1 Introduction

We are considering planetary motion with 2 and 3 body systems using numerical solutions and simulation. We begin by considering a simple 2 body system where gravitational attraction will affect a projectile launched from a fixed planet towards an orbiting moon.

For simplicity we set the planet's mass (m_{pl}) and radius (R_{pl}) to 1, as well as the Gravitational constant $G = 1$. In these units, the moon has mass $m_m = 0.1$ and radius $R_m = 0.25$ and orbits at a constant radius of $R_{orb} = 19$. The projectile we are launching has negligible mass and radius ($\ll 1$). We are considering our system to be fixed in the $x - y$ plane. In this system, where the system is centred on the planet, gravitational acceleration has the form,

$$\ddot{\mathbf{r}}_i = \sum_{j \neq i} \frac{m_j}{|\mathbf{r}_{ij}|^2} \hat{\mathbf{r}}_{ij}, \tag{1}$$

where $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$, and $\mathbf{r}_i = (x_i, y_i)$. We can write this as an ODE:

$$\partial_t \begin{pmatrix} \mathbf{r}_i(t) \\ \dot{\mathbf{r}}_i(t) \end{pmatrix} = f(\mathbf{r}_i(t)) = \begin{pmatrix} \mathbf{v}_i(t) \\ \ddot{\mathbf{r}}_i(t) \end{pmatrix} \tag{2}$$

which we aim to solve using Runge Kutte 4 (RK4) methods.

2 B1: Two-Body Approximation:

2.1 Equations of Motion and Initial Conditions

We aim to calculate the velocity required for the projectile to leave Earth's surface and have 0 velocity by the time it reaches the moon's surface if the moon does not apply any gravitational force to the projectile.

Under the 2 body approximation where the moon does not influence the projectile, its acceleration simplifies to

$$\ddot{\mathbf{r}_{proj}} = \frac{m_{pl}}{|\mathbf{r}_{proj} - \mathbf{r}_{planet}|^2} (\mathbf{r}_{planet} \hat{-} \mathbf{r}_{proj}) = -\frac{\mathbf{r}_{proj}}{|\mathbf{r}_{proj}|^3}$$

as the planet is fixed at the center of the system and its mass is 1.

We are taking this as a circularly symmetric scenario and so this simplifies down to 1 dimension.

In this situation we can now consider conservation of energy. At the Earths surface the projectile will have gravitational potential energy equal to

$$U_i = -G \frac{M_{pl} M_{proj}}{R_{pl}} = -M_{proj}$$

and kinetic energy equal to

$$KE_i = \frac{1}{2} M_{proj} v_0^2$$

where v_0 is the velocity we will launch the projectile at. At the Moon's surface, it will have 0 kinetic energy and gravitational potential energy equal to,

$$U_f = -\frac{M_{proj}}{18.75}$$

as the radius of the moon is 0.25 and it is orbiting at 19 units away from the planet. Equating by conservation of energy we find,

$$\frac{1}{2} M_{proj} v_0^2 - M_{proj} = -\frac{M_{proj}}{18.75}$$

cancelling the mass and rearranging we yield,

$$v_0 = \sqrt{\frac{2 \times 17.75}{18.75}} = \sqrt{\frac{142}{75}}$$

as required.

We can also see the moon's equation of motion would be,

$$\ddot{\mathbf{r}_{moon}} = -\frac{\mathbf{r}_{moon}}{19^3}$$

as it is in circular orbit at a radius of 19.

2.2 How long is the projectile travelling for?

First we should find out how fast the moon is going around the planet so that we can give it sensible initial conditions. The velocity of an orbiting body in circular motion is given by

$$v_{moon} = \sqrt{\frac{M_{pl}}{R_{orb}}} = \sqrt{\frac{1}{19}}$$

this hence gives us the magnitude of the velocity of the moon at all times. As such we can initialise it at any point and ensure that the velocity is perpendicular to its position vector and with the magnitude given above. We can also input our code for the initial conditions of the projectile on the planets surface and create the RK4 step function as well as the gravitational acceleration function in the functions.hpp class.

We can create a loop over a large number of time steps and check if the projectile is at a radius of 18.75 (radius of orbit minus radius of moon). We print out the the time when the projectile is at this radius and find that it takes 89.4349 units of dimensionless time for the projectile to get to the surface of the moon.

2.3 Making the projectile 'land' on the moon

Given it takes this 89.4349 units of time, we can find where the moon should start so that the projectile passes exactly by the moon. The ending position of the projectile is $(18.75, 0)$ which is directly on the x axis as we shoot it radially starting from $(1, 0)$.

