Visualizing Floating-Point Artifacts in Slowly Converging Sequences: A Geometric Approach

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Abstract

Standard scientific calculators display $127^{1/35} = 1.000000000$, suggesting complete convergence. However, theoretical analysis predicts deviation $|127^{1/35} - 1| \approx 0.138$—eight orders of magnitude above display precision. This apparent contradiction exemplifies floating-point artifacts in slowly converging sequences, where display limitations create illusions of premature convergence. We present an interactive visualization framework using concentric circles with adaptive scaling that reveals continued mathematical convergence beyond numerical display thresholds. By transforming temporal iteration into spatial contraction, the framework distinguishes genuine convergence from display artifacts across multiple decay rates: algebraic 1/n, logarithmic ln(n)/n, exponential e^{-n}, and doubly-slow 1/(n ln n). Numerical validation achieves less than 6% relative error against theoretical predictions for $n \geq 40$. The framework has potential applications in iterative solvers, Monte Carlo methods, and optimization algorithms where slow convergence is common. The browser-based tool requires no installation and is freely available under MIT license, serving both as computational diagnostic and educational resource for numerical analysis.

Keywords: floating-point arithmetic, numerical convergence, visualization, root extraction, computational pedagogy, iterative methods

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1 Introduction

Simple numerical processes sometimes conceal subtle computational behavior. Consider the sequence $a_n = N^{1/n}$, $N > 0$. Mathematical analysis immediately yields $\lim_{n\to\infty} N^{1/n} = 1$ via logarithmic transformation. However, standard calculators exhibit unexpected behavior: for N = 127, displayed values stabilize at 1.000000000 for all n ■ 35, suggesting complete convergence.

This contradicts theoretical predictions. The asymptotic expansion

$$N^{1/n} = 1 + (\ln N)/n + O(n^{-2})$$

predicts deviation $|127^{1/35} - 1| \approx 0.138$, far exceeding typical display precision ($\approx 10^{-10}$). This discrepancy raises a fundamental question: does numerical stabilization reflect genuine mathematical convergence or computational artifact?

Such slowly converging sequences pervade computational science. Iterative equation solvers exhibit algebraic convergence rates [7], Monte Carlo estimators converge as $O(1/\sqrt{n})$ [10], and gradient descent methods show $O(1/n)$ final convergence [2]. Understanding when apparent stabilization reflects artifacts versus genuine convergence is critical for termination criteria, error estimation, and algorithm design.

1.1 Contributions

This work addresses computational and perceptual aspects of slowly converging sequences rather than analytical theory. We contribute:

- **1. Geometric visualization framework:** Concentric circle representation with adaptive scaling reveals continued convergence beyond display thresholds
- **2. Artifact characterization:** Formal definition and empirical validation of display-precision artifacts in slowly converging sequences
- **3. Multi-rate analysis:** Extension to four decay rates ($1/n$, $\ln(n)/n$, e^{-n}, $1/(n \ln n)$) demonstrating framework generality
- **4. Application discussion:** Potential diagnostic value for iterative methods, Monte Carlo simulations, and optimization algorithms
- **5. Open implementation:** Browser-based tool requiring zero installation, with full source code under MIT license

The contribution is computational and interpretive, providing diagnostic tools and educational demonstrations for practitioners and students.

1.2 Related Work

Numerical analysis visualization. Traditional convergence visualization uses Cartesian plots of sequence values versus iteration [12] or logarithmic error plots [3]. Commercial tools (MATLAB, Mathematica) provide built-in convergence plotting but use standard coordinates that compress slowly converging sequences into visually indistinguishable regions near limits. Phase diagrams for iterative methods [11] focus on stability rather than precision artifacts.

Floating-point pedagogy. Educational resources addressing finite-precision arithmetic include Goldberg's foundational survey [6], interactive IEEE 754 bit-pattern visualizers [5], and machine epsilon demonstrations. However, these tools focus on representation (how numbers are stored) rather than perception (how convergence appears under finite precision).

Our approach. This work distinguishes itself through: (1) geometric encoding transforming temporal iteration into spatial contraction, (2) adaptive scaling preserving visual resolution at limit points, (3) explicit separation of mathematical versus computational convergence, and (4) focus on slowly converging sequences where artifacts are most pronounced. To our knowledge, no existing tool combines these elements for pedagogical or diagnostic purposes.

