MAT257 RSG 4

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1. What one would like to say is that if $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ is continuous and injective then f^{-1} (on the appropriate domain) is also continuous. This unfortunately does not hold true in general.

   (a) Give an example of some A and f where $f : A \to \mathbb{R}^m$ is continuous but f^{-1} is not.

   (b) Show that if A is compact, then the assertion is in fact true. That is, show that if $A \subset \mathbb{R}^n$ is compact and $f : A \to \mathbb{R}^m$ is continuous and injective, then f^{-1} (on the appropriate domain) is continuous.

2. Show that a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^m$ is convergent if and only if it is Cauchy (you may assume this is true for $m = 1$). (*Bonus*: Use this to conclude that every finite dimensional real inner product subspace is closed).

3. (Adapted from Munkres' *Analysis on Manifolds*). In lecture, we prove the fact that if $A \subset \mathbb{R}^n$ is open and $f : A \to \mathbb{R}$ is of class C^∞ then $D_iD_jf(a) = D_jD_if(a)$ for all $1 \le i, j \le n$ and all $a \in A$. Unfortunately we never proved it, so let us do that now.

   (a) We need only prove the case for $n = 2$ (Why?). Let $R = [a, a + h] \times [b, b + k]$ be a rectangle contained in A. Define

   $$\lambda(h, k) = f(a, b) - f(a + h, b) + f(a + h, b + k) - f(a, b + k)$$

   Show there exist points $p, q \in R$ such that

   $$\lambda(h, k) = D_1D_2f(p) \cdot hk$$
   $$\lambda(h, k) = D_2D_1f(q) \cdot hk$$

   (*Hint:* Use the Mean Value Theorem)

(b) Use the above fact to conclude that $D_1 D_2 f(a,b) = D_2 D_1 f(a,b)$.

Just for fun

4. Let $\mathcal{H}$ be a possibly infinite dimensional real inner product space, such that that a sequence converges in $\mathcal{H}$ if and only if it is Cauchy.

   (a) Let S be any subset of $\mathcal{H}$. Let $S^\perp := \{v \in \mathcal{H} : \forall u \in S\ \langle v, u \rangle = 0\}$. Show that $S^\perp$ is a closed, linear subspace of $\mathcal{H}$.

   (b) Show that $(S^\perp)^\perp = \overline{S}$.*

   (c) Show that if K is a closed linear subspace of $\mathcal{H}$ then $K \oplus K^\perp = \mathcal{H}$.*