||Sadguru's||™

EDUCATION CENTRE

DEDICATED TO EDUCATION IN SOCIETY

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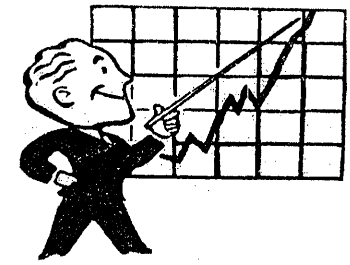
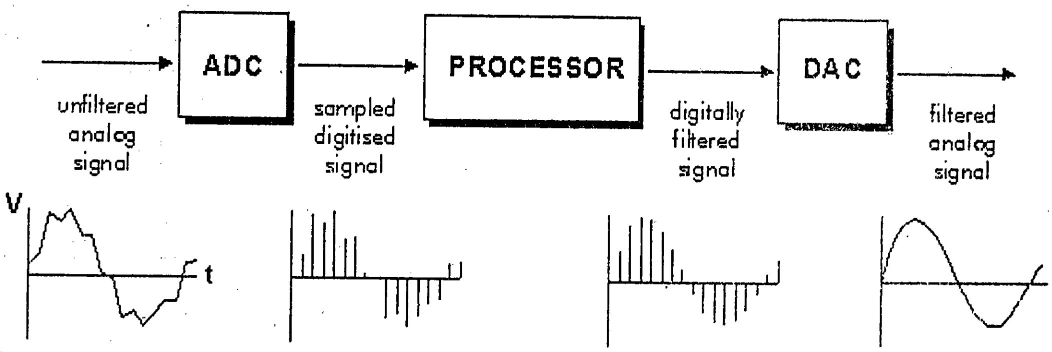


NAME :

DSP-DTSP (1)



I See and I Forget

I See, I Listen and I Remember

I See, I Listen, I Practice and then I Learn

..................Academic Year : 2008-09 ....

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Our New Branches At Borivali & Andheri

||Sadguru's||

.....always a step ahead of others

DISCRETE TIME SIGNAL

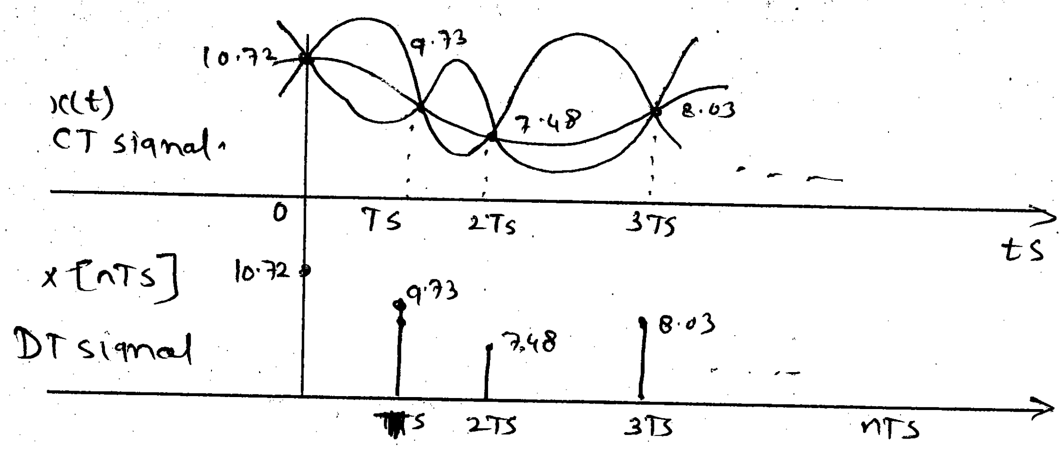
	TOPIC	
1.	**DISCRETE TIME SIGNAL**	
	1.1	DT Signal
	1.2	Standard DT Signal
		Concept of Digital Frequency W
	1.3	Linear Shifting of DT Signals
	1.4	Classification of DT Signals
	1.5	DSP System.
	1.6	Sampling
	1.7	Reconstruction
	1.8	Convolution
	1.9	Correlation

	EXAM	IT	ELX	COMP	EXTC	INSTRU
1	May-2004	16	15	--	--	--
2	Dec-2004	20	06	00	10	05
3	May-2005	04	05	10	08	05
4	Dec-2005	00	08	13	12	00
5	May-2006	10	04	20	06	05
6	Dec -2006	24	06	15	08	00
7	May-2007	04	04	05		
8	Dec -2007	08	13	00	00	
9	May-2008	24	13	00		
AVERAGE						

Ch-1 DISCRETE TIME SIGNAL

1.1 Discrete Time Signal

DT signal is obtained by sampling CT signal at regular intervals of time.



x[n] = $\{ \underset{\uparrow\, n=0}{10.72}, 9.73, 7.48, 8.03$

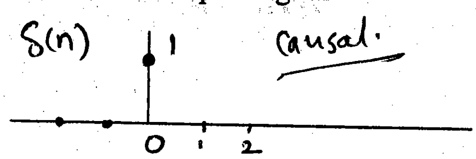
$\boxed{x(n)\big|_{n=1.5} = ?\ \text{Not defined}}$ — Viva

why not defined ?

1.2 Standard Discrete Time Signals

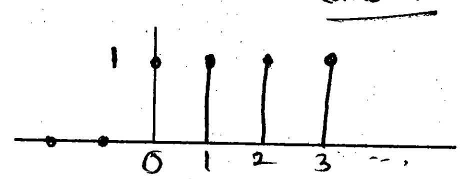
Ans- infinite posibility of CT.

[1] Delta Signal
Impulse Signal
Unit Sample Signal



$\delta(n) = \{ \underset{\uparrow}{1}, 0, 0, \ldots \}$

[2] Unit Step Signal



$u(n) = \{ \underset{\uparrow}{1}, 1, 1, 1 \ldots \}$

[3] Sinusoidal Signal

Consider a CT Sinusoidal signal **x(t) = cos (Ωt)** where Ω is Analog frequency in radians/sec such that **Ω = 2 π F** and F is Analog frequency in Hz.

BY Sampling, $x(t)\big|_{t=nTs} = \cos(\Omega t)\big|_{t = n/Fs}$

$$x(nTs) = \cos\left[2\pi F \frac{n}{Fs}\right]$$

$$x(n) = \cos\left[2\pi \left(\underset{\Downarrow}{\frac{F}{Fs}}\right) n\right]$$

$$= \cos[2\pi f n]$$

$$x(n) = \cos[\omega n]$$

A DT Sinusoidal signal **x[n] = cos (ωn)** where ω is Digital frequency in radians

such that $\omega = 2\pi f$ and **f** is Digital frequency

Here, $\boxed{f = \dfrac{F}{F_S}}$ F → Analog freq. in Hz; F_S → Sampling freq. in Hz.

f ↓ Digital freq. [No unit]

Similarly,

$$f = \frac{F}{F_S} \qquad \because \text{multiplying by } 2\pi$$

$$2\pi f = \frac{2\pi F}{F_S}$$

$$\therefore \boxed{\omega = \frac{\Omega}{F_S}}$$

$$\text{i.e. } \underset{\text{rad}}{\omega} = \underset{\text{rad/sec}}{\Omega}\ \underset{\text{sec}}{T_S}$$

Concept of Digital Frequency ω

Let (i) $x_1[n] = \cos(0.5\pi n)$ where $\omega_1 = 0.5\pi$

(ii) $x_2[n] = \cos(2.5\pi n)$ where $\omega_2 = 2.5\pi$

Now, $x_2[n] = \cos(2.5\pi n)$

$x_2[n] = \cos(0.5\pi n + 2\pi n)$

$= \cos(0.5\pi n)\cos(2\pi n) - \sin(0.5\pi n)\sin(2\pi n)$

$\because \cos 2\pi n = 1$

$\sin 2\pi n = 0.$

$\therefore x_2(n) = \cos(0.5\pi n)$

$\therefore x_2(n) = x_1(n)$

NOTE : (1) When digital frequencies are separated by multiples of $\pm 2\pi$
Then Discrete Time Signals are exactly Same..
(i.e. their Sample values are identical)

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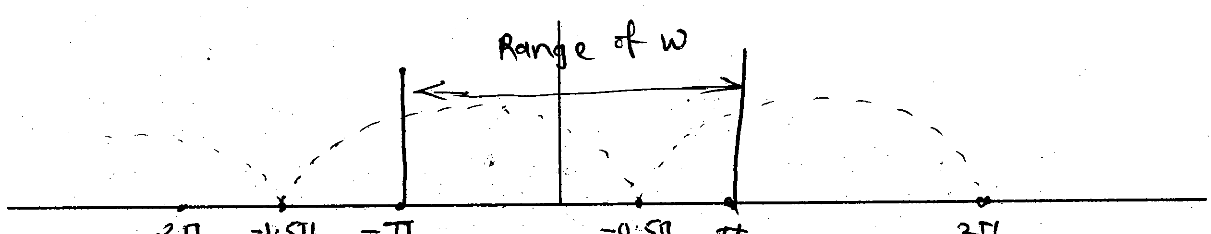
(2) Range of Digital frequency ω



Range of ω is $(-\pi, \pi]$

i.e $\boxed{-\pi < \omega \le \pi}$

↑ Digital freq. in radians.

(3) Range of Digital frequency f

Now, $\omega = 2\pi f$ so, $f = \dfrac{w}{2\pi}$

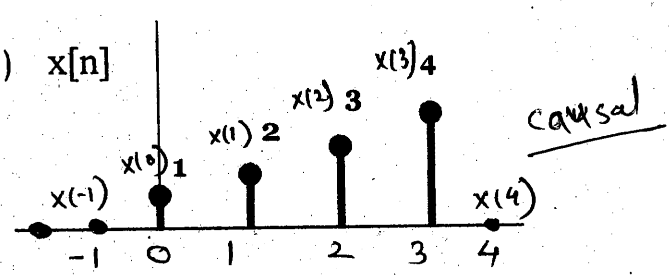
$$\text{Range of } f = \left[\frac{w_{\min}}{2\pi}, \frac{w_{\max}}{2\pi} \right]$$

$$= \left(\frac{-\pi}{2\pi}, \frac{\pi}{2\pi} \right]$$

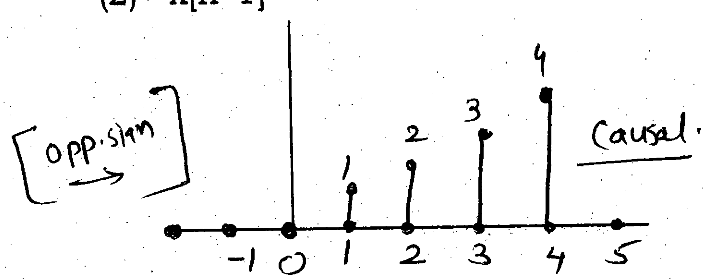
$$= \left(\frac{-1}{2}, \frac{1}{2} \right]$$

1.3 Linear Shifting of NON-Periodic DT Signals

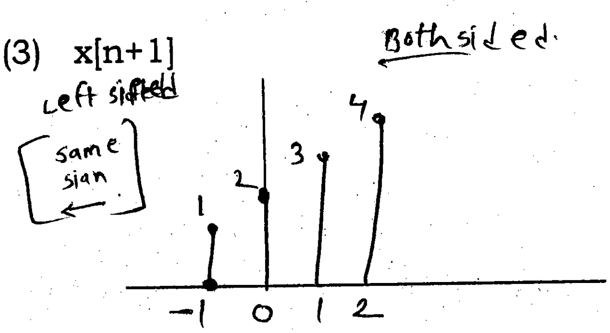
(1) x[n]



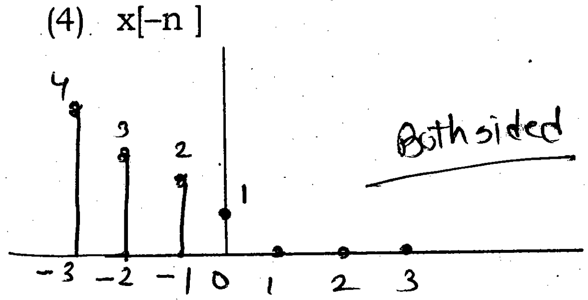
(2) x[n−1]



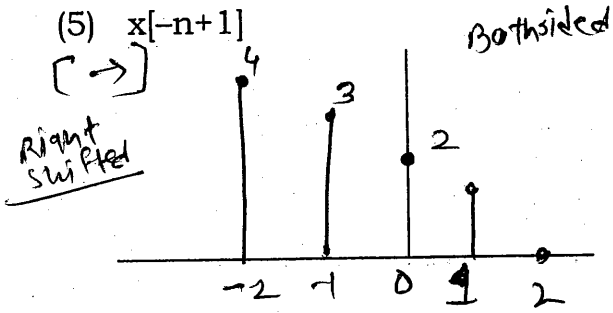
(3) x[n+1]



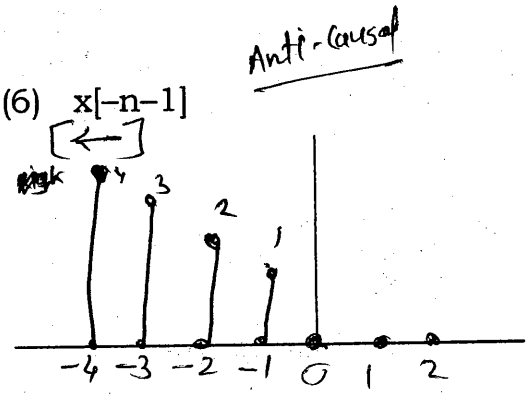
(4) x[−n]



(5) x[−n+1]



(6) x[−n−1]



1.4 Classification of Discrete Time signals

1. Finite length / Infinite length

If the number of samples are finite then signal is finite. Similarly if number of samples are infinite then signal is infinite.

Examples :

(i) Finite Length : $x[n]=\{\underset{\uparrow}{1}\ \ 2\ \ 3\ \ 4\}$

(ii)Infinite Length Signal : x[n] =u[n]

---

2. Causal / Anti-causal / Bothsided

If x[n] = 0 for all n < 0 then x[n] is causal signal.
If x[n] = 0 for all n ≥ 0 then x[n] is anticausal signal.
If x[n] is neither causal nor anticausal then x[n] is bothsided signal.

Examples :

(i) Causal signal : x[n] =u[n]
(ii) Anti-causal signal : x[n] =u[-n-1]
(iii) Both sided signal : x[n] + u[n] + u[-n-1]

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3. Periodic / Non-periodic

If the digital frequency of the signal is rational number then the signal is periodic. Otherwise signal is nonperiodic.

Examples :

(i) Periodic signal : x[n] = cos (0.2 π n) where w = 0.2π
f = 0.1
Here f is rational number ,so x[n] is periodic signal.

(ii) Non periodic signal : x[n] = cos(2n) where w = 2
f = 1/π
Here f is Not rational number ,so x[n] is not periodic signal.

---

4. Energy / Power / Neither Energy nor Power

Energy of signal is defined as, $E=\sum_{n=0}^{N-1}|x[n]|^2$

If Energy of DT signal is finite (0<E<∞) then x[n] is an energy signal.
If Energy is infinite then go for average power.

The average power of the x[n] is given as $P=\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2$

If P is finite and nonzero then x[n] is a power signal.

Examples :

(i) Energy signal :

$x[n] = (½)^n u[n]$ E = 2 (finite)
$x[n]=\{\underset{\uparrow}{1}\ \ 2\ \ 3\ \ 4\}$ E = 30 (finite)

(ii) Power Signal : x[n] = u[n]

5. Even / Odd

If $x[n] = x[-n]$ then $x[n]$ is even signal.
If $x[n] = -x[-n]$ then $x[n]$ is odd signal.

Examples :

(i) Even signal : $x[n] = \{ -1 \;\; -2 \;\; \underset{\uparrow}{3} \;\; -2 \;\; -1 \}$

(ii) Odd Signal $x[n] = \{ 1 \;\; 2 \;\; \underset{\uparrow}{0} \;\; -2 \;\; -1 \}$

(iii) Neither Even nor Odd signal : $x[n] = \{ \underset{\uparrow}{1} \;\; 2 \;\; 3 \;\; 4 \}$

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6. Causal Symmetric / Causal Antisymmetric

If $x[n] = x[N-1-n]$ then causal symmetric.
If $x[n] = -x[N-1-n]$ then causal antisymmetric.

Examples :

(i) Causal Symmetric signal : $x[n] = \{ \underset{\uparrow}{1} \;\; 2 \;\; 3 \;\; -2 \;\; -1 \}$

(ii) Causal Anti-symmetric Signal : $x[n] = \{ 1 \;\; 2 \;\; \underset{\uparrow}{0} \;\; -2 \;\; -1 \}$

Q(1) Determine whether the following DT signals are periodic or not. If periodic find the period.

a) $x[n] = \cos(0.3\pi n)$

b) $x[n] = \cos(0.3\pi n + \frac{\pi}{6})$

c) $x[n] = \cos(0.3\pi n) + \cos(\frac{1}{6}\pi n)$

d) $x[n] = \cos(\frac{n\pi}{4}) \cos(\frac{n\pi}{8})$

e) $x[n] = \cos(n\frac{\pi}{2}) - \sin(n\frac{\pi}{8}) + 3\cos(n\frac{\pi}{4} + \frac{\pi}{3})$

ANS : a) Periodic with N = 20
b) Periodic with N = 20
c) Periodic with N = 60
d) Periodic with N = 16
e) Periodic with N = 16

Q(2) Determine which of the following sinusoids are periodic and compute their fundamental period.

(a) $\cos 0.01\pi n$ (b) $\cos\left(\pi\frac{30n}{105}\right)$ (c) $\cos 3\pi n$ (d) $\sin 3n$ (e) $\sin\left(\pi\frac{62n}{10}\right)$

ANS : a) Periodic with N = 200
b) Periodic with N = 70
c) Periodic with N = 2
d) Not periodic.
e) Periodic with N = 20

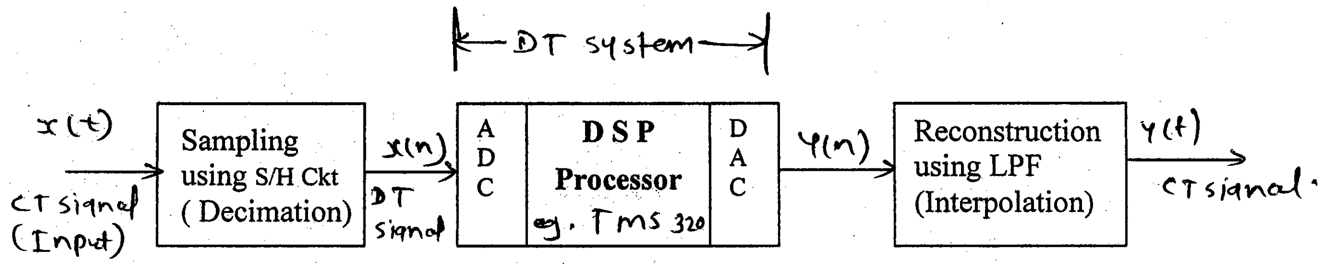
Q(3) Show that $\sum_{n=-\infty}^{\infty} x^2[n] = \sum_{n=-\infty}^{\infty} x_e^2[n] + \sum_{n=-\infty}^{\infty} x_o^2[n]$

Q(4) If $x[n] = 0$ for $n < 0$, derive an expression for $x[n]$ in terms of its even part $x_e[n]$.

Q(5) Show that any arbitrary signal $x[n]$ can be decomposed into its even part and odd part of the signal components. ie. $x[n] = x_e[n] + x_o[n]$

Q(6) Show that if $x[n]$ is an odd signal then $\sum_{n=-\infty}^{\infty} x[n] = 0$

1.4 DSP System

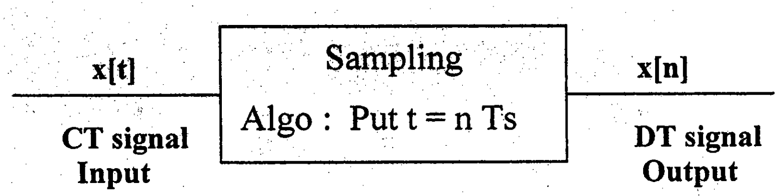


eg i) Voice signal
ii) Noise signal
iii) Bio-medical Signals
i.e. ECG, EEG, EMG.

DSP Algo
1) convolution
2) correlation.

voice signal.
Ex. of DSP system is
Digital Telephone System
$F_s = 8\,KHz$.
$N = 8000$ Samples/sec.
ADC with 8 bit resolution
Data Rate = 64 Kbps

1.5 SAMPLING



Eg-1 $x(t) = \cos(200\pi t)$ $\quad F_s = 500$ Hz

Solution : To find x[n] :

Put $t = n\,T_s = \dfrac{n}{F_s} = \dfrac{n}{500}$

$x[n\,T_s] = \cos\left(200\pi \dfrac{n}{500}\right)$

$x[n] = \cos[0.4\pi n]$, $\omega = 0.4\pi$

Ans

Eg-2 $x(t) = \cos(200\pi t)$ $\quad F_s = 80$ Hz

Solution : To find x[n] :

Put $t = n\,T_s = \dfrac{n}{F_s} = \dfrac{n}{80}$

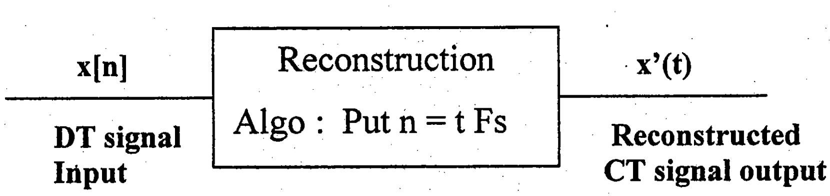
$x[n\,T_s] = \cos\left(200\pi \dfrac{n}{80}\right)$

$x[n] = \cos(2.5\pi n)$ $\quad \omega = 2.5\pi$ ✗

$x[n] = \cos(0.5\pi n)$ $\quad \omega = 0.5\pi$ ✓

Ans

1.6 RECONSTRUCTION



Eg-1 $x[n] = \cos(0.4\pi n)$ Fs = 500 Hz	**Eg-2** $x[n] = \cos(0.5\pi n)$ Fs = 80 Hz
Solution : To find x'(t):	**Solution :** To find x'(t) :
Put $n = t\,F_s = 500\,t$	Put $n = t\,F_s = 80\,t$
$x'(t) = \cos(0.4\pi\ 500\,t)$ $x'(t) = \cos(200\pi\ t)$	$x'(t) = \cos(0.5\pi\ 80\,t)$ $x'(t) = \cos(40\pi\ t)$
Here, x'(t) == x(t)	**Here,** $x'(t) \neq x(t)$

Q(7) Consider the analog signal $x(t) = 3\cos(100\pi t)$

a) Determine the minimum sampling rate required to avoid aliasing.
b) Suppose that the signal is sampled at fs = 200 Hz. What is the DT signal obtained after sampling.
c) Suppose that the signal is sampled at fs = 75 Hz. What is the DT signal obtained after sampling

ANS : a) Min Fs = 100 Hz b) $x[n] = 3\cos(n\frac{\pi}{2})$ c) $x[n] = 3\cos(n\frac{2\pi}{3})$

Q(8) Consider the signal, $x(t) = 3\cos(50\pi t) + 10\sin(300\pi t) - \cos(100\pi t)$
i) If the signal is sampled with Fs = Nyquist rate, What will be the DT signal obtained after sampling. ii) If ideal interpolation is used, what will be the reconstructed analog signal?

ANS : i) $x[n] = 3\cos(n\frac{\pi}{6}) + 10\sin(n\pi) - \cos(n\frac{\pi}{3})$
ii) $x'(t) = 3\cos(50\pi t) + 10\sin(300\pi t) - \cos(100\pi t)$

Q(9) Consider the signal, $x(t) = 3\cos(50\pi t) + 10\sin(300\pi t) - \cos(100\pi t)$
i) If the signal is sampled with Fs = 200 Hz, What will be the DT signal obtained after sampling. ii) If ideal interpolation is used, what will be the reconstructed analog signal?

ANS : i) $x[n] = 3\cos(n\frac{\pi}{4}) - 10\sin(n\frac{\pi}{2}) - \cos(n\frac{\pi}{2})$
ii) $x'(t) = 3\cos(50\pi t) - 10\sin(100\pi t) - \cos(100\pi t)$

Q(10) $x(t) = 7\cos(250\pi t)$. Determine the different analog signals that gives samples identical to that obtained by sampling x(t) with Fs = 200 Hz.

ANS : Hint Any signal with $\mathbf{F_k = F + k\,F_s}$ gives signal with identical samples.
i) $x_0(t) = 7\cos(250\pi t)$ ii) $x_1(t) = 7\cos(650\pi t)$ iii) $x_2(t) = 7\cos(1050\pi t)$

Q(11) $x(t) = \sin(480\pi t) + \sin(720\pi t)$ is sampled with Fs = 600 times per seconds.

a) Determine the Nyquist rate.
b) Determine the folding Frequency.
c) What are the Frequencies in radians in the resulting DT signal x[n] ?
d) If x[n] is passed through an ideal DAC what is the reconstructed signal.

ANS : a) Fs = 720 Hz b) Fo = 300 Hz c) $f = \frac{4\pi}{5}$ d) $x'[t] = -2\sin(480\pi\ t)$

Q(12) Consider the analog signal $x(t) = 3 \sin(100\pi t)$

a) The signal x(t) is sampled with a sampling frequency Fs = 300 samples/sec. Determine the frequency of the DT signal x[n] and show that x[n] is periodic.
b) Find the sampling rate Fs such that the signal x[n] reaches its peak value of 3. What is the minimum Fs suitable for this task ?

ANS : a) $f = 1/6$ b) Fs = 200 Hz

Q(13) A digital communication link carries binary coded words representing samples of an input signal.

$X_a(t) = 3\cos(600\pi t) + 2\cos(1000\pi t)$

The link is operated at 10000 bits/s and each input sample is quantized into 1024 different voltage levels.

i) What is sampling and folding frequency ?
ii) What is the nyquist rate for $X_a(t)$?
iii) What are the frequency in the resulting discrete time signal x(n) ?
iv) What is the resolutions Δ?

ANS : i) Fs = 1000 Hz and Fo = 500 Hz ii) $F_N = 1000$ Hz iii) $f_1 = 0.3$ and $f_2 = 0.5$ iv) $\Delta = 10 / 1024$

1.7 CONVOLUTION

To find o/p of digital freq.

Linear Convolution	Circular Convolution
(for Non-Periodic Signals)	(for Periodic Signals)

$$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{\infty} x(m)\, h(n-m)$$

$$y(n) = x(n) \circledast h(n) = \sum_{m=0}^{N-1} x(m)\, h(n-m)$$

Q(14) Given $x[n] = \{\underset{\uparrow}{1}\ 2\ 3\ 4\}$ and $h[n] = \{5\ \underset{\uparrow}{0}\ 6\}$ Find Convolution.

Solution : To find Linear Convolution

STEP-1 To find y[n] for n ≥ 0

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\, h[n-m] = \sum_{m=0}^{3} x[m]\, h[n-m]$$

$$y[0] = \sum_{m=0}^{3} x[m]\,h[-m] = (1)(0)+(2)(5)+(3)(\bullet)+(4)(\bullet) = \boxed{10}$$

$$y[1] = \sum_{m=0}^{3} x[m]\,h[1-m] = (1)(6)+(2)(0)+(3)(5)+(4)(\bullet) = \boxed{21}$$

$$y[2] = \sum_{m=0}^{3} x[m]\,h[2-m] = (1)(\cdot)+(2)(6)+(3)(0)+(4)(5) = \boxed{32}$$

$$y[3] = \sum_{m=0}^{3} x[m]\,h[3-m] = (1)(\cdot)+(2)(\cdot)+(3)(6)+(4)(0) = \boxed{18}$$

$$y[4] = \sum_{m=0}^{3} x[m]\,h[4-m] = (1)(\cdot)+(2)(\cdot)+(3)(\cdot)+(4)(6) = \boxed{24}$$

Rough Work

x[m]	**1**	**2**	**3**	**4**	**y[n]**
h[-m] 6	0	5			10
h[-m+1]	6	0	5		21
h[-m+2]		6	0	5	32
h[-m+3]			6	0	18
h[-m+4]				6	24

STEP-2 To find y[n] for n < 0

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\,h[n-m] = \sum_{m=0}^{3} x[m]\,h[n-m]$$

$$y[-1] = \sum_{m=0}^{3} x[m]\,h[-1-m] = (1)(5)+(2)(\cdot)+(3)(\cdot)+(4)(\cdot) = \boxed{5}$$

$$y[-2] = \sum_{m=0}^{3} x[m]\,h[-2-m] = (1)(\cdot)+(2)(\cdot)+(3)(\cdot)+(4)(\cdot) = \boxed{0}$$

Rough Work

x[m]	**1**	**2**	**3**	**4**	**y[n]**
h[-m] 6	0	5	.	.	
h(-m-1) 6 0	5	-	.	.	y[-1] = 5
h(-m-2) 6 0 5	.	-	-	.	y[-2] = 0

ANS : y[n] = { 5, 10, 21, 32, 18, 24 }

NOTE : In Linear Convolution, If input signals are Causal, Then Resultant output signal is also causal.

⇨ Application of convolution is to find output of Digital filter for any given input signal

1.8 CORRELATION

Auto Correlation	Cross Correlation
$y(n) = x(n) \circ x(n)$	$y(n) = x(n) \circ h(n)$
$y(n) = \sum_{m=-\infty}^{\infty} x(m)\, x(m-n)$	$y(n) = \sum_{m=-\infty}^{\infty} x(m)\, h(m-n)$

Q(15) Given $x[n]=\{\underset{\uparrow}{1}\ \ 2\ \ 3\ \ 4\}$ and $h[n]=\{\underset{\uparrow}{5}\ \ 0\ \ 6\}$ Find Correlation.

Solution : To find Correlation

STEP-1 To find y[n] for n ≥ 0

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\, h[m-n] = \sum_{m=0}^{3} x[m]\, h[m-n]$$

$$y[0] = \sum_{m=0}^{3} x[m]\, h[m] = (1)(5)+(2)(0)+(3)(6)+(4)(\cdot) = \boxed{23}$$

$$y[1] = \sum_{m=0}^{3} x[m]\, h[m-1] = (1)(\cdot)+(2)(5)+(3)(0)+(4)(6) = \boxed{34}$$

$$y[2] = \sum_{m=0}^{3} x[m]\, h[m-2] = (1)(\cdot)+(2)(\cdot)+(3)(5)+(4)(0) = \boxed{15}$$

$$y[3] = \sum_{m=0}^{3} x[m]\, h[m-3] = (1)(\cdot)+(2)(\cdot)+(3)(\cdot)+(4)(5) = \boxed{20}$$

Rough Work

x[m]	1 ↓	2	3	4	y[n]
h[m]	5	0	6	·	y(0) = 23
h(m-1)		5	0	6	y(1) = 34
h(m-2)			5	0	y(2) = 15
h(m-3)				5	y(3) = 20

STEP-2 To find y[n] for n < 0

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\, h[m-n] = \sum_{m=0}^{3} x[m]\, h[m-n]$$

$$y[-1] = \sum_{m=0}^{3} x[m]\, h[m+1] = (1)(0)+(2)(6)+(3)(\ \cdot\ )+(4)(\ \cdot\ ) = \boxed{12}$$

$$y[-2] = \sum_{m=0}^{3} x[m]\, h[m+2] = (1)(6)+(2)(\ \cdot\ )+(3)(\ \cdot\ )+(4)(\ \cdot\ ) = \boxed{6}$$

Rough Work

x[m]	1 ↓	2	3	4	y[n]
h[m]	5	0	6		
h[m+1] 5	0	6			y(-1) = 12
h[m+2] 5, 0	6				y(-2) = 6

ANS : $y[n] = \{6, 12, \underset{\uparrow}{23}, 34, 15, 20\}$

Application of correlation is to find degree of similarity between two signals.

For ex. Two signals are given say x[n] and p[n].

To find out whether they are similar signals are not :-

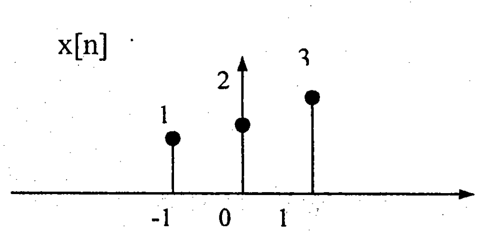
(i) Find a[n] = x[n] o x[n]

(ii) Find b[n] = x[n] o p[n]

(iii) Compare a[n] with b[n]

Q(16) Show that $x[n] = \sum_{m=-\infty}^{\infty} x[m]\,\delta[n-m]$

Solution : To develop above result, consider an arbitrary signal, x[n] = {1, 2, 3}



We decompose x[n] into component impulse function.

$\therefore\ x[n] = \delta[n+1] + 2\delta[n] + 3[n-1]$

More generally, we can express any DT signal as a sum of weighted impulses

$x[n] = \ldots + x[-2]\,\delta[n+2] + x[-1]\,\delta[n+1] + x[0]\,\delta[n] + x[1]\,\delta[n-1] + \ldots$

OR $x[n] = \sum_{m=-\infty}^{\infty} x[m]\cdot\delta[n-m]$

Now, $\delta[1-m]$ is defined only at m = 1 so, $x[m]\,.\,\delta[1-m]$ is defined only at m = 1 and is "0" elsewhere hence, the impulse for samples the particular value of x[n] and hence this is called "sifting property of the impulse".

Q(17) Prove that the auto correlation sequence at zero lag has highest magnitude with respect to magnitude at any other lag.

Autocorrelation of DT signal x[n] is defined as,

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\, x[m-n]$$

At n = 0, $y[0] = \sum_{m=-\infty}^{+\infty} x[m]\, x[m] = \sum_{m=-\infty}^{+\infty} x^2[m]$ ....(I)

Energy of the signal is defined as $E = \sum_{m=-\infty}^{+\infty} x^2[m]$ ....(II)

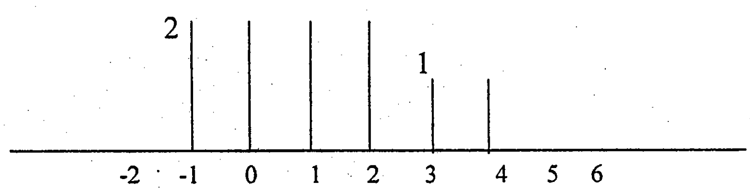
From equation (I) and (II) y[0]= E That means, Autocorrelation sequence at zero lag gives energy of the signal. That is the max value.

H.W. Questions only for COMPUTER Branch students....

Q(18) Sketch the following Discrete Time Signals –

1) $x[n] = 2\,\delta[n-3] - 3\,\delta[n+2]$
2) $x[n] = \delta[n] + \delta[n-1]$
3) $x[n] = u[n] + u[n-5] - u[n-8] - u[n-10]$)
4) $x[n] = \delta[n] + \delta[n-1] + u[n-2] - u[n-5]$)
5) $x[n] = \delta[n] + 2\,\delta[n-1] + 3\,\delta[n-2]$
6) $x[n] = u[n] - u[n-5]$
7) $x[n] = \cos(0.3\,\pi\, n)\, u[n]$
8) $x[n] = \sin(0.2\,\pi\, n)\, u[n]$

Q(19) A discrete time signal x[n] is shown below. Sketch each of the following signals :



1) $x[n-2]$
2) $x[4-n]$
3) $x[n+2]$
4) $x[n]\, u[2-n]$
5) $x[n]\, u[n-2]$
6) $x[n-1]\, \delta[n-3]$
7) $x[n^2]$
8) $x[2n]$
9) $x^2[n]$

Q(20) A discrete time signal is defined as,

$$x[n] = \begin{cases} 1 + n/3 & -3 \le n \le -1 \\ 1 & 0 \le n \le 3 \\ 0 & \text{otherwise} \end{cases}$$

a) Determine its value & sketch the signal x[n].
b) Express the signal x[n] in terms of δ[n].

Q(21) Express the sequence

$$x[n] = \begin{cases} 1 & n = 0 \\ 2 & n = 1 \\ 3 & n = 2 \\ 0 & \text{else} \end{cases}$$

as a sum of scaled and shifted a) delta sequences b) unit steps.

Priority 🔒 : **Definitely Everything** √ : **You must do this**
$: **You should not leave this** ◷ : **If possible, do it**

	EXAM	IT	ELX	COMP	EXTC	INSTRU
1	May-2004	52	50	--	--	--
2	Dec-2004	20	36	20	44	50
3	May-2005	31	24	40	40	48
4	Dec-2005	44	36	34	30	46
5	May-2006	30	40	35	39	53
6	Dec -2006	08		20	48	70
7	May-2007					
8	Dec -2007					
9	May-2008					
	AVERAGE					

Ch-2 DISCRETE FOURIER TRANSFORM AND FAST FOURIER TRANSFORM

2.1 DFT- IDFT EQUATION

(A) Discrete Fourier Transform of x[n] is defined as,

$$X[k] = \sum_{n=0}^{N-1} x(n)\, W_N^{nk}$$

Where i) N is Length of x[n]

ii) $W_N^1 = e^{-j\frac{2\pi}{N}}$

$\uparrow$

Twiddle (i.e. periodic) factor.

$$\underset{\text{Time domain}}{x(n)} \longrightarrow \boxed{\text{DFT}} \longrightarrow \underset{\text{freq. Domain}}{X(k)}$$

(B) Inverse Discrete Fourier Transform of X[k] is defined as,

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_N^{-nk}$$

Where i) N is Length of X[k]

ii) $W_N^{-1} = e^{j\frac{2\pi}{N}}$

Q(1) Let $x[n]=\{\underset{\uparrow}{1}\ \ 2\ \ 3\ \ 4\}$ Find DFT of x[n]. 4m

Solution : To find X[k]

By DFT, $X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$ *where* (i) $N = 4$ (ii) $W_N^1 = e^{-j\frac{2\pi}{N}}$

$$X[k] = X[0] + X[1]\, W_N^{k} + X[2]\, W_N^{2k} + X[3]\, W_N^{3k}$$

$$X[k] = 1 + 2W_N^{k} + 3W_N^{2k} + 4W_N^{3k}$$

(i) k=0, $X[0] = 1 + 2\,W_N^0 + 3\,W_N^0 + 4\,W_N^0 = 1+2+3+4 = \boxed{10}$

(ii) k=1, $X[1] = 1 + 2\,W_N^1 + 3\,W_N^2 + 4\,W_N^3$

Where,

$$W_N^1 = e^{-j\frac{2\pi}{N}} = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}}$$

$$= \cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right)$$

$$W_N^1 = -j$$

$$W_N^2 = W_N^1\, W_N^1 = (-j)(-j) = (j)^2 = -1$$

$$W_N^3 = W_N^1\, W_N^2 = (-j)(-1) = j$$

By substituting we get,

$$X[1] = \cancel{1+2W_N^1} = 1 + 2(-j) + 3(-1) + 4(j)$$

$$\therefore \boxed{X[1] = -2+2j}$$

(iii) k=2, $X[2] = 1 + 2W_N^2 + 3W_N^4 + 4W_N^6$

$X[2] = 1 + 2(-1) + 3(1) + 4(-1)$

$\therefore \boxed{X[2] = -2}$

(iv) k=3, $X[3] = 1 + 2W_N^3 + 3W_N^6 + 4W_N^9$

$X[3] = 1 + 2(j) + 3(-1) + 4(-j)$

$\therefore \boxed{X[3] = -2-2j}$

(v) k=4, $X[4] = 1 + 2W_N^4 + 3W_N^8 + 4W_N^{12}$

$X[4] = 1 + 2(1) + 3(+1) + 4(1)$

$\therefore \boxed{X[4] = 10}$

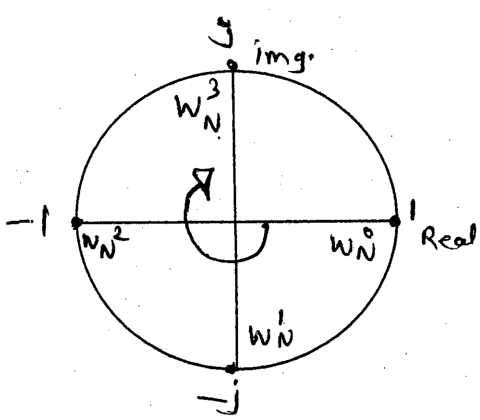
DFT results are periodic with period = N

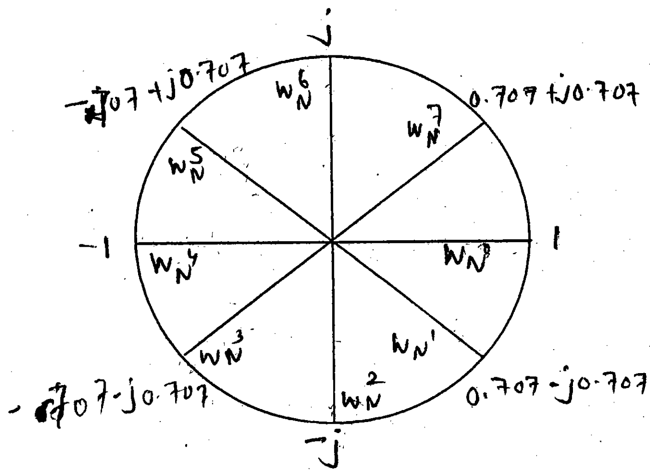
ANS
$$X[K] = \begin{bmatrix} 10 & K=0 \\ -2+2j & K=1 \\ -2 & K=2 \\ -2-2j & K=3 \end{bmatrix}$$

⇨ Cyclic Property of Twiddle factor W_N

Twiddle factor W_N is periodic with period = N.

Graphically, $\boxed{N=4}$ $\boxed{N=8}$





Q(2) Let $x[n]=\left\{ \underset{\uparrow}{1} \quad 2 \quad 3 \quad 4 \quad 0 \quad 0 \quad 0 \quad 0 \right\}$ Find 8 point DFT of x[n].

Solution :

By DFT, $X[k] = \sum_{n=0}^{N-1} x[n]\ W_N^{nk}$ *where* (i) $N = 8$ (ii) $W_N^1 = e^{-j\frac{2\pi}{N}}$

$X[k] = X[0]+ X[1]\ W_N^{k} + X[2]\ W_N^{2k} + X[3]\ W_N^{3k} + X[4]\ W_N^{4k} + X[5] W_N^{5k} + X[6]\ W_N^{6k} + X[7]\ W_N^{7k}$

$X[k] = 1 + 2\ W_N^{k} + 3\ W_N^{2k} + 4\ W_N^{3k}$

(i) $X[0] = 1 + 2 + 3 + 4$
$= 10$

(ii) $X[1] = 1 + 2\ W_N^{1} + 3\ W_N^{2} + 4\ W_N^{3}$
$= 1 + 2(0.707 - j\ 0.707) + 3(-j) + 4(-0.707 - j\ 0.707)$

$\therefore X[1] = 0.414 - j\ 7.242.$

(iii) $X[2] = 1 + 2\ W_N^{2} + 3\ W_N^{4} + 4\ W_N^{6}$
$= 1 + 2(-j) + 3(-1) + 4(j)$

$\therefore X[2] = -2+2j$

(iv) $X[3] = 1 + 2\ W_N^{3} + 3\ W_N^{6} + 4\ W_N^{9}$
$= 1 + 2(-0.707 - j0.707) + 3(j) + 4(0.707 - j0.707)$

$\therefore X[3] = 2.414 - j\ 1.242.$

(v) $X[4] = 1 + 2\ W_N^{4} + 3\ W_N^{8} + 4\ W_N^{12}$
$= 1 + 2(-1) + 3(1) + 4(-1)$

$\therefore X[4] = -2$

(vi) $X[5] = 1 + 2\ W_N^{5} + 3\ W_N^{10} + 4\ W_N^{15}$
$= 1 + 2(0.707 - j0.707) + 3(-j) + 4(0.707 + j0.707)$

$\therefore X[5] = 5.242 - j\ 1.586$

(vii) $X[6] = 1 + 2\ W_N^{6} + 3\ W_N^{12} + 4\ W_N^{18}$
$= 1 + 2(j) + 3(-1) + 4(-j)$

$\therefore X[6] = -2-2j$

(viii) $X[7] = 1 + 2\ W_N^{7} + 3\ W_N^{14} + 4\ W_N^{21}$
$= 1 + 2(0.707 + j0.707) + 3(j) + 4(-0.707 + j0.707)$

$\therefore X[7] = -0.414 + j\ 1.586.$

ANS : $X[k] = \begin{bmatrix} 10 \\ 0.414 - j\ 7.242 \\ -2+2j \\ 2.414 - j\ 1.242 \\ -2 \\ 5.242 - j\ 1.586 \\ -2-2j \\ -0.414 + j\ 1.586 \end{bmatrix}$

Q(3) Let $x[n]=\left\{\underset{\uparrow}{1} \quad 0 \quad 2 \quad 0 \quad 3 \quad 0 \quad 4 \quad 0\right\}$ Find DFT of x[n]. **HW**

Solution : To find X[k]

By DFT,

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk} \qquad \textit{where (i) } N = 8 \quad \textit{(ii) } W_N^1 = e^{-j\frac{2\pi}{N}}$$

$X[k] = X[0] + X[1]\, W_N^{k} + X[2]\, W_N^{2k} + X[3]\, W_N^{3k} + X[4]\, W_N^{4k} + X[5] W_N^{5k} + X[6]\, W_N^{6k} + X[7]\, W_N^{7k}$

$X[k] = 1 + 2\, W_N^{2k} + 3\, W_N^{4k} + 4\, W_N^{6k}$

(i) $X[0] = 1 + 2 + 3 + 4$
$= 10$

(ii) $X[1] = 1 + 2\, W_N^{2} + 3\, W_N^{4} + 4\, W_N^{6}$

$= 1 + 2(-j) + 3(-1) + 4(j)$

$\therefore X[1] = -2 + 2j$

(iii) $X[2] = 1 + 2\, W_N^{4} + 3\, W_N^{8} + 4\, W_N^{12}$

$= 1 + 2(-1) + 3(1) + 4(-1)$

$\therefore X[2] = -2$

(iv) $X[3] = 1 + 2\, W_N^{6} + 3\, W_N^{12} + 4\, W_N^{18}$

$= 1 + 2(j) + 3(-1) + 4(-j)$

$\therefore X[3] = -2 - 2j$

(v) $X[4] = 1 + 2\, W_N^{8} + 3\, W_N^{16} + 4\, W_N^{24}$

$= 1 + 2(1) + 3(-1) + 4(1)$

$\therefore X[4] = 4$

(vi) $X[5] = 1 + 2\, W_N^{10} + 3\, W_N^{20} + 4\, W_N^{30}$

$= 1 + 2(-j) + 3(-1) + 4(j)$

$\therefore X[5] = -2 + 2j$

(vii) $X[6] = 1 + 2\, W_N^{12} + 3\, W_N^{24} + 4\, W_N^{36}$

$= 1 + 2(-1) + 3(1) + 4(-1)$

$\therefore X[6] = -2$

(viii) $X[7] = 1 + 2\, W_N^{14} + 3\, W_N^{28} + 4\, W_N^{42}$

$= 1 + 2(j) + 3(-1) + 4(-j)$

$\therefore X[7] = -2 - 2j$

$$X[k] = \begin{bmatrix} 10 & k=0 \\ -2+2j & k=1 \\ -2 & k=2 \\ -2-2j & k=3 \\ 4 & k=4 \\ -2+2j & k=5 \\ -2 & k=6 \\ -2-2j & k=7 \end{bmatrix}$$

➤ Matrix Representation of DFT and IDFT

DFT and IDFT equations can be implemented using Matrices.

