Common subwords and subsequences

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Programming, Data Structures and Algorithms using Python

Week 9

Longest common subword

- Given two strings, find the (length of the) longest common subword
  - `"secret"`, `"secretary"` — `"secret"`, length 6
  - `"bisect"`, `"trisect"` — `"isect"`, length 5
  - `"bisect"`, `"secret"` — `"sec"`, length 3
  - `"director"`, `"secretary"` — `"ee"`, `"re"`, length 2

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- Formally
  - $u = a_0 a_1 \ldots a_{m-1}$
  - $v = b_0 b_1 \ldots b_{n-1}$

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  - $u = a_0 a_1 \ldots a_{m-1}$
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  - Common subword of length k — for some positions i and j, $a_i a_{i+1} a_{i+k-1} = b_j b_{j+1} b_{j+k-1}$

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  - `"director"`, `"secretary"` — `"ee"`, `"re"`, length 2
- Formally
  - $u = a_0 a_1 \ldots a_{m-1}$ 1st character of u
  - $v = b_0 b_1 \ldots b_{n-1}$
  - Common subword of length k — for some positions i and j, $a_i a_{i+1} a_{i+k-1} = b_j b_{j+1} b_{j+k-1}$
  - Find the largest such k — length of the longest common subword

Brute force

- $u = a_0 a_1 \dots a_{m-1}$
- $v = b_0 b_1 \dots b_{n-1}$
- Find the largest k such that for some positions i and j, $a_i a_{i+1} a_{i+k-1} = b_j b_{j+1} b_{j+k-1}$

Brute force

- $u = a_0 a_1 \ldots a_{m-1}$
- $v = b_0 b_1 \ldots b_{n-1}$
- Find the largest k such that for some positions i and j, $a_i a_{i+1} a_{i+k-1} = b_j b_{j+1} b_{j+k-1}$
- Try every pair of starting positions i in u, j in v
  - Match $(a_i, b_j), (a_{i+1}, b_{j+1}), \ldots$ as far as possible
  - Keep track of longest match

Brute force

- $u = a_0 a_1 \ldots a_{m-1}$
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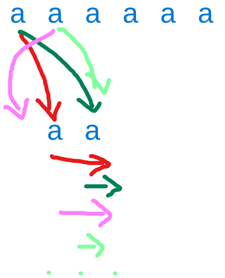
- Find the largest k such that for some positions i and j, $a_i a_{i+1} a_{i+k-1} = b_j b_{j+1} b_{j+k-1}$
- Try every pair of starting positions i in u, j in v
  - Match $(a_i, b_j), (a_{i+1}, b_{j+1}), \ldots$ as far as possible
  - Keep track of longest match
- Assuming $m > n$, this is $O(mn^2)$
  - mn pairs of starting positions
  - From each starting position, scan could be $O(n)$



Inductive structure

- $u = a_0 a_1 \ldots a_{m-1}$
- $v = b_0 b_1 \ldots b_{n-1}$
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- Find the largest k such that for some positions i and j, $a_i a_{i+1} a_{i+k-1} = b_j b_{j+1} b_{j+k-1}$
- $LCW(i,j)$ — length of longest common subword in $a_i a_{i+1} \ldots a_{m-1}$, $b_j b_{j+1} \ldots b_{n-1}$
  - If $a_i \neq b_j$, $LCW(i,j) = 0$
  - If $a_i = b_j$, $LCW(i,j) = 1 + LCW(i{+}1, j{+}1)$

    if at character at index i and j of u and v respectively are the same then increase the Longest Common Subword count by 1.

    Note LCW (i+1, j+1) simply means the longest common subword count with starting positions i+1 and j+1

Inductive structure

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- Find the largest k such that for some positions i and j,
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- $LCW(i,j)$ — length of longest common subword in $a_i a_{i+1} \ldots a_{m-1}$, $b_j b_{j+1} \ldots b_{n-1}$
  - If $a_i \neq b_j$, $LCW(i,j) = 0$
  - If $a_i = b_j$, $LCW(i,j) = 1 + LCW(i+1, j+1)$
  - Base case: $LCW(m,n) = 0$

