Linear Programming

Madhavan Mukund

https://www.cmi.ac.in/~madhavan

Programming, Data Structures and Algorithms using Python

Week 11

Optimization problems

- Many computational tasks involve optimization
  - Shortest path
  - Minimum cost spanning tree
  - Longest common subsequence

Optimization problems

- Many computational tasks involve optimization
  - Shortest path
  - Minimum cost spanning tree
  - Longest common subsequence
- . . . subject to constraints
  - Shortest path follows edges in the graph
  - Spanning tree is a subset of the given edges
  - Subsequence letters are from the given words

Optimization problems

- Many computational tasks involve optimization
  - Shortest path
  - Minimum cost spanning tree
  - Longest common subsequence
- ...subject to constraints
  - Shortest path follows edges in the graph
  - Spanning tree is a subset of the given edges
  - Subsequence letters are from the given words

Linear programming

- Constraints and objective to be optimized are linear functions
  - Constraints: $a_1x_1 + a_2x_2 + \cdots + a_mx_m \leq K$, $b_1x_1 + b_2x_2 + \cdots + b_mx_m \geq L$, ...
  - Objective: $c_1x_1 + c_2x_2 + \cdots + c_mx_m$

Example: Maximize profits

Grandiose Sweets sells cashew barfis and dry fruit halwa.

Example: Maximize profits

Grandiose Sweets sells cashew barfis and dry fruit halwa.

- Profit for each box of barfis is Rs 100
- Profit for each box of halwa is Rs 600

Example: Maximize profits

Grandiose Sweets sells cashew barfis and dry fruit halwa.

- Profit for each box of barfis is Rs 100
- Profit for each box of halwa is Rs 600
- Daily demand for barfis is at most 200 boxes
- Daily demand for halwa is at most 300 boxes

Example: Maximize profits

Grandiose Sweets sells cashew barfis and dry fruit halwa.

- Profit for each box of barfis is Rs 100
- Profit for each box of halwa is Rs 600
- Daily demand for barfis is at most 200 boxes
- Daily demand for halwa is at most 300 boxes
- Staff can produce 400 boxes a day, altogether

Example: Maximize profits

Grandiose Sweets sells cashew barfis and dry fruit halwa.

- Profit for each box of barfis is Rs 100
- Profit for each box of halwa is Rs 600
- Daily demand for barfis is at most 200 boxes
- Daily demand for halwa is at most 300 boxes
- Staff can produce 400 boxes a day, altogether
- What is the most profitable mix of barfis and halwa to produce?

Example: Maximize profits

Grandiose Sweets sells cashew barfis and dry fruit halwa.

- Profit for each box of barfis is Rs 100
- Profit for each box of halwa is Rs 600
- Daily demand for barfis is at most 200 boxes
- Daily demand for halwa is at most 300 boxes
- Staff can produce 400 boxes a day, altogether
- What is the most profitable mix of barfis and halwa to produce?

Linear programming model

Example: Maximize profits

Grandiose Sweets sells cashew barfis and dry fruit halwa.

- Profit for each box of barfis is Rs 100
- Profit for each box of halwa is Rs 600
- Daily demand for barfis is at most 200 boxes
- Daily demand for halwa is at most 300 boxes
- Staff can produce 400 boxes a day, altogether
- What is the most profitable mix of barfis and halwa to produce?

Linear programming model

- b boxes of barfi to produce per day
- h boxes of halwa to produce per day

Example: Maximize profits

Grandiose Sweets sells cashew barfis and dry fruit halwa.

- Profit for each box of barfis is Rs 100
- Profit for each box of halwa is Rs 600
- Daily demand for barfis is at most 200 boxes
- Daily demand for halwa is at most 300 boxes
- Staff can produce 400 boxes a day, altogether
- What is the most profitable mix of barfis and halwa to produce?

Linear programming model

- b boxes of barfi to produce per day
- h boxes of halwa to produce per day
- Profit: $100b + 600h$

Example: Maximize profits

Grandiose Sweets sells cashew barfis and dry fruit halwa.

- Profit for each box of barfis is Rs 100
- Profit for each box of halwa is Rs 600
- Daily demand for barfis is at most 200 boxes
- Daily demand for halwa is at most 300 boxes
- Staff can produce 400 boxes a day, altogether
- What is the most profitable mix of barfis and halwa to produce?

Linear programming model

- b boxes of barfi to produce per day
- h boxes of halwa to produce per day
- Profit: $100b + 600h$
- Demand constraints:
  - $b \leq 200$
  - $h \leq 300$

As demand for barfis is at most 200 boxes so there is no point in producing more than that, and thus b <= 200

Similarly for halwa

Example: Maximize profits

Grandiose Sweets sells cashew barfis and dry fruit halwa.

