SLP REPORT

Lagrange Coherent Structures

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Abstract

- This supervised learning project develops a theory & computation of lagrangian coherent structures using 2 approaches i.e.
  1. *lagrangian approach*
  2. *eulerian approach*
- Both these approaches have their own advantages and disadvantages.
- We will go through their approaches and deep mathematical relationship that exists between two of them and we will define new Eulerian diagnostic: Infinitesimal-time LCS (iLCS).
- iLCS will be shown to be the limit of LCS as $t \to 0$ and finally using the iLCS we will demonstrate the effectiveness of iLCS using double gyre and comparing the alteration rate field to FTLE. field.

Introduction

- These diagnostics can help to predict how particles spread in a fluid from over a certain time interval.
- However, when you look at the lagrangian methods, they rely on the integration of particle trajections which is time consuming & computationally expensive.
- But the eulerian approach, they use the eulerian rate of strain tensor which is calculated from the gradients of velocity i.e., here we analyze the system without integration which reduces time & computational power.
- Now let us look at both the approaches and compare them.

Literature Review - Lagrangian Approach

- Consider the Dynamical System,

$$\begin{gathered} \frac{d}{dt}(x(t)) = V(x(t), t) \\ x_0 = x(t_0) \\ x \in \mathbb{R}^n, t \in \mathbb{R} \end{gathered} \tag{1}$$

- For LCS, it is needed to compute Finite-Time-Lyapunov-Exponent (FTLE), with a FTLE for each grid point the structures can be plotted.
- FTLE is a scaler $\sigma_{t_0}^{T}(n)$ which represents structures of a fluid at a location. The maxima show the attracting (or repelling barriers). Let's say a particle at $x(t_0)$ goes to a new location after time T.

Literature Review - Lagrangian Approach

- Flow map of that point can be written as

$$F_{t_0}^{t}(x_0) = x_0 + \int_{t_0}^{T} V(x(t), t)\, dt \tag{2}$$

- Let's say there is another point close to $x(t_0)$ which is $y = x + \delta x(t_0)$. After a time interval T, distance between these 2 points becomes

$$\begin{aligned} \delta x(t_0 + T) &= F_{t_0}^{t_0+T}(y) - F_{t_0}^{t_0+T}(x) \\ &= \nabla F_{t_0}^{t_0+T}(x) \end{aligned} \tag{3}$$

- From above equation we can calculate the neon strain tensor,

$$C = \nabla F_{t_0}^{t_0+T}(x)^{T} \cdot \nabla F_{t_0}^{t_0+T}(n) \tag{4}$$

Literature Review - Lagrangian Approach

- Eigan values of which are $\lambda_1, \lambda_2, \ldots, \lambda_n$ & associated normalized eigenvectors, ϵ_{λ_i} $(i \in \{1, 2, 3, \ldots, n\})$
- $\therefore$ From the maximum eigenvalue of Chuchy-Green tensor FTLE can be calculated as
  - $$\sigma_{t_0}^{T}(x_0) = \frac{1}{2|T|}\log(\lambda_n) \tag{5}$$
  - $$\begin{aligned} \lambda_n &= max\ eig(C) \\ &= max\ eig(\nabla F_{t_0}^{t_0+T}(x)^{T} \cdot \nabla F_{t_0}^{t_0+T}(n)) \end{aligned} \tag{6}$$

Literature Review - Lagrangian Approach

- To complete FTLE, it is necessary to have locations of particles at initial state $t = t_0$ & at $t = t_0 + T$. $\therefore$ Flow map can be determined.

$$F_{t_0}^{t_0+T} = \begin{bmatrix} \frac{x_{i+1,j}(t_0+T)-x_{i-1,j}(t_0+T)}{x_{i+1,j}(t_0)-x_{i-1,j}(t_0)} & \frac{x_{i,j+1}(t_0+T)-x_{i,j-1}(t_0+T)}{y_{i,j+1}(t_0)-y_{i,j-1}(t_0)} \\ \frac{y_{i+1,j}(t_0+T)-y_{i-1,j}(t_0+T)}{x_{i+1,j}(t_0)-x_{i-1,j}(t_0)} & \frac{y_{i,j+1}(t_0+T)-y_{i,j-1}(t_0+T)}{y_{i,j+1}(t_0)-y_{i,j-1}(t_0} \end{bmatrix} \quad (7)$$

