Modeling Long-term Capital Market Dynamics to Enable Rigorous Investment Planning

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Abstract:

Investment planning often requires projections of future capital market behavior. While these markets are widely considered very hard to predict, modern econometric tools allow us to capture their longer-term dynamics, such as tendencies to cycle or persist, as well as interlinkages across asset classes. This work develops a Monte Carlo simulation to generate plausible ranges of inflation, interest rates, bond returns, and stock returns. The simulation engine is tuned to historical and theoretical relationships and initiated with current market conditions. Projections exhibit interestingly intuitive economic and statistical properties. In particular, the longer is one's investment horizon, the less riskier stocks are than bonds. The projections allow investors to undertake rigorous plans and pre-experience the range of potential outcomes.

Introduction:

Investment planning and asset allocation are routine activities for professionals who are entrusted with significant wealth. As one example, consider the endowment of St. Paul's School, which is the second-largest such portfolio among independent schools [1]. According to the school's financial reports, this portfolio was valued at $643 million on June 30th, 2018, compared to the $203 million value of the school's land and buildings on that date. Over the subsequent 2018-'19 school year, an income of $26.8 million was drawn from SPS' endowment portfolio, to cover 50% of the school's $53.8 million in expenses that year; tuition receipts (net of financial aid) contributed 40%. From these facts, we can see that the endowment is a critical resource of the SPS community, that is relied upon to significantly fund the operation. This means SPS needs to invest its endowment carefully. In fact, the endowment portfolio experienced growth of $23.6m over 2018-'19, a return of 3.7% that almost covered the 4.2% that was used for spending. As a result, by June 30th, 2019, the endowment had slightly dropped in value, to just under $641 million, or a decline of -0.3%. Such a slight decline in a given year is not surprising, though of course, the goal remains to preserve or even grow the endowment over time. In all, the school's financial report contains 7 pages (out of 26 total) containing various details about the endowment and its allocation [2]. In short, decisions about how to invest the SPS endowment, and how much to spend from it each year have a lot at stake.

When making such important, long-term financial decisions, one needs projections of both future income (such as dividends and bond interest) and returns (which also include price changes) from various classes of capital market assets. While these markets are notoriously hard to predict, we do know a lot more about their dynamics than is popularly understood. In particular, Benjamin Graham remarked, "in the short run, the market is a voting machine but in the long run, it is a weighing machine," [3]. What Graham meant is that stock markets contain a lot of short-term fluctuation that can appear like random movements, however, long-term outcomes can usually be explained by structural economic relationships. He also instructed investors to "buy not on optimism, but on arithmetic." Both of these quotes suggest the importance to Graham of a careful quantitative approach to investment decisions.

Although investment planning is widely practiced by financial professionals, (and this work requires projections of future asset class returns) according to Lee & Masters there is a lot of disagreement regarding those projections. They list 7 characteristics that such projections should contain: "projections should be globally integrated, complete, flexible, forward looking, dynamic, logical and humble," [4]. They then list 3 standard approaches and explain why each tends to fail. For instance:

> It can appear reasonable to assume that rates of return seen in the past will recur. More specifically, you might simulate each year's portfolio performance going forward by picking a year from the available past. [But] selecting at random from prior years for future scenarios can violate the temporal logic of actual past returns; moreover, it ignores the possibility of future returns being dramatically different from those already experienced. Put differently, whereas history provides but one path, the future contains endless possibilities.

Their solution is a comprehensive Monte Carlo simulation with plenty of short-term noise but longer-term correlations among, say, inflation, interest rates, and stock returns. The simulation engine is tuned to historical and theoretical relationships and initiated with current market conditions.

There is a key distinction between projections and forecasts. Forecasts are what most people talk about with regards to capital markets. Those forecasts tend to stem from false certainties and may result in incorrect predictions, as academic researchers like Burton Malkiel have shown [5]. Rather, I am aiming to present a realistic range of 10,000 plausible outcomes that may inform future decision-making. These projections draw upon the large existing body of investment science that has helped produce 7 Nobel prizes since 1990 [6]. That work implies certain structural relationships, such as the effect of rising inflation on bond yields. These relationships can be statistically estimated from historical data to establish simulation parameters. In addition, it is possible to acknowledge uncertainty and plurality by adding suitably calibrated random noise. Also, note that my focus is on the long-term, not the short-term (i.e., years not days, consistent with the first above quote from Benjamin Graham) [3].

This work has an immediate practical application in a fintech startup known as Path Matters LLC. This is an investment consulting firm started by 4 experts in investment portfolio construction. The firm specializes in solving broad asset allocation problems faced by holders of large portfolios (like SPS' endowment). Their business will rely upon the robust capital markets engine described here. (That engine is shown in the center of Figure 1.) If the client were SPS, their profile (shown at left) would include the school's tax-free status, plus its need to contribute specific amounts toward annual spending. Scenarios might include various allocations to stocks or bonds. By running this profile and scenarios through the 10,000 simulated market paths that run 20 years into the future, Path Matters can illustrate ranges of outcomes (shown at right) such as the possibility of keeping up with inflation, or of running out of money. Quantifying the tradeoffs in this way allows client decision-makers (like SPS trustees) to make sound decisions regarding their long-term investment policy while pre-experiencing the range of potential outcomes.

Data & Methods:

As with any project involving financial markets and empirical research, the first step is to obtain some historical data and chart key variables. My goal is to understand the univariate, bivariate, and time-series characteristics. The main standard dataset used is the SBBI ("Stocks, Bonds, Bills, and Inflation") dataset that is assembled by Ibbotson & Associates and provided free to CFA (Chartered Financial Analyst) members. Exploratory data analysis and visualization were conducted first in Microsoft Excel, then the statistical estimation and full Monte Carlo simulations were produced in MATLAB.

Plotting the SBBI data to show the growth of $1 invested in each major asset class from 1926 to the present, we obtain the chart in Figure 2 and observe as follows:

- **Stocks** (shown in green) represent ownership of corporations and exhibit the strongest long-term growth. $1 invested in stocks in January 1926 had grown to $10,000 by July 2020. However, this comes at the cost of significant shorter-term volatility. For instance, an initial run up to nearby $3 by August 1929 was reversed by a decline down to $0.50 by June 1932. More recently, the dips from March 2000 to September 2002 (from $2,913 to $1,639) and from October 2007 to March 2009 ($3,414-$1,675) are clearly visible. US stocks have recovered from every crash, but this recovery required patience and resilience on the part of investors.
- **Bonds** (shown in purple) represent lending over an intermediate-term (2-10 years) to corporations or the government and exhibit moderate levels of risk and return. $1 invested in bonds in January 1926 grew to a mere $5 by January 1980, whereas the same investment in stocks had already risen to $113. From there, the bond investment rose noticeably more sharply, reaching almost $200 today (a 40-fold increase) whereas stocks again rose 100-fold. It is notable that interest rates peaked in January 1980 and have fallen steadily since then.
- **T-bills** (shown in blue) represent lending to the government over a short term (typically 3 months). **Inflation** (shown in orange) represents the rise in prices of goods and services. Notice that these two lines have crossed multiple times, indicating that investing in t-bills does not ensure keeping up with inflation.

Many investors have examined this data and their usual takeaway lesson is just to "buy and hold stocks for the long run," [7]. This is consistent with the dated idea that markets simply follow a random walk [5] with no predictable patterns or structure. An investor who fully believes this would have no reason to invest any differently near market peaks (think August 1929, March 2000, or October 2007) than near the troughs (think June 1932, September 2002, or March 2009). A more astute investor would pay attention and vary their risk tolerance so that they can, in the words of legendary investor Warren Buffett, "be fearful when others are greedy, and greedy when others are fearful," [8]. Of course, the precise peak of the data series is only known long after the fact. But current valuation levels plus recent momentum can still be compared to past history, so when markets have run upward for a while, investors can reasonably exercise greater caution than after markets have run down. In investment terms, this translates into a reliance on the tendency of stock returns, in particular, to revert toward their long-term means – a characteristic on which we now focus.