For example consider $x[n]=\left\{ \underset{\uparrow}{1} \quad 2 \quad 3 \quad 2 \right\}$

Solution :

By DFT, $X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$ where $N = 4$ and $W_N^1 = e^{-j\frac{2\pi}{N}}$

$$X[k] = x(0)W_N^0 + x(1)\,W_N^k + x(2)\,W_N^{2k} + x(3)\,W_N^{3k}$$

$$\begin{array}{l} At \quad k=0, \\ At \quad k=1, \\ At \quad k=2, \\ At \quad k=3, \end{array} \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} w_N^0 & w_N^0 & w_N^0 & w_N^0 \\ w_N^0 & w_N^1 & w_N^2 & w_N^3 \\ w_N^0 & w_N^2 & w_N^4 & w_N^6 \\ w_N^0 & w_N^3 & w_N^6 & w_N^9 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

$$X[k] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}$$

$$X[k] = \begin{bmatrix} 8 & k=0 \\ -2 & k=1 \\ 0 & k=2 \\ -2 & k=3 \end{bmatrix}$$

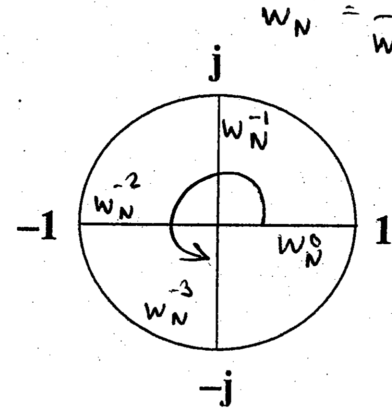
Q(4) Let $X[k] = \{\, 8, -2, 0, -2 \,\}$ Find x[n].

Solution :

By IDFT, $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, W_N^{-nk}$ where $N = 4$ and $W_N^1 = e^{-j\frac{2\pi}{N}}$

$$W_N^{-1} = \frac{1}{W_N^1} = \frac{1}{-j} = j$$

$$x[n] = \frac{1}{4} \begin{bmatrix} w_N^0 & w_N^0 & w_N^0 & w_N^0 \\ w_N^0 & w_N^{-1} & w_N^{-2} & w_N^{-3} \\ w_N^0 & w_N^{-2} & w_N^{-4} & w_N^{-6} \\ w_N^0 & w_N^{-3} & w_N^{-6} & w_N^{-9} \end{bmatrix} \begin{bmatrix} 8 \\ -2 \\ 0 \\ -2 \end{bmatrix}$$



$$x[n] = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix} \begin{bmatrix} 8 \\ -2 \\ 0 \\ 2 \end{bmatrix}$$

$$x[n] = \begin{bmatrix} 1 & n=0 \\ 2 & n=1 \\ 3 & n=2 \\ 2 & n=3 \end{bmatrix}$$

Q(5) **Find the DFT of the following sequences :**

a) $x[n]=\{\underset{\uparrow}{1}\ 1\ 1\ 1\}$ b) $x[n]=\{\underset{\uparrow}{1}\ 0\ 0\ 0\}$

Solution :

(a) By DFT, $X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$

where $N = 4$ and $W_N^1 = e^{-j\frac{2\pi}{N}}$

$$X[k] = \begin{bmatrix} w_N^0 & w_N^0 & w_N^0 & w_N^0 \\ w_N^0 & w_N^1 & w_N^2 & w_N^3 \\ w_N^0 & w_N^2 & w_N^4 & w_N^6 \\ w_N^0 & w_N^3 & w_N^6 & w_N^9 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

$$X[k] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$X[k] = \begin{bmatrix} 4 & k=0 \\ 0 & \\ 0 & \\ 0 & \end{bmatrix}$$

$$X[k] = 4\begin{bmatrix} 1 & k=0 \\ 0 & \\ 0 & \\ 0 & \end{bmatrix}$$

X[k] = 4 δ[k]

where

$$\delta[k] = \begin{bmatrix} 1 & k=0 \\ 0 & \\ 0 & \\ 0 & \end{bmatrix}$$

(b) By DFT, $X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$

where $N = 4$ and $W_N^1 = e^{-j\frac{2\pi}{N}}$

$$X[k] = \begin{bmatrix} w_N^0 & w_N^0 & w_N^0 & w_N^0 \\ w_N^0 & w_N^1 & w_N^2 & w_N^3 \\ w_N^0 & w_N^2 & w_N^4 & w_N^6 \\ w_N^0 & w_N^3 & w_N^6 & w_N^9 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

$$X[k] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X[k] = \begin{bmatrix} 1 & k=0 \\ 1 & \\ 1 & \\ 1 & \end{bmatrix}$$

X[k] = 1

(1) DFT { u[n]$_{N\,pt}$ } = $N\ \delta(k)$

$k = 0$

Where $\delta[k] = \begin{cases} 1 & \text{for } 0 \le k \le N-1 \\ 0 & \text{otherwise} \end{cases}$

(2) DFT { δ[n] } = 1

Q(6) Let $x[n] = \cos(n\frac{\pi}{2})\, u[n]$. Find X[k]. **HW**

Ans : $X[k] = \begin{bmatrix} 0 & k=0 \\ 2 & \\ 0 & \\ 2 & \end{bmatrix}$

2.2 Properties of DFT

[1] Scaling and Linearity property

If $x_1[n] \longleftrightarrow X_1[k]$

$x_2[n] \longleftrightarrow X_2[k]$

Then

(i) $\text{DFT}\{a\,x_1[n]\} = a\,X_1(k)$

(ii) $\text{DFT}\{a\,x_1[n] + b\,x_2[n]\} = a\,X_1(k) + b\,X_2(k)$

Q(7) Given x[n]= { 1,2,3,4 }.

(a) Find X[k].

(b) Let $p[n] = 2\,\delta[n]+x[n]$. Find P[k] using X[k].

(c) Let $q[n] = 2 + x[n]$ Find Q[k] using X[k].

Solution :

(a) To find X[k]

① formula
② matrix
③ •••• matrix substiⁿ
④ Calculation

⑤ANS
$$X[k] = \begin{bmatrix} 10 & k=0 \\ -2+2j & \\ -2 & \\ -2-2j & \end{bmatrix}$$

(b) To find P[k]

Given $p[n] = 2\,\delta[n]+x[n]$.

By Scaling and Linearity Property of DFT,

$P[k] = 2\,\text{DFT}\{\delta[n]\} + X[k]$

$P[k] = 2\{1\} + X[k]$

$\mathbf{P[k] = 2 + X[k]}$

k=0, $P[0] = 2 + X(0) = 2+10 = \boxed{12}$

k=1, $P(1) = 2 + X(1) = 2+(-2+2j) = \boxed{2j}$

k=2, $P(2) = 2 + X(2) = 2-2 = \boxed{0}$

k=3, $P(3) = 2 + X(3) = 2+(-2-2j) = \boxed{-2j}$

(c) To find Q[k]

Given $q[n] = 2 + x[n]$.

$q[n] = 2\{\underset{4pt}{u(n)}\} + x(n)$

By Scaling and Linearity Property of DFT,

$Q[k] = 2\,\text{DFT}\{\underset{4pt}{u(n)}\} + X(k)$

$Q[k] = 2\{4\,\delta(k)\} + X(k)$

$Q[k] = 8\,\delta k + X(k)$

$$Q[k] = 8\begin{bmatrix} 1 & k=0 \\ 0 & \\ 0 & \\ 0 & \end{bmatrix} + \begin{bmatrix} 10 & k=0 \\ -2+2j & \\ -2 & \\ -2-2j & \end{bmatrix}$$

$$Q[k] = \begin{bmatrix} 18 & k=0 \\ -2+2j & \\ -2 & \\ -2-2j & \end{bmatrix}$$

[2] Periodicity Property : Both DFT and IDFT equations produce periodic results with period = N.

> If $x[n] \longleftrightarrow X[k]$
> Then
> (i) $x[n] = x[n+N] = x[n \text{ MOD } N] = x[((n))]$
>
> (ii) $X[k] = X[k+N] = X[k \text{ MOD } N] = X[((k))_N]$

For example Let $x[n] = \{ \underset{\uparrow}{1} \ 2 \ 3 \ 4 \}$ Find x[0], x[3] and x[7].

Solution :

(i) $x[0] = 1$

(ii) $x[3] = 4$

To find $x(7)$

$x(n) = x(n \text{ MOD } N)$

$x(7) = x(7 \text{ MOD } 4)$

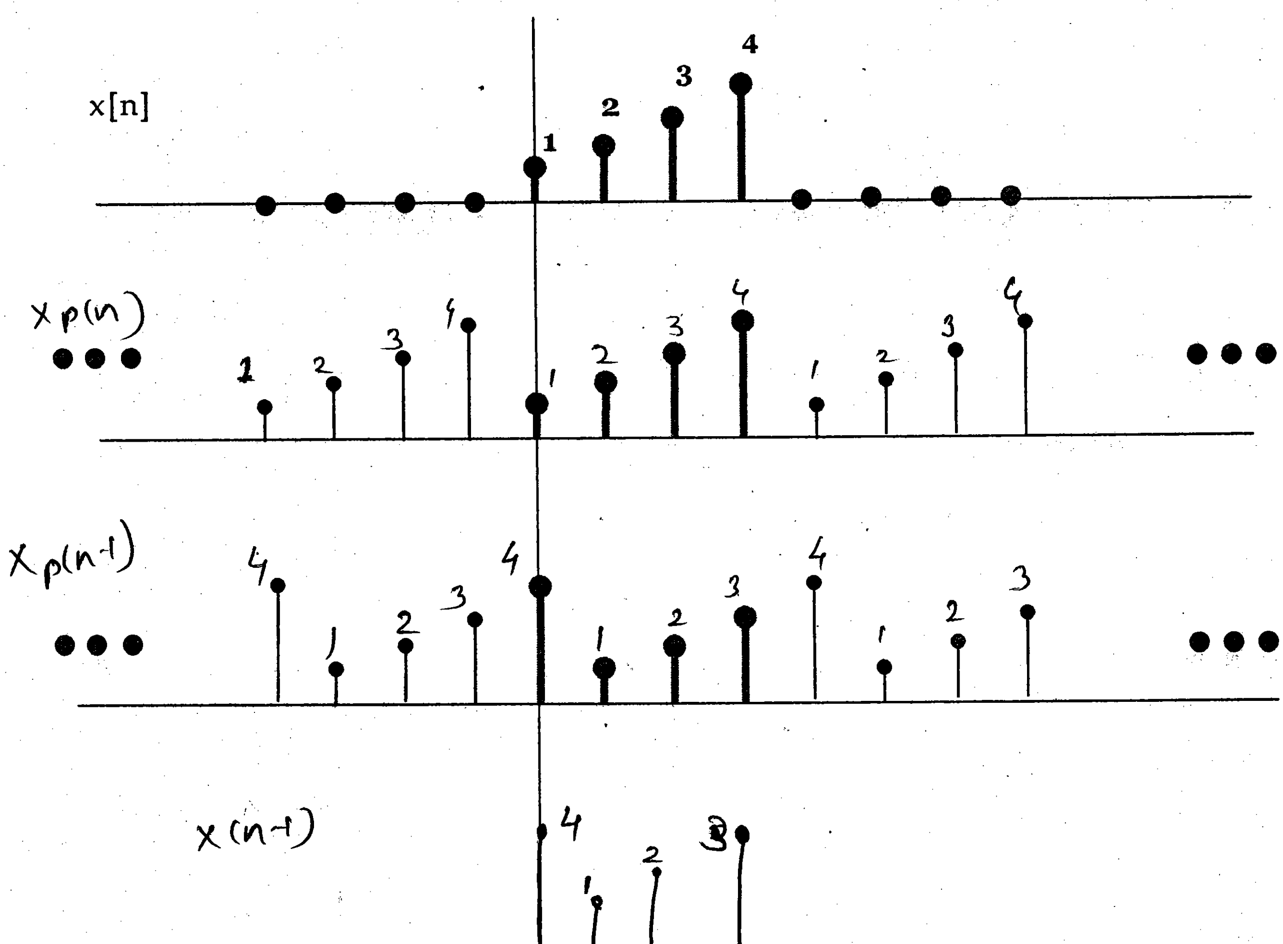
$x(7) = x(3)$

$\boxed{x(7) = 4}$

☆ **Circular Shifting of DFT input signal x[n]:**

Q(8) Given x[n] = { 1,2,3,4 }. Find x[((n-1))]

Solution : To find x[((n-1))]



Circular Shifting of Periodic Signals

1) $x[n] = \{1, 2, 3, 4\}$
2) $x[n-1] = \{4, 1, 2, 3\}$
3) $x[n+1] = \{2, 3, 4, 1\}$
4) $x[-n] = \{1, 4, 3, 2\}$
5) $x[-n+1] = \{2, 1, 4, 3\}$
6) $x[-n-1] = \{4, 3, 2, 1\}$

[3] Time Shift Property

If $x[n] \longleftrightarrow X[k]$

Then

(i) $DFT\{x[n-m]\} = W_N^{mk} X[k]$

(ii) $DFT\{x[n+m]\} = W_N^{-mk} X[k]$

[4] Frequency Shift property

If $x[n] \longleftrightarrow X[k]$

Then

(i) $DFT\{W_N^{-mn} x[n]\} = X[k-m]$

complex variable.

(ii) $DFT\{W_N^{mn} x[n]\} = X[k+m]$

[5] Time Reversal Property

If $x[n] \longleftrightarrow X[k]$

Then

$$DFT\{x[-n]\} = X[-k] = X[-k+N] = X[N-k]$$

Q(9) Given $x[n] = \{1, 2, 3, 4\}$

(a) Find $X[k]$ using DFT equation.

Solution :

• • • • • • • • • • •

$$X[k] = \begin{bmatrix} 10 & k=0 \\ -2+2j & \\ -2 & \\ -2-2j & \end{bmatrix} \text{ ANS}$$

(b) Let p[n] = { 4, 1, 2, 3 } Find P[k] using X[k].

Solution :

Let $p[n] = x(n-1)$

By Time Shift Property of DFT,

$$P[k] = W_N^k\, x(k)$$

Time Shift Property
$DFT\{x[n-m]\} = W_N^{mk} X[k]$

k=0, $P[0] = W_N^0\, x(0) = (1)(10) = \boxed{10}$

k=1, $P[1] = W_N^1\, x(1) = (-j)(-2+2j) = 2j - 2j^2 = \boxed{2j+2}$

k=2, $P[2] = W_N^2\, x(2) = (-1)(-2) = \boxed{2}$

k=3, $P[3] = W_N^3\, x(3) = (j)(-2-2j) = -2j - 2j^2 = \boxed{2-2j}$

(c) Let $q[n] = (-1)^n x[-n]$ Find Q[k] using ~~X[k]~~ x(n)

Solution :

Given $q[n] = (-1)^n x[+n]$

$$= (W_N^2)^n x(n)$$

$$= W_N^{2n} x(n)$$

Freq. Shift Property
$DFT\{ W_N^{-mn} x[n] \} = X[k-m]$

By Freq. shift property of DFT

$$q(k) = X[k+2]$$

$$q(k) = \begin{bmatrix} -2 & k=0 \\ -2-2j & k=1 \\ 10 & k=2 \\ -2+2j & k=3 \end{bmatrix}$$

Ans

(d) Let r[n] = { 1, 4, 3, 2 } Find R[k] using X[k].

Solution :

Let $r[n] = x(-n)$

By DFT, $R[k] = X(-k)$

By Time Reversal Property of DFT,

Time Reversal Property
$DFT\{x[-n]\} = X[-k] = X[N-k]$

$$R[k] = X[-k]$$

$$= \begin{bmatrix} 10 & k=0 \\ -2-2j & k=1 \\ -2 & k=2 \\ -2+2j & k=3 \end{bmatrix}$$

Ans

(e) Let $s[n] = \cos(\frac{n\pi}{2})\, x[n]$. Find $S[k]$ using $X[k]$.

Solution :

Given $s[n] = \cos(\frac{n\pi}{2})\ \ x[n]$

$$s[n] = \left(\frac{e^{j\frac{n\pi}{2}} + e^{-j\frac{n\pi}{2}}}{2}\right) x[n]$$

$$s[n] = \frac{1}{2}\left(e^{j\frac{n\pi}{2}} x[n] + e^{-j\frac{n\pi}{2}} x[n]\right)$$

Let $e^{j\frac{n\pi}{2}} = W_N^{-mn}$, $\qquad e^{-j\frac{n\pi}{2}} = W_N^{mn}$

$$\therefore\ s[n] = \frac{1}{2}\left[W_N^{-mn}\ \ x[n] + W_N^{mn}\ \ x[n]\right]$$

Frequency shift property

$W_N^{-mn}\ x[n] \longleftrightarrow X[k-m]$

By DFT frequency shift property.

$$\therefore\ S[k] = \frac{1}{2}(X[k-m] + X[k+m])$$

To find m :

$$e^{jn\frac{\pi}{2}} = W_N^{-mn} = \left(W_N^{1}\right)^{-mn} = \left(e^{-j\frac{\pi}{2}}\right)^{mn} = e^{jn\frac{\pi}{2}m}$$

By equating we get, m = 1.

By substituting m=1 in S[k] we get,

$$S[k] = \frac{1}{2}(X[k-1] + X[k+1])$$

$$S[k] = \frac{1}{2}\left(\begin{bmatrix} -2-2j \\ 10 \\ -2+2j \\ -2 \end{bmatrix} + \begin{bmatrix} -2+2j \\ -2 \\ -2-2j \\ 10 \end{bmatrix}\right)$$

$$S[k] = \frac{1}{2}\begin{bmatrix} -4 \\ 8 \\ -4 \\ 8 \end{bmatrix} \qquad \text{ANS} \quad \because\ S[k] = \begin{bmatrix} -2 & k=0 \\ 4 & \\ -2 & \\ 4 & \end{bmatrix}$$

---

[6] Symmetry Property

Imp

> If $x[n]$ is N point real valued sequence
> and $x[n] \longleftrightarrow X[k]$
> Then
> $$X[k] = X^*[-k]$$

For example consider 4 point sequence $x[n] = \{\ 1, 2, 3, 4\ \}$

Then By DFT,

$$X[k] = \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix} \qquad X^*[k] = \begin{bmatrix} 10 \\ -2-2j \\ -2 \\ -2+2j \end{bmatrix} \qquad X^*[-k] = \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

Here $X[k] == X^*[-k]$

NOTE : Simplified Symmetry Property

If x[n] is N point real valued sequence,

Then Real part of X[K] is symmetric about $K = \frac{N}{2}$ and imaginary part of X(K) is Antisymm. about $K = \frac{N}{2}$

For example consider 4 point sequence x[n] = { 1, 2, 3, 4 }

Then By DFT,

$$X[k] = \begin{bmatrix} 10 & k=0 \\ -2+2j & k=1 \\ -2 & \boxed{k=2} \\ -2-2j & k=3 \end{bmatrix}$$

$\because K = \frac{N}{2} = 2$

Symm

Anti-Symm

Q(10) The first five points of the eight point DFT of a real valued sequence are { 0.25, 0.125 - j 0.3018, 0, 0.125 - j 0.0518, 0 }. Determine the remaining three points.

$$X[k] = \begin{bmatrix} 0.25 & k=0 \\ 0.125 - j0.3018 & k=1 \\ 0 & k=2 \\ 0.125 - j0.0518 & k=3 \\ 0 & \boxed{k=4} \\ \boxed{0.125 + j0.0518} & k=5 \\ \boxed{0} & k=6 \\ \boxed{0.125 + j0.3018} & k=7 \end{bmatrix}$$

$K = \frac{N}{2} = 4$

Here, x(n) is real sequence,

Then by symmetry property of DFT

If $x(n) \longrightarrow X(K)$

then $X(K) = X^*(-K)$

$= X^*[N-K]$

$X[K] = X^*[8-K]$

$K = 5$, $X[5] = X^*[3] = 0.125 + j0.0518$

$K = 6$ $X[6] = X^*[2] = 0$

$K = 7$ $X[7] = X^*[1] = 0.125 + j.3018$

[7] DFT Property of Even Signal and Odd Signal

If $x[n] = x[-n]$ Then $X[k] = X[-k]$ i.e. If x[n] is EVEN Then X[k]	If $x[n] = -x[-n]$ Then $X[k] = -X[-k]$ i.e. If x[n] is ODD Then X[k] is ODD

Q(11) Let x[n]= { 1, 2, 3, 4 }

(a) Find X[k].

Solution :

$$X[k] = \begin{bmatrix} 10 & k=0 \\ -2+2j & \\ -2 & \\ -2-2j & \end{bmatrix}$$

Ans

For any DT signal (Real OR Complex)

$$x[n] = x_e[n] + x_o[n]$$

where

$$x_e[n] = \tfrac{1}{2}\{x[n] + x[-n]\}$$

$$x_o[n] = \tfrac{1}{2}\{x[n] - x[-n]\}$$

(b) Find DFT of $x_e[n]$ and $x_o[n]$ using X[k] and not otherwise

Solution (i) To find : $x_e[n]$

$x_e[n] = \tfrac{1}{2}(x[n] + x[-n])$

By Linearity Property of DFT,

$X_e[k] = \tfrac{1}{2}(X[k] + X[-k])$

$$X_e[k] = \tfrac{1}{2}\left\{\begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix} + \begin{bmatrix} 10 \\ -2-2j \\ -2 \\ -2+2j \end{bmatrix}\right\}$$

$$X_e[k] = \tfrac{1}{2}\begin{bmatrix} 20 & k=0 \\ -4 & \\ -4 & \\ -4 & \end{bmatrix} \qquad X_e[k] = \begin{bmatrix} 10 & k=0 \\ -2 & \\ -2 & \\ -2 & \end{bmatrix} = \text{Real}[X(k)]$$

Ans

* [Only for real valued x(n)]

Not for complex value

Solution (ii) To find DFT of $x_o[n]$

$x_o[n] = \tfrac{1}{2}(x[n] - x[-n])$

By Linearity property of DFT,

$\mathbf{X_o[k] = \tfrac{1}{2}(X[k] - X[-k])}$

$$X_o[k] = \tfrac{1}{2}\left\{\begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix} - \begin{bmatrix} 10 \\ -2-2j \\ -2 \\ -2+2j \end{bmatrix}\right\}$$

$$X_o[k] = \tfrac{1}{2}\begin{bmatrix} 0 & k=0 \\ 4j & \\ 0 & \\ -4j & \end{bmatrix} \qquad X_o[k] = \begin{bmatrix} 0 & k=0 \\ 2j & \\ 0 & \\ -2j & \end{bmatrix} = \text{Imaginary}[X(k)]$$

[Only for real valued x(n)]

NOTE :

(I) For any DT signal (Real OR Complex)

(1) $X_e[k] = \frac{1}{2}\{X[k] + X[-k]\}$

(2) $X_o[k] = \frac{1}{2}\{X[k] - X[-k]\}$

(3) $X[K] = X_e[K] + X_o[K]$

(II) If x[n] is Real Valued Sequence

Then

(1) $X_e[k] =$ Real $[X(K)]_*$

(2) $X_o[k] =$ Imaginary $[X(k)]_*$

(3) $X[k] = X^*[-K]_*$

[symmetry property]

3 marks

Q(12) Let x[n] = { j, -2j, 2j, j, 4j, j, 2j, -2j }. The first five coefficients of X[k] are { 7j, -1.58j, j, -4.4j, 3j }. Find the remaining three coefficients.

Solution : Given **x[n] = { j, -2j, 2j, j, 4j, j, 2j, -2j }.**

Here $x(n) = x(-n)$

$\therefore$ $x(n)$ is an even signal.

By DFT property of even signal

If $x(n) = x(-n)$

then,

$$x(k) = x(-k)$$

$$= x(N-k)$$

$$\therefore x(k) = x(8-k)$$

$\therefore$ For $K=5$, $x(5) = x(3) = \boxed{-4.4j}$

$K=6$, $x(6) = x(2) = \boxed{j}$

Ans $K=7$, $x(7) = x(1) = \boxed{-1.58j}$

$$X[K] = \begin{bmatrix} 7j & K=0 \\ -1.58j & K=1 \\ j & K=2 \\ -4.4j & K=3 \\ 3j & K=4 \\ -4.4j & K=5 \\ j & K=6 \\ -1.58j & K=7 \end{bmatrix}$$

[8] Complex Conjugate Property

If x[n] $\longleftrightarrow$ X[k]
Then
DFT { $x^*[n]$ } $= X^*[-K]$

Q(13) Given x[n] = { (1 + j), (2 + j2), (3 + j3), (4 + j2) }
(a) Find X[k]

Solution :

By DFT, $X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$ where $N = 4$ and $W_N^1 = e^{-j\frac{2\pi}{N}}$

$$X[k] = \begin{bmatrix} w_N^0 & w_N^0 & w_N^0 & w_N^0 \\ w_N^0 & w_N^1 & w_N^2 & w_N^3 \\ w_N^0 & w_N^2 & w_N^4 & w_N^6 \\ w_N^0 & w_N^3 & w_N^6 & w_N^9 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

$$X[k] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 + j \\ 2 + 2j \\ 3 + 3j \\ 4 + 2j \end{bmatrix}$$

$$X[k] = \begin{bmatrix} 10 + 8j & k=0 \\ -2 & k=1 \\ -2 & k=2 \\ -2-4j & k=3 \end{bmatrix}$$

(b) Let p[n] = { 1, 2, 3, 4 } and q[n] = { 1, 2, 3, 2 }. Find P[k] and Q[k] using X[k] and not otherwise. <u>IMP</u>

Solution : (i) To find P[k]

$x(n) = p(n) + jq(n)$ — (I)

By complex conjugate property,

$x^*(n) = p(n) - jq(n)$ — (II)

By eq^n (I) + (II), we get,

$x(n) + x^*(n) = 2\,p(n)$

$\therefore p(n) = \frac{1}{2}\left[x(n) + x^*(n)\right]$

By Linearity property,

$\therefore P(K) = \frac{1}{2}\left[X(K) + X^*(-K)\right]$

$$\therefore P[k] = \frac{1}{2}\left\{ \begin{bmatrix} 10+8j \\ -2 \\ -2 \\ -2-4j \end{bmatrix} + \begin{bmatrix} 10-8j \\ -2+4j \\ -2 \\ -2 \end{bmatrix} \right\}$$

$$\therefore P[k] = \frac{1}{2}\begin{bmatrix} 20 & k=0 \\ -4+4j & k=1 \\ -4 & k=2 \\ -4-4j & k=3 \end{bmatrix} = \begin{bmatrix} 10 & k=0 \\ -2+2j & k=1 \\ -2 & k=2 \\ -2-2j & k=3 \end{bmatrix} \text{ ANS}$$

(ii) To find Q[k]

Subtracting equation (2) from (1)

$2j\,q[n] = x[n] - x^*[n]$

$\therefore q[n] = \frac{1}{2j}\{ x[n] - x^*[n] \}$

By DFT

$$Q[k] = \frac{1}{2j}\left\{ X[k] - X^*[-k] \right\}$$

$$\therefore Q[k] = \frac{1}{2j}\left\{ \begin{bmatrix} 10+8j & k=0 \\ -2 & k=1 \\ -2 & k=2 \\ -2-4j & k=3 \end{bmatrix} - \begin{bmatrix} 10-8j & k=0 \\ -2+4j & \\ -2 & \\ -2 & \end{bmatrix} \right\}$$

$$\therefore Q[k] = \frac{1}{2j}\begin{bmatrix} 16j & k=0 \\ -4j & k=1 \\ 0 & k=2 \\ -4j & k=3 \end{bmatrix} = \begin{bmatrix} 8 & k=0 \\ -2 & k=1 \\ 0 & k=2 \\ -2 & k=3 \end{bmatrix} \text{ ANS}$$

Circular Convolution of Discrète Time Signals:

$$\boxed{y[n] = \sum_{m=0}^{N-1} x[m]\, h[n-m]}$$

Q(14) Given $x[n] = \{ \underset{\uparrow}{1}\ 2\ 3\ 4 \}$ $h[n] = \{ \underset{\uparrow}{5}\ 6 \cdot 7 \}$

(a) Find $y_1[n] = x[n] * h[n]$

Solution : (a) To find $y_1[n] = x[n] * h[n]$

$y_1[n] = \sum_{m=} x[m]\; h[n-m]$ where (i) $x[m] = \{ \underset{\uparrow}{1}, 2, 3, 4 \}$

(ii) $h[m] = \{ \underset{\uparrow}{5}, 6, 7 \}$

(iii) $h[-m] = \{ 7, 6, \underset{\uparrow}{5} \}$

$$y_1[0] = \sum_{m=0}^{3} x[m]\,h[-m] = (1)(5) + (2)(\bullet) + (3)(\bullet) + (4)(\bullet)$$
$$= \mathbf{5}$$

$$y_1[1] = \sum_{m=0}^{3} x[m]\,h[1-m] = (1)(6) + (2)(5) + (3)(\bullet) + (4)(\bullet)$$
$$= \mathbf{16}$$

$$y_1[2] = \sum_{m=0}^{3} x[m]\,h[2-m] = (1)(7) + (2)(6) + (3)(5) + (4)(\bullet)$$
$$= \mathbf{34}$$

$$y_1[3] = \sum_{m=0}^{3} x[m]\,h[3-m] = (1)(\bullet) + (2)(7) + (3)(6) + (4)(5)$$
$$= \mathbf{52}$$

$$y_1[4] = \sum_{m=0}^{3} x[m]\,h[4-m] = (1)(\bullet) + (7)(\bullet) + (3)(7) + (4)(6)$$
$$= \mathbf{45}$$

$$y_1[5] = \sum_{m=0}^{3} x[m]\,h[5-m] = (1)(\bullet) + (7)(\bullet) + (3)(\bullet) + (4)(7)$$
$$= \mathbf{28}$$

ANS : $y_1[n] = \left\{ \underset{\uparrow}{5}, \ 16, \ 34, \ 52, \ 45, \ 28 \right\}$ **for n ≥ 0**

Rough Work

x[m] ⇒	1	2	3	4	y_1[n]	
h[-m] 7 6	5				5	n=0
h[-m+1] 7	6	5			16	n=1
h[-m+2]	7	6	5		34	n=2
h[-m+3]		7	6	5	52	n=3
h[-m+4]			7	6	45	n=4
h[-m+5]				7	28	n=5

(b) Find $y_2[n] = x[n] \otimes h[n]$

Solution : To find Circular Convolution

$$y_2[n] = \sum_{m=0}^{N-1} x[m]\,h[n-m]$$

Where i) N = 4

ii) x[n] = {1, 2, 3, 4} periodic

iii). h[n] = {5, 6, 7, 0} periodic

iv). h[-n] = {5, 0, 7, 6}

Here '0' is meaningless
It just added to
make equal length

$$y_2[0] = \sum_{m=0}^{3} x[m]\,h[-m] = (1)(5) + (2)(0) + (3)(7) + (4)(6)$$
$$= \boxed{50}$$

$$y_2[1] = \sum_{m=0}^{3} x[m]\,h[1-m] = (1)(6) + (2)(5) + (3)(0) + (4)(7)$$
$$= \boxed{44}$$

$$y_2[2] = \sum_{m=0}^{3} x[m]\,h[2-m] = (1)(7) + (2)(6) + (3)(5) + (4)(0)$$
$$= \boxed{34}$$

$$y_2[3] = \sum_{m=0}^{3} x[m]\,h[3-m] = (1)(0) + (2)(7) + (3)(6) + (4)(5)$$
$$= \boxed{52.}$$

ANS : $y_2[n] = \{ 50, 44, 34, 52 \}$ for $n \geq 0$

Rough Work

$x[m] \Rightarrow$	1	2	3	4	$y_2[n]$	
$h[-m]$	5	0	7	6	50	$n=0$
$h[-m+1]$	6	5	0	7	44	$n=1$
$h[-m+2]$	7	6	5	0	34	$n=2$
$h[-m+3]$	0	7	6	5	52	$n=3$

(c) Find $y_3[n] = x[n] ⑤ h[n]$

Solution : To find Circular Convolution

$$y_3[n] = \sum_{m=0}^{N-1} x[m]\,h[n-m]$$

Where i) $N = 5$

* ii) $x[n] = \{ 1, 2, 3, 4, 0 \}$

* iii). $h[n] = \{ 5, 6, 7, 0, 0 \}$

* iv). $h[-n] = \{ 5, 0, 0, 7, 6 \}$

$$y_3[0] = \sum_{m=0}^{4} x[m]\,h[-m] = (1)(5) + (2)(0) + (3)(0) + (4)(7) + (0)(6)$$
$$= \boxed{33}$$

$$y_3[1] = \sum_{m=0}^{4} x[m]\,h[1-m] = (1)(6) + (2)(5) + (3)(0) + (4)(0) + (0)(7)$$
$$= \boxed{16}$$

$$y_3[2] = \sum_{m=0}^{4} x[m]\,h[2-m] = (1)(7)+(2)(6)+(3)(5)+(4)(0)+(0)(0)$$

$$= \boxed{34}$$

$$y_3[3] = \sum_{m=0}^{4} x[m]\,h[3-m] = (1)(0)+(2)(7)+(3)(6)+(4)(5)+(0)(7)$$

$$= \boxed{52}$$

$$y_3[4] = \sum_{m=0}^{4} x[m]\,h[4-m] = (1)(0)+(2)(0)+(3)(7)+(4)(6)+(0)(5)$$

$$= \boxed{45}$$

ANS : $y_3[n] = \{\ 33, 16, 34, 52, 45\ \}$ for $n \geq 0$

Rough Work

$x[m] \Rightarrow$	1	2	3	4	0	$y_3[n]$
$h[-m]$	5	0	0	7	6	33
$h[-m+1]$	6	5	0	0	7	16
$h[-m+2]$	7	6	5	0	0	34
$h[-m+3]$	0	7	6	5	0	52
$h[-m+4]$	0	0	7	6	5	45

(d) Find $y_4[n] = x[n] \circledast h[n]$

Solution : To find Circular Convolution

$$y_4[n] = \sum_{m=0}^{N-1} x[m]\,h[n-m]$$

Where i) $N = 6$

* ii) $x[n] = \{1, 2, 3, 4, 0, 0\}$

* iii). $h[n] = \{5, 6, 7, 0, 0, 0\}$

* iv). $h[-n] = \{5, 0, 0, 0, 7, 6\}$

$$y_4[0] = \sum_{m=0}^{5} x[m]\,h[-m] = (1)(5)+(2)(0)+(3)(0)+(4)(0)+(0)(7)+(0)(6)$$

$$= \boxed{5}$$

$$y_4[1] = \sum_{m=0}^{5} x[m]\,h[1-m] = (1)(5)+(2)(0)+(3)(0)+(4)(0)+(0)(7)+(0)(6)$$

$$= \boxed{16}$$

$$y_4[2] = \sum_{m=0}^{5} x[m]\,h[2-m] = (1)(6)+(2)(5)+(3)(0)+(4)(0)+(0)(0)+(0)(7)$$

$$= \boxed{34}$$

$$y_4[3] = \sum_{m=0}^{5} x[m]\, h[3-m]$$

$$= (1)(7)+(2)(6)+(3)(5)+(4)(0)+(0)(7)+(0)(0)$$

$$= \boxed{52}$$

$$y_4[4] = \sum_{m=0}^{5} x[m]\, h[4-m]$$

$$= (1)(0)+(2)(7)+(3)(6)+(4)(5)+(0)(0)+(0)(0)$$

$$= \boxed{45}$$

$$y_4[5] = \sum_{m=0}^{5} x[m]\, h[5-m]$$

$$= (1)(0)+(2)(0)+(3)(7)+(4)(6)+(0)(5)+(0)(0)$$

$$= \boxed{28}$$

ANS : $y_4[n] = \{ 5, 16, 34, 52, 45, 28 \}$ for $n \geq 0$,

Rough Work

x[m] ⇒	1	2	3	4	0	0	$y_4[n]$
h[-m]	5	0	0	0	7	6	5
h[-m+1]	6	5	0	0	0	7	16
h[-m+2]	7	6	5	0	0	0	34
h[-m+3]	0	7	6	5	0	0	52
h[-m+4]	0	0	7	6	5	0	45
h[-m+5]	0	0	0	7	6	5	28

(e) Find $y_5[n] = x[n]$ ⑦ $h[n]$ HW

Solution : To find Circular Convolution

$$y_5[n] = \sum_{m=0}^{N-1} x[m]\, h[n-m]$$

Where i) N = 7

*

ii) $x[n] = \{ 1, 2, 3, 4, 0, 0, 0 \}$

*

iii). $h[n] = \{ 5, 6, 7, 0, 0, 0, 0 \}$

*

iv). $h[-n] = \{ 5, 0, 0, 0, 0, 7, 6 \}$

$$y_5[0] = \sum_{m=0}^{6} x[m]\, h[-m] = (1)(5)+(2)(0)+(3)(0)+(4)(0)+(0)(0)+(0)(7)+(0)(6)$$

$$= \boxed{5}$$

$$y_5[1] = \sum_{m=0}^{6} x[m]\, h[1-m]$$

$$= (1)(6)+(2)(5)+(3)(0)+(4)(0)+(0)(0)+(0)(0)+(0)(7)$$

$$= \boxed{16}$$

$$y_5[2] = \sum_{m=0}^{6} x[m]\, h[2-m]$$

$$=(1)(7)+(2)(6)+(3)(5)+(4)(0)+(0)(0)+(0)(0)+(0)(0)$$

$$= \boxed{34}$$

$$y_5[3] = \sum_{m=0}^{6} x[m]\, h[3-m]$$

$$= (1)(0)+(2)(7)+(3)(6)+(4)(5)+(0)(0)+(0)(0)+(0)(0)$$

$$= \boxed{52}$$

$$y_5[4] = \sum_{m=0}^{6} x[m]\, h[4-m]$$

$$= (1)(0)+(2)(0)+(3)(7)+(4)(6)+(0)(5)+(0)(0)+(0)(0)$$

$$= \boxed{45}$$

$$y_5[5] = \sum_{m=0}^{6} x[m]\, h[5-m]$$

$$= (1)(0)+(2)(0)+(3)(0)+(4)(7)+(0)(6)+(0)(5)+(0)(0)$$

$$= \boxed{28}$$

$$y_5[6] = \sum_{m=0}^{6} x[m]\, h[6-m]$$

$$= (1)(0)+(2)(0)+(3)(0)+(4)(0)+(0)(7)+(0)(6)+(0)(5)$$

$$= \boxed{0}$$

ANS : $y_5[n]$ = { 5, 16, 34, 52, 45, 28, 0 } for $n \geq 0$,

Rough Work

x[m] ⇒	1	2	3	4	0	0	0	$y_5[n]$
h[-m]	5	0	0	0	0	7	6	5
h[-m+1]	6	5	0	0	0	0	7	16
h[-m+2]	7	6	5	0	0	0	0	34
h[-m+3]	0	7	6	5	0	0	0	52
h[-m+4]	0	0	7	6	5	0	0	45
h[-m+5]	0	0	0	7	6	5	0	28
h[-m+6]	0	0	0	0	7	6	5	0

NOTE : (I) Let L be the Length of x[n]
M be the Length of h[n]
N be the Length of y[n]

And $y[n] = x[n] * h[n]$

Then $\boxed{N = L+M-1}$

(II) To find Linear Convolution using Circular Convolution

Select $\boxed{N \geq L+M-1}$

(III) In Circular Convolution

If $\boxed{N \geq L+M-1}$

Then

CC Result = LC Result.

(IV) Algorithm to find Linear Convolution using Circular Convolution

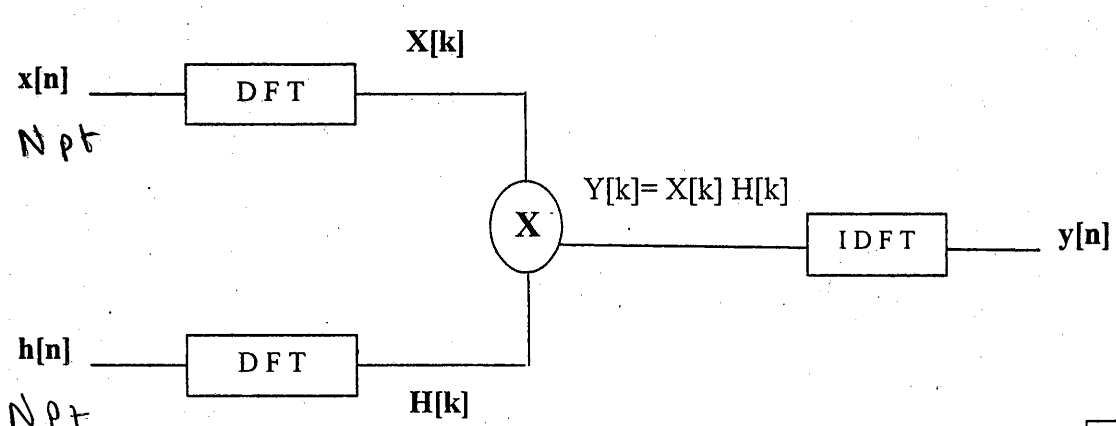
1. **Select N** such that $N \geq L + M - 1$.
2. Apply **zero padding.**
3. Perform **Circular Convolution.**

Ref DFT / FFT → Circular C.
CTFT
DTFT
ZT
LT } → Linear C.

[9] Circular Convolution Property

If $x[n] \longleftrightarrow X[k]$ and
$h[n] \longleftrightarrow H[k]$
Then
$$DFT\{ x[n] \otimes h[n] \} = X[k]\ H[k]$$

☆ Algorithm to find Circular Convolution using DFT-IDFT



Q(15) If x[n] = { 1, 2, 3, 4 } and h[n] = { 5, 6, 0, 0 }

(a) Find circular convolution using DFT-IDFT.

(b) Find circular convolution using Time Domain Method.

Solution : (a) To Find Circular Convolution using DFT

Step 1 : Find X[k]

By DFT, $X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$

$$X[k] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$X[k] = \begin{bmatrix} 10 \quad k=0 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

Step 2 : Find H[k]

By DFT, $H[k] = \sum_{n=0}^{N-1} h[n]\, W_N^{nk}$

$$H[k] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 11 \quad k=0 \\ 5-j6 \\ -1 \\ 5+j6 \end{bmatrix}$$

Step 3 : Find Y[k]

$Y[k] = X[k]\, H[k]$

$$Y[k] = \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix} \begin{bmatrix} 11 \\ 5-j6 \\ -1 \\ 5+j6 \end{bmatrix}$$

$$Y[k] = \begin{bmatrix} 110 \quad k=0 \\ 2+22j \\ 2 \\ 2-22j \end{bmatrix}$$

Step 4 : To find y[n]

By iDFT, $y[n] = \frac{1}{N} \sum_{k=0}^{3} W_N^{-nk}\; Y[k]$

$$y[n] = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 110 \\ 2+22j \\ 2 \\ 2-22j \end{bmatrix}$$

$$y[n] = \frac{1}{4} \begin{bmatrix} 116 \\ 64 \\ 108 \\ 39 \end{bmatrix} \qquad y[n] = \begin{bmatrix} 29 \\ 16 \\ 27 \\ 38 \end{bmatrix}$$ **ANS**

Solution : (b) To find Circular Convolution using Time Domain Method

$$y[n] = \sum_{m=0}^{N-1} x[m]\, h[n-m]$$

Where i) N = 4

ii) x[n] = { 1, 2, 3, 4 }

iii). h[n] = { 5, 6, 0, 0 }

iv). h[-n] = { 5, 0, 0, 6 }

$$y[0] = \sum_{m=0}^{3} x[m]\, h[-m] = (1)(5) + (2)(0) + (3)(0) + (4)(6) = \mathbf{29}$$

$$y[1] = \sum_{m=0}^{3} x[m]\, h[1-m] = (1)(6) + (2)(5) + (3)(0) + (4)(0) = \mathbf{16}$$

$$y[2] = \sum_{m=0}^{3} x[m]\, h[2-m] = (1)(0) + (2)(6) + (3)(5) + (4)(0) = \mathbf{27}$$

$$y[3] = \sum_{m=0}^{3} x[m]\, h[3-m] = (1)(0) + (2)(0) + (3)(6) + (4)(5) = \mathbf{38}$$

Rough Work

x[m]	1	2	3	4	$y_2[n]$
h[-m]	5	0	0	6	29
h[-m+1]	6	5	0	0	16
h[-m+2]	0	6	5	0	27
h[-m+3]	0	0	6	5	38

---

[10] Circular Correlation Property of DFT

If $x[n] \longleftrightarrow X[k]$ and
$h[n] \longleftrightarrow H[k]$
Then
DFT { x[n] o h[n] } = $X[k]\, H^*[k]$

Q(16) Let x[n] = (1, 2, 3, 2 } and h[n] = { 1, 2, 3, 4 }

(a) Find circular cross correlation using Time domain method.
(b) Find circular cross correlation using DFT.