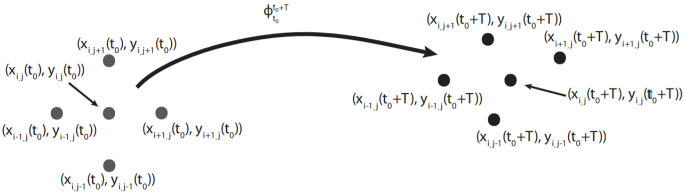

Figure: Flowmap used for computing the FTLE

Literature Review - Eulerian Approach

- The Eulerian role of strain Tensor is defined as

$$S(x,t) = \frac{1}{2}\nabla V(x,t) + \nabla V(x,t)^T \tag{8}$$

- And eigan values of which are $s_1 < s_2 < \cdots < s_n$ and associated normalized eigen vectors, $\epsilon_{S_i} \;\; i \in \{1, 2, \ldots, n\}$
- From the eigen values of eulerian rate of stain tensor, one can identify regions of flow which are more attracting & repelling.
- There s_1 $\rightarrow$minimum eigen values provides measure of attraction & s_2 maximum value of repulsion.

- Eigen value of S as FTLE limit as integration of time goes to 0.
- For small $|T|$, let us expand $C_{t_0}^{t}(x)$ as

$$C_{t_0}^{t}(n) = 1 + 2TS(x, t_0) + T^2 B(x, t_0) + \frac{1}{2} T^3 Q(x, t_0) + O(T^4) \tag{9}$$

- Where,

$$B(x, t_0) = \frac{1}{2}[\nabla a(x, t_0) + (\nabla a(x, t_0))^T] + \nabla V(x, t_0)^T \cdot \nabla V(x, t_0) \tag{10}$$

- Where acceleration field $a(x, t_0)$ is

$$a(x, t_0) = \frac{d}{dt} V(x, t_0) = \frac{\partial}{\partial t} V(x, t_0) + V(x, t_0) \cdot \nabla V(x, t_0) \tag{11}$$

$$\lambda_n = \lambda^{+}(C_{t_0}^{t}(x)) \text{for small, } T > 0 \tag{12}$$

- We can neglect $O(T^2)$ in

$$\therefore \lambda^+(C_{t_0}^t(x)) = 1 + 2T\lambda^+(S(x,t_0)) + O(T^2) \tag{13}$$

- $$\begin{aligned} \log(\lambda_n) &= \log(1 + 2T\lambda^+ S(x,t_0)) \\ &= 2T\lambda^+(S(x,t_0)) \\ &= 2Ts_n(x,t) \end{aligned} \tag{14}$$

- In the limit of small T, $\log(1+\epsilon) = \epsilon$

$$\begin{aligned} \sigma_{t_0}^T &= \frac{1}{2|T|}\log(\lambda_n) \\ &= \frac{1}{2T} \cdot 2T \cdot S(x,t_0) \\ &= s_n(x,t_0) \end{aligned} \tag{15}$$

- For $T < 0$, with small T

$$\lambda^+(C_{t_0}^t(x)) = 1 + 2T\lambda^-(S(x,t_0)) \tag{16}$$

$$\begin{aligned} \log(\lambda_n) &= 2T\lambda^-(S(x,t_0)) \\ &= 2Ts_1(x,t_0) \end{aligned} \tag{17}$$

- $\therefore |T| = -T \text{if } T < 0$

$$\sigma_{t_0}^t = \frac{1}{2|T|}\log(\lambda_n) = -s_1(x,t_0) \tag{18}$$

- $\therefore$ We can summarize as follows,

$$\sigma_{t_0}^{t} = \pm s^{\pm}(x, t_0) \text{ as } t - t_0 \to 0^{\pm} \tag{19}$$

$$\nabla V = \begin{bmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{bmatrix} \tag{20}$$

$$\nabla S = \begin{bmatrix} \frac{\partial U}{\partial x} & \frac{1}{2}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) & \frac{\partial V}{\partial y} \end{bmatrix} \tag{21}$$

-
$$\begin{aligned} S\epsilon_i &= S_i\epsilon_i \\ 2TS\epsilon_i + \epsilon_i &= 2TS_i\epsilon_i + \epsilon_i \\ (2TS+1)\epsilon_i &= (2TS_i+1)\epsilon_i \\ C\epsilon_i &= \lambda_i\epsilon_i \end{aligned} \tag{22}$$

- ϵ_i is an eigen vector of S then ϵ_i is an eigen vector of C.
- $\therefore$ We can say as T goes to 0, eigen vector of C is same as eigen vector of S.

