Linear Programming: Production Planning

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Programming, Data Structures and Algorithms using Python

Week 11

Linear programming

- Constraints and objective to be optimized are linear functions
  - Constraints: $a_1x_1 + a_2x_2 + \cdots + a_mx_m \leq K$, $b_1x_1 + b_2x_2 + \cdots + b_mx_m \geq L$, ...
  - Objective: $c_1x_1 + c_2x_2 + \cdots + c_mx_m$
- Defines a convex feasible region remember the resulting area has to be a convex shape

1. a_1, a_2, ..., b_1, b_2... are coefficients.

2. Remember in the Halwa, Barfi problem, we had a constraint that b + h < 400 in this constraint we have coefficients a_1 = 1, a_2 = 1

3. We also had a constraint that b >= 0, in this constraint we have coefficient b_1 = 1

4. NOTE: All the constraints were linear, in other words there was no constraint where b*h occurred

ps: b = number of barfi boxes, h = no. of halwa boxes

1. Also we had objective function which as again a linear function, 100b + 600h

2. 100Rs. per barfi and 600Rs. per halwa box is the profit

Linear programming

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- Defines a convex feasible region

Simplex algorithm

Which in this case is the profit

- Start at any vertex, evaluate objective
- If an adjacent vertex has a better value, move
- If current vertex is better than all neighbours, stop

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Simplex algorithm

- Start at any vertex, evaluate objective
- If an adjacent vertex has a better value, move
- If current vertex is better than all neighbours, stop
- Can be exponential, but efficient in practice
- Theoretically efficient algorithms exist

LP duality

When you have an LP: (Linear Programming Problem), with constraints and objective function then you...

- Can always construct a linear combination of constraints that tightly captures upper bound on objective function
- Dual LP problem
  - Minimize linear combination of constraints
  - Variables are multipliers for the linear combination
  - Implicit constraint: multipliers are non-negative
  - Optimum solution solves both the original (primal) and the dual LP

Linear programming duality allows transforming an LP problem into a dual problem by multiplying the constraints by constants and introducing new variables. These multipliers have constraints of being non-negative. Solving the dual problem yields a solution to the original problem.

STILL NOT CLEAR? SKIP THIS SLIDE...

Production planning

Handwoven carpets

Production planning

Handwoven carpets

- 30 employees,
  - Each produces 20 carpets a month
  - Salary Rs 20,000
  - Labour cost: Rs 1000 per carpet

Production planning

Handwoven carpets

- 30 employees,
  - Each produces 20 carpets a month
  - Salary Rs $20,000$
  - Labour cost: Rs 1000 per carpet
- Seasonal monthly demand
  - $d_1, d_2, \ldots, d_{12}$, demand from January to December

Like in the sweet shop case where the daily demand was the same, here the demand in each month changes.

Production planning

Handwoven carpets

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Coping with varying demand

So what are possible ways to managing production of these carpets with changing monthly demand?

Production planning

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Coping with varying demand

- Overtime
  - Pay 80% extra
  - Overtime limit is 30% per worker

1. Overtime means every carpet produced after 20th carpet by the worker is considered as overtime
2. We pay 80% extra for overtime, so 1800/carpet.
3. Overtime limit is 30% per worker, this means worker can do a maximum overtime of 30%*20 = 6 carpets. So a worker can produce a max of 26 carpet per month

Production planning

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Coping with varying demand

- Overtime
  - Pay 80% extra
  - Overtime limit is 30% per worker
- Hiring and firing
  - Hiring costs Rs 3200 per worker
  - Firing costs Rs 4000 per worker

Hiring when demand goes up and Firing when demand goes down

Production planning

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Coping with varying demand

- Overtime
  - Pay 80% extra
  - Overtime limit is 30% per worker
- Hiring and firing
  - Hiring costs Rs 3200 per worker
  - Firing costs Rs 4000 per worker
- Make surplus and store
  - Costs Rs 80 per carpet