- Profit for each box of barfis is Rs 100
- Profit for each box of halwa is Rs 600
- Daily demand for barfis is at most 200 boxes
- Daily demand for halwa is at most 300 boxes
- Staff can produce 400 boxes a day, altogether
- What is the most profitable mix of barfis and halwa to produce?

Linear programming model

- b boxes of barfi to produce per day
- h boxes of halwa to produce per day
- Profit: $100b + 600h$
- Demand constraints:
  - $b \leq 200$
  - $h \leq 300$
- Production constraint: $b + h \leq 400$

Example: Maximize profits

Grandiose Sweets sells cashew barfis and dry fruit halwa.

- Profit for each box of barfis is Rs 100
- Profit for each box of halwa is Rs 600
- Daily demand for barfis is at most 200 boxes
- Daily demand for halwa is at most 300 boxes
- Staff can produce 400 boxes a day, altogether
- What is the most profitable mix of barfis and halwa to produce?

Linear programming model

- b boxes of barfi to produce per day
- h boxes of halwa to produce per day
- Profit: $100b + 600h$
- Demand constraints:
  - $b \leq 200$
  - $h \leq 300$
- Production constraint: $b + h \leq 400$
- Implicit constraints:
  - $b \geq 0$
  - $h \geq 0$

as we cannot make -ve boxes of Barfis or Halwa right?

Linear program

Objective

- Maximize $100b + 600h$

Linear programming model

- b boxes of barfi to produce per day
- h boxes of halwa to produce per day
- Profit: $100b + 600h$
- Demand constraints:
  - $b \leq 200$
  - $h \leq 300$
- Production constraint: $b + h \leq 400$
- Implicit constraints:
  - $b \geq 0$
  - $h \geq 0$

Linear program

Objective

- Maximize $100b + 600h$

Constraints

- $b \leq 200$
- $h \leq 300$
- $b + h \leq 400$
- $b \geq 0$
- $h \geq 0$

Linear programming model

- b boxes of barfi to produce per day
- h boxes of halwa to produce per day
- Profit: $100b + 600h$
- Demand constraints:
  - $b \leq 200$
  - $h \leq 300$
- Production constraint: $b + h \leq 400$
- Implicit constraints:
  - $b \geq 0$
  - $h \geq 0$

