SLP REPORT

Lagrange Coherent Structures

Krith Sanvith Manthripragada
Prof.Vineeth Nair
Department of Aerospace Engineering
Indian Institute of Technology, Bombay

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List of Symbols

x	Position Vector
v	Velocity
t	Time
a	Acceleration
$F_{t_0}^{t}$	Position after being affected by flow
T	Integration Length
∇	Gradient
C	Cauchy-Green Deformation Tensor
λ	Eigen Value
S	Eulerian Rate of Strain Tensor
Ψ	Stream function

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1 Abstract

This supervised learning project develops a theory & computation of lagrangian coherent structures using 2 approaches i.e., one is *lagrangian approach* and the other is *eulerian approach*. Both these approaches have their own advantages and disadvantages. We will go through their approaches and deep mathematical relationship that exists between two of them and we will define new Eulerian diagnostic: Infinitesimal-time LCS (iLCS). iLCS will be shown to be the limit of LCS as $t \to 0$ and finally using the iLCS we will demonstrate the effectiveness of iLCS using double gyre and comparing the alteration rate field to FTLE. field.

2 Introduction

These diagnostics can help to predict how particles spread in a fluid from over a certain time interval. However, when you look at the lagrangian methods, they rely on the integration of particle trajections which is time consuming & computationally expensive but the eulerian approach, they use the eulerian rate of strain tensor which is calculated from the gradients of velocity i.e., here we analyze the system without integration which reduces time & computational power.

Now let us look at both the approaches and compare them.

3 Literature Review

This section contains review of papers in bibliography. From these we take the foundation & implement it for one example.

3.1 Lagrangian Approach

Consider the Dynamical System,

$$\begin{gathered}\frac{d}{dt}(x(t)) = V(x(t), t)\\ x_0 = x(t_0)\\ x \in \mathbb{R}^n, t \in \mathbb{R}\end{gathered} \tag{3.1}$$

For LCS, it is needed to compute Finite-Time-Lyapunov-Exponent (FTLE), with a FTLE for each grid point the structures can be plotted.

FTLE is a scaler $\sigma_{t_0}^{T}(n)$ which represents structures of a fluid at a location. The maxima show the attracting (or repelling barriers). Let's say a particle at $x(t_0)$ goes to a new location after time T.

Flow map of that point can be written as

$$F_{t_0}^{t}(x_0) = x_0 + \int_{t_0}^{T} V(x(t), t)\, dt \tag{3.2}$$

Let's say there is another point close to $x(t_0)$ which is $y = x + \delta x(t_0)$. After a time interval T, distance between these 2 points becomes

$$\begin{aligned}\delta x(t_0+T) &= F_{t_0}^{t_0+T}(y) - F_{t_0}^{t_0+T}(x) \\ &= \nabla F_{t_0}^{t_0+T}(x)\end{aligned} \tag{3.3}$$

From above equation we can calculate the neon strain tensor,

$$C = \nabla F_{t_0}^{t_0+T}(x)^T \cdot \nabla F_{t_0}^{t_0+T}(n) \tag{3.4}$$

Eigan values of which are $\lambda_1, \lambda_2, \ldots, \lambda_n$ & associated normalized eigenvectors, ϵ_{λ_i} $(i \in \{1,2,3,\ldots,n\})$

$\therefore$ From the maximum eigenvalue of chuchy green tensor FTLE can be calculated as

$$\sigma_{t_0}^T(x_0) = \frac{1}{2|T|}\log(\lambda_n) \tag{3.5}$$

$$\begin{aligned}\lambda_n &= max\ eig(C) \\ &= max\ eig(\nabla F_{t_0}^{t_0+T}(x)^T \cdot \nabla F_{t_0}^{t_0+T}(n))\end{aligned} \tag{3.6}$$

To complete FTLE, it is necessary to have locations of particles at initial state $t = t_0$ & at $t = t_0 + T$. $\therefore$ Flow map can be determined.