Solution (a) : To find Circular Correlation

$$y[n] = \sum_{m=0}^{N-1} x[m]\, h[m-n] = \sum_{m=0}^{3} x[m]\, h[m-n]$$

$$y[0] = \sum_{m=0}^{3} x[m]\, h[m] = (1)(1) + (2)(4) + (3)(3) + (2)(2) = \boxed{22}$$

$$y[1] = \sum_{m=0}^{3} x[m]\, h[m-1] = (1)(2) + (2)(1) + (3)(4) + (2)(3) = \boxed{22}$$

$$y[2] = \sum_{m=0}^{3} x[m]\, h[m-2] = (1)(3) + (2)(2) + (3)(1) + (2)(4) = \boxed{18}$$

$$y[3] = \sum_{m=0}^{3} x[m]\, h[m-3] = (1)(4) + (2)(3) + (3)(2) + (2)(1) = \boxed{18}$$

Rough Work

x[m]	**1** ↓	**2**	**3**	**2**	**y [n]**
h[-m]	**1**	**4**	**3**	**2**	**22**
h	**2**	**1**	**4**	**3**	**22**
	3	**2**	**1**	**4**	**18**
	4	**3**	**2**	**1**	**18**

Solution (b) : To find Circular Cross-Correlation using DFT

(i) Find X[k]

Given $x[n] = \{ \underset{\uparrow}{1} \quad 2 \quad 3 \quad 2 \}$ By DFT, $X[k] = \begin{bmatrix} 8 \; k=0 \\ -2 \\ 0 \\ -2 \end{bmatrix}$

(ii) Find H[-k]

Given $h\,[n] = \{ \underset{\uparrow}{1} \quad 2 \quad 3 \quad 4 \}$ By DFT, $H[k] = \begin{bmatrix} 10 & k = 0 \\ -2 + 2j & \\ -2 & \\ -2 - 2j & \end{bmatrix}$

(iii) Find Y[k] = X[k] H*[k]

$$Y[k] = \begin{bmatrix} 8 \\ -2 \\ 0 \\ -2 \end{bmatrix} \begin{bmatrix} 10 \\ -2 - 2j \\ -2 \\ -2 + 2j \end{bmatrix} = \begin{bmatrix} 80 \\ 4 + 4j \\ 0 \\ 4 - 4j \end{bmatrix}$$

(iv) Find y[n]

By Inverse DFT, $y[n] = \dfrac{1}{N} \sum_{k=0}^{3} W_N^{-nk} \; Y[k]$

$$y[n] = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 80 \\ 4+4j \\ 0 \\ 4-4j \end{bmatrix}$$

$$y[n] = \frac{1}{4} \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$ **ANS**

[11] Parseval's Energy Theorem

If $x[n] \longleftrightarrow X[k]$

Then $E = \sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$

Q(17) Let x[n] = (1, 2, 3, 2 }

(a) Find X[k].

(b) Find Energy of the signal using X[k].

(c) Find Energy of the signal using x[n].

Solution (a) To find X[k]

$$X[k] = \begin{bmatrix} 8 & k=0 \\ -2 & \\ 0 & \\ -2 & \end{bmatrix}$$

(b) To find Energy using X[k]

$$E = \sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} | X[k] |^2$$

E =

(c) To find Energy using x[n]

$$E = \sum_{n=0}^{N-1} |x[n]|^2 =$$

E =

Q(18) Let X[k] = {1, –2, 3, –4, 5, –6} without evaluating its DFT /iDFT compute the following (a) X[0] (b) X[3] (c) $\sum_{n=0}^{5} |X[k]|^2$

Solution :

(a) X[0] ?

$$X[0] = \sum_{n=0}^{N-1} x[n] = 1-2+3-4+5-6 = \mathbf{-3}$$

(b) X[3] ?

$$X[3] = \sum_{n=0}^{N-1} (-1)^n x[n] = (1)-(-2)+(3)-(-4)+5-(-6) = \mathbf{21}$$

(c) $\sum_{n=0}^{5} |X[k]|^2$?

According to Parseval's theorem, $\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$

$$\therefore \sum_{k=0}^{5} |X[k]|^2 = 6 \sum_{n=0}^{5} |x[n]|^2 \qquad [N=6]$$

$$= 6[(1)^2 + (-2)^2 + (3)^2 + (-4)^2 + (5)^2 + (-6)^2]$$

$$= \boxed{546}$$

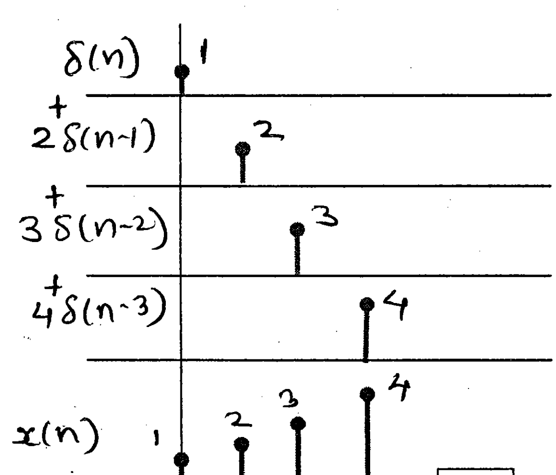
Summary of D F T Properties

	Properties of DFT		$X[k] = \sum_{n=0}^{N-1} x[n]\ W_N^{nk}$
1	Scaling	a x[n]]	a X[k]
2	Linearity	$\{a.x_1[n] + b x_2[n]\}$	$a X_1[k] + b\ X_2[k]$
3	Periodicity	x[n] = x[n + N] = x [n mod N] = x [((n))$_N$]	X[k] = X[k + N] = X [k mod N] = X [((k))$_N$]
4	Time Shift	$\{x[n-m]\}$	$W_N^{mk}\ x[k]$
5	Frequency Shift	$W^{-mn} x[n]$	$X[k-m]$
6	Time reversal	$\{x[-n]\}$	$X[-k]\ = X[N-k]$
7	Complex sequence	$x^*[n]$	$X^*[-k]$
8	Circular Convolution	$\{x[n] \otimes h[n]\}$	$X[k]\ H[k]$.
9	Circular Correlation	x[n] o y[n]	X[k] Y*[k]
10	Parseval's Theorem	$\sum_{n=0}^{N-1} \lvert x[n] \rvert^2$	$\frac{1}{N} \sum_{k=0}^{N-1} \lvert X[k] \rvert^2$
11.	Multiplication of Two Sequences	$x_1[n].\ x_2[n]$	$\frac{1}{N} X_1[k] \otimes X_2[k]$
12	If x[n] is real valued sequence Then i) $X[k] = X^*[-k]$ ii) Xe[k] = Real { X [k] } iii) Xo[k] = Imaginary { X [k] }		
13	If x[n] = x[- n] i.e. If x[n] is EVEN Then X[k] = X[-k] Then X[k] is also EVEN		
14	If x[n] = – x[- n] i.e. If x[n] is ODD Then X[k] = – X[-k] Then X[k] is also ODD		
15	**Sequence : x[n]**		**DFT : X[k]**
	Real & Even		Real & Even
	Real & Odd		Real & Odd
	Imaginary & odd		Imaginary & Odd
	Imaginary & Even		Imaginary & Even

Representation of DT signals using Shifted { δ[n] } signals.

Let $x[n] = \{ \underset{\uparrow}{1}\ \ 2\ \ 3\ \ 4 \}$

x[n]= δ[n]+2 δ[n-1]+3 δ[n-2]+4 δ[n-3]



Imp.

HW

Q(19) Given $x[n] = 2\,\delta[n] + 3\,\delta[n-1] + 4\,\delta[n-3]$. Let X[k] be six point DFT of x[n].

(a) Let $P[k] = W^{2k} X[k]$. Find p[n].

(b) Let $Q[k] = X[k-3]$. Find q[n].

(c) Let $R[k] = \text{Real}\{X[k]\}$. Find r[n].

Important

Solution :

Given $x[n] = 2\,\delta[n] + 3\,\delta[n-1] + 4\,\delta[n-3]$

$x[n] = \{2, 3, 0, 4, 0, 0\}$
6pt

Ref

$x(n-1) = \{0, 2, 3, 0, 4, 0\}$

$x(n-2) = \{0, 0, 2, 3, 0, 4\}$

(a) To find p[n]

Given $P[k] = W_N^{2K}\, X(K)$

By Time shift property of IDFT,

Ref

$$\left| \begin{array}{l} x(n-m) \rightarrow W_N^{mK} X(K) \\ W_N^{-mn} x(n) \rightarrow X(K-m) \end{array} \right|$$

$p(n) = x[n-2]$

Ans $p(n) = \{0, 0, 2, 3, 0, 4\}$

(b) To find q(n)

Given $Q(K) = X[K-3]$

By freq. shift property of IDFT,

$q(n) = W_N^{-3n}\, x(n)$

$= (W_N^{-3})^n\, x(n)$

where $W_N^{-3} = (W_N^1)^{-3}$

$= \left(e^{-j\frac{2\pi}{N}}\right)^{-3}$

for, $N = 6$

$= \left(e^{-j\frac{2\pi}{6}}\right)^{-3}$

$= e^{-j\pi}$

$= \cos(\pi) + j\sin(\pi)$

$W_N^{-3} = -1$ for $N = 6$

By substituting in q(n), we get

$\therefore q(n) = (-1)^n\, x(n)$

Solution: $q(n) = \begin{bmatrix} x(n) & \text{for } n \text{ even} \\ -x(n) & \text{for } n \text{ odd} \end{bmatrix}$

$$\therefore\ q(n) = \begin{bmatrix} 2 & n=0 \\ -3 & n=1 \\ 0 & n=2 \\ -4 & n=3 \\ 0 & n=4 \\ 0 & n=5 \end{bmatrix}$$

Ans

(c) To find $r(n)$

$$R(K) = \text{Real}\,[X[K]]$$
$$= X_e(K)$$

By IDFT,

$$r(n) = x_e(n)$$

$$r(n) = \frac{1}{2}\left[x(n) + x(-n)\right]$$

$$= \frac{1}{2}\left(\begin{bmatrix} 2 \\ 3 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \\ 4 \\ 0 \\ 3 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 4 \\ 3 \\ 0 \\ 8 \\ 0 \\ 3 \end{bmatrix}$$

$$r(n) = \begin{bmatrix} 2 \\ 3/2 \\ 0 \\ 4 \\ 0 \\ 3/2 \end{bmatrix}$$

Ans

NOTE

(1) $W_N^1 = e^{-j\frac{2\pi}{N}}$

(2) $W_N^{\pm\frac{N}{2}} = -1$

(3) $W_N^{\pm N} = 1$

Exercise

Q(20) Let x[n] be 4 point sequence with X[k] = {1, 2, 3, 4}. Find the DFT of the following sequences using X[k] and not otherwise. (a) p[n]= x[n-1] (b) q[n] = x[n+1]

ANS : (a) P[k] = { 1, –2j, –3, 4j } (b) Q[k] = { 1, 2j, –3, –4j }

Q(21) Let x[n] = {1, 2, 3, 4} and x[n] ←→ X[k].
Find inverse DFT of the following without using DFT/iDFT :
(a) $P[k]= e^{j\pi k} X[k]$ (b) $Q[k]=(-1)^k X[k]$

ANS : (a) p[n] = { 3, 4, 1, 2 } (b) q[n] = { 3, 4, 1, 2 }

Imp
By DFT,
$x(k) = 1 + 2W_N^{5k}$
$x(k) = 1 + 2(-1)^k$

Q(22) Consider the finite length sequence $x[n] = \delta[n] + 2\,\delta[n-5]$
a) Find 10-point DFT of x[n]
b) Find the sequence that has a DFT $P[k]= W^{-5k} X[k]$ where X[k] is 10 point DFT of x[n].

ANS : (a) X[k] = { 3, –1, 3, –1, 3, –1, 3, –1, 3, –1 }
(b) y[n] = { 2, 0, 0, 0, 0, 1, 0, 0, 0, 0 }

Q(23) Let x[n] = { 1, 2, 3, 4 } and x[n] ←→ X[k]. Find inverse DFT of the following without using DFT/iDFT. (a) P[k] = X[k-2] (b) Q[k] = X[k+2]

ANS : (a) p[n] = { 1, –2, 3, –4 } (b) q[n] = { 1, –2, 3, –4 }

Q(24) Let x[n] be 4 point sequence with X[k] = {1, 2, 3, 4}.Find the DFT of the following sequences using X[k] and not otherwise (a) x[–n+1] (b) x[–n–1]

ANS : (a) $DFT\{x[-n+1]\} = \{\ 1, -4j, -3, 2j\ \}$ (b) $DFT\{x[-n-1]\} = \{\ 1, 4j, -3, -2j\ \}$

Q(25) Let x[n] = { 1, 2, 3, 4 } and x[n] ←→ X[k]. Find inverse DFT of the following without using DFT/iDFT equations.
(a) X[–k+2] (b) X[–k–2]}

ANS : (a) $iDFT\{X[-k+2]\} = \{\ 1, -4, 3, -2\ \}$ (b) $DFT\{X[-k-2]\} = \{\ 1, -4, 3, -2\ \}$

Q(26) For the DFT of each real sequence compute boxed quantities
a) P[k] = { 0, □, 2+j, -1, □, j }
b) Q[k] = { 1, 2, □, □, 0, 1-j, -2, □ }

ANS : (a) P[1] = -j P[4]= 2-j
(b) Q[2] = -2 Q[3] =1+j Q[7] = 2

Q(27) For each DFT pair shown below, compute the values of the boxed quantities.
(a) p[n] = { □ , 3, -4, 0, 2 } P[k] = { 5, □, -1.28-j4.39 , □, 8.78-j1.4 }
(b) q[n] = { □ , 3, -4, 2, 0, 1} Q[k] = { 4, □, 4-j5.2, □ , □ , 4-j1.73 }

ANS : (a) p[0] = 4 P[1] = 8.78+j1.4 P[3]= -1.28+j4.39
(b) q[0] = 2, Q[1] = 4 + j1.73 Q[3] = - 8 Q[4] = 4 + j5.2

Q(28) Let X[k] = { 1, –2, 1–j, 2j, 0, ….} is the 8 point DFT of a real valued sequence. What is the 8 point DFT Y[k] such that $y[n] = (-1)^n\ x[n]$

ANS : X[k] = { 1, –2, 1–j, 2j, 0, –2j, 1+j, –2 }
Y[k] = X[k–4] = { 0, –2j, 1+j, –2, 1, –2, 1–j, 2j, }

Q(29) Let x[n] be the finite duration sequence of length 8. Its corresponding DFT X[k] is, X[k] = {(1), (4 + j2), (6 + j4), (2j), (6), (-2j), (6 – j4), (4 – j2)} . A new sequence p[n] of length 8 is defined as p[n] = ½ {x[n]+x[-n]} Find P[k] i.e. DFT of p[n] without performing DFT/ iDFT operations.

2.3 Fast Fourier Transform

Why DFT is Slow ?

By DFT, $X[k] = \sum_{n=0}^{N-1} x[n] w_N^{nk} = x(0) w_N^{0} + x(1) w_N^{k} + \cdots$

N pt

* Total Complex Multiplications = N^2
* Total Complex Additions = $N(N-1)$

$= N^2 - 1$

$\cong N^2$ For Large N.

---

[1]. **F F T Flowgraph for N = 2**

By DFT, $X[k] = \sum_{n=0}^{N-1} x[n] w_N^{nk}$

Put N = 2

$$X[k] = \sum_{n=0}^{1} x[n] w_N^{nk}$$

$$X[k] = x[0]\, w_N^{0} + x[1]\, w_N^{k}$$

$$X[k] = x[0] + x[1]\, w_N^{k}$$

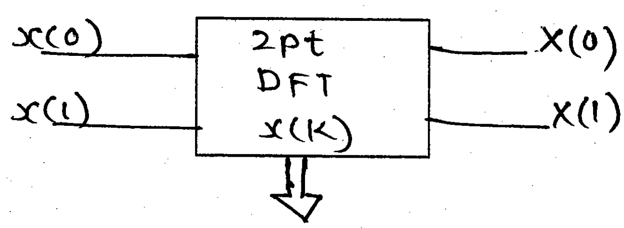
k=0, X[0] = $x(0) + x(1)$

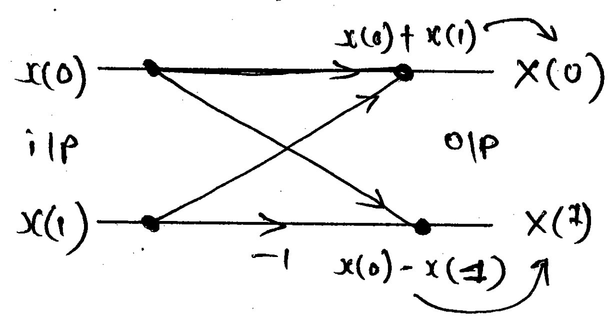
k=1, X [1] = $x(0) + x(1)\, w_N^{1}$

put $w_N^{1} = -1$ for $N = 2$

X [0] = $x(0) + x(1)$

X [1] = $x(0) - x(1)$





Prime factor F F T Algorithms

- D I T - F F T Algorithm
- D I F - F F T Algorithm

▶ Decimation means Sampling

VImp

> **Radix-2 DIT-FFT Algorithms**

2.3.1 Radix-2, DECIMATION IN TIME FFT (DIT-FFT) ALGORITHM
[Cooley & Tukey's DIT-FFT Algorithm]

Ref
For Radix-2 Algo
$N = 2^P \quad , P = 1, 2, 3, \ldots$
$N = \{2, 4, 8, 16, \ldots\}$

By DFT, $X[k] = \sum_{n=0}^{N-1} x[n] w_N^{nk}$

$$X[k] = \sum_{n \text{ even}} x[n] W_N^{nk} + \sum_{n \text{ odd}} x[n] W_N^{nk}$$

Put $n = 2r$ for n EVEN
$n = 2r+1$ for n ODD

$$X[k] = \sum_{r=0}^{\frac{N}{2}-1} x[2r]\, W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]\, W_N^{(2r+1)k}$$

$$\begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ \vdots \end{bmatrix} \qquad \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ \vdots \end{bmatrix}$$

meaning-less only for Reference.

$$W_N^2 = (W_N^1)^2 = \left(e^{-j\frac{2\pi}{N}}\right)^2$$
$$= \left(e^{-j\frac{\pi}{N/2}}\right)^2$$
$$= \left(e^{-j\frac{2\pi}{N/2}}\right)$$
$$\therefore \boxed{W_N^2 = W_{\frac{N}{2}}^1}$$

Where, (1) $W_N^{2rk} = W_{\frac{N}{2}}^{rk}$

(2) $W_N^{(2r+1)k} = W_N^{2rk+k}$
$$= W_N^{2rk}\, W_N^k$$
$$= W_{N/2}^{rk}\, W_N^k$$

$$X[k] = \sum_{r=0}^{\frac{N}{2}-1} x[2r]\, w_{\frac{N}{2}}^{rk} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]\, w_{\frac{N}{2}}^{rk}\, w_N^k$$

$$= \sum_{r=0}^{\frac{N}{2}-1} x[2r]\, w_{\frac{N}{2}}^{rk} + W_N^k \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]\, W_{\frac{N}{2}}^{rk}$$

$$= \text{DFT}\{x[2r]\} + W_N^k\, \text{DFT}\{x[2r+1]\}$$

Let $\mathbf{X[k]} = \mathbf{G[k]} + W_N^k\, \mathbf{H[k]}$ ☛ **DIT-FFT equation**

N pt DFT $\frac{N}{2}$ pt DFT $\frac{N}{2}$ pt DFT

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Discussion : OFU

To find X[k] ➔

(I) By DFT equation

$$X[k] = \sum_{n=0}^{N-1} x[n]\, w_N^{nk}$$

Total Complex Multiplications = N^2

(II) By DIT-FFT equation

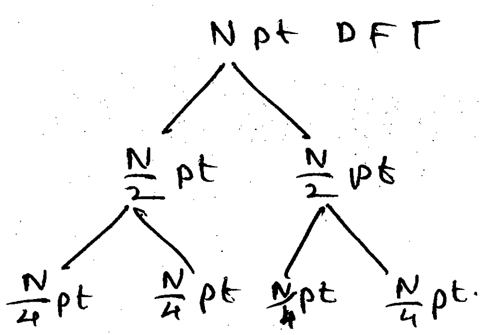
$$\underset{N\,pt}{\mathbf{X[k]}} = \underset{N/2\,pt}{\mathbf{G[k]}} + W_N^k\, \underset{N/2\,pt}{\mathbf{H[k]}}$$

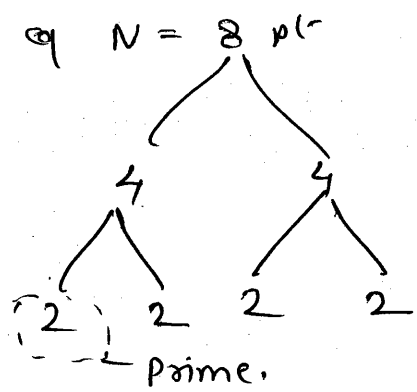
Total Complex Multiplications :

$$= \left(\frac{N}{2}\right)^2 + \left(\frac{N}{2}\right)^2 + N$$

$$= \frac{N^2}{4} + \frac{N^2}{4} + N$$

$$= \frac{N^2}{2} + N$$





Note

Decompose the DFT until prime found.

Radix-2 Algo is fastest in all Radix-Algo.

A] Radix-2, DIT-FFT FLOWGRAPH FOR N = 4

By DFT, $X[k] = \sum_{n=0}^{N-1} x[n] w_N^{nk}$

By Decomposing N point DFT into two $\frac{N}{2}$ pt DFT'S,

$$X[k] = \sum_{n \text{ even}} x[n] W_N^{nk} + \sum_{n \text{ odd}} x[n] W_N^{nk}$$

$$= \sum_{r=0}^{\frac{N}{2}-1} x[2r]\, W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]\, W_N^{(2r+1)k}$$

$$= \sum_{r=0}^{\frac{N}{2}-1} x[2r]\, w_{\frac{N}{2}}^{rk} + W_N^{k} \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]\, W_{\frac{N}{2}}^{rk}$$

Let $\underset{N\,pt}{X[k]} = \underset{\frac{N}{2}\,pt}{G[k]} + W_N^k \underset{\frac{N}{2}\,pt}{H[k]}$

Where $G[k] = \text{DFT } \{x[2r]\}$ and $H[k] = \text{DFT } \{X[2r+1]\}$

$$G[k] = DFT \begin{bmatrix} x(0) \\ x(2) \end{bmatrix} \qquad H[k] = DFT \begin{bmatrix} x(1) \\ x(3) \end{bmatrix}$$

To find X[k] :

$$\underset{4pt}{\mathbf{X[k]}} = \underset{2pt}{\mathbf{G[k]}} + W_N^k \underset{2pt}{\mathbf{H[k]}}$$

---

k=0, $X[0] = G(0) + W_N^0 H(0)$

k=1, $X[1] = G(1) + W_N^1 H(1)$

k=2, $X[2] = G(2) + W_N^2 H(2)$

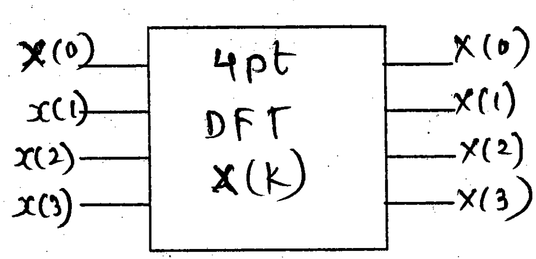
$= G(0) + W_N^2 H(0)$

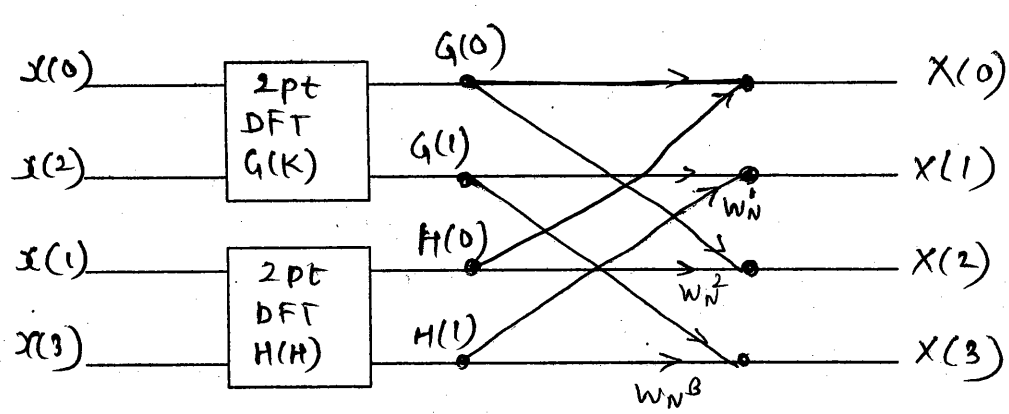
k=3, $X[3] = G(3) + W_N^3 H(3)$

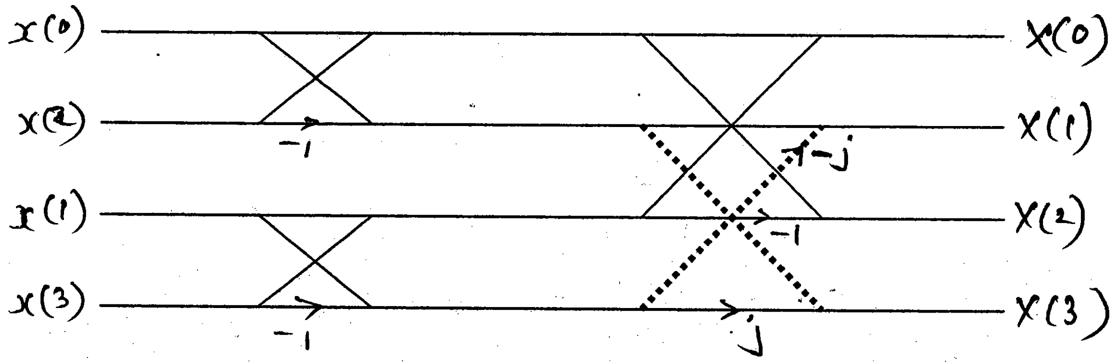
$= G(2) + W_N^3 H(1)$

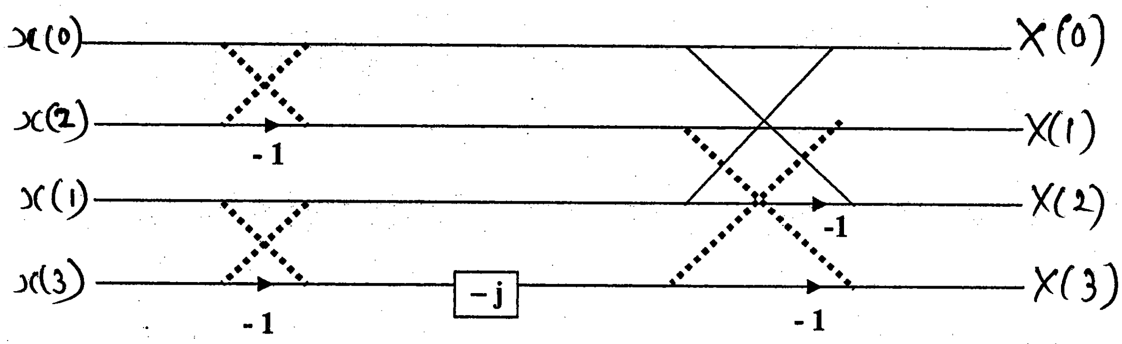
H[k] And G[k] are 2 pt & are periodic & circular

$\therefore G[2] = G[0]$









Note : By DIT-FFT Algorithm,

(1) Total Complex Multiplications $= \frac{N}{2} \log_2 N$

(2) Total Complex Additions $= N \log_2 N$

(3) Total Real Multiplications $=$ **$2 N \log_2 N$**

(4) Total Real Additions $=$ **$3 N \log_2 N$**

Q(20) Let x[n] = {1, 2, 3, 4}

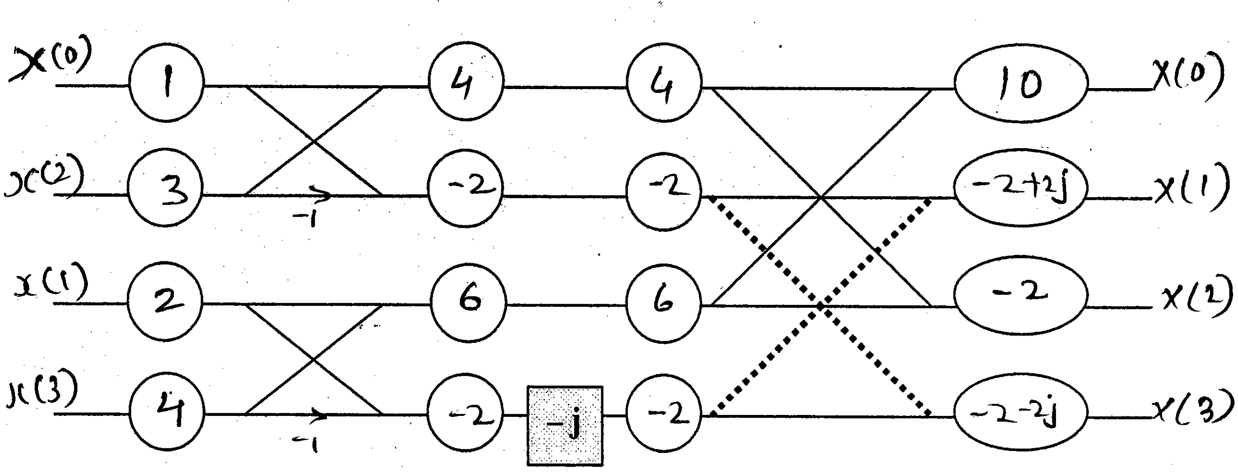
a) Find X[k] using DIT-FFT.

b) Let p[n] = {1, 0, 2, 0, 3, 0, 4, 0}. Find P[k] using X[k].

Solution :

At upper Addition
At lower Substraction — Imp

(a) To find X[k]



(b) **To find P[k] using X[k]**

Solution :

n $\Rightarrow$ $\boxed{0}$ 1 $\boxed{2}$ 3 $\boxed{4}$ 5 $\boxed{6}$ 7

$$p[n] = \{ 1, 0, 2, 0, 3, 0, 4, 0 \}.$$ Decomposing

By DIT,

Let $p[2r] = \{1, 2, 3, 4\} = x(n)$

$p[2r+1] = \{0, 0, 0, 0\} = 0$

To find P[k] using X[k],

$$P[k] = \sum_{n=0}^{\frac{N}{2}-1} p[2r] W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} p[2r+1] W_N^{(2r+1)k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} p[2r]\, W_{\frac{N}{2}}^{rk} + W_N^{k} \sum_{r=0}^{\frac{N}{2}-1} p[2r+1]\, W_{\frac{N}{2}}^{rk}$$

Let $P[k] = G[k] + W_N^k H[k]$ —— (1)

where

(i) $G(K) = DFT[p(2r)]$

$= DFT[x(n)]$

$= X[K]$

(ii) $H(K) = DFT[p(2r+1)]$

$= DFT[0]$

$= 0.$

By substituting in eqn. (1), we get

$\therefore p(K) = X(K) + 0$

$\because \underset{8pt}{P(K)} = \underset{4pt}{X(K)}$ periodic.

$$\therefore P(K) = \left[\begin{array}{ll} 10 & K=0 \\ -2+2j & K=1 \\ -2 & K=2 \\ -2-2j & K=3 \\ 10 & K=4 \\ -2+2j & K=5 \\ -2 & K=6 \\ -2-2j & K=7. \end{array}\right]$$

Q(21) Let x[n] = {1, 2, 3, 4}

(a) Find X[k] using DIT-FFT.

(b) Let p[n] = {0, 1, 0, 2, 0, 3, 0, 4}. Find P[k] using X[k].

Solution : **(b) To find P[k] using X[k]**

p[n] = {0, 1, 0, 2, 0, 3, 0, 4}.

Let p[2r] = { 0, 0, 0, 0} = 0
p[2r+1] = {1, 2, 3, 4} = x[n]

By DIT,

$$P[k] = \sum_{n=0}^{\frac{N}{2}-1} p[2r]W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} p[2r+1]W_N^{(2r+1)k}.$$

$$= \sum_{n=0}^{\frac{N}{2}-1} p[2r]\, W_{\frac{N}{2}}^{rk} + W_N^{k} \sum_{r=0}^{\frac{N}{2}-1} p[2r+1]\, W_{\frac{N}{2}}^{rk}$$

Let **P[k] = G[k] + W_N^k H[k]**

Where (i) G[k] = DFT { p[2r]} = DFT { 0 } = 0

(ii) H[k] = DFT { p[2r+1] } = DFT { x[n] } = X[k]

By substituting in P[k] we get,

P[k] = W_N^k X[k]

$$P[k] = \begin{bmatrix} 1 \\ 0.707 - j0.707 \\ -j \\ -0.707 - j0.707 \\ -1 \\ -0.707 + j0.707 \\ j \\ 0.707 + j0.707 \end{bmatrix} \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \\ 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix} == \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

Q(22) Let x[n] = {1, 2, 3, 4}

(a) Find X[k] using DIT-FFT.

(b) Let p[n] = {1, 1, 2, 2, 3, 3, 4, 4}. Find P[k] using X[k].

Solution : **(b) To find P[k] using X[k]**

Let p[2r] = { 1, 2, 3, 4 } = x[n]
p[2r+1] = { 1, 2, 3, 4 } = x[n]

By DIT, **P[k] = G[k] + W_N^k H[k]**

Where (i) G[k] = DFT { p[2r]} = DFT {x[n]} = X[k]

(ii) H[k] = DFT { p[2r+1] } = DFT { x[n] } = X[k]

By substituting in P[k] we get,

P[k] = X[k] + W_N^k X[k]

P[k] = (1 + W_N^k) X[k]

ANS : P[k] =

B] Radix-2, DIT-FFT FLOWGRAPH FOR N = 8

By DFT, $X[k] = \sum_{n=0}^{N-1} x[n] w_N^{nk}$

By Decomposing N point DFT into two $\frac{N}{2}$ pt DFT'S,

$$X[k] = \sum_{n \text{ even}} x[n] W_N^{nk} + \sum_{n \text{ odd}} x[n] W_N^{nk}$$

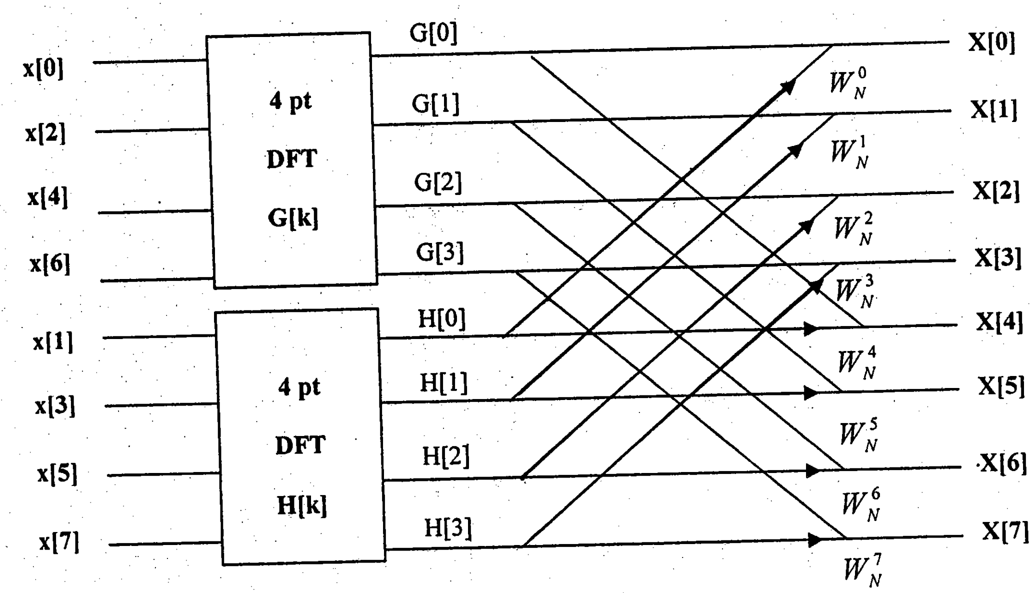
$$= \sum_{r=0}^{\frac{N}{2}-1} x[2r]\, W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]\, W_N^{(2r+1)k}$$

$$= \sum_{r=0}^{\frac{N}{2}-1} x[2r]\, w_{\frac{N}{2}}^{rk} + W_N^k \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]\, W_{\frac{N}{2}}^{rk}$$

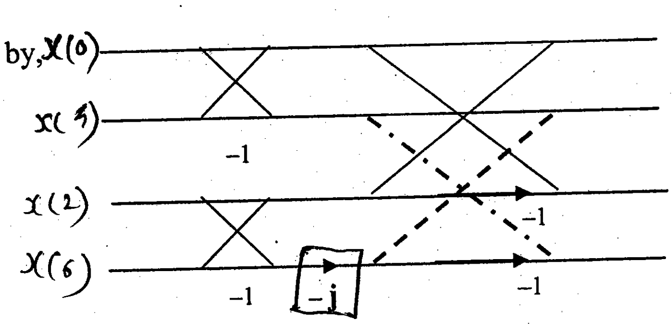
Let $X[k] = G[k] + W_N^k H[k]$

Where $G[k] = \text{DFT}\{x[2r]\}$ and $H[k] = \text{DFT}\{x[2r+1]\}$

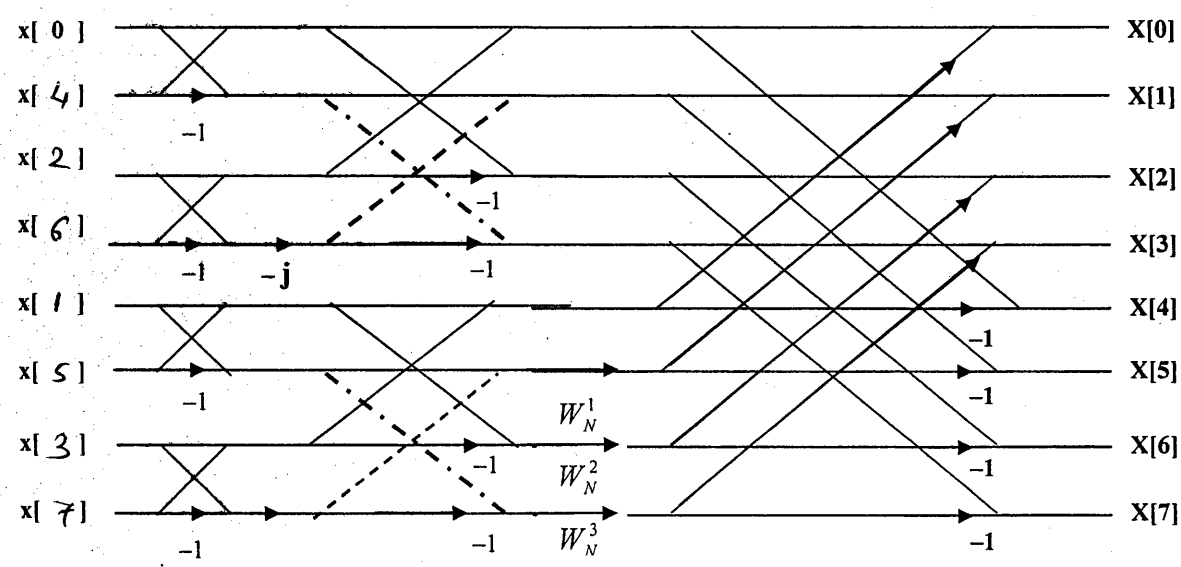
$$G[k] = DFT\begin{bmatrix} x[0] \\ x[2] \\ x[4] \\ x[6] \end{bmatrix} \qquad H[k] = DFT\begin{bmatrix} x[1] \\ x[3] \\ x[5] \\ x[7] \end{bmatrix}$$



DIT FFT flowgraph For N = 4 is given by,



By substituting we get,



Q(23) Let x[n] be a 8 pt sequence. How will you arrange the sequence for applying it as input to DIT-FFT flowgraph?

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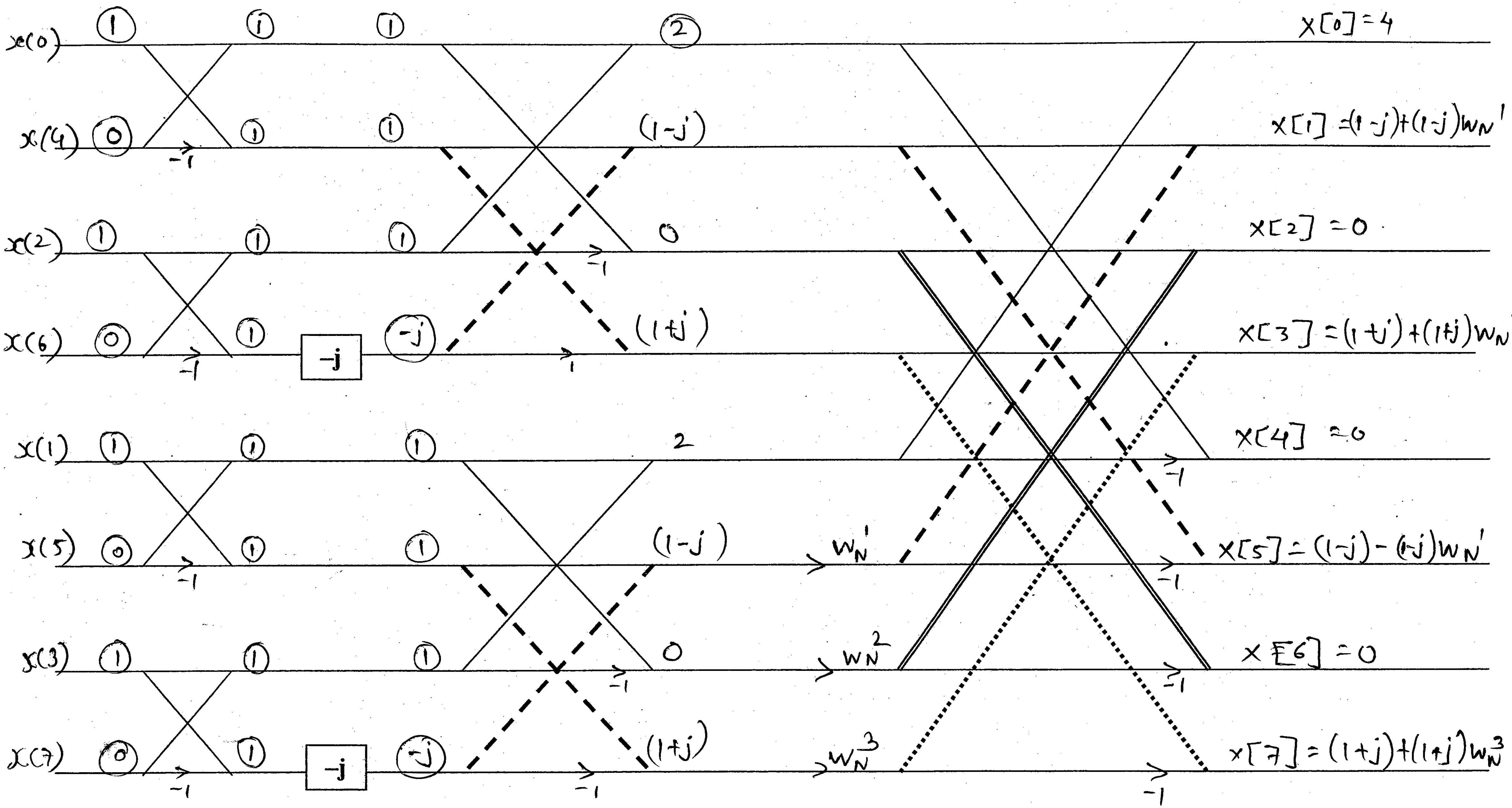
Solution--------- Bit Reversal technique - - - - - - - - -

Input sequence	Input with Index in Binary		Output with Index in Binary	Output sequence
x[0]	x[0 0 0]		X[0 0 0]	X[0]
x[4]	x[1 0 0]		X[0 0 1]	X[1]
x[2]	x[0 1 0]	DIT-FFT	X[0 1 0]	X[2]
x[6]	x[1 1 0]	Flowgraph	X[0 1 1]	X[3]
x[1]	x[0 0 1]	For	X[1 0 0]	X[4]
x[5]	x[1 0 1]	N = 8	X[1 0 1]	X[5]
x[3]	x[0 1 1]		X[1 1 0]	X[6]
x[7]	X[1 1 1]		X[1 1 1]	X[7]

Q(24) Given x(n) = { 1, 1 , 1, 1, 0, 0, 0, 0}. Find X[k] by using DIT-FFT.
Find DFT of the following sequences in terms of X[k].

