SLP REPORT

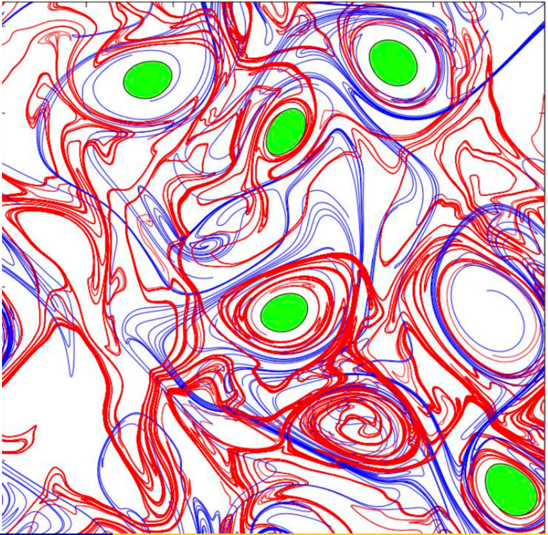
Lagrange Coherent Structures

Krith Sanvith Manthripragada
Guided by Prof.Vineeth Nair

Department of Aerospace Engineering
Indian Institute of Technology, Bombay

Introduction

What is an LCS?



Literature Review - Lagrangian Approach

Application of LCS.

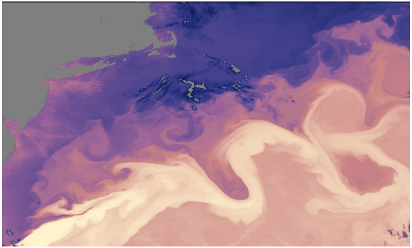


Figure: Sea surface temperature in gulf-stream parabolic LCS

Figure: Tornado off the Florida Keys 3D Elliptic LCS

Literature Review

- Lagrangian Approach

- Eulerian Approach

Literature Review - Lagrangian Approach

- Consider the Dynamical System,

$$\begin{gathered}\frac{d}{dt}(x(t)) = V(x(t), t) \\ x_0 = x(t_0) \\ x \in \mathbb{R}^n, t \in \mathbb{R}\end{gathered} \tag{1}$$

- For LCS, it is needed to compute Finite-Time-Lyapunov-Exponent (FTLE), with a FTLE for each grid point the structures can be plotted.
- FTLE is a scaler $\sigma_{t_0}^{T}(n)$ which represents structures of a fluid at a location. The maxima show the attracting (or repelling barriers). Let's say a particle at $x(t_0)$ goes to a new location after time T.

Literature Review - Lagrangian Approach

- Flow map of that point can be written as

$$F_{t_0}^{t}(x_0) = x_0 + \int_{t_0}^{T} V(x(t), t)\, dt \tag{2}$$

$$\begin{aligned} \delta x(t_0 + T) &= F_{t_0}^{t_0+T}(y) - F_{t_0}^{t_0+T}(x) \\ &= \nabla F_{t_0}^{t_0+T}(x) \end{aligned} \tag{3}$$

- From above equation we can calculate the neon strain tensor,

$$C = \nabla F_{t_0}^{t_0+T}(x)^{T} \cdot \nabla F_{t_0}^{t_0+T}(n) \tag{4}$$

Literature Review - Lagrangian Approach

- Eigan values of which are $\lambda_1, \lambda_2, \ldots, \lambda_n$ & associated normalized eigenvectors, ϵ_{λ_i} $(i \in \{1, 2, 3, \ldots, n\})$
- $\therefore$ From the maximum eigenvalue of Chuchy-Green tensor FTLE can be calculated as
  - $$\sigma_{t_0}^{T}(x_0) = \frac{1}{2|T|}\log(\lambda_n) \tag{5}$$
  - $$\begin{aligned} \lambda_n &= max\ eig(C) \\ &= max\ eig(\nabla F_{t_0}^{t_0+T}(x)^{T} \cdot \nabla F_{t_0}^{t_0+T}(n)) \end{aligned} \tag{6}$$

