Matrix Multiplication

Madhavan Mukund

https://www.cmi.ac.in/~madhavan

Programming, Data Structures and Algorithms using Python

Week 9

Multiplying matrices

- Multiply matrices A, B
  - $AB[i,j] = \sum_{k=0}^{n-1} A[i,k]B[k,j]$

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- Given n matrices $M_0 : r_0 \times c_0$, $M_1 : r_1 \times c_1$, . . . , $M_{n-1} : r_{n-1} \times c_{n-1}$

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  - Product $M_0 \cdot M_1 \cdots M_{n-1}$ can be computed
- Find an optimal order to compute the product
  - Multiply two matrices at a time
  - Bracket the expression optimally

Inductive structure

- Final step combines two subproducts
  $(M_0 \cdot M_1 \cdots M_{k-1}) \cdot (M_k \cdot M_{k+1} \cdots M_{n-1})$
  for some $0 < k < n$

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  - $M_0 \cdot M_1 \cdots M_{k-1}$ would decompose as $(M_0 \cdots M_{j-1}) \cdot (M_j \cdots M_{k-1})$
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in general a problem starts from j
and goes to some k

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  $\min_{j<\ell\leq k} [C(j, \ell{-}1) + C(\ell, k) + r_j r_\ell c_k]$
- Base case: $C(j, j) = 0$ for $0 \leq j < n$

when sequence consists of only one matrix then cost is 0 as no multiplication is needed

Subproblem dependency

- Compute $C(i,j)$, $0 \leq i,j < n$

	0	$\cdots$	i	$\cdots$	$\cdots$	j	$\cdots$	$n-1$
0								
$\cdots$								
i								
$\cdots$								
$\cdots$								
j								
$\cdots$								
$n-1$								

Subproblem dependency

- Compute $C(i,j)$, $0 \leq i,j < n$
  - Only for $i \leq j$
  - Entries above main diagonal

j

i

	0	1	i	$\cdots$	3	j	$\cdots$	$n-1$
0								
1								
i								
3								
$\cdots$								
j								
$\cdots$								
$n-1$								

C(1,3)

C(3,1)

Why is this not valid?

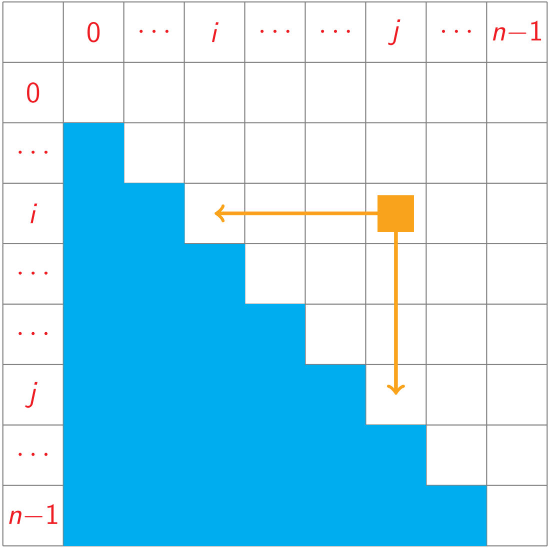
We are given that we can compute product of M0 * M1 but this does not mean we can compute M1 * M0 as matrix multiplication is not COMUTATIVE.

Thus, C(3, 1) means cost of computing M3 * M2 * M1 which is not possible.

But C(1, 3) is possible as we are given that M0 * M1 * ... *Mn-1 can be computed

Subproblem dependency

- Compute $C(i,j)$, $0 \leq i,j < n$
  - Only for $i \leq j$
  - Entries above main diagonal
- $C(i,j)$ depends on $C(i,k-1)$, $C(k,j)$ for every $i < k \leq j$



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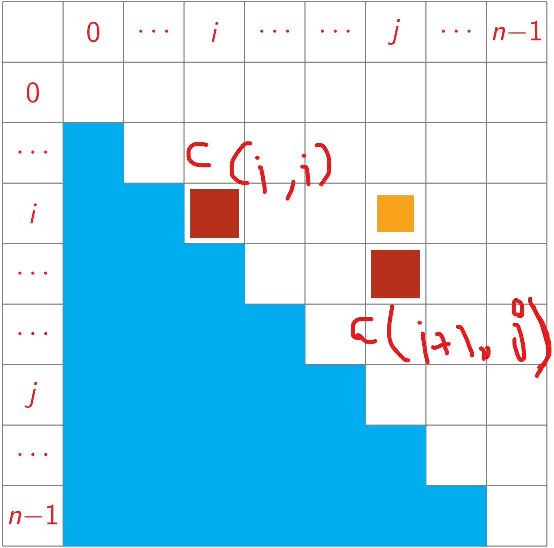
First we take k = i +1,
C(i, j) = C(i, i) + C(i+1, j) + Cost of multiplying both factors