A) a [n] = { 0, 0, 0, 0, 1, 1, 1, 1 }
B) b [n] = { 1, 0, 0, 0, 0, 1, 1, 1}
C) c [n] = { 1, 0, 0, 0, –1, 0, 0, 0}
D) d [n] = { 1, 1, 1, 1, –1, –1, –1, –1}
E) e [n] = { 1, 1, 1, 1, 1, 1, 1, 1}
F) f [n] = { 0, 0, 1, 1, 1, 1, 0, 0}
G) g [n] = { 1, –1, 1, –1, 0, 0, 0, 0}
H) p [n] = { 1, 0.5, 0.5, 0.5, 0, 0.5, 0.5, 0.5}

ANS:
A) $A[k] = (-1)^k X[k]$
B) $B[k] = X[-k]$
C) $C[k] = B[k] - A[k]$
D) $D[k] = X[k] - A[k]$
E) $E[k] = X[k] + A[k]$
F) $F[k] = W_N^{2k} X[k]$
G) $G[k] = X[k-4]$
H) $P[k] = X_e[k]$ = Real {X[k]}


x(0)
x(4)
x(2)
x(6)
x(1)
x(5)
x(3)
x(7)
1
0
1
0
1
0
1
0
-1
-j
2
(1-j)
0
(1+j)
2
(1-j)
0
(1+j)
W_N^1
W_N^2
W_N^3
X[0] = 4
X[1] = (1-j)+(1-j)W_N^1
X[2] = 0
X[3] = (1+j)+(1+j)W_N^2
X[4] = 0
X[5] = (1-j) - (1-j)W_N^1
X[6] = 0
X[7] = (1+j)+(1+j)W_N^3

(1) $\boxed{X[0] = 4}$

(2) $X[1] = (1-j) + (1-j)W_N^1$

$= (1-j) + (1-j)(0.707 - j0.707)$

$= 1-j + 0.707 - j0.707 - j0.707 - 0.707$

$= 1 - j2.414$

$\boxed{X[1] = 1 - j2.414.}$

(3) $\boxed{X[2] = 0}$

(4) $X[3] = (1+j) + (1+j)W_N^3$

$= (1+j) + (1+j)(-0.707 - j0.707)$

$= 1+j - 0.707 - j0.707 - j0.707 + 0.707$

$= 1 - j0.414.$

$\boxed{X[3] = 1 - j0.414.}$

(5) $\boxed{X[4] = 0}$

(6) $\boxed{X[5] = 1 + j0.414}$

(7) $\boxed{X[6] = 0}$

(8) $\boxed{X[7] = 1 + j2.414.}$

Q(25) Find the DFT of the following signals using DIT-FFT for N = 8.

a) x[n] = {1, 0, 2, 0, 3, 0, 2, 0} b) x[n] = {1, 2, 3, 2, 1, 2, 3, 2}

Q(26) Let x[n] = {1, 0, 2, 0, 3, 0, 2, 0}. Find X[k] using DIT-FFT. Using X[k] and not otherwise find the DFT of p[n] = {1, 2, 3, 2}

Q(27) Let x[n] be a 8 pt sequence. How will you arrange the sequence for applying it as input to DIT-FFT flowgraph?

2.3.2 Radix-2 Decimation In Frequency -FFT (DIF-FFT) Algorithm

Base **[Cooley & Tukey's DIF-FFT Algorithm]**

By DFT, $X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$

STEP – I. To find X [k] for k even. Put $k = 2r$

$$X[2r] = \sum_{n=0}^{N-1} x[n]\, W_n^{2rn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[n]\, W_N^{2rn} + \sum_{n=\frac{N}{2}}^{N-1} x[n]\, W_N^{2rn}$$

$$\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(\frac{N}{2}-1) \end{bmatrix} \qquad \begin{bmatrix} x(\frac{N}{2}) \\ x(\frac{N}{2}+1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

$$X[2r] = \sum_{n=0}^{\frac{N}{2}-1} x[n]\, W_N^{2rn} + \sum_{n=0}^{\frac{N}{2}-1} x\left[n+\frac{N}{2}\right] W_N^{2r\left(n+\frac{N}{2}\right)}$$

Where i) $W_N^{2rn} = W_{\frac{N}{2}}^{rn}$

ii) $W_N^{2r\left(n+\frac{N}{2}\right)} = W_N^{2rn+rN}$

$$= W_N^{2rn}\, W_N^{rN} \qquad \because W_N^N = 1$$

$$= W_{\frac{N}{2}}^{rn}\, (1)^r$$

$$X[2r] = \sum_{n=0}^{N/2-1} x[n]\, W_{\frac{N}{2}}^{rn} + \sum_{n=0}^{N/2-1} x\left[n+\frac{N}{2}\right] W_{\frac{N}{2}}^{rn}$$

$$X[2r] = \sum_{n=0}^{N/2-1} \left\{ x[n] + x\left[n+\frac{N}{2}\right] \right\} W_{\frac{N}{2}}^{rn}$$

$$X[2r] = \sum_{n=0}^{N/2-1} g[n]\, W_{\frac{N}{2}}^{rn}$$

$$\boxed{X[2r] = \mathrm{DFT}\,(g(n))}$$

STEP- II : To find X [k] For K odd. Put K = 2r + 1

$$x[2r+1] = \sum_{n=0}^{N-1} x[n]\, W_N^{n[2r+1]}$$

$$= \sum_{n=0}^{N/2-1} x[n]\, W_N^{n(2r+1)} + \sum_{n=N/2}^{N-1} x[n] W_N^{n(2r+1)}$$

$$= \sum_{n=0}^{N/2-1} x[n]\, W_N^{n(2r+1)} + \sum_{n=0}^{N/2-1} x\left[n+\frac{N}{2}\right] W_N^{(n+N/2)(2r+1)}$$

Where

i) $W_N^{n(2r+1)} = W_N^{2rn+n}$

$= W_N^{2rn}\, W_N^{n}$

$= W_{\frac{N}{2}}^{rn}\, W_N^{n}$

ii) $W_N^{(n+\frac{N}{2})(2r+1)} = W_N^{2rn+n+Nr+\frac{N}{2}}$

$= W_N^{2rn}\, W_N^{n}\, W_N^{Nr}\, W_N^{N/2}$

$= W_{\frac{N}{2}}^{rn}\, W_N^{n}\, (1)(-1)$

$= -\left(W_{\frac{N}{2}}^{rn}\right)\left(W_N^{n}\right)$

$$X[2r+1] = \sum_{n=0}^{N/2-1} x[n]\; W_N^{n}\, W_{N/2}^{rn} - \sum_{n=0}^{N/2-1} x\left[n+\frac{N}{2}\right] W_N^{n}\; W_{N/2}^{rn}$$

$$= \sum_{n=0}^{N/2-1} \left\{\left(x[n]-x\left[n+\frac{N}{2}\right]\right) W_N^{n}\right\}\; W_{N/2}^{rn}$$

$$X[2r+1] = \sum_{n=0}^{N/2-1} h[n] W_{N/2}^{rn}$$

$$\boxed{X[2r+1] = \mathrm{DFT}\,[h(n)]}$$

A] DIF-FFT flowgraph for N=4

STEP – I. To find X [k] for k even.

By DFT, $X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$

$$X[2r] = \sum_{n=0}^{N-1} x[n]\, W_N^{2rn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[n]\, W_N^{2rn} + \sum_{n=\frac{N}{2}}^{N-1} x[n]\; W_N^{2rn}$$

$$X[2r] = \sum_{n=0}^{\frac{N}{2}-1} x\,[n]\; W_N^{2rn} + \sum_{n=0}^{\frac{N}{2}-1} x\left[n+\frac{N}{2}\right]\; W_N^{2r\left(n+\frac{N}{2}\right)}$$

$$X[2r] = \sum_{n=0}^{N/2-1} x[n]\, W_{N/2}^{rn} + \sum_{n=0}^{N/2-1} x\left[n+\frac{N}{2}\right] W_{N/2}^{rn}$$

$$X[2r] = \sum_{n=0}^{N/2-1} \left\{ x[n] + x\left[n+\frac{N}{2}\right] \right\} W_{N/2}^{rn}$$

$$X[2r] = \sum_{n=0}^{N/2-1} g[n]\, W_{N/2}^{rn}$$

$$\boxed{X[2r] = DFT\{g[n]\}}$$

Where $g[n] = x[n] + x\left[n+\frac{N}{2}\right]$

$g[n] = x[n] + x[n+2]$

$n=0,\ g[0] = x(0) + x(2)$

$n=1,\ g(1) = x(1) + x(3)$

STEP- II To find X[k] For K odd. $\because N = 4,$

$$x[2r+1] = \sum_{n=0}^{N-1} x[n]\, W_N^{n[2r+1]}$$

$$= \sum_{n=0}^{N/2-1} x[n]\, W_N^{n(2r+1)} + \sum_{n=N/2}^{N-1} x[n] W_N^{n(2r+1)}$$

$$= \sum_{n=0}^{N/2-1} x[n]\, W_N^{n(2r+1)} + \sum_{n=0}^{N/2-1} x\left[n+\frac{N}{2}\right] W_N^{(n+N/2)(2r+1)}$$

$$= \sum_{n=0}^{N/2-1} x[n]\, W_N^{n}\, W_{N/2}^{rn} - \sum_{n=0}^{N/2-1} x\left[n+\frac{N}{2}\right] W_N^{n}\, W_{N/2}^{rn}$$

$$= \sum_{n=0}^{N/2-1} \left\{ \left(x[n] - x\left[n+\frac{N}{2}\right] \right) W_N^{n} \right\} W_{N/2}^{rn}$$

$$X[2r+1] = \sum_{n=0}^{N/2-1} h[n] W_{N/2}^{rn}$$

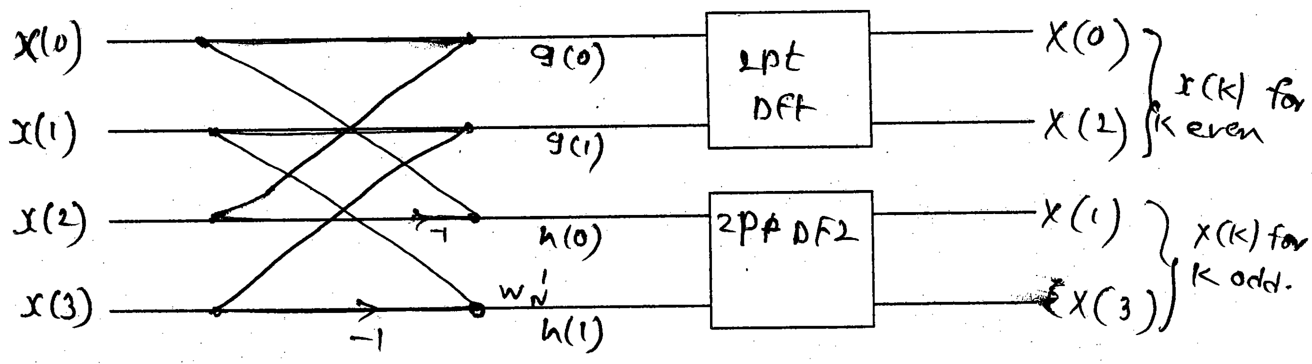
$$\boxed{X[2r+1] = DFT\{h[n]\}}$$

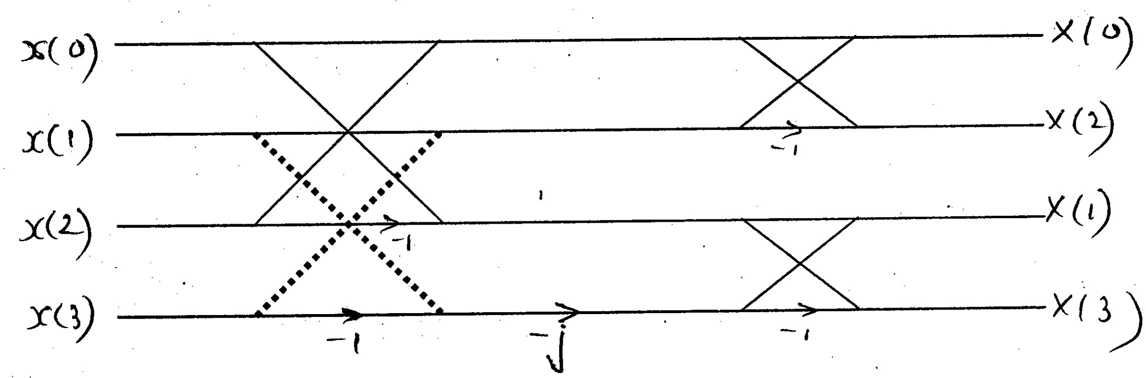
Where $h[n] = \left(x[n] - x\left[n+\frac{N}{2}\right] \right) W_N^{n}$

$h[n] = (x[n] - x[n+2])\, W_N^{n}$

$n = 0,\ h[0] = x(0) - x(2)\, W_N^{n}$

$n=1,\ h(1) = x(1) - x(3)\, W_N^{n}$





Note : By DIF-FFT Algorithm,

(1) Total Complex Multiplications $= \frac{N}{2} \log_2 N$

(2) Total Complex Additions $= N \log_2 N$

(3) Total Real Multiplications $= \mathbf{2\,N\,log_2 N}$

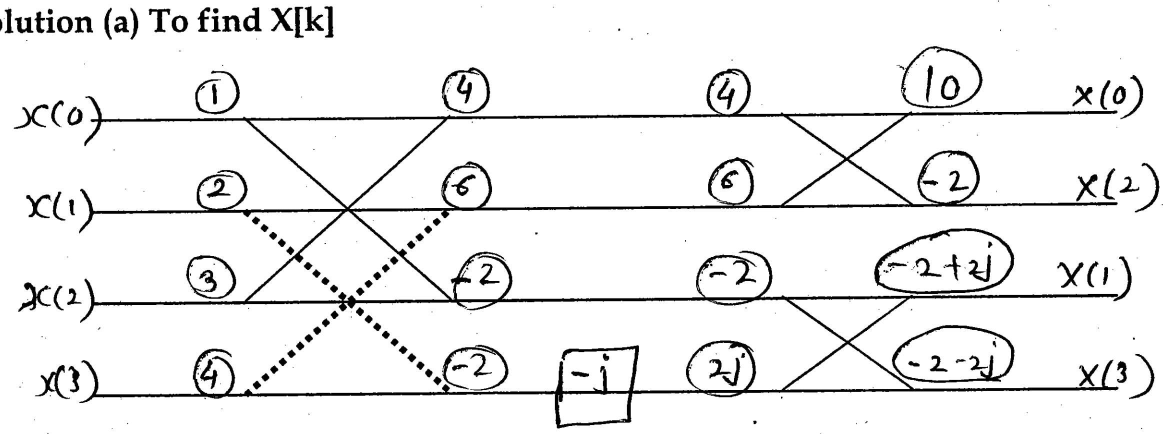
(4) Total Real Additions $= \mathbf{3\,N\,log_2 N}$

Q(28) Let x[n] = {1, 2, 3, 4}

(a) Find X[k] using DIF-FFT.

(b) Let p[n] = {1, 2, 3, 4, 1, 2, 3, 4}. Find P[k] using X[k].

Solution (a) To find X[k]



$$X[k] = \begin{bmatrix} 10 & k=0 \\ -2+2j & k=1 \\ -2 & k=2 \\ -2-2j & k=3 \end{bmatrix}$$

Solution (b) To find P[k]

By splitting

$$n \Rightarrow \quad 0 \quad 1 \quad 2 \quad 3 \mid 4 \quad 5 \quad 6 \quad 7$$

$$p[n] = \{ 1, 2, 3, 4, \mid 1, 2, 3, 4 \}.$$

By Decimation in Frequency,

Step-1 To find P[2r]

P[2r] = DFT { g[n] } = G[k]

Where $g[n] = p[n] + p[n+\frac{N}{2}]$

$$g[n] = p[n] + p[n+4]$$

n=0, g[0] = p[0] + p[4] == 2
n=1, g[1] = p[1] + p[5] == 4
n=2, g[2] = p[2] + p[6] == 6
n=3, g[3] = p[3] + p[7] == 8

$$g[n] = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}$$

$$g[n] = 2\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

g[n] = 2 x(n)

By DFT,

G[k] = 2 X[k]

By substituting in P[2r] we get,

P[2r] = 2 X[k]

$$\begin{array}{l} r=0 \\ r=1 \\ r=2 \\ r=3 \end{array} \begin{bmatrix} P[0] \\ P[2] \\ P[4] \\ P[6] \end{bmatrix} = \begin{bmatrix} 20 \\ -4+4j \\ -4 \\ -4-4j \end{bmatrix}$$

Step-2 To find P[2r+1]

P[2r+1] = DFT { h[n] }

Where $h[n] = (p[n] + p[n+\frac{N}{2}])\, W_N^n$

$$h[n] = (p[n] - p[n+4])\, W_N^n$$

n=0, $h[0] = (p[0] - p[4])\, W_N^0$ == 0

n=1, $h[1] = (p[1] - p[5])\, W_N^1$ == 0

n=2, $h[2] = (p[2] - p[6])\, W_N^2$ == 0

n=3, $h[3] = (p[3] - p[7])\, W_N^3$ == 0

h[n] = 0

By DFT,

H[k] = 0

By substituting in P[2r+1] we get,

P[2r+1] = 0

$$P[k] = \begin{bmatrix} 20 & k=0 \\ 0 & k=1 \\ -4+4j & k=2 \\ 0 & k=3 \\ -4 & k=4 \\ 0 & k=5 \\ -4-4j & k=6 \\ 0 & k=7 \end{bmatrix}$$

ANS

Q(29) Let x[n] = {1, 2, 3, 4} **HW**

(a) Find X[k] using DIF-FFT.
(b) Let p[n] = {1, 2, 3, 4, 0, 0, 0, 0}. Find P[k] using X[k] only for even values of k and not otherwise

Q(30) Find the DFT of the following signals using DIF-FFT for N = 8.

(a) x[n] = {1, 0, 2, 0, 3, 0, 2, 0} (b) x[n] = {1, 2, 3, 2, 1, 2, 3, 2}

B] DIF-FFT flowgraph for N=8

STEP – I. To find X [k] for k even.

By DFT, $$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$$

$$X[2r] = \sum_{n=0}^{N-1} x[n]\, W_n^{2rn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[n]\, W_N^{2rn} + \sum_{n=\frac{N}{2}}^{N-1} x[n]\, W_N^{2rn}$$

$$X[2r] = \sum_{n=0}^{\frac{N}{2}-1} x[n]\, W_N^{2rn} + \sum_{n=0}^{\frac{N}{2}-1} X\left[n+\frac{N}{2}\right] W_N^{2r\left(n+\frac{N}{2}\right)}$$

$$X[2r] = \sum_{n=0}^{N/2-1} X[n]\, W_{N/2}^{rn} + \sum_{n=0}^{N/2-1} X\left[n+\frac{N}{2}\right] W_{N/2}^{rn}$$

$$X[2r] = \sum_{n=0}^{N/2-1} \left\{x[n] + x\left[n+\frac{N}{2}\right]\right\} W_{N/2}^{rn}$$

$$X[2r] = \sum_{n=0}^{N/2-1} g[n]\, W_{N/2}^{rn}$$

$$\boxed{X[2r] = DFT\{g[n]\}}$$

STEP- II To find X[k] For K odd.

$$x[2r+1] = \sum_{n=0}^{N-1} x[n]\, W_N^{n[2r+1]}$$

$$= \sum_{n=0}^{N/2-1} x[n]\, W_N^{n(2r+1)} + \sum_{n=N/2}^{N-1} x[n] W_N^{n(2r+1)}$$

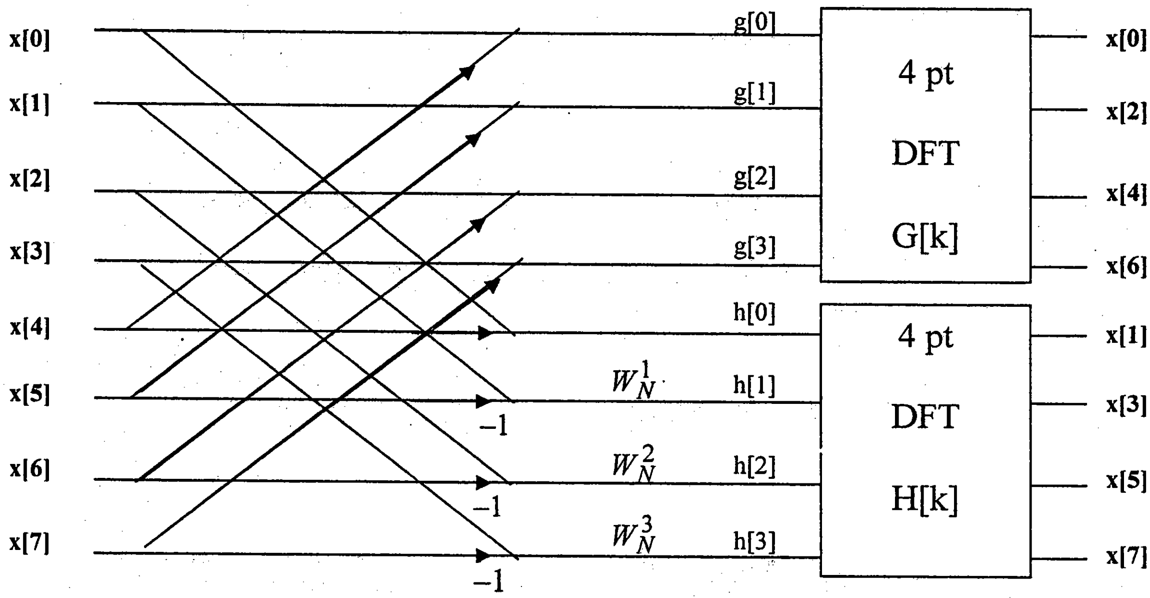
$$= \sum_{n=0}^{N/2-1} x[n]\, W_N^{n(2r+1)} + \sum_{n=0}^{N/2-1} x\left[n+\frac{N}{2}\right] W_N^{(n+N/2)(2r+1)}$$

$$= \sum_{n=0}^{N/2-1} x[n]\, W_N^{n}\, W_{N/2}^{rn} - \sum_{n=0}^{N/2-1} x\left[n+\frac{N}{2}\right] W_N^{n}\quad W_{N/2}^{rn}$$

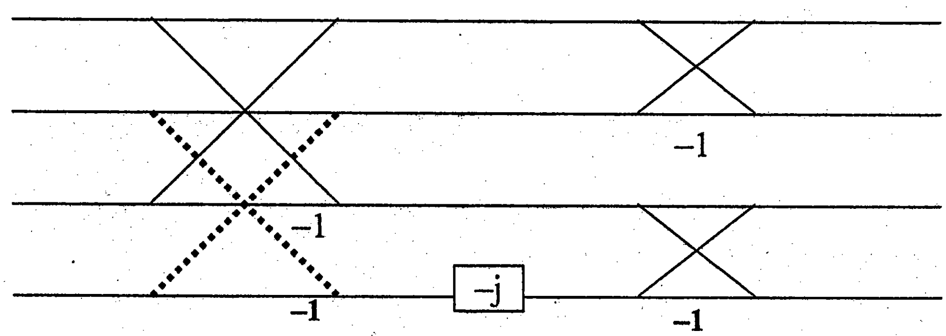
$$= \sum_{n=0}^{N/2-1} \left\{\left(x[n] - x\left[n+\frac{N}{2}\right]\right) W_N^{n}\right\} W_{N/2}^{rn}$$

$$X[2r+1] = \sum_{n=0}^{N/2-1} h[n] W_{N/2}^{rn}$$

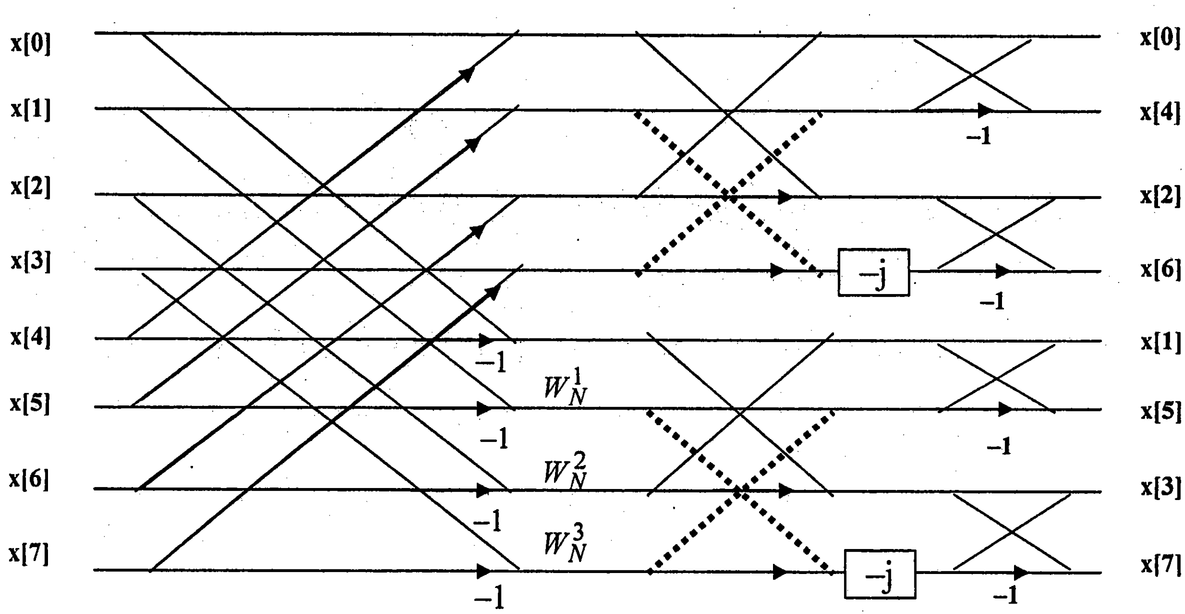
$$\boxed{X[2r+1] = DFT\{h[n]\}}$$



DIF-FFT flowgraph for N=4 is given by,



By substituting 4 point DIF-FFT flowgraph we get,



10 m

2.3.3 Radix-3 DIT-FFT Algorithm

A] DIT-FFT Algorithm for N=3

By DFT $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$ where N =3

$$X[k] = x[0] + x[1] W_N^{k} + x[2] W_N^{2k}$$

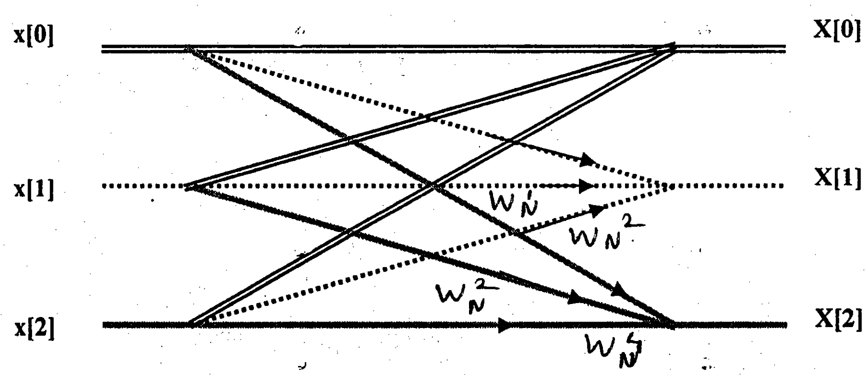
---

k= 0, $X[0] = x[0] + x[1] + x[2]$

k = 1, $X[1] = x[0] + x[1] W_N^{1} + x[2] W_N^{2}$

k = 2, $X[2] = x[0] + x[1] W_N^{2} + x[2] W_N^{4}$

FFT flowgraph for N = 3 is given by,



B] Radix-3, DIT-FFT ALGORITHM (Flowgraph) FOR N = 9

By DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

By Decomposing N pt DFT into three $\frac{N}{3}$ pt DFT's

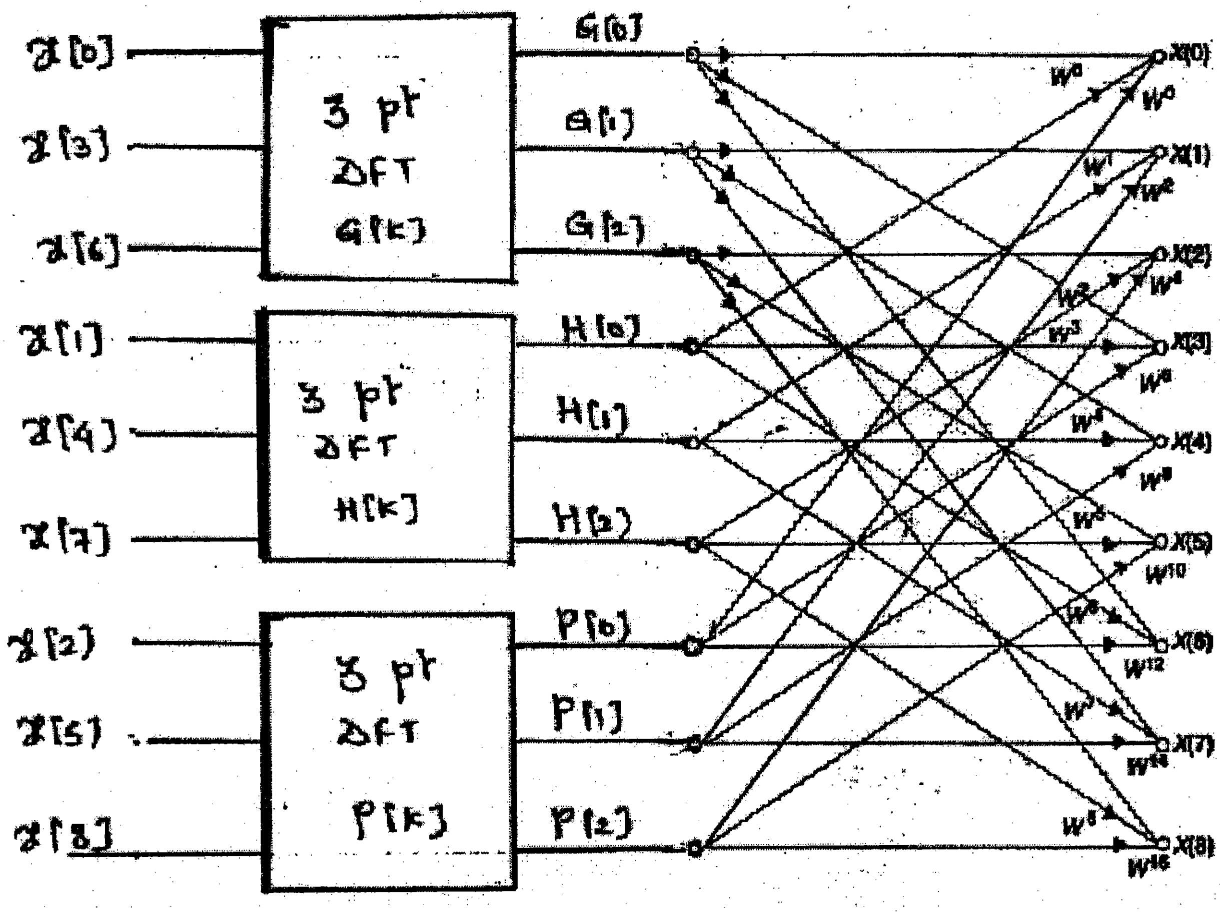
$$X[k] = \sum_{r=0}^{\frac{N}{3}-1} x[3r] \ W_N^{3rk} + \sum_{r=0}^{\frac{N}{3}-1} x[3r+1] W_N^{(3r+1)k} + \sum_{r=0}^{\frac{N}{3}-1} X[3r+2] W_N^{(3r+2)k}$$

$$= \sum_{r=0}^{\frac{N}{3}-1} x[3r] \ W_{\frac{N}{3}}^{rk} + W_N^k \sum_{r=0}^{\frac{N}{3}-1} x[3r+1] W_{\frac{N}{3}}^{rk} + W_N^{2k} \sum_{r=0}^{\frac{N}{3}-1} x[3r+2] W_{\frac{N}{3}}^{rk}$$

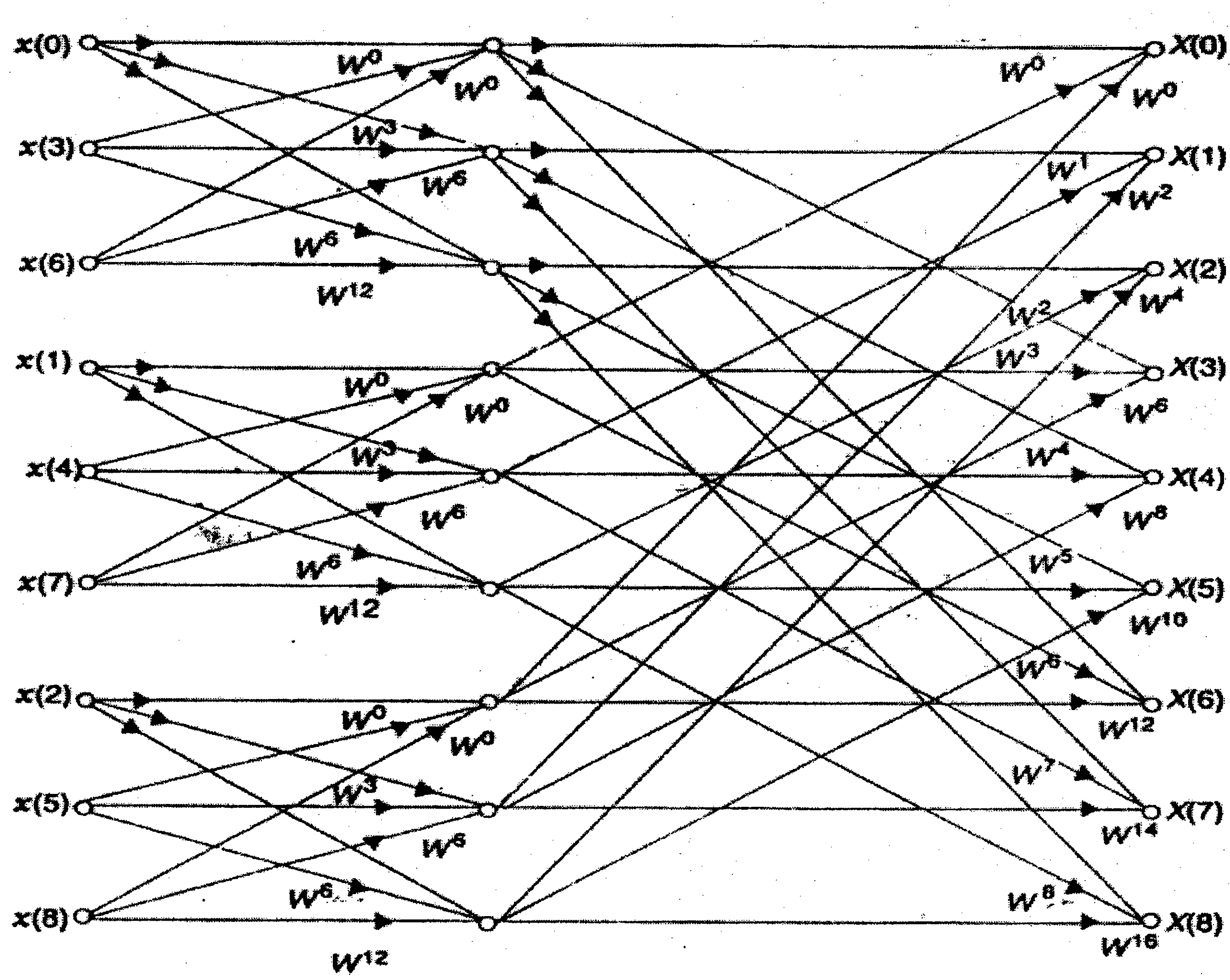
$$X[k] = G[k] + W_N^k H[k] + W_N^{2k} P[k].$$

Where $G[k] = DFT\{X[3r]\}$ $H[k] = DFT\{X[3r+1]\}$ $P[k] = DFT\{X[3r+2]\}$

$$G[k] = DFT\begin{bmatrix} X[0] \\ X[3] \\ X[6] \end{bmatrix} \quad H[k] = DFT\begin{bmatrix} X[1] \\ X[4] \\ X[7] \end{bmatrix} \quad P[k] = DFT\begin{bmatrix} X[2] \\ X[5] \\ X[8] \end{bmatrix}$$



By substituting FFT flowgraph for N=3 we get,



Q(31) Develop DIT FFT Algorithm for decomposing DFT for N= 6 and draw the flowgraph for the following two cases. (a) Using Two 3 pt DFT's (b) Using Three 2 pt DFT's

Solution :

(a) DIT FFT flowgraph for N = 6 using Two 3 pt DFT's

By DFT

$$X[k] = \sum_{n=0}^{N-1} X[n]\, W_N^{nk}$$

By Decomposing 6 pt DFT into Two 3pt DFT's

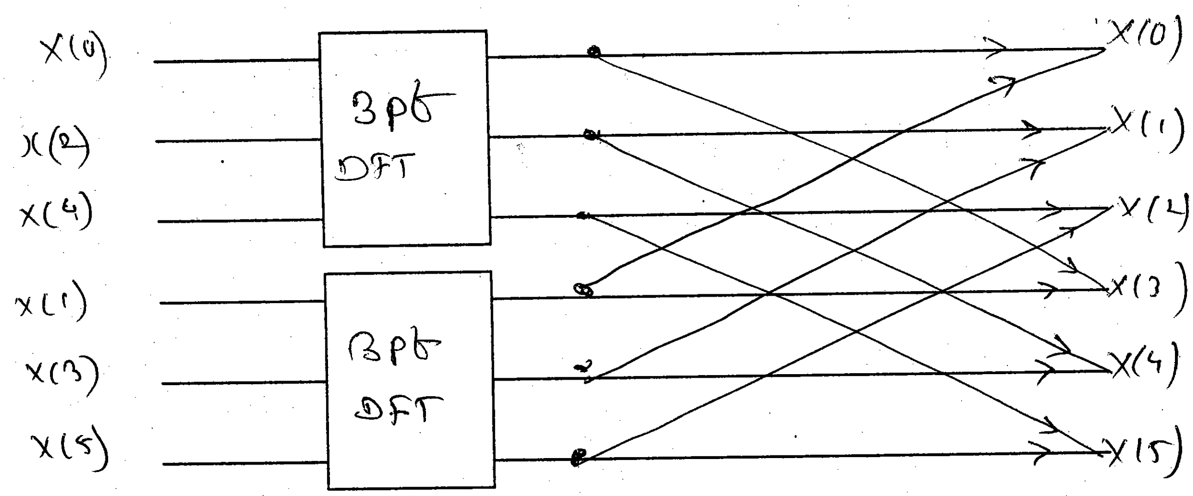
$$X[k] = \sum_{r=0}^{\frac{N}{2}-1} x[2r]\, W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]\, W_N^{(2r+1)k}$$

$$= \sum_{r=0}^{\frac{N}{2}-1} x[2r]\, W_{\frac{N}{2}}^{rk} + W_N^{k} \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]\, W_{\frac{N}{2}}^{rk}$$

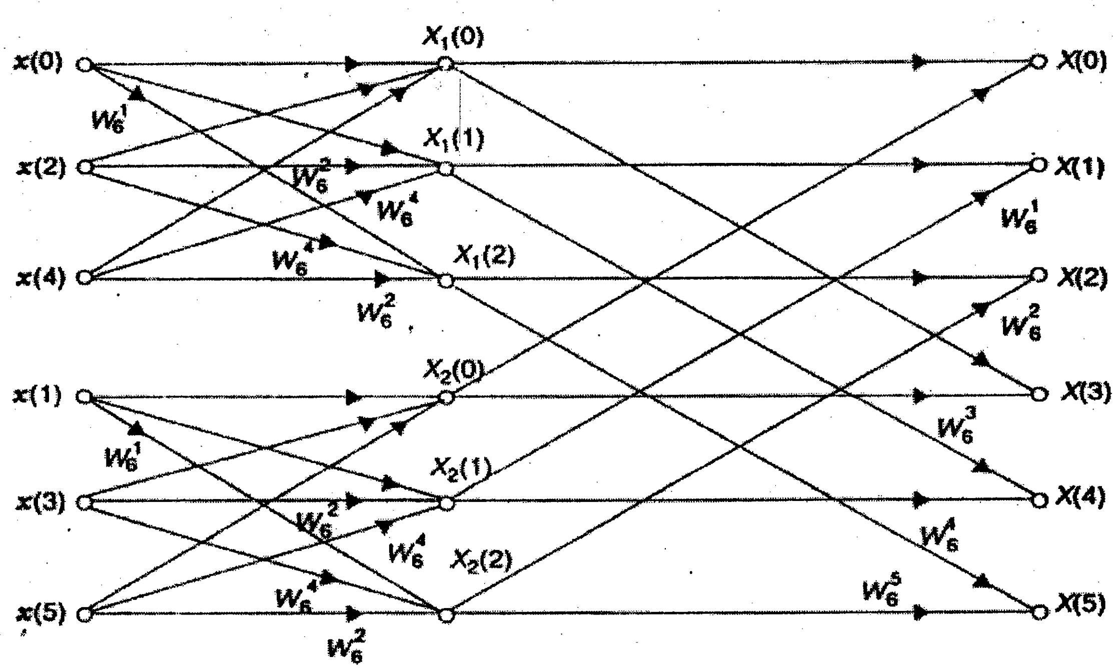
$$X[k] = G[k] + W_N^k\, H[k]$$

Where $G[k] = DFT\,\{X[2r]\}$ $\quad H[k] = DFT\,\{X[2r+1]\}$

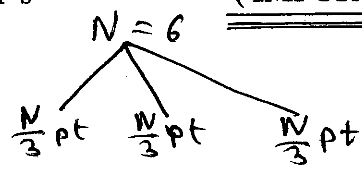
$$G[k] = DFT \begin{bmatrix} X[0] \\ X[2] \\ X[4] \end{bmatrix} \qquad H[k] = DFT \begin{bmatrix} X[1] \\ X[3] \\ X[5] \end{bmatrix}$$



By substituting FFT flowgraph for N=3 we get,



(b) DIT FFT flowgraph for N = 6 using Three 2 pt DFT's **(IMPORTANT)**



By DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$$

By Decomposing 6 pt DFT into Three 2 pt DFT's

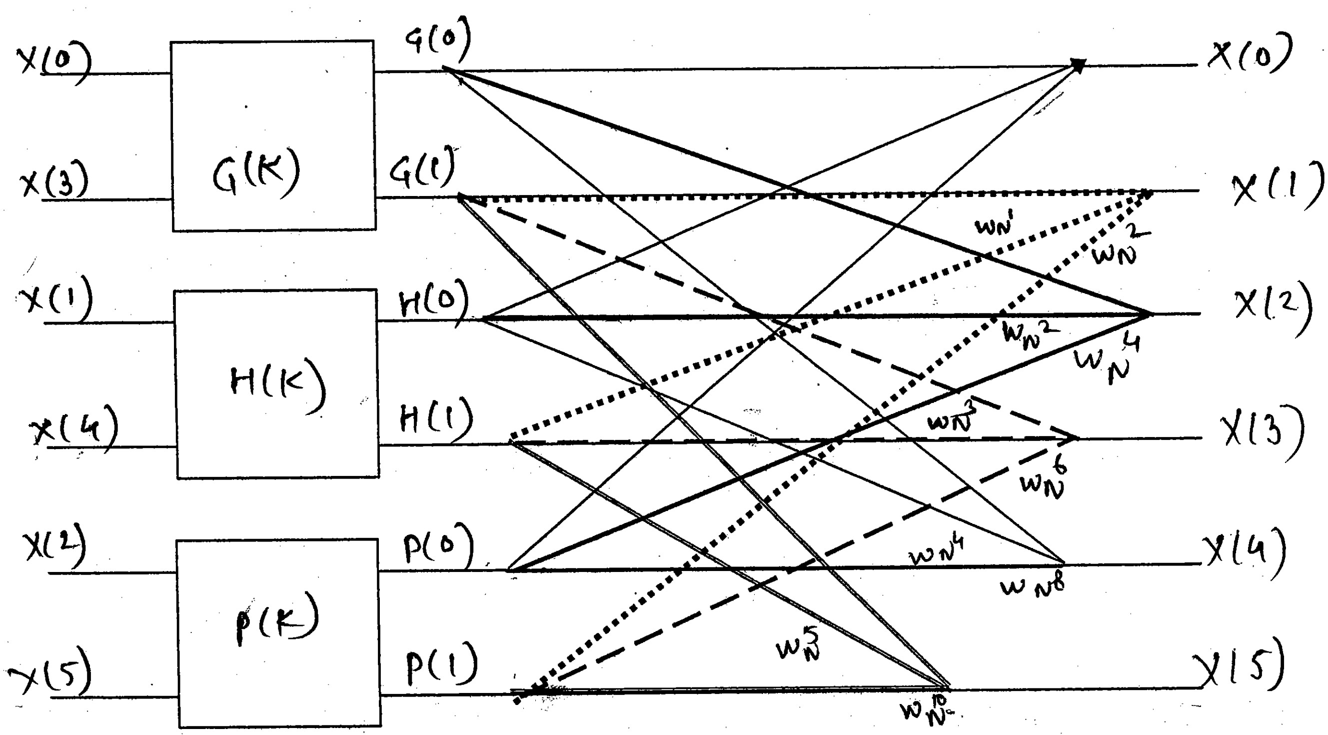
$$X[k] = \sum_{r=0}^{\frac{N}{3}-1} x[3r]\; W_N^{3rk} + \sum_{r=0}^{\frac{N}{3}-1} x[3r+1]\, W_N^{(3r+1)k} + \sum_{r=0}^{\frac{N}{3}-1} X[3r+2]\, W_N^{(3r+2)k}$$

$$= \sum_{r=0}^{\frac{N}{3}-1} x[3r]\; W_{\frac{N}{3}}^{rk} + W_N^{k} \sum_{r=0}^{\frac{N}{3}-1} x[3r+1]\, W_{\frac{N}{3}}^{rk} + W_N^{2k} \sum_{r=0}^{\frac{N}{3}-1} x[3r+2]\, W_{\frac{N}{3}}^{rk}$$

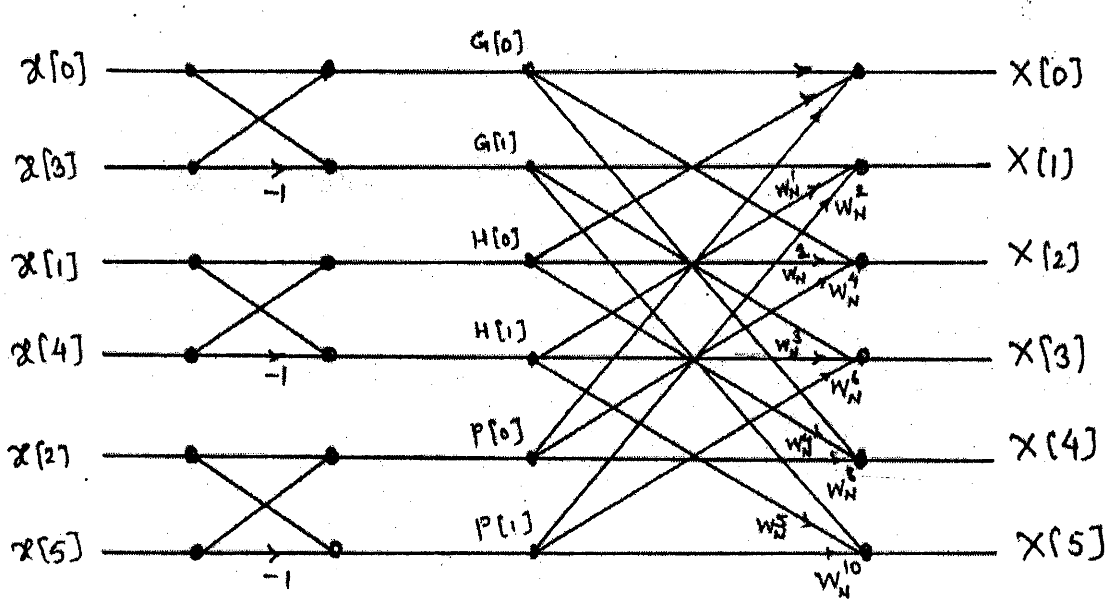
$$X[k] = G[k] + W_N^k\, H[k] + W_N^{2k}\, P[k].$$

Where $G[k] = DFT\{X[3r]\}$ $\quad$ $H[k] = DFT\{X[3r+1]\}$ $\quad$ $P[k] = DFT\{X[3r+2]\}$

$G[k] = DFT\begin{bmatrix} X[0] \\ X[3] \end{bmatrix}$ $\quad$ $H[k] = DFT\begin{bmatrix} X[1] \\ X[4] \end{bmatrix}$ $\quad$ $P[k] = DFT\begin{bmatrix} X[2] \\ X[5] \end{bmatrix}$



By substituting FFT flowgraph for N=2 we get,



2.4 INVERSE FFT

Derivation :

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\ W_N^{-nk}$$

By complex conjugate,

$$x^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X^*[k]\ W_N^{nk}$$

$$x^*[n] = \frac{1}{N}\ DFT\{\ X^*[k]\ \}$$

$$x^*[n] = \frac{1}{N}\ FFT\{\ X^*[k]\ \}$$

By complex conjugate,

$$x[n] = \frac{1}{N}\left(FFT\{\ X^*[k]\ \}\right)^* \longleftarrow \text{IFFT equation}$$

Ref -

$$W_N^{1} = e^{-j\frac{2\pi}{N}}$$

By complex conjugate

$$W_N^{-1} = e^{j\frac{2\pi}{N}}$$

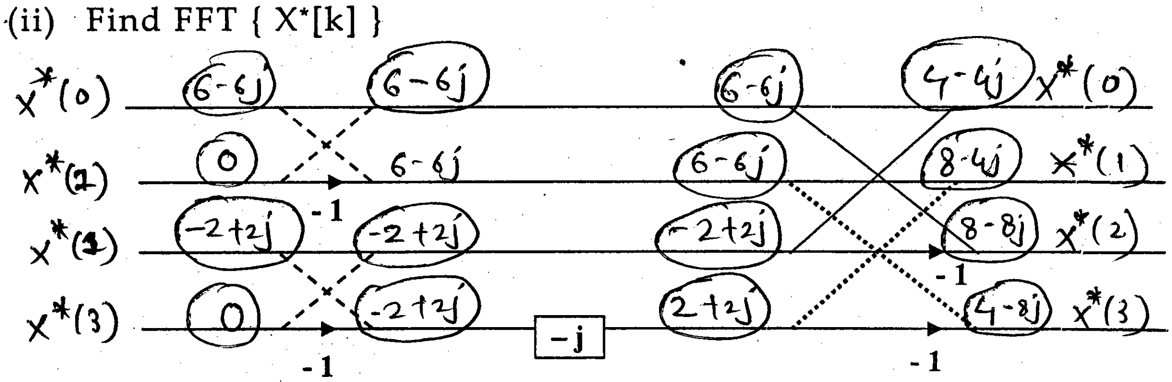
Q(32) Given $X[k] = \begin{bmatrix} 6+6j \\ -2-2j \\ 0 \\ 0 \end{bmatrix}$ Find x[n] by using fast algorithm.

