Intractability: Checking Algorithms

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Programming, Data Structures and Algorithms using Python

Week 11

Intractability refers to problems that are extremely difficult or practically impossible to solve within a reasonable amount of time using current computational resources.

Efficient algorithms

- Shortest path, minimum cost spanning tree, maximum flow, . . . have polynomial time algorithms

$n \log n, n^3, n^2 \ldots$

We know these have a big difference but generically algorithms with polynomial time algorithms are considered to be efficient algorithms

Efficient algorithms

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- Search space for solutions is exponential
  - All possible paths, all possible spanning trees, all possible subsets of edges, ...

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- Brute force: scan exponential possibilities and choose the best

We can always use a brute force to solve a problem... but not efficient and some times takes exponential time

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- Do all problems admit such efficient solutions?

Now the question is can we find clever tricks to reduce the time from exponential to polynomial every time?

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- Unfortunately not

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- Unfortunately not
- For a large class of "natural" problems, no shortcut is known to exist

Generating vs checking

- A teacher assigns homework:
  - Factorize a large number that is the product of two primes

Solving the problem

Checking if the solution is correct

Generating vs checking

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- Student: Given N, find p,q such that $pq = N$
  - Generate a solution

Generating Solution

The student needs to do the hard work of finding two primes which gives product N

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- Teacher: Given a student's solution p,q, verify that $pq = N$
  - Check a solution

Checking Solution

The teacher simply needs to check the solution by verifying product is N

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Checking algorithms

- Checking algorithm C for problem P

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Checking algorithms

- Checking algorithm C for problem P
- Takes an input instance I for P and a solution "certificate" S for I

Solution "certificate" is simply the solution for the problem P and input I

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In this case following is the checking algorithm

Checking algorithms

- Checking algorithm C for problem P
- Takes an input instance I for P and a solution "certificate" S for I
- C outputs yes if S represents a valid solution for I, no otherwise
- For factorization
  - I is N input
  - S is $\{p, q\}$ solution "certificate"
  - C involves verifying that $pq = N$

checking algorithm

Boolean satisfiability

- Boolean variables $x_1, x_2, x_3, \ldots$

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  - $\neg x_j$ — negation of x_j
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- Assign suitable values {True,False} to $x_1, x_2, x_3, \ldots$ so that the formula evaluates to True

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2) \wedge (x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3)$$

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- Now there is no satisfying assignment

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  - Try each possible assignment V to $x_1, x_2, \ldots, x_n$

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  - Given formula F and valuation $V(x)$ for each x, substitute into formula and evaluate

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$$(x_1 \wedge \neg x_2 \wedge x_3 \wedge \ldots \wedge x_k)$$

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- . . . and a formula is a disjunction of clauses $C_1 \vee C_2 \vee \ldots \vee C_m$

Now you just need one clause which is True as that will make the formula True

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- Each clause forces a unique valuation
- Try each clause in sequence

Travelling salesman

- A network of cities with distances between each pair

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- A complete graph $G = (V, E)$ with edge weights

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- A complete graph $G = (V, E)$ with edge weights
- Find the shortest tour that visits each city exactly once
- Simple cycle $x, y, z, \ldots, x$ of minimum cost, visiting all vertices

Simple cycle as same city must not be repeated. I need to find a cycle with minimum cost.
And the salesman want to visit all cities

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- Given a graph G and a proposed solution S we can
  - Verify that S is a cycle

We can verify that it is a simple cycle and it visits all the vertices

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- Checking algorithm must give a yes/no answer
- Given a graph G and a proposed solution S we can
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  - Compute its cost

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- Given a graph G and a proposed solution S we can
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  - How to check that S is the least cost cycle?

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- Transform the problem
- Is there a tour with cost at most K?

In other words can the sales man travel all vertices with at most K cost?