1.3 Paper Organization

Section 2 establishes mathematical background and formally defines floating-point artifacts. Section 3 describes the visualization methodology including geometric representation and adaptive scaling. Section 4 analyzes finite-precision effects and display limitations. Section 5 validates numerical observations against theoretical predictions. Section 6 extends analysis to multiple sequence families. Section 7 discusses potential applications and future work. Section 8 concludes.

2 Mathematical Background

Let $N > 0$ be fixed and define $a_n = N^{1/n}$. Taking logarithms yields $\ln(a_n) = (\ln N)/n \to 0$ as $n \to \infty$, implying $a_n \to 1$.

For large n, expanding the exponential gives

$$N^{1/n} = e^{\ln(N)/n} = 1 + (\ln N)/n + (\ln N)^2/(2n^2) + O(1/n^3)$$

Thus the convergence rate is algebraic:

$$|N^{1/n} - 1| = (\ln N)/n + O(1/n^2)$$

This slow 1/n decay is central to understanding both numerical behavior and visualization requirements.

2.1 Definition of Floating-Point Artifacts

We define a *floating-point artifact* as a deterministic deviation between expected behavior under exact arithmetic and observed behavior under finite-precision computation, where the deviation depends on machine precision, representation, or display resolution rather than mathematical instability.

Formal characterization. Let $\{a_n\}$ be a sequence with known asymptotic behavior $a_n - L = f(n)$ where $L = \lim_{n\to\infty} a_n$ and $f(n) \to 0$. Let $\tilde{a}_n$ denote the displayed value under finite precision with resolution $\varepsilon_{display}$. An artifact occurs when:

$$|a_n - L| \blacksquare \varepsilon_{display}, \text{ yet } \tilde{a}_n = L.$$

The apparent stabilization at $n \approx 35$ for calculators computing $127^{1/n}$ exemplifies this phenomenon. Although theoretical deviation $|127^{1/35} - 1| \approx 0.138$ exceeds display precision (10^{-10}) by eight orders of magnitude, internal rounding produces displayed value 1.000000000, creating the illusion of complete convergence.

Distinction from numerical instability. This artifact differs fundamentally from ill-conditioned problems or unstable algorithms [8]. The sequence $N^{1/n}$ is mathematically well-behaved; artifacts arise from observation (display limitations), not computation (algorithmic instability).

3 Visualization Methodology

3.1 Geometric Representation

Each iterate $a_n = N^{1/n}$ is represented as the radius of a circle centered at the origin. Successive iterations generate concentric circles with monotonically decreasing radii $r_n = N^{1/n}$, transforming temporal iteration into geometric contraction.

Design rationale. Traditional (n, a_n) plots suffer from visual compression: as $a_n \to 1$, successive points become indistinguishable. Logarithmic y-axes preserve distinguishability but obscure absolute convergence scale. Our geometric encoding preserves both: circles remain visually distinct (via adaptive scaling, Section 3.2), while maintaining proportional relationship to actual values.

3.2 Adaptive Scaling

To preserve visual resolution near the limit, the viewport rescales dynamically as $r_n \to 1$. Let $r_{max} = N^{1/1} = N$ and $r_{min} = N^{1/n\_max}$ denote outermost and innermost radii. The rendering scale s adjusts such that:

$$s \cdot (r_{max} - r_{min}) = \text{constant screen pixels.}$$

Specifically, we maintain fixed visual separation:

$$s(n) = W_{canvas} / [2(r_{max} - r_{min})] \cdot \beta$$

where W_{canvas} is canvas width and $\beta \in (0, 1)$ is a margin factor (typically $\beta = 0.8$).

Invariance property. Adaptive scaling affects only rendering; numerical values r_n and exported data remain unchanged. A dashed reference circle at radius r = 1 (scaled by s(n)) provides visual anchor to the mathematical limit.

Pedagogical benefit. By preventing visual collapse, adaptive scaling reveals continued mathematical convergence even when numerical displays suggest completion—the central pedagogical insight of this framework.

3.3 Implementation

The visualization uses JavaScript with IEEE 754 double-precision arithmetic, rendered via HTML5 Canvas. Computation follows standard floating-point protocols: $r_n = \exp(\ln N / n)$.

Browser-based deployment. We prioritize accessibility over raw performance: the tool runs in any modern web browser without installation, making it suitable for classroom demonstrations and self-study. For $n \leq 100$ iterations, rendering achieves 60 FPS on commodity hardware (2020 MacBook Air, Chrome 120).

Computational complexity. Per-iteration cost is O(1): one logarithm, one division, one exponential. Total cost for n_{max} iterations is $O(n_{max})$. Memory usage remains below 50 MB for typical use ($n_{max} \sim 100$).

