

$$V \leq \frac{\overline{m}}{2}\|\mathbf{s}\|^2 + \sum_{i=0}^{2}\frac{1}{2}(\hat{K}_i - K_i^*)^2. \tag{38}$$

Using (38), the condition (37) is further simplified to

$$\dot{V} \leq -\varrho V + \frac{1}{2}\sum_{i=0}^{2}\alpha_i K_i^{*2}, \tag{39}$$

where $\varrho \triangleq \dfrac{\min_i\{\lambda_{\min}(\boldsymbol{\Lambda})/n, \ \alpha_i/2\}}{\max\{\overline{m}/2, \ 1/2\}} > 0$ by design (cf. (27), (29b)). Defining a scalar $0 < \kappa < \varrho$, (39) simplifies to

$$\dot{V} \leq -\kappa V - (\varrho - \kappa)V + \frac{1}{2}\sum_{i=0}^{2}\alpha_i K_i^{*2}, \tag{40}$$

Defining a scalar $\mathcal{B} \triangleq \dfrac{\sum_{i=0}^{2}\alpha_i K_i^{*2}}{2(\varrho - \kappa)}$, and following similar lines of proof in Theorem 1, we have

$$V \leq \max\{V(0), \mathcal{B}\}, \quad \forall t \geq 0, \tag{41}$$

and the Lyapunov function enters in finite time inside the ball defined by $\mathcal{B}$. The definition of the Lyapunov function (33) yields $V \geq (\underline{m}/(2n))\|\mathbf{s}\|^2$, leading to the ultimate bound (30) on $\mathbf{s}$ which is global and uniform as it is independent of initial conditions. $\square$

Remark 5. Control laws (27)–(29) reveal that the proposed design does not require any knowledge of the systems dynamics parameters. This is in contrast with recent approaches for EL systems which require nominal knowledge of the mass matrix and of its upper bound $\overline{m}$ (Roy, Roy, & Kar, 2018; Roy, Roy, Lee, & Baldi, 2019; Shtessel et al., 2012).

5. Conclusions and outlook

This note addresses the long-standing challenge of adaptive sliding mode design when system uncertainties cannot be upper bounded by a constant a priori. An important future work, in line with Rodrigues and Oliveira (2018), could be to extend the proposed method via output feedback.

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[1] For the sake of uniformity, the ultimate bound is expressed in the 1-norm using the property $\|\mathbf{s}\|_2 \leq \|\mathbf{s}\| \leq \sqrt{n}\|\mathbf{s}\|_2$ where $\|\cdot\|_2$ denotes 2-norm; similar analysis, albeit a different bound, can be carried out in the 2-norm or other norms.

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