Solution : By IFFT, $x[n] = \frac{1}{N}\left(FFT\{\ X^*[k]\ \}\right)^*$

(i) Find X*[k]

$$X^*[k] = \begin{bmatrix} 6-6j \\ -2+2j \\ 0 \\ 0 \end{bmatrix}$$

(ii) Find FFT { X*[k] }

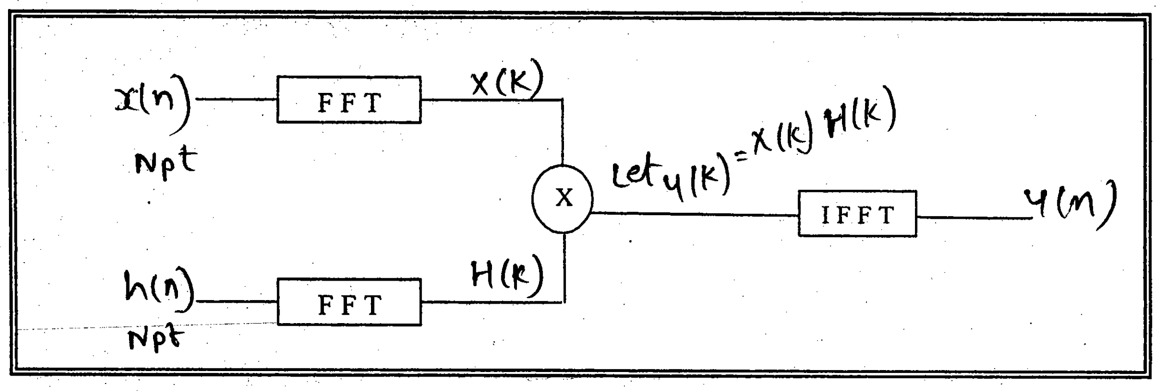


(iii) Find x[n]

$$x[n] = \frac{1}{N}\left(FFT\{\ X^*[k]\ \}\right)^*$$

$$x[n] = \frac{1}{4}\begin{bmatrix} 4-4j \\ 8-4j \\ 8-8j \\ 4-8j \end{bmatrix}^* = \frac{1}{4}\begin{bmatrix} 4+4j \\ 8+4j \\ 8+8j \\ 4+8j \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

CIRCULAR-CONVOLUTION Using FFT-IFFT



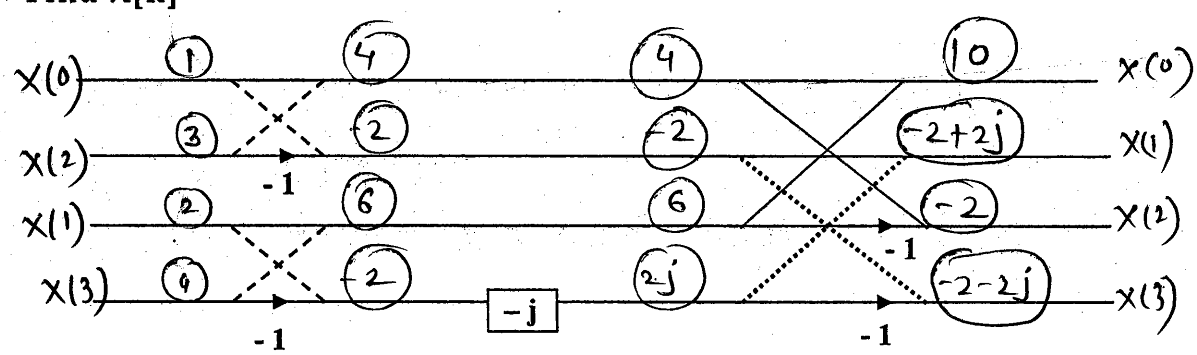
Q(33) If x[n] = { 1, 2, 3, 4 } and h[n] = { 5, 6, 0, 0 } Find circular convolution using FFT-IFFT

Solution :

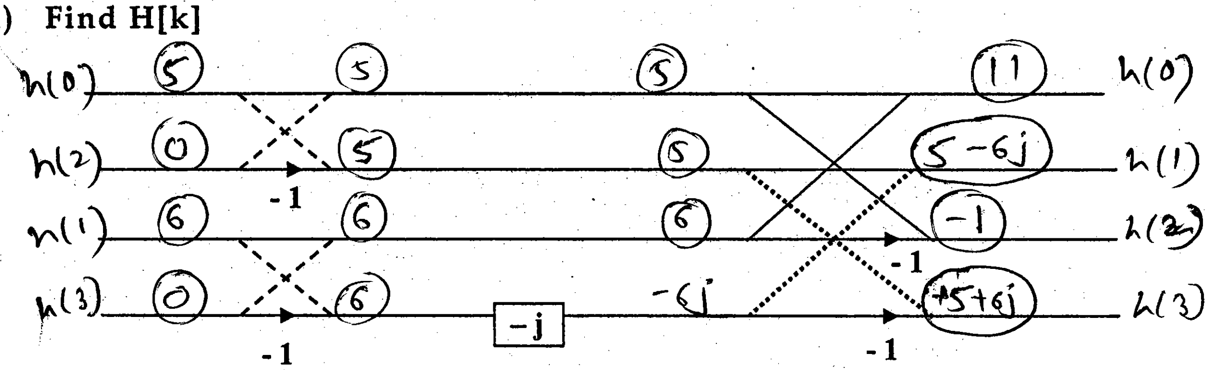
$$y[n] = x[n] \otimes h[n]$$

By DFT $$Y[k] = X[k]\ H[k]$$

(i) Find X[k]



(ii) Find H[k]

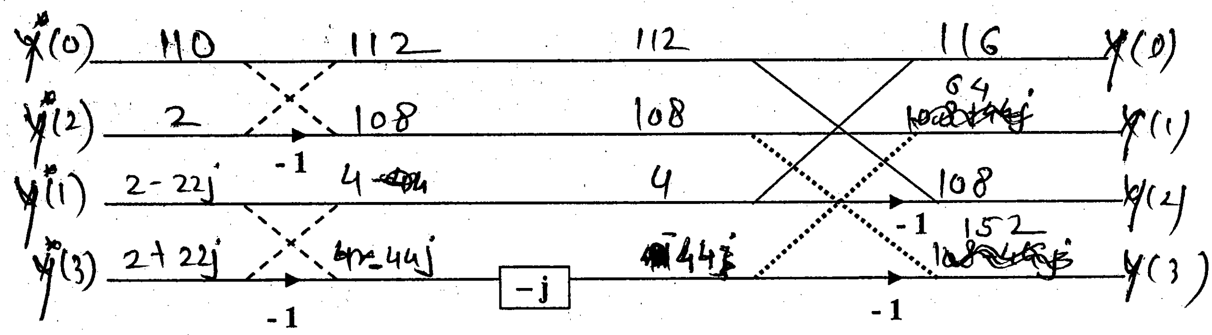


(iii) Find Y[k]

$$Y[k] = X[k]\ H[k] = \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix} \begin{bmatrix} 11 \\ 5-6j \\ -1 \\ +5+6j \end{bmatrix} = \begin{bmatrix} 110 \\ 2+22j \\ 2 \\ 2-22j \end{bmatrix}$$

(iv) Find y[n]

By Inverse FFT, $y[n] = \frac{1}{N}\left(FFT\{Y^*[k]\}\right)^*$ where $Y^*[k] = \begin{bmatrix} 110 \\ 2-22j \\ 2 \\ 2+22j \end{bmatrix}$



$$y[n] = \frac{1}{N}\left(FFT\{Y^*[k]\}\right)^* = \frac{1}{4}\begin{bmatrix} 116 \\ 64 \\ 108 \\ 152 \end{bmatrix}^* = \begin{bmatrix} 29 \\ 16 \\ 27 \\ 38 \end{bmatrix} \longleftarrow \text{ANS}$$

Q(34) Given x[n] = {x[0], x[1], x[2], x[3]} and h[n] = {h[0], h[1], h[2]}. Give step by step procedure to obtain Circular Convolution using FFT–IFFT.

Solution :

(i) Let 'L' be the length of x[n] and 'M' be the length of h[n]. Then L = 4 and M = 3

To find Circular Convolution,
Select $N \geq \max(L, M)$
$N \geq 4$

Let **N = 4** for radix-2 FFT algorithms,

(ii) Zero pad x[n] and h[n] to make their length equal to 4.

Let x[n] = {x[0], x[1], x[2], x[3]} and h[n] = {h[0], h[1], h[2], 0}.

(iii) Find FFT of both x[n] and h[n] using 4 point DIT-FFT algorithm.

(iv) Let Y[k] = X[k] H[k]

(v) Take IFFT of Y[k] to give y[n]

$$y[n] = \frac{1}{N}\left(FFT\{ Y^*[k] \}\right)^*$$

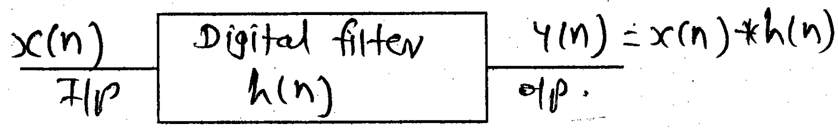
H.W.

Q(35) Let x[n] = { 1, 2, 3 } h[n] = { 5, 6 }

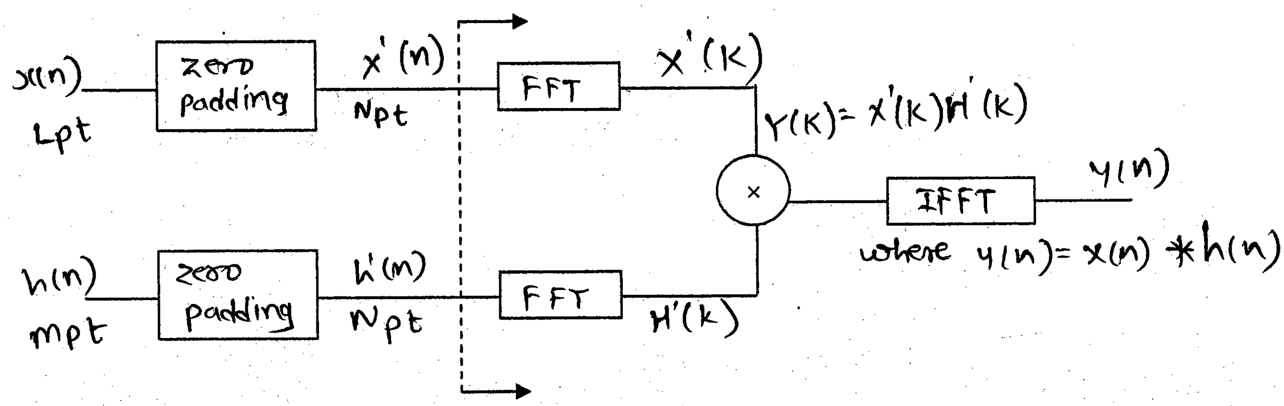
(a) Find Linear Convolution using FFT-iFFT.
(b) Find Linear Convolution using Time Domain Method.

➢ APLICATIONS OF D F T / F F T

(I) Linear Filtering : To find output of a digital filter for any given input say x[n].



Algo-1 : To find Linear Convolution using FFT



Q(36) Given x[n] = {x[0], x[1], x[2], x[3]} and h[n] = {h[0], h[1], h[2]}. Both are non-periodic finite length sequences. Give step by step procedure to obtain linear convolution using FFT–IFFT.

Solution :

(i) Let 'L' be the length of x[n] and 'M' be the length of h[n]. Then L = 4 and M = 3

To find Linear convolution using circular convolution using FFT-iFFT,

Select $N \geq L + M - 1$

$N \geq 4 + 3 - 1$

$N \geq 6$

Let **N = 8**, for radix-2 FFT algorithms,

(ii) Zero pad x[n] and h[n] to make their length equal to 8.

Let x'[n] ={x[0], x[1], x[2], x[3], 0, 0, 0, 0 } and h'[n] = {h[0], h[1], h[2] 0, 0, 0, 0, 0}.

(iii) Find FFT of both x'[n] and h'[n] using 8 point DIT-FFT algorithm.

(iv) Let Y[k] = X'[k] H'[k]

(v) Take IFFT of Y[k] to give y[n]

$$y[n] = \frac{1}{N}\left(\text{FFT}\{\, Y^*[k] \,\}\right)^*$$

- Limitations of Algo-1

1) **It is NOT suitable for Real Time Applications where entire input signal is not available.**
Examples include (i) ECG Monitoring system (ii) Digital Telephone System (iii) Weather Monitoring System

2) **It is NOT suitable for Long Data Sequence.**
Examples include Digital Song in the form of wave file Fs = 44.1 KHz + ECG/Weather Monitoring Systems. In most of the real Time applications data is Long sequence.

Q(37) Impulse response of 3rd order Linear Phase Low-Pass FIR filter is given by h[n]={ 1, 2, 2, 1 }. Give step by step procedure to find output of the filter to the input x[n] = { 1, 2, 3, 4} using FFT-IFFT.

Solution : Output of Digital filter is linear convolution of x[n] with h[n].
∴ Find LC by CC by FFT.

Algo-2 : To find Linear Convolution of Long Data Sequence using FFT

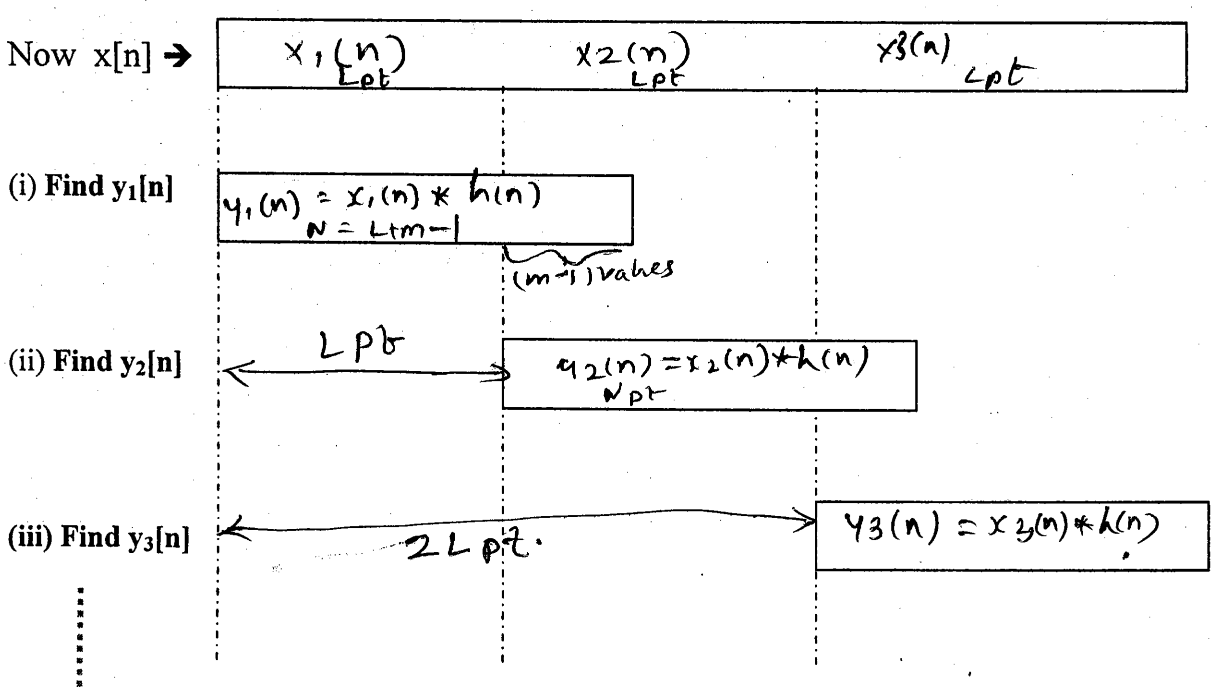
2.1 Overlap Save Method - - - - - -

- - - - - -

- - - - - -

2.2 Overlap Add method

Consider a digital filter with M pt h[n]. Consider long data sequence x[n].
By decomposing x[n] into L point sequences we get, $x_1[n]$, $x_2[n]$, $x_3[n]$, etc



(iv) Find y[n] $y(n) = y_1(n) + y_2(n-L) + y_3(n-2L)$

Filtering of Long Data Sequence

Overlap Add Method

Let x[n] be the sequence that is to be convolved with finite impulse response h[n] of length M.
In overlap add method x[n] is decomposed into non overlapping sequence of length L. Thus x[n] may be written a sum of shifted finite length sequences of length M such that $x[n] = \sum_{\ell=0}^{\infty} Xi[n-iL]$

$$\text{where } x_i[n] = \begin{cases} x[n+iL] & 0 \le n \le L-1 \\ 0 & otherwise \end{cases}$$

Therefore, the linear convolution of x[n] with h[n] is

$$y[n] = x[n] * h[n] = \sum_{i=0}^{\infty} Xi[n-iL] * h[n] = \sum_{i=0} yi[n-iL]$$

Where yi[n] is given by, $y_i[n] = xi[n] * h[n]$.

Overlap Add fast Convolution Algorithm :-

i) Decompose x[n] into L point sequences
ii) Append h[n] with (N-1) zeros and find H [k] using N pt FFT flowgraph.
iii) Append each input signal data block by (M-1) zeros and find DFT of each block using N pt FFT algorithm.
iv) Let Yi [k] = Xi [k] . H [k]   for i = 0,1.....∞
v) Obtain yi[n] by N pt iFFT Algorithm.
vi) Find y[n] ∴ $y[n] = \sum_{i=0}^{N-1} yi[n-iL] = y_0[n]+y_1[n-L]+y_2[n-2L]+..$

Q(38) Let $x[n]=\{\underset{\uparrow}{1}, 2, 3, 4, 5, 6, 7\}$ and $h[n]=\{\underset{\uparrow}{1}, 0, 2\}$

Find the convolution using overlap add method [Linear Convulation]

Solution (a) To find

Given $h[n] = \{\underset{\uparrow}{1}, 0, 2\}$, Length M = 3

Let L = 4 $x[n] = \{\underset{\uparrow}{1}, 2, 3, 4, \mid 5, 6, 7\}$

By decomposing x[n] into L=4 pt sequences we get,

$X_1[n] = \{\underset{\uparrow}{1}, 2, 3, 4\}$

$X_2[n] = \{\underset{\uparrow}{5}, 6, 7, 0\}$

(i) Find $y_1[n] = x_1[n] * h[n]$

$$y_1[n] = \sum_{m=-\infty}^{+\infty} x_1[m]\, h[n-m] \quad \text{where } x_1[n] = \{1, 2, 3, 4\} \quad h[n] = \{1, 0, 2\}$$

$$y_1[0] = \sum_{m=-\infty}^{+\infty} x_1[m]\, h[-m] = (1)(1) + (2)(\bullet) + (3)(\bullet) + (4)(\bullet) = \boxed{1}$$

$y_1[1] = \sum_{m=-\infty}^{+\infty} x_1[m]\, h[1-m] = (1)(0) + (2)(1) + (3)(\bullet) + (4)(\bullet) = \boxed{2}$

$y_1[2] = \sum_{m=-\infty}^{+\infty} x_1[m]\, h[2-m] = (1)(2) + (2)(0) + (3)(1) + (4)(\bullet) = \boxed{5}$

$y_1[3] = \sum_{m=-\infty}^{+\infty} x_1[m]\, h[3-m] = (1)(\bullet) + (2)(2) + (3)(0) + (4)(1) = \boxed{8}$

$y_1[4] = \sum_{m=-\infty}^{+\infty} x_1[m]\, h[4-m] = (1)(\bullet) + (2)(\bullet) + (3)(2) + (4)(0) = \boxed{6}$

$y_1[5] = \sum_{m=-\infty}^{+\infty} x_1[m]\, h[5-m] = (1)(\bullet) + (2)(\bullet) + (3)(\bullet) + (4)(2) = \boxed{8}$

Rough Work

$x_1[m]$	1	2	3	4	$y_1[n]$
h[-m] 2 0	1				1
2	0	1			2
	2	0	1		5
		2	0	1	8
			2	0	6
				2	8

(ii) Find $y_2[n] = x_2[n] * h[n]$

$y_2[n] = \sum_{m=-\infty}^{+\infty} x_2[m]\, h[n-m]$ where $x_2[n] = \{5, 6, 7, 0\}$ $h[n] = \{1, 0, 2\}$

$y_2[0] = \sum_{m=-\infty}^{+\infty} x_2[m]\, h[-m] = (5)(1) + (6)(\bullet) + (7)(\bullet) + (0)(\bullet) = \boxed{5}$

$y_2[1] = \sum_{m=-\infty}^{+\infty} x_2[m]\, h[1-m] = (5)(0) + (6)(1) + (7)(\bullet) + (0)(\bullet) = \boxed{6}$

$y_2[2] = \sum_{m=-\infty}^{+\infty} x_2[m]\, h[2-m] = (5)(2) + (6)(0) + (7)(1) + (0)(\bullet) = \boxed{17}$

$y_2[3] = \sum_{m=-\infty}^{+\infty} x_2[m]\, h[3-m] = (5)(\bullet) + (6)(2) + (7)(0) + (0)(1) = \boxed{12}$

$y_2[4] = \sum_{m=-\infty}^{+\infty} x_2[m]\, h[4-m] = (5)(\bullet) + (6)(\bullet) + (7)(2) + (0)(0) = \boxed{14}$

$y_2[5] = \sum_{m=-\infty}^{+\infty} x_2[m]\, h[5-m] = (5)(\bullet) + (6)(\bullet) + (7)(\bullet) + (0)(2) = \boxed{0}$

Rough Work

$x_2[m]$	5	6	7	0	$y_2[n]$
h[-m] 2 0	1				5
2	0	1			6
	2	0	1		17
		2	0	1	12
			2	0	14
				2	0

(iii) Find $y[n] = y_1[n] + y_2[n-L]$

Put L = 4

To find y[n] :

$$y_1[n] = \{\underset{\uparrow}{1}, 2, 5, 8, 6, 8\}$$

+

$$y_2[n-4] = \cancel{\{5, 6, 17, 12, 14, 0\}}\ \{0, 0, 0, 0, 5, 6, 17, 12, 14, 0\}$$

ANS : $y[n] = \{\underset{\uparrow}{1}, 2, 5, 8, 11, 14, 17, 12, 14, 0\}$

✪ Overlap Save Method

Let x[n] be the sequence that is to be convolved with finite impulse response h[n] of length M.
In overlap save method, input data sequence x[n] is decomposed into number of sequences. Each data block begins with the last (M-1) values in the previous data block, except the first data block which begins with (M-1) zeros.

Overlap Add fast Convolution Algorithm :-

i) Decompose x[n] into L point sequences

ii) Append h[n] with (L-1) zeros and find H[k] using N point FFT.

iii) Perform N – point FFT on the selected data block X_i [n] .

iv) Then Yi[k] = Xi[k] . H[k] .

v) Perform N point iFFT of Yi[k].

vi) Discard the first (M-1) values of y_i [n] and save the remaining values of y_i [n] .

vii) y[n] is obtained by concatenating all the saved values of y_i[n]

Q(39) **Let** $x[n]=\{\underset{\uparrow}{1}\ 2, 3, 4, 5, 6, 7\}$ **and** $h[n]=\{\underset{\uparrow}{1}\ 0\ 2\}$

Find the convolution using overlap save method

Solution :

Given h[n] = { 1, 0, 2}, Length M = 3

Let L = 4

To find Linear Convolution using Circular Convolution,
Let N = L+ M -1

For x[n] = - - - 0, 0, 0 { **1, 2, 3, 4,** | **5, 6, 7, 0** } 0, 0, 0, 0 - - -

By decomposing x[n] we get ,

$X_1[n] = \{0, 0, 1, 2, 3, 4\}$
N= 6pt

$X_2[n] = \{3, 4, 5, 6, 7, 0\}$
N=6pt

$X_3[n] = \{7, 0, 0, 0, 0, 0\}$
N=6pt

h[n] = { 1, 0, 2, 0, 0, 0 } periodic
h[-n] = { 1, 0, 0, 0, 2, 0 }

(i). Find $y_1[n] = x_1[n] \otimes h[n]$

$$y_1[n] = \sum_{m=0}^{N-1} x_1[m]\, h[n-m] \quad \text{where } x_1[n] = \{0, 0, 1, 2, 3, 4\} \quad h[n] = \{1, 0, 2, 0, 0, 0\}$$

$$y_1[0] = \sum_{m=0}^{N-1} x_1[m]\, h[-m] = (0)(1) + (0)(0) + (1)(0) + (2)(0) + (3)(2) + (4)(0) = \boxed{6}$$

$$y_1[1] = \sum_{m=0}^{N-1} x_1[m]\, h[1-m] = (0)(0) + (0)(1) + (1)(0) + (2)(0) + (3)(0) + (4)(2) = \boxed{8}$$

$$y_1[2] = \sum_{m=0}^{N-1} x_1[m]\, h[2-m] = (0)(2) + (0)(0) + (1)(1) + (2)(0) + (3)(0) + (4)(0) = \boxed{1}$$

$$y_1[3] = \sum_{m=0}^{N-1} x_1[m]\, h[3-m] = (0)(0) + (0)(2) + (1)(0) + (2)(1) + (3)(0) + (4)(0) = \boxed{2}$$

$$y_1[4] = \sum_{m=0}^{N-1} x_1[m]\, h[4-m] = (0)(0) + (0)(0) + (1)(2) + (2)(0) + (3)(1) + (4)(0) = \boxed{5}$$

$$y_1[5] = \sum_{m=0}^{N-1} x_1[m]\, h[5-m] = (0)(0) + (0)(0) + (1)(0) + (2)(2) + (3)(0) + (4)(1) = \boxed{8}$$

Rough Work

$x_1[m]$	0	0	1	2	3	4	$y_1[n]$
$h[-m]$	1	0	0	0	2	0	6
	0	1	0	0	0	2	8
	2	0	1	0	0	0	1
	0	2	0	1	0	0	2
	0	0	2	0	1	0	5
	0	0	0	2	0	1	8

(ii). Find $y_2[n] = x_2[n] \otimes h[n]$

$$y_2[n] = \sum_{m=0}^{N-1} x_2[m]\, h[n-m] \quad \text{where } x_2[n] = \{3, 4, 5, 6, 7, 0\} \quad h[n] = \{1, 0, 2, 0, 0, 0\}$$

$$y_2[0] = \sum_{m=0}^{N-1} x_2[m]\, h[-m] = (3)(1) + (4)(0) + (5)(0) + (6)(0) + (7)(2) + (0)(0) = \boxed{17}$$

$$y_2[1] = \sum_{m=0}^{N-1} x_2[m]\, h[1-m] = (3)(0) + (4)(1) + (5)(0) + (6)(0) + (7)(0) + (0)(2) = \boxed{4}$$

$$y_3[2] = \sum_{m=0}^{N-1} x_2[m]\, h[2-m] = (3)(2) + (4)(0) + (5)(1) + (6)(0) + (7)(0) + (0)(0) = \boxed{11}$$

$$y_4[3] = \sum_{m=0}^{N-1} x_2[m]\, h[3-m] = (3)(0) + (4)(2) + (5)(0) + (6)(1) + (7)(0) + (0)(0) = \boxed{14}$$

$$y_2[4] = \sum_{m=0}^{N-1} x_2[m]\, h[4-m] = (3)(0) + (4)(0) + (5)(2) + (6)(0) + (7)(1) + (0)(0) = \boxed{17}$$

$$y_2[5] = \sum_{m=0}^{N-1} x_2[m]\, h[5-m] = (3)(0) + (4)(0) + (5)(0) + (6)(2) + (7)(0) + (0)(1) = \boxed{12}$$

Rough Work

$x_2[m]$							$y_2[n]$
$h[-m]$	1	0	0	0	2	0	17
	0	1	0	0	0	2	4
	2	0	1	0	0	0	11
	0	2	0	1	0	0	14
	0	0	2	0	1	0	17
	0	0	0	2	0	1	12

(iii). Find $y_3[n] = x_3[n] \otimes h[n]$

$$y_3[n] = \sum_{m=0}^{N-1} x_3[m]\, h[n-m] \qquad \text{where } x_3[n] = \{7, 0, 0, 0, 0, 0\} \quad h[n] = \{1, 0, 2, 0, 0, 0\}$$

$$y_3[0] = \sum_{m=0}^{N-1} x_3[m]\, h[-m] = (7)(1) + (0)(0) + (0)(0) + (0)(0) + (0)(2) + (0)(0) = \boxed{7}$$

$$y_3[1] = \sum_{m=0}^{N-1} x_3[m]\, h[1-m] = (7)(0) + (0)(1) + (0)(0) + (0)(0) + (0)(0) + (0)(2) = \boxed{0}$$

$$y_3[2] = \sum_{m=0}^{N-1} x_3[m]\, h[2-m] = (7)(2) + (0)(0) + (0)(1) + (0)(0) + (0)(0) + (0)(0) = \boxed{14}$$

$$y_3[3] = \sum_{m=0}^{N-1} x_3[m]\, h[3-m] = (7)(0) + (0)(2) + (0)(0) + (0)(1) + (0)(0) + (0)(0) = \boxed{0}$$

$$y_3[4] = \sum_{m=0}^{N-1} x_3[m]\, h[4-m] = (7)(0) + (0)(0) + (0)(2) + (0)(0) + (0)(1) + (0)(0) = \boxed{0}$$

$$y_3[5] = \sum_{m=0}^{N-1} x_3[m]\, h[5-m] = (7)(0) + (0)(0) + (0)(0) + (0)(2) + (0)(0) + (0)(1) = \boxed{0}$$

Rough Work

$x_3[m]$							$y_3[n]$
$h[-m]$	1	0	0	0	2	0	7
	0	1	0	0	0	2	0
	2	0	1	0	0	0	14
	0	2	0	1	0	0	0
	0	0	2	0	1	0	0
	0	0	0	2	0	1	0

(iv) Find y[n]

$y_1(n) = \{\cancel{6}, \cancel{8}, 1, 2, 5, 8\}$ Discard first 2 values

$y_2(n) = \{\cancel{17}, \cancel{4}, 11, 14, 17, 12\}$

$y_3(n) = \{\cancel{7}, \cancel{0}, 14, 0, 0, 0\}$

we get

$y(n) = \{\underset{\uparrow}{1}, 2, 5, 8, 11, 14, 17, 12, 14, 0, 0, 0\}$

Q(40) **A 2^{nd} order FIR filter has impulse response $h[n]=\{\underset{\uparrow}{2},\ 2,\ 1\}$. Determine the output sequence response to the following input sequence $x[n]=\{\underset{\uparrow}{3}\ 0, -2, 0, 2, 1, 0, -2, -1, 0\}$ using (a) The overlap add method (b) The overlap save method.**

Solution.

(a) Overlap Add method

$$h[n] = \{2, 2, 1\}$$
$$x[n] = \{3, 0, -2, 0, 2, 1, 0, -2, -1, 0\}$$

Let $x_1[n] = \{3, 0, -2, 0, 2\}$ and $x_2[n] = \{1, 0, -2, -1, 0\}$

(i) Find $y_1[n] = x_1[n] * h[n]$

$$y_1[n] = \sum_{m=-\infty}^{+\infty} x_1[m]\, h[n-m]$$

$\therefore\ y_1[n] = \{6, 6, -1, -4, 2, 4, 2\}$

(ii) Find $y_2[n] = x_2[n] * h[n]$

$$y_2[n] = \sum_{m=-\infty}^{+\infty} x_2[m]\, h[n-m]$$

$\therefore\ y_1[n] = \{2, 2, -3, -6, -4, -1\}$

(iii) Find $y[n] = y_1[n] + y_2[n-5]$

$\therefore\ y[n] = \{6, 6, -1, -4, 2, (4+2), (2+2), -3, -6, -4, -1\}$

Ans : $y[n] = \{6, 6, -1, -4, 2, 6, 4, -3, -6, -4, -1\}$ for $n \geq 0$.

(b) Overlap save method

Solution : $h[n] = \{2, 2, 1, 0, 0, 0, 0\}$

Let $x_1[n] = \{0, 0, 3, 0, -2, 0, 2\}$
$x_2[n] = \{0, 2, 1, 0, -2, -1, 0\}$
$x_3[n] = \{-1, 0, 0, 0, 0, 0, 0\}$

(i) Find $y_1[n] = x_1[n] * h[n]$

$$= \sum_{m=0}^{N-1} x_1[m]\, h[n-m]$$

$\therefore\ y_1[n] = \{4, 2, 6, 6, -1, -4, 2\}$

(ii) Find $y_2[n] = x_2[n] * h[n]$

$$= \sum_{m=0}^{N-1} x_2[m]\, h[n-m]$$

$\therefore\ y_2[n] = \{-1, 4, 6, 4, -3, -6, -4\}$

(iii) Find $y_3[n] = x_3[n] * h[n]$

$$= \sum_{m=0}^{N-1} x_3[m]\, h[n-m]$$

$\therefore\ y_3[n] = \{-2, -2, -1, 0, 0, 0, 0\}$

(iv) To find y[n]

Discarding 1^{st} ($M-1 = 2$) values of $y_1[n]$, $y_2[n]$ and $y_3[n]$ we get ,

$y[n] = \{6, 4, -1, -4, 2, 6, 4, -3, -6, -4, -1\}$ for $n \geq 0$.

➤ **APLICATIONS OF DFT / FFT**

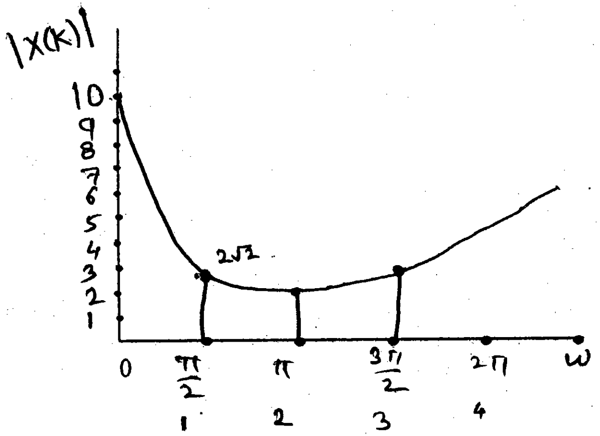
(II) **Spectral Analysis** : i.e. To find Magnitude Spectrum and Phase Spectrum of the signal.

Consider DT Signal $x[n] = \{ 1, 2, 3, 4 \}$ $N = 4$

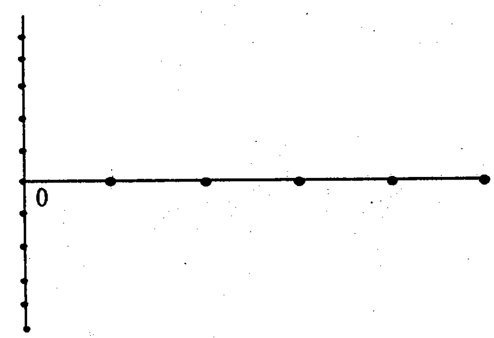
$$\text{Then} \quad X[k] = \begin{bmatrix} 10 & k=0 & w=0 \\ -2 + 2j & k=1 & w=\frac{\pi}{2} \\ -2 & k=2 & w=\pi \\ -2 - 2j & k=3 & w=\frac{3\pi}{2} \end{bmatrix} \quad \text{Freq spacing} = \frac{2\pi}{N} = \frac{\pi}{2}$$

$$\text{Where} \quad W = \frac{2\pi k}{N} = \frac{2\pi k}{4} = \frac{\pi}{2}k$$

Magnitude Spectrum



Phase Spectrum



NOTE : ① As N increases :-

i) Freq. spacing $= \frac{2\pi}{N}$ decreases

ii) Approx. Error decreases.

iii) Resolution of ~~spect~~ spectrum [i.e. no. of pixels per unit length] increases.

∴ Accuracy / quality increases.

② $x[nTs] \xrightarrow{DFT} X[\frac{2\pi}{N}k]$

Dt signal ⇓ $x(n)$ — simplified notation of DT signal

DFT of $x(nTs)$ ⇓ $x(k)$ — simplified notation of DFT results.

$x(n) \xrightarrow{DFT} x(k)$

③

Signal	Spectrum
periodic	Line i.e. Discrete.
Non-periodic	continuous

DFT assumes that signal are periodic

∴ DFT gives Lines spectrum.

To find DFT of Two N point Real Sequence using a single N point FFT.

(I) Let p[n] and q[n] be two real N – point sequences.

Let $x[n] = p[n] + j\, q[n]$ ______(i)

$x^*[n] = p[n] - j\, q[n]$ ______ (ii)

By eq^n (i) + eq^n (ii) we get,

$x[n] + x^*[n] = 2\, p[n]$

$$p[n] = \frac{1}{2}\left(x[n] + x^*[n]\right)$$

By DFT,

$$P[k] = \frac{1}{2}\left(X[k] + X^*[-k]\right)$$

By eq^n (i) - eq^n (ii) we get,

$x[n] - x^*[n] = 2j\; q[n]$

$$q[n] = \frac{1}{2j}\left(x[n] - x^*[n]\right)$$

By DFT

$$Q[k] = \frac{1}{2j}\left(X[k] - X^*[-k]\right)$$

(II) Find X[k] using N point FFT Flowgraph.

(III) Find P[k] and Q[k] using X[k] by evaluating the above derived equations.

ie. $P[k] = \frac{1}{2}\left(X[k] + X^*[-k]\right)$ and $Q[k] = \frac{1}{2j}\left(X[k] - X^*[-k]\right)$

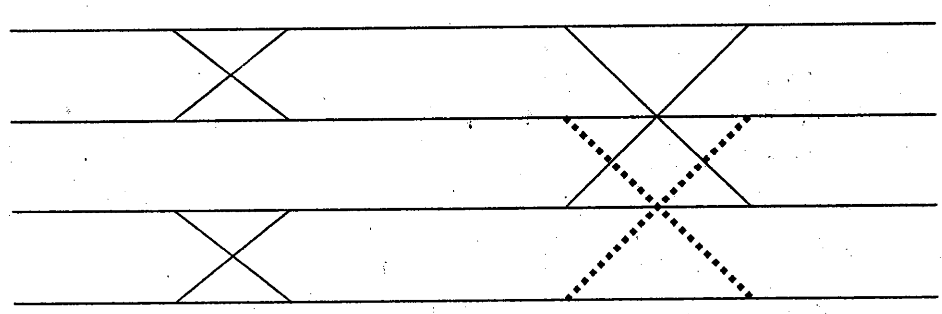
Q(41) Given that $x[n] = \{ (1 + 2j), (1 + j), (2 + j), (2 + 2j) \}$

(a) Find X [k] using DIT–FFT / DIF–FFT algorithm.

(b) Using the results in (a) and not otherwise find the DFT of p[n] and q[n] where $p[n] = \{ 1, 1, 2, 2 \}$ and $q[n] = \{ 2, 1, 1, 2 \}$.

Solution :

(a) Find X[k] by using 4 pt DIT-FFT flowgraph.



(b) i) **Find P[k]**

Let $x[n] = p[n] + j\, q[n]$ ______(i)

$x^*[n] = p[n] - j\, q[n]$ ______ (ii)

By eq^n (i) + eq^n (ii) we get,

$x[n] + x^*[n] = 2\, p[n]$

$$p[n] = \frac{1}{2}\left(x[n] + x^*[n]\right)$$

By DFT,

$$P[k] = \frac{1}{2}\left(X[k] + X^*[-k]\right)$$

$$\therefore P[k] = \begin{bmatrix} 6 + 6j & k = 0 \\ -1 + j & \\ 0 & \\ -1 - j & \end{bmatrix}$$

ii) **Find Q[k]**

By eq^n (i) - eq^n (ii) we get,

$$x[n]-x^*[n]=2j\ q[n]$$

$$q[n]=\frac{1}{2j}\left(x[n]-x^*[n]\right)$$

By DFT

$$Q[k]=\frac{1}{2j}\left\{X[k]-X^*[-k]\right\}$$

ANS : $\therefore Q[k]=\begin{bmatrix} 6 & k=0 \\ 1+j & \\ 0 & \\ 1-j & \end{bmatrix}$

Q(42) Let p[n] = {1, 2, 3, 4} and q[n]= {5, 6, 7, 8 }. Find DFT of each of the sequence using 4 pt FFT only once.

