MAT354: Complex Analysis I

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1 Complex Numbers

The set of complex numbers $\mathbb{C}$ is the collection of elements of the form $\alpha + i\beta$ where α, β are real numbers and i is (defined to be) a solution $z^2 + 1 = 0$. The other solution to this quadratic is $-i$. One can imagine that it is difficult to distinguish between the two solutions. One of them we call i and the other one then becomes $-i$ but it's not clear how to decide which one should be which. Ultimately, this is a somewhat arbitrary choice so we should be able to swap between the two solutions. This leads to the idea of conjugation. If $a = \alpha + i\beta$, we define the *conjugate* of a by

$$\overline{a} := \alpha - i\beta$$

The real numbers α and β are called the real and imaginary parts of a respectively. It is easy to verify that

$$\overline{a+b} = \overline{a} + \overline{b} \text{ and } \overline{ab} = \overline{a}\overline{b}$$

This means that addition and multiplication are not affected when we swap i and $-i$. This is good news since it means that our choice of calling one of the solutions i and the other $-i$ doesn't affect the algebraic structure in question.

Using the conjugate, we can find the real and imaginary parts of a complex number quite easily.

$$\mathrm{Re}(a) = \frac{a + \overline{a}}{2}, \ \mathrm{Im}(a) = \frac{a - \overline{a}}{2i}$$

Additionally, we have the *modulus* of a complex number

$$|a| = \sqrt{\alpha^2 + \beta^2} = \sqrt{a\overline{a}}$$

The familiar inequalities from analysis, the triangle inequality and Cauchy-Schwarz inequality still hold. So in particular we have

$$|a + b| \leq |a| + |b|$$
$$\left|\sum_{j=1}^{n} a_j b_j\right|^2 \leq \left(\sum_{j=1}^{n} |a_j|^2\right)\left(\sum_{j=1}^{n} |b_j|^2\right)$$

1.1 Operations of Complex Numbers

Addition of complex numbers is exactly vector addition (corresponding with the usual geometric intuition). Multiplication is a bit more interesting. Using the usual distributive laws and the definition of i we get

$$(\alpha + i\beta)(\gamma + i\delta) = (\alpha\gamma - \beta\delta) + i(\beta\gamma + \alpha\delta)$$

However, there is a nicer way of thinking about multiplication. First we note that any complex number $a = \alpha + i\beta$ can be written in polar coordinates with

$$\alpha = r\cos\theta$$
$$\beta = r\sin\theta$$

where $r = |a|$ and θ is the angle made with the positive real line (i.e. positive x-axis). We call θ the argument and note that it is only unique up to multiples of 2π.

If we have $a = r_1(\cos\theta_1 + i\sin\theta_1)$ and $b = r_2(\cos\theta_2 + i\sin\theta_2)$ then by using trigonometric identities, we find that

$$ab = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

Hence $\arg(ab) = \arg(a) + \arg(b)$ $(\mathrm{mod} 2\pi$ if necessary). The immediately shows that

$$a^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

for all integers n. To be precise, this has only been shown for non-negative integers, but we can see it holds for all integers by noting that

$$a^{-1} = r^{-1}\frac{1}{\cos\theta + i\sin\theta} = r^{-1}(\cos\theta - i\sin\theta) = r^{-1}(\cos(-\theta) + i\sin(-\theta))$$

The identity above also gives us de Moivre's formula

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta))$$

1.2 n-th roots of unity

The roots of the polynomial $z^n - 1$ are called the n-th roots of unity. Using de Moivre's formula, we see that one root of unity is

$$\omega := \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$$

and the remaining roots are $\omega^2, \omega^3, \ldots, \omega^n$ since

$$\omega^k = \cos\left(k \cdot \frac{2\pi}{n}\right) + i\sin\left(k \cdot \frac{2\pi}{n}\right)$$

2 Riemann Sphere

We will see that it is useful to work with $\mathbb{C}$ along with a point at ∞. In other words, we want to extend $\mathbb{C}$ to include ∞. There are some basic arithmetic properties we would like ∞ to have:

- $a + \infty = \infty = \infty + a$ for all $a \in \mathbb{C}$
- $a \cdot \infty = \infty = \infty \cdot a$ for $a \neq 0$
- $\frac{a}{\infty} = 0$ for $a \neq \infty$
- $\frac{a}{0} = \infty$ for $a \neq 0$

The way we are going to 'deal' with infinity is by using stereographic projection. Recall the plane is homeomorphic to the sphere without a point, say the north pole. Thus the north pole becomes the point at infinity (under the homeomorphism given by stereographic projection).

To be a bit more precise, we will consider 2 charts from S^2 to $\mathbb{C}$

$$z = \frac{x + iy}{1 - t},\ z' = \frac{x - iy}{1 + t}$$

where the first is used for $S^2 \backslash \{N\}$ and second is used for $S^2 \backslash \{S\}$. Note then that

$$zz' = \frac{x + iy}{1 - t} \cdot \frac{x - iy}{1 + t} = 1$$

Therefore the transition map from one chart to the other is

$$z' = \frac{1}{z}$$

To put it a bit informally, we will often think of $\infty = \frac{1}{0}$ so we will compute $f(\infty)$ by finding $f\left(\frac{1}{z}\right)$ and substituting $z = 0$.

In conclusion, the Riemann sphere will refer to the complex plane $\mathbb{C}$ with the point at infinity adjoined to it which we can identity with S^2 via stereographic projection which tells us the topology and such of the extended plane.

Before we move on to talking about complex functions, a brief aside on why stereographic projection is a particularly nice homeomorphism.

Lemma 2.1 *Any circle in S^2 (the Riemann sphere) corresponds (uniquely) to a circle or straight line in $\mathbb{C}$.*

Proof. Any circle in S^2 is the intersection of a plane with the sphere. Say this plane is given by the points (x, y, t) such that

$$ax + by + ct = d$$

Additionally we know that stereographic projection is given by

$$(x, y, t) \mapsto z := \frac{x + iy}{1 - t}$$

We would like to find out what happens to the points on the intersection above after they are mapped to the complex plane. In other words, we want to express the equation of the plane in terms of z as the resulting equation will capture how this circle is transformed. We will do this by expressing x, y, t entirely in terms of z. Another way of thinking about this is to note that by expressing the coordinates x, y, t in terms of z we are effectively computing the inverse of stereographic projection. Thus we are trying to determine what geometric objects on the complex plane are mapped to a circle on the sphere.

Then we see that

$$|z|^2 = z\overline{z} = \frac{x + iy}{1 - t} \cdot \frac{x - iy}{1 - t} = \frac{1 + t}{1 - t}$$

implying that

$$t = \frac{|z|^2 - 1}{|z|^2 + 1} \text{ or } 1 - t = \frac{2}{|z|^2 + 1}$$

Using these identities and considering the real/imaginary parts of z we see that

$$x = (1 - t) \cdot \frac{1}{2}(z + \overline{z}) = \frac{z + \overline{z}}{|z|^2 + 1}$$

and

$$y = (1 - t) \cdot \frac{1}{2i}(z - \overline{z}) = \frac{z - \overline{z}}{i(|z|^2 + 1)}$$

Now that we can substitute this into the equation of the plane above to get

$$a(z + \overline{z}) + \frac{b}{i}(z - \overline{z}) + c(|z|^2 - 1) = d(|z|^2 + 1)$$

We can substitute $z = u + iv$ to further simplify this to

$$(d - c)(u^2 + v^2) - 2au - 2bv + (d + c) = 0$$

If $d = c$ then we get a line and if $d \neq c$, we get a circle. In order to see the latter statement, note we can complete the square to rearrange the above to

$$\left(u - \frac{a}{d - c}\right)^2 + \left(v - \frac{b}{d - c}\right)^2 = \frac{a^2 + b^2 + c^2 - d^2}{(d - c)^2}$$

The right hand side is only negative if the plane is very far from the origin since the distance between the plane and the origin is

$$\frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$$

(see here).

Finally, note that the correspondence between circles on the sphere and circles/lines in the complex plane is one-to-one since any of the latter objects can be written in the above form. □

3 Rational functions

Ultimately what we want to build up to is of course differentiation of complex functions. But before that it is useful to study some simple examples of complex functions. The simplest examples of complex functions are polynomials and by extension their quotients, what we often call rational functions. Such functions can be differentiated 'like normal' using combinations of the chain, product and quotient rule (where perhaps we think of the derivative as a formal operator acting on rational functions).

3.1 Linear functions

The simplest possible complex functions are linear transformations, which we already know quite well from the study of linear algebra. For example, we know that all such maps T are of the form $T(z) = az$ for some fixed $a \in \mathbb{C}$. Note that such maps are a composition of rotation and dilation. Therefore it preserves angles and orientation. Such maps are called *homothetic*.

To be precise, the above maps are $\mathbb{C}$-linear. There are (of course) also maps that are not $\mathbb{C}$-linear but rather $\mathbb{R}$-linear. An example of such a map is the conjugation map $z \mapsto \overline{z}$. An $\mathbb{R}$-linear transformation of $\mathbb{R}^2$ is given by a matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

By identifying $\mathbb{R}^2$ with $\mathbb{C}$ we see that such a map is $\mathbb{C}$-linear if and only if $\alpha = \delta$ and $\gamma = -\beta$.

As an exercise let us find all angle-preserving $\mathbb{R}$-linear transformations of $\mathbb{C}$.

Proposition 3.1 *Suppose $T : \mathbb{C} \to \mathbb{C}$ is an $\mathbb{R}$-linear transformation such that it preserves angles. Then there is some $a \in \mathbb{C}$ such that $T(z) = az$ or $T(z) = a\overline{z}$.*

Proof. Suppose S is a homothetic linear transformation so that $S^{-1}T$ fixes $(1, 0)$. Since $S^{-1}T$ is also angle-preserving, we know that $(S^{-1}T)(0, 1) = (0, c)$ for some $c \neq 0$. Finally we know that $(S^{-1}T)(1, 1) = (1, c)$ since $S^{-1}T$ is linear. The fact that it is angle-preserving means that $c = \pm 1$. If $c = 1$ then $S^{-1}T = I$ so that $T(z) = S(z) = az$. If $c = -1$ then $(S^{-1}T)(z) = \overline{z}$ so that $T(z) = a\overline{z}$. □

3.2 Rational functions

Like in the real case, a rational function R is a function of the form

$$R(z) = \frac{P(z)}{Q(z)}$$

where P, Q are polynomials with no common factor. The zeroes of R are simply the zeroes of P while the zeroes of Q are called the *poles* of R. In other words, the poles of R are points z_0 in the extended complex plane (therefore z_0 could be ∞) so that

$$\lim_{z \to z_0} R(z) = \infty$$

Equivalently, the poles are the points that are mapped to the north pole on the Riemann sphere which hopefully justifies the name.

We also want to consider the order of a zeroes and poles. The order of a zero is simply its multiplicity (as a root of the polynomial). The order of a pole of R then is the order of the corresponding zero of the denominator. We see that the poles of $R'(z)$ are the same as the poles of R. This is easy to see using the quotient rule of derivative.

$$R'(z) = \frac{P'(z)Q(z) - P(z)Q'(z)}{Q(z)^2}$$

One might wonder what happens to the order of poles upon differentiation. Suppose $Q(z) = (z-a)^k\tilde{Q}(z)$ where $z-a$ does not divide $\tilde{Q}(z)$. Then $Q'(z) = k(z-a)^{k-1}\tilde{Q}(z) + (z-a)^k\tilde{Q}'(z)$. Then

$$\begin{aligned}R'(z) &= \frac{P'(z)(z-a)^k\tilde{Q}(z) - P(z)\cdot(k(z-a)^{k-1}\tilde{Q}(z) + (z-a)^k\tilde{Q}'(z))}{Q(z)^2} \\ &= \frac{(z-a)^{k-1}\cdot((z-a)P'(z)\tilde{Q}(z) - P(z)\cdot(k\tilde{Q}(z) + (z-a)\tilde{Q}'(z)))}{(z-a)^{2k}\tilde{Q}(z)^2} \\ &= \frac{(z-a)P'(z)\tilde{Q}(z) - P(z)\cdot(k\tilde{Q}(z) + (z-a)\tilde{Q}'(z))}{(z-a)^{k+1}\tilde{Q}(z)^2}\end{aligned}$$

Thus we see that the order of the pole a was k in R and $k+1$ in R'.

At this point, one might wonder how we would find the order of a pole at ∞. In this case we will consider $R_1(z) = R(\frac{1}{z})$. Then R has a pole at ∞ if and only if R_1 has a pole (the same one) at 0. So in particular the order of a pole of $R(z)$ at ∞ is the order of the pole of $R_1(z)$ at 0. We of course do the same thing if instead ∞ is a zero and we want to find its order.

Suppose

$$R(z) = \frac{a_m z^m + \cdots + a_0}{b_n z^n + \cdots + b_0}$$

with $a_m, b_n \neq 0$. Then

$$R_1(z) = R\left(\frac{1}{z}\right) = z^{n-m}\cdot\frac{a_0 z^m + a_1 z^{m-1} + \cdots + a_m}{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}$$

Therefore if $n > m$ then $R(z)$ has a 0 at ∞ of order $n-m$. R also has m finite zeroes, counted with multiplicity. Thus the total number of zeroes counted with multiplicity (including ∞) is n. If $n < m$, then $R(z)$ has a pole at ∞ of order $m-n$ and as before we can conclude that the total number of poles is m. Finally, if $n = m$ then $R(\infty) = \frac{a_n}{b_m}$ which we know is not 0.

The moral of the story is that the total number of zeroes/poles for a rational function is always $\max\{n, m\}$. We often call this number the order of the rational function. A consequence of this statement is that every equation $R(z) = a$ always has order R solutions.

3.2.1 Möbius Transformations

Consider rational functions of order 1. Such functions are of the form

$$S(z) = \frac{az+b}{cz+d}$$

with $ad - bc \neq 0$ (if $ad = bc$ then S simplifies to a constant function which would have order 0). Such maps are called fractional linear transformations or Möbius transformations.

We know that $S(z) = w$ has exactly 1 root for every $w \in \mathbb{C}$ implying that S is invertible. In fact we can compute this directly to find that

$$S^{-1}(z) = \frac{dz-b}{-cz+a}$$

3.2.2 Partial Fraction Decomposition

Partial fraction decomposition is a very useful tool since it allows us to turn a big product into a sum of simpler things. Essentially, partial decomposition allows us to 'deal with' the distinct poles of the rational function one at a time.

Theorem 3.2 (Partial fraction decomposition) *Given a rational function $R(z) = \frac{P(z)}{Q(z)}$ we can write it as*

$$R(z) = G(z) + \sum_{j=1}^{l} G_j\left(\frac{1}{z-\beta_j}\right)$$

where G and G_j are polynomials and β_j are zeroes of Q.

Proof. We have polynomials P, Q and wish to simplify

$$R(z) = \frac{P(z)}{Q(z)}$$

First we perform polynomial long division until $\deg(P) \leq \deg(Q)$. To be precise, we find polynomials $G, \widetilde{H}$ so that

$$P(z) = G(z)Q(z) + \widetilde{H}(z)$$

with $\deg(\widetilde{H}) \leq \deg(Q)$ and G is a polynomial without constant term (since the degree of $\widetilde{H}$ and Q are allowed to be the same, we can multiply the constant term in G by Q and absorb it into $\widetilde{H}$). We then have that

$$R(z) = G(z) + H(z)$$

where $H(z) := \frac{\widetilde{H}(z)}{Q(z)}$. Note that $H(z)$ is finite at ∞ (compare the degrees of $\widetilde{H}$ and Q). This means that the order of the pole of $R(z)$ at ∞ is simply the degree of G. Hence why we call G the singular part of $R(z)$ at ∞.

Hence we've gotten some handle on the infinite poles of R. Now we would like to do something similar with the finite poles. Let $\beta_1, \ldots, \beta_l$ be distinct finite poles of $R(z)$. Then $R\left(\beta_j + \frac{1}{\zeta}\right)$ is a rational function in ζ with a pole at ∞. Then by performing a change in coordinates we find that

$$R(z) = G_j\left(\frac{1}{z-\beta_j}\right) + H_j\left(\frac{1}{z-\beta_j}\right)$$

Recall that G_j has no constant term at H_j is finite at ∞.

Now consider

$$R(z) - G(z) - \sum_{j=1}^{l} G_j\left(\frac{1}{z-\beta_j}\right)$$

This is a rational function and we would like to determine its order by finding the number of poles it has.

We see that this rational function can have poles at most at $\beta_1, \ldots, \beta_l$ and ∞. For $z = \beta_j$ there are only two terms that can be infinity, namely $R(z)$ and $G_j(\frac{1}{z-\beta_j})$ but

$$R(z) - G_j\left(\frac{1}{z-\beta_j}\right) = H_j\left(\frac{1}{z-\beta_j}\right)$$

which we know is finite at β_j. A similar thing happens for $z = \infty$ by considering $R(z) - G(z)$. Thus we conclude that the above rational function has no poles and therefore must be of order 0 implying that it is a constant. By absorbing the constant into $G(z)$ we can write

$$R(z) = G(z) + \sum_{j=1}^{l} G_j\left(\frac{1}{z-\beta_j}\right)$$

□

Remark 3.3. The case with real polynomials can be deduced from the above by observing that roots of real polynomials appear in complex conjugate pairs.

3.3 Order 2 rational functions

As usual when mathematicians find a new thing to play with, they want to know how many of the thing there are. In this case, we want to know how many order 2 rational functions there are, up to a change of coordinates. Since we are working on the Riemann sphere the change of coordinate functions are given by fractional linear transformations. Remarkably (and rather delightfully), we will find that there are essentially 2 kinds of such functions.

Suppose we have a rational function $R(z)$ of order 2. We know then, it must have 2 poles. There are two cases to consider: either the function has one double pole β or it has two distinct poles a, b.

We begin by considering the first case of having a rational function with a double pole. First we move the pole to infinity by considering $R\left(\beta + \frac{1}{\zeta}\right)$ which is a rational function in ζ and has a double pole at ∞. A rational function with a double pole at infinity is simply a quadratic so now we have a function of the

$$w = az^2 + bz + c$$

To be precise we should have a function in ζ we have relabeled the variables for convenience. By completing the square we see that

$$w = a\left(z + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

We rearrange this to get

$$w + \frac{b^2}{4a} - c = a\left(z + \frac{b}{2a}\right)^2$$

If we relabel $w_1 := w + \frac{b^2}{4a} - c$ and $z_1 := \sqrt{a}\left(z + \frac{b}{2a}\right)$ we have an equation of the form

$$w_1 = z_1^2$$

This means that any rational function of order 2 with a double pole can be written as the simplest possible quadratic, $w = z^2$ (if we change coordinates appropriately).

Now suppose the rational function has distinct poles a, b. We first compose this function with

$$\frac{z-b}{z-a}$$

to move the poles to ∞ and 0 respectively. Such a rational function can be written in the form

$$w = Az + B + \frac{C}{z}$$

where A, B, C are some constants. We know this is the form of the resulting function by partial fraction decomposition. The partial fraction decomposition allows us to write a rational function as a sum of rational functions where each term in the sum only has one distinct pole. Thus the linear term comes from the pole at ∞ and the inverse term comes from the pole at 0. Suppose we take $z' = \sqrt{\frac{A}{C}}z$. Then

$$w = A\left(\frac{\sqrt{C}}{\sqrt{A}}z'\right) + B + C\left(\frac{\sqrt{A}}{\sqrt{C}z'}\right)$$
$$w - B = \sqrt{AC}\left(z' + \frac{1}{z'}\right)$$

Much like before we can simplify this to the case

$$w_1 = z_1 + \frac{1}{z_1}$$

by taking $w_1 := \frac{w-B}{\sqrt{AC}}$ and $z_1 := z'$.

Therefore every order 2 rational function is (basically) of the form $w = z^2$ or $w = z + \frac{1}{z}$.

3.4 Fractional Linear Transformations

Unfortunately the classification for order 1 rational functions, which is to say fractional linear transformations, is not nearly as nice (maybe this makes sense since these maps are homeomorphisms of the Riemann sphere which seems like a rather complicated, or dare I say complex, object). Besides, in this case the classification needs to be a bit different anyway since it wouldn't make sense to look at all fractional linear transformations up to fractional linear transformations. With all that out of the way, we *can* say some things about them.

Proposition 3.4 *Every fractional linear transformation is a composition of translation, homothety and inversion.*

Proof. Suppose we have

$$w = S(z) = \frac{az+b}{cz+d}$$

with $ad - bc \neq 0$. Suppose c is 0. Then we can write $w = az + b$ (with some relabeling of variables). This is clearly a composition of a homothety and translation.

Suppose $c \neq 0$. Then

$$\frac{az+b}{cz+d} = \frac{\frac{a}{c}\left(z+\frac{d}{c}\right)+\frac{bc^2-ad}{c^2}}{c\left(z+\frac{d}{c}\right)} = \frac{a}{c^2} + \frac{bc^2-ad}{c^2} \cdot \frac{1}{z+\frac{d}{c}}$$

In this formulation, it is apparent that the map is indeed a composition of translation, homothety and inversion. □

As mentioned previously, fractional linear transformations are also useful for understanding the geometry of the Riemann sphere. But first we need a few preliminary results.

Lemma 3.5 *Given any 3 distinct points z_2, z_3, z_4 in S^2, there is a unique fractional linear transformation S that maps these points to $(1, 0, \infty)$ respectively.*

Proof. Existence of such a fractional linear transformation can be seen by simply defining it:

$$S(z) = \left(\frac{z_2 - z_3}{z_2 - z_4}\right)^{-1} \frac{z - z_3}{z - z_4}$$

There are a few degenerate cases to consider (namely when one of the points is ∞).

$$\begin{aligned} z_2 = \infty, S(z) &= \frac{z - z_3}{z - z_4} \\ z_3 = \infty, S(z) &= \frac{z_2 - z_4}{z - 4} \\ z_4 = \infty, S(z) &= \frac{z - z_3}{z_2 - z_3} \end{aligned}$$

In order to see uniqueness, let T be any linear transformation that maps z_2, z_3, z_4 to $1, 0, \infty$ respectively. Then note $S \circ T^{-1}$ is also a fractional linear transformation, but one that fixes $1, 0$ and ∞. Since it is a fractional linear transformation, we can write

$$(S \circ T^{-1})(z) = \frac{az+b}{cz+d}$$

The fact that 0 is fixed implies that $b = 0$ and ∞ being fixed implies that $c = 0$. Thus we are left with a map of the form $(S \circ T^{-1})(z) = \alpha z$. By considering $(S \circ T^{-1})(1)$ we conclude that $\alpha = 1$ and $S \circ T^{-1}$ is the identity. Therefore $S = T$ as desired. □

Now we define a quantity used often in geometry.

Definition 3.6 (Cross-ratio)**.** Given 4 (distinct) points on S^2, we define

$$(z_1 : z_2 : z_3 : z_4) := S(z_1)$$

where S as before is the (unique) linear transformation sending z_2, z_3, z_4 to $1, 0, \infty$ respectively. This quantity is known as the cross-ratio

The cross-ratio has some delightful properties.