Linear program

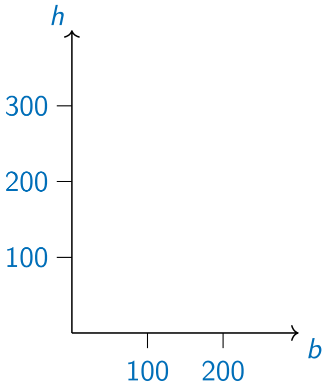
Objective

- Maximize $100b + 600h$

Constraints

- $b \leq 200$
- $h \leq 300$
- $b + h \leq 400$
- $b \geq 0$
- $h \geq 0$

Pictorially



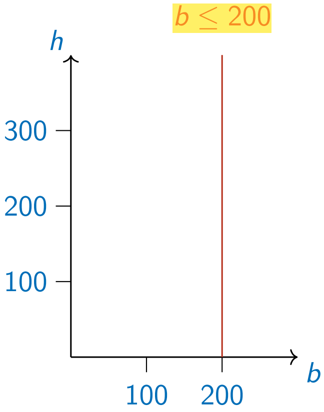
Objective

- Maximize $100b + 600h$

Constraints

- $b \leq 200$
- $h \leq 300$
- $b + h \leq 400$
- $b \geq 0$
- $h \geq 0$

Pictorially



Linear program

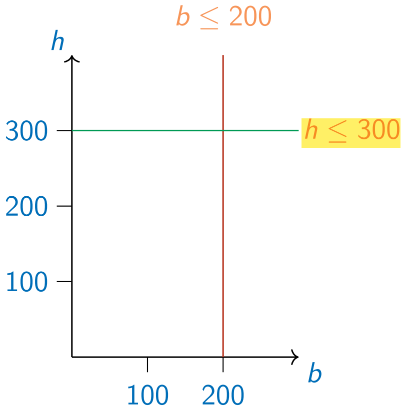
Objective

- Maximize $100b + 600h$

Constraints

- $b \leq 200$
- $h \leq 300$
- $b + h \leq 400$
- $b \geq 0$
- $h \geq 0$

Pictorially



Linear program

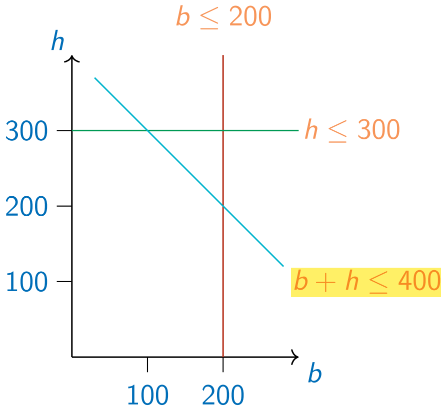
Objective

- Maximize $100b + 600h$

Constraints

- $b \leq 200$
- $h \leq 300$
- $b + h \leq 400$
- $b \geq 0$
- $h \geq 0$

Pictorially



Linear program

Objective

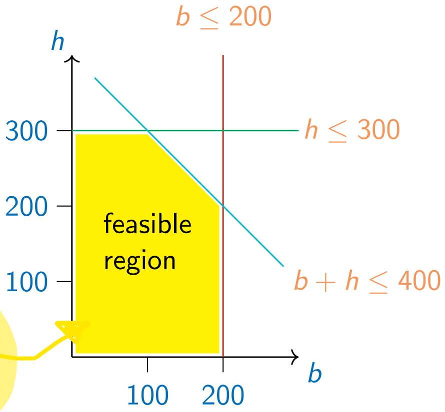
- Maximize $100b + 600h$

Constraints

- $b \leq 200$
- $h \leq 300$
- $b + h \leq 400$
- $b \geq 0$
- $h \geq 0$

So now all the values of b and h within this region are valid. And we need to maximize objective within this region