Example - Double Gyre

- Let's look at the double gyre flow. This flow comes from hamiltonian stream function.

$$\Psi(x, y, t) = A \sin(\pi f(x, t)) \sin(\pi y) \tag{23}$$

- Where,

$$f(x, t) = \epsilon \sin(wt) x^2 + (1 - 2\epsilon \sin(wt)) x \tag{24}$$

Example - Double Gyre

- We can calculate the velocity field $V = (U, V)$ as
  - $$\begin{aligned} \dot{x} &= U(x, y, t) \\ &= -A\pi \sin(\pi f(x, t)) cos(\pi y) \end{aligned} \tag{25}$$
  - $$\begin{aligned} \dot{y} &= v(x, y, t) \\ &= A\pi \cos(\pi f(x, t)) \sin(\pi y) \frac{\partial F}{\partial x} \end{aligned} \tag{26}$$

Example - Double Gyre

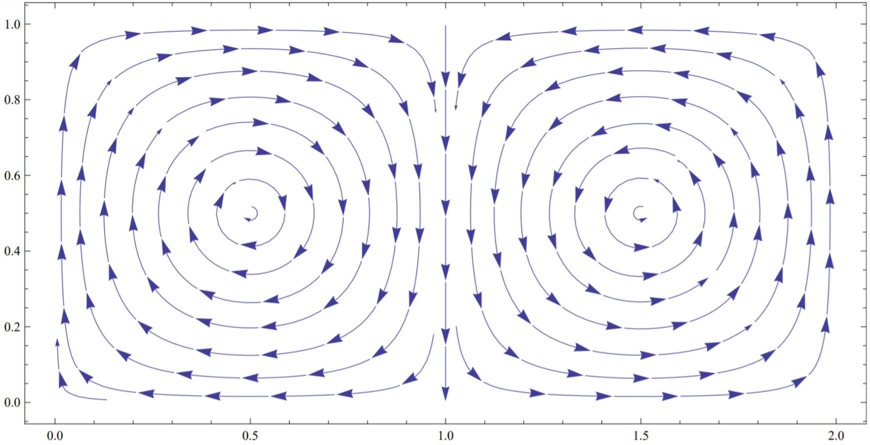

Figure: Streamline flow of Double Gyre

Example - Double Gyre

- We use the parameters, $A = 0.1$, $w = 0.2\pi$ and $\epsilon = 0.25$

$$\therefore \nabla V = \begin{bmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{bmatrix} = \begin{bmatrix} -\pi^2 A\cos(\pi f)\cos(\pi y)\frac{\partial f}{\partial x} & \pi^2 A\sin(\pi f)\sin(\pi y) \\ -\pi^2 A\sin(\pi f)\sin(\pi y)\frac{\partial f}{\partial x} + \pi A\cos(\pi f)\sin(\pi y)\frac{\partial^2 f}{\partial x^2} & \pi^2 A\cos(\pi f)\cos(\pi y)\frac{\partial f}{\partial x} \end{bmatrix} \tag{27}$$

- Now Eulerian Rate of Strain Tensor,

$$\therefore \nabla V = \begin{bmatrix} \frac{\partial U}{\partial x} & \frac{1}{2}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) & \frac{\partial V}{\partial y} \end{bmatrix} \tag{28}$$

Example - Double Gyre

- Now the below figure shows a comparison of the FTLE field for a short integration Time $T = -0.5$ first with an approximation to first order in T.

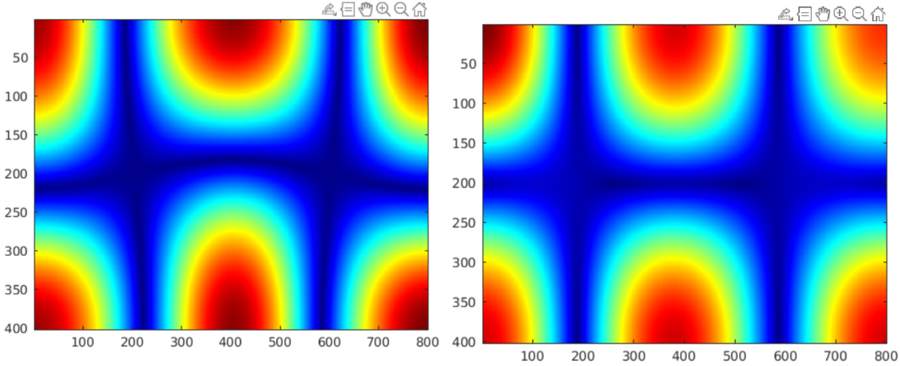

Figure: Left: FTLE field for the double-gyre flow for an integration period of T = -0.3. Right: approximation to the FTLE field to first-order in T. Parameters: A = 0.1, w = 0.2 e = 0.25, and to = 0.

Conclusion

- These mathematical connections prove that the attraction and repulsion rates are the limits of FTLE. as integration time goes to 0.
- Additionally this manuscript also shows that for small integration time $|T| << 1$ eigen vectors for cauchy-green strain tensor are equal to eulerian rate of strain tensor.