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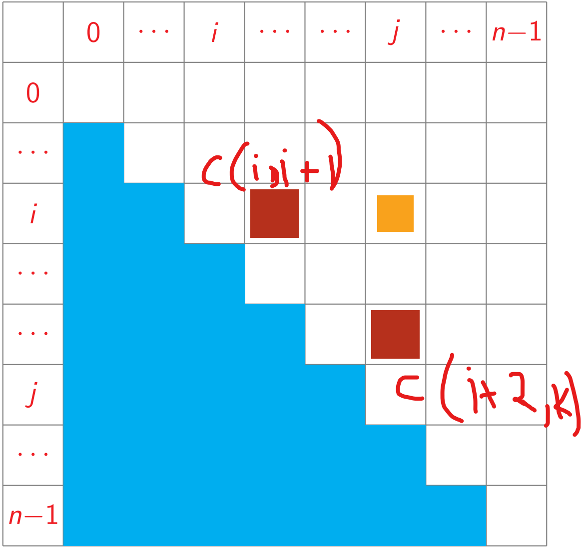
$k = i+2$



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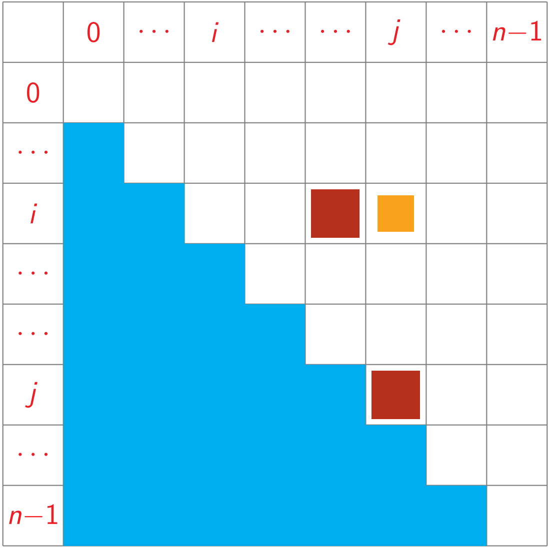

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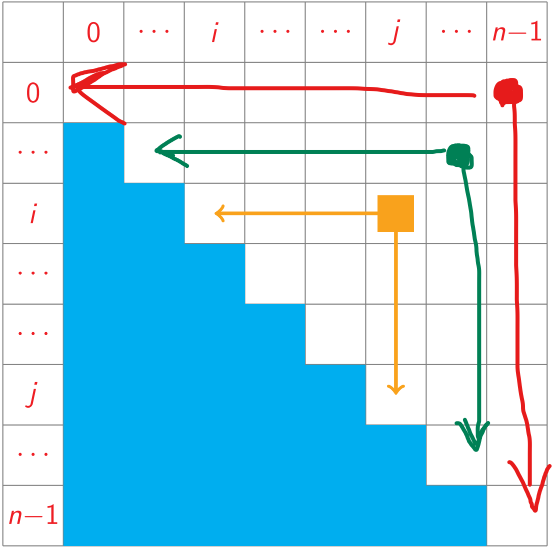
  - $O(n)$ dependencies per entry, unlike LCW, LCS and ED



The number of subproblems to evaluate for a given position is not fixed and it depends on the position (i, j).

And in general it could it could be as bad as of Order n, O(n)

Number of subproblems to evaluate for this is greater than

Number of subproblems to evaluate for this

Subproblem dependency

- Compute $C(i,j)$, $0 \leq i,j < n$
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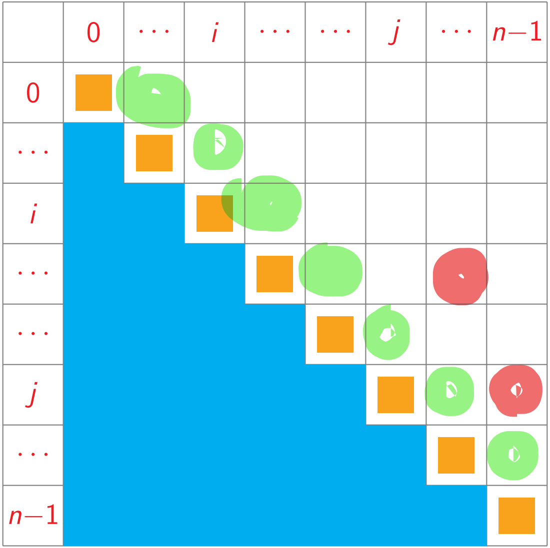
- Diagonal entries are base case

So now as we filled the base case values, what values can we fill?

