# Tutorial 3: Sample midterm

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**Problem 1.** a) Neither b) Neither c) Tautology d) Contradiction
**Problem 2.**

a) The negation is

$$\exists N > 0 \; \forall M > 0 : \exists x \in \mathbb{R}, \left( x > N \wedge |f(x)| \geq \frac{1}{M} \right)$$

b) There is at least one student who owns one dog that spends at least 3 hours each day walking it.

**Problem 3.**
By simple trial and error, one might find that 2 is a root of the polynomial. Thus we can divide by $x - 2$. The resulting polynomial is $x^2 + 7x + 12$ which is further factorised to $(x + 3)(x + 4)$. Thus we have that

$$x^3 + 5x^2 - 2x - 24 = (x - 2)(x + 3)(x + 4)$$

This equality can be verified by expanding the right hand side and using the fact that two polynomials are equal if and only if their coefficients are the same.

It is clear that the right hand side has exactly 3 roots (a product of real numbers is 0 only when one of the terms is 0 and each linear term in the factorised form is 0 at exactly one point). Therefore, the given polynomial must have exactly 3 roots: 2, $-3$ and $-4$.

**Problem 4.**

a) Consider the contrapositive which is $\sqrt{p} + \sqrt{q} \in \mathbb{Q} \Rightarrow \sqrt{pq} \in \mathbb{Q}$. Note that

$$\sqrt{pq} = \frac{(\sqrt{p} + \sqrt{q})^2 - p - q}{2}$$

Therefore if $\sqrt{p} + \sqrt{q}$ is rational then so is $\sqrt{pq}$.

b) Counterexample: $p = q = 2$

c) Counterexample: $x = \sqrt{2}, y = -\sqrt{2}$

**Problem 5.**

a)

$$\begin{aligned}
\sum_{j=1}^{n} (f(j+1) - f(j)) &= \sum_{j=1}^{n} f(j+1) - \sum_{j=1}^{n} f(j) \\
&= f(n+1) + \sum_{j=1}^{n-1} f(j+1) - \sum_{j=2}^{n} f(j) - f(1) \\
&= f(n+1) + \sum_{j=2}^{n} f(j) - \sum_{j=2}^{n} f(j) - f(1) \\
&= f(n+1) - f(1)
\end{aligned}$$

b) Note

$$3j^2 + 3j + 1 = (j+1)^3 - j^3$$

Therefore

$$\sum_{j=1}^{n} 3j^2 + 3j + 1 = \sum_{j=1}^{n} (j+1)^3 - j^3 = (n+1)^3 - 1$$