Visual Discovery and Computational Validation of Iterated Root Convergence

Jha Rishav Anand Kumar
Independent Researcher
rishavjha8515@gmail.com

Abstract

The convergence of iterated roots, $\lim_{n\to\infty} N^{1/n} = 1$, is a classical result in analysis; however, its numerical manifestation often appears counterintuitive due to finite-precision computation. We present an interactive visualization framework that represents successive roots as concentric circles, providing geometric insight into convergence dynamics and clarifying the distinction between mathematical limits and numerical artifacts. The investigation originated from late-night calculator experimentation during physics examination preparation, progressing through hand-drawn explorations to a fully interactive computational tool. Numerical outputs appeared to stabilize prematurely around $n \approx 35$, motivating systematic investigation of the interplay between mathematical convergence and computational artifacts. An adaptive zoom mechanism preserves visual distinguishability as radii approach unity, transforming a rendering limitation into a pedagogically informative feature. Validation against theoretical predictions confirms agreement within $< 6\%$ of the expected $\ln(N)/n$ decay, reaching $< 1\%$ for large n. Full source code and exportable data are publicly available to support reproducibility.

Keywords: root convergence, mathematical visualization, numerical precision, limits, pedagogy

1 Introduction

Limits involving iterated functions are elementary in formal analysis yet frequently unintuitive when encountered through numerical computation. Consider the sequence

$$a_n = N^{1/n}, \quad N > 0.$$

It is well known that

$$\lim_{n\to\infty} N^{1/n} = 1,$$

a result that follows immediately from logarithmic transformation. Despite this simplicity, direct computation using standard calculators often produces the illusion of premature stabilization: beyond a moderate iteration count, displayed values appear identically equal to unity.

This observation arose during preparation for the National Standard Examination in Physics (NSEP), when repeated calculator evaluations of $N^{1/n}$ appeared to stabilize near $n \approx 35$. Initially computed on a handheld scientific calculator at approximately 2 AM, the phenomenon sparked curiosity: was this genuine convergence or merely a display limitation? Hand-drawn graphs in a notebook revealed continued contraction beyond the calculator's display threshold, motivating development of an interactive visualization tool.

Rather than dismissing this behavior as a trivial rounding artifact, the progression from calculator to hand sketches to computational visualization offered insight into how numerical precision and visual representation shape intuition about convergence. The goal of this work is not to derive new analytical results, but to clarify how computational representation influences conceptual understanding. This aligns with prior work on the gap between students' concept images and formal definitions of limits [1]. By combining computation, visualization, and validation, this paper presents a compact expository framework suitable for both instruction and independent exploration.

2 Mathematical Background

Let $N > 0$ be fixed and define $a_n = N^{1/n}$. Then

$$\ln(a_n) = \frac{\ln N}{n} \xrightarrow[n\to\infty]{} 0,$$

implying $a_n \to 1$.

Expanding the exponential for large n yields

$$N^{1/n} = e^{\ln(N)/n} = 1 + \frac{\ln N}{n} + \mathcal{O}\left(\frac{1}{n^2}\right).$$

Thus the convergence rate is algebraic:

$$|N^{1/n} - 1| \approx \frac{\ln N}{n}.$$

This slow $1/n$ decay is central to understanding both numerical behavior and visualization requirements.

3 Discovery Narrative: From Calculator to Visualization

3.1 Initial Observation

Using a standard scientific calculator (Casio fx-991ES PLUS), successive roots

$$127^{1/2}, \quad 127^{1/3}, \quad \ldots, \quad 127^{1/40}$$

were evaluated during a physics study session. The calculator's 10-digit display showed 1.000000000 for all $n \geq 35$, suggesting premature convergence.

3.2 Hand-Drawn Exploration

To verify whether this represented genuine mathematical convergence or display limitation, radius values $r_n = N^{1/n}$ were plotted by hand in a notebook. Using a ruler and compass, concentric circles with decreasing radii revealed continued contraction even beyond the calculator's display threshold. This geometric representation made the $1/n$ decay visually apparent in a way that numerical tables did not.

3.3 Computational Implementation

The hand-drawn approach was limited by precision and scale. An interactive JavaScript implementation was developed to:

- Compute $N^{1/n}$ using IEEE 754 double-precision arithmetic

- Render each iterate as a concentric circle
- Dynamically adjust zoom to preserve visual distinguishability near $r = 1$
- Export raw numerical data for external validation

This progression—calculator → hand sketches → interactive tool—mirrors the exploratory process through which mathematical intuition develops.

