1)

i) To show NP:

Verification steps	Function	Run-time
Summing sizes	$\underbrace{\{S_1, S_2, \ldots S_m\}}$ $=\sum_{i:f(i)=j}$	$O(n)$
Check size	$S_j \leq C$	$O(m)$, $m \leq n$: $\therefore O(n)$

Since all the steps are polynomial time, it is NP

ii) To show NP:

Verification step: $\underbrace{\sum_{1}^{m} a_i = \sum_{m}^{n} a_i}_{\text{Solvable in } O(n)}$

$\therefore$ polynomial time: $O(n)$ and hence NP

iii) Instance of Partition: 1) Remove negatives since Binning requires nonnegative sums.

↳ Let k = lowest negative value

Then for every a_i, $a_i = a_i + k + 1$
This will set the lowest value to 1.

The adjusted total is $T' = T + nk$

Instance of Bin-Packing: 1) Set bins to 2 $(m=2)$

2) Set $C = \frac{T'}{2}$

∴ Partition is reduced to Bin-Packing as it is split into '2 bins' and will meet the criteria of $C > 0$ & $\sum_{i}^{m} s_i \leq C$ if the solution exists.

Additionally, all reductions are polynomial.

Therefore, Bin-Packing is at least as hard as Partition.

2)

i) <u>Vertex Cover</u>

Loop through all vertices and each time:

↳ temporarily remove the vertex and check whether the remaining vertices are connected

↳ Return subset if above is true

↳ Return empty graph if false

Verification is $O(n)$ at every vertex. $\therefore O(n^2)$ and polynomial

<u>Set Cover</u>

↳ Add all sets

↳ Remove one set at a time and iteratively verify if all elements are in the remaining sets.

ii)

↳ Let the edges e in Vertex Cover be the set of elements B.

$$B_e = E$$

↳ Each vertex would be reclassified as set $S_v \subseteq B$ by

$$S_v = \{ e \in E \mid e \text{ is incident on } v \}$$

$\therefore S_v$ consists of all edges that touch v

Therefore, finding the smallest subset of vertices so every edge is incident to an edge is equivalent to finding the fewest sets S_v such that all elements in B_e are covered.

3)

↳ For each set $S_i \subseteq B$, let integer vector

$$X: \quad x_j = \begin{cases} 1 & \text{if set } S_j \text{ is chosen,} \\ 0 & \text{otherwise} \end{cases}$$

$\therefore 0 \leq x_j \leq 1$ and is a nonnegative integer

↳ To verify that all elements $n \in B$ are covered:

$$\sum_{j: n \in S_j} x_j \geq 1$$

This ensures at least one set includes n.

↳ The n elements would be represented by a $n \times j$ matrix where each element holds a row consisting of 1 and 0 for which set j covers them.

Therefore, to find the minimum # of sets to cover all elements in B, we can minimize x where A is the n elements of B and $b = 1$ in the equation $Ax \geq 1$. To match ILP, $-Ax \leq 1$

$\therefore$ if $A = -B$ elements, $b = -1$, and x is a binary vector for set inclusion, then, the search version of set cover has been reduced to the search version of ILP.