Tutorial 2: Logic and Proofs

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Problem 1.

a) $\exists a \in A \ \forall b \in B : a \geq b$

b) Take A to be the even numbers and B to be the odd numbers

Problem 2.

a) By definition

$$\binom{n+1}{k} = |\{S \subset \{1, \ldots, n+1\} : |S| = k\}|$$

For every S in the above set, we have $n+1 \in S$ or $n+1 \notin S$ (also note these conditions are mutually exclusive) Therefore we get

$$\begin{aligned}
\binom{n+1}{k} &= \big|\{S \subset \{1, \ldots, n+1\} : |S| = k, n+1 \notin S\}\big| + \big|\{S \subset \{1, \ldots, n+1\} : |S| = k, n+1 \in S\}\big| \\
&= \big|\{S \subset \{1, \ldots, n\} : |S| = k\}\big| + \big|\{S \subset \{1, \ldots, n\} : |S| = k-1\}\big| \\
&= \binom{n}{k} + \binom{n}{k-1}
\end{aligned}$$

b) We prove this inductively. Define

$$P(n) := \text{“}\binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ for all } 0 \leq k \leq n\text{”}$$

Then $P(0)$ is true by convention. Suppose $P(n)$ holds. Then

$$\begin{aligned}
\binom{n+1}{k} &= \binom{n}{k} + \binom{n}{k-1} \\
&= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \\
&= n!\left(\frac{n-k+1}{k!(n-k+1)!} + \frac{k}{k!(n-k+1)!}\right) \\
&= \frac{n!(n+1)}{k!(n-k+1)!} \\
&= \frac{(n+1)!}{k!(n+1-k)!}
\end{aligned}$$

c) We use induction again. The statement is

$$P(n) := \text{“}(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\text{”}$$

Easy to see that $P(0)$ is true. Assume $P(n)$ is true. Then

$$\begin{aligned}
(1+x)^{n+1} &= (1+x)(1+x)^n \\
&= (1+x)\sum_{k=0}^{n}\binom{n}{k}x^k \\
&= \sum_{k=0}^{n}\binom{n}{k}x^k + \sum_{k=0}^{n}\binom{n}{k}x^{k+1} \\
&= 1 + \sum_{k=1}^{n}\binom{n}{k}x^k + \sum_{k=1}^{n}\binom{n}{k-1}x^k + x^{n+1} \\
&= 1 + \sum_{k=1}^{n}\left(\binom{n}{k} + \binom{n}{k-1}\right)x^k + x^{n+1} \\
&= 1 + \sum_{k=1}^{n}\binom{n+1}{k}x^k + x^{n+1} \\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}x^k
\end{aligned}$$

Problem 3. This will be another problem solved using induction. We will take the base case to be $n = 1$, i.e. with only one disk on the left (although you can start with $n = 0$ if you want to!). You need at least one move in order to move this singular disk and this one move is sufficient. So the minimum number of moves for $n = 1$, is 1 which is equal to $2^1 - 1$.

Assume by induction that the minimum number of moves needed to move n disks is $2^n - 1$. Then consider the game with $n + 1$ disks. We need to move the lowest disk somehow. This can only be done after moving all the disks above it. By assumption, this requires (at least) $2^n - 1$ moves. After all these smaller disks have been placed in the central rod, we need one move to place the largest disk on the last rod. Finally, we need a further $2^n - 1$ to move the n disks around the central rod, to the last rod. Thus the total number of moves comes to

$$2^n - 1 + 1 + 2^n - 1 = 2 \cdot 2^n - 1 = 2^{n+1} - 1$$

and by construction no fewer moves could have been made.