These values cannot be filled as dependencies are not yet filled

this value can be filled, as dependencies are already filled
(it has dependencies as base case)



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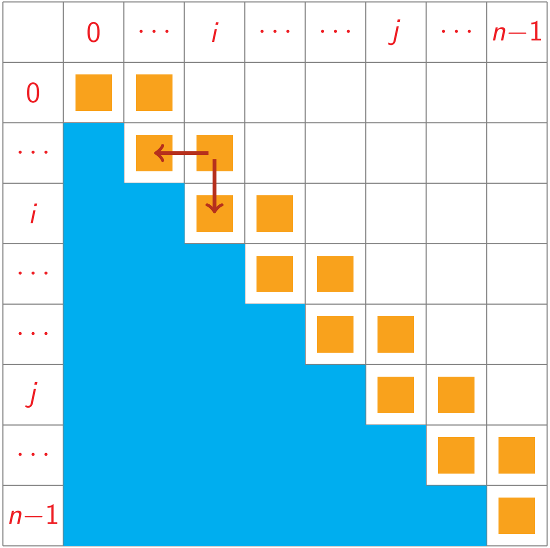

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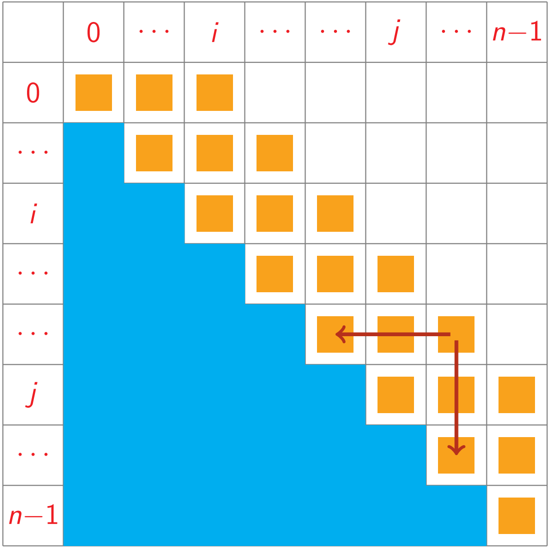

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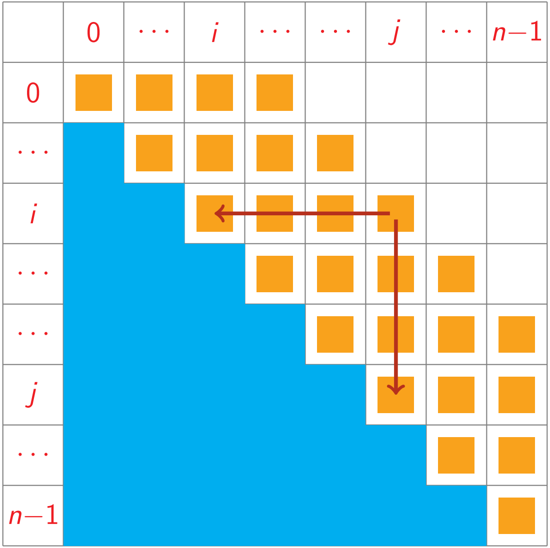

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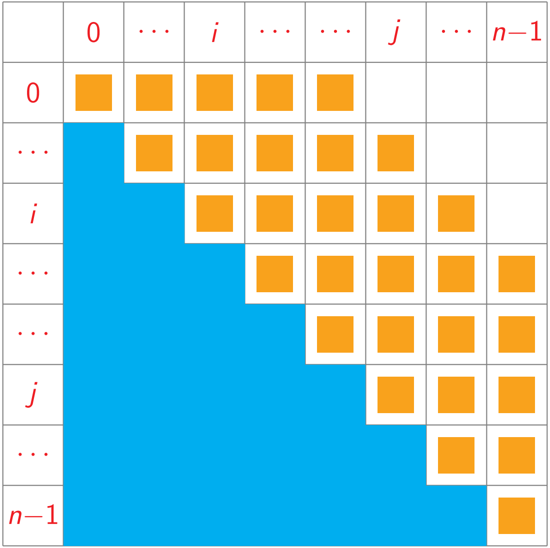