Theorem 3.7 *If $z_1, z_2, z_3, z_4 \in S^2$ are distinct points and T is a fractional linear transformation, then $(Tz_1 : Tz_2 : Tz_3 : Tz_4) = (z_1 : z_2 : z_3 : z_4)$.*

Proof. Let $S(z) = (z : z_2 : z_3 : z_4)$. Note then ST^{-1} maps Tz_2, Tz_3, Tz_4 to $(1, 0, \infty)$ respectively. Therefore

$$(Tz_1 : Tz_2 : Tz_3 : Tz_4) = ST^{-1}(Tz_1) = S(z_1) = (z_1 : z_2 : z_3 : z_4)$$

□

Theorem 3.8 *If $z_1, z_2, z_3, z_4 \in S^2$ are distinct points then $(z_1 : z_2 : z_3 : z_4)$ is real if and only if the four points lie on a circle or a line in the complex plane.*

Proof. Suppose we map $1, 0, \infty$ to the points z_2, z_3, z_4 respectively via a fractional linear transformation. As we've seen, fractional linear transformations are completely determined by the image of 3 points. Let us call the map above then T^{-1}. Consider the map $z \mapsto (z : z_2 : z_3 : z_4)$ which must be $T(z)$ (since $T(z_2) = 1, T(z_3) = 0$ and $T(z_4) = \infty$ by construction). Therefore $(z : z_2 : z_3 : z_4) = T(z)$ is real if $z \in T^{-1}(\mathbb{R})$ and non-real otherwise.

Thus we want to show that the image of the real line under a fractional linear transformation is a circle or a line. We do this by assuming T is a fractional linear transformation and consider what conditions are forced on w if Tw is real. In other words we have

$$\frac{aw + b}{cw + d} = \frac{\overline{aw} + \overline{b}}{\overline{cw} + \overline{d}}$$
$$(aw + b)(\overline{cw} + \overline{d}) = (\overline{aw} + \overline{b})(cw + d)$$

which we can rearrange this to

$$(a\overline{c} - \overline{a}c)\left|w\right|^2 + (a\overline{d} - \overline{b}c)w - (\overline{a}d - b\overline{c})\overline{w} + b\overline{d} - \overline{b}d = 0$$

We see that the coefficient of $|w|^2$ and the constant term is purely imaginary. The sum of the central two terms is also imaginary. So we can divide everything by i to get a purely real equation. Thus if $a\overline{c} - \overline{a}c$ is non-zero we get a circle and if it is zero, we get a line.

Of course, what we want is slightly stronger. We want to be able to map the real line to *any* line or circle. It is not immediate (at least to me) that this can be done. For this we can use the previous theorem which allows us to write any fractional linear transformations as a composition of simpler maps. It is easy to map the real line to any other line with $f(z) = az + b$ by choosing a, b appropriately. Next recall from Proposition 3.4 that we can effectively write a fractional linear transformation as

$$f(z) = \gamma + \beta \cdot \frac{1}{z + \alpha}$$

(with some relation between α, β, γ). First I claim that the map $\frac{1}{z+\alpha}$ sends the real line to a circle (assuming $\alpha \neq 0$) where the radius of the circle is purely determined by the imaginary part of α.

Suppose $\alpha := a + ib$. Then

$$\frac{1}{z+a+ib} = \frac{(z+a)}{(z+a)^2+b^2} + i\frac{-b}{(z+a)^2+b^2}$$

Calling the real and imaginary components u, v respectively we see that

$$u^2 + \left(v + \frac{1}{2b}\right)^2 = \frac{b^2}{4}$$

Thus we can obtain a circle of any radius by choosing α appropriately and move the center of this circle as desired by choosing γ appropriately. □

With these two facts about cross-ratios, we get the following theorem

Theorem 3.9 *The image of a circle or line under a fractional linear transformation T is a circle or line. Moreover, given a pair of circles or lines, there exists a fractional linear transformation taking one to the other.*

Proof. Suppose we are given a circle or line. Choose 4 distinct points on it. The cross-ratio of these 4 points is real by Theorem 3.8. By Theorem 3.7, we see that the cross-ratio remains real after applying a fractional linear transformation thus the image is also a circle or line. This shows the first statement.

The second statement is easily seen as well. Choose 3 points z_2, z_3, z_4 on the first circle/line and 3 points w_2, w_3, w_4 on the second circle/line. There is a fractional linear transformation T taking z_2, z_3, z_4 to $1, 0, \infty$ respectively and a fractional linear transformation S taking w_2, w_3, w_4 to $1, 0, \infty$ as well. Then $S^{-1}T$ mapping z_2, z_3, z_4 to w_2, w_3, w_4 is the desired map (we know the image of the map is a circle or line containing w_2, w_3, w_4. But a circle or line is uniquely determined by 3 points so we know the image must be the desired circle/line). □

4 Holomorphic Functions

Suppose f is a complex-valued function on an open set $\Omega \subset \mathbb{C}$. We say f is holomorphic at a point $z \in \Omega$ if

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists. In other words, we can write

$$f(z+h) - f(z) = ch + \phi(h)h$$

where $\lim_{h\to 0} \phi(h) = 0$.

If we write $z = x + iy, f = u + iv, c = a + ib$ and $h = \xi + i\eta$ then the derivative is given by the map

$$h \mapsto ch$$
$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

Note that the determinant of the Jacobian is $|f'(x)|^2$ Writing the Jacobian in terms of partial derivatives we get some differential equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

This means that being holomorphic is the same as being differentiable as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ *and* satisfying the above differential equation (we will see that is a very strong condition that has far-reaching consequences). These equations are known as the *Cauchy-Riemann equations.* We could equivalently write them more succinctly as

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$$

Recalling from multivariable calculus, the differential of f is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Taking f to be $f(z) = z$ and $f(z) = \overline{z}$ we find that

$$\begin{aligned} dz &= dx + idy \\ d\overline{z} &= dx - idy \end{aligned}$$

which allows us to write

$$dx = \frac{dz + d\overline{z}}{2}, dy = \frac{dy + d\overline{y}}{2}$$

This motivates the following definitions:

$$\begin{aligned} \frac{\partial f}{\partial z} &:= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \\ \frac{\partial f}{\partial \overline{z}} &:= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \end{aligned}$$

as it allows us to write

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}$$

Although one can can think of these definitions as notational shorthand, we will find that much like in the real case $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ span the tangent space while dz and $d\overline{z}$ span the cotangent space. Also like the real case, we will see that these bases are dual of one another.

The above definitions also mean that the Cauchy-Riemann equations are equivalent to saying

$$\frac{\partial f}{\partial \overline{z}} = 0$$

Roughly speaking, this means that holomorphic functions are independent of conjugation or only depend on z rather than $\overline{z}$.

Related to holomorphic functions are harmonic functions.

Definition 4.1 (Harmonic Functions)**.** A function f on $\mathbb{C}$ that is complex- or real-valued is called *harmonic* if f is C^2 and

$$\frac{\partial^2 f}{\partial z \partial \overline{z}} = 0$$

This is equivalent to saying

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

if we think of f as a function on $\mathbb{R}^2$.

Remark 4.2. The operator $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called the Laplacian or the Laplace operator.

One might think that holomorphic functions are harmonic and indeed we will find this to be the case. However, this is not immediate since we do not know that if a function is holomorphic it is also C^2.

Proposition 4.3 *Given a function f on $\mathbb{C}$ we have*

$$\frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow \frac{\partial \bar{f}}{\partial z} = 0$$

Proof. It suffices to show that

$$\overline{\frac{\partial f}{\partial \bar{z}}} = \frac{\partial \bar{f}}{\partial z}$$

By definition we have that

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

Taking conjugates of both sides we get

$$\overline{\frac{\partial f}{\partial \bar{z}}} = \frac{1}{2}\left(\overline{\frac{\partial f}{\partial x}} - i\overline{\frac{\partial f}{\partial y}}\right) = \frac{1}{2}\left(\frac{\partial \bar{f}}{\partial x} - i\frac{\partial \bar{f}}{\partial y}\right) = \frac{\partial \bar{f}}{\partial z}$$

□

Lemma 4.4 *If $f(z)$ is holomorphic on a connected open set and $f'(z) \equiv 0$ then f is constant.*

Proof. We have that

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}$$

Holomorphism implies that $\frac{\partial f}{\partial \bar{z}}$ is 0 and by assumption on the derivative we have that $\frac{\partial f}{\partial z}$ is 0. Therefore $df = 0$ and f is constant. □

Proposition 4.5 *Suppose f is a holomorphic function in a connected, open set Ω. Then*

1. *If $|f(z)|$ is constant, then f is constant.*
2. *If Re$(f(z))$ is constant, then f is constant.*

Proof. 1) By assumption $|f(z)|^2 = f(z)\overline{f(z)}$ is constant. Thus we can differentiate both sides with respect to z to conclude

$$0 = \frac{\partial f}{\partial z}\overline{f(z)} + f(z)\frac{\partial \bar{f}}{\partial z}$$

Using holomorphicity of f, we know the second term on the right is always 0, so

$$\frac{\partial f}{\partial z}\overline{f(z)} = 0$$

for all z. This means for any given point either $\frac{\partial f}{\partial z}$ is 0 or $\overline{f(z)}$ is 0.

By assumption $\overline{f(z)}$ is constant so it is either 0 everywhere or it is 0 nowhere. If it is 0 everywhere, then $f = 0$ and we are done. So suppose it is non-zero everywhere. Then $\bar{f}(z)$ is also non-zero everywhere. But this means that $\frac{\partial f}{\partial z}$ is 0 everywhere. By the previous lemma, this implies f is constant.

2) Suppose Re(f) is constant. We know that

$$\text{Re}(f(z)) = \frac{f(z) + \overline{f(z)}}{2}$$

thus we can, as before, differentiate both sides with respect to z to conclude

$$0 = \frac{1}{2}\left(\frac{\partial f}{\partial z} + \frac{\partial \overline{f}}{\partial z}\right)$$

which immediately gives us

$$\frac{\partial f}{\partial z} = 0$$

□

Remark 4.6. In complex analysis, we work in connected open sets often enough to give them a special name: domain.

4.1 Mapping geometries

We now consider how the geometry of the plane is manipulated by holomorphic functions. Suppose f is holomorphic at a point z_0. The tangent mapping of f at z_0 is simply $w = cz$ where $c = f'(z_0)$. We know from previous discussion that if $c \neq 0$ then the tangent mapping preserves angles and orientation (it's a homothety). To capture this fact, we say that f is *conformal* at z_0 if $f'(z_0) \neq 0$.

Consider $w = f(z)$ in a connected, open set Ω. Assume that f is continuously differentiable as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ and that it's Jacobian is invertible at every point. Suppose f preserves angles at every point of Ω. This means that the tangent mapping of f at any point is of the form $w = cz$ or $w = c\overline{z}$ (see Proposition 3.1). This implies that $\frac{\partial f}{\partial z} = 0$ or $\frac{\partial f}{\partial \overline{z}} = 0$ (the tangent mapping only depends on z or $\overline{z}$ and not both). Note that both partial derivatives cannot be 0 at a point since this would prevent the Jacobian from being invertible. This means that the sets

$$\left\{z \in \Omega : \frac{\partial f}{\partial z}(z) = 0\right\}, \left\{z \in \Omega : \frac{\partial f}{\partial \overline{z}}(z) = 0\right\}$$

are disjoint. Moreover, since the derivatives are assumed to be continuous we know both the sets above are closed (recall that $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$ are just linear combinations of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$). Hence we have two disjoint closed sets whose union is the connected set Ω. Therefore one of the sets must be empty allowing us to conclude that either $\frac{\partial f}{\partial z}$ is 0 on Ω or $\frac{\partial f}{\partial \overline{z}}$ is. We know that if we have the latter case f is holomorphic (since we only assume f to be continuously differentiable as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$, we don't immediately know that f is holomorphic) and if we have the former case, we say that f is *anti-holomorphic*. In conclusion, we can say that a C^1 function $f : \mathbb{R}^2 \to \mathbb{R}^2$ preserves angles at every point of Ω if and only if f is either holomorphic or anti-holomorphic.

One of the delightful things about studying complex functions is that many of the theorems we have with real numbers (specifically for $\mathbb{R}^2$) can be imported quite readily. The theorem below is one such example.

Theorem 4.7 (Inverse Function Theorem) *Suppose f is holomorphic in a neighbourhood of z_0 and $f'(z_0) \neq 0$. Then there are neighbourhoods U and V or z_0 and $w_0 := f(z_0)$ respectively such that $f|U$ is a homeomorphism onto V with inverse $g : f(U) \to U$. Moreover, g is holomorphic and*

$$g'(w) = \frac{1}{f'(g(w))} = \frac{1}{f'(z)}$$

where $w = f(z)$.

Proof. We can more or less use the theorem in the context of real numbers to prove the complex case. The one wrinkle is that we need to know that if a function is holomorphic it is continuously differentiable (this is one of the assumptions of the real inverse function theorem). So let us assume this is indeed the case (something we will prove later). All that we need to do is check that the inverse g is holomorphic.

Since f is holomorphic we know that

$$f'(z) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Then

$$g'(w) = [f'(z)]^{-1} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

This means g satisfies the Cauchy-Riemann equations (see Section 4) and is therefore holomorphic. □

5 Complex Power Series

As in the real case, a power series is simply a map of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

whenever the convergence makes sense. Since we are now working with complex numbers, we allow a_n to be any element of $\mathbb{C}$. There are some strong analogues to the real case. For example, we have the following theorem.

Theorem 5.1 *Given a series* $\sum_{n=0}^{\infty} a_n z^n$*, there exists* $0 \leq R \leq \infty$ *such that*

1. *For every* $r < R$, $\sum_{n=0}^{\infty} a_n z^n$ *converges uniformly and absolutely in disk* $|z| \leq r$
2. *If* $|z| > R$*, the series diverges and in fact, the terms of the series are unbounded*
3. *The derived series* $\sum_{n=1}^{\infty} n a_n z^{n-1}$ *has the same radius of convergence*

Finally, if we define a map f *where* $f(z) := \sum_{n=0}^{\infty} a_n z^n$ *for* $|z| < R$ *then* f *and is holomorphic and* $f'(z)$ *is given by the derived series.*

Consider the following examples:

- $\sum_{n=0}^{\infty} n! z^n$, $R = 0$ since the terms are unbounded otherwise
- $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, $R = \infty$
- $\sum_{n=0}^{\infty} z^n$, $R = 1$
- $\sum_{n=0}^{\infty} \frac{1}{n^k} z^n$, $R = 1$ for any positive integer k

For all the above, we can use the ratio test to find the radius of convergence.

5.1 Exponential

We define the exponential map using its Taylor series. To be precise, we write

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

We saw above that the radius of convergence is ∞ hence this function is defined everywhere. It's also clear that

$$\frac{\partial}{\partial z} e^z = e^z$$

The usual rules for exponentiation like $e^{z+w} = e^z e^w$ also follow by using the above property of the derivative of e^z. Namely consider $g(z) = e^z e^{c-z}$. Then we see that

$$g'(z) = e^z e^{c-z} + e^z(-e^{c-z}) = 0$$

by using the product and chain rules. Therefore $g(z)$ is constant. In fact, this constant is e^c which we find by evaluating g at 0. Taking $c = z + w$ gets us the desired result.

This seemingly simple result about exponentiation has very important consequences. Let $z = x + iy$. Then

$$e^z = e^{x+iy} = e^x e^{iy}$$

By substituting iy into the series expansion of e^z we find that $e^{iy} = \cos(y) + i\sin(y)$ (we simply group the real and imaginary parts of the series together and recognise the Taylor series for sin and cos). Note this immediately implies that $\left|e^{iy}\right| = 1$ for all $y \in \mathbb{R}$. Therefore

$$\left|e^{x+iy}\right| = \left|e^x e^{iy}\right| = e^x$$

In fact consider the map $\phi(y) = e^{iy}$ on the real numbers. This is a group homomorphism whose image is the circle. Additionally its kernel is $2\pi\mathbb{Z}$. Therefore by the first isomorphism theorem we conclude that $\mathbb{R}/2\pi\mathbb{Z} \cong S^1 \subset \mathbb{C}$. In fact this is a homeomorphism. We know its a homeomorphism because its a continuous, bijective map between compact Hausdorff spaces. In fact the inverse map, $S^1 \to R^2/2\pi\mathbb{Z}$ has a special name called arg. Note that in the real-numbers arg is *not* well-defined. It is only well-defined up to multiples of 2π. We will see that this has some interesting and important consequences. But first we will extend arg to be defined for all non-zero complex numbers by simply mapping those points to the unit circle. To be precise, we define

$$\arg(z) := \arg\left(\frac{z}{|z|}\right)$$

We can therefore write

$$z = |z|\, e^{i\arg(z)}$$

5.2 Trigonometric Functions

Suppose y is a real number. Then we know $e^{iy} = \cos(y) + i\sin(y)$. Therefore $\cos(y)$ is the real part of e^{iy} which we can find by taking the sum of e^{iy} with its conjugate and dividing the result by 2. This suggests a way of extending cos to $\mathbb{C}$. In particular, we define

$$\cos(z) := \frac{e^{iz} + e^{-iz}}{2}$$

Similarly

$$\sin(z) := \frac{e^{iz} - e^{iz}}{2i}$$

The usual statements about trigonometric functions still remain true. For example $\cos^2(z)+\sin^2(z) = 1$ and $\cos' = -\sin$ and $\sin' = \cos$. In fact the usual sum formulas hold as well

$$\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$$
$$\sin(z+w) = \cos(z)\sin(w) + \sin(z)\cos(w)$$

5.3 Complex Log

We want log to be the inverse of the exponential. In other words we want to $\log(z)$ to be a solution to

$$e^w = z$$

We see that

$$\begin{aligned} z &= |z|\, e^{i \arg(z)} \\ &= e^{\log(|z|)} e^{i \arg(z)} \\ &= e^{\log(|z|) + i \arg(z)} \end{aligned}$$

Therefore we want to say that

$$\log(z) := \log|z| + i \arg(z)$$

Unfortunately since arg is not well-defined, log cannot be well-defined either. It is only unique up to multiples of $2\pi i$. With this restriction, we get some of the usual properties of log such as

$$\log(zz') = \log(z) + \log(z') \mod 2\pi i$$

We would like to however have a continuous version of log (and indeed arg) which we can use for our analyses. This leads to the idea of branches.

Definition 5.2 (Branches)**.** Let $f(z)$ be a continuous function in a connected open set Ω. We say that $f(z)$ is a *branch* of $\log(z)$ if for all $z \in \Omega$ we have

$$e^{f(z)} = z$$

Later on, we will study what conditions on Ω allow branches of Ω to exist.

Lemma 5.3 *Suppose there is a branch $f(z)$ of $\log(z)$ in a connected open set Ω. Then any other branch has form $f(z) + 2k\pi i$ for some $k \in \mathbb{Z}$. Conversely, for all $k \in \mathbb{Z}$, $f(z) + 2k\pi i$ is a branch.*

Proof. We want to show that two branches of $\log(z)$ always differ by the same multiple of $2\pi i$. So let f and g be 2 branches. Then

$$h(z) = \frac{f(z) - g(z)}{2\pi i}$$

Since h is continuous on a connected set, we know its image is connected as well. Note that the image of h is in $\mathbb{Z}$ therefore there is only one point in the image implying that h must be constant as desired. $\square$

We can likewise define a branch of $\arg(z)$ to be a continuous map h on a connected open set such that

$$z = |z|\, e^{ih(z)}$$

Note that any branch of $\arg(z)$ defines a branch of $\log(z)$ and vice versa.

When a branch of log does exist, we want to say that it has many of the same properties that the usual log does. Hence we have the following proposition.

Proposition 5.4 *If $f(z)$ is a branch of $\log(z)$ in a connected open set Ω then $f(z)$ is holomorphic and moreover*

$$f'(z) = \frac{1}{z}$$

Proof. We simply use the definition of being holomorphic and confirm the limit is $\frac{1}{z}$.

$$\begin{aligned}
\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} &= \lim_{h\to 0}\frac{f(z+h)-f(z)}{z+h-z} \\
&= \lim_{h\to 0}\frac{f(z+h)-f(z)}{e^{f(z+h)}-e^{f(z)}} \\
&= \lim_{w\to f(z)}\frac{w-f(z)}{e^{w}-e^{f(z)}} \\
&= \frac{1}{e^{f(z)}} \\
&= \frac{1}{z}
\end{aligned}$$

where for the third line we note that as $h \to 0$ we have $w := f(z+h) \to f(z)$ and fourth line we use that the derivative of the exponential is itself. $\square$

5.4 Power Series Operations

We now consider given a formal power series what are some things we might do with it. Recall that a power series itself is just a formal expression

$$f(w) = a_0 + a_1 w + a_2 w^2 + \cdots = \sum_{n=0}^{\infty} a_n w^n$$

where w, rather than being a real or complex variable, is simply a formal symbol or an indeterminate. The fact that we're dealing with *complex* power series simply means we allow the coefficients to be complex. Then we might ask given two power series $f(w) = \sum_{n=0}^{\infty} a_n w^n$ and $g(z) = \sum_{p=0}^{\infty} b_p z^p$, does $f(g(z))$ make sense? We see that

$$f(g(z)) = a_0 + a_1(b_0 + b_1 z + \dots) + a_2(b_0 + b_1 z + \dots)^2 + \dots$$

In other words the coefficient of any z^n is an infinite series. So we need to think about convergence and such to make some sense of them. But because we want to deal with power series as formal objects we also don't want to deal with all the nuances that often accompany convergence and such. However one simple restriction we can make to bypass all this is to assert $b_0 = 0$. This means the coefficient of any z^n is now a finite sum which is certainly well-defined. Therefore we assert that we can compose power series f with g if g has no constant term.

We can also define a formal derivative for power series. We write

$$f'(w) := \sum_{n=1}^{\infty} n a_n w^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} w^n$$

and define

$$f(0) := a_0$$

Theorem 5.5 (Inverse Function Theorem for Formal Power Series) *Given a formal power series*

$$f(w) = \sum_{n=0}^{\infty} a_n w^n$$

there is a power series

$$g(z) = \sum_{p=0}^{\infty} b_p z^p$$

such that $b_0 = 0$ and $f \circ g = id$ (the identity being the series $id(z) = z$) if and only if $a_0 = 0$ and $f'(0) \neq 0$. In this case g is unique and $g \circ f$ is also the identity.
If f has a positive radius of convergence, then so does g.

Remark 5.6. Note the similarity of the condition $f'(0) \neq 0$ to the usual inverse function theorem.

Proof. We use the method of undetermined coefficients. We are are solving for b_p so that

$$a_0 + a_1(b_1 z + b_2 z^2 + \dots) + a_2(b_1 z + b_2 z^2 + \dots)^2 + \dots = z$$

Now we can solve for the b_p by equating coefficients of z^n. For example, we see must have $a_0 = 0$. Moreover we see that $a_1 b_1 = 1$ which can only have a solution if $a_1 = f'(0) \neq 0$. This shows that the condition $a_0 = 0$ and $a_1 \neq 0$ are necessary to have a solution. Conversely we see that these are sufficient to solve uniquely for the coefficents of g. For example, by comparing of coefficient of z^2 we see

$$a_1 b_2 + a_2 b_1^2 = 0$$

But since $b_1 = a_1^{-1}$ we must have $b_2 = -a_1^{-1}(a_2 b_1^2)$. Similarly we can recursively solve for all the b_n. This shows uniqueness of g.

Now note $g(0) = 0$ and $g'(0) \neq 0$ we can apply the first part of the theorem again to find $f_1(w)$ such that $g \circ f = \mathrm{id}$. Then

$$f_1 = \mathrm{id} \circ f_1 = (f \circ g) \circ f_1 = f \circ (g \circ f_1) = f \circ \mathrm{id} = f$$

Note we used associativity of composition of power series. Something we have not proven but can be verified.