Pictorially



Linear program

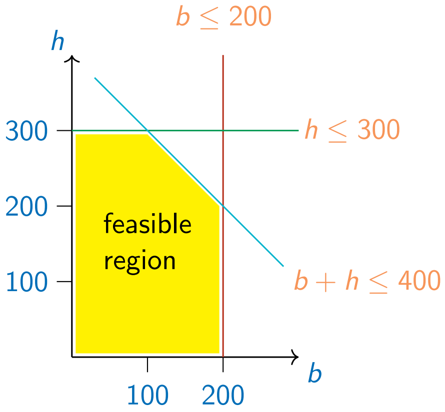
Objective

- Maximize $100b + 600h$

Constraints

- $b \leq 200$
- $h \leq 300$
- $b + h \leq 400$
- $b \geq 0$
- $h \geq 0$

Pictorially



Objective: $\boldsymbol{c = 100b + 600h}$

Linear program

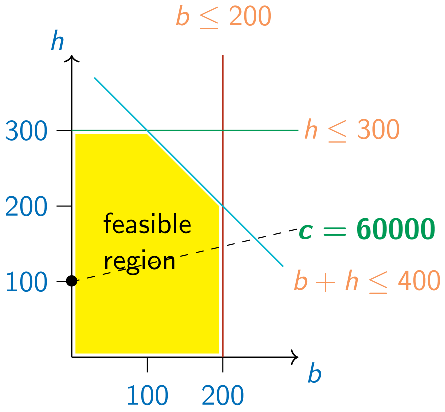
Objective

- Maximize $100b + 600h$

Constraints

- $b \leq 200$
- $h \leq 300$
- $b + h \leq 400$
- $b \geq 0$
- $h \geq 0$

Pictorially



Objective: $c = 100b + 600h$

Linear program

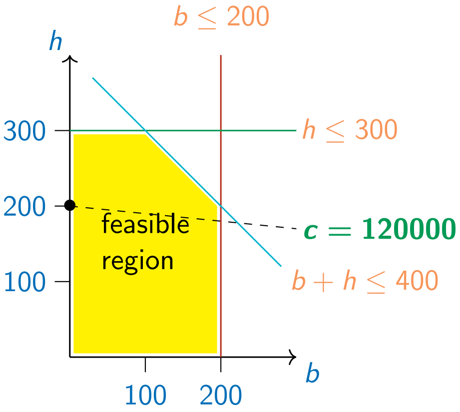
Objective

- Maximize $100b + 600h$

Constraints

- $b \leq 200$
- $h \leq 300$
- $b + h \leq 400$
- $b \geq 0$
- $h \geq 0$

Pictorially



Objective: $c = 100b + 600h$

Linear program

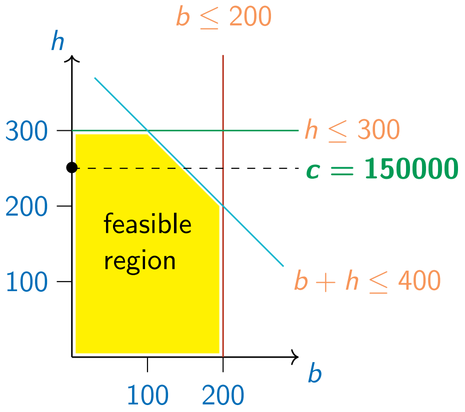
Objective

- Maximize $100b + 600h$

Constraints

- $b \leq 200$
- $h \leq 300$
- $b + h \leq 400$
- $b \geq 0$
- $h \geq 0$

Pictorially



Objective: $c = 100b + 600h$

Linear program

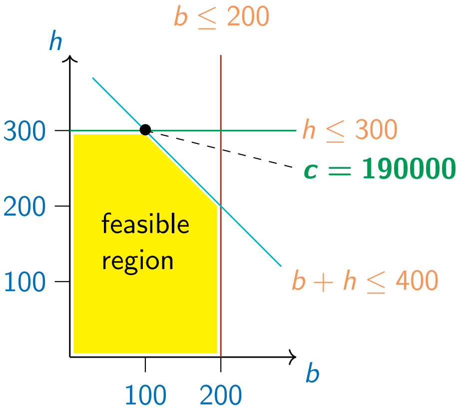
Objective

- Maximize $100b + 600h$

Constraints

- $b \leq 200$
- $h \leq 300$
- $b + h \leq 400$
- $b \geq 0$
- $h \geq 0$

Pictorially



Objective: $c = 100b + 600h$

Linear program

Objective

- Maximize $100b + 600h$

Constraints

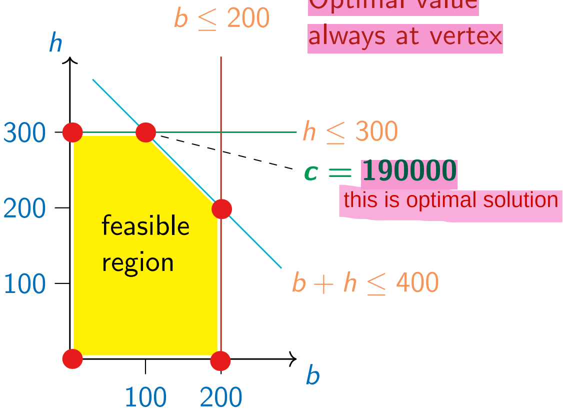
- $b \leq 200$
- $h \leq 300$
- $b + h \leq 400$
- $b \geq 0$
- $h \geq 0$

Optimal value always at some vertex in the graph

Pictorially

IMPORTANT!

Optimal value always at vertex



Objective: $c = 100b + 600h$

Solving linear programs

Simplex algorithm

- Start at any vertex, evaluate objective

Solving linear programs

Simplex algorithm

- Start at any vertex, evaluate objective
- If an adjacent vertex has a better value, move

Solving linear programs

Simplex algorithm

- Start at any vertex, evaluate objective
- If an adjacent vertex has a better value, move
- If current vertex is better than all neighbours, stop

Solving linear programs

Simplex algorithm

- Start at any vertex, evaluate objective
- If an adjacent vertex has a better value, move
- If current vertex is better than all neighbours, stop
- Can be exponential, but efficient in practice

Solving linear programs

Simplex algorithm

- Start at any vertex, evaluate objective
- If an adjacent vertex has a better value, move
- If current vertex is better than all neighbours, stop
- Can be exponential, but efficient in practice
- Theoretically efficient algorithms exist

Solving linear programs

Simplex algorithm

- Start at any vertex, evaluate objective
- If an adjacent vertex has a better value, move
- If current vertex is better than all neighbours, stop
- Can be exponential, but efficient in practice
- Theoretically efficient algorithms exist

Existence of solutions

Following conditions are must for the solution to exist

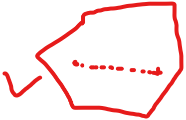
Solving linear programs

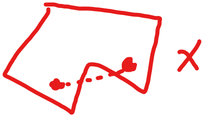
Simplex algorithm

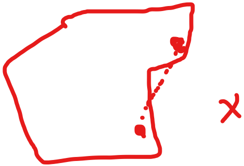
- Start at any vertex, evaluate objective
- If an adjacent vertex has a better value, move
- If current vertex is better than all neighbours, stop
- Can be exponential, but efficient in practice
- Theoretically efficient algorithms exist