The angular velocity of the moon can be found using Kepler's Law by noting that

$$T_{moon} = \sqrt{2\pi R_{orb}^{3/2}}$$

and so

$$\omega_{moon} = 2\pi/T = R^{-3/2} = \frac{1}{19^{3/2}}.$$

This means that the moon will travel a total of

$$89.4349 \times \frac{1}{19^{3/2}} = 1.0798828009$$

radians while the projectile is travelling. As such, to end on the x axis of our coordinate system, it must start 1.0799 radians below the x axis. With a radius of 19, this means its initial starting point must be

$$(\cos(1.0799) * 19, -\sin(1.0799) * 19)$$

and by perpendicular vectors and the fact we know the moon travels at a constant speed of $\frac{1}{\sqrt{19}}$, we know its initial speed vector must be

$$(\sin(1.0799) * \frac{1}{\sqrt{19}}, \cos(1.0799) * \frac{1}{\sqrt{19}}).$$

We can test this by simulating and plotting the path of the projectile and the moon as shown in figure 1

Overall, this question asks us to simplify the n-body problem to only gravitational attraction between 2 bodies at a time (moon and planet, planet and projectile). Under this simplification we can simulate this gravitational attraction by considering small time steps from an initial condition to numerically solve the relevant ODE via RK4. We show that after analytically finding the correct initial velocities and positions, we can make the projectile launch and stop at the surface of an orbiting moon. Clearly, this will not work if we consider the attraction between the moon and the projectile which is now the next portion of this project.

3 B2: Three-Body Problem

3.1 New Equations of Motion

Now that we are considering the 3 body interactions, we must expand our equation of motion to include the influence from the moon's gravitational pull.

We assume that the projectile exerts no gravitational pull on the moon nor the earth as it has a negligible mass in comparison. This means that the moon's motion does not depend on the projectile and hence its equation of motion is the same as before.

The projectile's equation of motion now expands to:

$$\ddot{\mathbf{r}_{proj}} = -\frac{\mathbf{r}_{proj}}{|\mathbf{r}_{proj}|^3} + \frac{m_{moon}}{|\mathbf{r}_{moon} - \mathbf{r}_{proj}|^3}(\mathbf{r}_{moon} - \mathbf{r}_{proj})$$

as we consider the moon's gravitational pull.

As required.

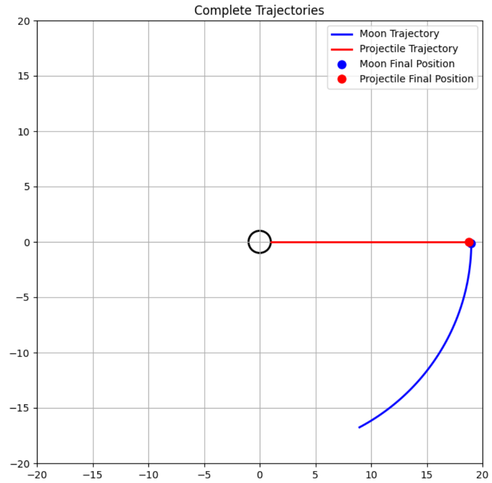


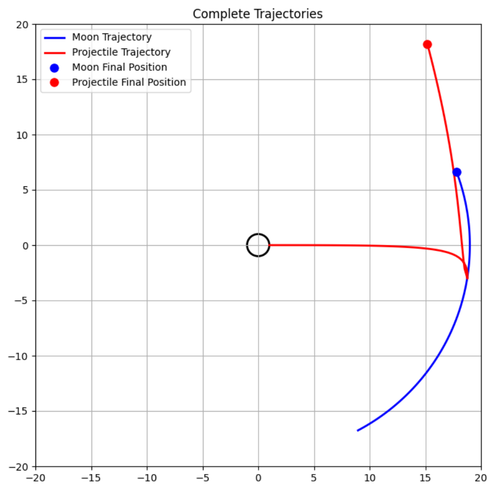
Figure 1: Plot of the trajectory of moon and projectile showing that the projectile is at the surface of the moon exactly as it passes by. Note this is not considering the gravitational force from the moon to the projectile.

3.2 Implementing 3 Body Gravitation

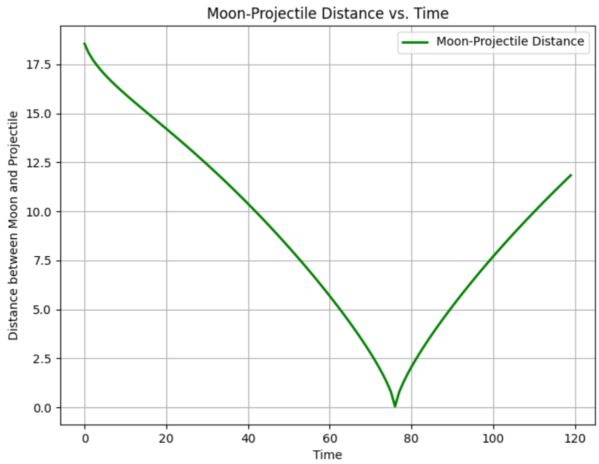
To implement this new interaction, we first change the gravitational attraction function to include 2 bodies which are applying gravitational force to the Particle object. To do this we use the new equation of motion for acceleration defined above. More explicitly we just add together the gravitational contributions of the 2 other Particle objects applied to the first particle object.