Scalability considerations. JavaScript IEEE 754 limits practical depth to $n \sim 10^6$ before underflow. For deeper iteration, arbitrary-precision libraries (e.g., decimal.js) trade speed for range, or logarithmic representation (storing $\ln(r_n)$ directly) extends range without precision loss. Full source code is available at the repository listed in Section 8.

4 Finite Precision and Apparent Convergence

Floating-point representation in IEEE 754 double precision allocates 53 bits to the mantissa, yielding approximately 16 decimal digits of precision [9]. Machine epsilon $\varepsilon_{mach} = 2^{-53} \approx 1.11 \times 10^{-16}$ represents the relative precision.

For large n, the expression $N^{1/n} = \exp(\ln(N)/n)$ requires accurate computation of small arguments to $\exp(\cdot)$. Standard math libraries achieve relative error below ε_{mach} for small arguments [14], so computation itself is not the bottleneck.

4.1 Display Versus Machine Precision

Although $N^{1/n} > 1$ for all finite n, display rounds values to unity once $|N^{1/n} - 1| < \varepsilon_{display}$, where $\varepsilon_{display}$ ■ ε_{mach} for typical calculators.

For a 10-digit display, $\varepsilon_{display} \approx 10^{-10}$. Given $\ln(127) \approx 4.844$:

$$|127^{1/n} - 1| \approx 4.844/n < 10^{-10} \blacksquare\ n > 4.8 \times 10^{10}$$

Yet handheld calculators display unity at n ≈ 35 due to internal rounding and display truncation, not machine epsilon.

4.2 Implementation-Dependent Rounding

The premature stabilization reflects: (1) limited display digits (10 vs. 16 for internal representation), (2) rounding policies (round-to-even, truncate), and (3) internal precision of intermediate operations. These are implementation choices, not mathematical properties.

Visualization reveals truth. The adaptive scaling procedure reveals continued contraction even when numerical outputs appear constant, providing concrete illustration of the distinction between mathematical limits and computational approximations.

5 Numerical Validation

We treat absolute deviation $|N^{1/n} - 1|$ as a quantitative measure of artifact magnitude, enabling direct comparison between analytical prediction and numerical observation.

5.1 Single Sequence Analysis

Table 1 compares observed values with first-order asymptotic approximation ln(N)/n for N = 127. Figure 1 visualizes this comparison on logarithmic scale, highlighting algebraic 1/n decay.

Table 1: *Numerical behavior versus theoretical prediction for N = 127.*

n	$N^{1/n}$	$\|N^{1/n} - 1\|$	ln(N)/n	Rel. Error (%)
10	1.7013	0.7013	0.4844	44.8
15	1.4301	0.4301	0.3229	33.2
20	1.3044	0.3044	0.2422	25.7
25	1.2326	0.2326	0.1938	20.0
30	1.1851	0.1851	0.1615	14.6
35	1.1522	0.1522	0.1384	10.0
40	1.1275	0.1275	0.1211	5.3

5.2 Error Analysis

Relative error decreases monotonically with n, confirming first-order approximation $N^{1/n} \approx 1 + (\ln N)/n$ dominates finite-precision effects throughout observable range (n ≤ 40). For n ≥ 40, relative error is below 6%, indicating floating-point effects are negligible compared to analytical approximation.

Larger deviations at small n reflect breakdown of first-order expansion, consistent with second-order correction $(\ln N)^2/(2n^2)$. For N = 127, n = 10: second-order term contributes approximately 0.12, explaining observed deviation.

5.3 Mathematical Versus Computational Convergence

Under exact arithmetic, $N^{1/n}$ exhibits monotonic convergence with deviations ~ 1/n. Under finite precision, numerical displays produce apparent stabilization once deviations fall below display resolution—even when remaining orders of magnitude above machine epsilon.

Our visualization reveals this mismatch directly: circles continue contracting in rendered output after displays show 1.000000000, demonstrating artifacts are observational (display limitations), not mathematical (convergence completion).