If you make surplus you need to spend 80Rs. for storing each carpet

Formulate a linear program

Target is to find the minimum cost I need to incur to meet these monthly demand

- 30 employees, each 20 carpets a month, salary Rs $20,000$, Rs 1000 per carpet
- Monthly demand $d_1, d_2, \ldots, d_{12}$
- Overtime: pay 80% extra, overtime limit is 30% per worker
- Hiring cost Rs 3200, firing cost Rs 4000
- Surplus storage cost: Rs 80 per carpet

Formulate a linear program

- w_i : workers in month i, $w_0 = 30$

  i = 1, 2,..., 12

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Formulate a linear program

- w_i : workers in month i, $w_0 = 30$
- x_i : carpets made in month i

Ideally you want to make d_i carpets

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So f_1 = 3 means in the month of Jan we have 27 workers

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- o_i : carpets made in overtime, month i
- h_i : workers hired at start of month i
- f_i : workers fired at start of month i
- s_i : surplus carpets after month i
  - $s_0 = 0$

We start the year with no inventory so s_0 is 0

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Formulate a linear program

$i = 1, 2 \ldots 12$

1 - w_i : workers in month i, $w_0 = 30$
2 - x_i : carpets made in month i
3 - o_i : carpets made in overtime, month i
4 - h_i : workers hired at start of month i
5 - f_i : workers fired at start of month i
6 - s_i : surplus carpets after month i
  - $s_0 = 0$
- 72 variables, plus w_0, s_0

$+$

$6 \times 12 = 72$

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Constraints

- All variables are nonnegative
  - $w_i, x_i, o_i, h_i, f_i, s_i \geq 0$

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Constraints

- All variables are nonnegative
  - $w_i, x_i, o_i, h_i, f_i, s_i \geq 0$
- Carpets made = regular + overtime
  - $x_i = 20w_i + o_i$

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- Carpets made = regular + overtime
  - $x_i = 20w_i + o_i$
- Number of workers match hiring/firing
  - $w_i = w_{i-1} + h_i - f_i$

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- Number of workers match hiring/firing
  - $w_i = w_{i-1} + h_i - f_i$
- Number of stored carpets connected to earlier stock, production, demand
  - $s_i = s_{i-1} + x_i - d_i$

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- Number of stored carpets connected to earlier stock, production, demand
  - $s_i = s_{i-1} + x_i - d_i$
- Overtime production at most 6 carpets per worker (30% of regular production)
  - $o_i \leq 6w_i$ Overall at most each worker can make 6 carpets in overtime

Formulate a linear program

Constraints

- $w_0 = 30$, $s_0 = 0$

For each $i \in \{1, 2, \ldots, 12\}$

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- $o_i \leq 6w_i$

Objective

- Minimize the cost

Salary costs for each month $20000(w_1 + w_2 + \cdots + w_{12}) +$

Hiring cost for each month $3200(h_1 + h_2 + \cdots + h_{12}) +$

firing cost for each month $4000(f_1 + f_2 + \cdots + f_{12}) +$

... $80(s_1 + s_2 + \cdots + s_{12}) +$

... $1800(o_1 + o_2 + \cdots + o_{12})$

...

Solving the linear program

- Run Simplex and find a solution

After running the algorithm you will get optimal value for each variable.
Remember in total you have 74 variables

Solving the linear program

- Run Simplex and find a solution
- Are we done?

Solving the linear program

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- Are we done?
- Optimum may have fractional values
  - Hire 10.6 workers in March

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Handling fractional solutions

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Handling fractional solutions

- Round off to 10 or 11 and recompute cost

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Handling fractional solutions

- Round off to 10 or 11 and recompute cost
- If values are "large", rounding does not affect quality of solution much

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Handling fractional solutions

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- If values are "large", rounding does not affect quality of solution much
- Values are "small", need more care when rounding

Solving the linear program

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Handling fractional solutions

- Round off to 10 or 11 and recompute cost
- If values are "large", rounding does not affect quality of solution much
- Values are "small", need more care when rounding
- Insisting on integer solutions makes the problem computationally intractable

  Integer Linear Programming