To hone in on the tendency of asset classes to mean revert, I examine annualized returns over 1, 5, 10, 20, and 30 year rolling periods since 1926. Rolling periods mean that we look at the 10-year period from January 1926 thru December 1935, then the period from February 1926 thru January

1936, and so on. These successive data points are obviously not fully independent, however they do allow us to examine the effect of varying start dates. Over each time horizon, we plot the highest, lowest, and average of the annualized returns, and also show their +/- 1-standard deviation range. We compare these results for stocks and bonds in Figure 3 and observe as follows:

- On a single-year horizon, US stocks have both made and lost a lot of money quickly: rolling 12-month returns have ranged from +54% to -43%, with a mean of 11.7% and a standard deviation of 20%. But as we scan toward longer investment horizons, the max/min bars move closer together. In no rolling 20-year period has the US stock market lost money. So a "Rip Van Winkel" (fictional character who sleeps for 20 years at a time) investor who experiences the stock market in 20-year increments would experience a far smaller 3% effective annualized standard deviation!
- Meanwhile, US government bonds are much less volatile over the short term (note the charts are drawn to the same scale and the 1-year standard deviation is below 6%), but their risk doesn't change much when you scan toward longer investment horizons. Here Rip Van Winkel experiences a far more modest reduction in his effective annualized standard deviation, to just under 3%.

These results indicate that stock returns exhibit mean reversion, whereas bond returns exhibit the opposite, known as momentum or persistence. Statistically, both characteristics may be present in the time series, but at different strengths. This has been observed by prior researchers, who also draw the obvious implications for dynamically varying the investor's prescribed asset allocation [9, 10]. Bear in mind that while it is common to speak informally about stock market cycles, most investors still do not possess and cannot access a calibrated econometric model that quantifies these effects to generate timely directions.

I can now outline the general form of the equations and hence my projections, using as an example the equation for intermediate-term government bond yields. Please refer to Figure 4, whose upper portion shows exactly the equation structure used by Lee & Masters [4]. Let's examine each term:

- First, notice that the equation produces a value of Yield(t,p) at every given time step t along a specified path p.
- Second, note that while history provides a single path, the future contains a plethora of possibilities. Each potential future path is a plausible continuation of the path to date and its pattern of cyclicality is roughly consistent with what came before.
- The equation structure resembles that of multi-variate regression, so parameters (like beta, the auto-correlation coefficient) can be estimated using standard techniques.
- The first term on the right captures persistence, also known as momentum. Where persistence is strong, beta is a value close to, and below, one. The resulting value of the variable (here Yield) stays close to its own lagged value along each path.
- The second term on the right captures mean reversion, also known as cyclicality. By definition, the parameter here is one minus beta. Where persistence is weak, beta is small and positive, so one minus beta is a value close to and below one. The resulting value of the variable (here Yield) tends to move in cycles, with large upward fluctuations eventually turning downward, and vice versa.
- The second term also refers to a "normal" level of the variable (here Yield) toward which it reverts. It is tempting to default to the historical sample average value, but that can easily prove myopic. Variables may have histories of different lengths, subjecting us to normal values that are mutually inconsistent. Moreover, for persistent variables (like Yield) even a long-term history may not reliably indicate the future. For these reasons, the historical

sample mean is replaced with a derived equilibrium level as explained in [4], though for simplicity, the random variations described there aren't applied.

- The third term on the right captures interrelationships among the variables. Per the chart in Figure 5, Inflation (t,p) is used in the calculation of Intermediate-term Bond Yield (t,p) which in turn is used in the calculation of Large Stock Return (t,p). These linkages are informed by economic theory and are validated by computing correlations over various subperiods and time horizons (per Figure 6), to ensure that only stable, logical linkages are incorporated into the model.
- The last term on the right contains pure noise. A naïve Random Walk model [5] would have only this term, which would thereby account for all of the variable's variance. Here the prior terms capture all knowable components of return, so this term accounts only for a fraction 1 – R^2 of variance, using the estimated R^2 of the regression equation shown.

The lower portion of Figure 4 showcases the equation at work. Intermediate-term government bond yields have fallen consistently over the past many decades (in fact ever since January 1980) so many people expect this variable to turn upward at some point. Each of the projections on the right shows a plausible continuation of the historical line containing the typical pattern or cyclicality of historical yields.

Figuring out the interrelationships aspect of an equation is the most involved. Identifying steady, indicative historical correlations between assets helps. On the left of Figure 6 is an 11-by-11 correlation matrix made in MATLAB. On the right is the same thing in Excel with different formatting and broken down into subperiods. Focusing on stock and corporate bond returns, in the lower left is a well-behaved scatter plot between these two variables with a clear, angled trendline. The panel on the right shows this relationship is steady through subperiods. Looking closely on the right, at the third last number in the left-most column of each grouping, the correlation between these two variables is fairly steady across different subperiods. Therefore, the stocks projection equation includes an interrelationship with corporate bonds.

Results:

Projections for all 4 asset classes are shown in Figure 7. Each asset class (and its interrelationships) are depicted in Figure 5. The upper right panel here is a repeat of the lower panel of Figure 4. Although 10,000 plausible paths were generated from each equation (for clarity only 20 paths are shown here). There appears to be a lot of noisy fluctuation in the projections, but in fact, each figure contains an asset class with a unique econometric signature. Starting in the upper left, inflation levels are fairly persistent yet there is some tendency to see sharp spikes, as was true in mid-2008 due to a sudden rise in oil prices; we see the same pattern of continuity with occasional spikes in the projections.

On the upper right, we have already covered the unique signature of US government bond yields with their long-term persistence (again, this is a variable that has been steadily falling for 40 years, might be due at last for a turn upward). Indeed, while most projected paths do indeed turn upward right away, others continue downward for a while. Note that negative bond yields are economically and mathematically possible. In fact, large amounts of government bonds in Japan, Switzerland, and the Eurozone currently exhibit negative yields. So the zero level of bond yields is not an absolute barrier, but it is still an unlikely one for US yields to cross.

Moving to Figure 8, we see projections for the growth of $1 invested in each of the 4 asset classes. Here the differences among these asset classes are even more pronounced. In the lower right are Large US Stocks, which given their rapid growth potential are shown on a log scale like Figure 2. Stocks appear reasonably valued today given there is no discernible change in the slope of the historical vs. projected curves as a group. Moreover, the projected curves don't appear to show any tendency to decline in the early years, as would be expected if stocks were initially overvalued. Stocks exhibit significant intermediate fluctuations, yet those risks seem relatively muted for investors having a horizon longer than 5 years. By contrast, intermediate-term treasury bonds in the upper right have risen comparatively so much since 2000 that their forward prospects appear more modest. There is a clear inflection between the steeper historical line and the flatter projected ones.

Lastly, Figure 9 contains box plots from MATLAB. These depict vertical slices of the charts in Figure 8, identifying specific percentiles of the projected cumulative wealth (i.e., growth of $1) at 5, 10, 15, and 20 years. This view provides the more precise measurements on which professional investors rely. The lowest 5% value, represented by the lower horizontal dash, is known as the 95% VAR (Value at Risk) number. This is frequently used to quantify a practical worst-case outcome. Although using a random number generator means that we can always project a far worse outcome than those shown, in practice an investor needs to be comfortable absorbing negative outcomes at the 95% VAR level shown. Note that due to the power of mean reversion in stocks, the 95% VAR level is roughly the same across all time horizons in the lower right panel. However, this is not true in the upper right panel. Given the persistence or momentum property of intermediate-term US government bonds, the 95% VAR level (or "worst-case") outcomes are significantly worse at longer time horizons. As we have seen, over a few years stocks tend to recover from even large shocks, but bond yields can persist in one direction for decades. This chart can be readily recomputed for mixes of stocks, bonds, and other asset classes, to depict the realistic choices that most investors (such as SPS trustees considering the school's endowment portfolio) might face.

These figures are just illustrative. 10,000 projected paths are available in MATLAB for practical use by Path Matters LLC. The firm will re-run the MATLAB code after every month-end, to generate a fresh set of projections that use initial market conditions from that latest month-end.

