PHYS4070

Week 13 Worksheet — Exact Diagonalization in Quantum Optical Systems (Part II)

A two-level atom driven by a classical oscillating electric field is described by the following Hamiltonian (in a frame rotating at the field frequency and after making the rotating-wave approximation):

$$\hat{\mathcal{H}} = -\Delta\hat{\sigma}_z + \hbar\Omega\hat{\sigma}_x\,. \tag{1}$$

Here, $\Omega \in \mathbb{R}$ is proportional to the amplitude of the driving field, $\Delta \in \mathbb{R}$ is the detuning between drive and atom, and

$$\hat{\sigma}_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\sigma}_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

This system evolves according to the Schrödinger equation,

$$i\,\frac{d\psi}{dt} = \hat{\mathcal{H}}\psi\,, \tag{3}$$

where $\psi \equiv \psi(t)$ is a two-component vector. The exact time dynamics are given by

$$\psi(t) = e^{-i\hat{\mathcal{H}}t}\psi(0) = Ue^{-iDt}U^\dagger\psi(0)\,, \tag{4}$$

where D is a diagonal matrix of the eigenvalues of $\hat{\mathcal{H}}$, and U is a matrix of the corresponding eigenvectors.

(b) Solve for the time dynamics of an atom initially in the state $\psi(0) = (1,\,0)^T$ by using Eq. (4). Compare the results with those obtained from Runge–Kutta integration.

Now, consider two adjacent two-level atoms. When the atoms are sufficiently close together (less than one wavelength apart), the oscillating dipole moment $\mathbf{d}(t)$ of one atom produces an electric field that drives the neighbouring atom. Ignoring decay, the Hamiltonian governing this interaction and an on-resonance drive can be written as follows:

$$\hat{\mathcal{H}} = \hbar\Omega\left[\hat{\sigma}_x^{(0)} + \hat{\sigma}_x^{(1)}\right] + \hbar g\hat{\sigma}_z^{(0)}\hat{\sigma}_z^{(1)}\,, \tag{5}$$

where $g \in \mathbb{R}$ is the dipole-dipole interaction strength.

(c) By using the general approach outlined above [i.e., Eq. (4) with $\psi \equiv \psi(t)$ now a four-component vector], solve for the dynamics of this system, starting with both atoms in the state $\psi(0) = (1,\,0)^T$. Once again, compare the results with those obtained from Runge–Kutta integration.

Note, you should already have code that is able to diagonalize a symmetric matrix; this gives you U and D. It may be helpful to include a matrix multiplication function inside your matrix class, and also to adapt your class to handle matrices with complex entries.