ECE 5371 ENGINEERING ANALYSIS

Assignment 4

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1 Problem 1

Given the line integral:

$$\int_C (y^2\,dx + x\,dy)$$

where C is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$, we parameterize the curve using y as the parameter, giving $x = 4 - y^2$ and $dx = -2y\,dy$. This transforms the integral into:

$$\int_{-3}^{2} (-2y^3 - y^2 + 4)\,dy$$

Evaluating the integral, we find:

$$\int (-2y^3 - y^2 + 4)\,dy = -\frac{1}{2}y^4 - \frac{1}{3}y^3 + 4y$$

Evaluating this from -3 to 2, we get:

$$\left(-\frac{1}{2}(2)^4 - \frac{1}{3}(2)^3 + 4(2)\right) - \left(-\frac{1}{2}(-3)^4 - \frac{1}{3}(-3)^3 + 4(-3)\right) = \frac{82}{3}$$

Thus, the solution to the integral is $\frac{82}{3}$.

2 Problem 2

Given the line integral to evaluate over the ellipse $\frac{x^2}{4} + y^2 = 1$ in the counterclockwise direction:

$$\int_C (3x - 5y)\,dx + (x - 6y)\,dy$$

The ellipse can be parametrized as $x = 2\cos(t)$, $y = \sin(t)$, with $0 < t < 2\pi$ because the ellipse is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a = 2$ and $b = 1$.

The derivatives with respect to t are $\frac{dx}{dt} = -2\sin(t)$ and $\frac{dy}{dt} = \cos(t)$.

Substituting these into the integral gives:

$$\int_0^{2\pi} [(3(2\cos(t)) - 5\sin(t))(-2\sin(t)) + ((2\cos(t)) - 6\sin(t))\cos(t)]\,dt$$

Simplifying the expression:

$$= \int_0^{2\pi} [(-6\cos(t)\sin(t) + 10\sin^2(t)) + (2\cos^2(t) - 6\sin(t)\cos(t))]\,dt$$

$$= \int_0^{2\pi} (10\sin^2(t) + 2\cos^2(t) - 12\sin(t)\cos(t))\,dt$$

Using trigonometric identities $\sin^2(t) = \frac{1-\cos(2t)}{2}$ and $\cos^2(t) = \frac{1+\cos(2t)}{2}$:

$$= \int_0^{2\pi} \left(5 - 5\cos(2t) + 1 + \cos(2t) - 12\sin(t)\cos(t)\right)\,dt$$

$$= \int_0^{2\pi} (6 - 4\cos(2t) - 12\sin(t)\cos(t))\,dt$$

$$= \int_0^{2\pi} 6\,dt - \int_0^{2\pi} 4\cos(2t)\,dt - \int_0^{2\pi} 12\sin(t)\cos(t)\,dt$$

The integrals of $4\cos(2t)$ and $12\sin(t)\cos(t)$ over a period 0 to 2π equal zero due to their periodic properties. Thus, the result of the integral simplifies to:

$$\int_0^{2\pi} 6\,dt = 12\pi$$

Therefore, the evaluation of the line integral over the given ellipse is 12π.

3 Problem 3

Consider the vector field $\mathbf{V} = y^2\mathbf{i} - xy\mathbf{j}$ along the curve $(x, 0.5x^2 + x - 3)$, with x ranging from 1 to 3. The vector line integral $\int \mathbf{V} \cdot d\mathbf{l}$ is given by

$$\int_C (y^2\,dx - xy\,dy)$$

where $y = 0.5x^2 + x - 3$. Therefore, $dx = dx$ and $dy = (x+1)\,dx$, as $dy/dx = x + 1$.