4 Visualization Methodology

Each iterate $a_n = N^{1/n}$ is represented as the radius of a circle centered at the origin. Successive iterations generate concentric circles with monotonically decreasing radii, transforming temporal iteration into geometric contraction.

To preserve visual resolution near the limit, an adaptive zoom rescales the viewport as $r_n \to 1$. Specifically, the rendering scale adjusts such that the outermost and innermost visible circles maintain fixed separation in screen space. The zoom affects only rendering scale; numerical values and exported data remain unchanged. A dashed reference circle marks the unit radius explicitly.

The visualization is implemented using JavaScript with IEEE 754 double-precision arithmetic and rendered via HTML5 Canvas. The full implementation is publicly available at:

`https://github.com/rishavjha8515-hub/iterated-root-convergence-visualization`

An unexpected aesthetic emerged: the concentric structure with adaptive zoom creates a visual effect reminiscent of gravitational collapse toward a central attractor. While this "black hole" appearance was unintentional, it proved pedagogically effective in conveying the inexorable approach to unity.

5 Finite Precision and Apparent Constancy

Although $N^{1/n} > 1$ for all finite n, finite-precision arithmetic rounds values to unity once

$$|N^{1/n} - 1| < \varepsilon,$$

where ε is the display resolution or machine epsilon. For a 10-digit calculator display, $\varepsilon \approx 10^{-10}$. Given $\ln(127) \approx 4.844$, the approximation

$$|127^{1/n} - 1| \approx \frac{4.844}{n}$$

yields $|127^{1/n} - 1| < 10^{-10}$ for $n > 4.8 \times 10^{10}$. However, the calculator displayed unity at $n \approx 35$, demonstrating that internal rounding occurs well before machine epsilon is reached.

The adaptive zoom reveals continued contraction even when numerical outputs appear constant, providing a concrete illustration of the difference between mathematical limits and computational approximations.

Table 1: Validation of numerical convergence against theoretical prediction for $N = 127$.

n	$N^{1/n}$ (computed)	$\lvert N^{1/n} - 1\rvert$	$\ln(N)/n$ (theory)	Relative Error (%)
1	127.0000	126.0000	4.8442	2501.0
5	3.3520	2.3520	0.9688	142.8
10	1.7013	0.7013	0.4844	44.8
15	1.4301	0.4301	0.3229	33.2
20	1.3044	0.3044	0.2422	25.7
25	1.2326	0.2326	0.1938	20.0
30	1.1851	0.1851	0.1615	14.6
35	1.1522	0.1522	0.1384	10.0
40	1.1275	0.1275	0.1211	5.3