  Both `m` and `n` are out of range of v and u, thus
  it acts as a base. As both `m` and `n` are out of range
  there exist no common subwords with starting position `m` and `n`

i = m

bisect

insect

j = n

Inductive structure

- $u = a_0 a_1 \ldots a_{m-1}$
- $v = b_0 b_1 \ldots b_{n-1}$
- Find the largest k such that for some positions i and j, $a_i a_{i+1} a_{i+k-1} = b_j b_{j+1} b_{j+k-1}$
- $LCW(i,j)$ — length of longest common subword in $a_i a_{i+1} \ldots a_{m-1}$, $b_j b_{j+1} \ldots b_{n-1}$
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  - Base case: $LCW(m,n) = 0$
  - In general, $LCW(i,n) = 0$ for all $0 \leq i \leq m$

Inductive structure

- $u = a_0 a_1 \ldots a_{m-1}$ len(u) = m
- $v = b_0 b_1 \ldots b_{n-1}$ len(v) = n
- Find the largest k such that for some positions i and j, $a_i a_{i+1} a_{i+k-1} = b_j b_{j+1} b_{j+k-1}$
- $LCW(i,j)$ — length of longest common subword in $a_i a_{i+1} \ldots a_{m-1}$, $b_j b_{j+1} \ldots b_{n-1}$
  - If $a_i \neq b_j$, $LCW(i,j) = 0$
  - If $a_i = b_j$, $LCW(i,j) = 1 + LCW(i+1, j+1)$
  - Base case: $LCW(m,n) = 0$
  - In general, $LCW(i,n) = 0$ for all $0 \leq i \leq m$
  - In general, $LCW(m,j) = 0$ for all $0 \leq j \leq n$

If starting position of LCW of either words is at the `m` or `n` respectively then this means there exit no common subwords

Subproblem dependency

- Subproblems are $LCW(i,j)$, for $0 \leq i \leq m$, $0 \leq j \leq n$

Subproblem dependency

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- Table of $(m+1)\cdot(n+1)$ values

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							
1	i							
2	s							
3	e							
4	c							
5	t							
6	•							

Subproblem dependency

DYNAMIC PROGRAMMING

- Subproblems are $LCW(i,j)$, for $0 \leq i \leq m$, $0 \leq j \leq n$
- Table of $(m+1) \cdot (n+1)$ values
- $LCW(i,j)$ depends on $LCW(i{+}1,j{+}1)$

Solving the problem using DYNAMIC PROGRAMMING will require to start from bottom up.

NOTE: In LCW(i, j) and LCW(i+1, j+1) the arguments are starting positions of u and v

Example:
u = "bisect"
v = "secret"

For getting longest common subword from here you need to know longest common subword starting from here

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							
1	i							
2	s							
3	e							
4	c							
5	t							
6	•							

Subproblem dependency

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- Table of $(m+1) \cdot (n+1)$ values
- $LCW(i,j)$ depends on $LCW(i+1,j+1)$
- Start at bottom right and fill row by row or column by column

As we are solving using dynamic programming we follow bottom up approach

(m, j)

		0	1	2	3	4	5	6
		s	e	c	r	e	t	• (i, n)
0	b							**0**
1	i							**0**
2	s							**0**
3	e							**0**
4	c							**0**
5	t							**0**
6	•	0	0	0	0	0	0	**0**

Subproblem dependency

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- Start at bottom right and fill row by row or column by column

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b						0	0
1	i						0	0
2	s						0	0
3	e						0	0
4	c						0	0
5	t						1	0
6	•						0	0

if a_i = b_j then
LCW(i, j) = 1 + LCW(i+1, j+1)

LCW(i+1, j+1) = 0

Subproblem dependency

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- Start at bottom right and fill row by row or column by column

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b					0	0	0
1	i					0	0	0
2	s					0	0	0
3	e					1	0	0
4	c					0	0	0
5	t					0	1	0
6	•					0	0	0

if a_i != b_j
then LCW = 0

Subproblem dependency

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b				**0**	**0**	**0**	**0**
1	i				**0**	**0**	**0**	**0**
2	s				**0**	**0**	**0**	**0**
3	e				**0**	**1**	**0**	**0**
4	c				**0**	**0**	**0**	**0**
5	t				**0**	**0**	**1**	**0**
6	•				**0**	**0**	**0**	**0**

