Tutorial I: Sets and Numbers

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Problem 1.

a) Let $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B \cup C$. If $y \in B$, then $(x, y) \in A \times B$ and if $y \in C$ then $(x, y) \in A \times C$. Either way $(x, y) \in (A \times B) \cup (A \times C)$.
Let $(x, y) \in (A \times B) \cup (A \times C)$. If $(x, y) \in (A \times B)$ then $x \in A$ and $y \in B$ otherwise $y \in C$. Regardless, $y \in B \cup C$ so $(x, y) \in A \times (B \cup C)$. Thus $A \times (B \cup C) = (A \times B) \cup (A \times C)$

b) An entirely analogous argument as above

c) Let $(x, y) \in A \times (B \backslash C)$. Then $x \in A$ and $y \in B \backslash C$. In particular $y \notin C$ so $(x, y) \notin A \times C$. On the other hand $y \in B$ so $(x, y) \in A \times B$, therefore $(x, y) \in (A \times B) \backslash (A \times C)$.

Problem 2.

a) Let d be a divisor of p. Then $\frac{p}{d}$ is an integer and clearly p divides $p = d \cdot \frac{p}{d}$. By assumption then $p|d$ or $p|\frac{p}{d}$. But both of these are smaller than p, so p can only divide them if $d = 1$. Thus 1 and p are the only divisors of p hence it must be prime.

b) Let n be the smallest number with two distinct prime factorisations $p_1 \cdots p_n = n = q_1 \cdots q_m$, where all p_i and q_j are prime. By previous question, we know $p_1|q_1$ or $p_1|q_2 \cdots q_m$. Doing this again and again, we conclude that p_1 must divide at least one of the q_j. Without loss of generality, we may assume that $p_1|q_1$ (we reorder the q_j if necessary). Since p_1 and q_1 are prime, they can only divide one another by being equal. Thus we can divide by them on both side to conclude that $p_2 \cdots p_n = q_2 \cdots q_m$. But this number is smaller than n and has two distinct prime factorisations which contradicts n being the smallest one.

Problem 3.

a) Suppose there is some $x \in \mathbb{Q}$ such that $x^2 - 5 = 0$. Then we know there exist coprime $p, q \in \mathbb{Z}$ such that $x = \frac{p}{q}$. This implies that $p^2 = 5q^2$. Since 5 is prime, we know it must divide p^2 (by previous question) implying 5 must also divide p (Why?). So we can write $p = 5k$ for some integer k. Thus $p^2 = 25k^2 = 5q^2$ which implies $5k^2 = q^2$. But this means 5 is also a factor of q contradicting p, q being coprime.

b) Note that $210 = 2 \cdot 3 \cdot 5 \cdot 7$. As before, suppose there exist coprime integers p, q such that $\left(\frac{p}{q}\right)^2 - 210 = 0$. This means that $p^2 = 210q^2$. As before we conclude that 2 is a factor of p so $4k^2 = 210q^2$ for some integer k. But this mean $2k^2 = 105q^2$. Since 2 is a factor on the left, it must also be a factor of the right. Clearly 2 cannot divide 105 so it must divide q^2. But this means q must also be divisible by 2 contradicting coprimeness of p and q.