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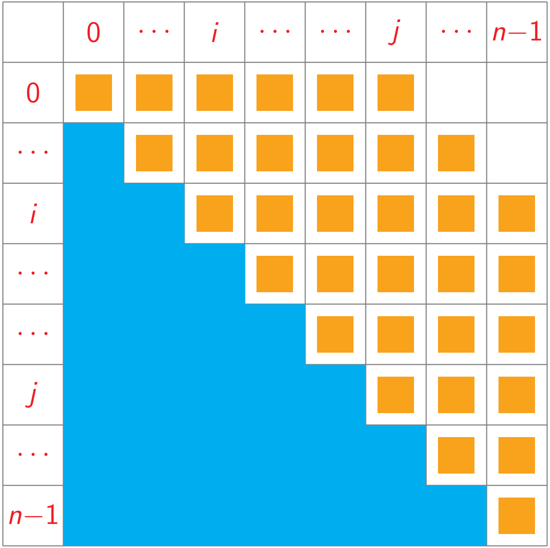

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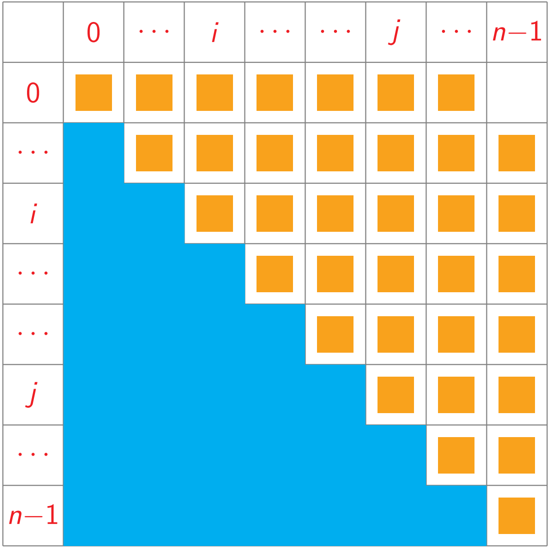

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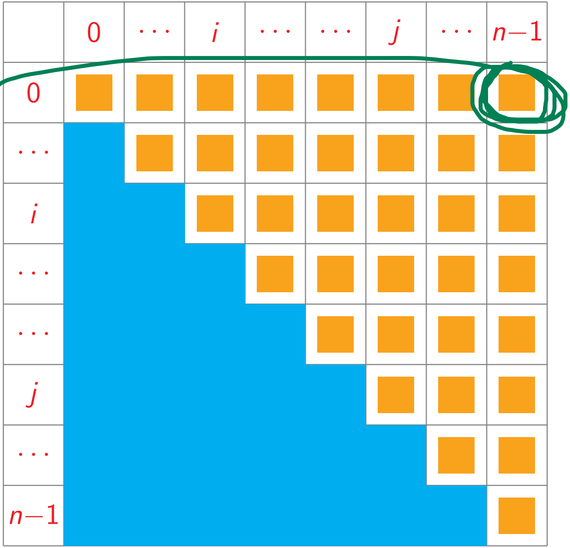


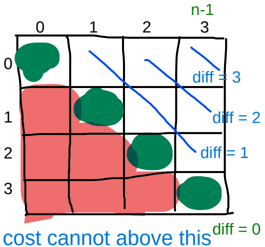
This is the answer for finding cost of multiplying n matrices, C(0, n-1)

Implementation

```
def C(dim):
  # dim: dimension matrix,
  #      entries are pairs (r_i,c_i)
  import numpy as np
  n = dim.shape[0]
  C = np.zeros((n,n))
  for i in range(n):
    C[i,i] = 0
  for diff in range(1,n):
    for i in range(0,n-diff):
      j = i + diff
      C[i,j] = C[i,i] +
               C[i+1,j] +
               dim[i][0]*dim[i+1][0]*dim[j][1]
      for k in range(i+1,j+1):
        C[i,j] = min(C[i,j],
                     C[i,k-1] + C[k,j] +
                     dim[i][0]*dim[k][0]*dim[j][1])
  return(C[0,n-1])
```

This is first term of the minimum sequence, minimum cost cannot above this

diff visualization



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Complexity

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- We have to fill a table of size $O(n^2)$

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Complexity

- We have to fill a table of size $O(n^2)$
- Filling each entry takes $O(n)$
- Overall, $O(n^3)$