$$F_{t_0}^{t_0+T} = \begin{bmatrix} \frac{x_{i+1,j}(t_0+T)-x_{i-1,j}(t_0+T)}{x_{i+1,j}(t_0)-x_{i-1,j}(t_0)} & \frac{x_{i,j+1}(t_0+T)-x_{i,j-1}(t_0+T)}{y_{i,j+1}(t_0)-y_{i,j-1}(t_0)} \\ \frac{y_{i+1,j}(t_0+T)-y_{i-1,j}(t_0+T)}{x_{i+1,j}(t_0)-x_{i-1,j}(t_0)} & \frac{y_{i,j+1}(t_0+T)-y_{i,j-1}(t_0+T)}{y_{i,j+1}(t_0)-y_{i,j-1}(t_0} \end{bmatrix} \tag{3.7}$$

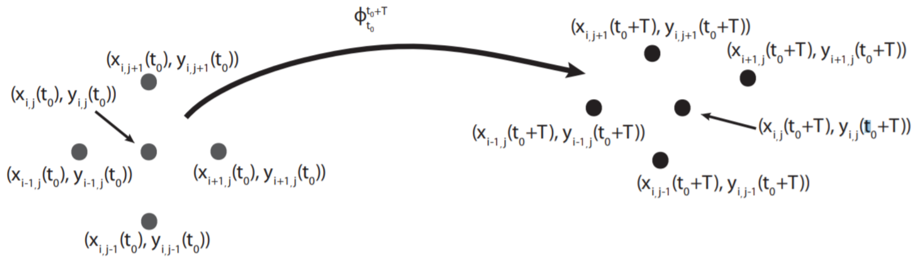

Figure 3.1: Flowmap used for computing the FTLE

Now let's see the Eulerian Approach.

3.2 Eulerian Approach

The Eulerian role of strain Tensor is defined as

$$S(x,t) = \frac{1}{2}\nabla V(x,t) + \nabla V(x,t)^T \tag{3.8}$$

and eigan values of which are $s_1 < s_2 < \cdots < s_n$ and associated normalized eigen vectors, ϵ_{S_i} $i \in \{1,2,\ldots,n\}$

From the eigen values of eulerian rate of stain tensor, one can identify regions of flow which are more attracting & repelling.

There s_1 →minimum eigen values provides measure of attraction & s_2 maximum value of repulsion.

3.2.1 Relation between Cauchy-Green Strain Tensor & Eulerian Rate of Strain Tensor

Eigen value of S as FTLE limit as integration of time goes to 0.

For small $|T|$, let us expand $C_{t_0}^t(x)$ as

$$C_{t_0}^t(n) = 1 + 2TS(x,t_0) + T^2 B(x,t_0) + \frac{1}{2}T^3 Q(x,t_0) + O(T^4) \tag{3.9}$$

Where,

$$B(x,t_0) = \frac{1}{2}[\nabla a(x,t_0) + (\nabla a(x,t_0))^T] + \nabla V(x,t_0)^T \cdot \nabla V(x,t_0) \tag{3.10}$$

Where acceleration field $a(x, t_0)$ is

$$a(x, t_0) = \frac{d}{dt} V(x, t_0) = \frac{\partial}{\partial t} V(x, t_0) + V(x, t_0) \cdot \nabla V(x, t_0) \tag{3.11}$$

$$\lambda_n = \lambda^+ (C_{t_0}^t(x)) \text{for small, } T > 0 \tag{3.12}$$

We can neglect $O(T^2)$ in

$$\therefore \lambda^+ (C_{t_0}^t(x)) = 1 + 2T\lambda^+ (S(x, t_0)) + O(T^2) \tag{3.13}$$

$$\begin{aligned} \log(\lambda_n) &= \log(1 + 2T\lambda^+ S(x, t_0)) \\ &= 2T\lambda^+ (S(x, t_0)) \\ &= 2Ts_n(x, t) \end{aligned} \tag{3.14}$$

In the limit of small T, $\log(1 + \epsilon) = \epsilon$

$$\begin{aligned} \sigma_{t_0}^T &= \frac{1}{2|T|} \log(\lambda_n) \\ &= \frac{1}{2T} \cdot 2T \cdot S(x, t_0) \\ &= s_n(x, t_0) \end{aligned} \tag{3.15}$$