Substituting y and differentials dx and dy into the integral, we obtain

$$\int_1^3 \left((0.5x^2 + x - 3)^2 - x(0.5x^2 + x - 3)(x+1)\right)\,dx$$

Expanding and simplifying the expression inside the integral yields

$$\int_1^3 \left(0.25x^4 + x^3 - 6x^2 + x^2 - 3x - x^3 - x^2(0.5x^2 + x - 3)\right)\,dx$$

$$= \int_1^3 \left(0.25x^4 - 6x^2 - 3x - 0.5x^4 - x^3 + 3x^2\right)\,dx$$

$$= \int_1^3 \left(-0.25x^4 - x^3 - 3x^2 - 3x\right)\,dx$$

Computing the integral, we find

$$\left[-\frac{0.25}{5}x^5 - \frac{1}{4}x^4 - x^3 - \frac{3}{2}x^2\right]_1^3$$

$$= \left(-\frac{0.05}{5}(3)^5 - \frac{1}{4}(3)^4 - (3)^3 - \frac{3}{2}(3)^2\right) - \left(-\frac{0.05}{5}(1)^5 - \frac{1}{4}(1)^4 - (1)^3 - \frac{3}{2}(1)^2\right)$$

Simplify to obtain the final value of the integral, which represents the vector line integral of $\mathbf{V}$ along the given curve.

4 Problem 4

Given the function $f = xyz$, we transform it into cylindrical coordinates (ρ, ϕ, z), where the relations between Cartesian and cylindrical coordinates are given by $x = \rho\cos(\phi)$, $y = \rho\sin(\phi)$, and $z = z$. Thus, the function f in cylindrical coordinates becomes $f(\rho, \phi, z) = \rho^2 z\cos(\phi)\sin(\phi)$.

The gradient of a scalar field in cylindrical coordinates is given by:

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}$$

For $f(\rho, \phi, z) = \rho^2 z \cos(\phi)\sin(\phi)$, the partial derivatives are:

$$\begin{aligned}
\frac{\partial f}{\partial \rho} &= 2\rho z \cos(\phi)\sin(\phi) \\
\frac{\partial f}{\partial \phi} &= \rho^2 z(-\sin^2(\phi) + \cos^2(\phi)) \\
\frac{\partial f}{\partial z} &= \rho^2 \cos(\phi)\sin(\phi)
\end{aligned}$$

Thus, the gradient in cylindrical coordinates is:

$$\nabla f = 2\rho z \cos(\phi)\sin(\phi)\hat{\rho} + \rho z(-\sin^2(\phi) + \cos^2(\phi))\hat{\phi} + \rho^2 \cos(\phi)\sin(\phi)\hat{z}$$

5 Problem 5

Given the vector field in cylindrical coordinates

$$\mathbf{F} = \rho^3 \mathbf{a}_\rho + \rho z \mathbf{a}_\theta + \rho z \sin(\theta)\mathbf{a}_z$$

where

$$\mathbf{a}_\rho = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}, \quad \mathbf{a}_\theta = -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j}, \quad \mathbf{a}_z = \mathbf{k}$$

The divergence in cylindrical coordinates is given by

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

Substituting $F_\rho = \rho^3$, $F_\theta = \rho z$, and $F_z = \rho z \sin(\theta)$, we find

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho}\frac{\partial(\rho^3)}{\partial \rho} + \frac{1}{\rho}\frac{\partial(\rho z)}{\partial \theta} + \frac{\partial(\rho z \sin(\theta))}{\partial z}$$

$$= 3\rho^2 + 0 + \sin(\theta)$$

Next, converting $\mathbf{F}$ to Cartesian coordinates:

$$\mathbf{F} = (\rho^3 \cos(\theta), \rho^3 \sin(\theta), \rho z \sin(\theta))$$

$$= (x^3, y^3, xz \sin(\arctan(\frac{y}{x})))$$

The divergence in Cartesian coordinates is given by

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Substituting $F_x = x^3$, $F_y = y^3$, and $F_z = xz\sin(\arctan(\frac{y}{x}))$, we compute

$$\nabla \cdot \mathbf{F} = 3x^2 + 3y^2 + \sin(\arctan(\frac{y}{x}))$$

Given $x = \rho\cos(\theta)$ and $y = \rho\sin(\theta)$, it follows that $x^2 + y^2 = \rho^2$, hence

$$\nabla \cdot \mathbf{F} = 3\rho^2 + \sin(\theta)$$

This shows that the divergence of $\mathbf{F}$ is consistent in both cylindrical and Cartesian coordinates, demonstrating the invariance of divergence under coordinate transformation.