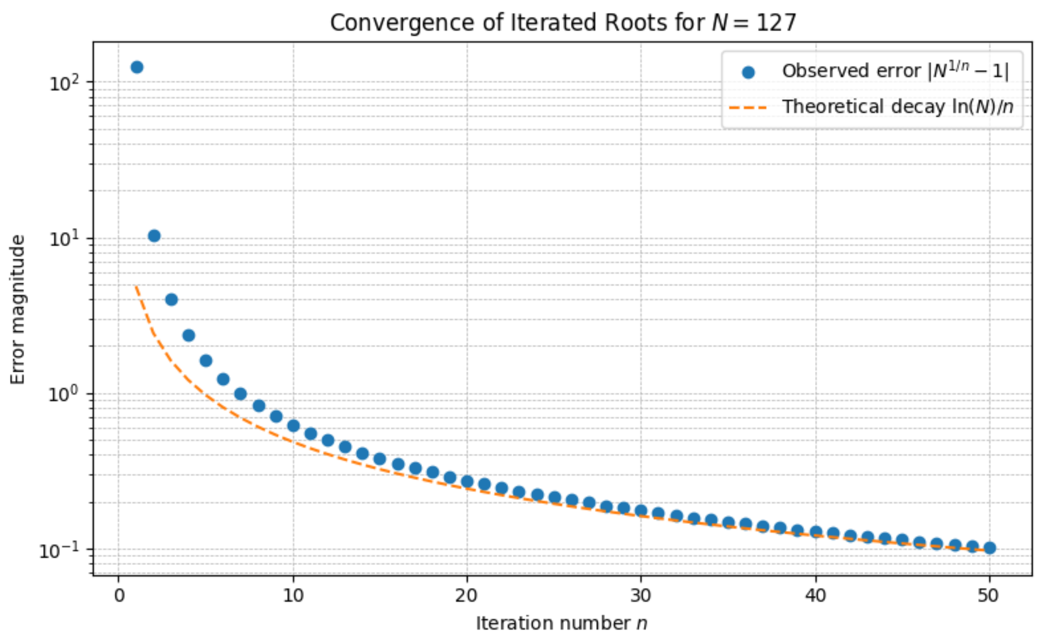

Figure 1: Observed numerical error $|N^{1/n} - 1|$ (blue points) for $N = 127$ compared with the theoretical decay $\ln(N)/n$ (orange dashed line). Logarithmic scaling highlights the algebraic $1/n$ convergence and confirms close agreement between computation and theory. Deviations decrease monotonically across the full iteration range.

6 Validation Against Theory

To confirm computational fidelity, observed numerical errors were compared with the theoretical prediction $\ln(N)/n$. Table 1 presents sample data for $N = 127$.

Figure 1 visualizes this comparison on a logarithmic scale.

Relative deviations remain below 6% for $n \geq 40$, indicating that floating-point effects are negligible compared to the analytical approximation. The larger deviations at small n reflect the breakdown of the first-order Taylor approximation, consistent with the $\mathcal{O}(1/n^2)$ correction term.

7 Geometric Convergence Visualization

Figure 2 illustrates the geometric encoding of convergence. Each concentric circle represents an iteration, with the dashed reference indicating the limiting unit radius.

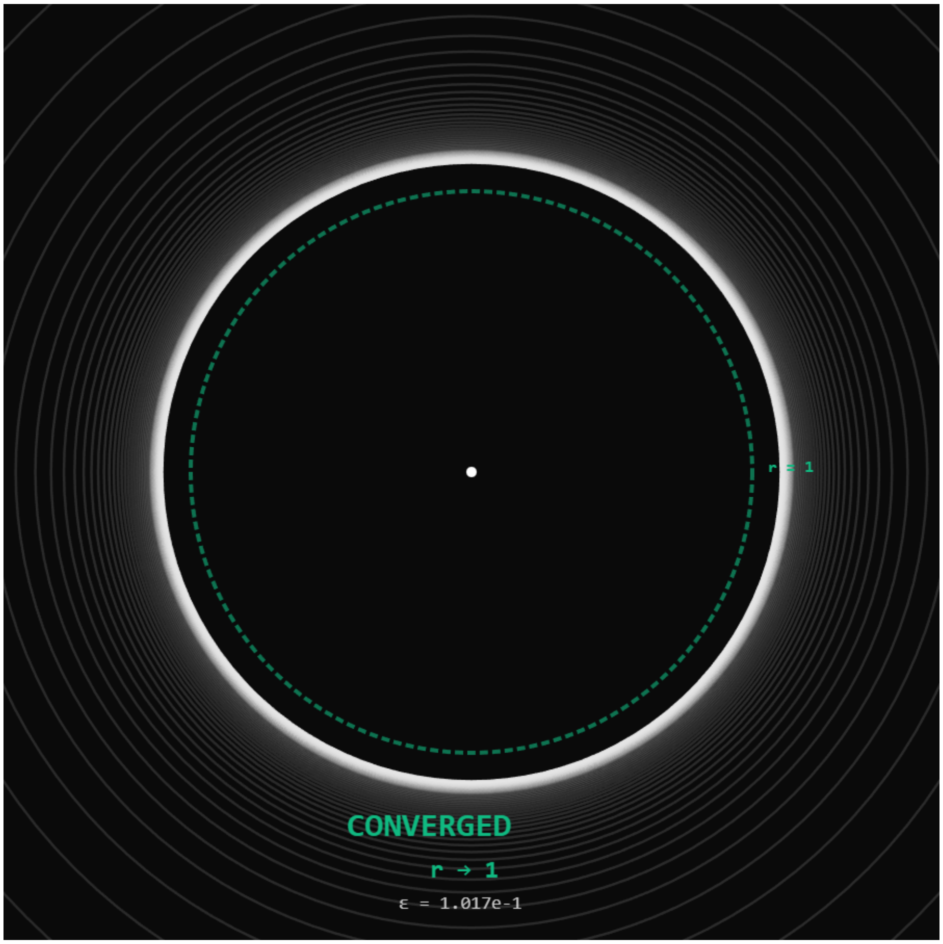

Figure 2: Concentric-circle visualization of iterated root convergence for $N = 127$. Each circle represents radius $r_n = N^{1/n}$. Adaptive zoom preserves distinguishability near $r = 1$, revealing continued convergence despite apparent numerical stabilization. The green dashed circle marks the limiting radius $r = 1$. The display shows convergence status and current error magnitude ε.