The argument about radius of convergence can be seen by computing estimates directly or from our usual Inverse Function Theorem (see Theorem 4.7), once we know that a holomorphic functions have Taylor series that converge and represent the function. $\square$

Now that we have some understanding about formal power series let us take a look at what happens with convergence.

Proposition 5.7 *If $f(w) = \sum_{n=0}^{\infty} a_n w^n, g(z) = \sum_{p=1}^{\infty} b_p z^p$ are convergent then so is $f \circ g$. In fact take $r > 0$ such that $\sum_{p=1}^{\infty} |b_p| r^p < R(f)$ where $R(f)$ is the radius of convergence for f. Then*

1. *$R(f \circ g) \geq r$*
2. *$|g(z)| < R(f)$ if $|z| < r$*
3. *$f(g(z)) = (f \circ g)(z)$*

Remark 5.8. For the last point above, $f(g(z))$ involves computing $f(w)$ where w is the value that $g(z)$ converges to. On other hand to compute $(f \circ g)(z)$, we first find $f \circ g$ as above and then find what value this converges to at z

Remark 5.9. We know that an r as described exists by using a simple continuity argument with $g(0) = 0$.

Proof. We see that

$$(f \circ g)(z) = \sum_{n=0}^{\infty} a_n \left(\sum_{p=0}^{\infty} b_p z^p \right)^n = \sum_{k=0}^{\infty} c_k z^k$$

We also note that

$$\left| \sum_{n=0}^{\infty} a_n \left(\sum_{p=0}^{\infty} b_p z^p \right)^n \right| \leq \sum_{n=0}^{\infty} |a_n| \left(\sum_{p=0}^{\infty} |b_p| \, |z|^p \right)^n = \sum_{k=0}^{\infty} \gamma_k \, |z|^k$$

Note that $|c_k| \leq \gamma_k$. In order to see this note that c_k is a polynomial in $a_0, \dots, a_k, b_1, \dots, b_k$ allowing us to write $c_k = P(a_0, \dots, a_k, b_1, \dots, b_k)$. Then $\gamma_k = P(|a_0|, \dots, |a_k|, |b_1|, \dots, |b_k|)$. By the triangle inequality we conclude $|c_k| \leq \gamma_k$. Then by using the limit comparison test (see Limit comparison test) we conclude that $(f \circ g)(z)$ converges absolutely if $|z| < r$. Therefore the radius of convergence for $(f \circ g)$ is at least r and moreover for such z we have $|g(z)| < R(f)$. The final statement is exactly an application of the fact that reordering the terms of an absolutely convergent series does not affect the sum. □

Theorem 5.10 (Reciprocal of a power series) *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $a_0 \neq 0$ then there is a unique power series $g(z)$ such that $f(z)g(z) = 1$. If f has a positive radius of convergence then so does g.*

Proof. We can assume that $a_0 = 1$ (we simply divide by it if necessary). Then we can write $f(z) = 1 - h(z)$ where $h(z)$ is a power series satisfying $h(0) = 0$. Then we know that $(1-w)^{-1} = 1 + \sum_{n=1}^{\infty} w^n$ so by substituting $w = h(z)$ we compute $g(z) := (1 - h(z))^{-1}$. The statement about the radius of convergence follows from the previous theorem. □

6 Principal Branches

We know that $\arg(z)$ is not well-defined on $\mathbb{C}$. However, we can fix this by restricting it to a subset of $\mathbb{C}$. The largest subset we can work with is $\mathbb{C} \setminus (-\infty, 0]$ on which arg has a unique value in $(-\pi, \pi)$. We call this Arg, the principal branch of arg.

In order to verify that Arg is continuous, it suffices to show that it is continuous on $S^1 \setminus \{-1\}$. We have a continuous map $z = e^{i\theta}$ that bijectively maps $(-\pi, \pi)$ to $S^1 \setminus \{-1\}$. In order to see that the inverse is continuous we can argue that the exponential map above is continuous on $[-\pi+\epsilon, \pi-\epsilon]$ for every $\epsilon > 0$. Since the domain is compact for all of these we know that the inverse maps are all going to be continuous as well. By taking smaller and smaller ϵ we eventually cover the whole space.

Since we have a principal branch for arg we can similarly have a principal branch for log which we define as $\log(|z|) + i\text{Arg}(z))$. This coincides with the real log on $(0, \infty)$.

Proposition 6.1 *For $|z| < 1$ the power series*

$$f(z) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}$$

converges and is equal to the principal branch of $\log(1 + z)$.

Proof. One can check that $f(z)$ and $g(w) = \sum_{n=1}^{\infty} \frac{w^n}{n!}$, the series expansion of $e^w - 1$, are inverses of another. Therefore $g(f(z)) = z$. This means that $e^{f(z)} = z + 1$. By definition, this means $f(z)$ is a branch of $\log(z + 1)$. We can check this is the principal branch by evaluation at f at 0. □

We can now define exponentiation using any complex number, not just rationals. Namely we say

$$z^\alpha = e^{\alpha \log z}$$

for any $\alpha \in \mathbb{C}$ and $z \neq 0$. Note that in general this is a many-valued function of z (interestingly it remains single-valued if $\alpha \in \mathbb{Z}$ this is because $\log z$ is only ill-defined up to integer multiples of 2π. If α is an integer then the power of the exponent can only vary by integer multiples of 2π but this is simply 1).

The fact that these functions, such as $z^{1/2}$, are multi-valued can be quite difficult to handle. So instead what we would like is to have a covering space X for $\mathbb{C}$ such that the multi-valued function lifts to a single-valued function.

For the function $w = z^{1/2}$ which we can equivalent write as $z = w^2$ there is a natural candidate for such a space. Consider $X := \{(z, w) \in \mathbb{C}^2 | w^2 = z\}$. This is necessary a manifold since it is the graph of a smooth function. Moreover, note the map $(z, w) \mapsto w$ on X gives exactly $z^{1/2}$.

Now we want to try and think about what X actually looks like. We know X looks roughly like 2 copies of $\mathbb{C}$ since every non-zero point z in $\mathbb{C}$ has two point w so that $w^2 = z$. In particular this means that X has two disjoint, homeomorphic copies of some neighbourhood of z for every z.

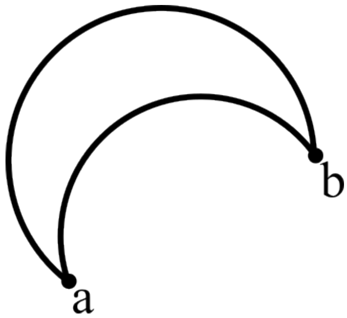
7 Conformal mappings

We know we can map the upper half-plane $\mathbb{H}^+$ to the open unit disk via the (conformal) map

$$w = \frac{z-i}{z+i}$$

We now ask whether we can use conformal maps to map various different geometrical objects to one another.

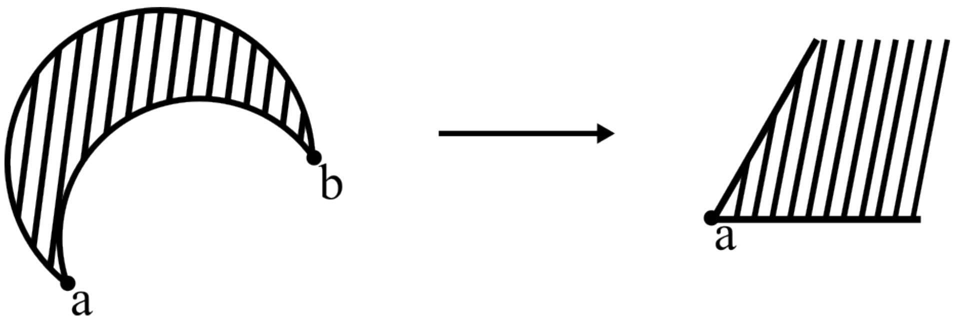
Suppose we have a circular wedge as below.



We ask whether or not we can map the interior of this shape to the upper half-plane and if so, how. First we can map a to 0 and b to infinity so that the two arcs become straight lines. We do this using

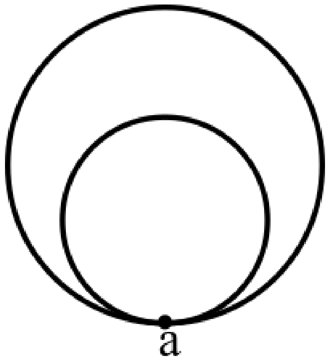
$$\zeta = \frac{z-a}{z-b}$$

This results in an infinite wedge like shown below



We can then 'fan' out the wedge by raising the result to some appropriate power (in particular if the angle is α then we rotate things so that one line lies along the positive real axis and then raise the result to the power of $\frac{\pi}{\alpha}$).

We might also have a degenerate wedge like the one below.



In this case we send a to infinity which results in 2 parallel lines. Then we recall that exp maps $\{x + iy \in \mathbb{C} : x \in \mathbb{R}, y \in [0, \pi]\}$ to the upper half-plane which gives us the desired result.

A similar exercise is to map the complement of a line-segment to the interior (or exterior) of the unit disk, conformally. A slightly easier task to consider is how we might map this to a half-plane (we know we can go back and forth between a half-plane and a disk easily). Moreover we don't even need to have a half-plane. Given a 'wedge' formed by any angle at all (like the one above) we can open or close the wedge (by raising it to some appropriate power) so that the result is a half-plane. With this in mind, we can find our conformal map quite easily.

Without loss of generality we can assume that the line segment is $[-1, 1]$ on the real line. First we do a change of coordinates

$$z_1 = \frac{z+1}{z-1}$$

so that we are instead working on $\mathbb{C} \setminus (-\infty, 0]$. The square root function allows us to fold this into a half-plane, in this case the right half-plane (where the real part of complex numbers is positive. Then we can use

$$w = \frac{z-1}{z+1}$$

to map this half-plane onto the unit disk. Overall the conformal map is $w = z - \sqrt{z^2 - 1}$.

7.1 Mapping properties of the exponential

We know that the map $w = e^z$ is periodic with period $2\pi i$. By noting that $e^{x+iy} = e^x(\cos(y) + i\sin(y))$ we see that horizontal lines are mapped to rays (approaching the origin) and vertical lines are mapped to circles (centered at the origin).

We can force e^z to be injective by restricting its domain. In particular the map is injective on domains of the form $\{z \in \mathbb{C} : a < \mathrm{Im}(z) < b\}$ with $b - a \leq 2\pi$. If $b - a = 2\pi$ then the image of this strip is $\mathbb{C} \setminus [0, \infty)$. If $b - a < 2\pi$ then the image of the strip is a sector of the plane (recall horizontal lines are mapped to rays).

We can do a similar exercise with log but first we would like to make it better behaved by considering it as a map from some appropriate covering space, i.e. Riemann surface. As before, we will take X to be the graph of the exponential. To be precise, $X := \{(z, w) \in \mathbb{C}^2 : z = e^w\}$. Then of course the log function is simply a mapping onto the second coordinate. Note in this case that small neighbourhoods in $\mathbb{C}$ have countably many copies of themselves in X, the Riemann surface for log. Thus X forms a spiral of sorts.

The log function then allows us to invert the transformations above. For example, we can transform $\mathbb{C} \setminus [0, \infty)$ to a horizontal strip between $\mathrm{Im}(z) = 2\pi i$ and $\mathrm{Im}(z) = 0$.

8 Analytic Functions

Definition 8.1 (Analytic functions)**.** A function f is analytic in an open set Ω if it has a convergent power series representation at every point in Ω. That is to say, f is analytic in Ω if for any $z_0 \in \Omega$, there is a convergent power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ so that $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ in the open disk $|z-z_0| < r$ for some r less than or equal to the radius of convergence for the power series.

One of the nice things about analytic functions is that it can sometimes be easier to do operations using the power series representation. For example, given an analytic function f, we can easily find its primitive g since if

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

then

$$g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1}$$

Moreover we know the two series have the same radius of convergence (see Theorem 5.1).

The first thing we want to verify that functions given by power series are indeed analytic (at least within their radius of convergence)

Proposition 8.2 *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has a convergent power series with radius of convergence R then $f(z)$ is analytic in the open disk $|z| < R$.*

Remark 8.3. Above we assume that the series expansion is centered at the origin. This is mostly to make the computations slightly easier and of course the theorem holds more generally for series centered at any point.

Proof. We will show that for any $|z_0| < R$, f has a convergent power series centered at z_0 and that the convergence is uniform and absolute in the closed disk $|z-z_0| \leq r$ for any $r < R - |z_0|$. We see that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n(z_0 + (z-z_0))^n \\ &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} z_0^{n-k}(z-z_0)^k \end{aligned}$$

We know that z above is in the radius of convergence for the series centered at the origin (this is a consequence of the triangle inequality since $|z| \leq |z-z_0| + |z_0| < R - |z_0| + |z_0| = R$). Convergence is absolute within the radius of convergence (again, see Theorem 5.1). In particular this means that we can rearrange the order of summation without changing its value. Doing this above, allows us to conclude that

$$f(z) = \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k} \right) (z-z_0)^k$$

□

8.1 Principle of Analytic Continuation

Theorem 8.4 *Let $f(z)$ be analytic in a domain (connected open set) Ω and $z_0 \in \Omega$. The following are equivalent:*

1. $f^{(n)}(z_0) = 0$ *for* $n = 0, 1, \ldots$
2. f *is identically 0 in a neighbourhood of* z_0
3. f *is 0 in* Ω

Proof. (3) $\Rightarrow$ (1) is trivial. (1) $\Rightarrow$ (2) follows immediately from the power series representation of f at z_0 (recall Taylor's theorem). The only direction we need to work on then is (2) $\Rightarrow$ (3).

Let $\Omega' = \{z \in \Omega : f \equiv 0 \text{ in a neighbourhood of } z \text{ in } \Omega\}$. We know that Ω' is non-empty sine z_0 is in Ω'. It is also immediate from the definition that Ω' is open. What we want to show then is that Ω' is closed. Let $z \in \overline{\Omega'}$ (closure of Ω'). Continuity of derivatives implies that $f^{(n)}(z) = 0$ for all $n \in \mathbb{N}$ (z can be expressed as the limit of points each of which has 0 derivatives). Therefore, by (1) $\Rightarrow$ (2) we know that f is identically 0 in a neighbourhood of z implying that $z \in \Omega'$. since $\Omega' = \overline{\Omega'}$ we conclude that Ω' is closed. Connected of Ω implies that $\Omega' = \Omega$. ☐

The reason that the above forms the principle of analytic continuation is due to the following corollary.

Corollary 8.5 *If f, g are analytic in a domain Ω and $f = g$ in a neighbourhood of some point, then $f = g$ in Ω.*

This means that if an analytic function can be continued, the continuation is unique. Another corollary of the theorem is that it gives the ring of analytic function on Ω a nice algebraic structure.

Corollary 8.6 *The ring of analytic function $\mathscr{A}(\Omega)$ on Ω form an integral domain.*

Proof. An integral domain is a ring where the product of two things being 0 implies that at least one of the things was 0. So suppose $fg = 0$ on Ω and $f \neq 0$. Then there is some $z_0 \in \Omega$ such that $f(z_0)$ is non-zero in a neighbourhood of z_0. Then g must be 0 on this neighbourhood and therefore all of Ω. ☐

8.2 Zeros and Poles of analytic functions

Suppose f is analytic in a neighbourhood of z_0. Then $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ provided z is close enough to z_0. Moreover, suppose $f(z_0) = 0$ but f is not identically 0. In this case $f(z) \neq 0$ for $0 < |z - z_0| < \epsilon$ for some $\epsilon > 0$. In other words the zeros of f are isolate.

Let k be the smallest integer such that $f^{(k)}(z_0) \neq 0$ (which is equivalent to saying $a_k \neq 0$). Then

$$f(z) = (z - z_0)^k g(z)$$

where g is analytic and $g(z_0) \neq 0$. Then we call k the *order* or *multiplicity* of the zero z_0 of f. Given the above expression, we can define a coordinate change to make f a very simple function, namely define $\zeta := (z - z_0) g(z)^{1/k}$. Then

$$f(z(\zeta)) = \zeta^k$$

Now we consider the quotient of analytic functions. So consider $\frac{f(z)}{g(z)}$ where g is not identically 0. If $g(z_0) \neq 0$ then $\frac{f(z)}{g(z)}$ is well-defined and analytic in a neighbourhood of z_0. Now suppose $g(z_0)$ is 0. Then we write $f(z) = (z - z_0)^k f_1(z)$ and $g(z) = (z - z_0)^l g_1(z)$ where as before we choose k and l so that $f_1(z_0)$ and $g_1(z_0)$ are non-zero. Then

$$\frac{f(z)}{g(z)} = (z - z_0)^{k-l} \frac{f_1(z)}{g_1(z)}$$

If $k \geq l$ then $\frac{f}{g}$ extends to be analytic at z_0. Otherwise, if $k < l$ then z_0 is a pole of $\frac{f}{g}$ of order/multiplicity $l - k$. Note in this case

$$\left|\frac{f(z)}{g(z)}\right| \to \infty$$

as $z \to \infty$. So $\frac{f}{g}$ makes sense as a function with values in Riemann sphere.

Definition 8.7 (Meromorphic Function)**.** A *meromorphic function* in an open set Ω is a function that is well-defined and analytic in the complement of a discrete set and expressible in a neighbourhood of *any* point in Ω as a quotient of analytic functions $\dfrac{f}{g}$ where g is not identically 0.

Although meromorphic functions are slightly less well-behaved than analytic functions we will find that they provide the benefit of forming a field.

9 Integration over curves

Let $\Omega \subset \mathbb{R}^2$ be open. A curve in Ω is a map $\gamma : [a, b] \to \Omega$ which we usually take to be C^1 (or piecewise C^1). Sometimes we might write $\gamma(t) = (x(t), y(t))$ if we are interested in using its components.

We will be integrating 1-forms over curves. A 1-form ω can be written as

$$\omega = Pdx + Qdy$$

where P, Q are continuous functions on Ω (they might be real or complex-valued). Then

$$\int_\gamma \omega = \int_a^b F(t)dt$$

where $F(t) = P(\gamma(t))x'(t) + Q(\gamma(t))y'(t)$. This follows from the usual pullback formula since

$$\gamma^*(Pdx + Qdy) = \gamma^*(P)\gamma^*(dx) + \gamma^*(Q)\gamma^*(dy)$$

We know that

$$\gamma^*(P) = P \circ \gamma$$

and

$$\gamma^*(dx) = d(\gamma^* x) = d(x \circ \gamma) = d(x(t)) = x'(t)$$

One thing we would like to know is that integration over a curve doesn't depend on how we parameterise the curve (loosely speaking integration should only depend on the image of the curve). Suppose $\delta(u) = \gamma(t(u))$ where $t : [c, d] \to [a, b]$ such that $t(c) = a$, $t(d) = b$ and $t' > 0$ in (c, d) (the derivative condition is to ensure that orientation is preserved). Then we see that

$$\delta^*\omega = (\gamma \circ t)^*\omega = t^*(\gamma^*\omega) = t^*(Fdu) = F(t(u))t'(u)du$$

Then

$$\int_\delta \omega = \int_c^d \delta^*\omega = \int_c^d F(t(u))t'(u)du = \int_a^b F(t)dt = \int_\gamma \omega$$

using the usual integration by substitution formula.

Therefore although $\displaystyle\int_\gamma \omega$ depends on ω and γ (as an oriented curve) it does not depend on the parameterisation of γ. Of course if we flip orientations, we simply pick up a negative sign.

Suppose given an interval $[a,b]$ we partition it using $t_0 < t_1 < \cdots < t_n$ where $t_0 = a$ and $t_n = b$. Then

$$\int_\gamma \omega = \sum_{i=1}^{n} \int_{\gamma_i} \omega \quad , \gamma_i = \gamma|_{[t_{i-1},t_i]}$$

Therefore $\int_\gamma$ makes sense even for piecewise C^1 curves. Keeping this statement in mind, it will be useful to have some more facts about C^1 curves.

Lemma 9.1 *Any two points in a domain $\Omega \subset \mathbb{R}^2$ can be joined by a piecewise C^1 curve.*

Proof. Fix $a \in \Omega$. Let $E := \{b \in \Omega : a, b$ can be joined by a piecewise C^1 curve $\}$. Note E is non-empty since $a \in E$. Moreover E is open since anything in an open disk can be connected to the center via a straight line. Now we show E is closed. Let b be a point in the closure. Take any neighbourhood of b. We know this intersects E by (a) definition of closure and therefore we can connect b to a point in E via a straight line (we might need to take a smaller neighbourhood to ensure that the line remains in E). Therefore $b \in E$ implying that E is closed. Since E is non-empty and clopen, $E = \Omega$ as desired. □

9.1 Primitives of forms

Given a (1-)form ω, a primitive of ω is a C^1 function F on Ω such that

$$\omega = dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

In this case

$$\int_\gamma \omega = \int_\gamma dF = \int_a^b (F \circ \gamma)'(t)dt = F(\gamma(b)) - F(\gamma(a))$$

A consequence of this fact for example is that if Ω is connected and $dF = 0$ then F is constant (as one might expect).

Given how easy it becomes to evaluate integrals using primitives, we might ask when can we find one. The proposition below gives a nice criteria.

Proposition 9.2 *A form ω has a primitive if and only if $\int_\gamma \omega = 0$ for every piecewise C^1 closed curve γ.*

Proof. If ω has a primitive then

$$\int_\gamma \omega = F(\gamma(b)) - F(\gamma(a)) = 0$$

for every closed curve γ using the fact that $\gamma(a) = \gamma(b)$.

Now we show the converse. Suppose the integral of ω over every closed curve is 0. We fix some $(x_0, y_0) \in \Omega$. Then we define

$$F(x,y) = \int_\gamma \omega$$

where γ is a piecewise C^1 path from (x_0, y_0) (we know such a path exists by the previous lemma). The fact that F is well-defined (i.e. independent of the choice of γ) follows from our assumption. Namely if δ is another path from (x_0, y_0) to (x, y) then going along γ and then back down δ forms a closed curve. Since the integral over this is 0, the integral over the 2 paths is equal. All that remains to show then is that $dF = \omega$.

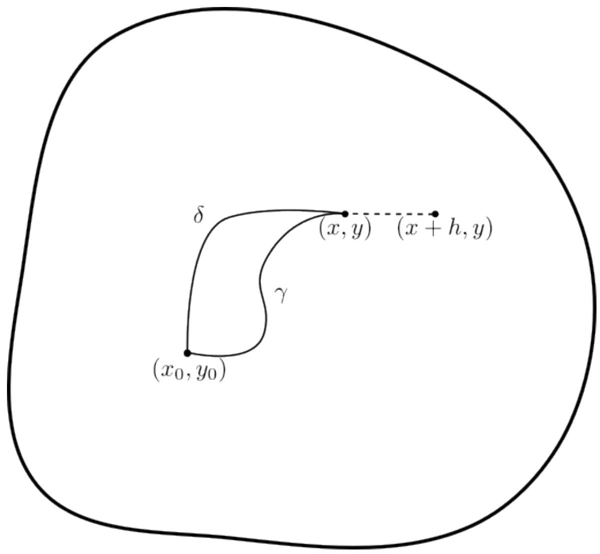


Figure 1: Define primitives by integrating along paths

Suppose $\omega = Pdx + Qdy$. We want to show that the partial derivatives of F are P and Q. In order to compute $\frac{\partial F}{\partial x}$, we need to compute the difference $F(x+h, y) - F(x, y)$. I claim that

$$F(x+h, y) - F(x, y) = \int_x^{x+h} P(t, y)dt$$

We can see this by choosing our paths cleverly. In order to compute $F(x+h, y)$ we need a path from (x_0, y_0) to $(x+h, y)$. The path we will choose will go to (x, y) first and then go to $(x+h, y)$ on a horizontal path σ. Then

$$\begin{aligned} F(x+h, y) - F(x, y) &= \int_{\gamma * \sigma} \omega - \int_\gamma \omega \\ &= \int_\sigma \omega \\ &= \int_0^1 \sigma^*(Pdx + Qdy) \\ &= \int_0^1 (P \circ \sigma) d(x \circ \sigma) + \int_0^1 (Q \circ \sigma) d(y \circ \sigma) \\ &= \int_x^{x+h} P(t, y)dt \end{aligned}$$

where in the final equality we use the fact that $d(y \circ \sigma) = 0$ since σ is constant in the y-direction and do a substitution for the first coordinate (namely we have $z = \sigma(x)$).