We can check this easily

We are just checking if it is possible or not

We are not interested in finding out the minimum cost

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- Is there a tour with cost at most K?
- Now, given a solution S, we can check it
- For the original problem, cost is at most the sum of all the edge weights in the graph

Each edge connects two city (or location)

Cost cannot be more than sum of all edge weights

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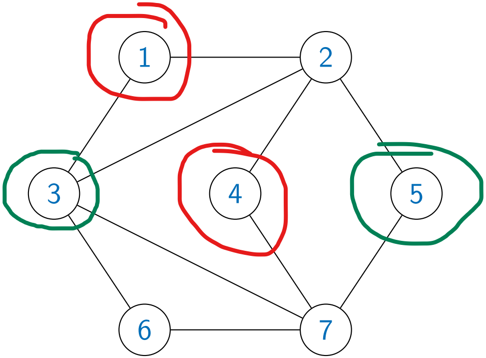
- Is there a tour with cost at most K?
- Now, given a solution S, we can check it
- For the original problem, cost is at most the sum of all the edge weights in the graph
- Find optimum K — test different values using binary search

Do binary search, take cost from 0 to sum of all edge weights which acts as an upper bound.
Example: 0.....1000, here 1000 is sum of all edge weights. Start, can tour in 500? if yes can we tour in 250? if yes can we tour in 125? if no then can we tour in... finally find the optimal k

Independent set

- u, v are independent if there is no edge (u, v)



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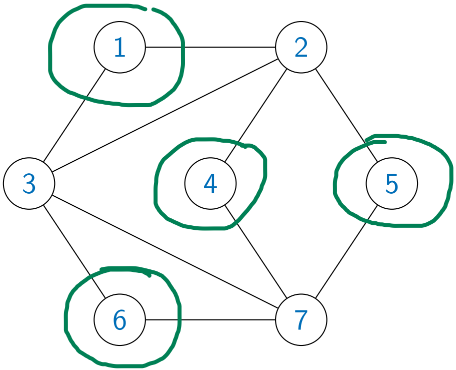

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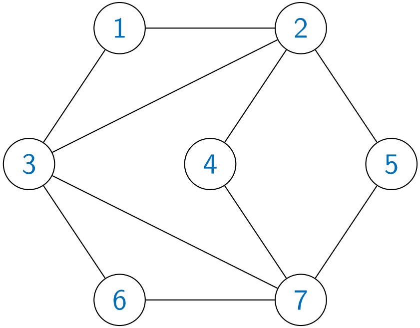
- Constitute a neutral committee where none of the members know each other



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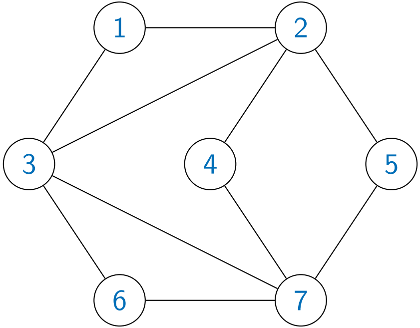

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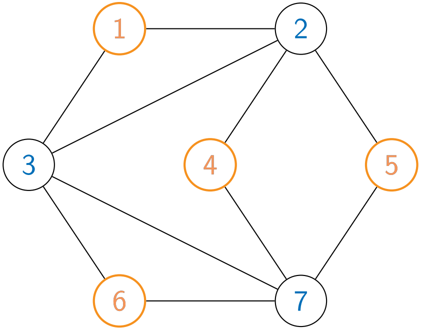

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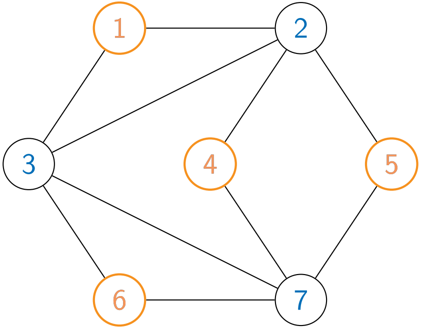
- Checking version: Is there an independent set of size K?