Discussion:

It's a paradox of the capital markets that, while they can appear utterly unpredictable over the short term, they still exhibit stable dynamics over longer horizons. This is exactly what Benjamin Graham meant when he said "in the short run, the market is a voting machine but in the long run, it is a weighing machine" [3]. Of course, when Graham said this decades ago he didn't have access to modern econometric tools (such as MATLAB toolboxes and internet databases) to extract that dynamic underlying structure. Nor had economists invented the theories that won 7 Nobel prizes since 1990 [6]! Fortunately, we can access all of this easily today, so classic problems (like how to allocate the SPS endowment portfolio) can now be tackled with reasonable precision to produce rigorous, intuitive answers.

I have demonstrated a model that captures the univariate, bivariate, and time series characteristics of 4 key investment variables, namely inflation, intermediate-term government bonds (including their yields), high yield bonds, and large stocks. Key to this was the ability to extract the mean reversion and persistence characteristics of each variable and then project those forward with realistic interdependence plus pure noise. Importantly, investors can prepare for the journey ahead by pre-experiencing a full projected range of outcomes.

Of course, there are many possible extensions of this work. Investors in the real world (including SPS' endowment) deal with many more than 4 investment choices. They may want to draw distinctions among US and international stocks, large, mid, and small stocks, bonds of many other maturities, or gold, oil, bitcoin, and other alternatives. Using the structure we have already laid out, modeling these additional asset classes should be reasonably straightforward. It would require data for all of these asset classes, as well as data on any intermediate variables, such as currency exchange rates or GDP growth rates. A longer list of asset classes will of course require a broader set of explanatory variables. Some of this is within immediate reach, as we initially assembled a larger data set than was ultimately used.

It is natural to ask, how good are the projections? And had we done them as of 1, 5, 10, or 20 years ago, how well would they predict what subsequently happened? There are a couple of issues with such tests. First, remember that we aren't making forecasts, but projecting 10,000 plausible projections. Technically that means we can declare success as long as what actually happened fell somewhere among the 10,000 possibilities, which does not seem like a high standard! Bear in mind that, as we said before, history provides but one path, whereas the future is a plethora of possibilities. Second, it is reasonable to ask whether the projections change by a reasonable amount when the initial conditions change. We know that interest yields have been falling since 1980 and that stocks were unusually expensive in 2000 and 2008. This is relatively easy to test and I plan to do so soon before presenting results to Path Matters. Doing so, however, does not constitute a tight test of the model, since 1980, 2000, and 2008 are all included in the sample data that was used for estimation. I would argue that we would not want to exclude any of those data from inclusion in the model estimation, since, given the long horizon nature of this model, we have relatively few time periods to begin with. In short, testing these projections is inherently difficult. We can fall back on the intuition and economic logic, which do check out from the intermediate bond and stock projections as discussed above.

Lastly, Path Matters LLC may want a more complete software package, where they can simply enter some details of the portfolio or client they are dealing with and get a full set of pre-formatted reports. This app would be largely distinct from the generation of capital market paths, which has been the focus to date. As long as the paths generated each month went into a database (rather than simply sitting in MATLAB matrices), it would be easy for an app to pull them.

Acknowledgments:

I'd like to thank Ms. Boylan, who has mentored me throughout the summer and fall and taught me the importance of disciplined habits, especially in conducting independent work. I'd like to thank Ms. Heitmiller for pointing me in the right direction in several crucial technical decisions. And I'd like to thank Mr. Gordon for his continued support.

Finally, thank you to Path Matters LLC for guiding me in this research and using the results in their applied client work.

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Final Reflections:

As I reach the end of my time in ASEP, I am reminded of the great thrills and experiences this program has offered me. Before ASEP, I had never thought that I could pursue high-level academic research during high school. Joining this exclusive group of motivated, capable high schoolers has inspired me to stretch myself academically in a truly meaningful way.

ASEP has hosted my transition from technical background to practical application. Such a transition has taught me what courses simply cannot, and also has built up my confidence in my skills to take on complex technical challenges.

I've also realized how so much of applying one's ability requires accumulating specific, targeted subjects. I came into this program thinking I'd be using what I already knew in new contexts, but I ended up learning machine learning, statistics, COVID-19 pathogenesis, capital markets, finance, econometrics, and Excel/VBA. As a result, I now appreciate the value and complexity of applying science to solving big questions we face. As such, I fondly look forward to Math Seminar next term.

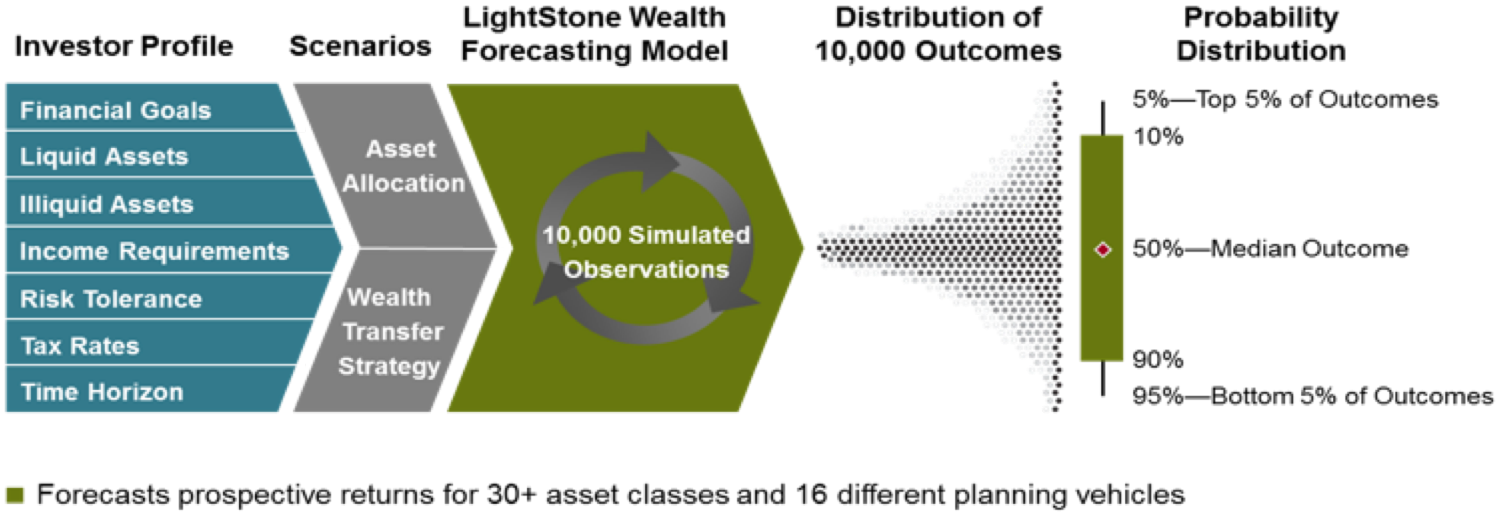
This has been an unpredictable year, to say the least, and my challenges have been many. Losing my internship at Harvard Medical School in mid-May was a big disappointment. Over the summer, I found some scholarly work published online that I developed. It was a lonely pursuit, as while researchers offered their work publicly, they were unresponsive to help otherwise. However, ultimately it taught me how to seek opportunities, pace myself, and dive headfirst into the world of academic research. Since I wrapped up my work on public health in the summer, I spent the fall on a brand-new project which posed additional challenges. This capstone asked me to pick up lots of background knowledge in a short period of time and perform rigorous statistical and econometric analysis with a large dataset. Never before have I been so aware of precision and detail orientation when manipulating such a large numerical dataset. Altogether, ASEP has improved my research ability in the specific fields I specialize in, but also improved my habits of study and discipline which apply to all my academic and personal endeavors.

However, I've also learned from my peers in the class and from the interesting and impactful ways they are using their skills to solve a variety of complex problems. Overall, the scope of applied science, mathematical modeling, and the scientific method have become abundantly clear!