Then

$$\frac{\partial}{\partial x} F(x, y) = \lim_{h \to 0} \frac{F(x+h, y) - F(x, y)}{h} = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} P(t, y)dt = P(x, y)$$

where the final equality comes from the Fundamental Theorem of Calculus. We can similarly conclude $\frac{\partial}{\partial y} F(x, y) = Q$ giving us the desired result. $\square$

9.2 Cauchy's Theorem

The previous proposition tells us that ω has a primitive if and only if its integral over every (piecewise C^1) closed curve is 0. This is a somewhat difficult condition to work with. However, if we are working in a disk, we can check a much simpler condition. In a disk, a form ω has a primitive if and only if its integral over the boundary of every rectangle (one whose sides are parallel to the axes) is 0. This is simply because in a disk we can connect any two points with a path that travels parallel to the axes. Thus although it might be difficult to determine whether a form has a primitive or not, doing this locally is not too hard. This motivates the following definition.

Definition 9.3 (Closed forms)**.** A form ω is *closed* if it locally has a primitive.

Remark 9.4. We will check soon that this definition of closed agrees with the usual definition of $d\omega = 0$.

By the prior discussion, it is easy to see that a form ω is closed if and only if the integral over the boundary of sufficiently small rectangles is 0. If the form locally has a primitive, then given a point there is a neighbourhood within which the form has a primitive and hence the integral rectangle boundaries contained in this neighbourhood is 0. On the other hand, if the integral over the boundaries of sufficiently small rectangles is 0, we can define primitives on disks containing these rectangles. In fact by noting that any rectangle can subdivided into a number of arbitrarily small rectangles, we can say that a form is closed if and only the integral over the boundary of *any* rectangle is 0.

It is important to note that the existence of a local primitives does not necessarily imply the existence of a global primitive (although we know in some cases it does such as in disks). As an example, consider $\Omega = \mathbb{C} \setminus \{0\}$ and the form

$$\omega = \frac{dz}{z}$$

We know ω is closed because locally at every point of Ω there's a branch of $\log z$, which forms a local primitives for ω. However we claim that there is no global primitive. In order to verify this, it is enough to find a closed curve γ on which $\int_\gamma \omega \neq 0$. We will take $\gamma = e^{it}$ for $t \in [0, 2\pi]$. Since $z = e^{it}$ we have $dz = ie^{it}dt$. Therefore

$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{it}dt}{e^{it}} = 2\pi i$$

Such an example need not be complex. For example if we write $z = x + iy$ then

$$\frac{d(x+iy)}{x+iy} = \frac{xdx + ydy}{x^2+y^2} + i\frac{xdy - ydx}{x^2+y^2}$$

Without any calculations then, by looking at the complex result, we can conclude that

$$\int_\gamma \underbrace{\frac{xdy - ydx}{x^2+y^2}}_{\eta} = 2\pi$$

Note that this particular form η is equal to dt where $t = \arctan(\frac{y}{x})$

9.3 *Aside: Green's Theorem*

Before we move on to proving Cauchy's theorem, let us quickly verify that our definition of closed forms agrees with the usual definition.

Suppose P, Q are continuous with continuous partials $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ in a neighbourhood of a closed rectangle A. Let γ denote the boundary of A. Then by Green's Theorem we know that

$$\int_\gamma Pdx + Qdy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

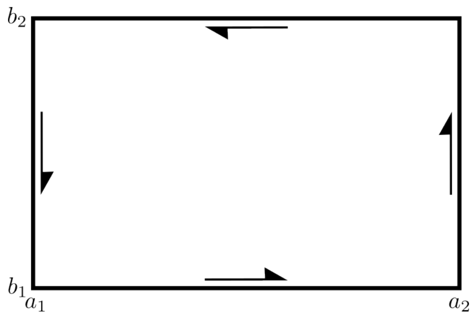


Figure 2: Integrating over the boundary of a rectangle

We can prove Green's Theorem by simple evaluation.

$$\begin{aligned}
\iint_A \frac{\partial Q}{\partial x} dx dy &= \int_{b_1}^{b_2} \left(\int_{a_1}^{a_2} \frac{\partial Q}{\partial x} dx \right) dy \\
&= \int_{b_1}^{b_2} Q(a_2, y) - Q(a_1, y) dy \\
&= \int_{a_1}^{a_2} Q(x, b_1) dx + \int_{b_1}^{b_2} Q(a_2, y) dy + \int_{a_2}^{a_1} Q(x, b_2) dx + \int_{b_2}^{b_1} Q(a_1, y) dy \\
&= \int_\gamma Q dy
\end{aligned}$$

where we use the fact that Q is constant on the horizontal sides. Doing a similar calculation with $\frac{\partial P}{\partial y}$ and summing the results, we get Green's Theorem.

Now suppose $\omega = P dx + Q dy$ is such that its integral over the boundary γ of any sufficently small rectangle A is 0. By Green's theorem this happens if and only if

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = 0$$

Thus if $d\omega = 0$ then it is obvious that the integral over the boundary of any rectangle is 0. In order to see the converse, suppose the integral over any small rectangle is 0 then the form must be 0 everywhere (if it was non-zero at some point, it would be non-zero in a neighbourhood and so the integral over a rectangle contained in this neighbourhood would be non-zero). Therefore we conclude

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = 0$$

but that is exactly saying $d\omega = 0$.

Now we can move on to proving Cauchy's Theorem.

Theorem 9.5 (Cauchy's Theorem) *If $f(z)$ is holomorphic in an open subset $\Omega \subset \mathbb{C}$ then $f(z)dz$ is closed.*

We first show a very simple proof in the case where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous. While we don't want to prove the statement with this assumption (in fact we will use Cauchy's theorem to show that the partials of holomorphic functions exist and are continuous), it is interesting to note how short and simple the proof is in this case.

Proof. We can write

$$f(z)dz = f(z)d(x+iy) = \underbrace{f(z)}_{P}\,dx + \underbrace{if(z)}_{Q}\,dy$$

By Green's formula, in order to show f is closed, it suffices to show that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Substituting the actual values of P and Q into this statement, we see we want to show that

$$\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial x}$$

But this is exactly what it means to be holomorphic. $\square$

A lovely proof that we cannot use. Instead we have the following.

Proof. In order to show f is closed we know it is sufficient to show that

$$\int_\gamma f(z)dz = 0$$

where γ is the boundary of any rectangle R in Ω. Let us call the value of the integral $\mu(R)$. Suppose we divide R into 4 equal parts R_i each with oriented boundary γ_i. Then

$$\int_\gamma f(z)dz = \sum_{i=1}^{4}\int_{\gamma_i} f(z)dz$$

Then there is at least one i such that

$$\left|\int_{\gamma_i} f(z)dz\right| \geq \frac{1}{4}\left|\int_\gamma f(z)dz\right|$$

Let us define $R_i := R^{(1)}$ and $\gamma_i := \gamma^{(1)}$. From here we can repeatedly apply this procedure to obtain a chain of rectangles $R \supset R^{(1)} \supset R^{(2)} \subset \ldots$. Then

$$\left|\mu(R^{(k)}\right| = \left|\int_{\gamma^{(k)}} f(z)dz\right| \geq \frac{1}{4^k}\left|\int_\gamma f(z)dz\right| = \frac{1}{4^k}\,|\mu(R)| \tag{9.1}$$

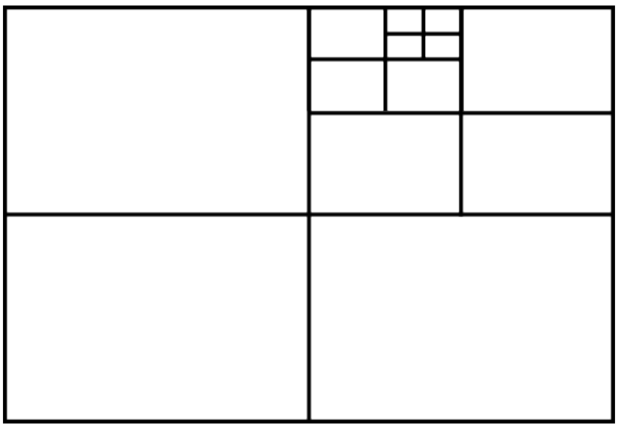


Figure 3: Divide R into a chain of rectangles

Note there exists a (unique) point z_0 in $\bigcap_{k=1}^{\infty} R^{(k)}$ (this is one of the characterisations of compact sets). Since f is holomorphic at z_0 we can write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \phi(z)\,|z - z_0|$$

where $\phi(z) \to 0$ as $z \to 0$. In particular this means that given any $\epsilon > 0$, there exists some $\delta > 0$ so that $|z - z_0| < \delta$ implies that $|\phi(z)| < \epsilon$. Then

$$\int_{\gamma^{(k)}} f(z)dz = \int_{\gamma^{(k)}} f(z_0)dz + \int_{\gamma^{(k)}} f'(z_0)(z - z_0)dz + \int_{\gamma^{(k)}} \phi(z)\,|z - z_0|\,dz$$

The first two terms are 0 since both forms have primitives ($f(z_0)z$ and $\frac{f'(z_0)}{2}(z - z_0)^2$ respectively) which are being integrated over a closed curve. What we want to show then is that the last term is also 0. Suppose we take k sufficiently large so that $|z - z_0| < \delta$ for all $z \in R^{(k)}$. Then

$$\left|\int_{\gamma^{(k)}} \phi(z)\,|z - z_0|\,dz\right| \leq \epsilon \int_{\gamma^{(k)}} |z - z_0|\,dz \leq \epsilon \ \mathrm{diam}(R^{(k)}) \ \mathrm{perm}(R^{(k)})$$

where $\mathrm{diam}(R^{(k)})$ is the maximum distance between two points in $R^{(k)}$ and $\mathrm{perm}(R^{(k)})$ is the perimeter of the rectangle. Both these quantities half as we iterate implying that

$$\left|\int_{\gamma^{(k)}} f(z)dz\right| \leq \frac{1}{4^k}\epsilon \ \mathrm{diam}(R) \ \mathrm{perm}(R)$$

Then by (9.1) we conclude

$$|\mu(R)| \leq 4^k \left|\int_{\gamma^{(k)}} f(z)dz\right| \leq \epsilon \ \mathrm{diam}(R) \ \mathrm{perm}(R)$$

Since ϵ was arbitrary, we conclude that $|\mu(R)| = 0$. □

Corollary 9.6 *Holomorphic functions $f(z)$ in open $\Omega \subset \mathbb{C}$ locally have a primitive which is holomorphic.*

Proof. By Cauchy's Theorem we know that holomorphic functions in open subsets of $\mathbb{C}$ are closed. This is equivalent to saying that they have a local primitive. Let F be such a local primitive for f. This means that

$$f(z)dz = dF = \frac{\partial F}{\partial z}dz + \frac{\partial F}{\partial \overline{z}}d\overline{z}$$

in some neighbourhood. Recall that dz and $d\overline{z}$ are linearly independent. Then since the coefficient of $d\overline{z}$ is 0 on the left hand side, it must also be 0 on the right hand 0 implying that

$$\frac{\partial F}{\partial \overline{z}} = 0$$

which is one of the (many) equivalent formulations for being holomorphic. □

Corollary 9.7 *Cauchy's Theorem remains true if $f(z)$ is continuous in Ω and holomorphic everywhere except possibly on a line. In particular then a continuous function that is holomorphic except on a finite number of points is holomorphic everywhere.*

Proof. We assume that the line on which f is not holomorphic is parallel to real axis since other cases are quite similar. We want to show that the integral over the boundary of a rectangle is still 0. If the rectangle does not intersect the line at all, we are done. The other possibilities are the line intersects the boundary at two points or the line goes through an edge.

Let us consider the latter case first. In this case, we can take another rectangle whose edge is ϵ away from the original rectangle, as shown in Figure 4. We know the integral over the boundary of the smaller rectangle is 0 for every $\epsilon > 0$ and as we let $\epsilon \to 0$, the integral converges to the integral over the boundary of the desired rectangle which therefore must be 0. of the If we are instead in a situation like Figure 5, then we can reduce it back to the prior case by integrating on the upper and lower halves separately and summing the values. □

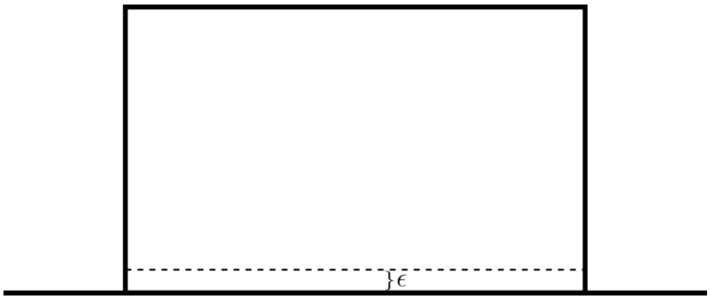


Figure 4: Rectangle with non-holomorphic edge

Figure 5: Rectangle intersecting non-holomorphic edge

Remark 9.8. It is important the function be continuous everywhere. For example, $f(z) = \frac{1}{z}$ is holomorphic everywhere except the origin so one might expect there to be a holomorphic extension of f on all of $\mathbb{C}$ based on the previous proposition. But in fact the proposition does not apply since f is not continuous at 0 (and indeed there is no holomorphic extension of f onto $\mathbb{C}$. We will be able to prove this quite easily with the tools discussed later).

Where the proof breaks down in the discontinuous case is in our assumption that as $\epsilon \to 0$, we have a convergence of the integrals. Obviously this can only hold true if we have some control over how much f can vary in small neighbourhoods.

10 Cauchy's Integral Formula

10.1 Primitives along curves

In general a closed differential form ω in an open set Ω need not have a primitive. However, we can always find what is called a *primitive along a curve*. We know that closed forms always have local primitives. So a primitive along a curve is simply a continuous function that agrees with all the local primitives along the curve. This definition is made precise in the following proposition.

Proposition 10.1 *Let $\Omega \subset \mathbb{C}$ be open and let ω be a closed form in Ω. Also suppose $\gamma : [a, b] \to \Omega$ is a continuous curve. Then there is a continuous functions $f(t)$ on $[a, b]$ such that for every $t_0 \in [a, b]$ there is a local primitive F of ω in a neighbourhood of $\gamma(t_0)$ such that*

$$f(t) = F(\gamma(t))$$

for all t in some neighbourhood of t_0. Moreover, f is uniquely determined up to the addition of a constant.

Proof. The uniqueness is easy to determine. Suppose f_1, f_2 are primitives of ω along γ. Then in a

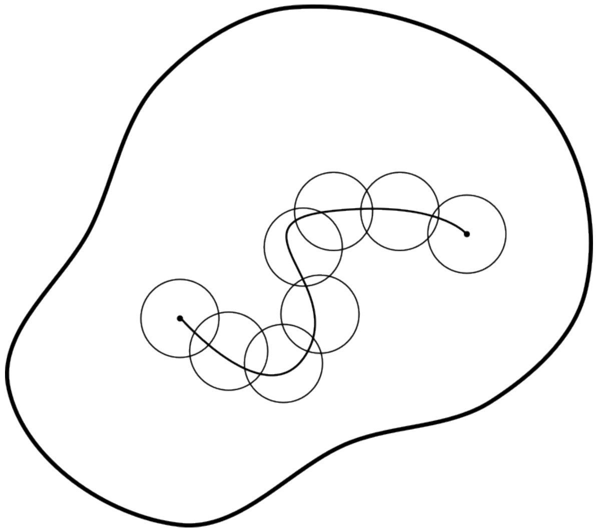


Figure 6: Local primitives are defined on each disk, just need to make them agree on the intersection

neighbourhood of t_0 we have

$$f_1(t) - f_2(t) = F_1(\gamma(t)) - F_2(\gamma(t))$$

where F_1, F_2 are two local primitives of ω. Since they are primitives we know they can only differ by a constant. But this means that $f_1(t) - f_2(t)$ is only constant on some neighbourhood of t_0. This means $f_1 - f_2$ is locally constant but since $[a, b]$ is connected we conclude that $f_1 - f_2$ is constant everywhere.

The slightly trickier thing to do is show existence of f. We would like to define f at a point to be simply be the value of the local primitive at that point. The problem is that a point might lie in the neighbourhood for two different primitives. Therefore what we will do is split the curve into a different parts where we have a primitive on each of the smaller parts and just show we can get agreement on the intersections.

So first we find a partition $a = t_0 < t_1 < \cdots < t_n = b$ of the interval so that every $\gamma([t_{i-1}, t_i])$ lies in an open disk U_i in which ω_i has a primitive F_i (the Lebesgue number lemma guarantees the existence of such a partition). Since the U_i are disks their intersection (if non-empty) is connected. This means that $F_i - F_{i-1}$ is constant on $U_i \cap U_{i-1}$ (again, primitives can only differ by a constant). Thus we simply adjust these constants one at a time for $i = 1, \ldots, n$ so that we have agreement on all the intersections. Finally, we define $f(t) = F_i(\gamma(t))$ where $t \in [t_{i-1}, t_i]$. $\square$

The primitive along a curve behaves at least somewhat like a genuine primitive. For example, we have the following.

Corollary 10.2 *Suppose $\gamma : [a, b] \to \Omega$ is a piecewise C^1 curve and f is a primitive along γ. Then*

$$\int_\gamma \omega = f(b) - f(a)$$

Proof. Using notation as in the previous proposition, define $\gamma_i := \gamma|_{[t_{i-1},t_i]}$. Then

$$\begin{aligned}
\int_\gamma \omega &= \sum_{i=1}^n \int_{\gamma_i} \omega \\
&= \sum_{i=1}^n F_i(\gamma(t_i)) - F_i(\gamma(t_{i-1})) \\
&= \sum_{i=1}^n f(t_i) - f(t_{i-1}) \\
&= f(b) - f(a)
\end{aligned}$$

□

Note that for the statement although the left-hand side requires γ to be C^1 the right-hand side makes even if γ is just continuous (we only needed continuity of γ in Proposition 10.1). This allows us to define

$$\int_\gamma \omega$$

even for continuous curves as $f(b) - f(a)$ where f is simply a primitive along γ (since primitives along curves only differ by a constant this is well-defined).

We can use this idea to get some very nice things very easily. For example suppose γ is a closed curve not containing 0. Then

$$\int_\gamma \frac{1}{z} dz = f(b) - f(a)$$

where, as usual, f is a primitive along γ. But we know that a primitive of $\frac{1}{z}dz$ is log so $f(b) - f(a)$ is the difference between two branches of log at $\gamma(a) = \gamma(b)$. Therefore we know that $f(b) - f(a) = 2\pi i n$ where n is some integer. Similarly we can conclude that

$$\int_\gamma \frac{xdy - ydx}{x^2 + y^2} = 2\pi n$$

(for the same n). One can see that it measures how the argument of z changes along γ. Thus we often call the integral on the left the "variation of arg(z) along γ".

10.2 Homotopy

Homotopy roughly talks about transforming one curve to another in some kind of continuous manner. In fact, we discuss two flavours of it depending on whether the curves share endpoints or whether they are closed curves.

Definition 10.3 (Homotopy of curves with fixed endpoints)**.** Suppose $\gamma_0, \gamma_1 : [0, 1] \to \Omega$ be continuous curves with the same endpoints (i.e. $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$). Then γ_0 and γ_1 are said to be *homotopic* (with fixed endpoints) if there is a continuous function $\gamma : [0, 1] \times [0, 1] \to \Omega$ such that

$$\begin{aligned}
\gamma(0, t) &= \gamma_0(t) \\
\gamma(1, t) &= \gamma_1(t) \\
\gamma(s, 0) &= \gamma_0(0) = \gamma_1(0) \\
\gamma(s, 1) &= \gamma_0(1) = \gamma_1(1)
\end{aligned}$$

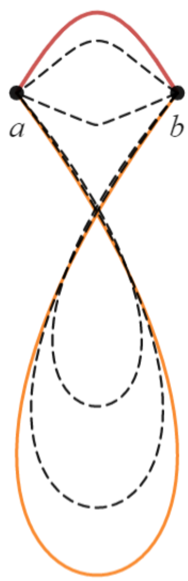


Figure 7: Homotopic curves with fixed endpoints. Dotted lines indicates the homotopy between the two curves (see desmos for interactive version).

Definition 10.4 (Homotopy of closed curves)**.** Suppose $\gamma_0, \gamma_1 : [0,1] \to \Omega$ be closed curves (i.e. $\gamma_0(0) = \gamma_0(1)$ and $\gamma_1(0) = \gamma_1(1)$ but it need not be true that $\gamma_0(0) = \gamma_1(0)$). Then γ_0 and γ_1 are said to be *homotopic* (as closed curved) if there is a continuous function $\gamma : [0,1] \times [0,1] \to \Omega$ such that for every $s, t \in [0,1]$ we have

$$\begin{aligned}\gamma(0,t) &= \gamma_0(t)\\ \gamma(1,t) &= \gamma_1(t)\\ \gamma(s,0) &= \gamma(s,1)\end{aligned}$$

We say γ_0 is homotopic to a point or *nullhomotopic* if γ_1 is a constant map.

See Figure 7 and Figure 8 for examples of the two kinds of homotopies.

The theorem we will prove is that integrals over homotopic curves are equal. Admittedly this seems a bit strange but really is just a restatement of the fact that the integral of closed forms over the boundary of rectangles is 0. The key idea is that integrals over homotopic curves can be computed by working over the boundary of $[0,1] \times [0,1]$ since we can pullback via the homotopy. Then it is just a matter of looking at the consequence of this. This is all made more precise in Theorem 10.6.

We also present a second proof of the same theorem for which we need the following lemma generalising the notion of primitives along curves.

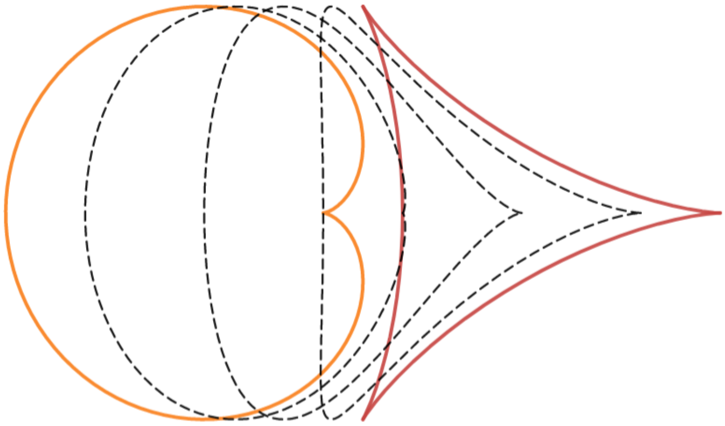


Figure 8: Homotopic closed curves. Dotted lines indicate homotopy between the curves (see desmos for interactive version).