Literature Review - Lagrangian Approach

- To complete FTLE, it is necessary to have locations of particles at initial state $t = t_0$ & at $t = t_0 + T$. $\therefore$ Flow map can be determined.

$$F_{t_0}^{t_0+T} = \begin{bmatrix} \frac{x_{i+1,j}(t_0+T) - x_{i-1,j}(t_0+T)}{x_{i+1,j}(t_0) - x_{i-1,j}(t_0)} & \frac{x_{i,j+1}(t_0+T) - x_{i,j-1}(t_0+T)}{y_{i,j+1}(t_0) - y_{i,j-1}(t_0)} \\ \\ \frac{y_{i+1,j}(t_0+T) - y_{i-1,j}(t_0+T)}{x_{i+1,j}(t_0) - x_{i-1,j}(t_0)} & \frac{y_{i,j+1}(t_0+T) - y_{i,j-1}(t_0+T)}{y_{i,j+1}(t_0) - y_{i,j-1}(t_0} \end{bmatrix} \tag{7}$$

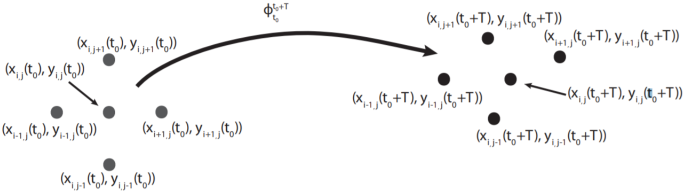


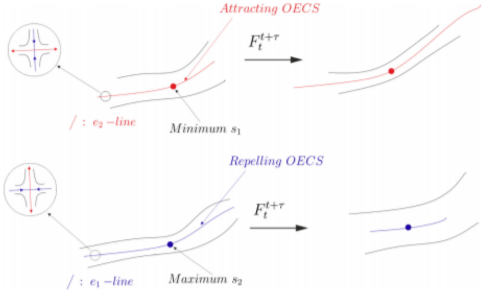
Figure: Flowmap used for computing the FTLE

Literature Review - Eulerian Approach

- The Eulerian role of strain Tensor is defined as

$$S(x,t) = \frac{1}{2}\nabla V(x,t) + \nabla V(x,t)^T \tag{8}$$

- There s_1 $\rightarrow$minimum eigen values provides measure of attraction & s_2 maximum value of repulsion.



Rel. b/w Cauchy-Green Strain Tensor & Eulerian Rate of Strain Tensor

- Eigen value of S as FTLE limit as integration of time goes to 0.
- For small $|T|$, let us expand $C_{t_0}^t(x)$ as

$$C_{t_0}^t(n) = 1 + 2TS(x, t_0) + T^2 B(x, t_0) + \frac{1}{2}T^3 Q(x, t_0) + O(T^4) \tag{9}$$

- Where,

$$B(x, t_0) = \frac{1}{2}[\nabla a(x, t_0) + (\nabla a(x, t_0))^T] + \nabla V(x, t_0)^T \cdot \nabla V(x, t_0) \tag{10}$$

- Where acceleration field $a(x, t_0)$ is

$$a(x, t_0) = \frac{d}{dt}V(x, t_0) = \frac{\partial}{\partial t}V(x, t_0) + V(x, t_0) \cdot \nabla V(x, t_0) \tag{11}$$

$$\lambda_n = \lambda^+(C_{t_0}^t(x)) \text{for small, } T > 0 \tag{12}$$

- We can neglect $O(T^2)$ in

$$\therefore \lambda^+(C_{t_0}^t(x)) = 1 + 2T\lambda^+(S(x,t_0)) + O(T^2) \tag{13}$$

- $$\begin{aligned} \log(\lambda_n) &= \log(1 + 2T\lambda^+ S(x,t_0)) \\ &= 2T\lambda^+(S(x,t_0)) \\ &= 2Ts_n(x,t) \end{aligned} \tag{14}$$