Figures

Figure 1

Practical Application of My Capstone



Source: PathMatters LLC

Figure 2

Stocks Have Won Over the Long Term

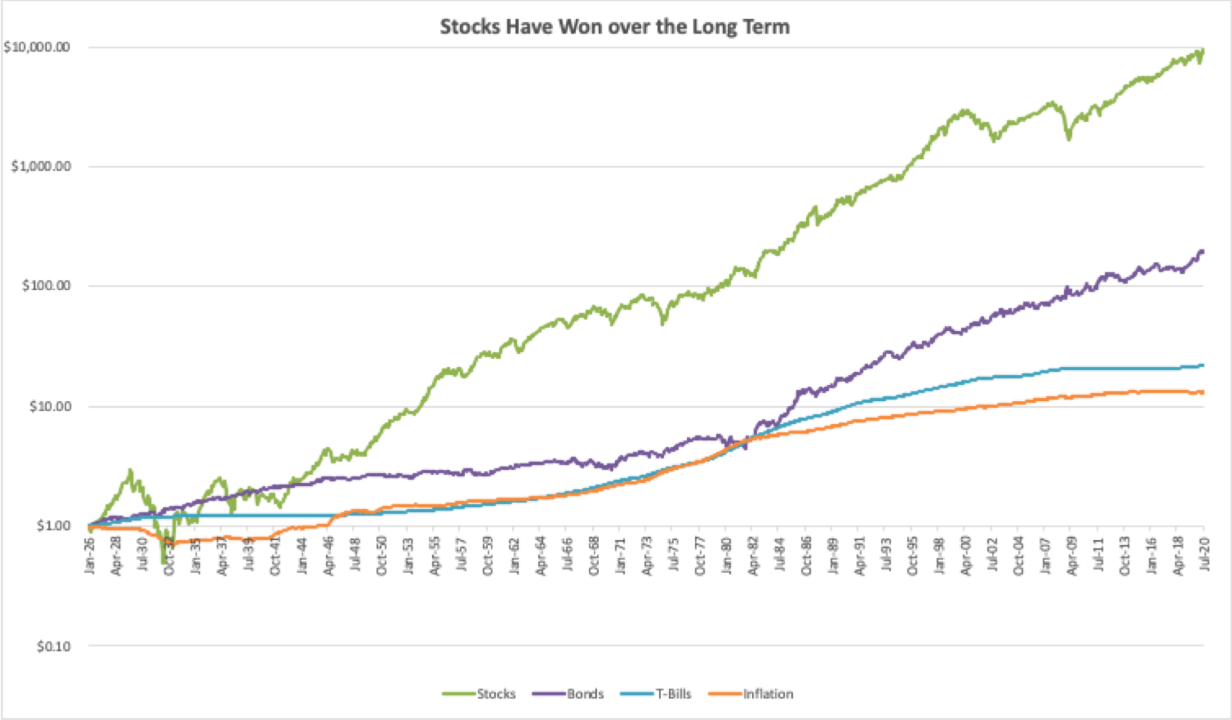
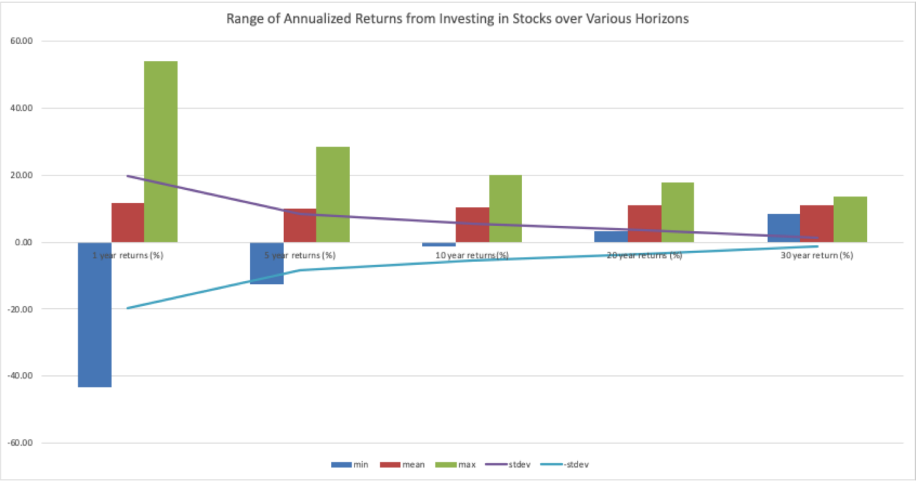
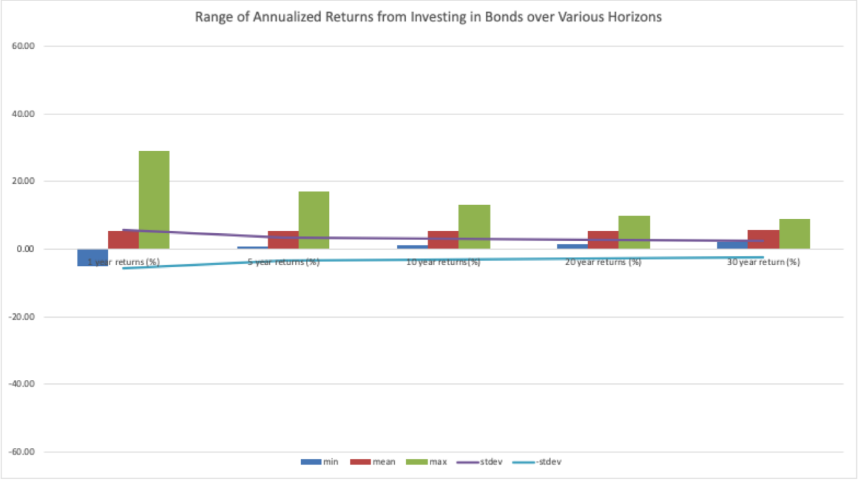


