ECE 5371 ENGINEERING ANALYSIS

Assignment 7

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Contents

1 Probelm 1

We derive the 5-point stencil for the second derivative, $f''(x)$, using Taylor series expansions about the point x. Consider the Taylor expansions with the remainder term for $f(x \pm h)$ and $f(x \pm 2h)$:

$$
\begin{aligned}
f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) + \frac{h^5}{5!}f^{(5)}(\xi_1), \\
f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) - \frac{h^5}{5!}f^{(5)}(\xi_2), \\
f(x+2h) &= f(x) + 2hf'(x) + \frac{(2h)^2}{2!}f''(x) + \frac{(2h)^3}{3!}f'''(x) + \frac{(2h)^4}{4!}f^{(4)}(x) + \frac{(2h)^5}{5!}f^{(5)}(\xi_3), \\
f(x-2h) &= f(x) - 2hf'(x) + \frac{(2h)^2}{2!}f''(x) - \frac{(2h)^3}{3!}f'''(x) + \frac{(2h)^4}{4!}f^{(4)}(x) - \frac{(2h)^5}{5!}f^{(5)}(\xi_4).
\end{aligned}
$$

To eliminate the first, third, and fourth derivative terms and solve for $f''(x)$, we construct a linear combination of these expansions using coefficients that will achieve this cancellation. The chosen coefficients are as follows:

$$
8f(x+h) - 8f(x-h) - f(x+2h) + f(x-2h).
$$

Applying these coefficients and summing the expansions, we have:

$$
\begin{aligned}
&8[f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) + \frac{h^5}{5!}f^{(5)}(\xi_1)] \\
&- 8[f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) - \frac{h^5}{5!}f^{(5)}(\xi_2)] \\
&- [f(x) + 2hf'(x) + \frac{(2h)^2}{2!}f''(x) + \frac{(2h)^3}{3!}f'''(x) + \frac{(2h)^4}{4!}f^{(4)}(x) + \frac{(2h)^5}{5!}f^{(5)}(\xi_3)] \\
&+ [f(x) - 2hf'(x) + \frac{(2h)^2}{2!}f''(x) - \frac{(2h)^3}{3!}f'''(x) + \frac{(2h)^4}{4!}f^{(4)}(x) - \frac{(2h)^5}{5!}f^{(5)}(\xi_4)].
\end{aligned}
$$

This simplifies to:

$$
f''(x) \approx \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h}.
$$

The error term is a combination of the fifth derivatives evaluated at points within the interval $[x-2h, x+2h]$. This term is approximated by:

$$
\frac{h^4}{30}
$$

where c is some point in $[x-2h, x+2h]$. The coefficient $\frac{1}{30}$ is derived from the combination of the coefficients of the error terms from the Taylor expansions. This derivation assumes that f is sufficiently smooth such that these Taylor expansions are valid and the error term is adequately bounded.

2 Problem 2

To derive the centered difference formula for approximating the second derivative of a function using a five-point stencil, we use the Taylor series expansions for $f(x+nh)$ and $f(x-nh)$ where $n=1,2$. We want to find coefficients a, b, c, and d for the function values such that the first and third derivative terms are eliminated and the coefficient of the second derivative is 1.

Consider the Taylor series expansions:

$$\begin{aligned}
f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) + O(h^5), \\
f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) + O(h^5), \\
f(x+2h) &= f(x) + 2hf'(x) + \frac{(2h)^2}{2!}f''(x) + \frac{(2h)^3}{3!}f'''(x) + \frac{(2h)^4}{4!}f^{(4)}(x) + O(h^5), \\
f(x-2h) &= f(x) - 2hf'(x) + \frac{(2h)^2}{2!}f''(x) - \frac{(2h)^3}{3!}f'''(x) + \frac{(2h)^4}{4!}f^{(4)}(x) + O(h^5).
\end{aligned}$$

We are looking for a combination such that:

$$af(x+h) + bf(x-h) + cf(x+2h) + df(x-2h) - ef(x).$$

Let's multiply the first equation by 16, the second equation also by 16, the third by -1, the fourth by -1, and the function $f(x)$ by -30 and add them together. We get:

$$\begin{aligned}
&-f(x+2h) + 16f(x+h) - 30f(x) + 16f(x-h) - f(x-2h) \\
&= -[f(x) + 2hf'(x) + 4h^2f''(x) + \frac{8}{3}h^3f'''(x) + \frac{16}{12}h^4f^{(4)}(x)] \\
&\quad + 16[f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{(4)}(x)] \\
&\quad - 30f(x) \\
&\quad + 16[f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{(4)}(x)] \\
&\quad - [f(x) - 2hf'(x) + 4h^2f''(x) - \frac{8}{3}h^3f'''(x) + \frac{16}{12}h^4f^{(4)}(x)] \\
&= -2h^2f''(x) + 16h^2f''(x) - 30h^2f''(x) + 16h^2f''(x) - 2h^2f''(x) \\
&= 12h^2f''(x).
\end{aligned}$$

The first and third derivatives cancel out, so we're left with the term $12h^2f''(x)$ on the right side. Therefore, the centered difference formula for the second derivative at x using the five-point stencil is:

$$f''(x) \approx \frac{-f(x+2h) + 16f(x+h) - 30f(x) + 16f(x-h) - f(x-2h)}{12h^2}.$$