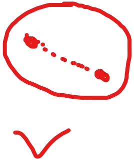
Existence of solutions

- Feasible region is convex









Solving linear programs

Simplex algorithm

- Start at any vertex, evaluate objective
- If an adjacent vertex has a better value, move
- If current vertex is better than all neighbours, stop
- Can be exponential, but efficient in practice
- Theoretically efficient algorithms exist

Existence of solutions

- Feasible region is convex
- May be empty — constraints are unsatisfiable, no solutions

b >= 250
h >= 250

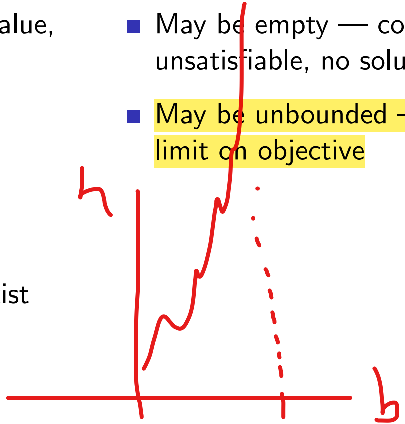
but
b + h <= 250

Simplex algorithm

- Start at any vertex, evaluate objective
- If an adjacent vertex has a better value, move
- If current vertex is better than all neighbours, stop
- Can be exponential, but efficient in practice
- Theoretically efficient algorithms exist

Existence of solutions

- Feasible region is convex
- May be empty — constraints are unsatisfiable, no solutions
- May be unbounded — no upper/lower limit on objective



Example, extended

Grandiose Sweets adds almond rasmalai

Example, extended

Grandiose Sweets adds almond rasmalai

- Profit per box: barfis – Rs 100, halwa – Rs 600, rasmalai – Rs 1300

Example, extended

Grandiose Sweets adds almond rasmalai

- Profit per box: barfis – Rs 100, halwa – Rs 600, rasmalai – Rs 1300
- Daily demand, in boxes: barfis – 200, halwa – 300, rasmalai – unlimited

Example, extended

Grandiose Sweets adds almond rasmalai

- Profit per box: barfis – Rs 100, halwa – Rs 600, rasmalai – Rs 1300
- Daily demand, in boxes: barfis – 200, halwa – 300, rasmalai – unlimited
- Production capacity: 400 boxes a day, altogether

Example, extended

Grandiose Sweets adds almond rasmalai

- Profit per box: barfis – Rs 100, halwa – Rs 600, rasmalai – Rs 1300
- Daily demand, in boxes: barfis – 200, halwa – 300, rasmalai – unlimited
- Production capacity: 400 boxes a day, altogether
- Milk supply is limited
  - 600 boxes halwa or 200 boxes rasmalai
  - Or any combination (rasmalai needs 3 times as much milk)

Example, extended

Grandiose Sweets adds almond rasmalai

- Profit per box: barfis – Rs 100, halwa – Rs 600, rasmalai – Rs 1300
- Daily demand, in boxes: barfis – 200, halwa – 300, rasmalai – unlimited
- Production capacity: 400 boxes a day, altogether
- Milk supply is limited
  - 600 boxes halwa or 200 boxes rasmalai
  - Or any combination (rasmalai needs 3 times as much milk)
- Most profitable mix to produce?

Example, extended

Grandiose Sweets adds almond rasmalai

- Profit per box: barfis – Rs 100, halwa – Rs 600, rasmalai – Rs 1300
- Daily demand, in boxes: barfis – 200, halwa – 300, rasmalai – unlimited
- Production capacity: 400 boxes a day, altogether
- Milk supply is limited
  - 600 boxes halwa or 200 boxes rasmalai
  - Or any combination (rasmalai needs 3 times as much milk)
- Most profitable mix to produce?

New linear program

Example, extended

Grandiose Sweets adds almond rasmalai

- Profit per box: barfis – Rs 100, halwa – Rs 600, rasmalai – Rs 1300
- Daily demand, in boxes: barfis – 200, halwa – 300, rasmalai – unlimited
- Production capacity: 400 boxes a day, altogether
- Milk supply is limited
  - 600 boxes halwa or 200 boxes rasmalai
  - Or any combination (rasmalai needs 3 times as much milk)
- Most profitable mix to produce?

New linear program

Objective

- Maximize $100b + 600h + 1300r$

Example, extended

Grandiose Sweets adds almond rasmalai

- Profit per box: barfis – Rs 100, halwa – Rs 600, rasmalai – Rs 1300
- Daily demand, in boxes: barfis – 200, halwa – 300, rasmalai – unlimited
- Production capacity: 400 boxes a day, altogether
- Milk supply is limited
  - 600 boxes halwa or 200 boxes rasmalai
  - Or any combination (rasmalai needs 3 times as much milk)
- Most profitable mix to produce?