Solution :

(i) Let x[n] = p[n] + j q[n]

$\therefore$ x[n] = { 1 + j5, 2+ j6, 3 + j7, 4 + j8 }

(ii) Find X[k] by four point FFT, flowgraph

$$\therefore X[k]=\begin{bmatrix} 10+j26 & k=0 \\ -4 & \\ -2-j2 & \\ -j4 & \end{bmatrix}$$

(iii) To find P[k]

Hint : Derive $\quad p[n]=\frac{1}{2}\{x[n]+x^*[n]\}$

Then By DFT, $\quad P[k]=\frac{1}{2}\left\{X[k]+X^*[-k]\right\}$

ANS : $\quad \therefore P[k]=\begin{bmatrix} 10 & k=0 \\ -2+2j & \\ -2 & \\ -2-2j & \end{bmatrix}$

(iv) To find Q[k]

Hint : Derive $\quad \therefore q[n]=\frac{1}{2j}\{x[n]-x^*[n]\}$

Then By DFT, $\quad Q[k]=\frac{1}{2j}\left\{X[k]-X^*[-k]\right\}$

ANS : $\therefore Q[k]=\begin{bmatrix} 26 & k=0 \\ -2+2j & \\ -2 & \\ -2-2j & \end{bmatrix}$

Q(43) Let p[n] = { 1, 1 , 2, 2 } and q[n] = { 2, 1, 1, 2 }. Find DFT of each of the sequence using 4 pt FFT only once.

Q(44) Let x [n] = { 2 + 3j, 1 + 2j, 1 + j, 2 + j } and a[n] = { 5, 3, 2, 3 }. Find DFT of each of the sequence using FFT only once.

To find 2N point DFT of real valued sequence using a single N point FFT algorithm.

1) Let a [n] be a real valued sequence of length 2N.

By DIT FFT equation, $A[k]= G[k]+W_N^k\ H[k]$ _______ I

where $G[k]=DFT\ \{g[n]\} = DFT\ \{a[2r]\}\ \ for\ \ r = 0,1.....(N-1)$

and $H[k] = DFT\ \{h[n]\} = DFT\ \{a[2r+1]\}$

2) Let $x[n] = g[n] + j\ h[n]$ _______ i

Then $x^*[n] = g[n] - j\,h\ [n]$ _______ ii

By eq^n (i) + eq^n (ii),

$x[n] + x^*[n] = 2g[n]$

$g[n] = \frac{1}{2}\ (x\ [n] + x^*[n])$

By DFT

$G[k] = \frac{1}{2}\ (X[k] + X^*[-k])$ _______ II

By eq^n (i) -eq^n (ii)

$x[n] - x^*[n] = 2jh[n]$

$h[n] = \frac{1}{2j}\ (x[n] - x^*[n])$

By DFT

$H[k] = \frac{1}{2j}\ (X[k] - X^*[-k])$ _______ III

3) Find X[k] using N pt FFT algorithm.
4) Obtain G[k] and H[k] using X[k] by evaluating the equation II and III respectively.
5) Find 2N pt A[k] using G[k] and H[k] by evaluating the equation I.

Q(45) Let a[n] ={ 1, 1, 2, 2, 3, 3, 4, 2 } Find A[k] using 4 pt FFT flowgraph.

Hint :

(i) Decompose a[n] into 4 pt sequences by Decimating in Time

a [n] is 8 pt real valued sequence.

By D I T, a[2r] = { 1, 2 ,3, 4 } and a[2r+1] = { 1, 2, 3, 2 }

(ii) Derive, DIT FFT equation $A[k]= G[k]+W_N^k\ H[k]$ _______ I

where $G[k]= DFT\ \{a[2r]\}$

and $H[k] = DFT\ \{a[2r+1]\}$

(iii) Obtain G[k] and H[k] using X[k] by evaluating the equation II and III respectively.

$$\therefore G[k] = \begin{bmatrix} 10 & k=0 \\ -2+2j & \\ -2 & \\ -2-2j & \end{bmatrix} \qquad \therefore H[k] = \begin{bmatrix} 8 & k=0 \\ -2 & \\ 0 & \\ -2 & \end{bmatrix}$$

(iv) Find 8 pt A[k] using G[k] and H[k] by evaluating the equation I.

ANS : $A[k]= G[k]+ W_N^k\ H[k]$

$$A[k] = \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \\ 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix} + \begin{bmatrix} 1 \\ 0.707-j0.707 \\ -j \\ -0.707-j0.707 \\ -1 \\ -0.707+j0.707 \\ j \\ 0.707+j0.707 \end{bmatrix}\begin{bmatrix} 8 \\ -2 \\ 0 \\ -2 \\ 8 \\ -2 \\ 0 \\ -2 \end{bmatrix} === \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

Q(46) Let a[n] ={ 1, 1, 2, 2, 3, 3, 4, 2 } Find A[k] using 4 pt FFT flowgraph only once.

Hint :

(i) Decompose a[n] into 4 pt sequences by decimating in Time

a [n] is 8 pt real valued sequence.
By D I T, a[2r] = { 1, 2 ,3, 4 } and a[2r+1] = { 1, 2, 3, 2 }

(ii) Derive, DIT FFT equation $A[k]= G[k]+W_N^k\ H[k]$ _______ I

where $G[k]= DFT\ \{a[2r\}$ Let g[n] = a[2r]

and $H[k] = DFT\ \{a[2r+1]\}$ Let h[n] = a[2r+1]

(v) Let $x[n] = g[n] + j\ h[n]$_____ i

Then $x^*[n] = g[n] - j h\ [n]$_____ ii

By eq^n (i) + eq^n (ii),

$$x[n]+x^*[n]=2g[n]$$

$$g[n]=\frac{1}{2}(x[n]+x^*[n])$$

By DFT

$$G[k]=\frac{1}{2}\left(X[k]+X^*[-k]\right)\_\_\_\_\_ II$$

By eq^n (i) − eq^n (ii)

$$x[n]-x^*[n]=2jh[n]$$

$$h[n]=\frac{1}{2j}\left(x[n]-x^*[n]\right)$$

By DFT

$$H[k]=\frac{1}{2j}\left(X[k]-X^*[-k]\right)\_\_\_\_\_ III$$

(vi) Find X[k] using N=4 pt FFT flowgraph.

$$\therefore X[k]=\begin{bmatrix} 10+8j & k=0 \\ -2 & \\ -2 & \\ -2-2j & \end{bmatrix}$$

(vii) Find G[k] and H[k] by using 4 pt FFT flowgraph.

$$\therefore G[k]=\begin{bmatrix} 10 & k=0 \\ -2+2j & \\ -2 & \\ -2-2j & \end{bmatrix} \quad \therefore H[k]=\begin{bmatrix} 8 & k=0 \\ -2 & \\ 0 & \\ -2 & \end{bmatrix}$$

(viii) Find 8 pt A[k] using G[k] and H[k] by evaluating the equation I.

ANS : $A[k]= G[k]+W_N^k\ H[k]$

$$A[k]=\begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \\ 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix} + \begin{bmatrix} 1 \\ 0.707-j0.707 \\ -j \\ -0.707-j0.707 \\ -1 \\ -0.707+j0.707 \\ j \\ 0.707+j0.707 \end{bmatrix}\begin{bmatrix} 8 \\ -2 \\ 0 \\ -2 \\ 8 \\ -2 \\ 0 \\ -2 \end{bmatrix} \equiv \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \ \end{bmatrix}$$

2.7 DFT COMPUTATION BY DIVIDE and CONQUER

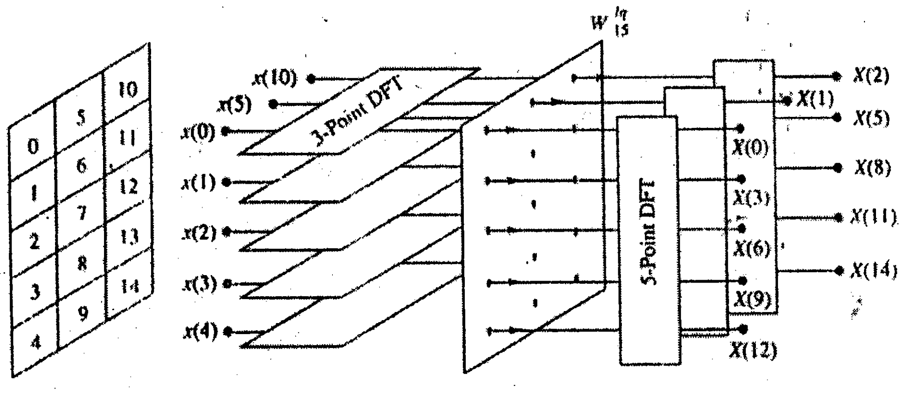
The divide and conquer approach of DFT, computation is based on the splitting of an N- point DFT into smaller DFTs.

Let us consider a N –point DFT . where N can be factored as product of two integers as N = L M
Now DT signal x [n] can be stored either one dimensional array or two dimensional array where two dimensional array is $\ell \times m$
and $(\ell = Number\ of\ rows)$ *and* $(m = Number\ of\ columns.)$

Similar arrangements are made for storing computed DFT values.
Data can be stored in ROW wise or COLUMN wise.

Number of computations required for matrix form.

- N (M + L + 1) complex multiplications than N^2 multiplications required for direct DFT computation.
- N (M + L - 2) complex additions than N (N-1) additions required for direct DFT computation.
  e.g. : N = 1000 say L = 2 and M = 500



In general number of computations required for DFT by matrix method is as follows.

Computations required for **Algorithm – 1**	Computations required for **Algorithm – 2**
1) Store the DT signal column-wise 2) Compute L-point DFT of each row. 3) Multiply the resulting array by corresponding twiddle factors. 4) Compute M point DFT of each column. 5) Read the resulting array row-wise.	1) Store the DT signal row-wise 2) Compute L-point DFT of each column. 3) Multiply the resulting array by corresponding twiddle factors. 4) Compute M point DFT of each row. 5) Read the resulting array column-wise.

(i) **CTFT** of CT signal x[t] is defined as, **X(w)** = $\int_{t=-\infty}^{\infty} x(t) e^{-j\omega t} dt$

(ii) **DTFT** of DT signal x[n] is defined as, **X(w)** = $\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$ for $-\pi < \omega \le \pi$ period 2π

$\uparrow$ continuous function of ω.

(iii) **Inverse DTFT** of X(w) is defined as, **x[n]** = $\frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} X(\omega) e^{jn\omega} d\omega$.

Q(47) Given x[n] = { 1, 2, 3, 4 }
(a) Find Fourier Transform of x[n].
(b) Plot magnitude spectrum of x[n].

	?	ANS
1	X(w)	DTFT
2	F.T.	DTFT
3	X[k]	DFT

Solution (a) To find DTFT:

By, DTFT, **X(w)** = $\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

$n = 4$

$$X(\omega) = \sum_{n=0}^{3} x(n) e^{-jn\omega}$$

$$= x(0) + x(1) e^{-j\omega} + x(2) e^{-j2\omega} + x(3) e^{-j3\omega}$$

Ans: $X(\omega) = 1 + 2e^{-j\omega} + 3e^{-j2\omega} + 4e^{-j3\omega}$

b) To plot magnitude spectrum

Put $e^{-j\omega} = \cos(\omega) - j\sin(\omega)$

$$X(\omega) = 1 + 2[\cos(\omega) - j\sin(\omega)] + 3[\cos(2\omega) - j\sin(2\omega)] + 4[\cos(4\omega) - j\sin(4\omega)]$$

$$X(\omega) = 1 + 2\cos(\omega) + 3\cos(2\omega) + 4\cos(3\omega) - j[2\sin(\omega) + 3\sin(2\omega) + 4\sin(3\omega)]$$

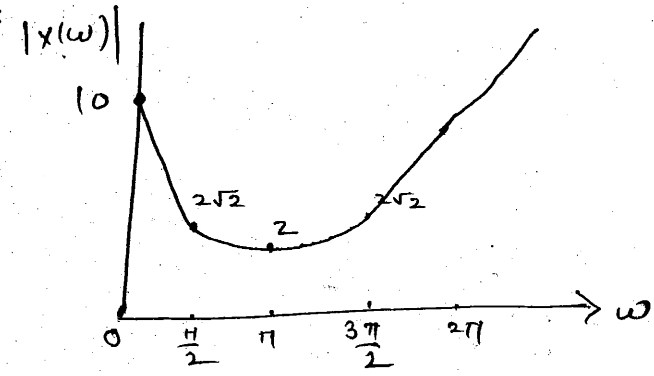
i) At $\omega = 0$, $X(0) = 10$
ii) At $\omega = \frac{\pi}{2}$, $X(1) = -2+2j$
iii) At $\omega = \pi$, $X(2) = -2$
iv) At $\omega = \frac{3\pi}{2}$, $X(3) = -2-2j$

} DFT coeficient

$$\boxed{\underset{\text{DFT}}{X(K)} = \underset{\text{DTFT}}{X(\omega)} \quad \Big| \quad \omega = \frac{2\pi K}{N}}$$

Note – DFT is freq. sampling of DTFT

Solution :



Derivation of DFT eq^n

By DTFT;

$$x(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jn\omega}$$

Put $\omega = \frac{2\pi K}{N}$

$$x\left[\frac{2\pi K}{N}\right] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jn\left(\frac{2\pi K}{N}\right)}$$

$$x(K) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi}{N}(nk)}$$

Put $W_N^1 = e^{-j\frac{2\pi}{N}}$

$$x(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{nK}$$

↑

DFT of $x(n)$

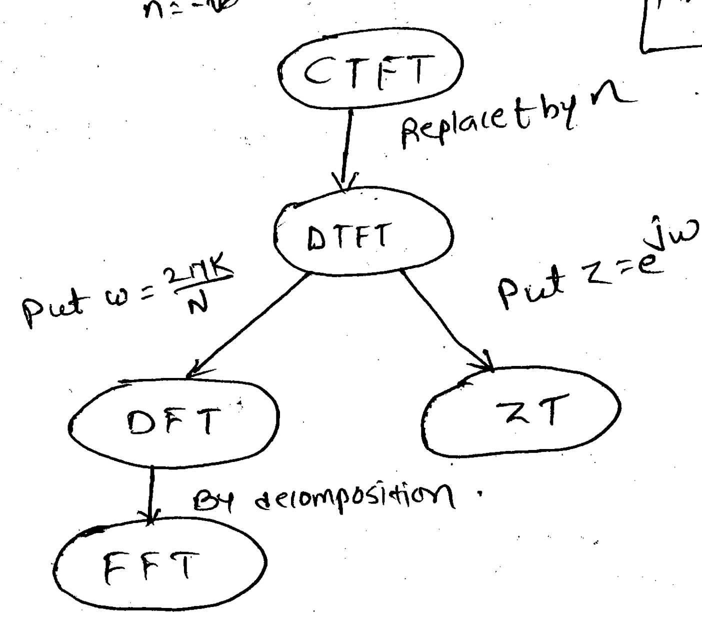
Ref By DTFT,

$$x(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jn\omega}$$

Put $z = e^{j\omega}$

$$x(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

family of Fourier Transfor



Q(48) Given $x[n]=\{\underset{\uparrow}{1} \ 2 \ 3 \ 4\}$

(a) Find X[k] by using DFT equation.

(b) Find X[k] by using DTFT equation.

Solution :

(a) By DFT,

****

$$X[k]=\begin{bmatrix} 10 & k=0 \\ -2+2j & \\ -2 & \\ -2-2j & \end{bmatrix} \quad \textbf{ANS.}$$

(b) To find X[k] from DTFT X(W)

By DTFT, $X(w)=\sum_{n=-\infty}^{\infty} x[n] \ e^{-jnw}$

$$X(w)=x[0]+x[1] \ e^{-jw}+x[2] \ e^{-j2w}+x[3] \ e^{-j3w}$$
$$X(w)=1+2 \ e^{-jw}+3 \ e^{-j2w}+4 \ e^{-j3w}$$

$$X(w)=[1+2\cos(w)+3\cos(2w)+4\cos(3w)]-j[2\sin(w)+3\sin(2w)+4\sin(3w)]$$
$$X[k]=X(w)\big|_{w=\frac{2\pi k}{N}}$$

where N =4

$$X[k]=X(w)\big|_{w=\frac{\pi k}{2}}$$

$$X[k]=\left[1+2\cos\left(\frac{\pi k}{2}\right)+3\cos(\pi k)+4\cos\left(\frac{3\pi k}{2}\right)\right]-j\left[2\sin\left(\frac{\pi k}{2}\right)+3\sin(\pi k)+4\sin\left(\frac{3\pi k}{2}\right)\right]$$

$$X[k]=\begin{bmatrix} 10 & k=0 \\ -2+2j & \\ -2 & \\ -2-2j & \end{bmatrix}$$

Imp

Q(49) Find DFT of the following signals and plot magnitude spectrum.

(a) $x_1[n]=\{\underset{\uparrow}{1} \ 2 \ 3 \ 4\}$ (b) $x_2[n]=\{1 \ \underset{\uparrow}{2} \ 3 \ 4\}$ (c) $x_3[n]=\{1 \ 2 \ \underset{\uparrow}{3} \ 4\}$

$x(\omega)=\sum_{n=-1}^{2} x(n)e^{j n \omega}$

ENERGY DENSITY SPECTRUM OF DT APERIODIC SIGNALS

The energy of DT signal x[n] is E = $\sum_{n=-\infty}^{\infty} |x[n]|^2$

According to parseval's theorem, E = $\sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x(w)|^2 dw$

Let Sx (w) = $|x(w)|^2$ = x (w) x* (w)

Sx (w) is the function of frequency and it is called energy density spectrum of x [n]. $E=\sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} Sx(w) \ dw.$

Q(50) $x[n] = a^n u[n] \quad -1 < a < 1$ **Find DTFT and Energy Density Spectrum.**

Solution : By DTFT, $X[w] = \sum_{n=-\infty}^{\infty} x[n]e^{-jnw} = \sum_{n=-\infty}^{\infty} a^n u[n]e^{-jnw}$

$$= \sum_{n=0}^{\infty} a^n e^{-jnw}$$

$$x(w) = \frac{1}{1 - ae^{-jw}} \qquad \because |ae^{-jw}| < |$$

$$X(w) = \frac{1}{1 - ae^{jw}}$$

Let a = 0.5

$$X(w) = \frac{1}{1 - 0.5e^{jw}} = \frac{1}{1 - 0.5\,(\cos(w) + j\sin(w))} = \frac{1}{[1 - 0.5\cos(w)] - j\,[0.5\sin(w)]}$$

Energy spectrum = $X(w) \cdot X^*(w)$

$$S_x(w) = \left(\frac{1}{1 - ae^{-jw}}\right)\left(\frac{1}{1 - ae^{jw}}\right)$$

$$= \frac{1}{1 - ae^{jw} - ae^{-jw} + a^2}$$

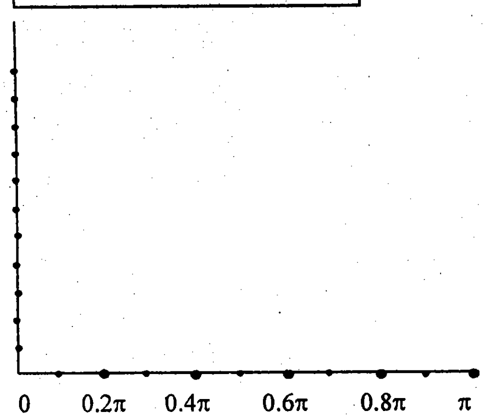
$$= \frac{1}{(1 - a^2) - a(e^{jw} + e^{-jw})}$$

$$= \frac{1}{(1 - a^2) - 2a\cos w}.$$

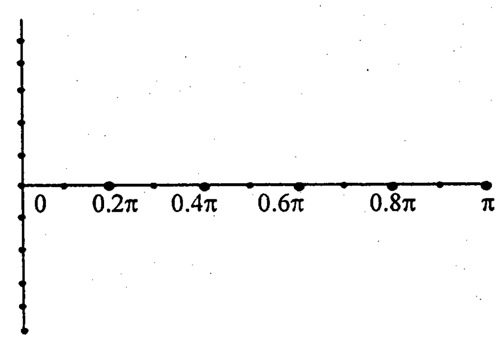
For a = 0.5, $S_x(w) = \dfrac{1}{0.75 - \cos w}.$

Sr No.	Freq W	Magnitude | X(w) |	Phase ∅	Sx(w)
1	0			
2	0.2 π			
3	0.4 π			
4	0.6 π			
5	0.8 π			
6	π			

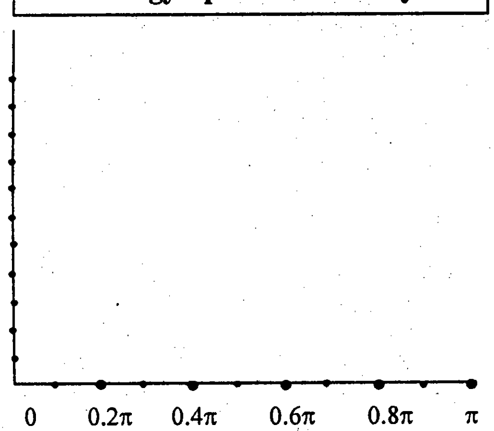
Magnitude Spectrum



Phase Spectrum



Energy Spectrum density



2.9 DISCRETE FOURIER SERIES

The Fourier series of a periodic sequence $x_p[n]$ with period N is given by,

$$\boxed{x_p[n] = \sum_{n=0}^{n-1} C_K\, e^{jwnk} \quad where\ w = \frac{2\pi}{N} \quad -\infty < n < \infty}$$

The Fourier series coefficients are given by,

$$C_k = \frac{1}{N} \sum_{N=0}^{N-1} x_p[n]\, e^{-jwnk} \quad where \quad w = \frac{2\pi}{N}$$

POWER DENSITY SPECTRUM OF PERIODIC DT SIGNALS

The average power of periodic DT signal is given by $P = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$

According to Parseval's theorem, $\boxed{P = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |C_k|^2}$

The coefficients $|C_k|^2$ for k =0, 1, 2.....N-1 is the distribution of power as a function of frequency. It is called the power density spectrum of the DT periodic signal.

Q(51) Given x[n] = { 1, 2, 3, 4 } periodic

(a) Find the Fourier series coefficients.

(b) Plot Magnitude Spectrum and Poer Density Spectrum.

Solution : Fourier series coefficients C_k are given by

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p[n] e^{-j\frac{2\pi nk}{N}} \qquad \text{Where } N = 4$$

$$\therefore C_k = \frac{1}{4} \sum_{n=0}^{3} x[n] e^{-j\frac{\pi nk}{4}}$$

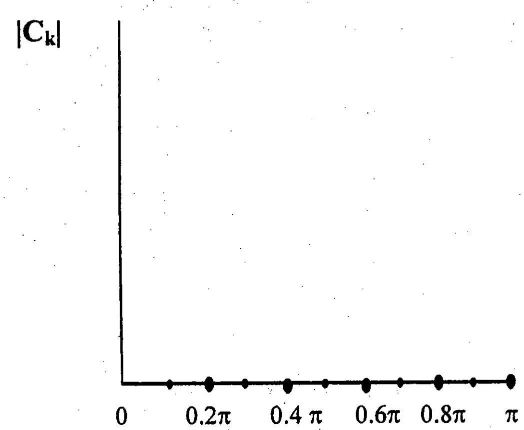
$$\therefore C_k = \frac{1}{4} \left\{ x[0]\, e^{-j0} + x[1]\, e^{-j\frac{\pi k}{4}} + x[2]\, e^{-j\frac{2\pi k}{4}} + x[3]\, e^{-j\frac{3\pi k}{4}} \right\}$$

$$C_k = \frac{1}{4} \left\{ (1) + (2)\, e^{-j\frac{\pi k}{4}} + (3)\, e^{-j\frac{2\pi k}{4}} + (4)\, e^{-j\frac{3\pi k}{4}} \right\}$$

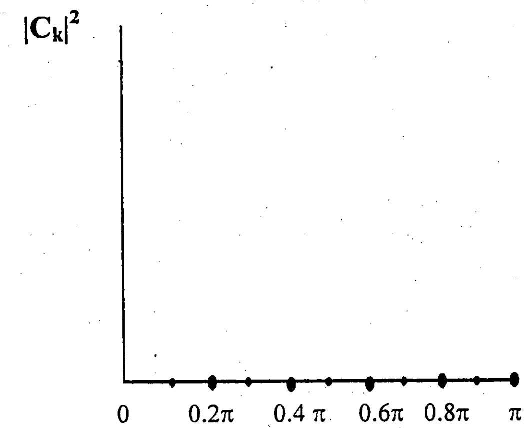
$$C_k = \frac{1}{4} \left\{ \begin{array}{c} \left[(1) + (2)\cos\left(\frac{\pi k}{4}\right) + (3)\cos\left(\frac{2\pi k}{4}\right) + (4)\cos\left(\frac{3\pi k}{4}\right)\right] \\ -j\left[(2)\sin\left(\frac{\pi k}{4}\right) + (3)\sin\left(\frac{2\pi k}{4}\right) + (4)\sin\left(\frac{3\pi k}{4}\right)\right] \end{array} \right\}$$

k=0	C_0 =	
k=1	C_1 =	
k=2	C_2 =	
k=3	C_3 =	

Magnitude Spectrum



Power Density Spectrum



Q(52) Develop the relationship between Discrete Time Fourier Series of a periodic sequence $x_p[n]$ and DFT of a sequence x[n].

Solution : Relationship between DTFS and DFT

The fourier series representation of a periodic sequence $x_p[n]$ with fundamental period N is given by,

$$x_p[n] = \sum_{k=0}^{N-1} C_k\, e^{j\frac{2\pi nk}{N}}, \quad -\infty \le n \le \infty$$

Where the fourier series coefficients are given by,

$$C_k = \frac{1}{N} \sum x_p[n] e^{\frac{-j2\pi nk}{N}}, \qquad 0 \le k \le n-1$$

By DFT, $X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk} = \sum_{n=0}^{N-1} x[n]\, e^{\frac{-j2\pi nk}{N}}$

By comparing the above equations, If $x[n] = x_p[n]$ then $\mathbf{X[k] = N.C_k}$.

Q(53) Develop the relationship between DTFT and DFT of Discrete Time Signal x[n]

Solution : Relationship between DTFT and DFT.

Let x[n] be an a periodic finite Energy sequence.

The DTFT is given by, $X(w) = \sum_{n=0}^{N-1} x[n]\, e^{-jnw}$

If X(w) is sampled at N equally sampled frequencies,

Then

$$X[k] = X(w)\Big|_{w=\frac{2\pi k}{N}} \qquad 0 \le k \le N-1$$

$$X[k] = \sum_{N=0}^{N-1} x[n]\, e^{\frac{-j2\pi nk}{N}}$$

The original DT signal x[n] can be recovered without aliasing provided length of x[n] is less than or equal to N. The spectral components $\{X[k]\}$ corresponds to the spectrum of a periodic sequence of period N.

Q(54) Develop the relationship between ZT and DFT of Discrete Time Signal x[n]

Solution : Relationship between ZT and DFT.

The ZT of a sequence x [n] is given by,

$$X[z]=\sum_{-\infty}^{\infty} x[n]z^{-n}$$

If X(z) is sampled at N equally spaced points on the unit circle,

$$Z_k = e^{j2\pi k/N} \qquad 0 \le k \le N-1$$

We get, $X(z)\Big|_{z=e^{j2\pi k/N}} = \sum_{n=0}^{N-1} x[n]\, e^{\frac{-j2\pi nk}{N}} = X[k].$

That means,

$$X[k]=X(z)\Big|_{Z=e^{\frac{j2\pi k}{N}}}$$

$X(z)=\sum_{n=0}^{N-1} X[n]z^{-n}$ when x [n] is N point sequence

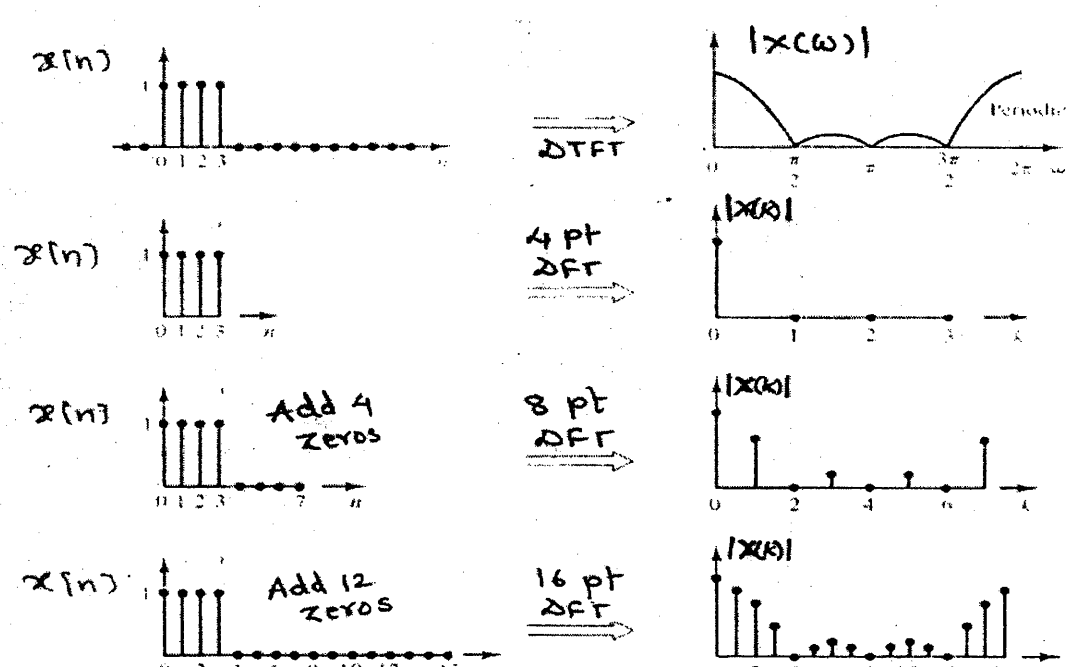
DFT computes the values of the Z transform for evenly spaced points around the unit circle for a given sequence.

Q(55) What is the effect of zero padding ?

Effect of adding or padding zeros to the end of a sequence: -

Let x[n] be N_1 pt sequence. If DFT of x[n] us taken then the sampled values of the fourier transform are spaced $\frac{2\pi}{N_1}$ apart. If x [n] is padded with N_2 zeros to give sequence of N values and if N pt DFT is taken then, the sampled values of fourier transform are spaced $\frac{2\pi}{N_1 + N_2}$ apart. As more zeros are added, the DFT points are closely spaced samples of the furrier transform of the original sequence thus giving a better displayed version of the fourier Transform. I.e. just a better display of available information.

The following figure gives spectrum of 4, 8 and 16 pt DFT of the zero padded original sequence.



Q(56) Explain **Goertzel Algorithm**

The Goertzel Algorithm makes use of the periodicity of the sequence W_N^{nk} to reduce the computation involved in calculating DFT.

By DFT, $X[k]=\sum_{n=0}^{N-1} x[n]\, W_N^{nk} = \sum_{r=0}^{N-1} x[r]\, W_N^{rk}$

Multiplying by W_N^{-kN}

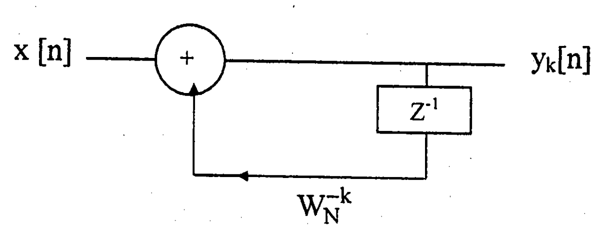
$$X[k] = W_N^{-kN} \sum_{r=0}^{N-1} x[r]\ W_N^{rk} = \sum_{r=o}^{N-1} x[r]\ W_N^{rk}\ W_N^{-kN}$$

$$= \sum_{r=0}^{N-1} x[r]\ W_N^{-k(N-r)}$$

Let $y_k[n] = \sum_{r=0}^{N-1} x[r]\, W_N^{-k(n-r)} = x[n] * W_N^{-kn}\ u[n].$

$y_k[n]$ can be veined as the response of the system with impulse response, $h_k[n] * W_N^{-nk}\ u[n].$

By ZT, $H_k(z) = \dfrac{Z}{Z - W_N^{-k}}$



(i) To find difference equation,

$$\frac{Y_k(z)}{X_k(z)} = \frac{1}{1 - W_N^{-k}\ Z^{-1}}$$

$$Y_k(z) - W_N^{-k} z^{-1}\ Y_k(z) = X_k(z)$$

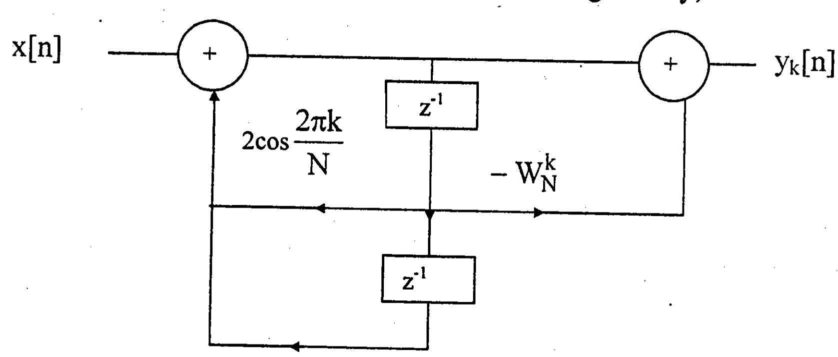
By iZT,

$$y_k[n] = W_N^{-k}\ y_k[n-1] + x_k[n].$$

Now yk [n] can be computed by the difference equation rerecursively. The desired output is $X[k] = y_k[n]$, $k = 0, 1 \ldots N-1$. The complex Multiplications and additions can be avoided by combining two single pole filters consisting of complex conjugate poles.

$$H_k(z) = \frac{z}{(z - w_N^{-k})} \frac{z - w_N^k}{z - w_N^k} = \frac{z^2 - w_N^k z}{z^2 - 2\cos\left(\frac{2\pi k}{N}\right) z + 1}$$

(ii) Realization diagram of second order system is given by,



Q(57) Explain **Chirp Z–Transform**

The chirp Z–Transform is an efficient algorithm for evaluating the Z–Transform of a finite length sequence at spaced samples along a generalized couture in the z plane.

Let x [n] be N pt sequence, the ZT of x [n] is given by $X(z) = \sum_{n=0}^{N-1} x[n]z^{-n}$

The chirp ZT algorithm compute the samples of the ZT on a spiral contour equally spaced in angle over same portion of the contour.

Using the chirp ZT algorithm X[z] can be computed at the points Zk given by ,

$Z_k = AB^{-k}$ k = 0,1 … (m-1).

Where $B = Bo\ e^{-j\phi_0}$
$A = Ao\ e^{j\theta_0}$

This contour is spiral in the z plane
The parameter Bo controls the rate at which the contour spirals.
If Bo is less than unity, the contour spirals outward as k increases.
If Bo is greater than unity, the contour spirals towards the origin as k increases.
The parameters Ao and θo are the location in radius and angle respectively of the first sample i.e. k = 0 . The remaining samples are located along the spiral contour with an angular spacing of ϕ_0.

$Z_k = AB^{-k}$.

$$X(Z_k) = \sum_{n=0}^{N-1} X[n](AB^{-k})^{-n}$$

$$= \sum_{n=0}^{N-1} x[n]A^{-n}B^{nk}$$

According to Bluestein, $nk = \left[n^2 + k^2 - (k-n)^2\right]$

$$X(Z_k) = \sum_{n=0}^{N-1} x[n]\,A^{-n}\,B^{\frac{1}{2}[n^2+k^2-(k-n)^2]}$$

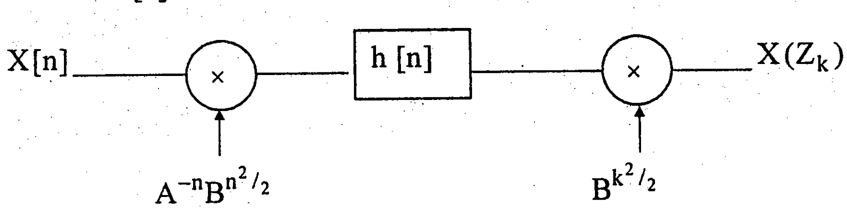
$$= \sum_{n=0}^{N-1} x[n]\quad A^{-n}\ B^{\frac{n^2}{2}}\ B^{\frac{k^2}{2}}\ B^{\frac{-(k-n)^2}{2}}$$

Let $g[n] = x[n]\ A^{-n}\ B^{n^2/2}$

$$X(Zk) = B^{k^2/2}\sum_{n=o}^{N-1} g[n]\ B^{\frac{-(k-n)^2}{2}}$$

Let $X(Zk) = B^{\frac{k^2}{2}}\sum_{n=0}^{N-1} g[n]\ h[k-n]$ ________________ I

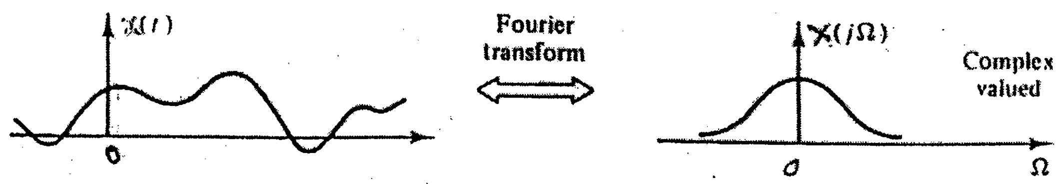
Where $h[n] = B^{\frac{n^2}{2}}$



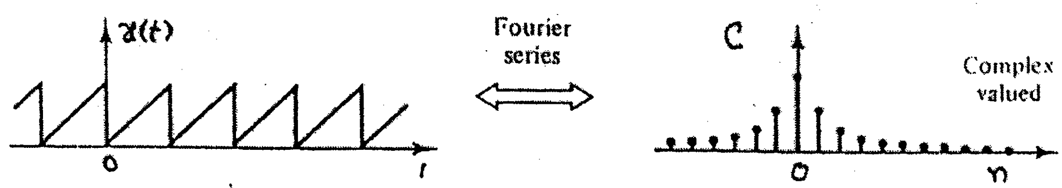
The Z-Transform evaluated by eq^n (I) ie $X(z_k)$ as a convolution for the response y[n] of linear filter is – called chirp Z Transform.

• Summary

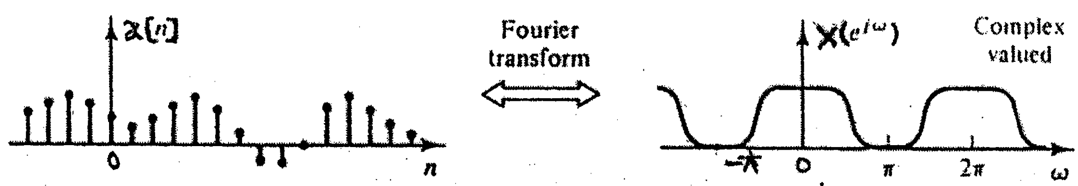
NONPERIODIC CONTINUOUS-TIME



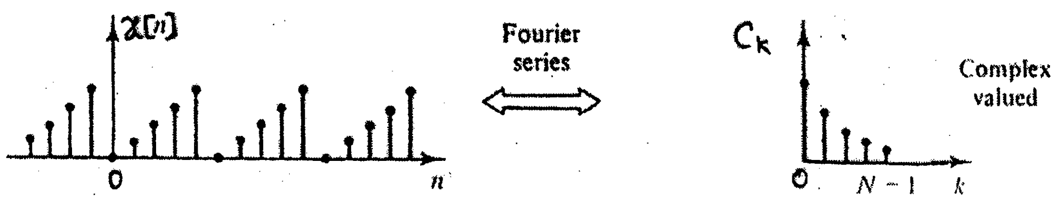
PERIODIC CONTINUOUS-TIME



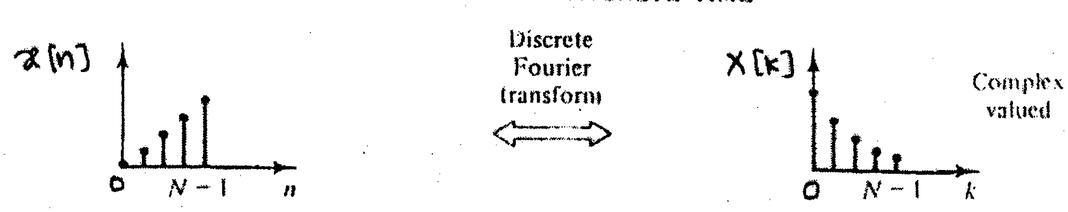
NONPERIODIC DISCRETE-TIME



PERIODIC DISCRETE-TIME



FIXED LENGTH DISCRETE-TIME



Priority 🔒 : **Definitely Everything** √ : **You must do this**
$: **You should not leave this** ⏲ : **If possible, do it**

	EXAM	IT	ELX	COMP	EXTC	INSTRU
1	May-2004	54	68	--	--	---
2	Dec-2004	80	70	62	20	15
3	May-2005	71	72	70**	14	27
4	Dec-2005	64	72	61	52	23
5	May-2006	70	90	48*	29	52
6	Dec -2006	45		60	26	05
7	May-2007	54	77	57		
8	Dec -2007	46	87	60	40	
9	May-2008	34	83	53		
AVERAGE						

3.1. Bilateral Z Transform

The bilateral Z – Transform of DT sequence X[n] is defined as,

$$X(z) = \sum_{-\infty}^{\infty} x[n]\ z^{-n} \quad \text{where } Z = re^{j\omega} \text{ is a complex variable.}$$

Q(1) Find the Z-Transform of the following sequence :

a) $x[n] = \{ \underset{\uparrow}{1}\ 2\ 3\ 4 \}$ b) $x[n] = \{ 1\ 2\ 3\ \underset{\uparrow}{0} \}$

c) $x[n] = a^n u[n]$ d) $x[n] = a^n u[-n-1]$

Solution:

(a) To find X(z)

Given $x[n] = \{ \underset{\uparrow}{1}\ 2\ 3\ 4 \}$ causal finite length

By ZT, $X(z) = \sum_{n=-\infty}^{\infty} x[n]\ z^{-n}$

$= \sum_{n=0}^{3} x^{n} z^{-n}$

$= x(0) + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$

Ans $\boxed{X(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}}$

By IZT,

$\boxed{x(n) = \{ \underset{\uparrow}{1}, 2, 3, 4 \}}$ Causal

Note If we put $z = re^{j\omega}$ at $r = 1$ it is DTFT eqn.

(b) To find X(z)

Given $x[n] = \{ 1\ 2\ 3\ \underset{\uparrow}{0} \}$ Anti causal finite length.

By ZT, $X(z) = \sum_{n=-\infty}^{\infty} x[n]\ z^{-n}$

$X(z) = \sum_{n=-1}^{-3} x(n) z^{-n}$

$= x(-1) + x(-2) z^{2} + x(-3) z^{3}$

Ans $\boxed{X(z) = 3z + 2z^{2} + z^{3}}$

By IZT,

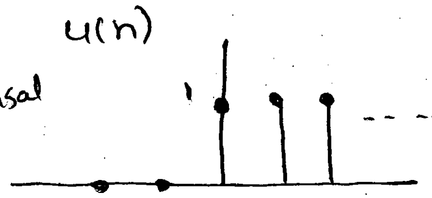
$x(n) = \{ 1, 2, 3, \underset{\uparrow}{0} \}$ Anticausal

If it is causal n varies from 0 to n

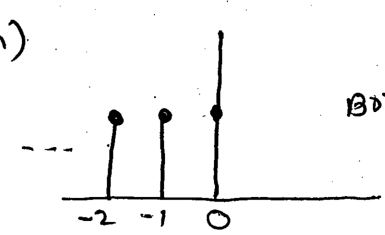
& If it is anticausal n varies from −1 to −n.

u(n)

causal



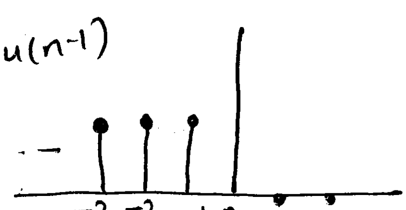
u(−n)



Both sided

u(n−1)

Anti causal



(c) To find X(z)

Given $x[n] = a^n u[n]$ Causal Infinite length

By ZT, $X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$

$$X(z) = \sum_{n=0}^{\infty} a^n u(n) z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-1} \quad (\because u(n)=1)$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n$$

But $a z^{-1} = \gamma$

$$\therefore \sum_{n=0}^{\infty} \gamma^n = \begin{bmatrix} \frac{1}{1-\gamma} \text{ finite} & \text{if } |\gamma|<1 \\ \infty & \text{if } |\gamma| \geq 1 \end{bmatrix}$$

Let, $X(z) = \left[\frac{1}{1-az^{-1}}\right]^*_{\text{finite}}$

Provided $|az^{-1}|<1$

i.e. $\left|\frac{a}{z}\right|<1$

$\therefore |a|<|z|$

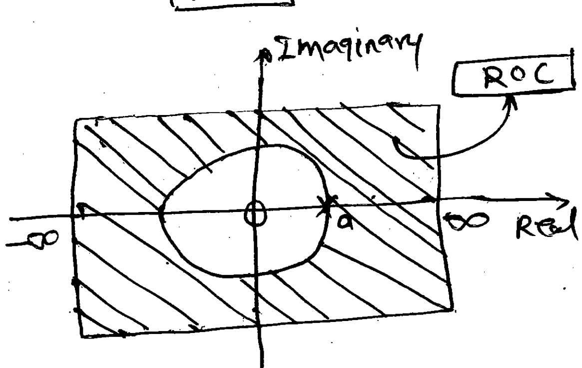
$$\therefore X(z) = \frac{z}{z-a} \quad \boxed{|z|>|a|}$$

Note :- $ZT\{a^n u(n)\} = \begin{bmatrix} \frac{z}{z-a} & |z|>|a| \\ \infty & \text{otherwise} \end{bmatrix}$
causal
Infinite length

To find ROC: $X(z) = \frac{z}{z-a}$ $|z|>|a|$
Now

(1) Pole - $z=?$ $X(z)=\infty$
Let $z-a=0$
$\therefore \boxed{z=a}$

(2) Zero : $z=?$ $x(z)=0$
$\therefore \boxed{z=0}$



$\longleftarrow$ Z-plane $\longrightarrow$

ROC - Region of convergence.