Subproblem dependency

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b			**0**	**0**	**0**	**0**	**0**
1	i			**0**	**0**	**0**	**0**	**0**
2	s			**0**	**0**	**0**	**0**	**0**
3	e			**0**	**0**	**1**	**0**	**0**
4	c			**1**	**0**	**0**	**0**	**0**
5	t			**0**	**0**	**0**	**1**	**0**
6	•			**0**	**0**	**0**	**0**	**0**

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b		0	0	0	0	0	0
1	i		0	0	0	0	0	0
2	s		0	0	0	0	0	0
3	e		2	0	0	1	0	0
4	c		0	1	0	0	0	0
5	t		0	0	0	0	1	0
6	•		0	0	0	0	0	0

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	0	0	0	0	0	0	0
1	i	0	0	0	0	0	0	0
2	s	3	0	0	0	0	0	0
3	e	0	2	0	0	1	0	0
4	c	0	0	1	0	0	0	0
5	t	0	0	0	0	0	1	0
6	•	0	0	0	0	0	0	0

Subproblem dependency

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Reading off the solution

- Find entry (i,j) with largest LCW value

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1	i	**0**	**0**	**0**	**0**	**0**	**0**	**0**
2	s	**3**	**0**	**0**	**0**	**0**	**0**	**0**
3	e	**0**	**2**	**0**	**0**	**1**	**0**	**0**
4	c	**0**	**0**	**1**	**0**	**0**	**0**	**0**
5	t	**0**	**0**	**0**	**0**	**0**	**1**	**0**
6	•	**0**	**0**	**0**	**0**	**0**	**0**	**0**

Subproblem dependency

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- Table of $(m+1) \cdot (n+1)$ values
- $LCW(i,j)$ depends on $LCW(i+1,j+1)$
- Start at bottom right and fill row by row or column by column

Reading off the solution

- Find entry (i,j) with largest LCW value
- Read off the actual subword diagonally

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	0	0	0	0	0	0	0
1	i	0	0	0	0	0	0	0
2	s	3	0	0	0	0	0	0
3	e	0	2	0	0	1	0	0
4	c	0	0	1	0	0	0	0
5	t	0	0	0	0	0	1	0
6	•	0	0	0	0	0	0	0

Subproblem dependency

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Reading off the solution

- Find entry (i,j) with largest LCW value
- Read off the actual subword diagonally

on projecting this diagonal on either side
you will get LCW

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	0	0	0	0	0	0	0
1	i	0	0	0	0	0	0	0
2	s	3	0	0	0	0	0	0
3	e	0	2	0	0	1	0	0
4	c	0	0	1	0	0	0	0
5	t	0	0	0	0	0	1	0
6	•	0	0	0	0	0	0	0

Implementation

```
def LCW(u,v):
  import numpy as np
  (m,n) = (len(u),len(v))
  lcw = np.zeros((m+1,n+1))

  maxlcw = 0

  for c in range(n-1,-1,-1):
    for r in range(m-1,-1,-1):
      if u[r] == v[c]:
        lcw[r,c] = 1 + lcw[r+1,c+1]
      else:
        lcw[r,c] = 0
      if lcw[r,c] > maxlcw:
        maxlcw = lcw[r,c]

  return(maxlcw)
```

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Complexity

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Complexity

- Recall that brute force was $O(mn^2)$

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Complexity

- Recall that brute force was $O(mn^2)$
- Inductive solution is $O(mn)$, using dynamic programming or memoization

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```

Complexity

- Recall that brute force was $O(mn^2)$
- Inductive solution is $O(mn)$, using dynamic programming or memoization
  - Fill a table of size $O(mn)$
  - Each table entry takes constant time to compute

Longest common subsequence

- Subsequence — can drop some letters in between
- Given two strings, find the (length of the) longest common subwsequence
  - "secret", "secretary" — "secret", length 6
  - "bisect", "trisect" — "isect", length 5
  - "bisect", "secret" — "sect", length 4
  - "director", "secretary" — "ectr", "retr", length 4

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  - `"director"`, `"secretary"` — `"ectr"`, `"retr"`, length 4
- LCS is the longest path connecting non-zero LCW entries, moving right/down