Vertex cover

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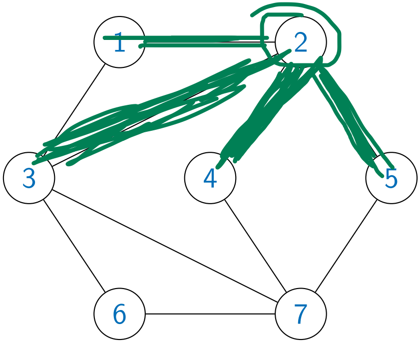

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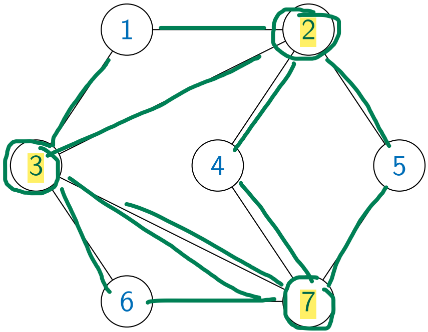

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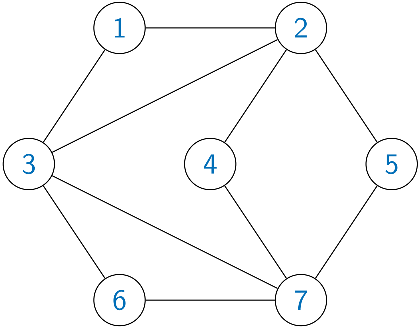
- Position surveillance cameras at intersections to watch all roads



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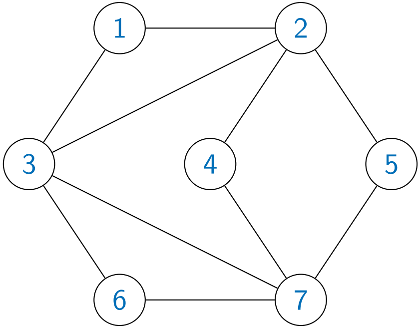

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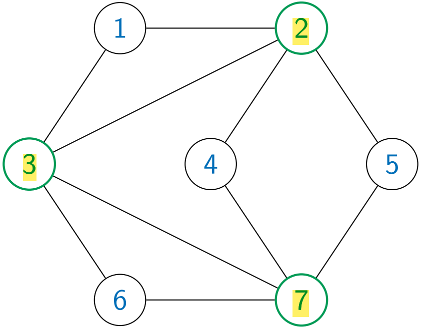

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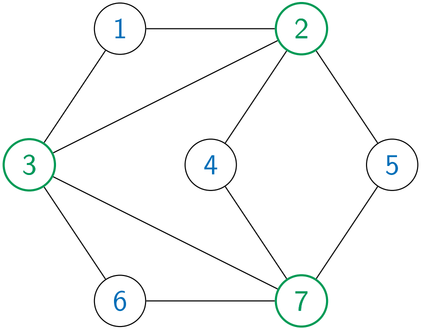
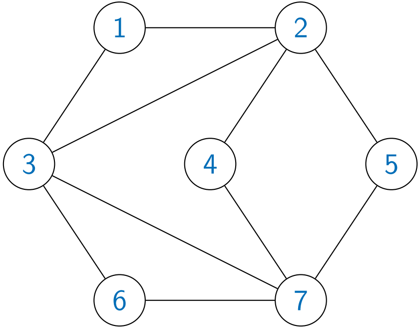


Connecting independent set, vertex cover

- U is an independent set of size K iff $V \setminus U$ is a vertex cover of size $N - K$



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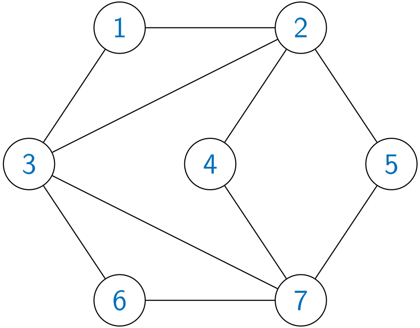
($\Rightarrow$) Every edge (u, v) has at most one end point in U, so at least one end point in $V \setminus U$