Importantly, as the projectile has negligible force on the moon, we set its mass to 0. Additionally, we wish to simultaneously update the moon and projectiles motion, meaning that in the RK4 function with 3 bodies, we simultaneously calculate the 4 K values based on the positions of the moon and projectile at each increment step.

Making this change gives us figure 2(a) where we use the same initial conditions as before, and see that the moon now 'flings' the projectile due to its gravitational attraction.



((a)) Trajectory of the projectile and moon under the same initial conditions with the addition of gravitational attraction between the moon and the projectile.



((b)) Distance between the moon and projectile over time where we clearly see the projectile being flung by the moon. We can use this data to find when the projectile first is 0.5 units away from the moon.

Figure 2

Clearly, we see that the addition of the 3 body interaction changes the course of the projectile but does not change the moon's orbit (it looks different as we are running the simulation for slightly longer).

3.3 Distance between Moon and Projectile

We can calculate the euclidean distance between the moon and the projectile in the 2D plane and we yield figure 2(b).

With this simulation and the same initial conditions as before we can test when the distance is first equal to 0.5 and we find that this first occurs at 75.4959 in our dimensionless time. The code prints out this value in the terminal.

3.4 Analytically finding the kick for orbital transfer

With this interaction now in place, it is natural to consider how we can make an orbital transfer using instantaneous thrusts. We can analytically find the kick required so that the projectile enters a circular orbit around the moon at radius 0.5 from the moon. This kick will be applied at the first moment the projectile is at 0.5 units away from the moon. Note that we assume this radius is small enough that we can neglect the gravitational attraction from the planet to the projectile.

We can calculate what this kick must be by considering what we would like the velocity to be after the kick is applied.

After the kick is applied we want $\mathbf{v}_{proj} + \Delta\mathbf{v} = \mathbf{v}'_{proj} = \mathbf{v}_{moon} + \mathbf{v}_{circular}$

which is the addition of the moons velocity (so the projectile is in the moon's reference frame) and the velocity required for circular stable orbit.

Rearranging we see,

$$\Delta\mathbf{v} = \mathbf{v}_{moon} - \mathbf{v}_{proj} + \mathbf{v}_{circular}$$

and so the problem simplifies to finding what the velocity vector of a perfectly circular orbit is for a projectile at a distance r_{mp} and an angular separation θ_{mp} away from a moon of mass m_{moon}. We can see this situation in the diagram below (figure 3). From the diagram we can see that the velocity vector is

$$\mathbf{v}_{circular} = v_{circ}(\cos(90 - \theta_{mp}), \sin(90 - \theta_{mp}))$$

where we take θ_{mp} to always be a positive angle. Next we can calculate that for a circular orbit, the speed it must travel at is,

$$v_{circ} = \sqrt{\frac{m_{moon}}{r_{mp}}}.$$

Next, we use the complement angle identity for trig functions to yield that the velocity for the projectile in circular orbit is

$$\mathbf{v}_{circular} = \sqrt{\frac{m_{moon}}{r_{mp}}}(\sin(|\theta_{mp}|), \cos(|\theta_{mp}|))$$

and hence the kick required to send the projectile into circular orbit is given by,

$$\Delta\mathbf{v} = \mathbf{v}_{moon} - \mathbf{v}_{proj} + \sqrt{\frac{m_{moon}}{r_{mp}}}(\sin(|\theta_{mp}|), \cos(|\theta_{mp}|))$$

as required. This is a velocity vector and hence has both magnitude and direction, as required.

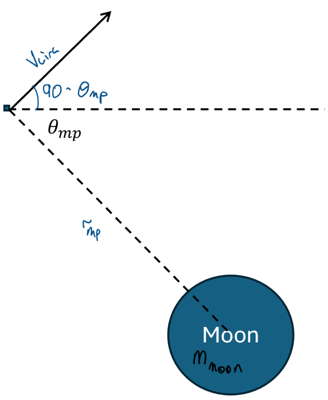


Figure 3: Diagram of the projectile in circular orbit around the moon with specified radial and angular separation. We use this to find the vector $\mathbf{v}_{circular}$ required for the projectile to enter circular motion.