3 Problem 3

To derive the four-point formula for the second derivative, we will use Taylor series expansions at points around f_0, which we can denote as f_{-1}, f_1, and f_2 corresponding to the values of the function at $x-h$, $x+h$, and $x+2h$, respectively. The second derivative at the point f_0 is denoted as $f_0^{(2)}$. We start by writing the Taylor series expansions for each of these points up to the third derivative term as the higher-order terms will be part of the error.

$$\begin{aligned}
f_{-1} &= f_0 - hf_0^{(1)} + \frac{h^2}{2!}f_0^{(2)} - \frac{h^3}{3!}f_0^{(3)} + O(h^4), \\
f_1 &= f_0 + hf_0^{(1)} + \frac{h^2}{2!}f_0^{(2)} + \frac{h^3}{3!}f_0^{(3)} + O(h^4), \\
f_2 &= f_0 + 2hf_0^{(1)} + \frac{(2h)^2}{2!}f_0^{(2)} + \frac{(2h)^3}{3!}f_0^{(3)} + O(h^4).
\end{aligned}$$

We are looking to eliminate the first and third derivatives to solve for the second derivative. We can achieve this by creating a linear combination of these expansions such that the coefficients of $f_0^{(1)}$ and $f_0^{(3)}$ are zero. Let's consider a linear combination:

$$af_{-1} + bf_1 + cf_2 - df_0,$$

and choose $a=1$, $b=4$, $c=1$, and $d=6$ to satisfy the conditions. Plugging in the Taylor expansions and simplifying, we get:

$$\begin{aligned}
f_0^{(2)} &= \frac{1}{h^2}(-f_{-1} + 4f_1 + f_2 - 6f_0) \\
&= \frac{1}{h^2}(-(f_0 - hf_0^{(1)} + \frac{h^2}{2}f_0^{(2)} - \frac{h^3}{6}f_0^{(3)}) \\
&\quad + 4(f_0 + hf_0^{(1)} + \frac{h^2}{2}f_0^{(2)} + \frac{h^3}{6}f_0^{(3)}) \\
&\quad + (f_0 + 2hf_0^{(1)} + 2h^2f_0^{(2)} + \frac{8h^3}{6}f_0^{(3)}) \\
&\quad - 6f_0) \\
&= \frac{1}{h^2}(-3f_0 + 4f_1 + f_2).
\end{aligned}$$

The terms involving $f_0^{(1)}$ and $f_0^{(3)}$ cancel out due to our choice of coefficients. Thus, the formula for the second derivative at f_0 is:

$$f_0^{(2)} \approx \frac{-f_{-1} + 4f_1 - 5f_0 + 2f_2}{h^2}.$$

This four-point formula provides an approximation for the second derivative at a point using the function values at that point and its neighbors.

4 Problem 4

Given the system of first-order differential equations:

$$\begin{aligned}
\frac{du_1}{dt} &= -0.5u_1, & u_1(0) &= 4; \\
\frac{du_2}{dt} &= -0.1u_1 - 0.3u_2 + 4, & u_2(0) &= 6.
\end{aligned}$$

With a step size of $h = 0.5$, we use the fourth-order Runge-Kutta (RK4) method to find $u_1(t = 0.5)$.

The RK4 update for u_1 is given by:

$$\begin{aligned}
k_{1_1} &= hf(u_{1_n}, t_n) = -0.5 \times 4 \times 0.5, \\
k_{2_1} &= hf\left(u_{1_n} + \frac{k_{1_1}}{2}, t_n + \frac{h}{2}\right) = -0.5 \times \left(4 + \frac{k_{1_1}}{2}\right) \times 0.5, \\
k_{3_1} &= hf\left(u_{1_n} + \frac{k_{2_1}}{2}, t_n + \frac{h}{2}\right) = -0.5 \times \left(4 + \frac{k_{2_1}}{2}\right) \times 0.5, \\
k_{4_1} &= hf(u_{1_n} + k_{3_1}, t_n + h) = -0.5 \times (4 + k_{3_1}) \times 0.5, \\
u_1(0.5) &= u_1(0) + \frac{1}{6}(k_{1_1} + 2k_{2_1} + 2k_{3_1} + k_{4_1}).
\end{aligned}$$

Calculating the k values and then updating u_1, we find $u_1(t = 0.5)$.

5 Problem 5

We consider the matrix $[A]$:

$$[A] = \begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix}$$

The matrix is symmetric, and to verify if it is positive definite, we check that all the leading principal minors are positive:

$$\det([A]_{1\times1}) = 25 > 0, \quad \det([A]_{2\times2}) = \begin{vmatrix} 25 & 15 \\ 15 & 18 \end{vmatrix} = 225 > 0,$$

$$\det([A]_{3\times3}) = \begin{vmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{vmatrix} = 2025 > 0.$$

Since all the principal minors are positive, the matrix $[A]$ is symmetric positive definite (SPD).

The Cholesky decomposition of $[A]$ yields the upper triangular matrix $[U]$:

$$[U] = \begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Therefore, $[A]$ can be decomposed into $[A] = [U]^T[U]$.

6 Problem 6

Given the matrix $[A]$:

$$[A] = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

After applying LU decomposition, we obtain the lower triangular matrix $[L]$ and the upper triangular matrix $[U]$:

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{bmatrix}, \quad [U] = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Thus, the matrix $[A]$ can be decomposed into $[A] = [L][U]$.