- In the limit of small T, $\log(1+\epsilon) = \epsilon$

$$\begin{aligned} \sigma_{t_0}^T &= \frac{1}{2|T|}\log(\lambda_n) \\ &= \frac{1}{2T} \cdot 2T \cdot S(x,t_0) \\ &= s_n(x,t_0) \end{aligned} \tag{15}$$

Rel. b/w Cauchy-Green Strain Tensor & Eulerian Rate of Strain Tensor

- For $T < 0$, with small T

$$\lambda^{+}(C_{t_0}^{t}(x)) = 1 + 2T\lambda^{-}(S(x, t_0)) \tag{16}$$

$$\begin{aligned} \log(\lambda_n) &= 2T\lambda^{-}(S(x, t_0)) \\ &= 2Ts_1(x, t_0) \end{aligned} \tag{17}$$

- $\therefore |T| = -T \text{if } T < 0$

$$\sigma_{t_0}^{t} = \frac{1}{2|T|}\log(\lambda_n) = -s_1(x, t_0) \tag{18}$$

- $\therefore$ We can summarize as follows,

$$\sigma_{t_0}^{t} = \pm s^{\pm}(x, t_0) \text{ as } t - t_0 \rightarrow 0^{\pm} \tag{19}$$

$$\nabla V = \begin{bmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{bmatrix} \tag{20}$$

$$\nabla S = \begin{bmatrix} \frac{\partial U}{\partial x} & \frac{1}{2}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) & \frac{\partial V}{\partial y} \end{bmatrix} \tag{21}$$

- $$\begin{aligned} S\epsilon_i &= S_i\epsilon_i \\ 2TS\epsilon_i + \epsilon_i &= 2TS_i\epsilon_i + \epsilon_i \\ (2TS+1)\epsilon_i &= (2TS_i+1)\epsilon_i \\ C\epsilon_i &= \lambda_i\epsilon_i \end{aligned} \tag{22}$$
- ϵ_i is an eigen vector of S then ϵ_i is an eigen vector of C.
- $\therefore$ We can say as T goes to 0, eigen vector of C is same as eigen vector of S.

Validation - Double Gyre

- Let's look at the double gyre flow. This flow comes from hamiltonian stream function.

$$\Psi(x, y, t) = A\sin(\pi f(x, t))\sin(\pi y) \tag{23}$$

- Where,

$$f(x, t) = \epsilon\sin(wt)x^2 + (1 - 2\epsilon\sin(wt))x \tag{24}$$

- We can calculate the velocity field $V = (U, V)$ as
  -
  $$\begin{aligned} \dot{x} &= U(x, y, t) \\ &= -A\pi\sin(\pi f(x, t))cos(\pi y) \end{aligned} \tag{25}$$
  -
  $$\begin{aligned} \dot{y} &= v(x, y, t) \\ &= A\pi\cos(\pi f(x, t))\sin(\pi y)\frac{\partial F}{\partial x} \end{aligned} \tag{26}$$