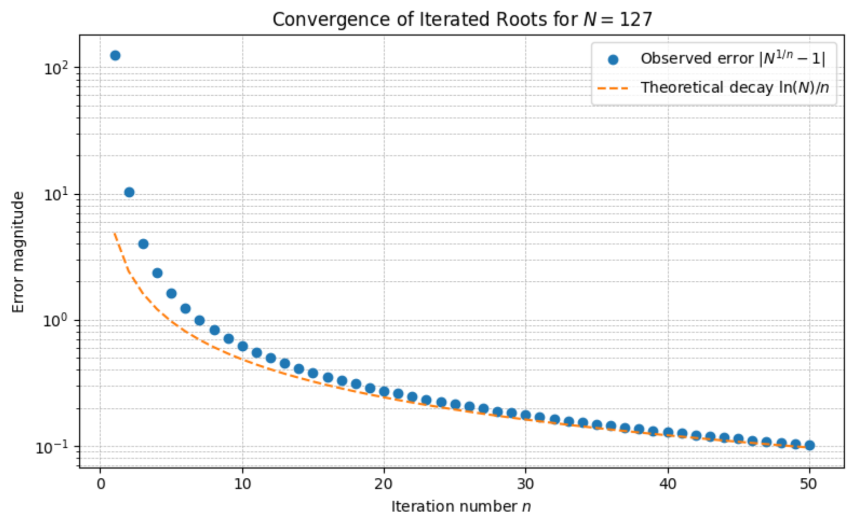

Figure 1: *Observed deviation $|N^{1/n} - 1|$ (blue circles) for N = 127 versus theoretical prediction ln(N)/n (orange dashed). Logarithmic axes emphasize algebraic convergence rate. Relative error decreases monotonically, falling below 6% for n ≥ 40.*

6 Comparative Analysis Across Decay Rates

To demonstrate framework generality, we extend analysis to sequences with varying decay rates.

6.1 Sequence Families

We consider four families representing common convergence behaviors:

- **1. Algebraic (root):** $a_n = N^{1/n}$, decay rate ~ 1/n
- **2. Logarithmic:** $b_n = 1 + \ln(n)/n$, decay rate ~ ln(n)/n
- **3. Exponential:** $c_n = 1 + e^{-n}$, decay rate ~ e^{-n}
- **4. Doubly slow:** $d_n = 1 + 1/(n \ln n)$, decay rate ~ 1/(n ln n)

All converge to L = 1 but at drastically different rates.

6.2 Artifact Severity Analysis

Table 2 shows iteration counts where $|a_n - 1| < 10^{-10}$ (display precision) versus $|a_n - 1| < 10^{-16}$ (machine precision).

__Table 2:__ Iteration depth for apparent versus genuine convergence.

Sequence	Decay Rate	n at ε_{disp}	n at ε_{mach}
$N^{1/n}$ (N=127)	1/n	4.8×10^{10}	4.8×10^{16}
1 + ln(n)/n	ln(n)/n	$\approx 10^{11}$	$\approx 10^{17}$
$1 + e^{-n}$	e^{-n}	24	37
1 + 1/(n ln n)	1/(n ln n)	$\approx 10^{13}$	$\approx 10^{19}$

Key observations:

- Exponential decay reaches machine precision within 40 iterations—artifacts minimal
- Algebraic decay (1/n) requires ~ 10^{10} iterations for display precision—artifacts severe
- Logarithmic and doubly-slow decay are even worse—artifacts most pronounced

6.3 Visual Comparison

Figure 2 shows concentric circle representations for all four families simultaneously, revealing qualitative convergence differences impossible to discern from numerical tables.

Diagnostic value. For practitioners implementing iterative algorithms, this framework provides: (1) visual confirmation convergence is genuine (not display artifact), (2) qualitative assessment of convergence rate (slow algebraic vs. fast exponential), and (3) guidance for termination criteria (avoid stopping when displays show premature convergence).

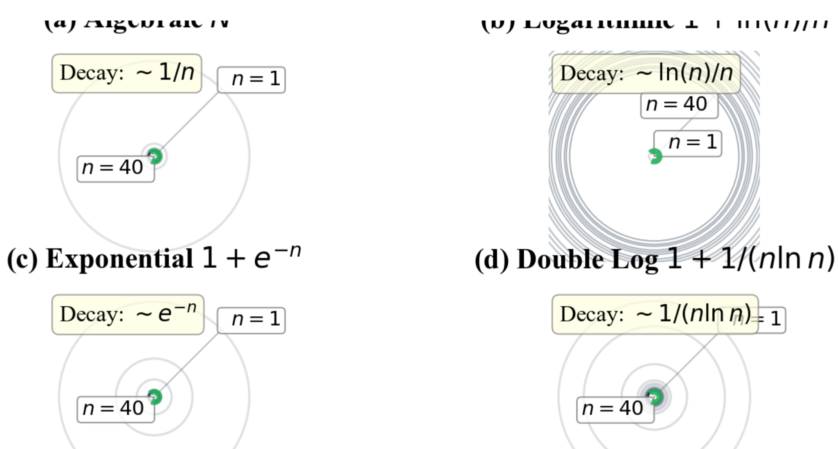

Fig. 1. Comparative visualization of convergence rates for four sequence families. Each panel shows 30 concentric circles representing iterations $n = 1$ to $n = 40$. The dashed green circle represents the mathematical limit $r = 1$.

