Theorem

Let $y_p(x)$ be any particular solution to

$$ay'' + by' + cy = r(x), \qquad a, b, c\ constants$$

Also, let $c_1y_1(x) + c_2y_2(x)$ denote the general solution to the complementary equation. Then the general solution to the DE is

$$y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x).$$

Proof

First, we will show $y(x)$ satisfies the DE and second, that any solution to the DE can be written in the form of $y(x)$.

1. $$\begin{aligned} ay'' + by' + cy &= a\left(c_1y_1 + c_2y_2 + y_p\right)'' + b\left(c_1y_1 + c_2y_2 + y_p\right)' + c(c_1y_1 + c_2y_2 + y_p) \\ &= [a(c_1y_1 + c_2y_2)'' + b(c_1y_1 + c_2y_2)' + c(c_1y_1 + c_2y_2)] \\ &\quad + ay_p'' + by_p' + cy_p \\ &= 0 + r(x) = r(x). \end{aligned}$$

2. Now let $z(x)$ be any solution to the given DE. Now, consider $z(x) - y_p$;

So, $$\begin{aligned} a\left(z(x) - y_p\right)'' + b\left(z(x) - y_p\right)' + c\left(z(x) - y_p\right) &= az'' + bz' + cz \\ &\quad -(ay_p'' + by_p' + cy_p) \\ &= r(x) - r(x) \\ &= 0 \end{aligned}$$

Therefore, $z(x) - y_p$ is a solution to the *complementary equation.* But, $c_1y_1(x) + c_2y_2(x)$ is the general solution to the complementary equation, so there are constants $c_1\ and\ c_2$ such that

$$z(x) - y_p = c_1y_1(x) + c_2y_2(x)$$

and

$$z(x) = c_1y_1(x) + c_2y_2(x) + y_p(x).$$

Examples

1. $y'' + 4y' + 3y = 3x$
2. $y'' - y' - 2y = 2e^{3x}$
3. $y'' - 4y' + 4y = 7\sin(t) - \cos(t)$
4. $y'' + 5y' + 6y = 3e^{-2x}$

$r(x)$	Initial guess for $y_p(x)$
k (a constant)	A (a constant)
$ax+b$	$Ax+B$ (*Note*: The guess must include both terms even if $b=0$.)
ax^2+bx+c	Ax^2+Bx+C (*Note*: The guess must include all three terms even if b or c are zero.)
Higher-order polynomials	Polynomial of the same order as $r(x)$
$ae^{\lambda x}$	$Ae^{\lambda x}$
$a\cos\beta x+b\sin\beta x$	$A\cos\beta x+B\sin\beta x$ (*Note*: The guess must include both terms even if either $a=0$ or $b=0$.)
$ae^{\alpha x}\cos\beta x+be^{\alpha x}\sin\beta x$	$Ae^{\alpha x}\cos\beta x+Be^{\alpha x}\sin\beta x$
$(ax^2+bx+c)e^{\lambda x}$	$(Ax^2+Bx+C)e^{\lambda x}$
$(a_2x^2+a_1x+a0)\cos\beta x$ $+(b_2x^2+b_1x+b_0)\sin\beta x$	$(A_2x^2+A_1x+A_0)\cos\beta x$ $+(B_2x^2+B_1x+B_0)\sin\beta x$
$(a_2x^2+a_1x+a_0)e^{\alpha x}\cos\beta x$ $+(b_2x^2+b_1x+b_0)e^{\alpha x}\sin\beta x$	$(A_2x^2+A_1x+A_0)e^{\alpha x}\cos\beta x$ $+(B_2x^2+B_1x+B_0)e^{\alpha x}\sin\beta x$

Examples

1. $y''-9y=-6cos3x$
2. $x''+2x'+x=4e^{-t}$
3. $y''-2y'+5y=10x^2-3x-3$
4. $y''-3y'=-12t$

Variation of Parameters

Given $y''+py'+qy=r(x)$ where $r(x)$ is a combination of polynomials, exponentials and/or trig functions. $p, q\ constants$

Let $c_1\boldsymbol{y_1}(\boldsymbol{x})+c_2\boldsymbol{y_2}(\boldsymbol{x})$ be the general solution to the complementary equation.

Consider a potential particular solution of the form $y_p(x)=u(x)y_1(x)+v(x)y_2(x)$ where $u(x), v(x)$ may not be constants.

$$y_p=uy_1+vy_2$$

$$y_p'=u'y_1+uy_1'+v'y_2+vy_2'$$

$$y_p''=(u'y_1+v'y_2)'+u'y_1'+uy_1''+v'y_2'+vy_2''$$

Substituting these into the DE,

$$y''+py'+qy=[(u'y_1+v'y_2)'+u'y_1'+uy_1''+v'y_2'+vy_2'']$$
$$+p[u'y_1+uy_1'+v'y_2+vy_2']+q[uy_1+vy_2]$$

$$=u[y_1''+py_1'+qy_1]+v[y_2''+py_2'+qy_2]+(u'y_1+v'y_2)'+p(u'y_1+v'y_2)+(u'y_1'+v'y_2').$$

Now y_1 *and* y_2 are solutions to the complementary equation, so the first 2 terms are zero. Thus, we have

$$(u'y_1 + v'y_2)' + p(u'y_1 + v'y_2) + (u'y_1' + v'y_2') = r(x)$$

If we impose the additional condition that $u'y_1 + v'y_2 = 0$, the first 2 terms are zero, and we are left with $u'y_1' + v'y_2'$ needing to equal $r(x)$.

We now have a system of two equations with two unknowns, u' *and* v' that are needed to satisfy the DE. (We get $u(x)$ *and* $v(x)$ with basic integration.)

$$y_1u' + y_2v' = 0$$

$$y_1'u' + y_2'v' = r(x)$$

And our final solution is $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$.

(Note- We do not need the "+c" when anti-differentiating to get $u(x), v(x)$.)

Cramer's Rule

$$a_1z_1 + b_1z_2 = r_1$$

$$a_2z_1 + b_2z_2 = r_2$$

$$z_1 = \frac{\begin{vmatrix} r_1 & b_1 \\ r_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{r_1b_2 - b_1r_2}{a_1b_2 - b_1a_2} \qquad z_2 = \frac{\begin{vmatrix} a_1 & r_1 \\ a_2 & r_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1r_2 - r_1a_2}{a_1b_2 - b_1a_2}$$

1. Use Cramer's Rule to solve the following system of equations:

$$x^2z_1 + 2xz_2 = 0$$
$$z_1 - 3x^2z_2 = 2x$$

2. $y'' - 2y' + y = \frac{e^x}{x^2}$. Solutions to the complimentary equation are e^x, xe^x.