Figure 3

The Longer your Horizon is, the Less Riskier Stocks are than Bonds

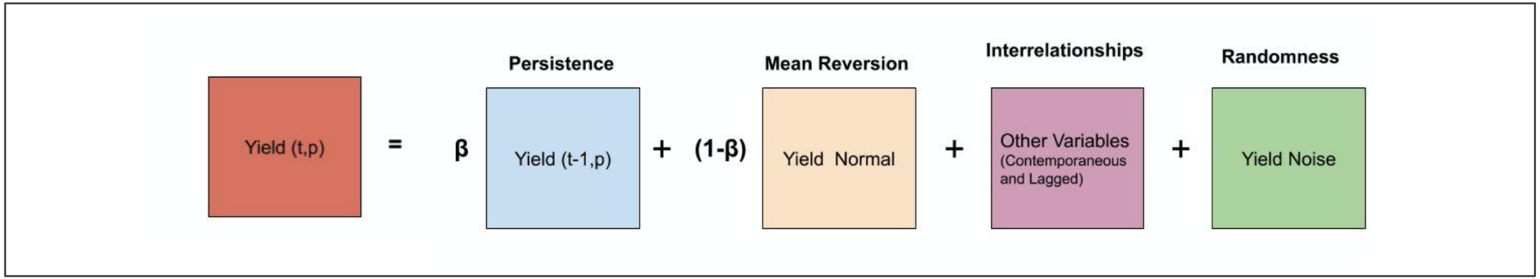




Data from Ibbotson via CFA Institute

Figure 4

Econometric Projection Details



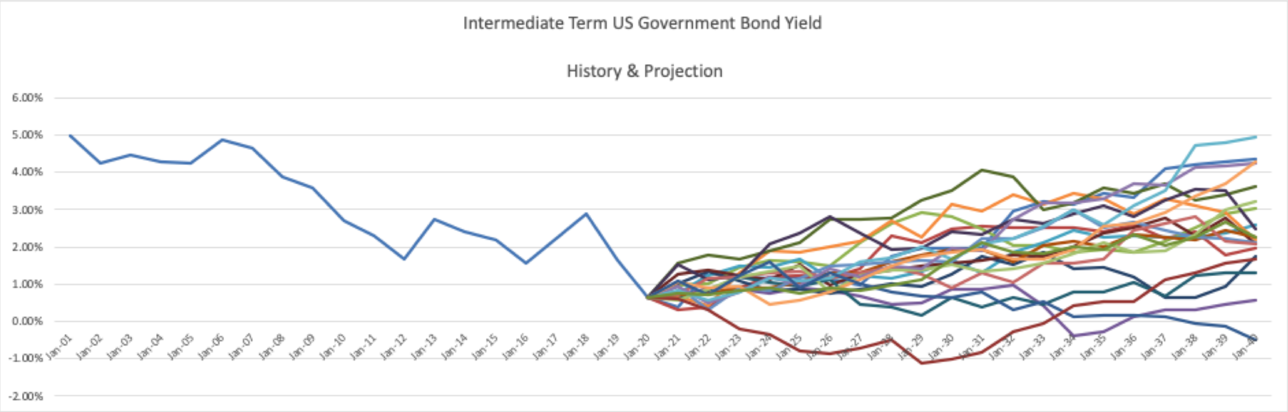


Figure 5

Structural Mapping Follows Economic Theory

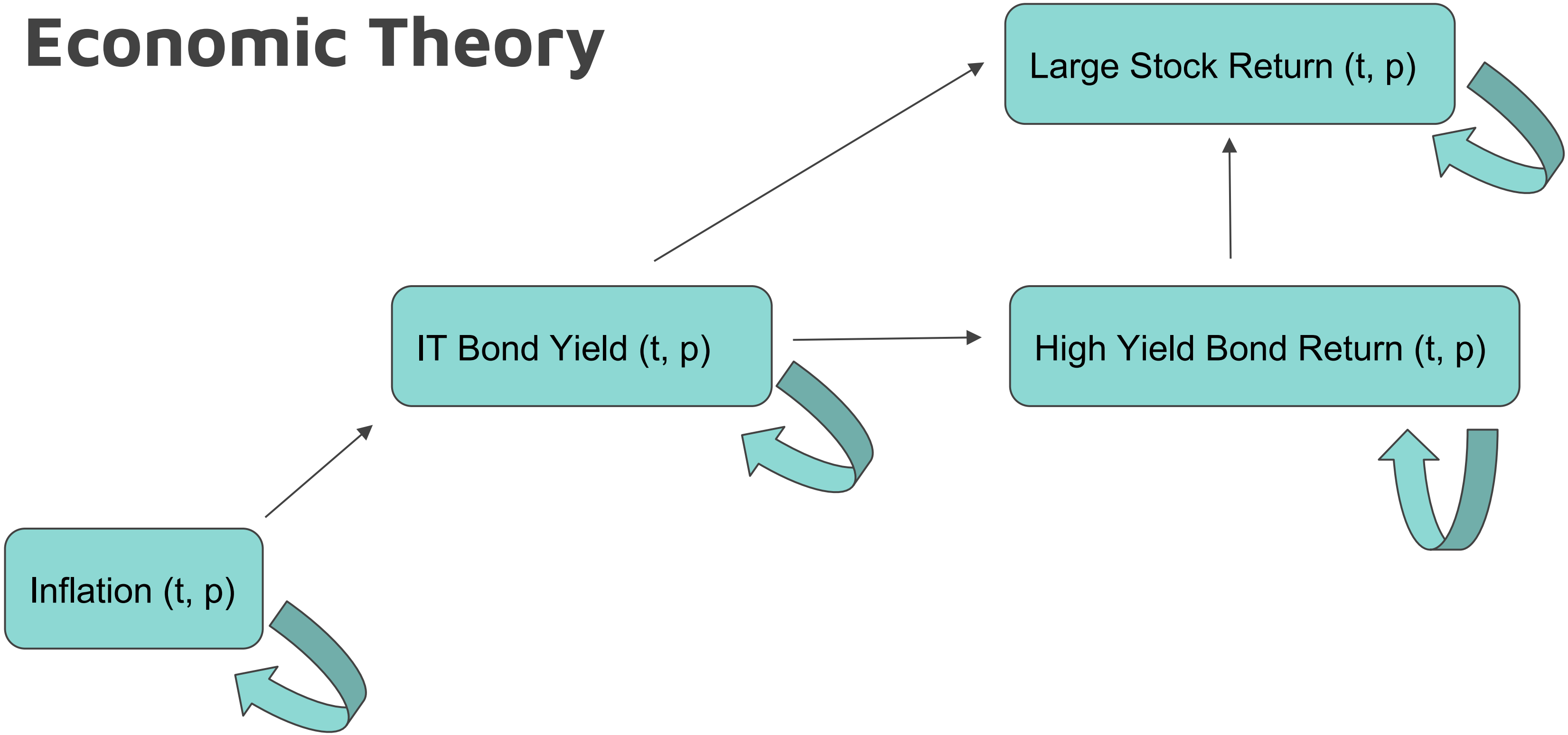
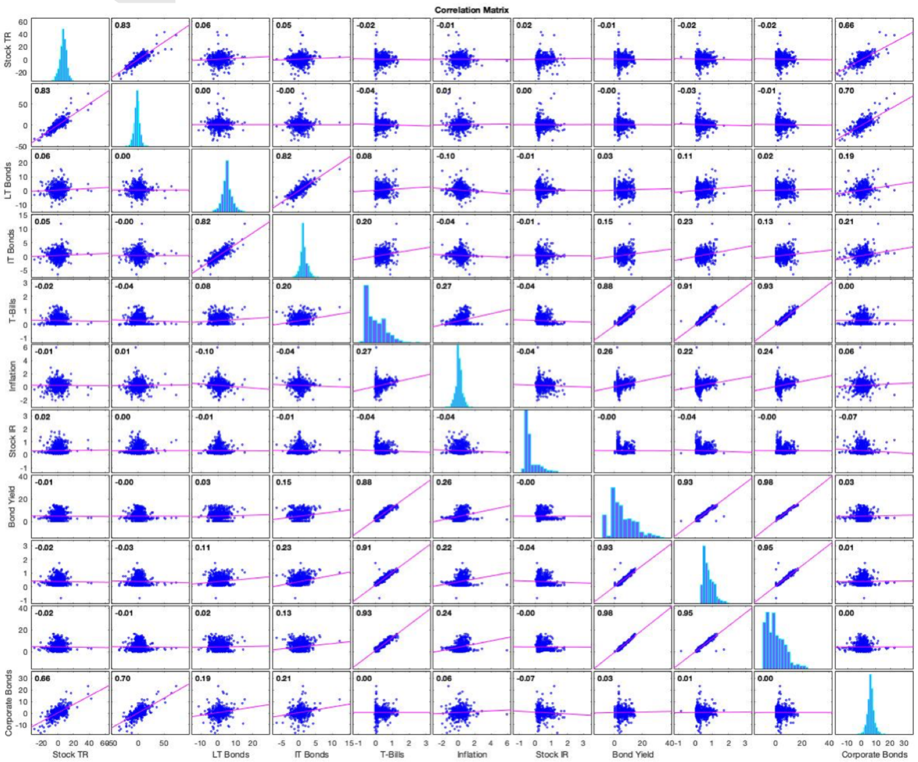