6 Problem 6

The gradient of a scalar field f in spherical coordinates (r, θ, φ) is given by:

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \varphi}\hat{\varphi}$$

Given the scalar field $f = r^2\sin(\theta + \varphi)$, we compute the partial derivatives with respect to r, θ, and φ: 1. $\frac{\partial f}{\partial r} = 2r\sin(\theta + \varphi)$ 2. $\frac{\partial f}{\partial \theta} = r^2\cos(\theta + \varphi)$ 3. $\frac{\partial f}{\partial \varphi} = r^2\cos(\theta + \varphi)$
Substituting these derivatives into the gradient formula yields:

$$\nabla f = 2r\sin(\theta + \varphi)\hat{r} + \frac{1}{r}r^2\cos(\theta + \varphi)\hat{\theta} + \frac{1}{r\sin(\theta)}r^2\cos(\theta + \varphi)\hat{\varphi}$$

Simplifying the expression gives us the gradient of f in spherical coordinates:

$$\nabla f = 2r\sin(\theta + \varphi)\hat{r} + r\cos(\theta + \varphi)\hat{\theta} + \frac{r}{\sin(\theta)}\cos(\theta + \varphi)\hat{\varphi}$$

This is the gradient of the function $f = r^2\sin(\theta + \varphi)$ in spherical coordinates.

7 Problem 7

The prolate spheroidal coordinates (ξ, η, ϕ) are related to Cartesian coordinates (x, y, z) by the transformations:

$$\begin{aligned} x &= a\sinh(\xi)\sin(\eta)\cos(\phi), \\ y &= a\sinh(\xi)\sin(\eta)\sin(\phi), \\ z &= a\cosh(\xi)\cos(\eta). \end{aligned}$$

To find the metric coefficients h_ξ, h_η, and h_ϕ, we compute the derivatives of the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ with respect to ξ, η, and ϕ respectively.

For h_ξ:

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial \xi} &= \frac{\partial x}{\partial \xi}\mathbf{i} + \frac{\partial y}{\partial \xi}\mathbf{j} + \frac{\partial z}{\partial \xi}\mathbf{k} \\
&= a\cosh(\xi)\sin(\eta)\cos(\phi)\mathbf{i} + a\cosh(\xi)\sin(\eta)\sin(\phi)\mathbf{j} + a\sinh(\xi)\cos(\eta)\mathbf{k}, \\
\left|\frac{\partial \mathbf{r}}{\partial \xi}\right| &= a\left[\sinh^2(\xi) + \sin^2(\eta)\right]^{1/2}.
\end{aligned}$$

Similarly, for h_η:

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial \eta} &= \frac{\partial x}{\partial \eta}\mathbf{i} + \frac{\partial y}{\partial \eta}\mathbf{j} + \frac{\partial z}{\partial \eta}\mathbf{k} \\
&= a\sinh(\xi)\cos(\eta)\cos(\phi)\mathbf{i} + a\sinh(\xi)\cos(\eta)\sin(\phi)\mathbf{j} - a\cosh(\xi)\sin(\eta)\mathbf{k}, \\
\left|\frac{\partial \mathbf{r}}{\partial \eta}\right| &= a\left[\sinh^2(\xi) + \sin^2(\eta)\right]^{1/2}.
\end{aligned}$$

And for h_ϕ:

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial \phi} &= \frac{\partial x}{\partial \phi}\mathbf{i} + \frac{\partial y}{\partial \phi}\mathbf{j} + \frac{\partial z}{\partial \phi}\mathbf{k} \\
&= -a\sinh(\xi)\sin(\eta)\sin(\phi)\mathbf{i} + a\sinh(\xi)\sin(\eta)\cos(\phi)\mathbf{j} + 0\mathbf{k}, \\
\left|\frac{\partial \mathbf{r}}{\partial \phi}\right| &= a\sinh(\xi)\sin(\eta).
\end{aligned}$$

Therefore, the metric coefficients are given as:

$$\begin{aligned}
h_\xi &= a\left[\sinh^2(\xi) + \sin^2(\eta)\right]^{1/2}, \\
h_\eta &= a\left[\sinh^2(\xi) + \sin^2(\eta)\right]^{1/2}, \\
h_\phi &= a\sinh(\xi)\sin(\eta).
\end{aligned}$$