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	0	0	0	0	0	0	0
1	i	0	0	0	0	0	0	0
2	s	3	0	0	0	0	0	0
3	e	0	2	0	0	1	0	0
4	c	0	0	1	0	0	0	0
5	t	0	0	0	0	0	1	0
6	•	0	0	0	0	0	0	0

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	0	0	0	0	0	0	0
1	i	0	0	0	0	0	0	0
2	s	3	0	0	0	0	0	0
3	e	0	2	0	0	1	0	0
4	c	0	0	1	0	0	0	0
5	t	0	0	0	0	0	1	0
6	•	0	0	0	0	0	0	0

Applications

- Analyzing genes
  - DNA is a long string over A, T, G, C
  - Two species are similar if their DNA has long common subsequences

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	0	0	0	0	0	0	0
1	i	0	0	0	0	0	0	0
2	s	3	0	0	0	0	0	0
3	e	0	2	0	0	1	0	0
4	c	0	0	1	0	0	0	0
5	t	0	0	0	0	0	1	0
6	•	0	0	0	0	0	0	0

Applications

- Analyzing genes
  - DNA is a long string over A, T, G, C
  - Two species are similar if their DNA has long common subsequences
- diff command in Unix/Linux
  - Compares text files
  - Find the longest matching subsequence of lines
  - Each line of text is a “character”

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1	i	**0**	**0**	**0**	**0**	**0**	**0**	**0**
2	s	**3**	**0**	**0**	**0**	**0**	**0**	**0**
3	e	**0**	**2**	**0**	**0**	**1**	**0**	**0**
4	c	**0**	**0**	**1**	**0**	**0**	**0**	**0**
5	t	**0**	**0**	**0**	**0**	**0**	**1**	**0**
6	•	**0**	**0**	**0**	**0**	**0**	**0**	**0**

Inductive structure

- $u = a_0 a_1 \dots a_{m-1}$
- $v = b_0 b_1 \dots b_{n-1}$

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- If $a_i = b_j$, $LCS(i,j) = 1 + LCS(i+1, j+1)$
  - Can assume (a_i, b_j) is part of LCS

secret

bisect

Inductive structure

- $u = a_0 a_1 \ldots a_{m-1}$
- $v = b_0 b_1 \ldots b_{n-1}$

If both "c" and "s" becomes part of LCS then LCS (i, j) (2, 2) will be "cet" and "set" but this wrong right?

- $LCS(i,j)$ — length of longest common subsequence in $a_i a_{i+1} \ldots a_{m-1}$, $b_j b_{j+1} \ldots b_{n-1}$
- If $a_i = b_j$, $LCS(i,j) = 1 + LCS(i+1, j+1)$
  - Can assume (a_i, b_j) is part of LCS
- If $a_i \neq b_j$, a_i and b_j cannot both be part of the LCS

secret

bisect

here i = 2 and j = 2
for i = 3 and j = 3 we already have computed LCS
so LCS(3, 3) = 2 ("et")

Inductive structure

- $u = a_0 a_1 \ldots a_{m-1}$
- $v = b_0 b_1 \ldots b_{n-1}$
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  - Can assume (a_i, b_j) is part of LCS
- If $a_i \neq b_j$, a_i and b_j cannot both be part of the LCS
  - Which one should we drop?

secret

bisect

if we drop "s" we get two possible LCS "et" or "ct".
If we drop "c" we get LCS "et"
In both cases LCS length is 2 so we can drop either "c" or "s"

Inductive structure

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  - Can assume (a_i, b_j) is part of LCS
- If $a_i \neq b_j$, a_i and b_j cannot both be part of the LCS
  - Which one should we drop?
  - Solve $LCS(i, j+1)$ and $LCS(i+1, j)$ and take the maximum

secret

bisect

So, LCS(i, j) = max(LCS(i, j+1), LCS(i+1, j))

implicitly dropping "c" in other words considering this part of "secret"

implicitly dropping "s" in other words considering this part of "bisect"

Inductive structure

- $u = a_0 a_1 \ldots a_{m-1}$
- $v = b_0 b_1 \ldots b_{n-1}$
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- If $a_i = b_j$, $LCS(i,j) = 1 + LCS(i{+}1, j{+}1)$
  - Can assume (a_i, b_j) is part of LCS
- If $a_i \neq b_j$, a_i and b_j cannot both be part of the LCS
  - Which one should we drop?
  - Solve $LCS(i, j{+}1)$ and $LCS(i{+}1, j)$ and take the maximum
- Base cases as with LCW
  - $LCS(i, n) = 0$ for all $0 \leq i \leq m$
  - $LCS(m, j) = 0$ for all $0 \leq j \leq n$