Figure 6

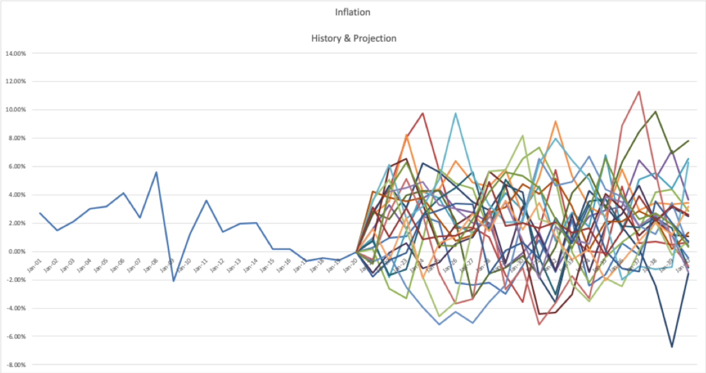
Correlations Tell a Structural Story

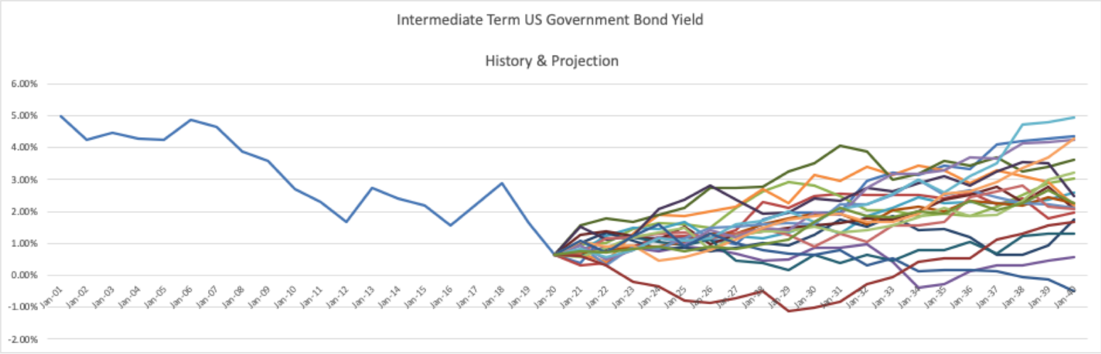


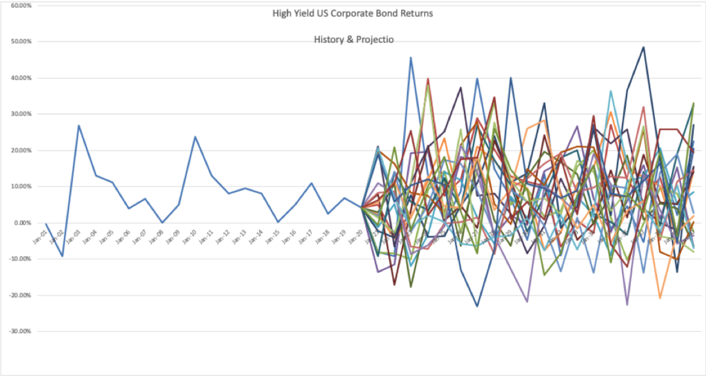
	Ibbotson SBBI US Large Stock TR USD (%Total Return)	Ibbotson SBBI US Large Stock IR	U.S. Small Stk TR (%Total Return)	U.S. IT Gvt TR (%Total Return)	U.S. IT Gvt Yld (%Yield)	U.S. 30 Day TBill TR (%Total Return)	U.S. LT Gvt TR (%Total Return)	U.S. LT Gvt Yld (%Yield)	IA Barclays US HY Corporate Bonds	Gold TR (%Price Return)	U.S. Inflation (%Price Re
FULL PERIOD											
Ibbotson SBBI US Large Stock TR USD (%Total Return)	100%	2%	83%	5%	-2%	-2%	6%	-1%	66%	1%	-1%
Ibbotson SBBI US Large Stock IR	2%	100%	0%	-1%	0%	-4%	-1%	0%	-7%	3%	-4%
U.S. Small Stk TR (%Total Return)	83%	0%	100%	0%	-1%	-4%	0%	0%	70%	0%	1%
U.S. IT Gvt TR (%Total Return)	**5%**	-1%	0%	100%	13%	20%	82%	15%	21%	5%	-4%
U.S. IT Gvt Yld (%Yield)	-2%	0%	-1%	13%	100%	93%	2%	98%	0%	-4%	24%
U.S. 30 Day TBill TR (%Total Return)	-2%	-4%	-4%	20%	93%	100%	8%	88%	0%	-4%	27%
U.S. LT Gvt TR (%Total Return)	**6%**	-1%	0%	82%	2%	8%	100%	3%	19%	4%	-10%
U.S. LT Gvt Yld (%Yield)	-1%	0%	0%	15%	98%	88%	3%	100%	3%	-3%	26%
IA Barclays US HY Corporate Bonds	66%	-7%	70%	21%	0%	0%	19%	3%	100%	8%	6%
Gold TR (%Price Return)	1%	3%	0%	5%	-4%	-4%	4%	-3%	8%	100%	11%
U.S. Inflation (%Price Return)	-1%	-4%	1%	-4%	24%	27%	-10%	26%	6%	11%	100%
1940-1979	Period of rising interest rates										
Ibbotson SBBI US Large Stock TR USD (%Total Return)	100%	-2%	81%	12%	-9%	-9%	20%	-8%	56%	-7%	-10%
Ibbotson SBBI US Large Stock IR	-2%	100%	-6%	0%	-20%	-19%	0%	-16%	-11%	4%	-2%
U.S. Small Stk TR (%Total Return)	81%	-6%	100%	12%	-6%	-6%	18%	-3%	62%	-11%	-7%
U.S. IT Gvt TR (%Total Return)	**12%**	0%	12%	100%	10%	17%	77%	13%	32%	-2%	0%
U.S. IT Gvt Yld (%Yield)	-9%	-20%	-6%	10%	100%	96%	-2%	98%	-12%	6%	19%
U.S. 30 Day TBill TR (%Total Return)	-9%	-19%	-6%	17%	96%	100%	3%	93%	-11%	10%	22%
U.S. LT Gvt TR (%Total Return)	**20%**	0%	18%	77%	-2%	3%	100%	0%	37%	-5%	-2%
U.S. LT Gvt Yld (%Yield)	-8%	-16%	-3%	13%	98%	93%	0%	100%	-9%	8%	22%
IA Barclays US HY Corporate Bonds	56%	-11%	62%	32%	-12%	-11%	37%	-9%	100%	4%	-4%
Gold TR (%Price Return)	-7%	4%	-11%	-2%	6%	10%	-5%	8%	4%	100%	17%
U.S. Inflation (%Price Return)	-10%	-2%	-7%	0%	19%	22%	-2%	22%	-4%	17%	100%
1980-present	Period of falling interest rates										
Ibbotson SBBI US Large Stock TR USD (%Total Return)	100%	9%	78%	2%	1%	1%	2%	1%	59%	5%	-4%
Ibbotson SBBI US Large Stock IR	9%	100%	6%	7%	51%	47%	7%	49%	7%	-4%	20%
U.S. Small Stk TR (%Total Return)	78%	6%	100%	-11%	2%	-2%	-11%	3%	59%	5%	-1%
U.S. IT Gvt TR (%Total Return)	**2%**	7%	-11%	100%	13%	21%	84%	14%	20%	8%	-8%
U.S. IT Gvt Yld (%Yield)	1%	51%	2%	13%	100%	93%	1%	98%	4%	-10%	43%
U.S. 30 Day TBill TR (%Total Return)	1%	47%	-2%	21%	93%	100%	7%	88%	4%	-12%	42%
U.S. LT Gvt TR (%Total Return)	**2%**	7%	-11%	84%	1%	7%	100%	1%	15%	9%	-19%
U.S. LT Gvt Yld (%Yield)	1%	49%	3%	14%	98%	88%	1%	100%	6%	-8%	41%
IA Barclays US HY Corporate Bonds	59%	7%	59%	20%	4%	4%	15%	6%	100%	12%	1%
Gold TR (%Price Return)	5%	-4%	5%	8%	-10%	-12%	9%	-8%	12%	100%	3%
U.S. Inflation (%Price Return)	-4%	20%	-1%	-8%	43%	42%	-19%	41%	1%	3%	100%
2000-present	Modern era										
Ibbotson SBBI US Large Stock TR USD (%Total Return)	100%	8%	79%	-37%	-11%	-11%	-30%	-11%	67%	4%	-1%
Ibbotson SBBI US Large Stock IR	8%	100%	4%	-3%	-35%	-32%	8%	-30%	-1%	2%	-14%
U.S. Small Stk TR (%Total Return)	79%	4%	100%	-38%	1%	-7%	-32%	2%	65%	4%	4%
U.S. IT Gvt TR (%Total Return)	**-37%**	-3%	-38%	100%	8%	15%	80%	10%	-27%	23%	-15%
U.S. IT Gvt Yld (%Yield)	-11%	-35%	1%	8%	100%	78%	-1%	94%	-5%	5%	27%
U.S. 30 Day TBill TR (%Total Return)	-11%	-32%	-7%	15%	78%	100%	4%	60%	-11%	1%	16%
U.S. LT Gvt TR (%Total Return)	**-30%**	8%	-32%	80%	-1%	4%	100%	-2%	-20%	21%	-22%
U.S. LT Gvt Yld (%Yield)	-11%	-30%	2%	10%	94%	60%	-2%	100%	-1%	8%	26%
IA Barclays US HY Corporate Bonds	67%	-1%	65%	-27%	-5%	-11%	-20%	-1%	100%	15%	10%
Gold TR (%Price Return)	4%	2%	4%	23%	5%	1%	21%	8%	15%	100%	6%
U.S. Inflation (%Price Return)	-1%	-14%	4%	-15%	27%	16%	-22%	26%	10%	6%	100%