New linear program

Objective

- Maximize $100b + 600h + 1300r$

Constraints

- $b \leq 200$
- $h \leq 300$

Example, extended

Grandiose Sweets adds almond rasmalai

- Profit per box: barfis – Rs 100, halwa – Rs 600, rasmalai – Rs 1300
- Daily demand, in boxes: barfis – 200, halwa – 300, rasmalai – unlimited
- Production capacity: 400 boxes a day, altogether
- Milk supply is limited
  - 600 boxes halwa or 200 boxes rasmalai
  - Or any combination (rasmalai needs 3 times as much milk)
- Most profitable mix to produce?

New linear program

Objective

- Maximize $100b + 600h + 1300r$

Constraints

- $b \leq 200$
- $h \leq 300$
- $b + h + r \leq 400$

Example, extended

Grandiose Sweets adds almond rasmalai

- Profit per box: barfis – Rs 100, halwa – Rs 600, rasmalai – Rs 1300
- Daily demand, in boxes: barfis – 200, halwa – 300, rasmalai – unlimited
- Production capacity: 400 boxes a day, altogether
- Milk supply is limited
  - 600 boxes halwa or 200 boxes rasmalai
  - Or any combination (rasmalai needs 3 times as much milk)
- Most profitable mix to produce?

New linear program

Objective

- Maximize $100b + 600h + 1300r$

Constraints

- $b \leq 200$
- $h \leq 300$
- $b + h + r \leq 400$
- $h + 3r \leq 600$

Example, extended

Grandiose Sweets adds almond rasmalai

- Profit per box: barfis – Rs 100, halwa – Rs 600, rasmalai – Rs 1300
- Daily demand, in boxes: barfis – 200, halwa – 300, rasmalai – unlimited
- Production capacity: 400 boxes a day, altogether
- Milk supply is limited
  - 600 boxes halwa or 200 boxes rasmalai
  - Or any combination (rasmalai needs 3 times as much milk)
- Most profitable mix to produce?

New linear program

Objective

- Maximize $100b + 600h + 1300r$

Constraints

- $b \leq 200$
- $h \leq 300$
- $b + h + r \leq 400$
- $h + 3r \leq 600$
- $b \geq 0,\ h \geq 0,\ r \geq 0$

Example, extended

New linear program

Objective

- Maximize $100b + 600h + 1300r$

Constraints

- $b \leq 200$
- $h \leq 300$
- $b + h + r \leq 400$
- $h + 3r \leq 600$
- $b \geq 0,\ h \geq 0,\ r \geq 0$

Example, extended

New linear program

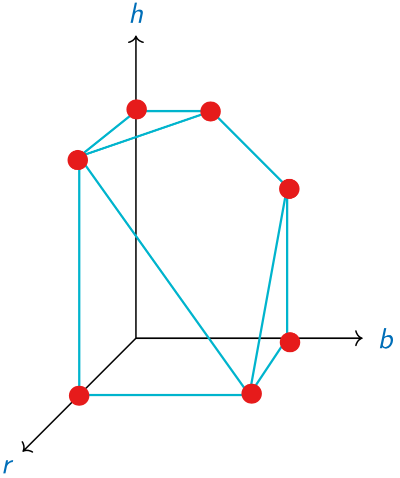
Objective

- Maximize $100b + 600h + 1300r$

Constraints

- $b \leq 200$
- $h \leq 300$
- $b + h + r \leq 400$
- $h + 3r \leq 600$
- $b \geq 0,\ h \geq 0,\ r \geq 0$