Validation - Double Gyre

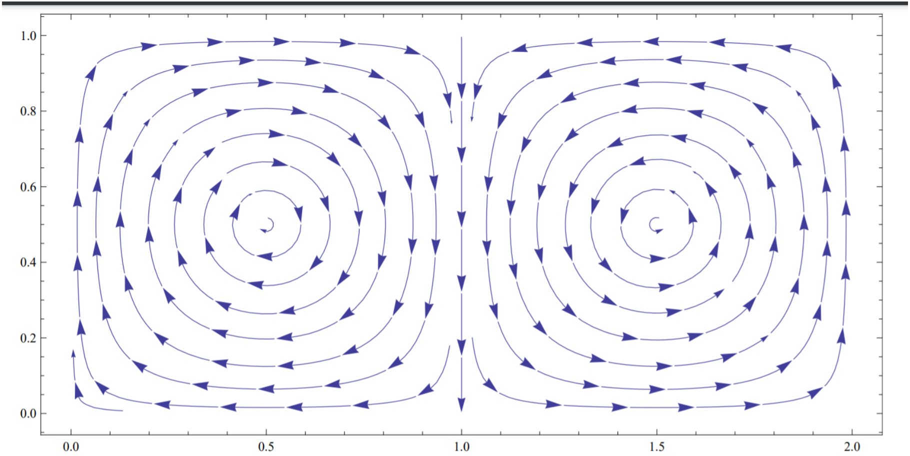


Figure: Streamline flow of Double Gyre

Validation - Double Gyre

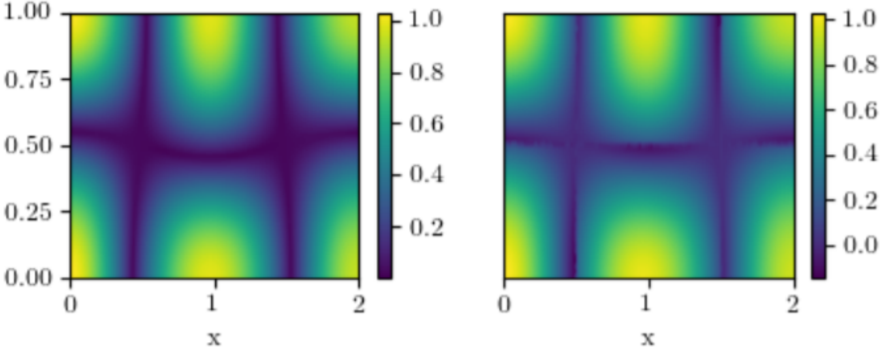
- We use the parameters, $A = 0.1$, $w = 0.2\pi$ and $\epsilon = 0.25$

$$\begin{aligned}\therefore \nabla V &= \begin{bmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} -\pi^2 A\cos(\pi f)\cos(\pi y)\frac{\partial f}{\partial x} & \pi^2 A\sin(\pi f)\sin(\pi y) \\ -\pi^2 A\sin(\pi f)\sin(\pi y)\frac{\partial f}{\partial x} + \pi A\cos(\pi f)\sin(\pi y)\frac{\partial^2 f}{\partial x^2} & \pi^2 A\cos(\pi f)\cos(\pi y)\frac{\partial f}{\partial x} \end{bmatrix}\end{aligned} \tag{27}$$

- Now Eulerian Rate of Strain Tensor,

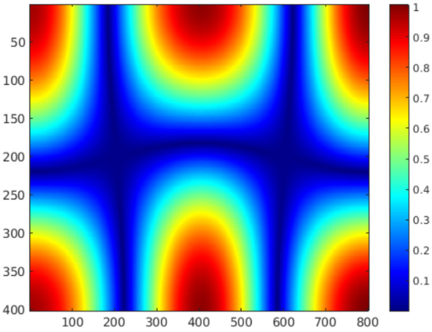
$$\therefore \nabla V = \begin{bmatrix} \frac{\partial U}{\partial x} & \frac{1}{2}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) & \frac{\partial V}{\partial y} \end{bmatrix} \tag{28}$$

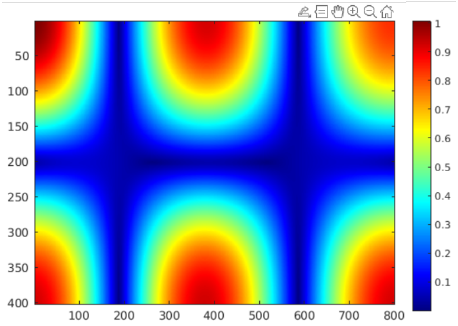
Validation - Double Gyre



Validation - Double Gyre

- Now the below figure shows a comparison of the FTLE field for a short integration Time $T = -0.5$ first with an approximation to first order in T.





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