For $T < 0$, with small T

$$\lambda^+ (C_{t_0}^t(x)) = 1 + 2T\lambda^- (S(x, t_0)) \tag{3.16}$$

$$\begin{aligned} \log(\lambda_n) &= 2T\lambda^- (S(x, t_0)) \\ &= 2Ts_1(x, t_0) \end{aligned} \tag{3.17}$$

$\therefore |T| = -T \text{if } T < 0$

$$\sigma_{t_0}^t = \frac{1}{2|T|} \log(\lambda_n) = -s_1(x, t_0) \tag{3.18}$$

$\therefore$ We can summarize as follows,

$$\sigma_{t_0}^t = \pm s^{\pm}(x, t_0) \text{ as } t - t_0 \to 0^{\pm} \tag{3.19}$$

$$\nabla V = \begin{bmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{bmatrix} \tag{3.20}$$

$$\nabla S = \begin{bmatrix} \frac{\partial U}{\partial x} & \frac{1}{2}(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}) \\ \\ \frac{1}{2}(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}) & \frac{\partial V}{\partial y} \end{bmatrix} \tag{3.21}$$

3.2.2 Equality of Eigen Vectors of S & C

$$\begin{aligned} S\epsilon_i &= S_i\epsilon_i \\ 2TS\epsilon_i + \epsilon_i &= 2TS_i\epsilon_i + \epsilon_i \\ (2TS+1)\epsilon_i &= (2TS_i+1)\epsilon_i \\ C\epsilon_i &= \lambda_i\epsilon_i \end{aligned} \tag{3.22}$$

$\Rightarrow \epsilon_i$ is an eigen vector of S then ϵ_i is an eigen vector of C.

$\therefore$ We can say as T goes to 0, eigen vector of C is same as eigen vector of S.

4 Examples

4.1 Double Gyre

Let's look at the double gyre flow. This flow comes from hamiltonian stream function.

$$\Psi(x, y, t) = A\sin(\pi f(x, t))\sin(\pi y) \tag{4.1}$$

Where,

$$f(x, t) = \epsilon\sin(wt)x^2 + (1 - 2\epsilon\sin(wt))x \tag{4.2}$$

We can calculate the velocity field $V = (U, V)$ as

$$\begin{aligned} \dot{x} &= U(x, y, t) \\ &= -A\pi\sin(\pi f(x, t))cos(\pi y) \end{aligned} \tag{4.3}$$

$$\begin{aligned} \dot{y} &= v(x, y, t) \\ &= A\pi\cos(\pi f(x, t))\sin(\pi y)\frac{\partial F}{\partial x} \end{aligned} \tag{4.4}$$