Again, optimal solutions are always on a vertex



Example, extended

New linear program

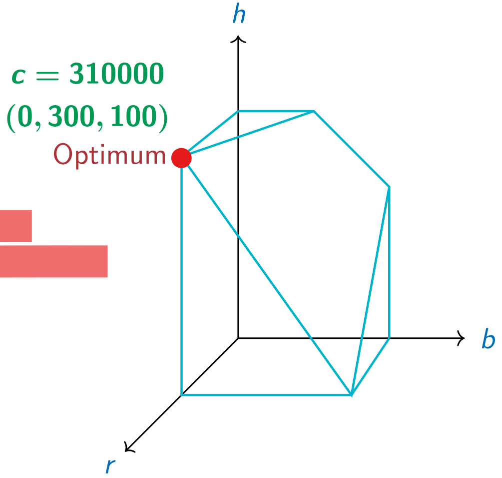
Objective

- Maximize $100b + 600h + 1300r$

Constraints

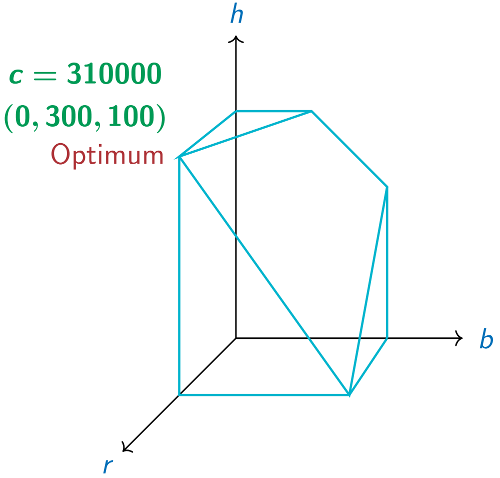
- $b \leq 200$
- $h \leq 300$
- $b + h + r \leq 400$
- $h + 3r \leq 600$
- $b \geq 0, \; h \geq 0, \; r \geq 0$

0 barfi
300 halwa
100 rasmalai



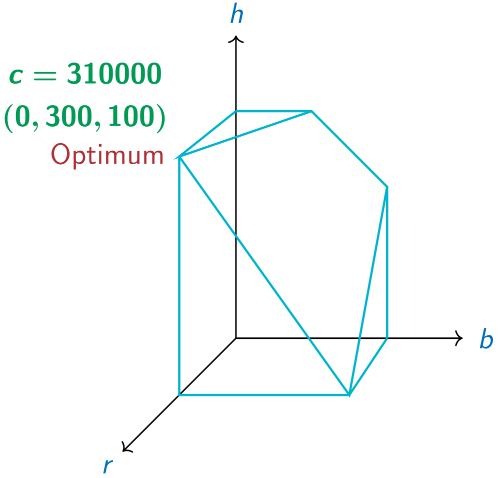
Example, extended

- Why is $(0, 300, 100)$ optimal?



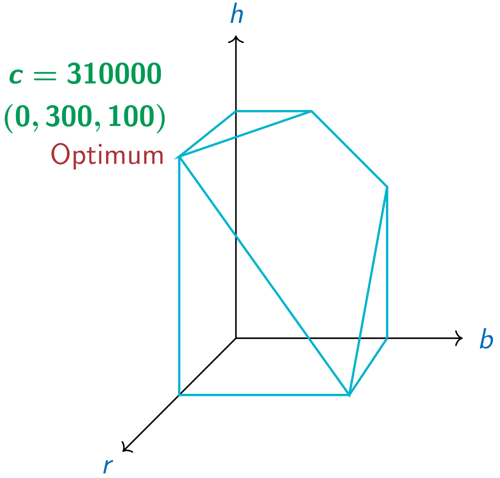
Example, extended

- Why is $(0, 300, 100)$ optimal?
- Profit is $100b + 600h + 1300r$



Example, extended

- Why is $(0, 300, 100)$ optimal?
- Profit is $100b + 600h + 1300r$
- Consider the following constraints
  - (A) $h \leq 300$
  - (B) $b + h + r \leq 400$
  - (C) $h + 3r \leq 600$



Example, extended

- Why is $(0, 300, 100)$ optimal?
- Profit is $100b + 600h + 1300r$
- Consider the following constraints

  (A) $h \leq 300$

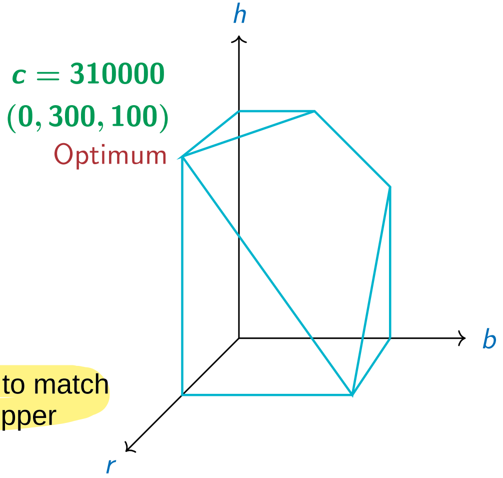
  (B) $b + h + r \leq 400$

  (C) $h + 3r \leq 600$

- Combine as

  $100 \cdot (A) + 100 \cdot (B) + 400 \cdot (C)$

Trying to make a linear combination of constraints to match the resultant with our objective function to get an upper bound



Example, extended

- Why is $(0, 300, 100)$ optimal?
- Profit is $100b + 600h + 1300r$
- Consider the following constraints