Subproblem dependency

- Subproblems are $LCS(i,j)$, for $0 \leq i \leq m$, $0 \leq j \leq n$

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							
1	i							
2	s							
3	e							
4	c							
5	t							
6	•							

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- $LCS(i,j)$ depends on $LCS(i{+}1, j{+}1)$, $LCS(i, j{+}1)$,$LCS(i{+}1, j)$,

Remember LCW(i, j) depended on LCW(i+1, j+1)

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							
1	i							
2	s							
3	e							
4	c							
5	t							
6	•							

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							**0**
1	i							**0**
2	s							**0**
3	e							**0**
4	c							**0**
5	t							**0**
6	•							**0**

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b						**0**	**0**
1	i						**0**	**0**
2	s						**0**	**0**
3	e						**0**	**0**
4	c						**0**	**0**
5	t						**1**	**0**
6	•						**0**	**0**

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b					1	0	0
1	i					1	0	0
2	s					1	0	0
3	e					1	0	0
4	c					1	0	0
5	t					1	1	0
6	•					0	0	0

take maximum

diagonal + 1
as its a match

Subproblem dependency

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reminder: we are computing from bottom to top

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b				1	1	0	0
1	i				1	1	0	0
2	s				1	1	0	0
3	e				1	1	0	0
4	c				1	1	0	0
5	t				1	1	1	0
6	•				0	0	0	0

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THINK: If bisect and secret has LCS of 2 then of course bisect and secret has minimum LCS of 2

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b			2	1	1	0	0
1	i			2	1	1	0	0
2	s			2	1	1	0	0
3	e			2	1	1	0	0
4	c			2	1	1	0	0
5	t			1	1	1	1	0
6	•			0	0	0	0	0

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b		3	2	1	1	0	0
1	i		3	2	1	1	0	0
2	s		3	2	1	1	0	0
3	e		3	2	1	1	0	0
4	c		2	2	1	1	0	0
5	t		1	1	1	1	1	0
6	•		0	0	0	0	0	0

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Your answer LCS length is always on the index (0, 0)

That is because LCS(0, 0) means LCS of bisect and secret. And LCS(1, 1) means LCS of "ecret" and "isect"

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	3	2	1	1	0	0
1	i	4	3	2	1	1	0	0
2	s	4	3	2	1	1	0	0
3	e	3	3	2	1	1	0	0
4	c	2	2	2	1	1	0	0
5	t	1	1	1	1	1	1	0
6	•	0	0	0	0	0	0	0

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Reading off the solution

- Trace back the path by which each entry was filled

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	3	2	1	1	0	0
1	i	4	3	2	1	1	0	0
2	s	4	3	2	1	1	0	0
3	e	3	3	2	1	1	0	0
4	c	2	2	2	1	1	0	0
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on every diagonal a new element is added to the LCS

Reading off the solution

- Trace back the path by which each entry was filled
- Each diagonal step is an element of LCS

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	3	2	1	1	0	0
1	i	4	3	2	1	1	0	0
2	s	4	3	2	1	1	0	0
3	e	3	3	2	1	1	0	0
4	c	2	2	2	1	1	0	0
5	t	1	1	1	1	1	1	0
6	•	0	0	0	0	0	0	0

Implementation

```
def LCS(u,v):
  import numpy as np
  (m,n) = (len(u),len(v))
  lcs = np.zeros((m+1,n+1))

  for c in range(n-1,-1,-1):
    for r in range(m-1,-1,-1):
      if u[r] == v[c]:
        lcs[r,c] = 1 + lcs[r+1,c+1]
      else:
        lcs[r,c] = max(lcs[r+1,c],
                       lcs[r,c+1])
  return(lcs[0,0])
```

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Complexity

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Complexity

- Again $O(mn)$, using dynamic programming or memoization

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```

Complexity

- Again $O(mn)$, using dynamic programming or memoization
  - Fill a table of size $O(mn)$
  - Each table entry takes constant time to compute