Lemma 10.5 *Let ω be a closed form in Ω. Let $\gamma : [a,b] \times [c,d] \to \Omega$ be a continuous map. Then there is a continuous function $f : [a,b] \times [c,d] \to \mathbb{C}$ such that for every $(s_0,t_0) \in [a,b] \times [c,d]$ that there is a primitive F of ω defined on a neighbourhood of $\gamma(s_0,t_0)$ such that*

$$f(s,t) = F(\gamma(s,t))$$

for all (s,t) in a neighbourhood of (s_0,t_0). Moreover, f is unique up the addition of a constant.

Proof. The proof of uniqueness is the exact same as before (see Proposition 10.1). In particular, the difference of two different primitives along γ is the difference of two genuine primitives of ω in a neighbourhood. So the difference is locally constant but since the domain is connected it must be constant everywhere.

The proof of existence will be quite similar to the previous proposition as well. First we choose partitions $\{s_i\}$ and $\{t_j\}$ of $[a,b]$ and $[c,d]$ respectively so that $\gamma([s_{i-1},s_i] \times [t_{j-1},t_j])$ is contained in an open disk U_{ij} in which ω has a primitive F_{ij}.

Suppose we fix a j. Then two primitives $F_{i,j}$ and $F_{i+1,j}$ defined on U_{ij} and $U_{i+1,j}$ respectively differ by a constant in the intersection of the two disks. Thus we can adjust the constants to ensure that they agree on $\bigcup_i [s_{i-1},s_i] \times [t_{j-1},t_j]$. Doing this for all j, we find primitives f_j along $\gamma|_{[a,b]\times[t_{j-1},t_j]}$ by defining $f_j = F_{i,j} \circ \gamma$ in $[s_{i-1},s_i] \times [t_{j-1},t_j]$. The functions f_j and f_{j+1} may be different on the curve $[a,b] \times \{t_j\}$ but they are primitives along a curve so once again they only differ by a constant. Thus we can adjust the constants to define f on all of $[a,b] \times [c,d]$. □

Theorem 10.6 *Let ω be a closed form in an open set $\Omega \subset \mathbb{C}$. Let $\gamma_0, \gamma_1 : [0,1] \to \mathbb{C}$ be homotopic, continuous curves (they might be homotopic with fixed endpoints or homotopic as closed curves). Then*

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$$

Proof. Suppose γ_0, γ_1 are homotopic with fixed endpoints. Let $\gamma : [0,1] \times [0,1] \to \Omega$ be the homotopy between them. This means that $\gamma(0,t) = \gamma_0(t), \gamma(1,t) = \gamma_1(t)$ and $\gamma(s,0) = \gamma_0(0) = \gamma_1(0)$ and $\gamma(s,1) = \gamma_0(1) = \gamma_1(1)$.

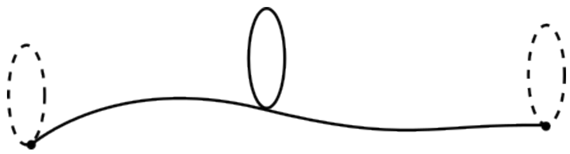


Figure 9: Homotopy given by σ until time s, followed by $\gamma(s,-)$, followed by σ until the end

First we observe that

$$\int_{\gamma(\partial I^2)} \omega = \int_{\partial I^2} \gamma^*\omega = 0$$

since $\gamma^*\omega$ is closed (one way of seeing this is to note that $d(\gamma^*\omega) = \gamma^*(d\omega) = 0$).

On other hand let us compute what this integral is without pulling back. Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ denote the bottom, right, top, and left edge of $[0,1]\times[0,1]$ all oriented as the boundary of the square. Then we see that

$$\int_{\gamma(\partial I^2)} \omega = \int_{\gamma(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} \omega = \int_{\gamma(\sigma_1)} \omega + \int_{\gamma(\sigma_2)} \omega + \int_{\gamma(\sigma_3)} \omega + \int_{\gamma(\sigma_4)} \omega$$

Since γ is a homotopy (between curves with fixed endpoints) we know $\gamma(\sigma_1)$ and $\gamma(\sigma_3)$ are constant while $\gamma(\sigma_2) = \gamma_1$ and $\gamma(\sigma_4) = -\gamma_0$. Thus we get

$$\int_{\gamma_1} \omega - \int_{\gamma_0} \omega = 0$$

For closed curves we would find that the integral over σ_1 and σ_3 cancel out leading us to the same conclusion. $\square$

Proof. Suppose γ_0, γ_1 are homotopic with fixed endpoints. Let $\gamma : [0,1]\times[0,1] \to \Omega$ be the homotopy between them. This means that $\gamma(0,t) = \gamma_0(t), \gamma(1,t) = \gamma_1(t)$ and $\gamma(s,0) = \gamma_0(0) = \gamma_1(0)$ and $\gamma(s,1) = \gamma_0(1) = \gamma_1(1)$.

Let f be a primitive of ω along γ. Note that $f(s,0)$ and $f(s,1)$ is constant as we vary s. This is because locally $f = F \circ \gamma$ where F is some local primitive of ω. But we know γ is constant as we vary the first component, given that the second component is 0 or 1. In particular this means that $f(0,0) = f(1,0)$ and $f(0,1) = f(1,1)$. Thus we find

$$\int_{\gamma_0} \omega = f(0,1) - f(0,0) = f(1,1) - f(1,0) = \int_{\gamma_1} \omega$$

where we use the fact that $f(0,t)$ and $f(1,t)$ are primitives along γ_0 and γ_1 respectively.

The result for closed curves can actually be deduced from the above case. Let γ_0 and γ_1 be two homotopic closed curves. Let γ be the homotopy between them. Let $\sigma(t)$ denote the path $\gamma(t,0)$ which connects $\gamma_0(0)$ and $\gamma_1(0)$. Then $\gamma_0 * \sigma$ is also a path from $\gamma_0(0)$ to $\gamma_1(0)$ while $\sigma * \gamma_1$ is another such path. These two paths are obviously homotopic (the homotopy is given by following σ until time s, then following $\gamma(s,-)$, and then following σ until the end, see Figure 9). Thus the integral over the two paths is the same. Both paths have the integral over σ which we can then cancel allowing us to conclude that the integral over the closed curves is equal.

$\square$

Corollary 10.7 *In a simply connected open set every closed form has a primitive.*

Proof. Recall that a simply connected set is one that is connected and where every closed curve is homotopic to a point. Furthermore, a form has a primitive if and only if the integral over every closed curve is 0 (see Proposition 9.2). Hence we will show that the integral over any closed curve is 0.

Let γ be a closed curve in the simply connected open set. We know γ is homotopic to a constant curve and by the previous theorem we know that integral over homotopic curves are equal. Moreover, the integral over a constant curve is necessarily 0 (we are effectively integrating over a point, or to put it more precisely the 'path' is constant so when we compute the pullback it becomes 0). Therefore the integral over γ is 0 as desired. □

Some examples of simply connected open sets are disks and rectangles. A slightly less familiar example (which in fact includes the previous two) is a star-shaped open set. A star shaped set is a set S which contains a point a so that for any point x in S the line connecting a and x is also contained in S. In this case it is clear that any closed curve γ_0 in S is homotopic to the point a. In fact, we can even explicitly give the homotopy $\gamma(s,t) = (1-s)a + s\gamma_0(t)$.

This also means we can define a branch of $\log z$ in any simply connected set not containing 0 by

$$\log z = w_0 + \int_{z_0}^{z} \frac{dz}{z}$$

where z_0 is some fixed point in the set and w_0 is such that $e^{w_0} = z_0$ (to be fair, the above is improper notation since the bounds of the integral may be complex. What we mean of course by the integral is to choose a path from z_0 to z and to integrate over that. We know the choice of path is irrelevant which justifies the notation).

We can also use this to conclude that $\mathbb{C} \setminus \{0\}$ is not simply connected since

$$\int_{S^1} \frac{dz}{z} = 2\pi i \neq 0$$

10.3 Cauchy's Integral Formula

Cauchy's integral formula allows us to compute the value of a holomorphic function at a point by integrating along the boundary of a region containing that point. This relationship between evaluating the function and integrating it is exactly what we need in order to show that holomorphic functions are analytic.

Definition 10.8 (Winding Number)**.** Let γ be a closed curve in an open set Ω and let $a \in \Omega$ be a point lying outside the curve. Then the winding number of γ with respect to a is given by

$$w(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z-a} dz$$

Note that $w(\gamma, a)$ is always an integer.

As the name suggests, the winding number counts the number of times that the curve winds around the point a. Thus for example $w(e^{2\pi i t}, 0) = 1$ while $w(e^{4\pi i t}, 0) = 2$.The winding number has some very nice properties.

1. Fix some a. Then $w(\gamma, a)$ is invariant under homotopies of γ that do not pass through a. This follows from Theorem 10.6.

2. In particular, if γ lies in a simply connected open set not containing a then $w(\gamma, a) = 0$. This follows from the same theorem since we can create a homotopy to a point.

3. Fix the curve γ. Then $w(\gamma, a)$ is constant on connected components of the complement of γ. In order to prove this, it suffices to show that $w(\gamma, -)$ is locally constant. Note that shifting a so that it remains in the same connected component is the same as shifting γ (via a homotopy not passing through a). Thus by the first point, we know $w(\gamma, -)$ is locally constant.

4. A specific but useful case of the previous point is the following, γ is a circle described in the positive sense (i.e. $w(\gamma, center) = 1$) then $w(\gamma, a) = 1$ if a is inside the circle and 0 otherwise.

With the winding number in hand, we can compute certain integrals very easily.

Theorem 10.9 (Cauchy's Integral Formula) *Let Ω be an open subset of $\mathbb{C}$ and let f be a holomorphic function on Ω. Let γ be a nullhomotopic closed curve in Ω and let a be a point in Ω that is not on γ. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz = w(\gamma, a)\cdot f(a)$$

Proof. We define

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & \text{if } z \neq a \\ f'(a) & \text{if } z = a \end{cases}$$

Note that g is continuous on Ω and holomorphic on $\Omega \setminus \{a\}$. Then we know that $g(z)dz$ is closed by Cauchy's Theorem (see Corollary 9.7). Therefore

$$0 = \int_\gamma g(z)dz = \int_\gamma \frac{f(z)-f(a)}{z-a}dz$$

where for the first equality we use the fact that γ is nullhomotopic. Thus we conclude that

$$\int_\gamma \frac{f(z)}{z-a}dz = \int_\gamma \frac{f(a)}{z-a}dz = 2\pi i f(a) w(\gamma, a)$$

□

Corollary 10.10 *If $f(z)$ is holomorphic in the neighbourhood of a closed disk D and γ is the boundary of D (in the positive sense) then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz = \begin{cases} f(a) & \textit{if } a \textit{ is inside the circle} \\ 0 & \textit{if } a \textit{ is outside the circle} \end{cases}$$

Proof. This follows from the previous lemma by noting that $w(\gamma, a)$ is 1 if a is inside the circle and 0 if it's outside. □

A consequence of Cauchy's Integral Formula is that holomorphic functions are in fact infinitely differentiable. For example suppose f is holomorphic in an open disk D. Let γ be the boundary of some circle just slightly smaller than D. Then for any z inside this smaller circle, we know by the corollary above that

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta$$

But this means that

$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}d\zeta$$

and more generally

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta$$

Let us summarise everything we have learned so far.

Proposition 10.11 *Suppose f is a continuous function in the open set Ω. Then the following are equivalent:*

1. *f is holomorphic in Ω*
2. *$f(z)dz$ is closed*
3. *Given a closed disk D in Ω and γ its oriented boundary, we have*

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta$$

for every z in the interior of D.

Proof. We know that (1) $\Rightarrow$ (2) is simply Cauchy's theorem (see Theorem 9.5) and (1) $\Rightarrow$ (3) is exactly Cauchy's Integral formula. We just checked (3) $\Rightarrow$ (1) above since you can swap the integration and differentiation which is what gave us all the derivatives of f. Thus we only need to show (2) $\Rightarrow$ (1) to finish the proof. That particular statement is sometimes called Morera's Theorem.

Since $f(z)dz$ is closed, we know it locally has a primitive, $g(z)$. Moreover, we know that g is holomorphic. This means that $g'(z) = f(z)$ must also be holomorphic since the derivative of a holomorphic function is itself holomorphic. □

Corollary 10.12 *A continuous function which is holomorphic except on a line is holomorphic everywhere.*

Proof. Suppose $f(z)$ is a continuous function that is holomorphic everywhere except maybe on a line. Then we know by Cauchy's Theorem that $f(z)dz$ is closed but this means that $f(z)$ is holomorphic by the above. □

10.4 Applications of Cauchy's Formula

A very important fact that we can now show quite easily is that holomorphic functions always have a convergent power series expansion (at least locally), which is to say that being holomorphic and analytic are equivalent conditions in the complex setting.

Theorem 10.13 *Suppose $f(z)$ is a holomorphic function in the disk $|z| < R$. Then f has a convergent power series expansion in this disk.*

Proof. Let z be a point inside the disk and let r be such that $|z| < r < R$. Suppose ζ is such that $|\zeta| = r$. Then

$$\frac{1}{\zeta - z} = \frac{1}{\zeta}\left(1 - \frac{z}{\zeta}\right)^{-1} = \frac{1}{\zeta}\left(1 + \frac{z}{\zeta} + \frac{z^2}{\zeta^2} + \cdots\right)$$

Then we can use this in Cauchy's Integral Formula to obtain the power series expansion for f.

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta \\ &= \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty} \frac{z^n f(z)}{\zeta^{n+1}}d\zeta \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta^{n+1}}d\zeta\right) z^n \end{aligned}$$

We can swap the summation and integration because the convergence of the series is uniform (being a geometric series).

We know that the coeffecients a_n in a Taylor expansion (centered at 0) are simply given by $\frac{f^{(n)}(0)}{n!}$. We have computed $f^{(n)}(z)$ previously and using this we would find the coefficients should be

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta^{n+1}}d\zeta$$

which is exactly what we found above. □

By using polar coordinates we find that

$$f(re^{i\theta}) = \sum_{m=0}^{\infty} a_m r^m e^{im\theta}$$

Then

$$e^{-in\theta}f(re^{i\theta}) = \sum_{m=0}^{\infty} a_m r^m e^{(m-n)i\theta}$$

We can then integrate both sides from $\theta = 0$ to $\theta = 2\pi$. Almost all the terms on the right evaluate to 0 under this integral except when $m = n$. Therefore we find that

$$a_n r^n = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})e^{-in\theta}d\theta$$

This also allows us to bound the size of the coefficients. Let $M(r) := \sup_{\theta\in[0,2\pi]}\left|f(re^{i\theta})\right|$. Then we can immediately conclude that

$$|a_n| \le \frac{M(r)}{r^n}$$

These are known as Cauchy's inequalities. In fact we can use this to prove Liouville's Theorem which gives us the Fundamental Theorem of Algebra as a corollary.

Theorem 10.14 (Liouville's Theorem) *A bounded holomorphic function in $\mathbb{C}$ is constant.*

Proof. There is some M so that $M(r) \le M$ for all r. Therefore

$$|a_n| \le \frac{M}{r^n}$$

for any $r > 0$. As we let $r \to \infty$ we see that $|a_n| \to 0$. Therefore for all $n \ge 1$ we conclude that $a_n = 0$. Thus we get $f(z) = a_0$. □

Corollary 10.15 (Fundamental Theorem of Algebra) *Every non-constant polynomial has a root.*

Proof. Suppose a polynomial $P(z)$ has no root. Then $\frac{1}{P(z)}$ is holomorphic in $\mathbb{C}$ and is bounded. Therefore by Liouville's theorem, it must be constant. □

Proposition 10.16 (Schwartz's Reflection Principle) *Suppose $\Omega \subset \mathbb{C}$ is open and symmetric with respect to the real axis. Define $\Omega^+ := \{z \in \Omega : Im(z) \ge 0\}$ and similarly $\Omega^- := \{z \in \Omega : Im(z) < 0\}$. Let f be a continuous function on Ω^+ that is real on $\Omega \cap \mathbb{R}$ and holomorphic on $\Omega \cap \{Im(z) > 0\}$. Then we can extend f to a holomorphic function on Ω. The extension is unique by the principle of analytic continuation.*

Proof. We define the extension by reflection. In particular, we define

$$g(z) = \begin{cases} f(z), z \in \Omega^+ \\ \overline{f(\overline{z})}, z \in \Omega^- \end{cases}$$

We see that g is holomorphic everywhere except possibly $\Omega \cap \mathbb{R}$. But we know a continuous function that is holomorphic everywhere except maybe on a line is in fact holomorphic everywhere. $\square$

There are many generalisations of Schwartz's reflection principle. For example, the domain can be symmetric with respect to any line, not necessarily the reals and because of the strong correspondence between lines and circles in the complex plane, any use of the word 'line' in the previous statements can be replaced with 'circle'.

Cauchy's Integral Formula also tells us that holomorphic functions have the mean value property.

Definition 10.17 (Mean Value Property)**.** If a function f has the mean value property, then mean value of f along the boundary of any disk is equal to the value of f at the center. In other words

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z + re^{i\theta})d\theta$$

for sufficiently small r.

An important consequences of the mean value property is the maximum modulus principle.

Lemma 10.18 (Maximum Modulus Principle) *Let f be a continuous function in $\Omega \subset \mathbb{C}$ which has the mean value property. Then if $|f|$ has a local max at some point a in Ω, then f is constant in a neighbourhood of a.*

Proof.

$$M(r) := \sup_{\theta \in [0,2\pi]} \left|f(a + re^{i\theta})\right| \leq f(a) \leq M(r)$$

The first inequality follows from the local max being at a. The second inequality follows from the mean value property. Namely, $f(a)$, being the mean value of the boundary of the disk, is at most as large as the largest value on the boundary. But this means that $f(a) = M(r)$.

Then we define $g(z) = \text{Re}(f(a) - f(z)) = f(a) - \text{Re}(f(z))$ since $f(a)$ is real. There are two important things to note about g. First it is non-negative on the circle $|z - a| = r$. This follows from the fact $f(a) = M(r)$ and the fact that $\text{Re}(f(z)) \leq |f(z)|$. Thus we have $0 \leq f(a) - |f(z)| \leq f(a) - \text{Re}(f(z))$.

The second important thing about g is $g(z) = 0$ if and only if $f(z) = f(a)$. It is clear that if $f(z) = f(a)$ then $g(z) = 0$ since $f(a)$ is real. Conversely suppose $g(z) = 0$. This means that $f(a) = \text{Re}(f(z))$. Since $f(a) = M(r)$, this means there is some z' on the circle $|z' - a| = r$ such that $\text{Re}(f(z)) = |f(z')|$. This immediately means that $|f(z)| \geq |f(z')|$ (again the modulus is at least as big as the real part). On the other hand by definition, z' is where $|f|$ reaches its largest modulus on the circle $|z - a| = r$. Therefore we also have $|f(z)| \leq |f(z')|$ which gives us

$$|f(z)| = |f(z')| = f(a)$$

Since $\text{Re}(f(z)) = f(a)$ and $|f(z)| = f(a)$ we necessarily have $f(z) = f(a)$ (in particular the imaginary part of $f(z)$ must be 0).

Finally, we observe that $g(a) = 0$. This means that the mean value on the circle $|z - a| = r$ is 0 (this uses the fact that g also has the mean value property. This is because if f satisfies the mean value property then so do $\text{Re}(f)$ and $\text{Im}(f)$ and $f + c$ for any constant c). But since g is continuous and non-negative on the circle, the mean value can only be 0 if it is identically 0 on the circle. By the previous paragraph, this means that $f(z) = f(a)$ on $|z - a| = r$. Since this holds for all sufficiently small r, we find that f is locally constant on a small disk. $\square$

The following is a useful corollary of the maximum modulus principle and is in fact how we use the principle most often.

Corollary 10.19 *Suppose Ω is a bounded domain in $\mathbb{C}$. Let $f(z)$ be a continuous function on $\overline{\Omega}$ the closure of Ω. Suppose also that f has the mean value property in Ω. Define*

$$M := \sup_{z \in \mathrm{Bd}(\Omega)} |f(z)|$$

Then $|f(z)| \leq M$ for all $z \in \overline{\Omega}$ and if $|f(z_0)| = M$ for some $z_0 \in \Omega$ then f is constant.

Proof. Let $M' := \sup_{z \in \overline{\Omega}} |f(z)|$. Since $\overline{\Omega}$ is compact, we know that the supremum is actually achieved. In other words, there is some $a \in \overline{\Omega}$ such that $|f(a)| = M'$. If a is on the boundary $\mathrm{Bd}(\Omega)$, then we are done. So suppose a is in Ω. Then by the maximum modulus principle, we get that $S := \{z \in \mathbb{C} : f(z) = f(a)\}$ is an open set. But this is also a closed set. Therefore $S = \Omega$. $\square$

A very important result in complex analysis is Schwarz's lemma. Roughly speaking, it tells us that holomorphic functions from the disk to itself can't change distances too much. The lemma even allows us to classify all biholomorphisms from the disk to itself.

Theorem 10.20 (Schwarz's lemma) *Suppose $f(z)$ is a holomorphic function from the unit disk to itself such that $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $|z| < 1$ and $|f'(0)| \leq 1$. Moreover, if $|f(z_0)| = |z_0|$ for some non-zero z_0 or $|f'(0)| = 1$ then $f(z) = \lambda z$ for some λ on the unit circle.*

Proof. Since $f(0) = 0$, we conclude that $\frac{f(z)}{z}$ is holomorphic on the unit disk. This follows from the fact that $\frac{f(z)}{z}$ has a power series expansion. Namely if $f(z) = \sum_{n=1}^{\infty} a_n z^n$ (note the constant term is necessarily 0 since $f(0) = 0$) then $\frac{f(z)}{z} = \sum_{n=0}^{\infty} a_{n+1} z^n$). Thus this fuction evaluates to $f'(0)$ at 0.

This means that if $|z| = r < 1$ then

$$\left| \frac{f(z)}{z} \right| = \frac{|f(z)|}{r} < \frac{1}{r}$$

By the corollary above, we conclude that this inequality holds for every $|z| \leq r$ (not just on the circle). Taking the limit of both sides as $|z| = r \to 1$, we conclude that

$$\left| \frac{f(z)}{z} \right| \leq 1$$

Moreover if this function achieves the maximum modulus, i.e. if $|f(z_0)| = |z_0|$ or $|f'(0)| = 1$, it does so in the interior of the unit disk and therefore by the maximum modulus principle we know it is constant. Therefore $|f(z)| = |z|$ implying that $f(z) = \lambda z$ with $|\lambda| = 1$. $\square$

10.5 Laurent Expansions

Suppose f is any function on an open subset of the complex plane. Then we say f has a Laurent expansion if there exist a_n for $n \in \mathbb{Z}$ so that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

One might wonder how to interpret this sum and what it means for such a sum to be convergent. One natural way to think about this sum is to consider the positive and negative indices separately. In other words, we will say that

$$\sum_{n=-\infty}^{\infty} a_n z^n = \sum_{n\geq 0} a_n z^n + \sum_{n<0} a_n z^n$$

Remark 10.21. Note the similarity to $\int_{-\infty}^{\infty}$.

The convergence of the second term above can be understood by considering whether it converges in ζ for $z = \frac{1}{\zeta}$ (this gets us back to a series with positive powers).