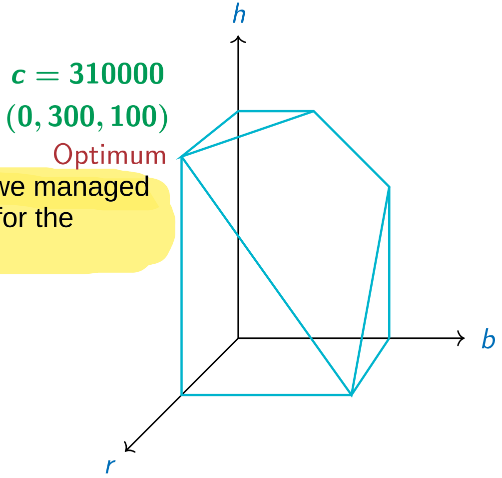
  (A) $h \leq 300$

  (B) $b + h + r \leq 400$

  (C) $h + 3r \leq 600$

- Combine as
  $100 \cdot (A) + 100 \cdot (B) + 400 \cdot (C)$
- Result is
  $100b + 600h + 1300r \leq 310000$

Both are same and we managed to find upper bound for the objective (profit)



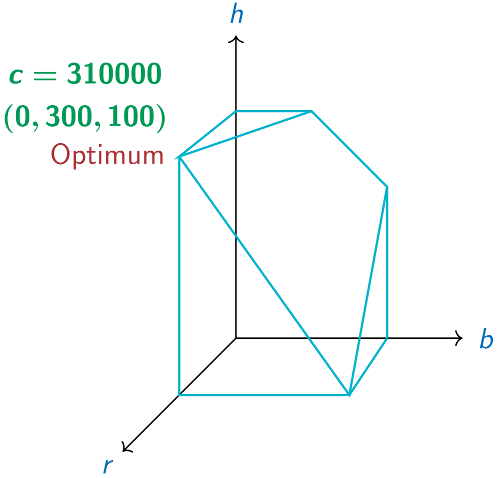
Example, extended

- Why is $(0, 300, 100)$ optimal?
- Profit is $100b + 600h + 1300r$
- Consider the following constraints

  (A) $h \leq 300$

  (B) $b + h + r \leq 400$

  (C) $h + 3r \leq 600$
- Combine as
  $100 \cdot (A) + 100 \cdot (B) + 400 \cdot (C)$
- Result is
  $100b + 600h + 1300r \leq 310000$
- LHS is profit, so value at $(0, 300, 100)$ matches upper bound on profit



LP Duality

- We derived an upper bound on the objective through a linear combination of constraints

- Why is $(0, 300, 100)$ optimal?
- Profit is $100b + 600h + 1300r$
- Consider the following constraints

  (A) $h \leq 300$

  (B) $b + h + r \leq 400$

  (C) $h + 3r \leq 600$
- Combine as
  $100 \cdot (A) + 100 \cdot (B) + 400 \cdot (C)$
- Result is
  $100b + 600h + 1300r \leq 310000$
- LHS is profit, so value at $(0, 300, 100)$ matches upper bound on profit

LP Duality

- We derived an upper bound on the objective through a linear combination of constraints
- This is always possible!

- Why is $(0, 300, 100)$ optimal?
- Profit is $100b + 600h + 1300r$
- Consider the following constraints

  (A) $h \leq 300$

  (B) $b + h + r \leq 400$

  (C) $h + 3r \leq 600$
- Combine as $100 \cdot (A) + 100 \cdot (B) + 400 \cdot (C)$
- Result is $100b + 600h + 1300r \leq 310000$
- LHS is profit, so value at $(0, 300, 100)$ matches upper bound on profit

LP Duality

- We derived an upper bound on the objective through a linear combination of constraints
- This is always possible!
- Dual LP problem
  - Minimize linear combination of constraints
  - Variables are multipliers for the linear combination
  - Implicit constraint: multipliers are non-negative
  - Optimum solution solves both the original (primal) and the dual LP

This was not explained clearly,
if needed see 2min video below
https://youtu.be/3YPqsJpYQJU?t=1560

- Why is $(0, 300, 100)$ optimal?
- Profit is $100b + 600h + 1300r$
- Consider the following constraints
  - (A) $h \leq 300$
  - (B) $b + h + r \leq 400$
  - (C) $h + 3r \leq 600$
- Combine as $100 \cdot (A) + 100 \cdot (B) + 400 \cdot (C)$
- Result is $100b + 600h + 1300r \leq 310000$
- LHS is profit, so value at $(0, 300, 100)$ matches upper bound on profit