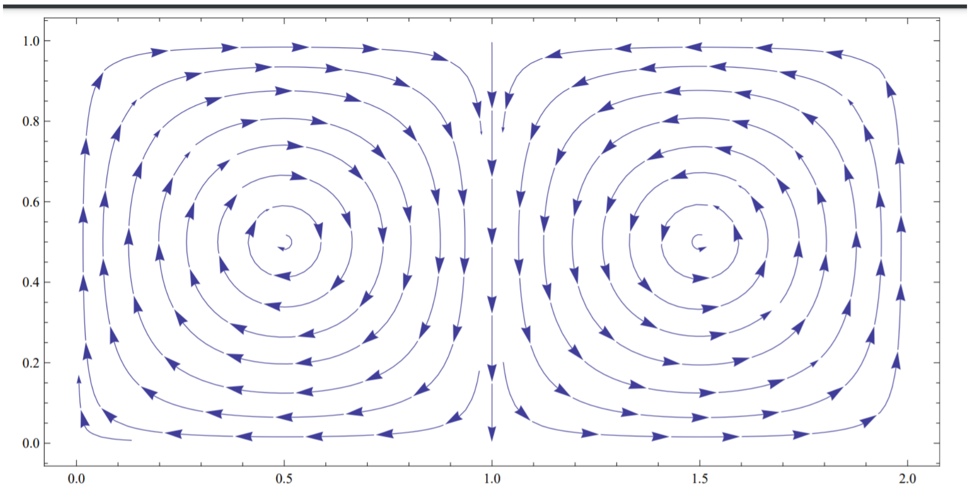

Figure 4.1: Streamline flow of Double Gyre

We use the parameters, $A = 0.1$, $w = 0.2\pi$ and $\epsilon = 0.25$

$$\therefore \nabla V = \begin{bmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{bmatrix}$$
$$= \begin{bmatrix} -\pi^2 A\cos(\pi f)\cos(\pi y)\frac{\partial f}{\partial x} & \pi^2 A\sin(\pi f)\sin(\pi y) \\ \\ -\pi^2 A\sin(\pi f)\sin(\pi y)\frac{\partial f}{\partial x} + \pi A\cos(\pi f)\sin(\pi y)\frac{\partial^2 f}{\partial x^2} & \pi^2 A\cos(\pi f)\cos(\pi y)\frac{\partial f}{\partial x} \end{bmatrix} \tag{4.5}$$

Now Eulerian Rate of Strain Tensor,

$$\therefore \nabla V = \begin{bmatrix} \frac{\partial U}{\partial x} & \frac{1}{2}(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}) \\ \\ \frac{1}{2}(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}) & \frac{\partial V}{\partial y} \end{bmatrix} \tag{4.6}$$

Now the below figure shows a comparison of the FTLE field for a short integration Time $T = -0.5$ first with an approximation to first order in T.

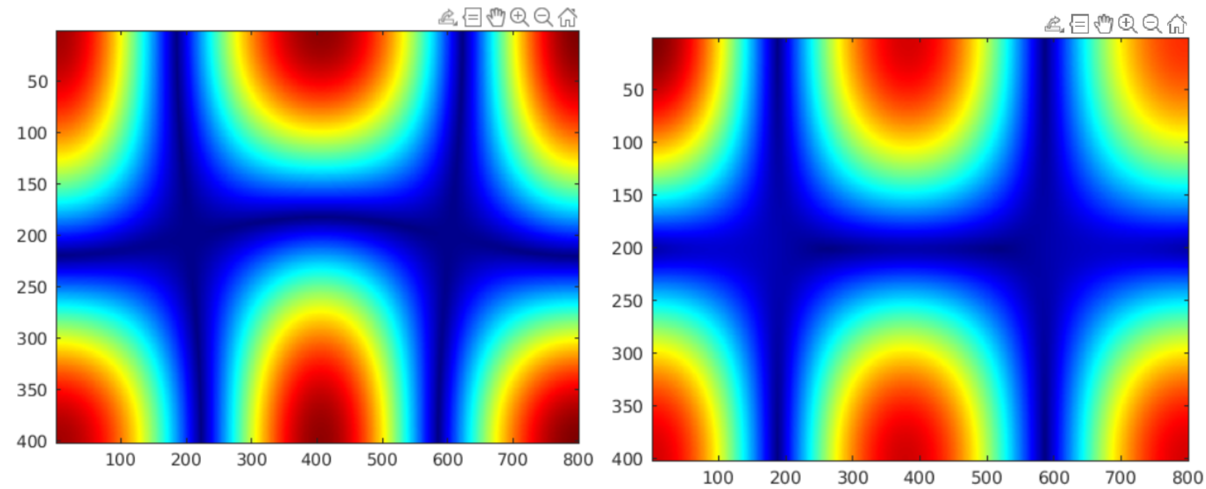

Figure 4.2: Left: FTLE field for the double-gyre flow for an integration period of T = -0.3. Right: approximation to the FTLE field to first-order in T. Parameters: A = 0.1, w = 0.2 e = 0.25, and to = 0.

5 Conclusion

These mathematical connections prove that the attraction and repulsion rates are the limits of FTLE. as integration time goes to 0. Additionally this manuscript also shows that for small integration time $|T| << 1$ eigen vectors for cauchy-green strain tensor are equal to eulerian rate of strain tensor.