Tutorial 4: Functions and Trigonometry

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7-3) Suppose $x^3 = y^3$. This means that

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2) = 0$$

This means that either $x - y = 0$ or $x^2 + xy + y^2 = 0$. If $x - y = 0$, we can immediately conclude that f is injective. So suppose not. Then $x^2 + xy + y^2 = 0$. This means that $x^2 + y^2 = -xy$. Since the left hand side is always non-negative we conclude that that x, y always have opposing sign (unless they are both 0). But cubing maintains signs so they could not be equal.

7-4) Suppose $f(x) = f(y)$ and $x \neq y$. Then either $x < y$ or $x > y$. But both of these contradict the fact that f is increasing so we must have $x = y$.

7-5) Let $x, y \in f(I)$ such that $x < y$. Then there exist (unique) $a, b \in I$ such that $x = f(a)$ and $y = f(b)$. Note this implies that $a = f^{-1}(x)$ and $b = f^{-1}(y)$. Since a and b must be distinct (otherwise we would contradict injectivity), we must have $a < b$ or $a > b$. But $a > b$ contradicts f being increasing. So we have $f^{-1}(x) = a < b = f^{-1}(y)$.

7-13)

a) We want to show that $A \supset f^{-1}(f(A))$. So let $x \in f^{-1}(f(A))$. Define $y = f(x)$. We immediately see that $y \in f(A)$. By definition of $f(A)$, there exists some $z \in A$ such that $y = f(z)$. By definition of injectivity, $x = z$ implying that $x \in A$.

b) We want to show that $A \subset f(f^{-1}(A))$. Let $x \in A$. Since f is surjective, $f^{-1}(A)$ is non-empty. So there exists $y \in f^{-1}(A)$ such that $f(y) = x$. But since $y \in f^{-1}(A)$, we must that $x = f(y) \in f(f^{-1}(A))$.

7-14)

a) Note $f(2x) = f(x + x) = f(x) + f(x) = 2f(x)$. Then proceed by induction. This covers the positive integers. For 0, note $f(0) = f(0 + 0) = 2f(0)$ implying that $f(0) = 0$. Finally, $0 = f(x - x) = f(x) + f(-x)$ so $f(-x) = -f(x)$. Then proceed by induction to cover the negative integers.

b) Suppose $q \in \mathbb{Q}$ such that $q = \frac{m}{n}$. Then

$$f\left(\frac{m}{n} \cdot x\right) = m \cdot \frac{n}{n} f\left(\frac{1}{n} x\right) = \frac{m}{n} f\left(n \cdot \frac{1}{n} x\right) = q f(x)$$

c) The simplest examples are the identity and zero maps ($f(x) = 0$ for all x). The less trivial examples $f(x) = rx$ where r is any real numbers (the previous examples are $r = 1$ and $r = 0$ respectively).

9-19)

a)

$$\cot(x) + \tan(x) = \frac{\cos(x)}{\sin(x)} + \frac{\sin(x)}{\cos(x)} = \frac{\cos^2(x) + \sin^2(x)}{\sin(x)\cos(x)} = \sec(x)\csc(x)$$

b)

$$\sin^4(x) - \cos^4(x) = (\sin^2(x) + \cos^2(x))(\sin^2(x) - \cos^2(x)) = \sin^2(x) - (1 - \sin^2(x)) = 2\sin^2(x) - 1$$

c)

$$\frac{\sec^2(x) - \tan^2(x)}{\cos(x)} = \frac{1 - \sin^2(x)}{\cos^3(x)} = \frac{1}{\cos(x)} = \cos(x)\sec^2(x)$$

d)

$$\tan(x) - \frac{1}{2}\sin(2x) = \tan(x) - \frac{1}{2}(2\sin(x)\cos(x)) = \frac{\sin(x)}{\cos(x)} - \sin(x)\cos(x) = \frac{\sin(x)}{\cos(x)}(1 - \cos^2(x)) = \tan(x)\sin(x)$$

e)

$$(\tan(x))(\cot(x) + \tan(x)) = 1 + \tan^2(x) = \frac{1}{\cos^2(x)}(\cos^2(x) + \sin^2(x)) = \frac{\sin^2(x)}{\cos^2(x)} \cdot \frac{1}{\sin^2(x)} = \frac{\tan^2(x)}{\sin^2(x)}$$

f)

$$\frac{1 - 2\sin^2(x)}{\cos(x) + \sin(x)} = \frac{1 - \sin^2(x) - \sin^2(x)}{\cos(x) + \sin(x)} = \frac{\cos^2(x) - \sin^2(x)}{\cos(x) + \sin(x)} = \cos(x) - \sin(x)$$