Figure 7

Resulting Paths Follow Economic Laws







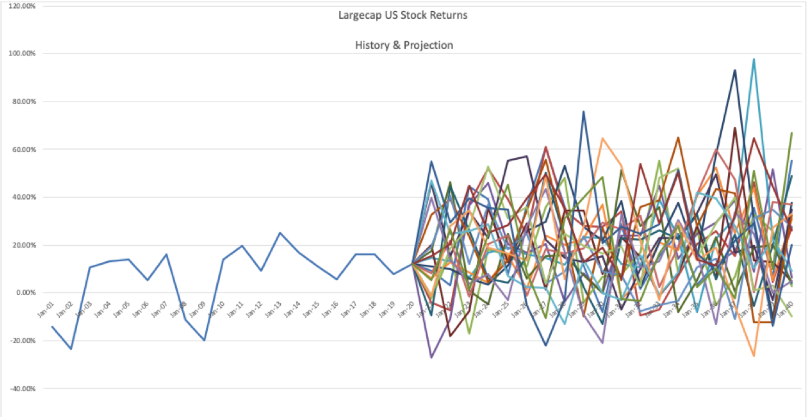
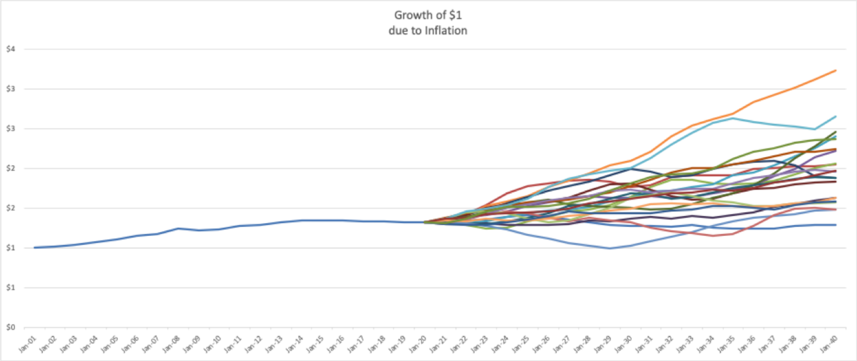
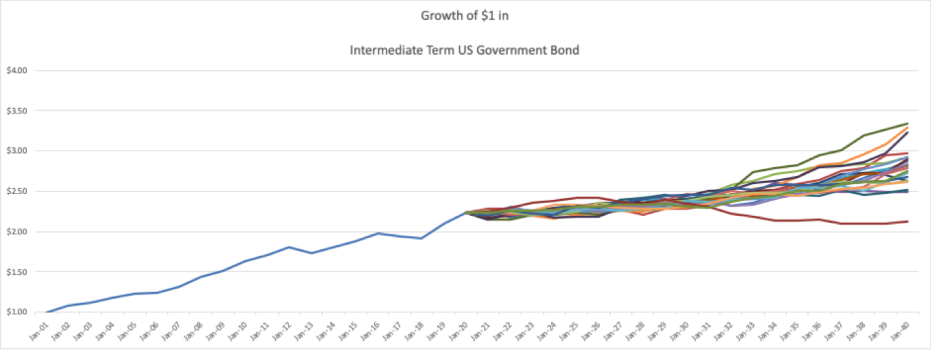
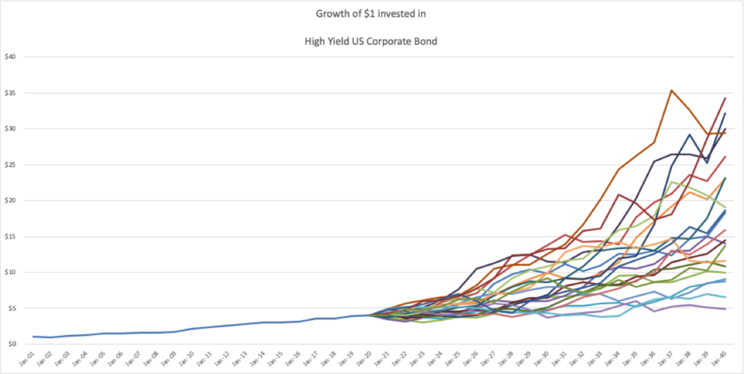


Figure 8

Investors Can Pre-Experience Wealth, Growth, and Volatility







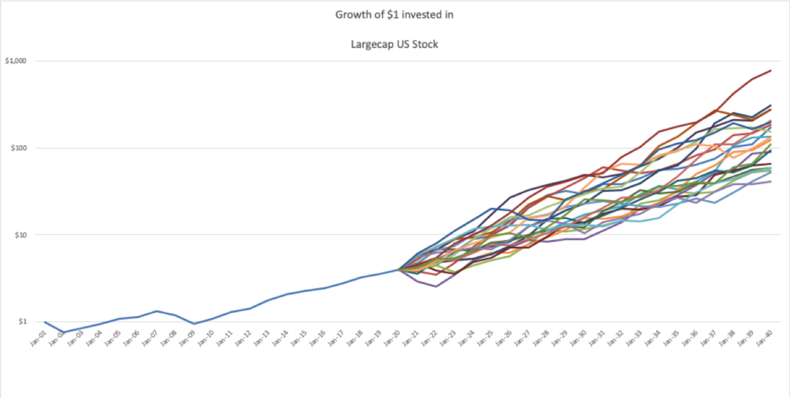
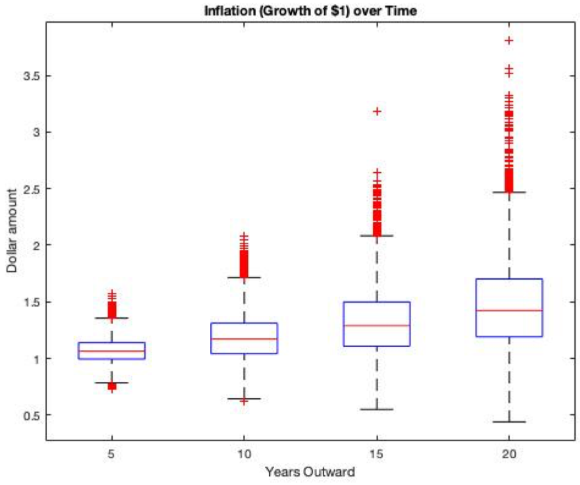
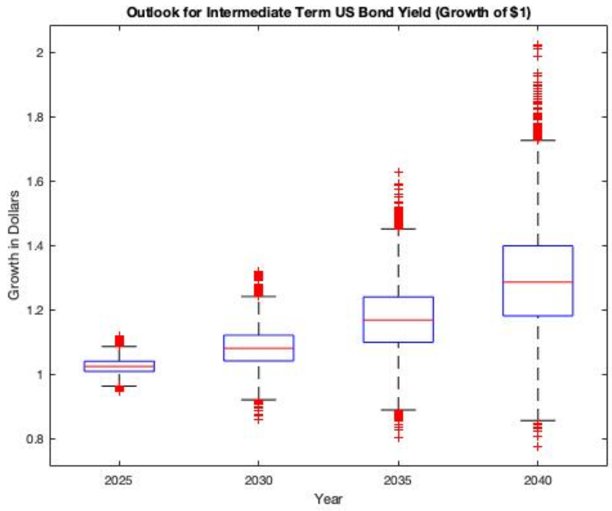
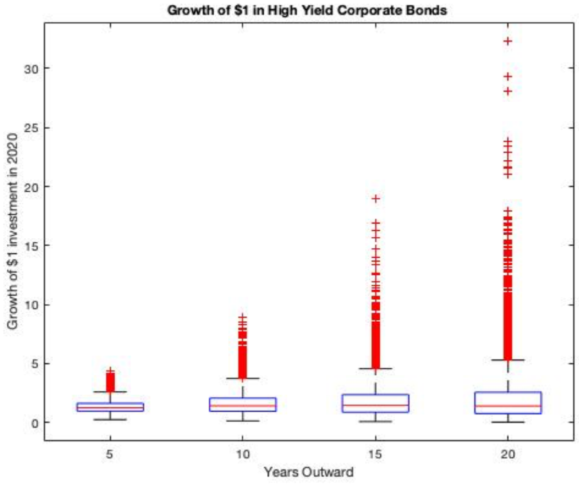