Proposition 10.22 *Suppose f is a holomorphic function on an annulus $A := \{0 \leq R_2 < |z| < R_1 \leq \infty\}$. Then f has a convergent Laurent expansion in the annulus.*

Proof. Suppose z is any point in A. Choose r_1, r_2 such that $R_2 < r_2 < |z| < r_1 < R_1$. Let K denote the closed annulus corresponding to this choice of r_1, r_2. In other words, $K := \{z \in \mathbb{C} : r_2 \leq |z| \leq r_1\}$. Let D be a small closed disk centered at z that is contained in K. Then $\frac{f(\zeta)}{\zeta - z}$ is holomorphic on $K \setminus D$. Therefore by Cauchy's Theorem we get

$$\int_{\partial(K\setminus D)} \frac{f(\zeta)}{\zeta - z} d\zeta = 0$$

($\frac{f(\zeta)}{\zeta - z} d\zeta$ is closed so by Stokes' Theorem we get that the above integral is 0). In particular, then we conclude that

$$\int_{\partial K} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta = f(z) 2\pi i$$

where the final equality follows from Cauchy's Integral Formula. Thus overall we conclude that

$$f(z) = \frac{1}{2\pi i} \int_{\partial K} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{|\zeta|=r_1} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{|\zeta|=r_2} \frac{f(\zeta)}{\zeta - z} d\zeta$$

Let γ_1 denote the circle of radius r_1 and γ_2 the circle of radius r_2.

We know the first integral can be written as an infinite sum since

$$\begin{aligned}
\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta &= \int_{\gamma_1} \frac{f(\zeta)}{\zeta} \cdot \frac{1}{1 - \frac{z}{\zeta}} d\zeta \\
&= \int_{\gamma_1} \frac{f(\zeta)}{\zeta} \sum_{n=0}^{\infty} \left(\frac{z}{\zeta}\right)^n d\zeta \\
&= \sum_{n=0}^{\infty} \left(\int_{\gamma_1} \frac{f(\zeta)}{\zeta^{n+1}} \right) z^n
\end{aligned}$$

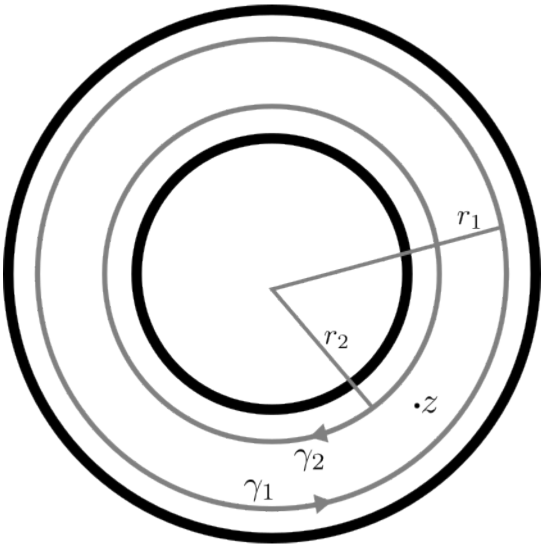


Figure 10: Laurent expansion in an annulus

For the second integral, we do something similar

$$\begin{aligned}
-\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z} d\zeta &= \int_{\gamma_2} -\frac{f(\zeta)}{z} \cdot \frac{1}{1 - \frac{\zeta}{z}} d\zeta \\
&= \int_{\gamma_2} \frac{f(\zeta)}{z} \sum_{n=0}^{\infty} \left(\frac{\zeta}{z}\right)^n d\zeta \\
&= \int_{\gamma_2} f(\zeta) \sum_{n=0}^{\infty} \frac{\zeta^n}{z^{n+1}} d\zeta \\
&= \int_{\gamma_2} f(\zeta) \sum_{n=1}^{\infty} \frac{\zeta^{n-1}}{z^n} d\zeta \\
&= \int_{\gamma_2} f(\zeta) \sum_{n<0} \frac{\zeta^{-n-1}}{z^{-n}} d\zeta \\
&= \int_{\gamma_2} f(\zeta) \sum_{n<0} \frac{z^n}{\zeta^{n+1}} d\zeta \\
&= \sum_{n<0} \left(\int_{\gamma_2} \frac{f(\zeta)}{\zeta^{n+1}} \right) z^n
\end{aligned}$$

We see that the coefficients are (almost) the exact same. The only difference is the curve we integrate over. Thus we can write the Laurent expansion quite succinctly as

$$f(z) \sum_{n=-\infty}^{\infty} a_n z^n$$

where

$$a_n = \frac{1}{2\pi i} \int_{\gamma_i} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta$$

so that $i = 1$ if $n \geq 0$ and $i = 2$ if $n < 0$. This series converges uniformly and absolutely for $r_2 \leq |z| \leq r_1$ where $R_2 < r_2 < r_1 < R_1$. $\square$

Above we looked at a general annulus but a particularly interesting and important case is that of the punctured neighbourhood, i.e. when $R_2 = 0$. For example, we might have a holomorphic function $f(z)$ in a punctured disk centered at 0. Then we say that $f(z)$ has an isolated singularity at $z = 0$ if it can't be extended to a holomorphic function on the entire disk. The statement below tells us exactly when this is possible.

Proposition 10.23 *Suppose f is a holomorphic function on the punctured disk $0 < |z| < R$. Then f extends to a holomorphic function on the entire disk $|z| < R$ if and only if f is bounded in some neighbourhood of 0.*

Proof. First it is clear that if f extends holomorphically to the entire disk then it is bounded in a neighbourhood of 0 by simple continuity. The more interesting problem is to show the converse.

By the above, we know that f has a Laurent expansion say

$$f(z) = \sum_{n=-\infty}^{n=\infty} a_n z^n$$

We will show that all the negative index coefficients must be 0. Thus we can use this analytic function to extend f to be defined at 0.

In order to find the coeffecients, we will use the standard trick of multiplying by $e^{-in\theta}$ and integrating.

$$f(re^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n r^n e^{in\theta}$$

$$\int_0^{2\pi} f(re^{i\theta})e^{-in\theta} d\theta = a_n r^n$$

Thus once again we get Cauchy's inequalities

$$|a_n| \leq \frac{M(r)}{r^n}$$

where $M(r) := \sup_{|z|=r} f(z)$. If f is bounded, there is some M so that $M(r) \leq M$ for all M. Therefore $|a_n| \leq Mr^{-n}$. Note that the right hand side goes to 0 as $r \to 0$ for negative n. But this means that $|a_n| = 0$ for negative n as desired. $\square$

11 Poles and Essential singularities

Of course we can't always extend a holomorphic function on a punctured neighbourhood to the entire neighbourhood (and indeed the previous statement tells us we cannot have such an extension exactly when f is unbounded). In this case we have essentially two different kinds of behaviour, which are characterised by the Laurent expansion. If the Laurent expansion of f has finitely many coefficients with negative indices then, we say f has a *pole*. Otherwise, f is said to have an *essential singularity.*

One can guess that poles should be better behaved that essential singularities and indeed this is true. For example f has a pole (at 0 say), then $z^n f(z)$ is holomorphic for some n. This means that you can express f as the quotient of holomorphic functions implying that f is meromorphic. In such cases it makes sense to say $f(0) = \infty$ since $\lim_{z\to 0} f(z) = \infty$ as points on the Riemann sphere S^2. Equivalently, we can say that a meromorphic function into $\mathbb{C}$ is (or can be viewed as) a holomorphic function into the Riemann sphere S^2. The same cannot be said if f had an essential singularity instead. The theorem below tells us that small neighbourhoods around 0, don't map to 'small neighbourhood around ∞' (small neighbourhood around ∞ are complements of large neighbourhoods of 0, one way to see why this is the case is by considering the images of such sets under stereographic projection), $\lim_{z\to 0} f(z)$ does not exist.

Theorem 11.1 (Weirstrass' Theorem) *If 0 is an essential singularity of f then for any $\epsilon > 0$, $f(0 < |z| < \epsilon)$ is dense in $\mathbb{C}$.*

Proof. Suppose $f(0 < |z| < \epsilon)$ is not dense in $\mathbb{C}$. Then we will show that f cannot have essential singularities.

If the image of the punctured disk is not dense in $\mathbb{C}$ then there is some $a \in \mathbb{C}$ such that $|f(z) - a| > \delta$ for some $\delta > 0$ and all $0 < |z| < \epsilon$. Then consider the function

$$g(z) = \frac{1}{f(z) - a}$$

which is holomorphic on $0 < |z| < \epsilon$ and is bounded by $\frac{1}{\delta}$. Therefore by Proposition 10.23 we know that g is holomorphic on the entire disk $|z| < \epsilon$. But then this means that

$$f(z) = \frac{1}{g(z)} + a$$

can be written as the quotient (and sum) of holomorphic functions and is therefore meromorphic. But we have seen above that meromorphic function only have poles (since they are holomorphic functions on S^2) and no essential singularities. □

In fact there is a considerably stronger theorem from Picard which says that for any $\epsilon > 0$, $f(0 < |z| < \epsilon)$ is all of $\mathbb{C}$ except maybe a single point. This is known as Picard's Big Theorem.

We can also talk about having poles/essential singularities at infinity. You use the usual 'trick' by changing coordinates $z' = \frac{1}{z}$. For example, suppose $f(z)$ is holomorphic on $|z| > R$. Then we say that f has a pole/essential singularity at ∞ if $f(1/z)$ has a pole/essential singularity (respectively) at 0 in the disk $|z| < R$. We can also detect this from the Laurent expansion. Namely if $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ is holomorphic in $|z| > R$ then f has a pole at ∞ if there are only finitely many coefficients for the positive indices. Otherwise, you have an essential singularity at infinity.

12 Residues

Given a holomorphic function we can evaluate its integral over a closed curve quite easily by using its Laurent expansion. In particular, suppose that as usual

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

Then

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \frac{1}{2\pi i}\int_\gamma \sum_{n=-\infty}^{\infty} a_n z^n dz = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi i}\int_\gamma a_n z^n dz = \frac{1}{2\pi i}\int_\gamma \frac{a_{-1}}{z} dz = a_{-1} w(\gamma, 0)$$

In particular, if we choose γ to be a circle oriented in the positive sense then

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = a_{-1}$$

The left hand side is known as the *residue* of the differential form $f(z)dz$ at 0, although quite often one simply says 'the residue of f at 0'. As we can see, residues are very easy to calculate using the Laurent expansion.

The residue at ∞ can be defined in the exact same manner, the only change being that γ needs to be a small circle oriented positively with respect to infinity. Changing coordinates to $z' = \frac{1}{z}$, we see that a small circle around ∞ in the z-plane is a large circle around 0 in the z'-plane. Moreover, the orientation is reversed. Putting everything together we get that

$$res(f,\infty) = \frac{1}{2\pi i}\int_\gamma f(z)dz = -\frac{1}{2\pi i}\int_{\gamma'} \frac{1}{z'^2} f\left(\frac{1}{z'}\right) dz'$$

By considering the Laurent expansion of $\frac{1}{z'^2} f\left(\frac{1}{z'}\right)$ we find that if $res(f,0) = a_{-1}$ then $res(f,\infty) = -a_{-1}$.

12.1 Residue Theorem

Theorem 12.1 (Residue Theorem) *Let Ω be an open subset of the Riemann sphere S^2 (in particular Ω might contain the point at ∞) and suppose $f(z)$ is a function on Ω that is holomorphic everywhere except maybe at isolated points. Let K be a compact set in Ω with piecewise C^1 oriented boundary Γ in Ω such that Γ contains no singularities or the point at infinity. Then*

$$\frac{1}{2\pi i}\int_\Gamma f(z)dz = \sum_{z_k \in S} res(f, z_k)$$

where S is the set of singularities of f that lie in K.

Remark 12.2. Note that compactness of K implies that there can only ever be finitely many singularities within it since otherwise you could form a cover of K which has not finite subcover.

Proof. We will consider 2 cases. The first case is when ∞ is not in K. For each $z_k \in S$ consider a closed disk D_k (contained in the interior of K of course) and let γ_k be its boundary. Consider $K' = K \setminus \bigcup_k \operatorname{Int}(D_k)$. Then f is holomorphic in a neighbourhood of K'. Then by Green's theorem we have

$$\int_{\partial K'} f(z)dz = \int_{K'} d(f(z)dz) = 0$$

where we use the fact that $f(z)dz$ is closed (Cauchy's Theorem) hence its differential is 0. But $\partial K' = \Gamma - \sum \gamma_k$. Therefore substituting this back in, we conclude that

$$\int_\Gamma f(z)dz = \sum_k \int_{\gamma_k} f(z)dz = 2\pi i \sum_{z_k \in S} res(f, z_k)$$

Now suppose ∞ is indeed in K. Then first we choose an r so that $|z| > r$ is contained in the interior of K. We choose r sufficiently large so that $f(z)$ is holomorphic on $|z| > r$, except maybe at ∞. Let γ be the circle $|z| = r$ oriented positively with respect to ∞. Then we define $K'' = K \setminus \{|z| > r\}$. Then $\partial K'' = \Gamma - \gamma$ (the $-$ comes from the fact that γ is oriented negatively with respect to 0). Since K'' does not contain ∞ we can use the previous result to conclude that

$$\int_\Gamma f(z)dz - \int_\gamma f(z)dz = 2\pi i \sum_{z_k \in S} res(f, z_k)$$

The second term on the left is exactly $2\pi i res(f,\infty)$ thus taking it to the other side gets us the desired result. $\square$

We will be using the Residue Theorem to evaluate many integrals. For this we will need to calculate residues so it's useful to get some practice with that first. A very simple example is the case when the boundary Γ is empty. In this case the Residue theorem tells us that the sum of the residues of a function that is holomorphic on the Riemann sphere, such as a rational function, is 0.

Example 12.1. For another, less trivial example suppose $f(z)$ is a meromorphic function with a simple pole z_0. Then

$$f(z) = \frac{1}{z - z_0} g(z)$$

where g is some holomorphic function such that $g(z_0) \neq 0$. Then by writing

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

we see that the coefficient of $(z - z_0)^{-1}$ in the expansion for g is $a_0 = g(z_0)$. We can easily compute this, giving us

$$res(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

If we write f as the quotient of two holomorphic functions $f = \frac{P}{Q}$ then we find that

$$res(f, z_0) = \lim_{z \to z_0} (z - z_0) \frac{P(z)}{Q(z)} = \lim_{z \to z_0} \frac{P(z)}{\frac{Q(z) - Q(z_0)}{z - z_0}} = \frac{P(z_0)}{Q'(z_0)}$$

■

Example 12.2. We generalise the above to compute the residue at a pole of any order. Suppose f has a pole of order n at z_0. This means we can write

$$f(z) = \frac{1}{(z - z_0)^n} g(z)$$

where $g(z)$ is holomorphic near z_0 and $g(a) \neq 0$. Since g is holomorphic we can write

$$g(z) = \sum_{m=0}^{\infty} a_m (z - z_0)^m$$

Then we know that $res(f, z_0) = a_{n-1}$. We also know that

$$a_{n-1} = \frac{g^{(n-1)}(z_0)}{(n-1)!}$$

Since $g(z) = (z - z_0)^n f(z)$ we get

$$res(f, z_0) = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} [(z - z_0)^n f(z)] \bigg|_{z = z_0} = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} [(z - z_0)^n f(z)]$$

■

Example 12.3. Let us consider a very concrete example. Suppose we have

$$f(z) = \frac{e^{iz}}{z(z^2 + 1)^2}$$

and we want to compute $res(f, i)$. Since i is not a simple pole, we can't use the simple formula found in Example 12.1. We can, however, compute residues by finding the Laurent expansion of f (which is also a fairly common and occasionally easy way of computing residues).

First we substitute $z = i + \zeta$ so that we can center everything at 0. Then we see that

$$\begin{aligned} f(i + \zeta) &= \frac{e^{i(i+\zeta)}}{(i + \zeta)((i + \zeta)^2 + 1)^2} \\ &= \frac{e^{-1+i\zeta}}{(i + \zeta)(-1 + 2i\zeta + \zeta^2 + 1)^2} \\ &= \frac{e^{-1+i\zeta}}{\zeta^2 (i + \zeta)(2i + \zeta)^2} \end{aligned}$$

We will the expansion for each expression individually. For example,

$$e^{-1+i\zeta} = e^{-1}e^{i\zeta} = e^{-1}\left(1 + i\zeta + \frac{(i\zeta)^2}{2} + \cdots\right)$$

Similarly

$$(i+\zeta)^{-1} = -i(1-i\zeta)^{-1} = -i(1 + i\zeta + (i\zeta)^2 + \cdots)$$

Finally we have

$$\begin{aligned}
(2i+\zeta)^{-2} &= -\frac{1}{4}\left(1 - \frac{i}{2}\zeta\right)^{-2} \\
&= -\frac{1}{4} \cdot -2i\frac{d}{d\zeta}\left(1 - \frac{i}{2}\zeta\right)^{-1} \\
&= -\frac{1}{4} \cdot -2i\frac{d}{d\zeta}\left(1 + \frac{i}{2}\zeta + \left(\frac{i}{2}\zeta\right)^2 + \cdots\right) \\
&= -\frac{1}{4}\frac{d}{d\zeta}(-2i + \zeta + \frac{i}{2}\zeta^2 + \cdots) \\
&= -\frac{1}{4}(1 + i\zeta + \cdots)
\end{aligned}$$

Then the product of the three expansions above is

$$\frac{i}{4e}(1 + 3i\zeta + \cdots)$$

Thus when we multiply this with ζ^{-2} (the only remaining expression in f) we see that the coefficient of ζ^{-1} is

$$\frac{i}{4e} \cdot 3i = -\frac{3}{4e}$$

Thus we conclude that

$$res(f, i) = -\frac{3}{4e}$$

■

Example 12.4. Suppose f is a meromorphic function in a neighbourhood of $z = a$ and we want to compute $res(\frac{f'}{f}, a)$. Since f is meromorphic we know that $f(z) = (z-a)^k g(z)$ for some integer k and some holomorphic function g where $g(a) \neq 0$. Then using the product rule we compute that

$$\frac{f'}{f} = \frac{k}{z-a} + \frac{g'}{g}$$

From this it is easy to compute the residue of $\frac{f'}{f}$ at a. Recall that we simply need to find the a_{-1} coefficient which is the coefficient of $(z-a)^{-1}$ in this case. We know that $\frac{g'}{g}$ is holomorphic in a neighbourhood of a since $g(a) \neq 0$. Therefore its expansion around a can only have non-zero coefficients for the positive indices. This means the only contribution for $(z-a)^{-1}$ comes from the first term. Hence we conclude that

$$res\left(\frac{f'}{f}, a\right) = k$$

■

Note in the final example that if f has a zero at a then k is positive an if f has a pole at a, k is negative. This more or less immediately gives us the following result.

Theorem 12.3 (Argument Principle) *Let $f(z)$ be a non-constant meromorphic function in an open set $\Omega \subset \mathbb{C}$. Let K be a compact set with oriented boundary Γ in Ω. Suppose $f(z)$ does not take the value a on Γ and that it has no poles on Γ. Then*

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)-a}dz = Z - P$$

where Z is the number of zeroes of $f(z)-a$ in K (counted with multiplicity) and P is the number of poles of $f(z)$ in K (also counted with multiplicity).

Proof. The proof follows quite readily from the Residue Theorem and the last example above. By the Residue Theorem, we know that the integral of a meromorphic function over the boundary of a compact set can be found summing the residues on the singular points that lie in the compact set. We want to apply this result to the meromorphic function $\frac{f'(z)}{f(z)-a}$. We see that the singularities of this function are exactly the zeroes of $f(z)-a$ and the poles of $f(z)$. By the above example, the zeroes of $f(z)-a$ contribute a positive amount to the residue (counted with multiplicity) and the poles contribute a negative amount (again, with multiplicity). □

The argument principle highlights why integrating $\frac{f'}{f}$ is important. Another way of seeing why its important is to consider what the integral actually is. Suppose γ is an oriented circle centered at a point. Then

$$\int_\gamma \frac{f'}{f}dz = \int_{f\circ\gamma}\frac{1}{z}dz = 2\pi i w(f\circ\gamma, 0)$$

where the first equality follows from substituting $w = f(z)$ and the second equality is a definition. Thus the integral gives us a way to compute the number of zeroes that lie within the disk or any region whose boundary is a simple closed curve.

Theorem 12.4 *Let $f(z)$ be a non-constant holomorphic function in a neighbourhood of $z = z_0$ where z_0 is a root of order k of $f(z)-a$ for some $a \in \mathbb{C}$. Then for every sufficiently small neighbourhood U of z_0 and every b sufficiently close to a (with $b \neq a$), $f(z)-b$ has k simple roots in U.*

Proof. We take U small enough so that $f(z)-a$ has no zeroes but z_0 and $f'(z) \neq 0$ for $z \neq z_0$ in a closed disk centered at z_0 . Let γ be the boundary of this disk.

Note that for any b

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-b}dz = \frac{1}{2\pi i}\int_{f\circ\gamma}\frac{1}{w-b}dw$$

where we substitute $w = f(z)$. This is simply the winding number $w(f\circ\gamma, b)$. But we know that the winding number is constant along connected components in the complement of the closed curve. Thus we need b close enough to z_0 to that the integral above remains constant.

In this case we know that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-b}dz = k$$

since that is the value of the integral when $b = z_0$. Then by the argument principle, we conclude that $f(z) = b$ has k roots inside γ. Furthermore these roots have to be simple since we assume that $f'(z)$ was non-zero for $z \neq z_0$. □

Theorem 12.5 (Rouché's Theorem) *Suppose $f(z), g(z)$ are holomorphic in an open subset Ω. Let K be a compact set with oriented boundary Γ in Ω. If*

$$|f(z)| > |g(z)|$$

on Γ then $f(z)$ and $f(z)+g(z)$ have the same number of zeros in K, counted with multiplicity (in other words the dominant function determines the number of zeroes of the sum).

Proof. We of course want to use the argument principle but that would only tell us about zeros in the interior of K. In principle, it seems like we might miss zeros that lie on Γ itself. However, the inequality guarantees that f and g have no zeros on Γ.

Define $h = f + g$. Then we know that

$$|f - h| < |f|$$

Dividing both sides of the inequality by f (which recall is non-zero on Γ), we see that

$$\left|1 - \frac{h(z)}{f(z)}\right| < 1$$

on Γ. This means that $F(z) := \frac{h(z)}{f(z)}$ takes values in $|z-1| < 1$ the unit disk centered at 1. Now consider

$$\frac{1}{2\pi i}\int_\Gamma \frac{F'}{F}dz$$

On one hand, by the argument principle we know this is the number of zeros of F minus the number of poles of F. But the zeros of F are exactly the zeros of h and the poles of F are exactly the zeros of f. Thus if we could show that the above integral is necessarily 0 we would be done.

We know that Γ is the disjoint union of closed curves. Thus the integral above is the sum over the integral over closed curves γ for every γ in Γ. Moreover

$$\frac{1}{2\pi i}\int_\gamma \frac{F'}{F}dz = \frac{1}{2\pi i}\int_{F\circ\gamma}\frac{d\zeta}{\zeta}$$

substituting $\zeta = F(z)$. This is exactly $w(F\circ\Gamma, 0)$. Since $F\circ\Gamma$ lies in $|z-1| < 1$, the origin will never lie within the curve and hence the winding number is always 0. Thus the integral above is indeed 0, as desired. $\square$

13 Residue Calculus

Evaluating integrals by way of computing residues can allow us to compute certain integral that would difficult, or in some case impossible, by the usual methods of integration (i.e finding primitives).