This representation emphasizes contraction toward a universal attractor independent of the initial value. The visual metaphor of "spiraling inward" toward unity, while geometrically approximate, effectively conveys the monotonic approach to the limit.

8 Educational Implications

The visualization supports intuitive understanding of limits by making convergence observable rather than inferred. Students can:

- Explore parameter dependence (varying N shows identical convergence rate structure)
- Confront numerical artifacts (recognizing when displays stabilize prematurely)

- Bridge procedural computation with theoretical reasoning (connecting $\ln(N)/n$ to visual decay)

The framework aligns with Tall and Vinner's concept image theory [1]: the geometric visualization provides a mental image that students can manipulate independently of the formal ε–δ definition.

9 Connection to Broader Convergence Phenomena

While the functional forms differ, the pedagogical challenge of visualizing approach to equilibrium appears across disciplines. In prior work on quantum decoherence near black holes [2], we examined how quantum coherence decays due to gravitational and thermal effects—a physically distinct but conceptually parallel process of equilibration.

Both contexts illustrate a general principle: systems approaching equilibrium (whether mathematical limits or physical steady states) exhibit dynamics that are mathematically exact yet computationally subtle. In quantum decoherence, environmental coupling drives pure states toward mixed states; in root convergence, iteration drives arbitrary initial values toward unity. While the mechanisms differ fundamentally (exponential vs. algebraic, physical vs. mathematical), the pedagogical challenge is shared: naive numerical observation suggests premature stabilization in both cases.

Visualization frameworks that adapt to the approach toward equilibrium can clarify dynamics that would otherwise remain opaque in numerical tables. This cross-domain observation is structural rather than mathematical; root convergence and quantum decoherence share no functional relationship. However, the utility of adaptive visualization applies broadly to any convergence process where computational artifacts obscure the underlying dynamics.

10 Discussion and Limitations

While root convergence is elementary, its numerical perception is not. This work demonstrates how visualization can clarify subtle computational effects even in simple analytical settings.

10.1 Primary Limitations

1. **Floating-point precision:** IEEE 754 double precision limits iteration depth. Arbitrary-precision arithmetic (e.g., Python's `mpmath`) would extend the observable range but at computational cost.

2. **Two-dimensional restriction:** The current framework visualizes real-valued roots. Extension to complex N or fractional iteration indices would require higher-dimensional representations.

3. **Empirical validation scope:** While Table 1 confirms theoretical predictions within $< 6\%$ for $n \leq 40$, systematic exploration of parameter space (varying N across orders of magnitude) remains incomplete.

10.2 Future Directions

Potential extensions include:

- Comparative visualization of convergence rates for different functional families (e.g., $N^{1/n}$ vs. e^{-n} vs. $1/n$)
- Interactive exploration of complex-valued roots, revealing spiral convergence in the complex plane
- Integration with symbolic computation (e.g., Mathematica, SageMath) for arbitrary-precision validation
- Pedagogical assessment through classroom deployment to measure impact on student understanding of limits

11 Conclusion

We have presented an interactive computational visualization of iterated root convergence that clarifies the distinction between mathematical limits and numerical artifacts. By combining geometric encoding, adaptive zoom, and theoretical validation, the framework transforms a simple analytical result into an instructive computational narrative.

The progression from calculator curiosity to hand-drawn sketches to interactive tool mirrors the exploratory process central to mathematical discovery. While the underlying mathematics is classical, the visualization approach offers a concrete pedagogical tool for bridging the gap between formal definitions and intuitive understanding.

All implementation details, numerical data, and source code are publicly available to support reproduction and extension.

Data Availability

All source code, figures, and exportable numerical data are publicly available at:

`https://github.com/rishavjha8515-hub/iterated-root-convergence-visualization`

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