(d) To find X(z)

Given $x[n] = a^n u[-n-1]$

By ZT, $X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$

$$x(z) = \sum_{n=-1}^{-\infty} a^n . u(-n-1) . z^{-n}$$

$$x(z) = \sum_{n=-1}^{-\infty} a^n z^{-n}$$

Put $n=-m$

$$x(z) = \sum_{m=1}^{\infty} a^{-m} z^m$$

$$= \sum_{m=1}^{\infty} (a^{-1} z)^m$$

$$x(z) = \sum_{m=1}^{\infty} \gamma^m = \begin{bmatrix} \frac{\gamma}{1-\gamma} \text{ finite} & \text{if } |\gamma|<1 \\ \infty & \text{if } |\gamma| \geq 1 \end{bmatrix}$$

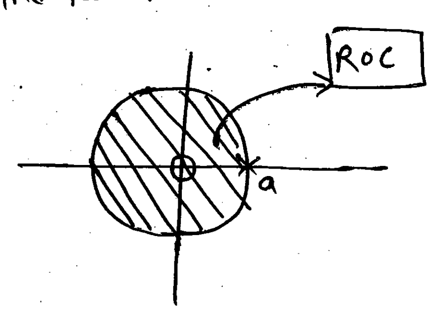
Let $x(z) = \left[\frac{a^{-1}z}{1-a^{-1}z}\right]^*$ Provided $|a^{-1}z|<1$
i.e. $\left|\frac{z}{a}\right|<1$
i.e. $|z|<|a|$

$$x(z) = \frac{-z}{z-a}$$

Note :

$ZT\{a^n u(-n-1)\} = \begin{bmatrix} \frac{-z}{z-a} & |z|<|a| \\ \infty & \text{otherwise.} \end{bmatrix}$

To find ROC :



If it is Causal, ROC is outside the circle.
& If it is anticausal, ROC is inside the circle.
Note

Eg - 3. $\quad x[n] = a^n u[n] + b^n u[-n-1]$

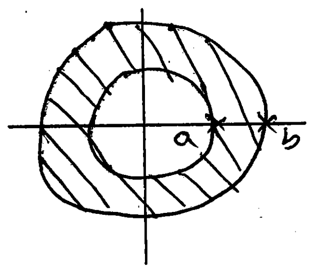
Bothsided

$$X[z] = \frac{z}{z-a} + \frac{-z}{z-b}$$

$\boxed{|z| > |a|}$ Causal $\qquad \boxed{|z| < |b|}$ Anticausal.

To find ROC :

$\boxed{|b| > |z| > |a|}$ Bothsided.



Bounded bet^n two poles.

Eg - 4. $\quad x[n] = 2^n u(n) + 3^n u(-n-1) + 4^n u(-n-1)$

Bothsided

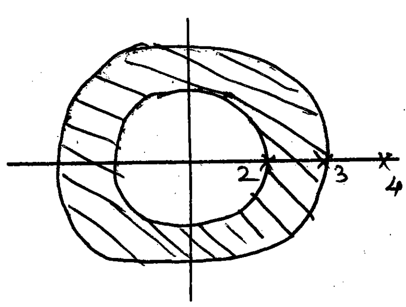
$$X[z] = \frac{z}{z-2} + \frac{-z}{z-3} + \frac{-z}{z-4}$$

$\boxed{|z| > 2}$ Causal $\qquad \boxed{|z| < 3} \qquad \boxed{|z| < 4}$

To find ROC :

$\boxed{|z| < 3}$ simplified version of anticausal.

$\boxed{3 > |z| > 2}$



Eg - 5. $\quad x[n] = 2^n u(n) + 3^n u(n) + 4^n u(-n-1)$

Bothsided

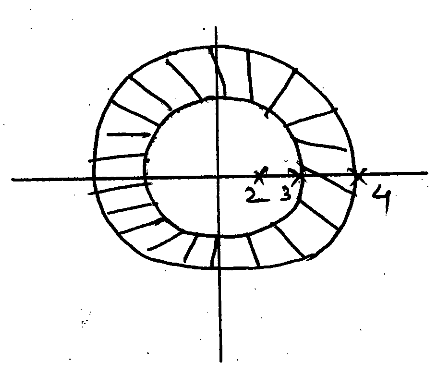
$$X[z] = \frac{z}{z-2} + \frac{z}{z-3} + \frac{-z}{z-4}$$

$\boxed{|z| > 2} \qquad \boxed{|z| > 3} \qquad \boxed{|z| < 4}$ Anticausal.

To find ROC :

$\boxed{|z| > 3}$ Causal.

$\boxed{4 > |z| > 3}$



NOTE :

1) **If x[n] is right handed sequence, the ROC extends outward from the outermost finite pole in** $X(z)$ *to* $z = \infty$

	Sequence	ROC
1	$x[n] = \{ \underset{\uparrow}{1}, 0, 0, 0 \}$	Entire Z-plane
2	$x[n] = \{ \underset{\uparrow}{1}, 2, 3, 4 \}$	$\|Z\| > 0$
3	$x[n] = a^n\ u[n]$	$\|Z\| > \|a\|$
4	$x[n] = a^n\ u[n] + b^n\ u[n]$	$\|Z\| > \max\{ \|a\|, \|b\| \}$
5	$x[n] = (-3)^n\ u[n] + (2)^n\ u[n]$	$\|Z\| > 3$

2) **If x[n] is Left handed sequence, the ROC extends inward from the innermost finite pole in** $X(z)$ to $z = 0$

	Sequence	ROC
1	$x[n] = \{ 1, 2, 3, \underset{\uparrow}{0} \}$	$\|Z\| < \infty$
2	$x[n] = a^n\ u[-n-1]$	$\|Z\| < \|a\|$
3	$x[n] = a^n\ u[-n-1] + b^n\ u[-n-1]$	$\|Z\| < \min\{ \|a\|, \|b\| \}$
4	$x[n] = (-3)^n u[-n-1] + (2)^n u[-n-1]$	$\|Z\| < 2$

3) **If x[n] is two sided sequence, the ROC consist of a ring in the Z plane, bounded by interior and exterior pole.**

	Sequence	ROC
1	$x[n] = a^n\ u[n] + b^n\ u[-n-1]$	$\|b\| > \|z\| > \|a\|$
2	$x[n] = (2)^n\ u[n] + (3)^n\ u[-n-1]$	$3 > \|z\| > 2$
3	$x[n] = (3)^n\ u[n] + (2)^n\ u[-n-1]$	Not possible $\therefore X(z) = \infty$
4	$x[n] = (2)^n u[n] + (3)^n u[n] + (4)^n u[-n-1] + (5)^n u[-n-1]$	$4 > \|z\| > 3$

Q(2) Find the ZT of the following signals and determine its ROC.

(a) $x[n] = \left(\frac{-1}{2}\right)^{n-1} u[n] + \left(\frac{-1}{3}\right)^{n} u[n]$ (b) $x[n] = \{3(2)^{-n} - 4(3)^{-n}\}\mu[n]$

Solution :

a) $x[n] = \left(\frac{-1}{2}\right)^{n-1} u[n] + \left(\frac{-1}{3}\right)^{n} u[n]$

$$= \left(\frac{-1}{2}\right)^{-1} \left(\frac{-1}{2}\right)^{n} u[n] + \left(\frac{-1}{3}\right)^{n} u[n]$$

$$X(z) = -2 \cdot \frac{z}{(z+1/2)} + \frac{z}{(z+1/3)}$$

$R.O.C.: \ |z| > \frac{1}{2}$

(b) $x[n] = \{3(2)^{-n} - 4(3)^{-n}\}\mu[n]$

$$= 3(2)^{-n} .u[n] - 4(3)^{-n} u[n]$$

$$= 3(\tfrac{1}{2})^n u[n] - 4(1/3)^n u[n]$$

$$X(z) = \frac{3z}{Z-1/2} - \frac{4z}{Z-1/3}$$

$\text{ROC} : \ |Z| > \frac{1}{2}$

[2] Time Shift Property

If $x[n] \longleftrightarrow X(z)$

Then

(i) $ZT\{x[n-m]\} = z^{-m}X(z)$

(ii) $ZT\{x[n+m]\} = z^{m}X(z)$

Q(3) Find the ZT of the following signals and determine its ROC.

a) $x[n] = 2^n u[n-10]$ b) $x[n] = (½)^n \{u[n]-u[n-10]\}$

c) $x[n] = (2)^{n-2} u[n-2]$ d) $x[n] = (2)^{n+2} u[n-2]$

Solution :

(a)	b)
(a) Given $x[n] = 2^n u[n-10]$ $= 2^{n-10} \cdot 2^{10} u[n-10]$ By ZT Time shift property, $X(z) = 2^{10} \cdot z^{-10} \cdot \dfrac{Z}{Z-2}$ $X(z) = \dfrac{2^{10} \cdot z^{-9}}{z-2} = \dfrac{2^{10}}{z^n(z-2)}$ ROC : $\lvert Z \rvert > 2$	b) Given $x[n] = (½)^n \{u[n]-u[n-10]\}$ $\therefore x[n] = \left(\dfrac{1}{2}\right)^n u[n] - \left(\dfrac{1}{2}\right)^{n-10}\left(\dfrac{1}{2}\right)^{10} u[n-10]$ Time shift property, $X(Z) = \dfrac{Z}{Z-1/2} - \left(\dfrac{1}{2}\right)^{10} \dfrac{1}{Z^9\left(Z-\dfrac{1}{2}\right)}$ ROC : $\lvert Z \rvert > \dfrac{1}{2}$

[3] Time Reversal Property

If $x[n] \longleftrightarrow X(z)$

Then

$ZT\{x[-n]\} = X\left(\dfrac{1}{z}\right)$

[4] Scaling in Z-Domain

If $x[n] \longleftrightarrow X(z)$

Then

$ZT\{r^n X[n]\} = X\left(\dfrac{z}{r}\right)$

[5] Differentiation Property

If $x[n] \longleftrightarrow X(z)$

Then

$ZT\{n\,x[n]\} = -Z\,\dfrac{d\,X(z)}{d\,z}$

Q(4) $y[n] = n\,a^n u[n]$. Find $Y(z)$.

Solution : Let $y[n] = n\,x[n]$ where $x[n] = a^n u[n]$

And $X[z] = \dfrac{z}{z-a}$ ROC : $|z| > |a|$

By ZT and Differentiation property,

$$Y[z] = ZT\{\,n\,x[n]\,\}$$
$$= -Z\,\frac{d\,X(z)}{d\,z}$$
$$= -Z\frac{d}{dz}\left[\frac{z}{z-a}\right] = -Z\left[\frac{(z-a)(1)-(z)(1)}{(z-a)^2}\right]$$
$$= -Z\left[\frac{z-a-z}{(z-a)^2}\right]$$
$$X(z) = \left[\frac{az}{(z-a)^2}\right]$$

$$\boxed{\text{NOTE} - \quad ZT\{n a^n u(n)\} = \begin{cases} \dfrac{az}{(z-a)^2} & |z|>|a| \\ \infty & \text{otherwise.} \end{cases}}$$

Q(5) Find the ZT of the following signals and determine its ROC.

a) $x[n] = (n+1)\,a^n u[n]$

Solution :

Given $x[n] = (n+1)a^n\ u[n]$

$x[n] = n\,.\,a^n\,.u[n] + a^n u[n]$

By Z.T., $X(z) = \dfrac{az}{(z-a)^2} + \dfrac{z}{z-a} = \dfrac{z^2}{(z-a)^2}$ ROC : $|z| > |a|$

---

[6] Convolution Property

If $x[n] \longleftrightarrow X(z)$

$h[n] \longleftrightarrow H(z)$

Then

$ZT\{\,x[n] * h[n]\,\} = X(z).\,H(z)$

Proof of Convolution Property :

Let $y[n] = x[n] * h[n] = \sum\limits_{m=-\infty}^{\infty} x[m]\,h[n-m]$

By ZT,

$$ZT\{x[n]*h[n]\} = \sum_{-\infty}^{\infty} y[n]\,.\,z^{-n} = \sum_{n=-\infty}^{\infty}\ \sum_{m=-\infty}^{\infty} x[m]\,h[n-m]\,z^{-n}$$
$$= \sum_{m=-\infty}^{\infty} x[m] \sum_{n=-\infty}^{\infty} h[n-m]z^{-n}$$
$$= \left(\sum_{-\infty}^{\infty} x(m)\,z^{-m}\right)H[z]$$
$$= X(z)\,.\,H(z) \quad \text{Proved.}$$

Q(6) Let $x[n]=\{\underset{\uparrow}{1}\ \ 2\ \ 3\ \ 4\}$ and $h[n]=\{\underset{\uparrow}{5}\ \ 6\ \ 0\ \ 0\}$

a) Find Linear Convolution using Z–Transform and convolution property.
b) Prove your result using Time domain method.

Solution : (a) Let $y[n] = x[n] * h[n]$
By Linear Convolution property of Z-Transform,
$\therefore\ Y(z) = X(z) . H(z)$

(i) Find X(z) $X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$ $X(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$	**(iii) Let Y(z) = X(z) H(z)** $Y(z) = (1 + 2z^{-1} + 3z^{-2} + 4z^{-3})\ (5 + 6z^{-1})$ $\therefore\ Y(z) = 5 + 16z^{-1} + 27z^{-2} + 38z^{-3} + 24z^{-4}$
(ii) Find H(z) $H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$ $H(z) = 5 + 6z^{-1}$	**(iv) Find y[n]** $Y(z) = 5 + 16z^{-1} + 27z^{-2} + 38z^{-3} + 24z^{-4}$ By iZT, **ANS :** $y[n]=\{\underset{\uparrow}{5}\ \ 16\ \ 27\ \ 38\ \ 24\}$

Solution : (b) To find Linear Convolution using Time domain Method

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\, h[n-m] \qquad \text{where } x[n] = \{1, 2, 3, 4\}\ \ h[n] = \{5, 6, 0, 0\}$$

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\, h[n-m] = \sum_{n=0}^{3} x[m]\, h[n-m]$$

$$y[0] = \sum_{n=0}^{3} x[m]\, h[-m] = (1)(5) + (2)(0) + (3)(0) + (4)(0) = [\ \mathbf{05}\ ]$$

$$y[1] = \sum_{n=0}^{3} x[m]\, h[1-m] = (1)(6) + (2)(5) + (3)(0) + (4)(0) = [\ \mathbf{16}\ ]$$

$$y[2] = \sum_{n=0}^{3} x[m]\, h[2-m] = (1)(0) + (2)(6) + (3)(5) + (4)(0) = [\ \mathbf{27}\ ]$$

$$y[3] = \sum_{n=0}^{3} x[m]\, h[3-m] = (1)(0) + (2)(0) + (3)(6) + (4)(5) = [\ \mathbf{38}\ ]$$

$$y[4] = \sum_{n=0}^{3} x[m]\, h[4-m] = (1)(0) + (2)(0) + (3)(0) + (4)(6) = [\ \mathbf{24}\ ]$$

ANS : $y[n]=\{\underset{\uparrow}{5}\ \ 16\ \ 27\ \ 38\ \ 24\}$

Q(7) Let $x[n] = (0.2)^n u[n]$ and $h[n] = (0.3)^n u[n]$

a) Find Linear Convolution using Z–Transform and convolution property.
b) Prove your result using Time domain method. [Compare at least first four values]

Solution (a) :

Let $y[n] = x[n] * h[n]$
By Linear Convolution property of Z-Transform,
$\therefore\ Y(z) = X(z) . H(z)$

(i) Find X(z) $x[n] = (0.2)^n u[n]$ By ZT, $X(z) = \frac{z}{z-0.2}$ $\quad \lvert Z \rvert > \lvert 0.2 \rvert$	**(iii) Let Y(z) = X(z) H(z)** $Y(z) = \frac{z}{(z-0.2)} \frac{z}{(z-0.3)}$
(ii) Find H(z) $h[n] = (0.3)^n u[n]$ By ZT, $H(z) = \frac{z}{z-0.3}$ $\quad \lvert Z \rvert > \lvert 0.3 \rvert$	**(iv) Find y[n]** $Y(z) = \frac{z^2}{(z-0.2)(z-0.3)}$ By IZT, y[n] =

Solution : (b) To find convolution using Time domain Method

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\, h[n-m] = \sum_{m=0}^{+\infty} (0.2)^m (0.3)^{n-m} u[n-m]$$

Evaluate the formula and find at least four values of y[n].

ANS : {

[7] Correlation Property

If $x[n] \longleftrightarrow X(z)$

$h[n] \longleftrightarrow H(z)$

Then

$$ZT\{ x[n] \circ h[n] \} = X(z)\ H(z^{-1})$$

Proof of Correlation Property :

$$r_{xh}[n] = x[n] \circ h[n]$$
$$= x[n] * h[-n]$$

By ZT,

$$ZT\{r_{xh}[n]\} = ZT\{x[n]\} . ZT\{h[-n]\}$$
$$ZT\{ x[n] \circ h[n] \} = X(z) . H(z^{-1})$$

Q(8) Let $x[n] = \{ \underset{\uparrow}{1}, 2, 3, 4 \}$ and $h[n] = \{ \underset{\uparrow}{5}, 6, 0, 0 \}$

a) Find correlation using Z-Transform and correlation property.

b) Prove your result using Time domain method.

Solution : (a) Let $y[n] = x[n] \circ h[n]$

$= x[n] * h[-n]$

By Linear correlation property of Z-Transform,

$\therefore\ Y(z) = X(z) . H(z^{-1})$

(i) Find X(z) $X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$ $= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$	**(iii) Let Y(z) = X(z) H(z⁻¹)** $= (1 + 2z^{-1} + 3z^{-2} + 4z^{-3})(5 + 6z)$ $\therefore Y(z) = 6z + 17 + 28z^{-1} + 39z^{-2} + 24z^{-3}$
(ii) Find $H(z^{-1})$ $H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$ $= 5 + 6z^{-1}$ $\therefore H(z^{-1}) = 5 + 6z$	**(iv) Find y[n]** $Y(z) = 6z + 17 + 28z^{-1} + 39z^{-2} + 24z^{-3}$ By iZT, ANS : $y[n] = \{ \underset{\uparrow}{6}, 17, 28, 39, 24 \}$

Solution : (b) To find correlation using Time domain Method

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\, h[m-n] \qquad \text{where } x[n] = \{1, 2, 3, 4\} \;\; h[n] = \{5, 6, 0, 0\}$$

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\, h[m-n] = \sum_{m=0}^{3} x[m]\, h[m-n]$$

Step-1 : To find y[n] for $n \geq 0$

$$y[0] = \sum_{m=0}^{3} x[m]\, h[m] = (1)(5) + (2)(6) + (3)(0) + (4)(0) = \boxed{17}$$

$$y[1] = \sum_{m=0}^{3} x[m]\, h[m-1] = (1)(0) + (2)(5) + (3)(6) + (4)(0) = \boxed{28}$$

$$y[2] = \sum_{m=0}^{3} x[m]\, h[m-2] = (1)(0) + (2)(0) + (3)(5) + (4)(6) = \boxed{39}$$

$$y[3] = \sum_{m=0}^{3} x[m]\, h[m-3] = (1)(0) + (2)(0) + (3)(0) + (4)(5) = \boxed{20}$$

Step-2 : To find y[n] for $n < 0$

$$y[-1] = \sum_{m=0}^{3} x[m]\, h[m+1] = (1)(6) + (2)(0) + (3)(0) + (4)(0) = \boxed{6}$$

$$y[-2] = \sum_{m=0}^{3} x[m]\, h[m+2] = (1)(0) + (2)(0) + (3)(0) + (4)(0) = \boxed{0}$$

ANS : $y[n] = \{\ 6,\ \underset{\uparrow}{17}\ 28,\ 39,\ 20\ \}$

Q(9) Let $x[n] = (0.2)^n u[n]$ and $h[n] = (0.3)^n u[n]$

a) Find correlation using Z-Transform and correlation property.
b) Prove your result using Time domain method. [Compare at least first four values]

Solution : (a) Let $y[n] = x[n] \circ h[n]$
$= x[n] * h[-n]$

By Linear Convolution property of Z-Transform,

$\therefore\ Y(z) = X(z) \,.\, H(z^{-1})$

(i) Find X(z) $x[n] = (0.2)^n u[n]$ By ZT, $X(z) = \dfrac{z}{z-0.2}$ **ROC :**	**(iv) Find y[n]** $\dfrac{Y(z)}{z} = \dfrac{-0.3}{(z-0.3)(z-0.2)}$
(ii) Find H(z^{-1}) $h[n] = (0.3)^n u[n]$ By ZT, $H(z) = \dfrac{z}{z-0.3}$ ROC : Then $H(z^{-1}) = \dfrac{z^{-1}}{z^{-1}-0.3} = \dfrac{-0.3}{z-0.3}$ **ROC :** **(iii) Let Y(z) = X(z) H(z^{-1})** $Y(z) = \dfrac{-0.3\,z}{(z-0.3)(z-0.2)}$	

(i) Find X(z) $x[n] = (0.2)^n u[n]$ By ZT, $X(z) = \dfrac{z}{z-0.2} \quad \lvert Z \rvert > \lvert 0.2 \rvert$	**(iii) Let Y(z) = X(z) H(z)** $Y(z) = \dfrac{z}{(z-0.2)} \dfrac{z}{(z-0.3)}$
(ii) Find H(z) $h[n] = (0.3)^n u[n]$ By ZT, $H(z) = \dfrac{z}{z-0.3} \quad \lvert Z \rvert > \lvert 0.3 \rvert$	**(iv) Find y[n]** $Y(z) = \dfrac{z^2}{(z-0.2)(z-0.3)}$ By IZT, y[n] =

Solution : (b) To find convolution using Time domain Method

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\, h[n-m] = \sum_{m=0}^{+\infty} (0.2)^m (0.3)^{n-m} u[n-m]$$

Evaluate the formula and find at least four values of y[n].

ANS : {

[7] Correlation Property

If $x[n] \longleftrightarrow X(z)$
$h[n] \longleftrightarrow H(z)$

Then

$$ZT\{ x[n] \circ h[n] \} = X(z)\; H(z^{-1})$$

Proof of Correlation Property :

$r_{xh}[n] = x[n] \circ h[n]$
$= x[n] * h[-n]$

By ZT,
$ZT\{r_{xh}[n]\} = ZT\{x[n]\} \,.\, ZT\{h[-n]\}$
$ZT\{ x[n] \circ h[n] \} = X(z) \,.\, H(z^{-1})$

Q(8) Let $x[n] = \{ \underset{\uparrow}{1}, 2, 3, 4 \}$ and $h[n] = \{ \underset{\uparrow}{5}, 6, 0, 0 \}$

a) Find correlation using Z-Transform and correlation property.
b) Prove your result using Time domain method.

Solution : (a) Let $y[n] = x[n] \circ h[n]$
$= x[n] * h[-n]$

By Linear correlation property of Z-Transform,

$\therefore\; Y(z) = X(z) \,.\, H(z^{-1})$

(i) Find X(z) $X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$ $= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$	**(iii) Let Y(z) = X(z) H(z⁻¹)** $= (1 + 2z^{-1} + 3z^{-2} + 4z^{-3})(5 + 6z)$ $\therefore\; Y(z) = 6z + 17 + 28z^{-1} + 39z^{-2} + 24z^{-3}$
(ii) Find H(z⁻¹) $H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$ $= 5 + 6z^{-1}$ $\therefore\; H(z^{-1}) = 5 + 6z$	**(iv) Find y[n]** $Y(z) = 6z + 17 + 28z^{-1} + 39z^{-2} + 24z^{-3}$ By iZT, ANS : $y[n] = \{ \underset{\uparrow}{6}, 17, 28, 39, 24 \}$

Solution : (b) To find correlation using Time domain Method

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\,h[m-n] \qquad \text{where } x[n] = \{1, 2, 3, 4\} \;\; h[n] = \{5, 6, 0, 0\}$$

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\,h[m-n] = \sum_{m=0}^{3} x[m]\,h[m-n]$$

Step-1 : To find y[n] for $n \geq 0$

$$y[0] = \sum_{m=0}^{3} x[m]\,h[m] = (1)(5) + (2)(6) + (3)(0) + (4)(0) = \boxed{17}$$

$$y[1] = \sum_{m=0}^{3} x[m]\,h[m-1] = (1)(0) + (2)(5) + (3)(6) + (4)(0) = \boxed{28}$$

$$y[2] = \sum_{m=0}^{3} x[m]\,h[m-2] = (1)(0) + (2)(0) + (3)(5) + (4)(6) = \boxed{39}$$

$$y[3] = \sum_{m=0}^{3} x[m]\,h[m-3] = (1)(0) + (2)(0) + (3)(0) + (4)(5) = \boxed{20}$$

Step-2 : To find y[n] for $n < 0$

$$y[-1] = \sum_{m=0}^{3} x[m]\,h[m+1] = (1)(6) + (2)(0) + (3)(0) + (4)(0) = \boxed{6}$$

$$y[-2] = \sum_{m=0}^{3} x[m]\,h[m+2] = (1)(0) + (2)(0) + (3)(0) + (4)(0) = \boxed{0}$$

ANS : $y[n] = \{\; 6,\; \underset{\uparrow}{17}\; 28,\; 39,\; 20 \;\}$

Q(9) Let $x[n] = (0.2)^n u[n]$ and $h[n] = (0.3)^n u[n]$

a) Find correlation using Z-Transform and correlation property.
b) Prove your result using Time domain method. [Compare at least first four values]

Solution : (a) Let $y[n] = x[n] \circ h[n]$
$= x[n] * h[-n]$

By Linear Convolution property of Z-Transform,

$\therefore\; Y(z) = X(z) \,.\, H(z^{-1})$

(i) Find X(z)

$x[n] = (0.2)^n u[n]$

By ZT, $X(z) = \dfrac{z}{z-0.2}$

ROC :

(ii) Find H(z⁻¹)

$h[n] = (0.3)^n u[n]$

By ZT,

$H(z) = \dfrac{z}{z-0.3}$ ROC :

Then

$H(z^{-1}) = \dfrac{z^{-1}}{z^{-1}-0.3} = \dfrac{-0.3}{z-0.3}$

ROC :

(iii) Let Y(z) = X(z) H(z⁻¹)

$$Y(z) = \frac{-0.3\,z}{(z-0.3)(z-0.2)}$$

(iv) Find y[n]

$$\frac{Y(z)}{z} = \frac{-0.3}{(z-0.3)(z-0.2)}$$

Solution : (b) To find correlation using Time domain Method

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\,h[m-n] = \sum_{m=0}^{+\infty} (0.2)^{m} (0.3)^{m-n} u[m-n]$$

Evaluate the formula and find at least four values of y[n].

ANS :

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[8] Complex sequence Property of ZT

> If $x[n] \longleftrightarrow X(z)$
> Then
> $\text{DFT}\{x^*[n]\} = X^*(z^*)$

[9] Initial Value Theorem

If $x[n] \longleftrightarrow X(z)$
Then

$$x[0] = \lim_{z\to\infty} X(z)$$

[10] Final Value Theorem

If $x[n] \longleftrightarrow X(z)$
Then

$$x[\infty] = \lim_{z\to 1} (1-z^{-1})\, x(z)$$

Q(10) Find the initial and final values of x[n] for the following causal signals

a) $X(z) = \dfrac{z}{z^2 + z - 1}$ b) $X(z) = \dfrac{2z^2+1}{z^2 - 0.5z - 0.5}$

Solution :

(a) Given $X(z) = \dfrac{2Z^2+1}{Z^2 - 0.5z - 0.5}$

Solution :

(i) $x[0] = \lim\limits_{z\to\infty} X(z)$

$$= \lim_{z\to\infty} \frac{2+\frac{1}{z^2}}{1-\frac{0.5}{z}-\frac{0.5}{z^2}}$$

$\therefore$ **x[0] = 2 ANS**

(ii) $x[\infty] = \lim\limits_{z\to 1} (1-z^{-1})X(z)$

$$= \lim_{z\to 1} \frac{(1-z^{-1})2z^2+1}{z^2-0.5z-0.5}$$

$$= \lim_{z\to 1} \frac{2z^2-2z+1-z^{-1}}{z^2-0.5z-0.5}$$

$$= \frac{2-2+1-1}{1-0.5}$$

$x[\infty] = 0$ **ANS**

(b) Given $X(z) = \dfrac{Z}{z^2+z-1}$

Solution :

(i) $x[0] = \lim\limits_{z\to\infty} X(z)$

$$\therefore x[0] = \lim_{\to\infty} \frac{Z}{z^2+z-1}$$

$$\therefore x[0] = \lim_{\to\infty} \frac{1/Z}{1+\frac{1}{z}-\frac{1}{z^2}}$$

$\therefore$ **x[0] = 0 ANS**

(ii) $x(\infty) = \lim\limits_{z\to 1} (1-z^{-1})\, X(z)$

$$= \lim_{z\to 1} \frac{(z-1)}{z} \frac{Z}{(z-1)^2}$$

$$= \lim_{z\to 1} \frac{1}{z-1}$$

$x[\infty] = \infty$ **ANS**

.... Z-Transform Table ....

	X[n]	X(Z)	ROC
1	$\delta[n]$	1	Entire Z plane
	$\delta[n-m]$	z^{-m}	Entire Z plane
	$\delta[n+m]$	z^{m}	Entire Z plane
2	$a^n u[n]$	$\frac{z}{z-a}$	$\|Z\| > \|a\|$
	$a^n u[-n-1]$	$\frac{-z}{z-a}$	$\|Z\| < \|a\|$
3	$u[n]$	$\frac{z}{z-1}$	$\|Z\| > 1$
	$u[-n-1]$	$\frac{-z}{z-1}$	$\|Z\| < 1$
4	$n a^n u[n]$	$\frac{az}{(z-a)^2}$	$\|Z\| > \|a\|$
	$n a^n u[-n-1]$	$\frac{-az}{(z-a)^2}$	$\|Z\| < \|a\|$
5	$r^n \cos(n\,\omega)\ u[n]$	$\frac{z^2 - r z \cos(w)}{z^2 - 2 r z \cos(w) + r^2}$	$\|Z\| > \|r\|$
	$r^n \sin(n\,\omega)\ u[n]$	$\frac{r z \sin(w)}{z^2 - 2 r z \cos(\omega) + r^2}$	$\|Z\| > \|r\|$

➢ The Z-Transform of Switched Periodic Signals

Q(11) Find the ZT of $x[n] = \{ \underset{\uparrow}{1}, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, ------\}$

Solution :

Let $p[n] = \{ \underset{\uparrow}{1}, 2, 3 \}$

So, $P(z) = 1 + 2Z^{-1} + 3Z^{-2}$

Then $x[n] = p[n] + p[n-3] + p[n-6] + +$

$$x[n] = \sum_{m=0}^{\infty} p[n - mN]$$

By ZT

$$X(z) = \sum_{m=0}^{\infty} Z^{-mN}\ P(z)$$

$$X(z) = P(z) \sum_{m=0}^{\infty} \left(Z^{-3}\right)^m$$

$$X(z) = P(z) \left[\frac{1}{1 - z^{-3}}\right]$$

$$X(z) = \left[\frac{1 + 2z^{-1} + 3z^{-2}}{1 - z^{-3}}\right]$$

3.2. Inverse Z-Transform

$$X(z) = \frac{z}{z-a}$$

$|z| > |a|$ $\qquad$ $|z| < |a|$

$x(n) = a^n u(n)$ $\qquad$ $x(n) = -a^n u(-n-1)$

Causal $\qquad$ Anticausal.

Inverse ZT Methods

Power Series Expansion Method i.e. Long Division Method	Partial Fraction Expansion Method
Only for FIR Filters	Only for IIR Filters

Q(12) Find the iZT of the following by using Power Series Expansion :-

a) $X(z) = 1 + z^{-1} + 6z^{-3} + 8z^{-5} + 4z^{-10}$ By iZT $x[n] = \{\underset{\uparrow}{1}, 1, 0, 6, 0, 8, 0, 0, 0, 0, 4\}$	b) $X(z) = z^{-3} + 2z^{-2} + 3z^{-1} + 4z + 3z^2$ By iZT, $x[n] = \{3, 4, \underset{\uparrow}{0}, 3, 2, 1\}$
c) $X(z) = \dfrac{z}{z-2}$ $\quad |z| > 2$ Signal is causal $\begin{array}{r} 1+2z^{-1}+4z^{-2}+8z^{-3} \\ z-2 \,\overline{)\, z \qquad\qquad\qquad} \\ \underline{z-2} \\ 2 \\ \underline{-2-4z^{-1}} \\ +4z^{-1} \\ \underline{-4z^{-1}-8z^{-2}} \\ +8z^{-2} \\ \underline{-8z^{-2}-16z^{-3}} \\ +16z^{-3}\ldots \end{array}$ $\therefore$ x[n] = {1, 2, 4, 8, …} for n >= 0 $x[n] = (2)^n u[n]$	d) $X(z) = \dfrac{z}{z-2}$ $\quad |z| < 2$ By power-series expansion, $\begin{array}{r} \left(-\frac{1}{2}\right)z - \left(-\frac{1}{4}\right)z^2 - \left(-\frac{1}{8}\right)z^3 - \ldots \\ -2+z \,\overline{)\, z \qquad\qquad\qquad\qquad} \\ \underline{z(-)\frac{1}{2}z^2} \\ \frac{1}{2}z^2 \\ \underline{+\frac{1}{2}z^2\ (-)\ \frac{1}{4}z^3} \\ \frac{1}{4}z^3 \\ \underline{\frac{1}{4}z^3\ (-)\ \frac{1}{8}z^4} \\ \frac{1}{8}z^4\ldots \end{array}$ $\therefore x[n] = \left\{\ldots, \frac{-1}{8}, \frac{-1}{4}, \frac{-1}{2}, 0\right\}$ for n < 0 $\therefore x[n] = -(2)^n u[-n-1]$

Q(13) Find the iZT of the following by using PFE :-

(a) $X(z) = \dfrac{z+1}{(z+2)(z+3)}$ $m=1$, $n=2$

Solution (a): $X(z) = \dfrac{z+1}{(z+2)(z+3)}$

$$\therefore \frac{X(z)}{z} = \frac{z+1}{z(z+2)(z+3)}$$

By PFE, $\therefore \dfrac{X(z)}{z} = \dfrac{A}{z} + \dfrac{B}{z+2} + \dfrac{c}{z+3}$

Where (i) $A = \left.\dfrac{X(z)}{z}(z)\right|_{z=0} = X(z)|_{z=0} = \boxed{1/6}$

(ii) $B = \left.\dfrac{X(z)}{z}(z+2)\right|_{z=-2} = \left.\dfrac{z+1}{z(z+3)}\right|_{z=-2} = \boxed{1/2}$

(iii) $C = \left.\dfrac{X(z)}{z}(z+3)\right|_{z=-3} = \left.\dfrac{z+1}{z(z+2)}\right|_{z=-3} = \boxed{-\dfrac{2}{3}}$

From equation-1 we get,

$$\therefore X(z) = A + B\left[\frac{z}{z+2}\right] + C\left[\frac{z}{z+3}\right]$$

$$\therefore X(z) = \frac{1}{6} + \frac{1}{2}\left[\frac{z}{z+2}\right] - \frac{2}{3}\left[\frac{z}{z+3}\right]$$

$\delta[n] \longrightarrow 1$

$\frac{1}{6}\delta(n) \xrightarrow{ZT} \frac{1}{6}$

IZT

By IZT, **x[n]** $= \frac{1}{6}\delta(n) + \frac{1}{2}(-2)^n u(n) - \frac{2}{3}(-3)^n u(n)$

(b) $Y(z) = \dfrac{z+1}{z^2(z+2)(z+3)}$

If poles are at origin break it.

Solution (b): $Y(z) = \dfrac{z+1}{z^2(z+2)(z+3)}$

$$= \frac{z+1}{z^2 (z+2)(z+3)}$$

$$= \frac{1}{z^2}\left[\frac{z+1}{(z+2)(z+3)}\right]$$

Poles :
$p_1 = 0$
$p_2 = 0$
$p_3 = -2$ $p_4 = -3$

Let $Y(z) = Z^{-2} X(z)$

By IZT
$y(n) = x(n-2)$

Time Shift Property

$ZT\{x[n-m]\} = Z^{-m} X(z)$

To find x[n] : Let $X(z) = \frac{z+1}{(z+2)(z+3)}$

--

By IZT, $x[n] = \frac{1}{6}\delta[n] + \frac{1}{2}(-2)^n u[n] - \frac{2}{3}(-3)^n u[n]$

But y[n] = x[n–2]

$y[n] = \frac{1}{6}\delta(n-2) + \frac{1}{2}(-2)^{n-2}u(n-2) - \frac{2}{3}(-3)^{n-2}u(n-2)$ Ans

---

(c) $X(z) = \frac{z+1}{(z+2)(z+3)^2}$

poles :
$P_1 = -2$
$P_2 = -3$
$P_3 = -3$ } repeated non-zero poles

Solution (c): $X(z) = \frac{z+1}{(z+2)(z+3)^2}$

∴ Divide by z

$\therefore \frac{X(z)}{z} = \frac{z+1}{z(z+2)(z+3)^2}$

By PFE, $\therefore \frac{X(z)}{z} = \frac{A}{z} + \frac{B}{z+2} + \frac{C}{z+3} + \frac{D}{(z+3)^2}$ ------Equation- 1

Where (i) $A = \frac{X(z)}{z}(z)\Big|_{z=0} = \boxed{\frac{1}{18}}$

(ii) $B = \frac{X(z)}{z}(z+2)\Big|_{z=-2} = \boxed{1/2}$

(iii) $D = \frac{X(z)}{z}(z+3)^2\Big|_{z=-3} = \boxed{\frac{-2}{3}}$

(iv) To find C

From Equation--- 1

$\therefore \frac{X(z)}{z} = \frac{1/18}{z} + \frac{1/2}{z+2} + \frac{C}{z+3} + \frac{-2/3}{(z+3)^2}$

But $Z \neq 0$
$\neq -2$
$\neq -3$

Let $Z = \boxed{-1}$ $C = \boxed{\frac{-5}{9}}$

~~By IZT,~~

$\cancel{x(n)} = \frac{1}{18}\cancel{\delta(n)} + \cancel{\frac{1}{2}(-2)^n} u$

From Equation 1

$\therefore X(z) = A + B\left[\frac{z}{z+2}\right] + C\left[\frac{z}{z+3}\right] + D\left[\frac{z}{(z+3)^2}\right]$

$\therefore X(z) = \frac{1}{18} + \frac{1}{2}\left[\frac{z}{z+2}\right] - \frac{5}{9}\left[\frac{z}{z+3}\right] - \frac{2}{3}\left[\frac{z}{(z+3)^2}\right]$

$n a^n u(n) \rightarrow \frac{az}{(z-a)^2}$

$\frac{1}{a} n a^n u(n) \xrightarrow{ZT} \frac{z}{(z-a)^2}$

$n a^{n-1} u(n)$

By IZT,

$x[n] = \frac{1}{18}\delta(n) + \frac{1}{2}(-2)^n u(n) - \frac{5}{9}(-3)^n u(n) - \frac{2}{3} n(-3)^{n-1} u(n)$

Ans

..............Inverse Z-Transform Table..................

	X(z)	X[n]
1	A	$A\delta(n)$
	z^{-m}	$\delta(n-m)$
	z^{m}	$\delta(n+m)$
2	**Only For Causal Signal**	
	$\frac{z}{z-a}$ $\lvert z\rvert > \lvert a\rvert$	$a^{n}u(n)$
	$\frac{z}{(z-a)^2}$ $\lvert z\rvert > \lvert a\rvert$	$na^{n-1}u(n)$
	$\frac{z}{(z-a)^3}$ $\lvert z\rvert > \lvert a\rvert$	$\frac{(n)(n-1)}{2!}u(n)a^{n-2}$
3	**Only For Anti-Causal Signal**	
	$\frac{z}{z-a}$ $\lvert z\rvert < \lvert a\rvert$	$-a^{n}u(-n-1)$
	$\frac{z}{(z-a)^2}$ $\lvert z\rvert < \lvert a\rvert$	$-na^{n-1}u(-n-1)$
	$\frac{z}{(z-a)^3}$ $\lvert z\rvert < \lvert a\rvert$	$-\frac{n(n-1)}{2!}a^{n-2}u(-n-2)$

Q(14) Find the iZT of X(z) for the following three possible cases.

$$X(z) = \frac{z}{(z+2)(z+3)}$$

a) $|z| > 3$
b) $|z| < 2$
c) $2 < |z| < 3$

ANS :

(a) For $|z| > 3$, signal is causal

By iZT $\therefore$ $x[n] = (-2)^n u[n] - (-3)^n u[n]$

(b) $|z| < 2$ $\Rightarrow$ signal is anticausal

By iZT, $x[n] = -(-2)^n u[-n-1] + (-3)^n u[-n-1]$

(c) $2 < |z| < 3$ $\Rightarrow$ signal is both sided

By iZT, $x[n] = (-2)^n u[n] + (-3)^n u[-n-1]$

3.1.2. One Sided Z-Transform

The one sided or unilateral ZT of x[n] is defined as, $X(z)=\sum_{n=0}^{\infty}x[n]z^{-n}$

➤ TIME SHIFT PROPERTY :

(1) Time Delay :-

If $x[n] \leftrightarrow X(z)$

Then $ZT\{x[n-m]\} = Z^{-m}\left[X(z)+\sum_{n=1}^{m}x[-n]\,Z^{n}\right]$

Proof :

$$ZT\{x[n-m]\}=\sum_{n=0}^{\infty}x[n-m]\,Z^{-n}$$

Take $m=3$, $ZT\{x[n-3]\}=\sum_{n=0}^{\infty}x[n-3]z^{-n}$

$$=x[-3]z^{0}+x[-2]z^{-1}+x[-1]z^{-2}+x[0]z^{-3}+x[1]z^{-4}+x[2]z^{-5}+x[3]z^{-6}+-----$$

$$=x[-3]+x[-2]Z^{-1}+x[-1]Z^{-2}+Z^{-3}[x[0]+x[1]Z^{-1}+x[2]Z^{-3}+---]$$

$$=Z^{-3}\left[x(-3)Z^{3}+x[-2]Z^{2}+x[-1]Z\right]+Z^{-3}X(z)$$

$$=Z^{-3}X(z)+Z^{-3}\left[x(-1)Z+x[-2]Z^{2}+x[-3]Z^{3}\right]$$

$$ZT\{x[n-3]\}=Z^{-3}X(z)+Z^{-3}\sum_{n=1}^{3}x[-n]Z^{n}$$

$$=Z^{-3}\left[X(z)+\sum_{n=1}^{3}x[-n]z^{n}\right]$$

In General, $ZT\{x[n-m]\}=Z^{-m}\left[X(z)+\sum_{n=1}^{m}x[-n]\,Z^{n}\right]$

---

(2) Time Advance

If $x[n] \leftrightarrow X(z)$

Then $ZT\{x[n+m]\}=Z^{m}\left[X(z)+\sum_{n=0}^{m-1}x[n]\,Z^{-n}\right]$

Proof :

$$ZT\{x[n+m]\}=\sum_{n=0}^{\infty}x[n+m]\,Z^{-n}$$

Let $m=3$, $ZT\{x[n+3]\}=\sum_{n=0}^{\infty}x[n+3]z^{-n}$

$$=x[3]z^{0}+x[4]z^{-1}+x[5]z^{-2}+x[6]z^{-3}+-----$$

$$=Z^{3}\left[x[3]\,Z^{-3}+x[4]\,Z^{-4}+x[5]\,Z^{-5}+-------\right]$$

$$=Z^{3}(x[0]-x[0]+x[1]Z^{-1}-x[1]Z^{-1}+x[2]Z^{-2}-x(2)Z^{-2}+x[3]Z^{-3}+x[4]Z^{-4}+—)$$

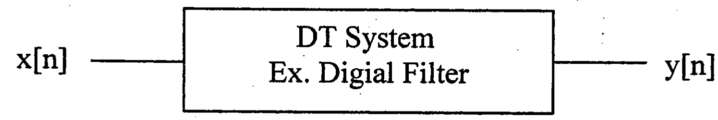
$$=Z^{3}\left[(x[0]+x[1]\,Z^{-1}+x[2]\,Z^{-2}+.......)-(x(0)+x[1]\,Z^{-1}+x[2]\,Z^{-2})\right]$$

$$ZT\{x[n+3]\}=Z^{3}\left[X(z)-\sum_{n=0}^{2}x[n]\,Z^{-n}\right]$$

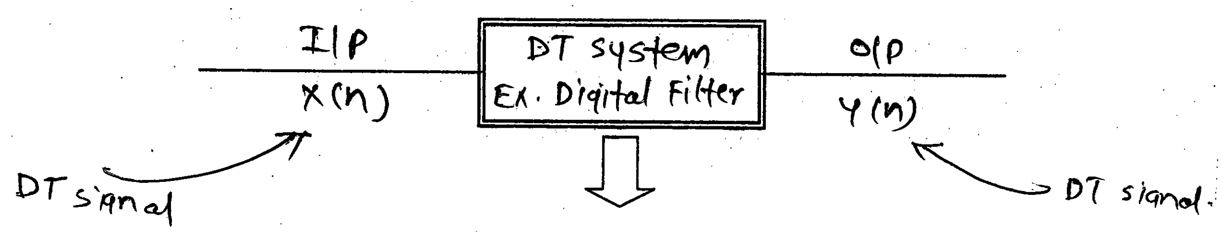
In General, $ZT\{x[n+m]\}=Z^{m}\left[X(z)-\sum_{n=0}^{m-1}x[n]Z^{-n}\right]$

3.2 DISCRETE TIME SYSTEM

A DT system is a device or algorithm that operates on a DT signal according to some well defined rule, to produce another DT signal. In general a DT system can be thought as a set of operations performed on the input signal x[n] to produce the output signal y[n].

