MAT257 RSG 2

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1. We say that a collection of subsets $\{A_\alpha\}$ has the *finite intersection property* if any finite intersection of element in the collection is non-empty.

   (a) Give an example of a collection of closed subsets in $X = (0, 1)$ with the finite intersection property such that their intersection is empty (*Note:* A set F is closed in a subset X if $F = X \cap E$ where E is closed in $\mathbb{R}^n$).

   (b) Show that a space X is compact if and only if for any collection $\{F_\alpha\}$ of closed subsets of X with the finite intersection property, we have that
   $$\bigcap_\alpha U_\alpha \neq \phi$$

2. The following is one my favourite statements from linear algebra. We partially answered it in the past two 257 homeworks but we can certainly generalise things.

   Let V be a (possibly infinite dimensional) inner product space and let $T : V \to V$ be a linear operator. Show that the following are all equivalent:

   (a) T is bounded (i.e there exists some $M > 0$ such that $|T(h)| \leq M|h|$)

   (b) T is continuous at 0

   (c) T is continuous at a point

   (d) T is continuous everywhere

3. Daddy Dror mentioned that in order to find find derivatives of limit points in a non-necessarily-open set you find an extension of the function where you *can* take the derivative and take this to be the differen-

tial. This of course requires that all eligible extensions agree on what the derivative should be. Let us formalise this and prove it.

Let A be a not-necessarily-open subset of $\mathbb{R}^n$ and let $a \in A$ be a limit point. Suppose we are give a map $f : A \to \mathbb{R}^m$. Let U, V be open sets containing a and $g : U \to \mathbb{R}^m$ and $h : V \to \mathbb{R}^m$ such that $f(x) = g(x) = h(x)$ for all $x \in U \cap V \cap A$. Suppose additionally that g and h are differentiable at a. Show that $g'(a) = h'(a)$. Show that it may not be true that $g(x) = h(x)$ for all $x \in U \cap V$.

4. Show that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, then its graph Γ_f is closed in $\mathbb{R}^{2n}$. Does the converse hold?

5. Now that we have continuity, we should show that some of our basic functions are continuous. Show that the projection maps $\pi_i : \mathbb{R}^n \to \mathbb{R}$, given by $\pi_i(x_1, \ldots, x_n) = x_i$ are continuous.

   We also have natural maps going the other way, namely the inclusion maps. Define $\iota : \mathbb{R} \to \mathbb{R}^n$, given by $\iota(x) = (x, 0, \ldots, 0)$. Is ι continuous?

6. (Pederson, *Analysis Now*, E 1.4.6) Given $f : \mathbb{R}^2 \to \mathbb{R}$, we say that f is *separately continuous in each variable* if for every $x', y' \in \mathbb{R}$, the maps $f_{x'}(y) = f(x', y)$ and $f_{y'}(x) = f(x, y')$ are continuous.

   (a) Show if $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, then f is separately continuous in each variable

   (b) Show that $f : \mathbb{R}^2 \to \mathbb{R}$ being separately continuous on each variable does *not* imply that f is continuous. (*Hint:* Consider `www.math3d.org/OR9Feg3Y`)

7. (Our lord and saviour Saied) Suppose we have a map $f : \mathbb{R} \to \mathbb{R}^n$.

   (a) Show that there exist $f_1 \ldots, f_n : \mathbb{R} \to \mathbb{R}^n$ such that

   $$f(x) = (f_1(x), \ldots, f_n(x))$$

   (b) Show that f is continuous if and only if all the $f_1, \ldots, f_n$ are continuous.

8. Show that there is no continuous map from $\mathbb{R}$ onto S^1 (i.e. surjective map), where $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ (*Hint:* There's a one line answer!).

9. Show that there is a continuous, bijective function from $\mathbb{R}$ to $S^1 \backslash \{(1, 0)\}$.