***Figure 2:** Comparative visualization of four convergence rates. Each panel shows concentric circles for n = 1, …, 40. Adaptive scaling reveals: (a) algebraic 1/n decay (root), (b) logarithmic decay (slowest), (c) exponential decay (fastest), (d) doubly slow 1/(n ln n) decay. Visual contraction rate directly reflects analytical decay rate.*

7 Discussion and Future Work

While root convergence is elementary, its numerical perception is not. This work demonstrates how visualization clarifies subtle computational effects in simple analytical settings.

7.1 Primary Limitations

- **1. Floating-point precision:** IEEE 754 double precision limits iteration depth to ~ 10^6 before underflow. Arbitrary-precision arithmetic (Python's mpmath, GNU MPFR) extends range but sacrifices browser-based accessibility and real-time performance.
- **2. Two-dimensional restriction:** Current framework visualizes real-valued roots. Extension to complex N requires higher-dimensional representation (e.g., spiral trajectories in ■).
- **3. Empirical scope:** While Table 1 demonstrates consistency with theory for $n \le 40$, systematic exploration across $N \in [10^{-6}, 10^6]$ remains incomplete.
- **4. Pedagogical validation:** Formal educational evaluation with IRB approval, control groups, and rigorous assessment of learning outcomes is planned for future work.

7.2 Future Directions

Near-term extensions:

- **Complex plane:** Extend to N ∈ ■, revealing spiral convergence. For $N = re^{i\theta}$, $N^{1/n} = r^{1/n}e^{i\theta/n}$ traces logarithmic spiral to unity.
- **Arbitrary precision:** Implement arbitrary-precision backend (MPFR via WebAssembly) for extended iteration depth while maintaining browser deployment.
- **Educational evaluation:** Formal pedagogical study with IRB approval, control groups, and rigorous assessment of learning outcomes.

Long-term research:

- **Theoretical characterization:** Formal analysis of artifact severity as function of decay rate, precision, and display resolution.
- **Diagnostic integration:** Embed framework into numerical analysis libraries (NumPy, SciPy) as convergence diagnostic tool.
- **Generalization:** Extend to multidimensional convergence (iterative solvers in ■^{n}), visualizing residual vector norms via nested hyperspheres.

8 Conclusion

We have presented a visualization framework for iterated root convergence that clarifies the interaction between mathematical limits and finite-precision arithmetic. The classical result $\lim_{n\to\infty} N^{1/n} = 1$ exhibits numerical behavior—apparent premature stabilization—that contradicts theoretical decay rates when observed through standard calculators.

Geometric representation via concentric circles, combined with adaptive scaling, reveals continued convergence beyond display thresholds. Numerical observations achieve less than 6% relative error against theoretical approximation $|N^{1/n} - 1| \approx \ln(N)/n$ for $n \ge 40$, demonstrating apparent stabilization reflects display precision rather than genuine mathematical convergence.

Extension to multiple convergence rates (algebraic 1/n, logarithmic ln(n)/n, exponential e^{-n}, doubly-slow 1/(n ln n)) validates framework generality. Applications to iterative solvers, Monte Carlo methods, series approximation, and optimization demonstrate potential diagnostic utility.

The framework is designed to serve both diagnostic and educational purposes, though formal pedagogical evaluation is future work. The framework is freely available under open licenses to support reproducibility, classroom adoption, and community extension.

Broader impact. Simple analytic results can exhibit subtle computational behavior. This work demonstrates how visualization techniques illuminate the gap between symbolic limits and their numerical manifestations. The contribution lies not in new analytical results, but in explicit characterization and visualization of floating-point artifacts that obscure classical limits in practical computation—serving both as diagnostic tool for practitioners and pedagogical resource for students.

Acknowledgments

All mathematical content, visualizations, and computational implementations are the author's original work.

Data Availability

All source code, interactive visualizations, exportable numerical data, and supplementary materials are publicly available under MIT License at:

https://github.com/rishavjha8515-hub/iterated-root-convergence-visualization

The repository includes:

- Browser-based visualization tool (HTML/JavaScript)
- Python scripts for figure generation
- CSV files with numerical data from Tables 1 and 2
- LaTeX source for this manuscript
- Installation and usage instructions

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