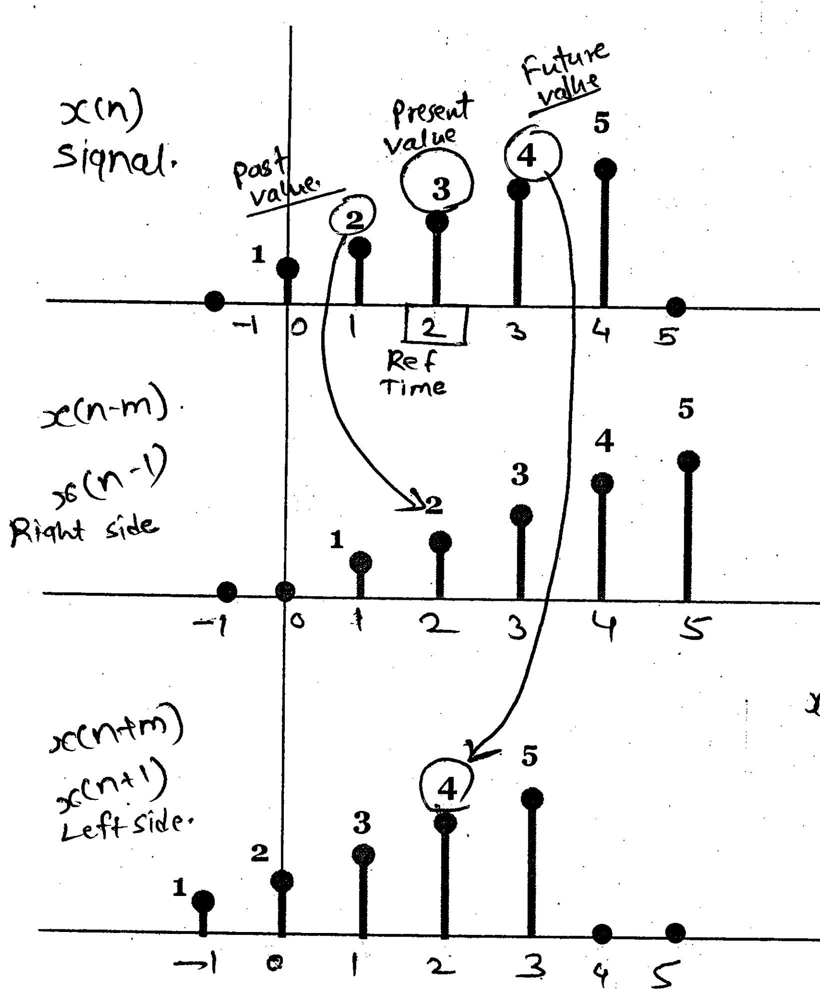


y[n] = T { x[n]} Where the symbol T denotes the transformation or operation performed by the system on x [n] to produce y[n].



mathematical Representation of DTS.
- (I) **Impulse Response**
- (II) **Transfer Function**
- (III) **Difference Equation**
- (IV) **Realization Diagram**
- (V) **Pole Zero Plot**

Representation of Input–Output signal values from system point of view.



x(n) → Present value of ip to corresponding to instant of time.

x(n-m) → Delayed x(n)
↳ Past value of I/P ..

x(n+m) → Advanced x(n)
↳ future value of I/P . –

3.2.1. CLASSIFICATION OF DT SYSTEMS :-

(1) Static (Memoryless) / Dynamic (Memory System) :-

A DT system is called static or memoryless if it output at any instant depends on the input sample at the same time and not on past or future samples of the input. If the system is not static then it is dynamic.

(2) Linear / Non Linear System.

A system that satisfies the superposition principle is called Linear System.
If a system is Linear then,
$T\{a . x_1[n] + b\, x_2[n]\} = a_1\ T\{x_1[n]\} + a_2\, T\{x_2[n]\}$
If a system does not satisfy the superposition principle then it is Non Linear System.

Examples :

(a) $y(n) = x[n] + 10$

$x_1(n) \longrightarrow \boxed{} \longrightarrow y_1(n) = x_1(n) + 10$

$x_2(n) \longrightarrow \boxed{} \longrightarrow y_2(n) = x_2(n) + 10$

$x_1(n) + x_2(n) \longrightarrow \boxed{} \longrightarrow y(n) =$

where $y[n] = \{x_1(n) + x_2(n)\} + 10$
$= x_1(n) + x_2(n) + 10$
$y(n) \neq y_1(n) + y_2(n)$

$\therefore$ system is Not linear system

(b) $y(n) = n\, x[n]$

$x_1(n) \longrightarrow \boxed{} \longrightarrow y_1(n) = n\, x_1(n)$

$x_2(n) \longrightarrow \boxed{} \longrightarrow y_2(n) = n\, x_2(n)$

$x_1(n) + x_2(n) \longrightarrow \boxed{} \longrightarrow y(n)$

where $y[n] = n\{x_1(n) + x_2(n)\}$
$= n\, x_1(n) + n\, x_2(n)$
$y(n) = y_1(n) + y_2(n)$

$\therefore$ system is linear system

(c) $y(n) = x(n^2)$

$x_1(n) \longrightarrow \boxed{} \longrightarrow y_1(n) = x_1(n^2)$

$x_2(n) \longrightarrow \boxed{} \longrightarrow y_2(n) = x_2(n^2)$

$x_1(n) + x_2(n) \longrightarrow \boxed{} \longrightarrow y(n)$

where $y[n] = x_1(n^2) + x_2(n^2)$

$y(n) = y_1(n) + y_2(n)$

$\therefore$ system is linear system

(d) $y(n) = x^2(n)$

$x_1(n) \longrightarrow \boxed{} \longrightarrow y_1(n) = x_1^2(n)$

$x_2(n) \longrightarrow \boxed{} \longrightarrow y_2(n) = x_2^2(n)$

$x_1(n) + x_2(n) \longrightarrow \boxed{} \longrightarrow y(n)$

$y(n) = [x_1(n) + x_2(n)]^2$

$y[n] = x_1^2(n) + x_2^2(n) + 2x_1(n)x_2(n)$
$y(n) \neq y_1(n) + y_2(n)$

$\because$ system is Non Linear.

(3) Causal / Non Causal System

A system is said to be causal if the output of the system at any time n depends only on present and past values of input and does not depend on future values of input.
If the system is not causal then it is Non casual. For non causal system output depends on future values of input.

(4) Time Invariant / Time Variant System.

A system is called Time Invariant if a time shift in the input signal causes a time shift in the output signal.

ie If $x[n] \longrightarrow \boxed{} \longrightarrow y[n]$

Then $x[n-m] \longrightarrow \boxed{} \longrightarrow y[n-m].$

Otherwise the system is Time Variant System.

Examples :

(a)

$y[n] = n\,x[n]$
o/p $= n$ I / P

$x[n] \longrightarrow \boxed{} \longrightarrow y[n] = n\,x[n]$
Delay by k
$x[n-k] \longrightarrow \boxed{} \longrightarrow y[n] = nx[n-k]$

Let $y[n] = n\,x[n]$
Delay by k,
$y[n-k] = (n-k)\,x\,(n-k)$
$y'[n] \neq y[n-k]$
$\because$ System is Time Variant.

(b)

$y[n] = x[2n]$
o/p I/p
$x[n] \longrightarrow \boxed{} \longrightarrow y[n] = x[2n]$
Delay by k,
$x[n-k] \longrightarrow \boxed{} \longrightarrow y'[n] = x[2(n-k)]$

Let $y[n] = x[2n]$
Delay by k,
$y[n-k] = x[2(n-k)]$
Since $y'[n] = y[n-k]$
System is Time Invariant.

(c)

$y[n] = e^{x[n]}$
$x[n] \longrightarrow \boxed{} \longrightarrow y[n] = e^{x[n]}$
Delay by k,
$x[n-k] \longrightarrow \boxed{} \longrightarrow y'[n] = e^{x[n-k]}$

Let $y[n] = e^{x[n-k]}$
Delay by k,
$y[n-k] = e^{x[n-k]}$
Since $y'[n] = y[n-k]$
System is Time Invariant.

(d)

$y[n] = \cos[x[n]]$
$x[n] \longrightarrow \boxed{} \longrightarrow y[n] = \cos(x[n])$
Delay by k,
$x[n-k] \longrightarrow \boxed{} \longrightarrow y'[n] = \cos(x[n-k])$

Let $y[n] = \cos(x[n])$
Delay by k,
$y[n-k] = \cos(x[n-k])$
Since $y'[n] = y[n-k]$
System is Time Invariant.

(5)Stable / Unstable system.

A system is said to be bounded input, bounded o/p stable if and only if every bounded input produces a bounded output.

Q(15) Classify the following systems :

a) $y[n] = n\,x[n]$ c) $y[n] = e^{x[n]}$ e) $y[n] = x^2[n]$
b) $y[n] = x[n] + 10$ d) $y[n] = x[2n]$ f) $y[n] = x[n^2]$

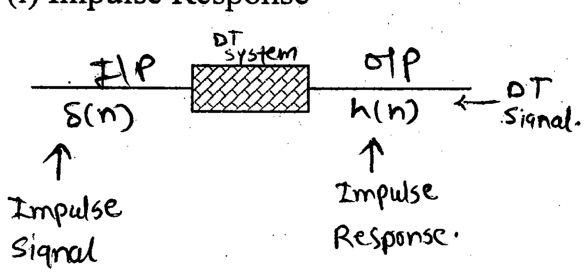
ANS:

a) $y[n] = n\,.\,x[n]$	static,	linear,	causal,	time-variant
b) $y[n] = x[n] + 10$	static,	non-linear,	causal,	time-invariant
c) $y[n] = e^{x[n]}$	static,	non-linear,	causal,	time-invariant
d) $y[n] = x[2n]$	dynamic,	linear,	non-causal,	time-variant
e) $y[n] = x^2[n]$	static,	non-linear,	causal,	time-invariant
f) $y[n] = x[n^2]$	dynamic,	linear,	non-causal,	time variant

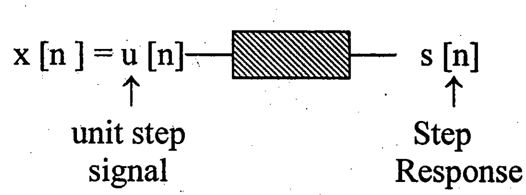
3.2.2 Linear Time Invariant (LTI) System [also called as Linear Shift Invariant (LSI) System]

I IMPULSE RESPONSE / STEP RESPONSE OF DT SYSTEM

(i) Impulse Response



(ii) Step response



Q(16) Prove that, output y[n] due to the input x[n] of LTI system is given by

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]\ h[n-m]$$

Consider a Discrete Time LTI System.

$x[n]$	LTI	$y[n]$	
$\delta[n]$	LTI	$h[n]$	Impulse Response of the system.
$\delta[n-m]$	LTI	$h[n-m]$	Time Invariant system.
$2\delta[n-m]$	LTI	$2\ h[n-m]$	Scaling
$x[m]\ \delta[n-m]$	LTI	$x[m]\ h[n-m]$	Scaling
$\sum_{-\infty}^{\infty} x[m]\delta[n-m]$	LTI	$\sum_{m=-\infty}^{\infty} x[m]\ h[n-m]$	Linearity Properly.

But $x[n] = \sum_{-\infty}^{\infty} x[m]\ \delta[n-m]$ Then $y[n] = \sum_{m=-\infty}^{\infty} x[m]\ h[n-m]$

* * * The convolution theorem provides a major cornerstone of linear systems theory. It implies, for example, that any LTI system (recursive or nonrecursive) can be implemented by convolving the input signal with the impulse response of the filter

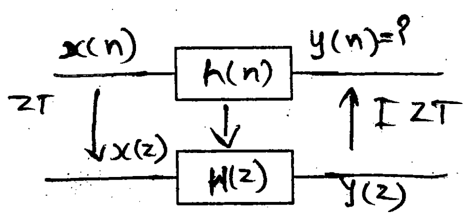
➢ RESPONSE OF LTI SYSTEM [i.e. output of system]

When the system is Linear and Time Invariant, the Zero stale output y[n] due to the input x[n] is given by, $y[n] = \sum_{m=-\infty}^{\infty} x[m]\ h[n-m]$

Ex-1. Given h[n] = $(0.25)^n$ u[n]. 4m

Find the response of the system to the input x[n]=$(0.5)^n$ u[n]

Solution :



(i) Find X(z)

$x[n] = (0.5)^n u[n]$

By ZT,

$X(z) = \frac{z}{z-0.5}$ $|z| > 0.5$

(ii) Find H(z)

$h[n] = (0.25)^n u[n]$

By ZT,

$H(z) = \frac{z}{z-0.25}$ $|z| > 0.25$

(iii) Let Y(z) = X(z) H(z)

$$Y(z) = \frac{z}{(z-0.5)}\ \frac{z}{(z-0.25)}$$

$$Y(z) = \frac{z^2}{(z-0.5)(z-0.25)}$$

(iv) Find y[n]

$$\frac{Y(z)}{z} = \frac{z}{(z-0.5)(z-0.25)}$$

By PFE, $\frac{Y(z)}{z} = \frac{A}{z-0.5} + \frac{B}{z-0.25}$ Where $A = \frac{Y(z)}{z}(z-0.5)\Big|_{z=0.5} = \boxed{2}$

$B = \frac{Y(z)}{z}(z-0.25)\Big|_{z=0.25} = \boxed{-1}$

By substituting,

$$Y(z) = 2\left[\frac{z}{z-0.5}\right] - \left[\frac{z}{z-0.25}\right]$$

ROC is not given ∴ causal

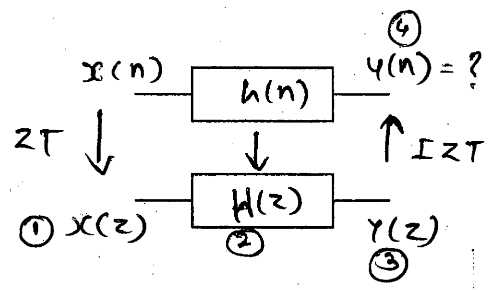
By IZT, $y[n] = 2(0.5)^n u(n) - (0.25)^n u(n)$

Ex-2. Given h[n] = $(0.5)^n$ $0 \le n \le 2$

4m

Find the response of the system to the input $x[n] = \cos\left(n\frac{\pi}{3}w\right)u[n]$

Solution :



$\cancel{H(z)}$ $h(n) = (0.5)^n$ $0 \le n \le 2$

i.e. 0, 1, 2

$h(n) = \{\underset{\uparrow}{1}, 0.5, 0.25\}$ causal Finite lengths

By Z.T.,

$H(z) = 1 + 0.5z^{-1} + 0.25z^{-2}$

$x(n) = \cos\left(n\frac{\pi}{3}\omega\right)u(n)$

$Y(z) = X(z)\,H(z)$

$= \left[1 + 0.5z^{-1} + 0.25z^{-2}\right] X(z)$

$Y(z) = X(z) + X(z)\,0.5z^{-1} + X(z)\,0.25z^{-2}$

By IZT,

$y(n) = x(n) + 0.5\,x(n-1) + 0.25\,x(n-2)$

$= \cos\left(\frac{n\pi}{3}\right)u(n) + 0.5\cos\left[(n-1)\frac{\pi}{3}\right]u(n-1)$

$+ 0.5\cos\left[(n-2)\frac{\pi}{3}\right]u(n-2)$

Note

$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$

	$x(n)$	$h(n)$	
case i)	Finite length	Finite length	calculate X(z)
case ii)	Infinite length	Infinite length	calculate H(z)
case iii)	Finite length	Infinite length	Do not substitute ZT of Infinite length signal.
case iv)	Infinite length	Finite length	

Ex-① (cases i, ii)

Ex-② (cases iii, iv)

Exercise :

Q(17) Given $h[n] = \{ \underset{\uparrow}{0.5}, 1, 0.5 \}$ Find the step response of the system using Time Domain Method.

HINT : To find step response take x[n] = u[n]. Then find Linear Convolution of u[n] with x[n]. Since u[n] is of infinite length, length of y[n] will be infinite. In that case, calculate at least four values of output signal. ANS : $y[n] = \{ \underset{\uparrow}{0.5}, 1.5, 2, 2, 2, 2, ----- \}$

Q(18) Given $h[n] = (\frac{1}{2})^n u[n]$. Find the response of the system to the input $x[n]=(\frac{1}{4})^n u[n]$ using time domain method.

HINT : Response means output. Response using time domain method means using Linear Convolution formula method. Length of x[n] and h[n] is infinite, therefore length of y[n] will be infinite. In that case, calculate at least four values of output signal.
ANS : $y[n] = : 2 (\frac{1}{2})^n u[n] - (\frac{1}{4})^n u[n]$

Q(19) Given $h[n] = (\frac{1}{2})^n \quad 0 \le n \le 3$ Find the response of the system to the input $x[n] = (\frac{1}{4})^n u[n]$.

HINT : Find output y[n] using Z Transform Method (i.e.To find LC by ZT) Length of h[n] is finite and length of x[n] in infinite. Do not substitute ZT of x[n].
ANS : $y[n] = (\frac{1}{4})^n u[n] + \frac{1}{2} (\frac{1}{4})^{n-1} u[n-1] + \frac{1}{4} (\frac{1}{4})^{n-2} u[n-2] + \frac{1}{8} (\frac{1}{4})^{n-3} u[n-3]$

Q(20) A discrete time system has an impulse response $h[n] = \{ \underset{\uparrow}{1}, 2, 3, 4 \}$.
If the output $y[n] = \{ \underset{\uparrow}{5}, 16, 27, 38, 24 \}$ what is the applied input x[n]?

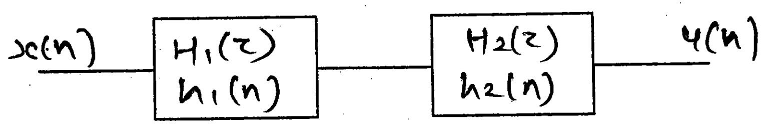
ANS : $x[n] = \{ \underset{\uparrow}{5}, 6 \}$

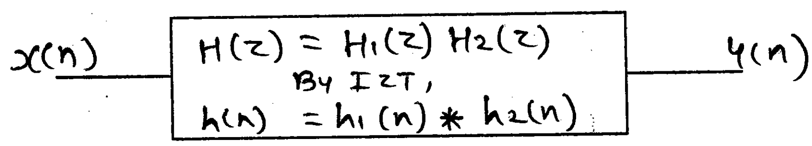
Q(21) If the step response of the system is given by $s[n] = (\frac{1}{2})^n u[n]$, find an impulse response of the system a) using ZT and iZT b) without using ZT and iZT technique.

ANS : a) $h[n] = 2 u[n] - (\frac{1}{2})^n u[n]$, b) $h[n] = (\frac{1}{2})^n u[n] - (\frac{1}{2})^{n-1} u[n-1]$

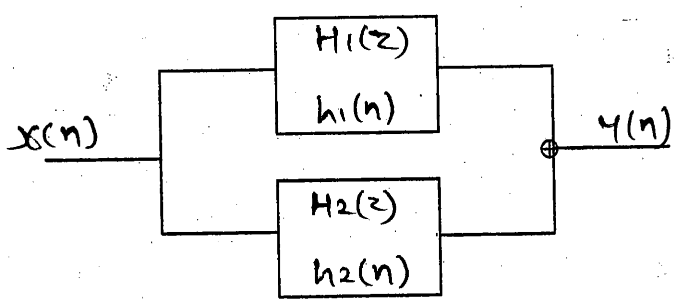
Interconnection of DT systems

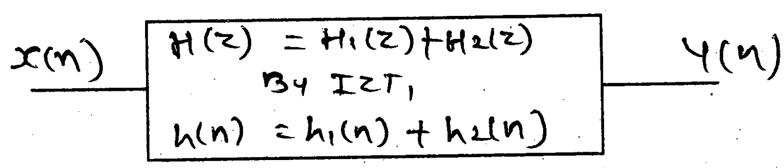
(a) Cascade / Series Connection



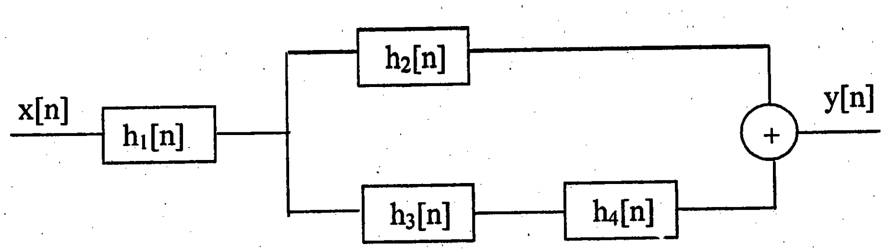


(b) Parallel Connection





Q(22) Consider the interconnection of LTI system as shown below,

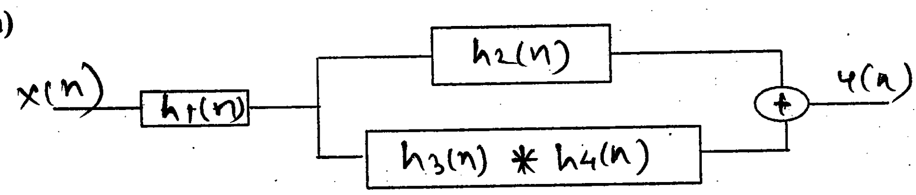


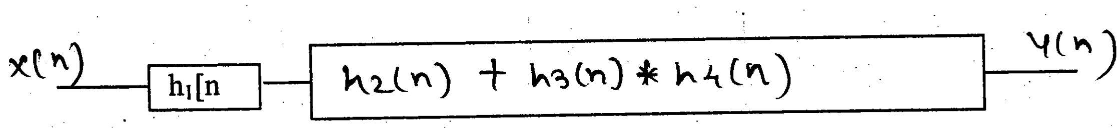
a) Express the overall impulse response in terms of $h_1[n]$, $h_2[n]$, $h_3[n]$ and $h_4[n]$.
b) Determine impulse response when

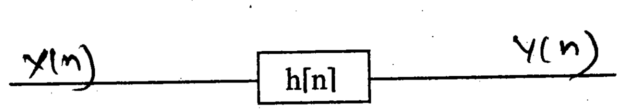
$h_1[n] = \{ \underset{\uparrow}{0.5}\ 0.25, 0.5 \}$, $h_2[n] = h_3[n] = (n+1)\,u[n]$, $h_4[n] = \delta[n-2]$

Solution :

(a)







Where $h[n] = h_1(n) * [h_2(n) + h_3(n) * h_4(n)]$

$h(n) = h_1(n) * h_2(n) + h_1(n) * h_3(n) * h_4(n)$ Ans

(b) To find impulse response

$$h[n] = h_1[n] * \{ h_2[n]+h_3[n]*h_4[n] \}$$

By ZT, $H(z) = H_1(z)\,[H_2(z)+H_3(z).(H_4(z)]$

(i) Now, $h_1[n] = \left\{ \underset{\uparrow}{\tfrac{1}{2}}, \tfrac{1}{4}, \tfrac{1}{2} \right\}$ finite length.

By ZT, $H_1(z) = \tfrac{1}{2} + \tfrac{1}{4} z^{-1} + \tfrac{1}{2} z^{-2}$

(ii) $h_2[n] = (n+1)\,u[n]$ infinite length.

(iii) $h_3[n] = (n+1)\,u[n]$ infinite length.

(iv) Now, $h_4[n] = \delta(n-2)$ finite length

By ZT, $H_4(z) = z^{-2}$

Now, $H(z) = \left(\frac{1}{2}+\frac{1}{4}z^{-1}+\frac{1}{2}z^{-2}\right)\left[H_2(z)+H_3(z)z^{-2}\right]$

$$\therefore H(z) = \frac{1}{2}H_2(z)+\frac{1}{4}z^{-1}H_2(z)+\frac{1}{2}z^{-2}H_2(z)+\frac{1}{2}z^{-2}H3(z)\,\frac{1}{4}z^{-3}H3(z)+\frac{1}{2}z^{-4}H3(z)$$

$$h[n] = \frac{1}{2}h_2[n]+\frac{1}{4}h_2[n-1]+\frac{1}{2}h_2[n-2]+\frac{1}{2}h_3[n-3]+\frac{1}{4}h_3[n-3]+\frac{1}{2}h_3[n-4]$$

$$\therefore h[n] = \frac{1}{2}(n+1)\,u[n]+\frac{1}{4}n\,u[n-1]+\frac{1}{2}(n-1)\,u[n-2]+\frac{1}{4}(n-2)u[n-3]+\frac{1}{2}(n-3)u[n-4]$$

Exercise :

Q(23) Find the Transfer function of a DT system that has two zeros at $z_1 = 0$ and $z_2=2$ and two complex conjugate poles at $P_1, P_2 = 0.8\,e^{\pm j2\pi/3}$.

ANS : $H[z] = \dfrac{z(z-2)}{z^2 - 1.6\,Cos(2\frac{\pi}{3}) + 1}$

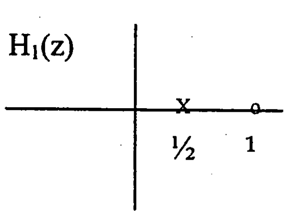
Q(24) A DT system has two zeros at origin and complex conjugate poles P_1 and P_2 at $0.25\ e\pm^{j45}$ The DC gain is 5. Write the transfer function of the system.

HINT : DC Gain means Magnitude response at w=0. i.e. | H(0) | = 0.

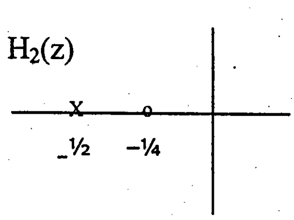
Let $\therefore H(z) = \dfrac{G(z-z_1)\,(z-z_2)}{(z-p_1)\,(z-p_2)}$ To find G put z =1 at w =0.

Q(25) The pole-zero diagram of $H_1(z)$ and $H_2(z)$ is shown below :

$H_1(z)$



$H_2(z)$



Find the transfer function of the overall system if $H_1(z)$ and $H_2(z)$ are connected in series and in parallel.

Solution : $\therefore H(z) = \dfrac{G_1(z-1)\,(z+\frac{1}{4})}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{2}\right)}$ $\therefore H(z) = \dfrac{G_1(z-1)}{\left(z-\frac{1}{2}\right)} + \dfrac{G_2(z+\frac{1}{4})}{\left(z+\frac{1}{2}\right)}$

III DIFFERENCE EQUATION

The difference equation with constant coefficient is given by,

$\sum_{k=0}^{N} a_k \; y[n-k] = \sum_{k=0}^{M} b_k \; x[n-k]$ Where, the order of the system refers to the largest delay (N or M) appearing in the equation.

$$a_0 \, y[n] \; + \; \sum_{k=1}^{n} a_k \, y[n-k] \; = \; \sum_{k=0}^{M} b_k \, x[n-k]$$

$$y[n] \; = \frac{1}{a_0} \sum_{k=0}^{M} b_k \, x[n-k] \; - \frac{1}{a_0} \; \sum_{k=1}^{n} a_k \, y[n-k]$$

- **When Initial conditions are NOT specified.**

Ex-1. Given $H(z) = \dfrac{z\left(z-\frac{1}{2}\right)}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{4}\right)}$ Determine the difference equation of the system.

Solution : To find Difference equation :

(i) Write H(z) in -ve powers of z

$$H(z) = \frac{z\left(z-\frac{1}{2}\right)}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{4}\right)} = \frac{z^2 - \frac{1}{2} z}{z^2 - \frac{7}{4} z + \frac{1}{12}}$$

Dividing by z^2

$$H(z) = \frac{1 - \frac{1}{2} z^{-1}}{1 - \frac{7}{4} z^{-1} + \frac{1}{12} z^{-2}}$$

(ii) Let $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1 - \frac{1}{2} z^{-1}}{1 - \frac{7}{4} z^{-1} + \frac{1}{12} z^{-2}}$

(iii) Cross Multiply

$$Y(z) - \frac{7}{12} z^{-1} Y(z) + \frac{1}{12} z^{-2} Y(z) = X(z) - \frac{1}{2} z^{-1} X(z)$$

(iv) Take Inverse ZT

$$y(n) - \frac{7}{12} \, y(n-1) \; + \; \frac{1}{12} \; y(n-2) \; = \; x(n) \; - \; \frac{1}{2} \; x(n-1)$$

$$\underbrace{y[n]}_{\text{o/p}} = \underbrace{\frac{7}{12} y(n-1)}_{\text{Past value}} - \underbrace{\frac{1}{12} y(n-2)}_{\text{Past value}} + \underbrace{x(n)}_{\text{Present value}} - \underbrace{\frac{1}{2} x(n-1)}_{\text{past value.}}$$

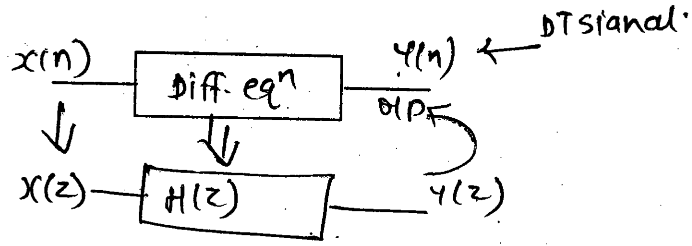
Difference Equation of DT system.

> ❁ **Causal System :** If output of the system depends on Past &/ Present, Input &/Output values [But No future values] Then system is called causal system.
>
> ❁ **All practical systems are causal.**

Ex-2. Given $y(n) = \frac{7}{12} y(n-1) - \frac{1}{12} y(n-2) + x(n) - \frac{1}{2} x(n-1)$

Determine the response of the system to the input $x[n] = (½)^n u[n]$.

Solution :



(i) $x[n] = (½)^n u[n]$

By ZT, $X(z) = \dfrac{z}{z - ½}$

(ii) To find H(z)

Now, $y(n) = \frac{7}{12} y(n-1) - \frac{1}{12} y(n-2) + x(n) - \frac{1}{2} x(n-1)$

By ZT, $Y(z) = \frac{7}{12} z^{-1} Y(z) - \frac{1}{12} z^{-2} Y(z) + X(z) - \frac{1}{2} z^{-1} X(z)$

$$Y(z)\left(1 - \frac{7}{12} z^{-1} + \frac{1}{12} z^{-2}\right) = X(z)\left(1 - \frac{1}{2} z^{-1}\right)$$

$$\frac{Y(z)}{X(z)} = \frac{\left(1 - ½ z^{-1}\right)}{\left(1 - \frac{7}{12} z^{-1} + \frac{1}{12} z^{-2}\right)}$$ multiply by z^2

$$H(z) = \frac{z^2 - \frac{1}{2} z}{z^2 - \frac{7}{4} z + \frac{1}{12}}$$

$$H(z) = \frac{z\left(z - \frac{1}{2}\right)}{\left(z - \frac{1}{3}\right)\left(z - \frac{1}{4}\right)}$$

(iii) Y(z) = H(z) X(z)

$$Y(z) = \frac{z\left(z - \frac{1}{2}\right)}{\left(z - \frac{1}{3}\right)\left(z - \frac{1}{4}\right)} \quad \frac{z}{\left(z - \frac{1}{2}\right)}$$

$$Y(z) = \frac{z^2}{\left(z - \frac{1}{3}\right)\left(z - \frac{1}{4}\right)}$$

(iv) To find y[n]

$$\frac{Y(z)}{z} = \frac{z}{\left(z - \frac{1}{3}\right)\left(z - \frac{1}{4}\right)}$$

By PFE,

By IZT.

$$y[n] = 4\left(\frac{1}{3}\right)^n u[n] - 3\left(\frac{1}{4}\right)^n u[n]$$

Q(26) The output of the oscillator is given below. Find the difference equation of the system. a) $y[n] = \sin(n\frac{\pi}{2})$ b) $y[n] = \cos(n\frac{\pi}{2}) - \sin(n\frac{\pi}{2})$

Hint : Take $x[n] = \delta[n]$ and assume that y[n] is causal infinite length sequence.

ANS : a) $y[n] =$

Q(27) Find the difference equation of the IIR filter, which has an impulse response

$$h[n] = \begin{cases} 1 & \text{for even } n \\ -1 & \text{for odd } n \end{cases}$$

ANS : Let $h[n] = (-1)^n u[n]$ $y[n] = x[n] - y[n-1]$

Q(28) Find the difference equation of the system, which generates the following output.
$y[n] = \{1, 0, 1, 0, 1, 0, 1, 0 ------\}$ for $n \geq 0$

Solution : To find difference Equation,

Let $y[n] = \delta[n] + \delta[n-2] + \delta[n-4] + \delta[n-6] + + + +$ eq-(1)

$y[n-2] = \delta[n-2] + \delta[n-4] + \delta[n-6] + \delta[n-8] + + +$ eq-(2)

By eq-(1) – eq-(2)

$y[n] - y[n-2] = \delta[n]$

Let $x[n] = \delta[n]$

Then $y[n] - y[n-2] = x[n]$

ANS : $y[n] = y[n-2] + x[n]$

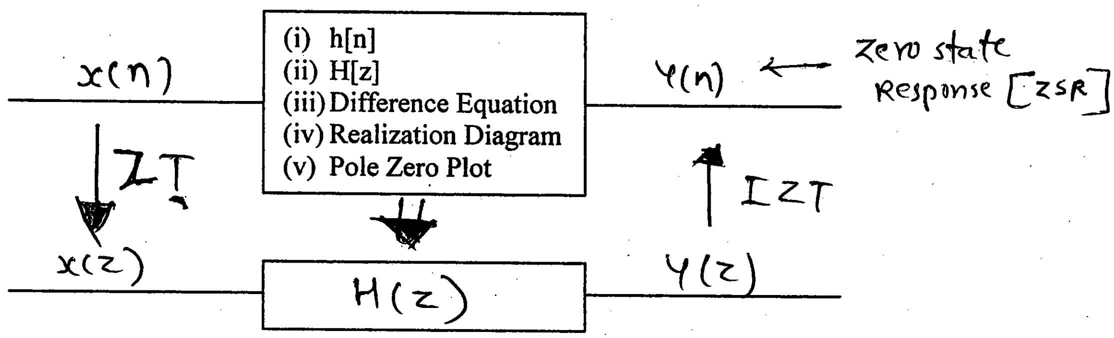
Q(29) Find the difference equation of the system, which generates the following output.
$y[n] = \{1, 1, 2, 3, 5, 8, 13 -----\}$ for $n \geq 0$

ANS : $y[n] = y[n-1] + y[n-2] + x[n]$

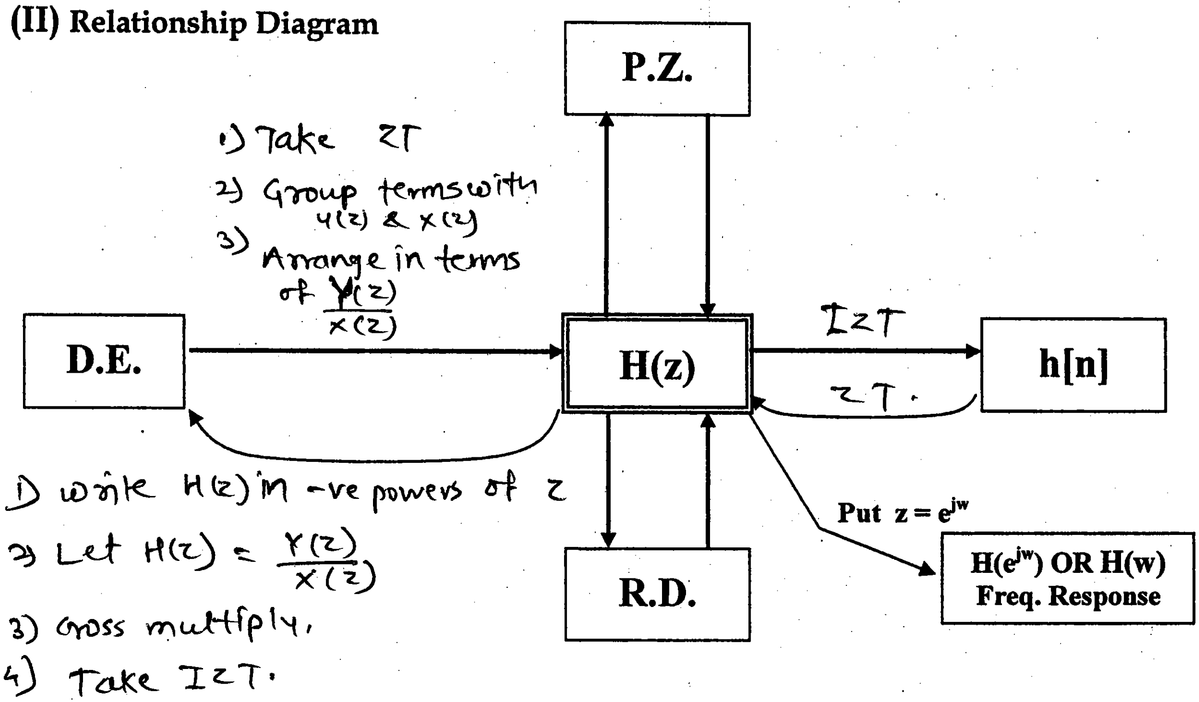
Q(30) Find the difference equation of the system, which generates the following output.
$y[n] = \{1, 0, 2, 0, 3, 0, 4, 0 -----\}$ for $n \geq 0$

ANS : $y[n] = 2y[n-2] + y[n-4] + x[n]$

NOTE (I) To find Zero State Response of the System [i.e. Normal Response]



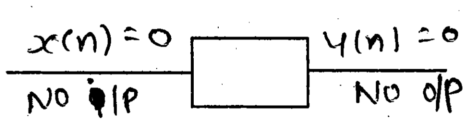
(II) Relationship Diagram



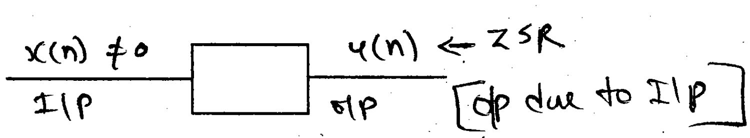
(III) When Initial Conditions are NOT given

- Initial conditions are **Zero**
- Initial state of the system is **Zero**
- Initially, system is at **Relaxed position**

Case -1



Case -2



(IMP) Time shift Property of Z T

$$ZT\{\ y[n\text{-}m]\ \} = Z^{-m}\,Y(z)$$

---

(1) $ZT\{\ y[n-1]\ \} = z^{-1}\,Y(z)$

(2) $ZT\{\ y[n-2]\ \} = z^{-2}\,Y(z)$

(IV) When Initial Conditions are given

- Initial conditions are **Zero**
- Initial state of the system is **Zero**
- Initially, system is at **Relaxed position**

Case -1 $x(n) = 0$ (No i/p) → [ ] → $y(n)$ (o/p) ← Zero i/p Response. [ZIR]

Case -2 $x(n) \neq 0$ (I/p) → [ ] → $y(n)$ (o/p) ← ZIR + ZSR.

(IMP) Time shift Property of Z T

$$ZT\{ y[n-m] \} = Z^{-m} \left[Y[z] + \sum_{n=1}^{m} y[-n]\, z^{n} \right]$$

---

eq. $ZT\{ y[n-1] \} = z^{-1}\left[Y(z) + \sum_{n=1}^{1} y(-n)\, z^{n} \right]$

$$= z^{-1} \left(Y(z) + y(-1)\, z^{1} \right]$$

$$\boxed{ZT\{ y[n-1] \} = z^{-1} Y(z) + \underbrace{y(-1)}_{}}$$ Initial conditions

$$\boxed{ZT\{ y[n-2] \} = z^{-2} Y(z) + \underbrace{z^{-1} y(-1) + y(-2)}_{}}$$

- **When Initial conditions are specified.**

Q(31) A DT system is represented by the following difference equation
10m
$y[n] = \frac{1}{4} y[n-1] + \frac{1}{8} y[n-2] + x[n] + 1/3\, x[n-1]$ and $y[-1] = 1$.
Find the response of the system to the input $x[n] = (-1/3)^n u[n]$.

Solution :

$x(n)$ → [D.E.] → $y(n) = ZIR + ZSR$?

Initial Condition $y[-1] = 0$

(i) Find X(z)

$x[n] = (-1/3)^n u[n]$

$X(z) = \dfrac{z}{z + 1/3}$

(ii) Find H(z)

$y[n] = \frac{1}{4} y[n-1] + \frac{1}{8} y[n-2] + x[n] + 1/3\, x[n-1]$

By ZT,

$Y(z) = \frac{1}{4} ZT\{ y[n-1] \} + \frac{1}{8} ZT\{ y[n-2] \} + X(Z) + 1/3\ ZT\{ x[n-1] \}$

$$Y(z) = \frac{1}{4}\left[z^{-1}Y(z) + y(-1)\right] + \frac{1}{8}\left[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)\right]$$

$$+ x(z) + \frac{1}{3}\left[z^{-1}x(z) + x(-1)\right]$$

$$Y(z) = \frac{1}{4}\left[z^{-1}Y(z) + 1\right] + \frac{1}{8}\left[z^{-2}Y(z) + z^{-1} + 0\right]$$

$$+ X(z) + \frac{1}{3}\left[z^{-1}X(z) + 0\right]$$

$$\underline{Y(z)} = \underline{\frac{1}{4}z^{-1}Y(z)} + \frac{1}{4} + \underline{\frac{1}{8}z^{-2}Y(z)} + \frac{1}{8}z^{-1} + X(z) + \frac{1}{3}z^{-1}X(z)$$

$$Y(z)\left(1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}\right) = \left(\frac{1}{4} + \frac{1}{8}z^{-1}\right) + X(z)\left(\frac{1}{3}z^{-1} + 1\right)$$

$$Y(z) = \frac{\frac{1}{4} + \frac{1}{8}z^{-1}}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}} + \frac{X(z)\left(\frac{1}{3}z^{-1} + 1\right)}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}$$

multiply by z^2

$$= \frac{\frac{1}{4}z^2 + \frac{1}{8}z}{z^2 - \frac{1}{4}z - \frac{1}{8}} + \left(\frac{z^2 + \frac{1}{3}z}{z^2 - \frac{1}{4}z - \frac{1}{8}}\right)X(z)$$

Let $Y(z) = Y_{zi}(z) + Y_{zs}(z)$

By IZT,

$y(n) = y_{zi}(n) + y_{zs}(n)$

ZIR ZSR

When $X(z) = x(n) = 0$ calculate ZIR

i) To find ZIR

$$Y_{zi}(z) = \frac{\frac{1}{4}z^2 + \frac{1}{8}z}{z^2 - \frac{1}{4}z - \frac{1}{8}}$$

By PFE;

By IZT,

$y_{zi}(n) =$

ii) $$Y_{zs}(z) = \left(\frac{z^2 + \frac{1}{3}z}{z^2 - \frac{1}{4}z - \frac{1}{8}}\right)X(z)$$

By PFE,

By IZT,

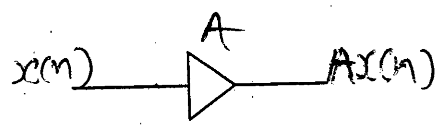
$y_{zs}(n) =$

iii) $y(n) = y_{zi}(n) + y_{zs}(n).$

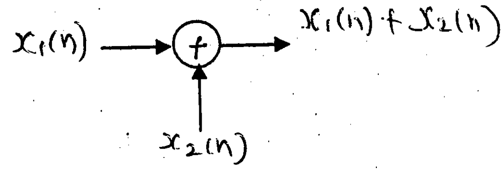
IV REALIZATION DIAGRAM OF DIGITAL FILTERS

- Basic Building Blocks

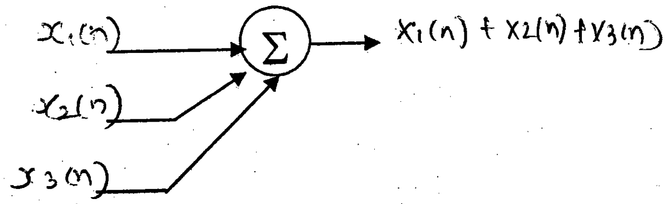
(1) Scaling (i.e. Multiplier)



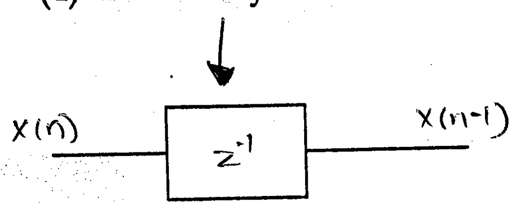
(2) Adder



(3) