Why Transform?

1. makes import information more accessible
2. Operations on signals become easier to perform

- high frequency will correspond to abrupt changes - sharp transitions

- low frequency will correspond to smooth changes - smooth transition.

- The fourier transform makes frequency accessible.

- Z, laplace, Fourier, wavelet all make frequency accessible.
- frequency is very important because all our senses depend on it touch, hearing, vision

convolution in the fourier Domain is just multiplication

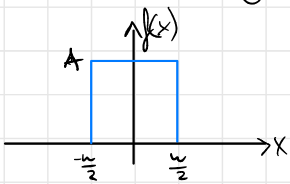
- The fourier transform's biggest short comming is that it does not give access to the time/spacial & frequency simultaneously

Continuous Fourier Transform of a signal, $f(x)$

$$\underbrace{F(u)}_{\text{Fourier frequency Domain signal}} = \int_{-\infty}^{\infty} \underbrace{f(x)}_{\text{Spacial Domain Signal (Image)}} \overbrace{e^{-2j\pi u x}}^{\text{eulers sinusoid identity}} dx$$

$\longrightarrow$ x: spatial coordinate (units of length)

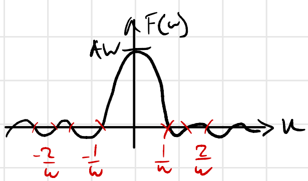
u: spatial frequency (units of cycles/length)



$$F(u) = \int_{-\frac{w}{2}}^{w/2} A e^{-j2\pi u x}\, dx = A\cancel{w} \frac{\sin(\pi u w)}{\pi u \cancel{w}}$$

$$= A w \operatorname{sinc}(\pi u w)$$

$! \ \frac{\sin(x)}{x} = \operatorname{sinc}(x)$



- The fourier transform of a rectangular pulse is a sinc function

Quiz 1 Notes: Will never ask something we have not discussed Directly in class. Will always apply topic to new question in test

Question 1: a)
- Histogram equalization integrates and is summing so the curve should reflect that
  - monotonically increasing, $0 < t < L-1$ Doesn't matter
- The sum should increase rapidly where the histogram values are high
- The slope will slow down in the sum where the histogram values are low

Question 1: b) The mapping shows the average intensity value must increase because the low values of the map get set high.

Question 2:) The image will be binary because

$$S_0 = \frac{n_{r_0}}{N}(L-1) \qquad S_1 = \frac{n_{r_0}+n_{r_1}}{N}(L-1) = L-1$$

Image will be binary but not the same

Question 3:) Use the gradient direction and mask the pixels with that direction.

- The fourier transform has a duality property.
  - $\mathcal{F}$ of squarewave is sinc func
  - $\mathcal{F}^{-1}$ of squarewave is sincfunc

Prooving

$F(-\omega) = F^*(\omega)$

$x(t) = a + bj$

$\overline{x(t)} = a - bj$

1. $F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$
2. $F(-\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j(-\omega)t}\, dt$
3. $F(-\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{j\omega t}\, dt$

complex component

$\overline{F(\omega)} = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$

real component (a)

$\overline{F(\omega)} = \int_{-\infty}^{\infty} f(t) e^{j\omega t}\, dt$

$=$

$\therefore F(-\omega) = \overline{F(\omega)}$

$f(x) \xrightarrow{\tilde{F}} F(u)$

$\hookrightarrow$ $F(u)$ is generally a complex quanitty $\rightarrow$ The sinc function example above is only real

$|F(u)| = \sqrt{a^2 + b^2} \leftarrow$ magnitude $\leftarrow$ (Spectrum)

$\angle F(u) = \tan^{-1}\left(\frac{b}{a}\right) \leftarrow$ Phase (Angle)

① If $f(x)$ is real then $\underbrace{F(-u) = F^*(u)}_{\text{Proove this}}$ then $\underbrace{|F(-u)| = |F(u)|}_{\text{Even symmetric}}$

$(a+bi)^* = a-bi$

- Realness and even symmetry are fourier transform pairs

- We start with real images

② $f(ax) \xrightarrow{\tilde{F}} \frac{1}{|\alpha|} F\left(\frac{u}{a}\right)$ $a>1$ contracts frequency
$a<1$ expands frequency

1-D Discrete Fourier Transform

Can be in either DFT or IDFT but not both

DFT $$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad u = 0, 1, 2, \cdots L-1$$

IDFT $$f(x) = \sum_{x=0}^{M-1} F(u) e^{j2\pi ux/M} \qquad x = 0, 1, 2, 3, \cdots L-1$$

- The input and output have the same length

- The Fourier transform of an image is almost always complex - why?

  - The image would have to be symmetric for the transform to be real

$e^{j\omega t} = \underset{\substack{\uparrow \\ a}}{\cos(\omega t)} + j \underset{\substack{\uparrow \\ b}}{\sin(\omega t)}$

$a \rightarrow$ cos component is in phase

$b \rightarrow$ sin component is out of phase

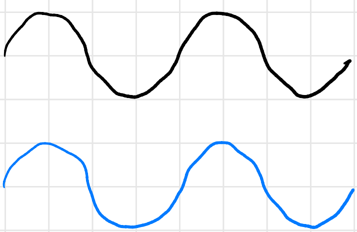
Magnitude tells amplitude of freq components

Phase tells shift of freq components

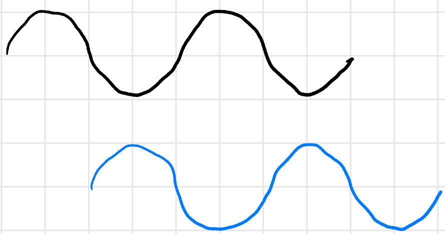
- If a freq component has a phase shift it will be shown as a complex # because sin & cos are not alligned (forming a complex valued result

- Imgs typically have spatial asymmetries (from lighting, edges, textures, etc) which dont exibt perfect symmetry the irregularities result in a mix of positive or negative frequency components which manifest in the FT as complex #'s

- If an Image has perfect symmetry the FT may be real at some frequences but in general, natural imgs do not have perfect symmetry.



In phase



Out of phase

- In phase components occur when the image has symmetry or smooth consistent patterns
  - Allignment of frequency components is uniform accross the image

- Out of phase components occure when sharp transitions like edges, corners, & abrupt changes in intensity occure
  - causes mis allignment of freq components leading to phase shift.

$$F(0) = \frac{1}{M} \sum_{x=1}^{M-1} f(x)$$ ← Mean values of signal

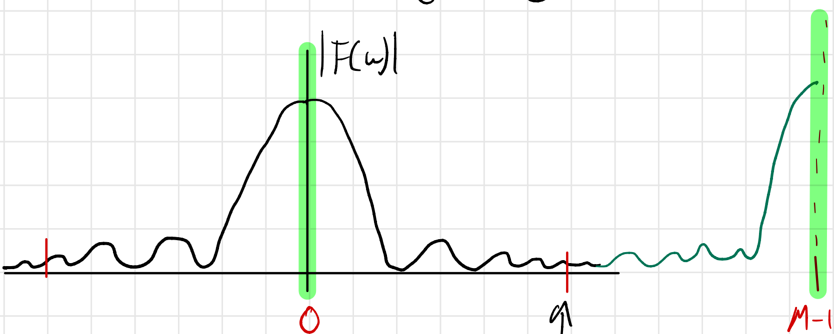
$$\boxed{F(u+M) = F(u)}$$ ← Proove this, if is because F(u) is periodic & M is the period

$$\underbrace{f(x) \otimes g(x)}_{\text{Instead of convolving in the Spatial Domain}} \overset{\tilde{F}}{\Longrightarrow} \underbrace{F(u) \cdot G(u)}_{\text{Multiply in the frequency Domain}}$$

$$f(x) \overset{F}{\equiv} F(u)$$

$$\odot \overset{g^{-1}}{\Longrightarrow} f(x) \otimes g(x)$$

$$g(x) \overset{F}{\equiv} g(u)$$



$f(x) \overset{F}{\Longrightarrow} F(u)$

$f(x)$ is real

- Does this plot make sense? - yes because of its symmetry. → real

fft (fast fourier transform)

fft(f(x))

fft shift fixes this

- The formula computes from 0 → M-1

2D Discrete Fourier Transform

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

$u = 0, 1, \ldots, M-1$
$v = 0, 1, \ldots, N-1$

Output Image (M×N) u: 0 .. N-1, v: 0 .. M-1

Input Img M×N

$$F(u,v) = \underbrace{\sum_{x=0}^{N-1} \underbrace{\left(\sum_{y=0}^{M-1} f(x,y)\, e^{-j\frac{2\pi v y}{M}} \right)}_{\text{1D DFT along columns}} e^{-j\frac{2\pi u x}{N}}}_{\text{1D DFT along rows}}$$

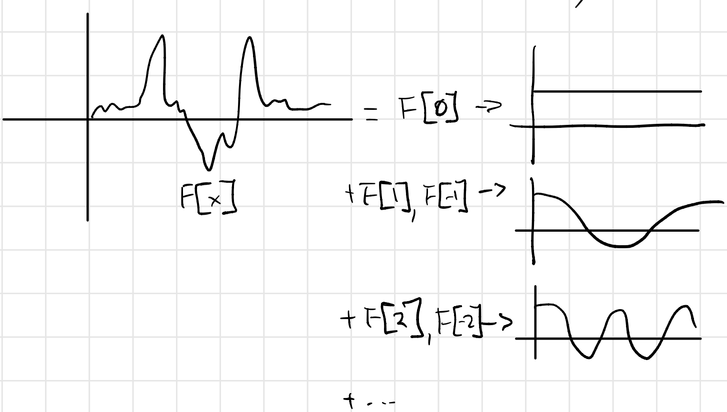
The 2D DFT is separable

$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{N-1} \sum_{v=0}^{M-1} F[u,v]\, e^{j\frac{2\pi u x}{N} + j\frac{2\pi v y}{M}}$$

! if $M=N$ you may see N^2

As with the 1D DFT, 2D DFT is like a decomposition of an image into complex exponentials (sines & cosines)

$$e^{-j\frac{2\pi u x}{N}} = \cos\left(\frac{2\pi u x}{N}\right) - j\sin\left(\frac{2\pi u x}{N}\right)$$



FT Properties

<u>Shift:</u> $g(x,y) = F(x-a, y-b)$

$$G(u,v) = F(u,v)\, \underbrace{e^{-2\pi j\left[\frac{au}{N} + \frac{bv}{M}\right]}}_{\text{complex phase shift but the Magnitude remains the same}}$$

$$|G(u,v)| = |F(u,v)|$$

<u>Scaling/Flip:</u> $g(x,y) = aF(x,y)$ $\qquad g(x,y) = F(ax, by)$

$$G(u,v) = aF(u,v) \qquad G(u,v) = \frac{1}{|ab|} F\left(\frac{u}{a}, \frac{v}{b}\right)$$

If a or $b = -1$ → Flip in spacial = Flip in freq. Domain

If a or $b < 1$ & $a/b > 0$, compress spatial, expand freq

If a or $b > 1$, expand spatial, compress freq

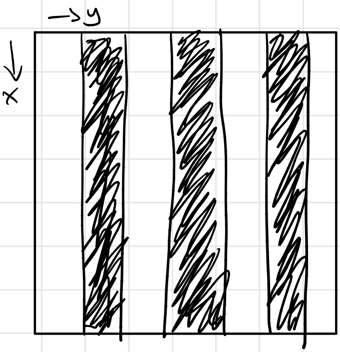
<u>Rotation:</u> If $g(x,y) = F(x,y)$ rotated by $\theta°$ CCW

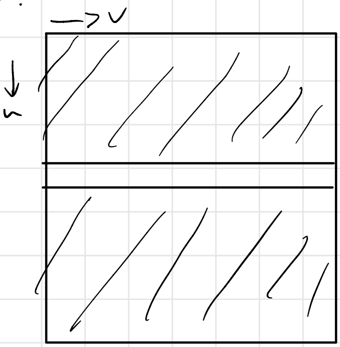
Then $G(u,v) = F(u,v)$ rotated by $\theta°$ CCW

<u>Convolution:</u> $h(x,y) = F(x,y) \otimes g(x,y)$

$$H(u,v) = F(u,v) \cdot G(u,v)$$

Why Do Strong Edges in images at $\theta°$ show a strong contribution to the DFT at $\theta° + 90°$?





[255 0 255 0 255 0]

DFT →

DC

[0 0 0 1 0 0 0 0 0 1 0 0 0]

DC

→

$$\begin{bmatrix} 0 \\ \text{Content} \\ 0 \end{bmatrix}$$

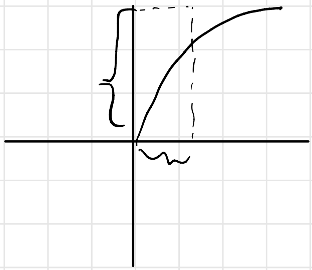
! DFT is seperable

$f: \boxed{M \times N} \longrightarrow \boxed{FFT} \longrightarrow F: \boxed{M \times N}$

↑ you can not directly display this because it is a complex number

you can view the spectrum (magnitude) or the phase

- To view magnitude, you must scale it (use linear scale)
- The low frequencies typically overwhelm the high frequencies so use logarithmic scaling



`log(1+abs(fftshift(fft2(t))))`

- Typically phase isnt used visually but it does have important info

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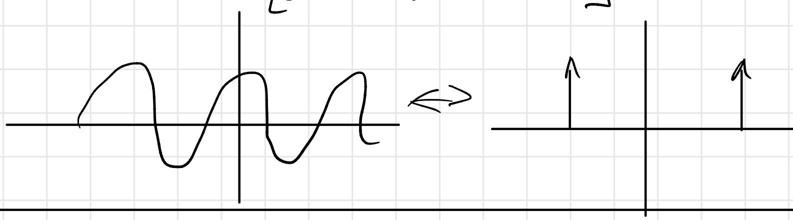
Do MATLAB Exercises w/ Synthetic Images

$\cos(w x) \overset{\mathcal{F}}{\Longleftrightarrow} [\delta(w+w_0) + \delta(w - w_0)]$



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fft vs. DFT

- Seperability means no-one does a 2D DFT, you do two 1D DFTs

$f(x,y) \rightarrow \underset{\text{1D FFT}}{\boxed{\text{columns}}} \longrightarrow \underset{\text{1D FFT}}{\boxed{\text{rows}}} \rightarrow F(u,v)$

Do row wise 1D FFT to the column wise 1D FFT

- Dont Do 1D FFT, its fast, and precise (not approximating)

This reduces computation from $M \times M$ to $M \cdot 2$

frequency Domain filtering (blurring & sharpening)

Convolution $\xRightarrow{F}$ Multiplying

$$g(x) = f(x) \otimes h(x)$$

out   in   kernal

-> FT both sides

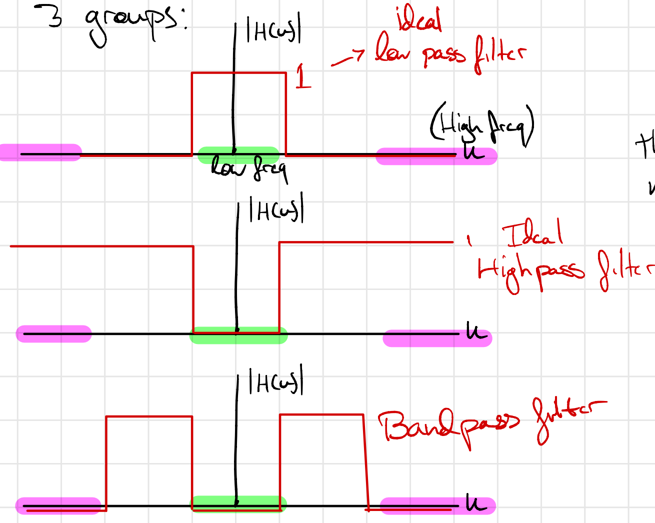
$$G(u) = F(u) \odot H(u)$$

↑ element by element multiplication.

! Multiply to complex quantity not the Magnitude of phase!

what does H(u) look like?

3 groups:



these values do not need to be zero & 1

Otsu's thresholding techique -> assumes bimodal hist & Places thresh in valley of the two nodes

↳ Binary thresh technique.

- Binarization first was important because it isolates the object right away which reduces noise.

$f(x,y) \xrightarrow{\mathcal{F}} F(x,y) \triangleq |F(u,v)| \cdot \angle F(u,v) \Rightarrow 1 \cdot \angle F(u,v) \rightarrow f(x,y)$

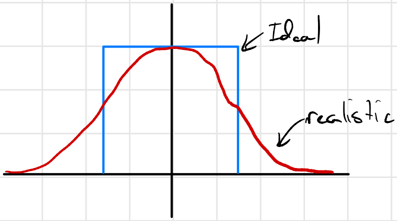
↳ shows phase structure

Phase gives structure info

↳ $|F(u,v)| \cdot \angle 0 \rightarrow f(x,y)$

Magnitude gives intensity information

↳ shows structureless magnitude.



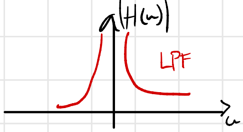
- computers can do the discretized Ideal filter but we don't actually want to do that.

Integration w/ F.T. and filters

integration of $f(x) \xrightarrow{\mathcal{F}} \frac{F(w)}{jw}$

differentiation of $f(x) \xrightarrow{\mathcal{F}} jw F(w)$

$G(w) = F(w) \cdot \frac{1}{jw} \longrightarrow$



would be integral

- integration is a lowpass filter
  - Reduces noise, abrupt change

$G(w) = F(w) \cdot jw \longrightarrow$

- Differentiation is a highpass filter
  - Details, texture has high frequency

→ to make band pass shift by u_0

Niquist theorem decides frequency range to analyze based on sample rate

Magnitude spectrum is not histogram

Real signal in spatial makes even symmetry in Fourier

You want to design filters symmetrical around the origin

Apply Gaussian for filter design

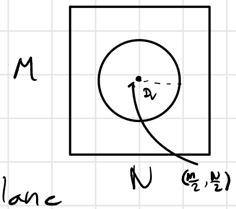
Lowpass filter Design

$H(u,v)$ is filter, D_0 is cutoff freq., n is order

$$D(u,v) = \left[\left(u - \frac{M}{2}\right)^2 + \left(v - \frac{N}{2}\right)^2\right]^{1/2}$$

for $M \times N$ image

Equation of a circle in u,v plane



Ideal

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$$

your filter is image, with disk in center, outside disk is 0
inside disk is 1
if things have been shifted so center is DC component
this is a low pass filter

- Ideal is transition between 1 and 0 is abrupt

FT → shift → multiply by filter → shift → IFT

- Smaller circle gets more blurring (smaller freq. range passed)
- Lowpass blurs and removes details
- Artifacts are occurring w/ stronger butterworth because of the Ideal filter

Ideal →

as this expands

This shrinks

Side bands cause the artifacts *

"ringing artifacts

So the ring artifact shrinks when the ideal filter expands

Butterworth filter:

$$H(u,v) = \frac{1}{1 + [D(u,v)/D_0]^{2n}}$$

As n approaches ∞, The filter becomes Ideal

D_0 is radius

↑

* How would you know if this is high pass or lowpass

Gaussian:

$$H(u,v) = e^{(-D^2(u,v))/(2D_0^2)}$$

↑ why does D_0 appear here? *

Highpass filter:

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$

Ideal

$$H(u,v) = \frac{1}{1 + [D_0/D(u,v)]^{2n}}$$

Butterworth

$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$$

Gaussian

Video Segment:

for High pass filter design, subtract lp from Identity filter

$$\cdots \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdots - lp$$

↙ gaussian or Ideal

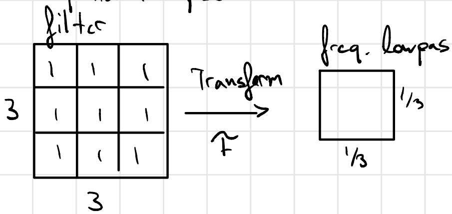
Anti-Aliasing:

- Its not removing aliasing - thats not possible
- It blurs a digital Image before downsampling to prevent aliasing/Artifacts

Original Img → $\boxed{\text{blur LPF}}$ → Downsample

Spatial -> Spatial Frequency Domain filter

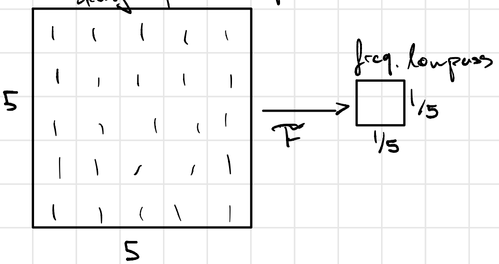
A spatial low pass filter



← shrinks in the Freq. Domain because

$f(at) = \frac{1}{|a|} \cdot F\left(\frac{u}{a}\right)$ There is an inverse relationship

Stronger spatial lowpass



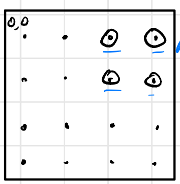
tighter lowpass ∴ more blurring because less high freq. which hold detail are passed

Binary Image Morphology

Morphology -> Structure, form, or shape of objects

- let A be a set in Z^2 If $a = (a_1, a_2)$ is an element in A, the $a \in A$

Ordered pair of positive integers -> a_1, a_2 is not the same as a_2, a_1



· -> 0's

⊙ -> 1's

$$A = \{(0,2), (0,3), (1,2), (1,3)\}$$

- Expression $C = \{w \mid w = -d \ \forall \ d \in D\}$

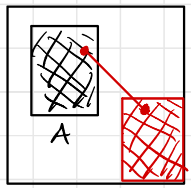
↳ means that C is a set w/ elements w such that w is formed by multiplying each of the two coordinates of all elements of set D by -1.

Notation: $A \subseteq B \rightarrow A$ is contained in B
$A \cup B \rightarrow A$ union B
$A \cap B \rightarrow A$ intersect w/ B

Basic operations: let A & B be sets in Z^2 with components $a = (a_1, a_2)$ and $b = (b_1, b_2)$, respectively

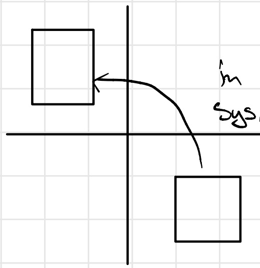
- Translation of A by $x = (x_1, x_2)$ is denoted by:

$$(A)_x = \{c \mid c = a + x \quad \forall a \in A\}$$



- Reflection of B is defined as

$$\hat{B} = \{x \mid x = -b \quad \forall b \in B\}$$



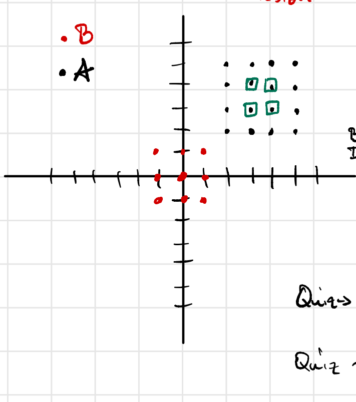
in cartesian sys. this rotates by 180°

- Two fundamental operations in BIM

Think of ANDing to set origin → - Erosion: $A \ominus B = \{x \mid (B)_x \subseteq A\}$ ← other equivalent definitions of erosion $\ominus$

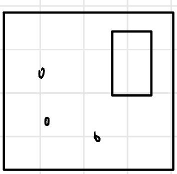
↑ Erosion

"consider the translations where B is contained in A"



$A \ominus B$ ▫
Binary Image
structuring element
result
usually not interchangable

only need to check x values in A



is $\boxed{A \ominus B \overset{?}{\subseteq} A}$ → yes, erosion just peels away a but still $\subseteq A$

Quiz: is there always an erosion of A by B? *
↳ not if A is smaller than B

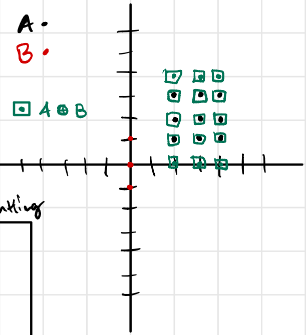
Quiz → if B was shifted to the left would you still get the same erosion? → import def. & see * think of ORing struct element to set origin →

- Dilation: $A \oplus B = \bigsqcup_{b \in B} (A)_b$

Union

Dilated

Union of all elements when A is translated by B



- Note: $A \oplus B = B \oplus A$

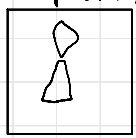
$A \ominus B \neq B \ominus A$

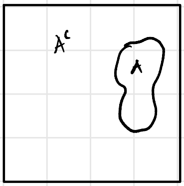
Duality of Erosion:

$$(A \ominus B)^c = A^c \oplus \hat{B}$$

$$(A \oplus B)^c = A^c \ominus \hat{B}$$

use for limiting

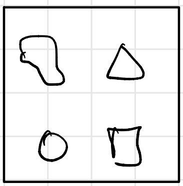




you can erode A by Dilating A^c or erode A^c by Dilating A

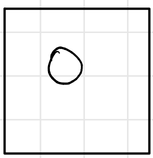
?