Figure 9

Box Plots Capture Finer Tradeoffs







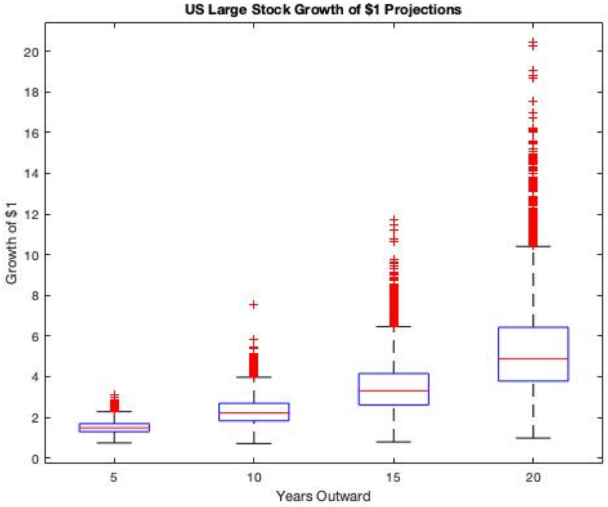


Figure 10

Equations I Estimated

$$\ln\left(1+inflation(time\ t, path\ p)\right) = 0.6359*\ln\left(1+inflation(t-12,p)\right)+0.3641*\ln(1.02)+0.5092*0.0394*normrnd(0,1)$$

$$Intermediate\ Term\ Government\ Bond\ Yield(t,p) = 0.9544*Intermediate\ Term\ Government\ Bond\ Yield(t-12,p)+0.0456*\ln(1.035)+0.0014*normrnd(0,1)$$

$$\text{Intermediate Term Government Bond Return} = Intermediate\ Term\ Government\ Bond\ Yield(t-12,p)-5*[Intermediate\ Term\ Government\ Bond\ Yield(t,p) - Intermediate\ \text{Term Government Bond Yield }(\text{t}-12,\text{p})]$$

$$\ln\left(1+High\ Yield\ Bond\ Return(t,p)\right) = 0.0722*\ln\left(1+High\ Yield\ Bond\ Return(t-12,p)\right)+0.9278*\ln(1.075)-3.2978*[Intermediate\ Term\ Government\ Bond\ Yield(t,p) - Intermediate\ Term\ Government\ Bond\ Yield(t-12,p) + 0.1129*normrnd(0,1)$$

$$\ln\left(1+Large\ Stock\ Return(t,p)\right) = 0.0546*\ln\left(1+Large\ Stock\ Return(t-12,p)\right)+0.9454*\ln(1.02)+1.162*\ln\left(1+High\ Yield\ Bond\ Return(t,p)\right)\ +0.09728*normrnd(0,1)$$

Figure 11

Sample Code from MATLAB

```
%INFLATION BEGIN

%calculate 12 month rolling period returns
inflation = table.USInflationPriceReturn;
inflationproj = zeros(20, 10000);

rolling = zeros(1132, 1);
rolling2 = zeros(1132, 1);
logconv = zeros(12, 1);
for a = 9:1132
    for b = 1:12
        logconv(b) = log(1+inflation(a + b-1)/100);
    end
    rolling(a-8) = sum(logconv);
    rolling2(a+3) = rolling(a-8);
end

%calculate R2
R = corrcoef(rolling(12:1124),rolling2(12:1124));
Rsq = R(1,2).^2;
%calculate standard deviation
sd = std(rolling(1:1124));
%linear regression with lag values
beta = [ones(1113,1) rolling(12:1124)] \ rolling2(12:1124);

%first projection
%inflationproj(1) = beta(2)*inflation(1143) + (1-beta(2))*log(1.02) + (1-Rsq)*sd*normrnd(0,1);

for f = 1:10000
    inflationproj(1,f) =  beta(2)*sum(logconv) + (1-beta(2))*log(1.02) + (1-Rsq)*sd*normrnd(0,1);
end

%10k projections
for d = 1:10000
    %inflationproj(1, d) = beta(2)*sum(logconv) + (1-beta(2))*log(1.02) + (1-Rsq)*sd*normrnd(0,1);
    for c = 2:20
        inflationproj(c, d) = beta(2)*inflationproj(c-1,d) + (1-beta(2))*log(1.02) + (1-Rsq)*sd*normrnd(0,1);
    end
end
%output below
inflationproj = (exp(inflationproj) - 1);

%growth of $1
inf_growth = zeros(21, 10000);
inf_growth(1,:) = 1;
for d = 1:10000
    for e = 2:21
        inf_growth(e,d) = inf_growth(e-1, d) * (1+inflationproj(e-1, d));
    end
end
%boxplots
forplot=(inf_growth([6 11 16 21],:))';
clearvars a b c d beta R sd Rsq;
%INFLATION END
```

intyieldproj

20x10000 double

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0.0028	0.0060	-0.0046	0.0068	0.0039	7.3216e-...	0.0035	0.0042	0.0018	0.0030	0.0034	-0.0016	0.005
2	0.0052	0.0030	-0.0048	0.0025	0.0061	-0.0014	0.0022	0.0032	0.0057	0.0096	0.0055	-0.0031	0.007
3	0.0034	0.0056	-0.0053	0.0062	0.0043	7.9518e-...	0.0089	0.0046	0.0092	0.0146	0.0061	0.0065	0.008
4	0.0065	0.0046	-0.0043	0.0123	0.0108	0.0013	0.0106	0.0076	0.0151	0.0166	0.0105	0.0110	0.006
5	0.0108	0.0097	-0.0024	0.0117	0.0082	0.0033	0.0161	0.0050	0.0170	0.0129	0.0103	0.0152	0.002
6	0.0095	0.0136	-0.0027	0.0120	0.0115	0.0066	0.0104	0.0073	0.0153	0.0116	0.0104	0.0142	0.004
7	0.0106	0.0142	-0.0018	0.0094	0.0042	0.0072	0.0142	0.0101	0.0177	0.0114	0.0104	0.0141	0.005
8	0.0116	0.0187	4.9390e-...	0.0059	0.0093	0.0076	0.0146	0.0173	0.0173	0.0123	0.0139	0.0151	0.006
9	0.0163	0.0195	0.0033	0.0146	0.0148	0.0107	0.0168	0.0170	0.0218	0.0121	0.0141	0.0179	0.002
10	0.0160	0.0196	0.0067	0.0227	0.0117	0.0080	0.0202	0.0177	0.0249	0.0144	0.0152	0.0184	-0.001
11	0.0174	0.0193	0.0022	0.0199	0.0186	0.0072	0.0188	0.0159	0.0235	0.0124	0.0164	0.0236	-0.002
12	0.0137	0.0220	0.0070	0.0183	0.0121	0.0045	0.0184	0.0182	0.0224	0.0081	0.0156	0.0293	2.5930e-.
13	0.0199	0.0250	0.0083	0.0163	0.0110	0.0095	0.0128	0.0181	0.0204	0.0041	0.0093	0.0282	-0.003
14	0.0216	0.0280	0.0044	0.0189	0.0106	0.0109	0.0148	0.0183	0.0172	0.0030	0.0148	0.0270	0.007
15	0.0209	0.0302	0.0065	0.0185	0.0093	0.0191	0.0196	0.0162	0.0215	0.0065	0.0138	0.0294	0.003
16	0.0177	0.0298	0.0102	0.0175	0.0079	0.0176	0.0201	0.0186	0.0286	0.0081	0.0134	0.0341	0.004
17	0.0184	0.0266	0.0120	0.0235	0.0127	0.0213	0.0218	0.0184	0.0321	0.0117	0.0101	0.0421	0.008
18	0.0245	0.0284	0.0145	0.0252	0.0162	0.0211	0.0205	0.0186	0.0233	0.0159	0.0081	0.0409	0.007
19	0.0225	0.0281	0.0111	0.0235	0.0180	0.0166	0.0260	0.0149	0.0221	0.0172	0.0080	0.0360	0.007
20	0.0222	0.0256	0.0084	0.0172	0.0180	0.0147	0.0276	0.0198	0.0211	0.0147	0.0076	0.0302	0.009
21													
22													