13.1 Rational functions of sin and cos

Suppose $R(x, y)$ is a (real) rational function (in two coordinates) with no poles on the unit circle and we wish to evaluate

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)d\theta$$

In principle, we could handle this by usual methods of integration and residue calculus is not strictly required. However, we will see that this makes the task a lot easier.

First note that since we are integrating on the unit circle, we can substitute $z = e^{i\theta}$. Then we can write $\cos\theta$ and $\sin\theta$ in terms of z and z^{-1}.

$$\int_0^{2\pi} R(\cos\theta, \sin\theta) d\theta = -i \int_{|z|=1} R\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right) \frac{dz}{z} = 2\pi \sum_{|z|<1} res\left(g(z)\right)$$

where

$$g(z) = \frac{1}{z} R\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)$$

The sum is actually a sum over the residue of the poles of g that lie inside the unit disk (in other words we are exactly applying the Residue Theorem). Such notation will be consistently used in this section.

For a concrete example, suppose we want to compute

$$\int_0^{\pi} \frac{d\theta}{a + \cos\theta}$$

where $a > 1$ is a real number. We cannot apply the procedure above directly since we are integrating from 0 to π instead of 0 to 2π (so in other words when we interpret this integral in the complex plane we don't actually form a closed curve). However, since $\cos(\theta) = \cos(2\pi - \theta)$ we get that

$$\int_0^{\pi} \frac{d\theta}{a + \cos\theta} = \int_{\pi}^{2\pi} \frac{d\theta}{a + \cos\theta}$$

and therefore

$$\int_0^{\pi} \frac{d\theta}{a + \cos\theta} = \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{a + \cos\theta}$$

The integral on the right hand side can be compute as above. We substitute $z = e^{i\theta}$ giving us

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{a + \cos\theta} &= -i \int_{\gamma} \frac{1}{a + \frac{1}{2}\left(z + \frac{1}{z}\right)} \cdot \frac{dz}{z} \\ &= -2i \int_{\gamma} \frac{1}{z^2 + 2az + 1} dz\end{aligned}$$

where γ is the unit circle oriented positively. As mentioned, we will use the Residue Theorem to calculate this. For this we need to know the poles of the rational function in the integrand that lie in the unit disk. The poles of the integrand are exactly the zeroes of $z^2 + 2az + 1$ which are $-a \pm \sqrt{a^2 - 1}$. Since we assumed a to be real and greater than 1, it is clear that both roots are real. However, only one of them lies inside the unit disk, namely $-a + \sqrt{a^2 - 1}$.

Therefore we need to compute the residue of the integrand at $-a + \sqrt{a^2 - 1}$. We see that this is a simple pole (there are only 2 poles and the other pole is obviously distinct). This makes it easy to compute the residue to be

$$\left.\frac{1}{2(z + a)}\right|_{z = -a + \sqrt{a^2 - 1}} = \frac{1}{2\sqrt{a^2 - 1}}$$

(see Example 12.1). Therefore by the Residue Theorem

$$\int_{\gamma} \frac{1}{z^2 + 2az + 1} dz = 2\pi i \cdot \frac{1}{2\sqrt{a^2 - 1}}$$

Therefore

$$\int_0^{\pi} \frac{d\theta}{a + \cos\theta} = \frac{1}{2} \cdot -2i \cdot \frac{2\pi i}{2\sqrt{a^2 - 1}} = \frac{\pi}{\sqrt{a^2 - 1}}$$

13.2 Rational functions over the real line

Suppose $R(x)$ is a rational function with no real poles and we wish to compute

$$\int_{-\infty}^{\infty} R(x)dx$$

Recall that this integral is defined to be

$$\int_{-\infty}^{0} R(x)dx + \int_{0}^{\infty} R(x)dx$$

Thus the integral over the real line exists if and only if both of the integrals in the sum above exist. These integrals exist if and only if R has zeros of (at least) order 2 at ∞ and $-\infty$. This is equivalent to saying that

$$\lim_{x\to\pm\infty} xR(x) = 0$$

In order to evaluate this integral we will evaluate over a certain curve in the complex plane that includes a part of the real line. We will see that the portion of the curve that does not lie on the real axis contributes negligibly to the integral, especially when we consider larger and larger curves. Hence, the value of the integral over the entire curve is very well approximated by the value of the integral over the real line and in the limit the two are equal. On the other hand, the integral over the entire curve can be calculated easily by the Residue Theorem. Hence this gives us a fairly easy way to compute the integral of the rational function over the entire real line.

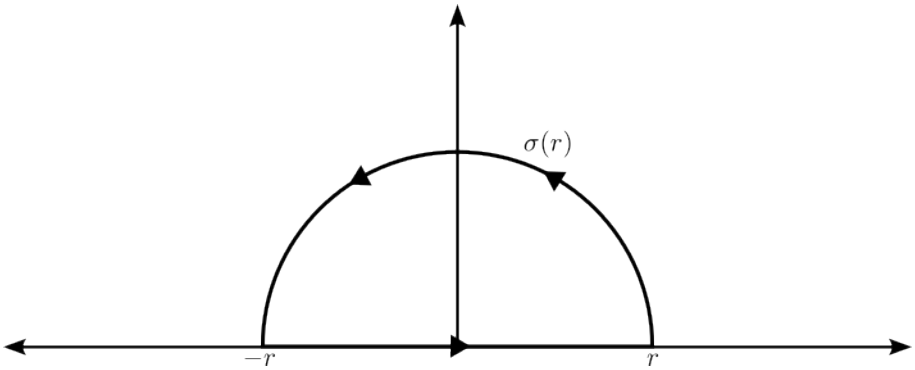


Figure 11: We integrate over a semicircle in the (closed) upper half-plane

Now, to make the above outline more precise. Let $\gamma(r)$ be a semicircle in the upper half-plane centered at 0 and of radius r (including the diameter formed by the interval $[-r, r]$). Let $\sigma(r)$ denote the circular arc itself (not including $[-r, r]$) so that $\gamma(r) = \sigma(r) + [-r, r]$. If we take r large enough then all the poles of R that are in the upper half-plane are inside $\gamma(r)$. Then we get

$$\int_{\gamma(r)} R(z)dz = \int_{-r}^{r} R(x)dx + \int_{\sigma(r)} R(z)dz = 2\pi i \sum_{\mathrm{Im}(z)>0} res(R(z))$$

I claim that as $r \to \infty$, the integral over $\sigma(r)$ goes to 0 which would imply that

$$\int_{-\infty}^{\infty} R(x)dx = 2\pi i \sum_{\mathrm{Im}(z)>0} res(R(z))$$

In order to see that the claim holds let $M(r)$ be the supremum of $R(re^{i\theta})$ for $\theta \in [0, \pi]$. Then

$$\int_{\sigma(r)} R(z)dz \leq M(r) \int_{\sigma(r)} dz = M(r) \cdot -2r$$

Since we know that $zR(z) \to 0$ as $z \to \infty$ we conclude that the above integral goes to 0 as $r \to \infty$.

Note that we could also integrate in the lower half-plane if we so wished by simply reflecting the contour across the real axis. However in order to integrate along the real axis in the usual direction we would need to consider this contour with a negative orientation.

For a concrete example, we compute

$$\int_0^\infty \frac{dx}{1+x^6}$$

Again the integral is over the non-negative reals instead over the entire real line as we calculated above but we will do the same thing as before by computing half the integral over the entire real line (note that the integrand is even).

We see that $\frac{1}{1+z^6}$ has 6 poles, all of which lie on the unit circle. There are 3 poles that lie in the upper half-plane: $e^{\pi i/6}, e^{\pi i/2}$ and $e^{5\pi i/6}$. The residue at each pole α is

$$\frac{1}{6\alpha^5} = \frac{-\alpha}{6}$$

where we use the fact that $\alpha^6 = -1$. Therefore

$$\begin{aligned}
\int_0^\infty \frac{dx}{1+x^6} &= \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{1+x^6} \\
&= \frac{1}{2} \cdot 2\pi i \left(\frac{-e^{\pi i/6}}{6} + \frac{-e^{\pi i/2}}{6} + \frac{-e^{5\pi i/6}}{6} \right) \\
&= \frac{\pi}{6}\left(2\sin\frac{\pi}{6} + 1\right) \\
&= \frac{\pi}{3}
\end{aligned}$$

13.3 Rational functions and trigonometric functions

Suppose we want to compute an integral of the form

$$\int_{-\infty}^\infty R(x)\cos x dx$$

Note that this is simply the real part of

$$\int_{-\infty}^\infty R(x)e^{ix} dx$$

And of course, we could do the same for $\sin x$ instead of $\cos x$ if we considered the imaginary part instead. Once again if R has a zero of order 2 at ∞ we can use the same argument as above (using the fact that $\left|e^{iz}\right| = e^{-y}$ is bounded in the upper half-plane). This gives us

$$\int_{-\infty}^\infty R(x)e^{ix} dx = 2\pi i \sum_{\mathrm{Im}(z)>0} res(R(z)e^{iz})$$

In fact, this equation holds even when R only has a simple pole at ∞. This means that $|zR(z)|$ is bounded as $z \to \infty$. In this case, it's not even immediate that the integral exists. In order to show that this integral exists and verify that it is given by the sum of residues as given above, we will integrate over the boundary of a rectangle whose vertices are $(r_1, 0), (r_1, s), (-r_2, s), (-r2, 0)$ where $r_1, r_2, s > 0$.

Let C be a constant such that

$$|R(z)| \le \frac{C}{|z|}$$

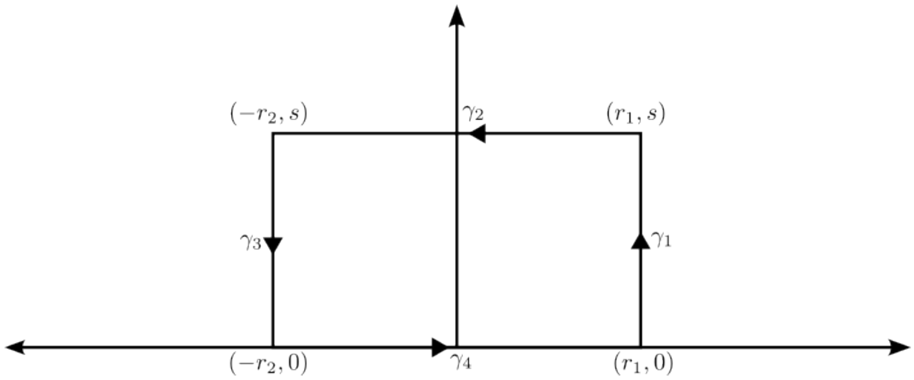


Figure 12: We integrate over the boundary of this rectangle

Let γ_1 denote the right side of the rectangle (i.e. the straight path from $(r_1, 0)$ to (r_1, s)). Then

$$\begin{aligned}
\left|\int_{\gamma_1} R(z)e^{iz}dz\right| &\leq \int_{\gamma_1} |R(z)| \left|e^{iz}\right| dz \\
&\leq C\int_{\gamma_1} \left|\frac{e^{iz}}{z}dz\right| \\
&\leq C\int_0^s \left|\frac{e^{i(r_1+iy)}}{r_1+iy}idy\right| \\
&\leq C\int_0^s \frac{e^{-y}}{\sqrt{r_1^2+y^2}}dy \\
&\leq \frac{C}{r_1}\int_0^s e^{-y}dy \\
&= \frac{C}{r_1}(1-e^{-s}) \\
&< \frac{C}{r_1}
\end{aligned}$$

Similarly by integrating along the left side of the rectangle, we can conclude that it is less than $\frac{C}{r_2}$. Now we wish to bound the rectangle along the top side of the rectangle. Let γ_2 denote the straight path from (r_1, s) to $(-r_2, s)$. Then we have

$$\begin{aligned}
\left|\int_{\gamma_2} R(z)e^{iz}dz\right| &\leq C\int_{\gamma_2} \frac{\left|e^{i(x+is)}\right|}{|z|}dz \\
&\leq Ce^{-s}\int_{-r_2}^{r_1} \frac{1}{\sqrt{x^2+s^2}}dx \\
&< Ce^{-s}\int_{-r_2}^{r_1} \frac{1}{s}dx \\
&= \frac{Ce^{-s}(r_1+r_2)}{s}
\end{aligned}$$

Then we see that as $s \to \infty$ (and we fix r_1, r_2), the integral over γ_2 goes to 0. We know by the residue theorem that integral over the boundary of the rectangle is given by the sum of the residues in the upper half plane (assuming we take r_1, r_2, s sufficiently large). Thus we get

$$\begin{aligned}
\left|\int_\gamma R(x)e^{ix}dx - \int_{-r_2}^{r_1} R(x)e^{ix}dx\right| &= \left|\int_{\gamma_1} R(x)e^{ix}dx + \int_{\gamma_2} R(x)e^{ix}dx + \int_{\gamma_3} R(x)e^{ix}dx\right| \\
&< \frac{C}{r_1} + \frac{Ce^{-s}(r_1+r_2)}{s} + \frac{C}{r_2}
\end{aligned}$$

Fixing r_1, r_2 we can send $s \to \infty$ which removes the middle term. Then by sending r_1, r_2 to ∞ we conclude that

$$\int_{-\infty}^{\infty} R(x)e^{ix}dx = \sum_{\mathrm{Im}(z)>0} res(R(z)e^{iz})$$

Similarly we can evaluate integrals of $R(x)\cos(mx)$ and $R(x)\sin(mx)$ by considering e^{imx} and even $R(x)\cos^m(x)$ and $R(x)\sin^m(x)$ by writing powers of $\sin/\cos$ as a linear combination of $\sin(kx)$ and $\cos(kx)$ for integers $k < m$.

For a concrete example consider

$$\int_0^{\infty} \frac{\cos(mx)}{x^2+1}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(mx)}{x^2+1}dx$$

Thus we need to consider the real part of

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{x^2+1}dx$$

Substituting $z = mx$ we get

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{imx}}{x^2+1}dx &= \frac{1}{m}\int_{-\infty}^{\infty} \frac{e^z}{(z/m)^2+1}dz \\ &= \frac{2\pi i}{m}\sum_{\mathrm{Im}(z)>0} res\left(\frac{m^2e^{iz}}{z^2+m^2}\right)\end{aligned}$$

The only pole of the function in the upper half-plane if im at which point the residue is $\frac{m^2e^{-m}}{2im}$. Hence

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{x^2+1}dx = \frac{2\pi i}{m}\cdot\frac{m^2e^{-m}}{2im} = \pi e^{-m}$$

Since this is real we can immediately conclude

$$\int_0^{\infty} \frac{\cos(mx)}{x^2+1}dx = \frac{\pi e^{-m}}{2}$$

13.4 Poles on the real axis

There are times when the rational function $R(x)$ has a pole on the real axis but $R(x)\sin x$ or $R(x)\cos(x)$ is still integrable over the real line. A primary example of this is $\frac{\sin x}{x}$. Note in this case our previous contour will not work since a pole lies on the curve itself. In order to avoid this, we go around 0 by including a small semicircle of radius δ on the contour.

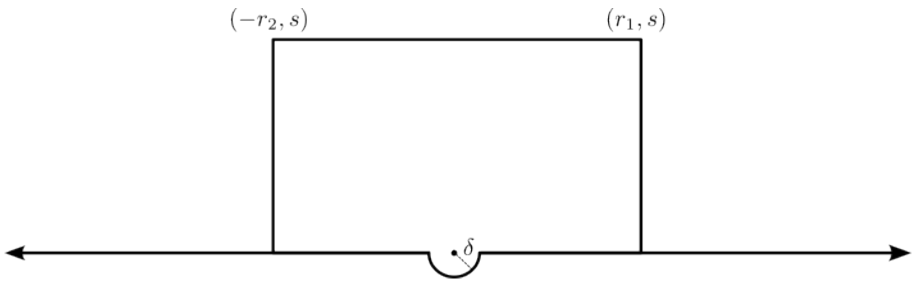


Figure 13: We go around the pole at 0

Since $R(z)e^{iz}$ has a simple pole at 0, we know that $zR(z)e^{iz}$ must be analytic at 0. Thus we can write

$$R(z)e^{iz} = \frac{B}{z} + R_0(z)$$

where B is the residue of $R(z)e^{iz}$ at 0 and $R_0(z)$ is a function that is analytic at 0. Then we see that the integral of the first term over the small semicircle is πiB (one can verify this by direct computation if they so desire) and the integral of the second term depends on δ. Therefore it goes to 0 as $\delta \to 0$ (for sufficiently small δ we can find a local primitive F defined on an open set containing the semicircle. The integral is given by $F(\delta) - F(-\delta)$ which goes to 0 as δ goes to 0 by continuity of F). Hence we conclude that

$$\int_{-\infty}^{\infty} R(x)e^{ix}dx = 2\pi i \sum_{\text{Im}(z)>0} res(R(z)e^{iz}) + \pi i \sum_{\text{Im}(z)=0} res(R(z)e^{iz})$$

Let us in fact consider the example

$$\int_0^{\infty} \frac{\sin x}{x} dx$$

We see that $z^{-1}e^{iz}$ has no poles in the upper half-plane and only has a simple pole at 0. By considering the Taylor series of e^{iz} and then multiplying it with z^{-1} we can immediately conclude that the residue at 0 is 1. Therefore

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{z} dz = \pi i \cdot 1 = \pi i$$

Since $\sin x$ corresponds to the imaginary part, we get

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{1}{2} \cdot \pi = \frac{\pi}{2}$$

13.5 Fractional powers of x

So far we've been working with fairly well-behave functions but there are times when we wish to integrate functions like $\sqrt{x}$ which are perfectly well-defined on the real line (or the positive real line to be precise) but not so on the complex plane. The problem, of course, is the fact that this is a multivalued functions so we need to be quite careful with how do we things. In fact we will exploit the multivaluedness to find the answer in a rather clever way.

First we make things precise. Suppose we want to evaluate something of the form

$$\int_0^{\infty} \frac{R(x)}{x^{\alpha}} dx$$

for some $0 < \alpha < 1$ where $R(x)$ has no poles on $[0, \infty)$. First we consider a contour as seen below. In order for this integral to converge, R must have a zero of (at least) order 2 at ∞ and (at most) a simple pole at 0.

In order to evaluate this integral we first need to choose a branch of x^{α} which in turn requires us to choose a domain on which to specify the branch. We will choose our domain to be $\mathbb{C} \setminus [0, \infty)$ (it might seem odd that we have exclude exactly the region we want to integrate over, we will see that this is precisely what allows us to evaluate the integral). Choosing a branch of x^{α} is equivalent to choosing a branch of $\arg z$ on this domain; we choose $\arg z$ to lie in $(0, 2\pi)$. Wit this set up, we can evaluate the given integral in the complex plane over the contour $\Gamma(\epsilon, r)$ as specified in Figure 14.

In particular, we have a small circle of radius ϵ called $\gamma(\epsilon)$ and a large circle of radius r which we call $\gamma(r)$. We connect the two circles via the interval on the real line, $[\epsilon, r]$.

Taking r sufficiently large and ϵ sufficiently small, we can write write

$$\int_{\Gamma(\epsilon,r)} \frac{R(z)}{z^{\alpha}} dz = 2\pi i \sum_{\mathbb{C}\setminus[0,\infty)]} res\left(\frac{R(z)}{z^{\alpha}}\right)$$

We can split in the integral on the left into its separate components

$$\int_{\Gamma(\epsilon,r)} = \int_{\gamma(r)} + \int_{\gamma(\epsilon)} + \int_{\epsilon}^{r} + \int_{r}^{\epsilon}$$

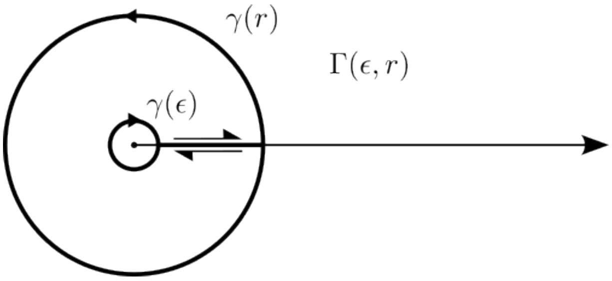


Figure 14: Contour $\Gamma(\epsilon, r)$ for integrating fractional powers of x

Under our assumptions for R we know that the integral over $\gamma(r)$ and $\gamma(\epsilon)$ both tend to 0 as $r \to \infty$ and $\epsilon \to 0$. Moreover, after travelling along $\gamma(r)$, $z^\alpha = e^{2\pi i\alpha} |z|^\alpha$. Therefore we conclude

$$2\pi i \sum_{\mathbb{C}\setminus[0,\infty)} res\left(\frac{R(z)}{z^\alpha}\right) = \int_0^\infty \frac{R(x)}{x^\alpha}dx - \int_0^\infty e^{-2\pi i\alpha}\frac{R(x)}{x^\alpha}dx = (1 - e^{-2\pi i\alpha})\int_0^\infty \frac{R(x)}{x^\alpha}dx$$

As an example, suppose we want to evaluate

$$\int_0^\infty \frac{dx}{x^\alpha(1+x)}$$

for $0 < \alpha < 1$. The integrand only has one pole in $\mathbb{C} \setminus [0, \infty)$, at -1. By our standard methods for calculating residues (for example we can write $\frac{1}{z^\alpha(1+z)} = \frac{z^{-\alpha}}{1+z}$ and use Example 12.1), we compute that the residue of $1/z^\alpha(1 + z)$ at -1 is $(-1)^\alpha = e^{\pi i\alpha}$ (by our choice of arg). Therefore by the above result we get

$$(1 - e^{-2\pi i\alpha})\int_0^\infty \frac{dx}{x^\alpha(1+x)} = 2\pi i \cdot \frac{1}{e^{\pi i\alpha}}$$

Therefore

$$\int_0^\infty \frac{dx}{x^\alpha(1+x)} = \frac{2\pi i}{e^{\pi i\alpha}(1 - e^{-2\pi i\alpha})} = \frac{2\pi i\alpha}{e^{\pi i\alpha} - e^{-\pi i\alpha}} = \frac{2\pi i\alpha}{2i\sin(\pi\alpha)} = \frac{\pi}{\sin\pi\alpha}$$

13.6 Rational functions and logarithms

The final example we consider is that of the humble logarithm which is also a multivalued function. So suppose we wish to evaluate

$$\int_0^\infty R(x)\log(x)dx$$

where $R(x)$ is a rational function with no poles on the non-negative real axis and $xR(x) \to 0$ as $x \to \infty$ (the final condition ensures convergence of the integral).

We might start by trying something similar to last time. This time what we find is that when the argument of z is 2π we get $\log(z) = \log(|z|) + 2\pi i$. Once again the integrals over $\gamma(r)$ and $\gamma(\epsilon)$ go to 0 in the limit so we are left with

$$\int_0^\infty R(x)\log(x)dx - \int_0^\infty R(x)(\log(x) + 2\pi i)dx = 2\pi i\int_0^\infty R(x)dx$$

In particular, the integral we are interested in cancels out. Thus we cannot simply work with $R(x)\log(x)$. Instead what we can do is integrate $R(x)\log(x)^2$. Then we would conclude

$$\begin{aligned}2\pi i \sum_{\mathbb{C}\setminus[0,\infty)} res(R(x)\log(x)^2) &= \int_0^\infty R(x)\log(x)^2dx - \int_0^\infty R(x)(\log(x) + 2\pi i)^2dx \\ &= -4\pi i\int_0^\infty R(x)\log(x)dx - (2\pi i)^2\int_0^\infty R(x)dx\end{aligned}$$

We can simplify this to write

$$\sum_{\mathbb{C}\backslash[0,\infty)} res(R(x)\log(x)^2) = -2\int_0^\infty R(x)\log(x)dx - 2\pi i\int_0^\infty R(x)dx$$

If $R(x)$ is real-valued then we can equate things by their real and imaginary parts. Finally then, we can say

$$\int_0^\infty R(x)\log(x)dx = -\frac{1}{2}\mathrm{Re}\left(\sum_{\mathbb{C}\backslash[0,\infty)} res(R(x)\log(x)^2)\right)$$
$$\int_0^\infty R(x)dx = -\frac{1}{2\pi}\mathrm{Im}\left(\sum_{\mathbb{C}\backslash[0,\infty)} res(R(x)\log(x)^2)\right)$$

14 Harmonic Functions

We now discuss harmonic functions which we will see are an important class of functions and are in fact intimately linked with holomorphic functions.