→ Connected Component Analysis

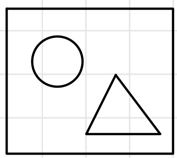


How many objects are there and how many pixels are in each object

- Quiz Question *

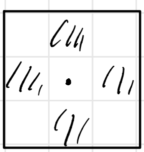


Easy

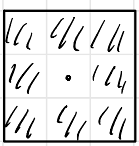


Hard

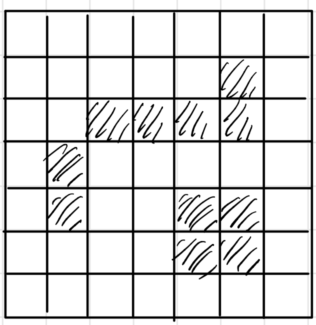
- if you dont know the number of objects it is very tough



4 connected neighbors of center pixel



8 connected neighbors of center pixel



# of objects?

4-conn = 3

8-conn = 2

Algorithm for CCA

- let A be a set containing one or more connected components. form a Matrix X_0 of the same size as A whose elements are zero except at each point known to correspond to a point in each connected component.
Following the iterative procedure:

$$X_K = (X_{K-1} \oplus B) \cap A \quad K = 1, 2, 3$$

will find all connected components of A when

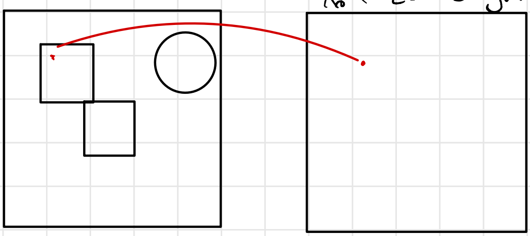
$$X_K = X_{K-1}$$

B is an appropriate structuring element

$$B_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad B_8 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

4 con   8 con

A $X_0 \leftarrow$ zero everywhere except for 1 point inside of one object



the process will keep expanding that point with dilation and intersects with the original object to limit the growth

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Opening $A \circ B = (A \ominus B) \oplus B$

Eroding by B then dilating by B

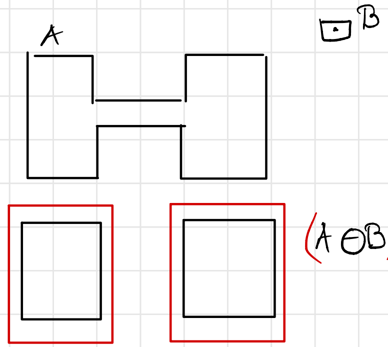
$$= \bigcup \{ (B)_z \mid (B)_z \subseteq A \}$$

← keeping whole set not just the point

- the union of all translations of z of B which are contained in A

Erosion: $\{ x \mid (B)_x \subseteq A \}$ ← keeps just the point

- Doesn't keep just the origin point it keeps all the points where the struct. fits



$A \ominus B$ $(A \ominus B) \oplus B \rightarrow A \circ B$

- opening is idempotent
  - if you erode → dilate → erode → Dilate, the idempotency holds, but if you erode, erode, erode → dilate, dilate, dilate it is not idemp

Opening breaks narrow bridges and eliminates thin structures

Closing: $A \cdot B = (A \oplus B) \ominus B$

- Look into Example why structuring element ... origin
- The rounding in photo ex. occurs b.c. circular Struct. element
- Opening can help smoothing
- what kinds of shapes will open / dilate / erode differently
- re-opening will give new results - look into why
- can create a skeleton of an object using erosion struct sizes
- you can have even or odd shaped structuring element but pay attention to what the origin should be.