Connecting independent set, vertex cover

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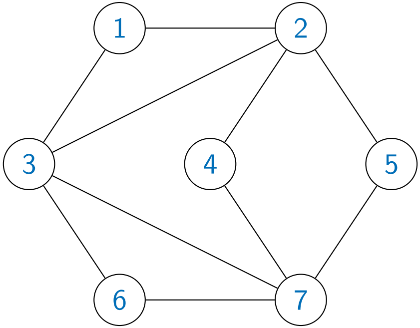

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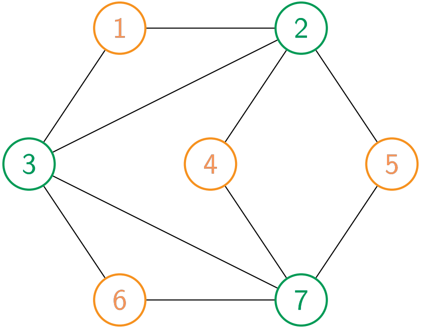
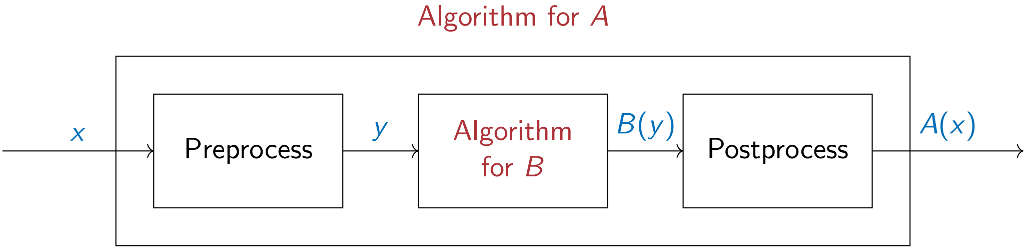


Reductions

- Independent set and vertex cover reduce to each other



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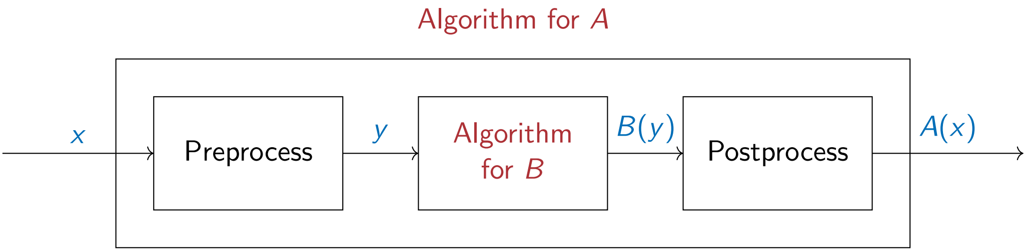
- Recall: if A reduces to B and A is intractable, so is B



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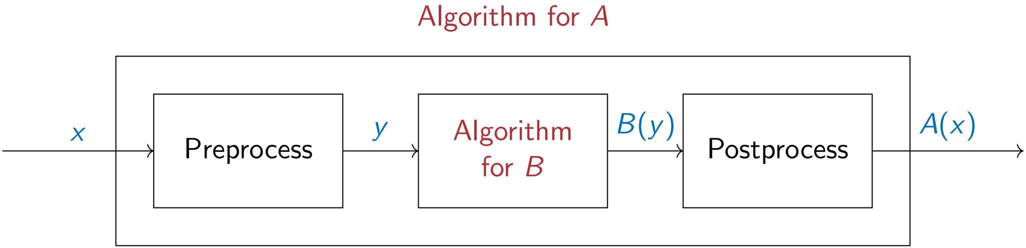

Reductions

- Independent set and vertex cover reduce to each other
- Recall: if A reduces to B and A is intractable, so is B
- Many pairs of checkable problems are inter-reducible
- All "equally" hard