Recall by definition, that a function $f(x,y)$ (or $f(z)$) that is complex- or real-valued is said to be harmonic if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

which is equivalent to saying

$$\frac{\partial^2 f}{\partial z \partial \overline{z}} = 0$$

This immediately tells us that holomorphic functions are harmonic since if f is holomorphic then $\frac{\partial f}{\partial \overline{z}} = 0$.

Moreover, by linearity of the derivative, we conclude that a (complex-valued) function is harmonic if and only if its real and imaginary parts are harmonic. In particular, suppose we have $f(x,y) = u(x,y) + iv(x,y)$. Then

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + i\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Thus the left-hand side is 0 if and only if both the real and imaginary parts on the right are 0. In fact, all real-valued harmonic functions are the real (or imaginary) part of a holomorphic function, at least locally. In order to see this, suppose g is a harmonic function. By definition, this means that

$$\frac{\partial^2 g}{\partial z \partial \overline{z}} = 0$$

This means that $\frac{\partial g}{\partial z}$ is holomorphic and therefore $\frac{\partial g}{\partial z}dz$ locally has a holomorphic primitive $f(z)$. By definition this means that

$$df = \frac{\partial g}{\partial z}dz$$

Taking conjugates of both sides and using the fact that g is real-valued we conclude that

$$d\overline{f} = \frac{\partial g}{\partial \overline{z}}d\overline{z}$$

We can see this easily by writing out everything explicitly. For example $\overline{f}(z) = \overline{f(z)}$ (by definition).

Therefore $d\overline{f} = \overline{df}$. Then

$$\begin{aligned}
\overline{df} &= \overline{\frac{\partial g}{\partial z} dz} \\
&= \overline{\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) g \cdot (dx + idy)} \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) g \cdot (dx - idy) \\
&= \frac{\partial g}{\partial \overline{z}} d\overline{z}
\end{aligned}$$

Then we get

$$d(f + \overline{f}) = df + d\overline{f} = \frac{\partial g}{\partial z} dz + \frac{\partial g}{\partial \overline{z}} d\overline{z} = dg$$

Therefore $g = 2\text{Re}(f) + c$ where c is some arbitrary constant.

We know that if f is a function that satisfies the maximum modulus principle then so do the real and imaginary parts of f. Then since holomorphic functions satisfy the maximum modulus principle and harmonic functions are the real part of holomorphic functions, they too must satisfy the principle (in fact we will soon see that any continuous function satisfying the maximum modulus principle is harmonic).

One may wonder if given a harmonic function $g(x, y)$, we can work out what the corresponding holomorphic function f would be. We can do this by using the fact that holomorphic functions always have a local power series representation. So suppose f is given by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

within some radius of convergence R. Without loss of generality we can assume a_0 is real (recall that f is only unique up to the addition of a constant so we can easily add something to make a_0 real). Then for any $r < R$, we can work out the real part of $f(re^{i\theta})$ to see what $g(r\cos\theta, r\sin\theta)$ would need to be and then use that to work out the coefficients.

To be precise, we see that

$$g(r\cos\theta, r\sin\theta) = \text{Re}(f(z)) = a_0 + \frac{1}{2}\sum_{n=1}^{\infty} r^n a_n (e^{in\theta} + e^{-in\theta})$$

Then integrating both sides form $\theta = 0$ to $\theta = 2\pi$ we get

$$\frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta) d\theta = a_0$$

We can work out the remaining coefficients by using our usual 'trick' of multiplying by $e^{-in\theta}$ and integrating. Thus we get

$$\frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta) e^{-in\theta} d\theta = \frac{1}{2} r^n a_n \Rightarrow a_n = \frac{1}{\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta) r^{-n} e^{-in\theta} d\theta$$

This means that

$$\begin{aligned}
f(z) &= a_0 + \sum_{n=1}^{\infty} a_n z^n \\
&= \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta) d\theta + \sum_{n=1}^{\infty}\left(\frac{1}{\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta) r^{-n} e^{-in\theta} d\theta\right) z^n \\
&= \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta)\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{z}{re^{i\theta}}\right)^n\right] d\theta
\end{aligned}$$

The inner sum is just a standard geometric series which we can evaluate quite easily. Simplifying everything we can write

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} g(r\cos\theta, r\sin\theta) \frac{re^{i\theta} + z}{re^{i\theta} - z} d\theta$$

We can also find g in this form by considering the real part of this integral. First note that

$$\frac{re^{i\theta} + z}{re^{i\theta} - z} \cdot \frac{re^{-i\theta} - \overline{z}}{re^{-i\theta} - \overline{z}} = \frac{r^2 - |z|^2}{|re^{-i\theta} - z|^2} + \frac{-re^{i\theta}\overline{z} + zre^{-i\theta}}{|re^{-i\theta} - z|^2}$$

The first term is obviously real and is called the Poisson kernel. The second term is purely imaginary since it is the difference between a complex number and its conjugate. Therefore for $|z| < r$, we get

$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} g(r\cos\theta, r\sin\theta) \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} d\theta$$

By taking g to be the function that is identically 1 we conclude that

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} = 1$$

for any z.

14.1 Dirichlet problem for a disk

Quite often we wish to extend functions beyond their original domain of definition and a question we often ask is what conditions we can impose on the extension (as an example, the principle of analytic continuation allows us to extend analytic functions analytically). The Dirichlet problem asks whether we can extend a continuous function to a harmonic function (in this case to a harmonic function on a disk). The theorem below tells us that this can indeed be done and what's more, can be done uniquely.

Theorem 14.1 *Let $f(\theta)$ be a continuous, periodic function defined on the circle of radius r centered at 0 with period 2π. Then there exists a function $F(z)$ that is continuous on the closed disk $|z| \leq r$ and harmonic in the interior $|z| < r$ such that*

$$F(re^{i\theta}) = f(\theta)$$

Moreover, this F is unique.

Proof. First note that it suffices to show this for real-valued f since if f is complex-valued we can consider the real and imaginary parts separately.

The uniqueness of F is easy to see. Suppose F_1 and F_2 are two harmonic extensions of f. Then $F_1 - F_2$ is 0 on the circle $|z| = r$. Then by the Maximum Modulus Principle (see Corollary 10.19) we conclude that $F_1 - F_2$ is 0 on the entire disk $|z| \leq r$ implying that $F_1 = F_2$. The proof of existence is a bit more finicky.

We know from our previous results what F would need to be if it were to exist. So let us simply define it as such

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} d\theta$$

We know that F is the real part of the holomorphic function

$$\frac{1}{2\pi} \int_0^{2\pi} f(\theta) \frac{re^{i\theta} + z}{re^{i\theta} - z} d\theta$$

and is therefore harmonic. The only thing we need to show is that this is continuous on the boundary (continuity everywhere else is clear). In other words we want to show that

$$\lim_{z \to re^{i\theta_0}} = f(\theta_0)$$

We first need to prove a little lemma before proceeding.

Lemma 14.2 *Let $\eta > 0$ be arbitrary and fix some $\theta_0 \in [0, 2\pi)$. Let γ denote the arc of the circle of radius r where $|\arg(z) - \theta_0| > \eta$. Then*

$$\frac{1}{2\pi} \int_\gamma \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} d\theta$$

tends to 0 as $z \to re^{i\theta_0}$.

Proof. Let $z = \rho e^{i\alpha}$ (where of course $\rho < r$). We want to try and bound the denominator of the integrand $|re^{i\theta} - z|$, at least for z close to $re^{i\theta_0}$, so that we can factor it out of the integral which only leaves $r^2 - |z|^2 = r^2 - \rho^2$ in the integral. Clearly then the integral would go to 0 as $z \to re^{i\theta_0}$. Moreover, it seems apparent that if z is close to $re^{i\theta_0}$ then it would have to be some minimal distance away from all the points that lie on γ which would give us the desired bound. Let us make everything mentioned here more precise.

First by the triangle inequality we see that if $|\theta - \theta_0| > \eta$ and $|\alpha - \theta_0| < \frac{\eta}{2}$ then necessarily $|\alpha - \theta| \geq \frac{\eta}{2}$ (in fact the inequality is probably strict). It is easy to see that $re^{i(\theta_0 + \eta)}$ is at least $r \sin\left(\frac{\eta}{2}\right)$ units away from z (see Figure 15, right $re^{i(\theta_0+\eta)}$ is on the intersection of the ray σ with the circle). For arbitrary θ satisfying $|\theta - \theta_0| > \eta$ we see that the circle of radius $r \sin \frac{\eta}{2}$ centered at $re^{i\theta}$ does not intersect the sector $|\theta_0 - \arg(w)| < \frac{\eta}{2}$ (once again consider Figure 15 and look at the figure on the right). Thus once again $|z - re^{i\theta}|$ is at least $r \sin \frac{\eta}{2}$. Thus we have

$$|z - re^{i\theta}| \geq r \sin \frac{\eta}{2}$$

for all $re^{i\theta}$ on γ. This means that

$$\frac{1}{2\pi} \int_\gamma \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} d\theta \leq \frac{1}{2\pi} \int_\gamma \frac{r^2 - |z|^2}{r^2 \sin^2 \frac{\eta}{2}} d\theta < \frac{r^2 - \rho^2}{r^2 \sin^2 \frac{\eta}{2}}$$

Then clearly as $z \to re^{i\theta_0}$ we have $\rho \to r$ and hence the above integral tends to 0.

$\square$

Given this lemma we now consider (using the fact that the integral of the Poisson kernel is 1).

$$\begin{aligned} F(z) - f(\theta_0) &= \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \frac{r^2 - |z|^2}{|re^{i\theta_0} - z|^2} d\theta - \int_0^{2\pi} \frac{1}{2\pi} f(\theta_0) \frac{r^2 - |z|^2}{|re^{i\theta_0} - z|^2} d\theta \\ &= \frac{1}{2\pi} \int_{|\theta - \theta_0| \leq \eta} (f(\theta) - f(\theta_0)) \frac{r^2 - |z|^2}{|re^{i\theta_0} - z|^2} d\theta + \frac{1}{2\pi} \int_{|\theta - \theta_0| > \eta} (f(\theta) - f(\theta_0)) \frac{r^2 - |z|^2}{|re^{i\theta_0} - z|^2} d\theta \end{aligned}$$

We can do this split for any η so now we need to decide what a good choice of η should be. Suppose we are given some $\epsilon > 0$. By continuity of f, we know $\sup_{|\theta - \theta_0| \leq \eta} |f(\theta) - f(\theta_0)|$ can be made as small as we like by choosing η appropriately. So in particular we can easily choose an η so the first integral is less than $\frac{\epsilon}{2}$ (we just integrate over a sufficiently small arc). With this choice of η we get

$$\left| \frac{1}{2\pi} \int_{|\theta - \theta_0| > \eta} (f(\theta) - f(\theta_0)) \frac{r^2 - |z|^2}{|re^{i\theta_0} - z|^2} d\theta \right| \leq M \cdot \frac{1}{2\pi} \int_{|\theta - \theta_0| > \eta} \frac{r^2 - |z|^2}{|re^{i\theta_0} - z|^2} d\theta$$

which we can make arbitrary small by the lemma above. In particular we can make it smaller than $\frac{\epsilon}{2}$ which gives us continuity of F on the boundary. $\square$

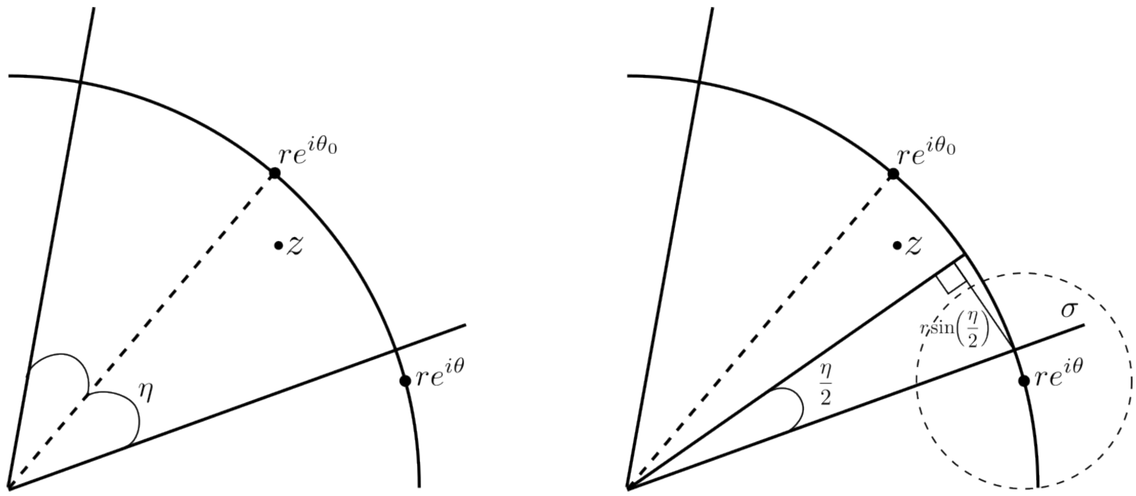


Figure 15: Distance between z and $re^{i\theta}$ is at least $r \sin\left(\frac{\eta}{2}\right)$

In fact, we can use this to completely characterise harmonic functions via the mean value property.

Theorem 14.3 *If f is a continuous function on an open set Ω such that it satisfies the Mean Value Property, then f is harmonic.*

Proof. It suffices to show that f is locally harmonic at every point. Let $z \in \Omega$ be arbitrary and D be a disk centered at z so that $D \subset \Omega$. We know that $f|_{\partial D}$ is continuous and therefore by the above theorem there exists a continuous F which agrees with f on the boundary of D and is harmonic on the interior of D. Since F and f satisfy the maximum modulus principle so does $F - f$. But since $F - f$ is 0 on the boundary, it must be identically 0 on D, implying that $f = F$ is harmonic. $\square$

15 Runge's Approximation Theorem

We end with a powerful statement that uses Cauchy's Integral Formula in a rather interesting way.

The question we wish to ask is whether a holomorphic function on a compact set can be uniformly approximated using polynomials (we know the analogous statement is true for real numbers by Stone-Weirstrass Theorem). In some cases, this can be done. For example, if we have a power series expansion on the entire compact set, then the partial sum will approximate the function uniformly. On the other hand, we have cases where there is definitely no polynomial approximation. Consider for example the function $f(z) = z^{-1}$ which we might wish to approximate on the unit circle S^1. Suppose $p_n(z)$ form a sequence of polynomials that uniformly converge to f. But we see that the integral over S^1 of f is $2\pi i$ while the integral over S^1 of $p_n(z)$ is 0 for all n. Thus p_n could not have converged to f. We will see that this is more or less the only thing that can go wrong. In other words, we can always approximate holomorphic functions by polynomials or by rational functions with poles outside the compact set.

Before stating and proving the theorem, let us first prove the following lemma.

Lemma 15.1 *If $\mathbb{C} \setminus K$ is connected and $z_0 \notin K$ then $\frac{1}{z-z_0}$ can be approximated on K by polynomials.*

Proof. Let D be a disk centered at 0 containing K. The first case is when z_0 is not in D. Then

$$\frac{1}{z - z_0} = -\frac{1}{z_0} \cdot \frac{1}{1 - \frac{z}{z_0}} = -\sum_{n=0}^{\infty} \frac{z^n}{z_0^{n+1}}$$

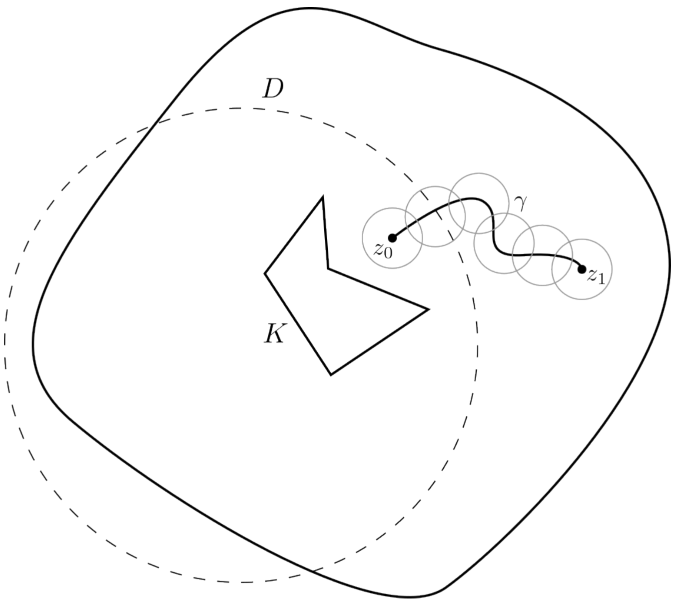


Figure 16: Successively approximate by polynomials in $\frac{1}{z-w}$ along the curve

So the partial sum of the series above form the desired polynomials (we need z_0 outside D to ensure that $\left|\frac{z}{z_0}\right| < 1$).

For the second case, suppose z_0 is in D. By the first case, it suffices to show that $\frac{1}{z-z_0}$ can be uniformly approximated on K by polynomials in $\frac{1}{z-z_1}$ where z_1 is some point outside D. Since $\frac{1}{z-z_1}$ itself can be approximated by polynomials this allows us to do the same for $\frac{1}{z-z_0}$.

Let γ be a path joining z_0 and z_1 (such a path exists by connectedness of $\mathbb{C} \setminus K$). Choose a sequence of points $\{w_1, \ldots, w_k\}$ on the curve such that $|w_{i+1} - w_i| < \frac{1}{2}d(\gamma, K) =: \delta$. Finally we reduce the problem even further to showing that if $w \in \gamma$ and $|w - w'| < \delta$ then $\frac{1}{z-w}$ can be approximated uniformly in K by polynomials in $\frac{1}{z-w'}$. This is easy to see by our usual geometric series argument since

$$\frac{1}{z-w} = \frac{1}{z-w'+w'-w} = \frac{1}{z-w'} \cdot \frac{1}{1-\frac{w-w'}{z-w'}} = \sum_{n=0}^{\infty} \frac{(w-w')^n}{(z-w')^{n+1}}$$

□

Theorem 15.2 (Runge's Approximation Theorem) *Let $\Omega \subset \mathbb{C}$ be open and let K be a compact subset of Ω. Let f be a holomorphic function on Ω. Then*

1. *f can be approximated on K uniformly by rational functions with poles in $\mathbb{C} \setminus K$*
2. *If $\mathbb{C} \setminus K$ is connected, then f can be uniformly approximated on K by polynomials instead*

Proof. First, we choose a grid of squares of side length less than $d(K, \mathbb{C} \setminus \Omega)$ so that any square intersecting K lies inside Ω. Let $\mathcal{Q} := \{Q_1, \ldots, Q_M\}$ be squares that intersect K (all with positively oriented boundaries). Let $\gamma_1, \ldots, \gamma_n$ be boundary segments of Q_j that don't belong to two adjacent squares in $\mathcal{Q}$. Then each γ_l is contained in Ω (because of how we chose the grid) and does not intersect K (if it did there would be 2 adjacent squares containing γ_l).

Then we claim that

$$f(z) = \frac{1}{2\pi i} \sum_{l=1}^{n} \int_{\gamma_l} \frac{f(\zeta)}{\zeta - z} d\zeta$$

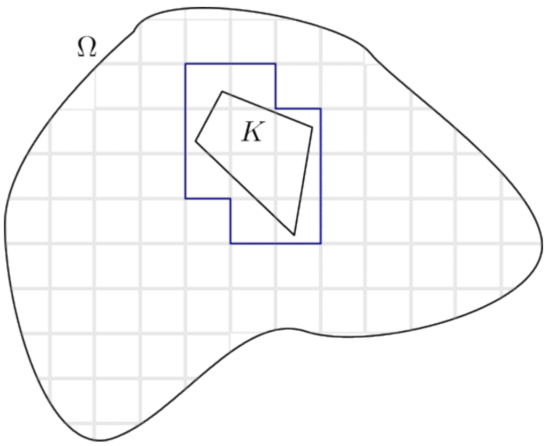


Figure 17: Draw a fine grid so that squares intersecting K are entirely inside Ω. The blue line indicates boundary elements of Q_j that do not belong to two adjacent squares

In order to verify this, first suppose z is an element of $Q_1 \cup \cdots \cup Q_M$ such that it does not lie on the boundary of any Q_j. Then if $z \in Q_j$ we see by Cauchy's Integral Formula that

$$\frac{1}{2\pi i} \int_{\partial Q_m} \frac{f(\zeta)}{\zeta - z} d\zeta = \begin{cases} f(z) & \text{if } m = j \\ 0 & \text{otherwise} \end{cases}$$

This means that

$$f(z) = \frac{1}{2\pi i} \sum_{m=1}^{M} \int_{Q_m} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \sum_{n=1}^{N} \int_{\gamma_n} \frac{f(\zeta)}{\zeta - z} d\zeta$$

where the last inequality follows from the fact that if a boundary is shared by two of the Q_j then the integral over it cancels out. The statement holds for z even if they lie on the boundary of some Q_j by continuity.

Then we can consider each term in the sum separately reducing the problem to the lemma below which more or less immediately tells us when polynomial approximations can be found.

Lemma 15.3 *Suppose γ is a line segment in $\Omega \setminus K$. Then*

$$\int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta$$

can be uniformly approximated on K by rational functions with poles on γ.

Proof. We see that

$$\int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_0^1 \frac{f(\gamma(t))}{\gamma(t) - z} \gamma'(t) dt$$

The integrand $F(z,t)$ is a continuous function on a compact set $K \times [0,1]$ and is therefore uniformly continuous. Therefore for any $\epsilon > 0$ we can find some δ such that whenever $|t_1 - t_2| < \delta$, we have

$$\sup_{z \in K} |F(z, t_1) - F(z, t_2)| < \epsilon$$

This exactly means that the Riemann sums of $\int_0^1 F(z,t)dt$ converge to the integral uniformly on K. But each term in the Riemann sum is of the form

$$\frac{f(\gamma(t_i))}{\gamma(t_i) - z} \gamma'(t_i) \cdot (t_{i+1} - t_i)$$

which is a rational function (in fact a very simple one since it is of the form $\frac{A}{B+z}$) with a pole $\gamma(t_i)$. This is true for each term of the sum and since the sum of rational functions is rational we conclude that the Riemann sums are rational functions with poles on γ that uniformly approximate the holomorphic function f. $\square$

The previous lemma, Lemma 15.1, tells us that under nice conditions (namely $\mathbb{C} \setminus K$ being connected) rational functions of the form $\frac{1}{z-z_0}$ can be approximated by polynomials. Above we have shown that any holomorphic function can be approximated by polynomials in $\frac{1}{z-z_0}$. Thus if $\mathbb{C} \setminus K$ is connected we can approximate f using polynomials. $\square$