Data Science, 2022 Tutorial - 4

Independent Component Analysis

1. Mixing statistically independent sources

A. Variance of the mixture is given as

$$\mathrm{var}(x) = \langle (x - \langle x \rangle)^2 \rangle$$

$$= \langle x^2 \rangle - \langle x \rangle^2$$

$$= \left\langle \left(\sum_i \omega_i s_i \right)^2 \right\rangle - \left\langle \sum_i \omega_i s_i \right\rangle^2$$

$$= \left\langle \left(\sum_i \omega_i s_i \right)^2 \right\rangle - \left(\sum_i \omega_i \langle s_i \rangle \right)^2$$

$$= \left\langle \left(\sum_i \omega_i s_i \right) \left(\sum_j \omega_j s_j \right) \right\rangle - \left(\sum_i \omega_i \langle s_i \rangle \right) \left(\sum_j \omega_j \langle s_j \rangle \right)$$

$$= \left\langle \sum_{i,j} \omega_i \omega_j s_i s_j \right\rangle - \sum_{i,j} \omega_i \omega_j \langle s_i \rangle \langle s_j \rangle$$

$$= \sum_{i,j} \omega_i \omega_j \langle s_i, s_j \rangle - \sum_{i,j} \omega_i \omega_j \langle s_i \rangle \langle s_j \rangle$$

$$= \sum_i \omega_i \omega_j \left(\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle \right)$$

$$+ \sum_{i,j \; i \neq j} \omega_i \omega_j \left(\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle \right)$$

$$= \sum_i \omega_i^2 \left(\langle s_i s_j \rangle - \langle s_i \rangle^2 \right) + \sum_{i,j \; i \neq j} \omega_i \omega_j \left(\langle s_i \rangle \langle s_j \rangle - \langle s_i \rangle \langle s_j \rangle \right)$$

s_i & s_j are statistically independent for
$i \neq j \Rightarrow \langle s_i \rangle \langle s_j \rangle - \langle s_i \rangle \langle s_j \rangle = 0$

& $Var(s_i) = 1$

$\therefore Var(x) = \sum_i \omega_i^2$

To guarantee that the mixture has unit variance

$Var(x) = 1$

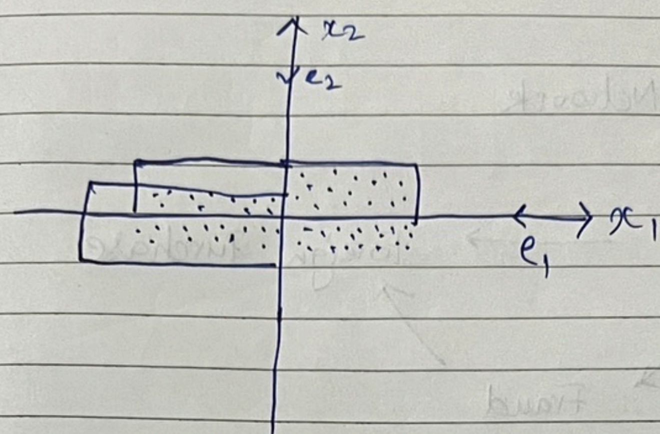
$\therefore \sum_i \omega_i^2 = 1$

$\therefore$ The following constraint has to be imposed on the weights ω_i for the mixture to have unit variance.

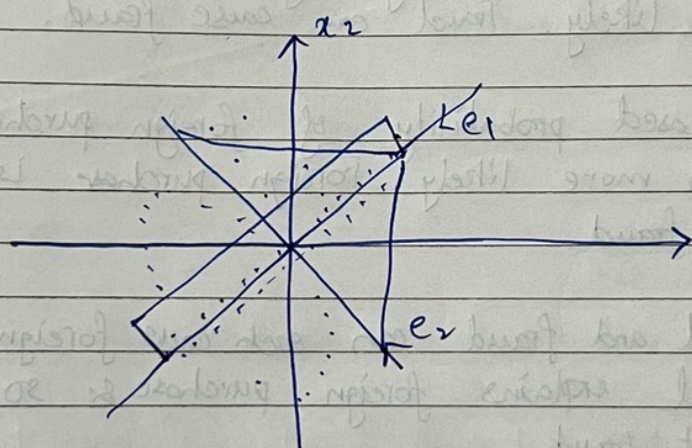
$$\boxed{\sum_i \omega_i^2 = 1}$$

Q2

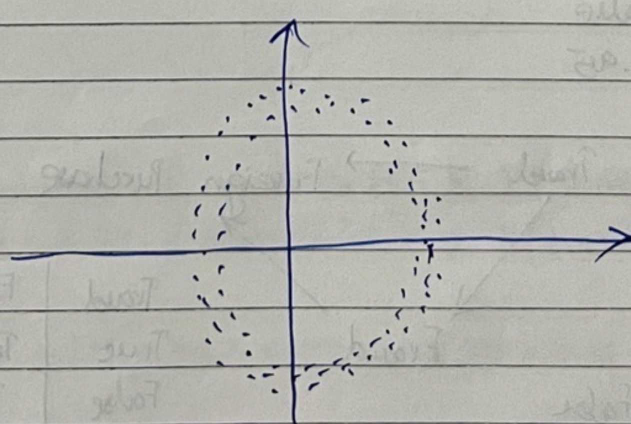
a)



b)



c)



$\Rightarrow$ Cannot be seperated into independent components.