3.5 Demonstrating the kick works for orbital transfer

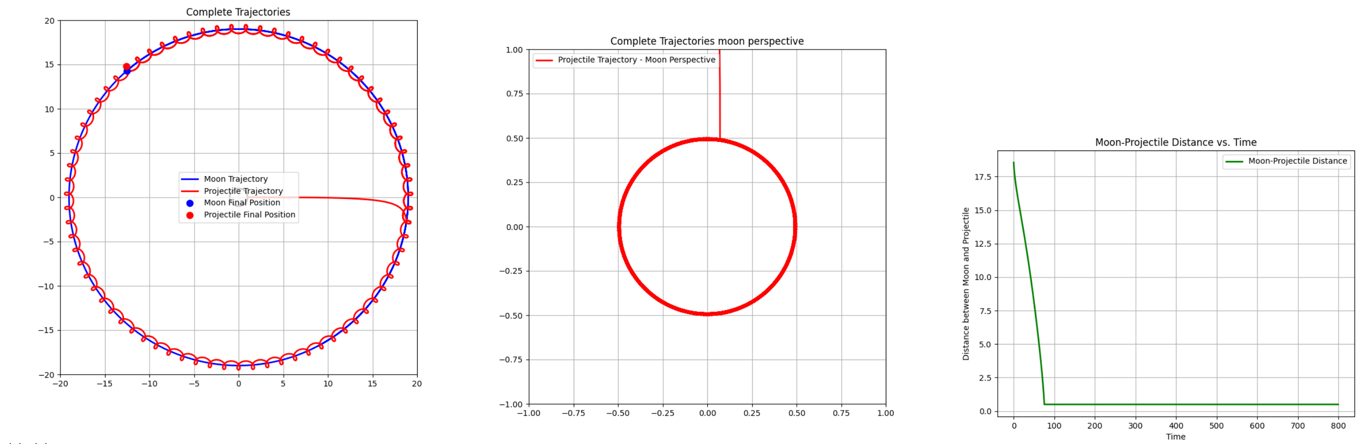
If we were to actually implement this, we would substitute in values for the mass and radial separation and then find that the kick is dependent on the situation (i.e. velocities and angular separation) when it is 0.5 units away from the moon but simplifies a bit to,

$$\Delta \mathbf{v} = \mathbf{v}_{moon} - \mathbf{v}_{proj} + \frac{1}{\sqrt{5}}(\sin(|\theta_{mp}|), \cos(|\theta_{mp}|)).$$

We can implement this into our code by activating this kick when the projectile first comes within 0.5 units of the moon and ensuring it only activates once. We can use the quantities of the moon and projectile to calculate the kick by finding the angular separation and the velocities required.

Doing so gives us the trajectories shown in figure 4(a) where we see the projectile enters stable circular orbit around the moon after a kick when it is nearby the moon. To see how circular the orbit is, we can calculate the motion of the projectile from the moon's reference frame by taking the moon's position from the projectile's position. This gives us figure 4(b) which is clearly a circular orbit centered around the moon with radius 0.5.

We can also plot the distance between the moon and the projectile and see that the kick makes the projectile in perfectly stable circular orbit around the moon at a fixed radius. This is shown in figure 4(c).



((a)) Trajectory of the moon and projectile in planet-centered coordinates. The projectile is transferred to a circular orbit around the moon via a kick.

((b)) Projectile's trajectory in the moon's coordinates, showing a clear circular orbit around the moon.

((c)) Distance between the moon and the projectile, showing how the kick results in a perfect circular orbit.

Figure 4: Visualization of the projectile's trajectory and final circular orbit around the moon.

4 Conclusion

Overall we explored numerical solutions of 2 and 3 body systems using c++ simulations.

We began by considering a projectile fired radially from the surface of a planet with a moon orbiting in circular motion. In this case we only considered the gravitational attraction between the moon and the planet and the projectile and the planet.

We were able to analytically calculate the initial conditions of both the projectile and the moon so that the projectile reached the moon's surface right as it passes by. This was demonstrated to work by using RK4 and a 2 body gravitational acceleration function.

When we consider 3 body acceleration we had to change the gravitational acceleration function to consider the attraction between one Particle object and 2 other Particle objects. Consequently, to modify RK4 we have to just change the acceleration function to use the 3 body function. Under the same initial conditions, the projectile is attracted to the moon and flung around it. Applying a kick which we can find analytically to the projectile allows it to enter perfect circular orbit around the moon. We demonstrated this numerically via a number of visual and quantitative metrics.

Overall, while we still make strong assumptions, such as the projectile having negligible mass and the kick occurring instantaneously, we have demonstrated how one may numerically solve ODEs in planetary systems starting from simple cases and building to a more complicated scenario. We also demonstrated how one may use the simulation to test theoretical results such as conditions for orbit transfers. In all, this project was demonstrated introductory topics to C++ coding and planetary simulations.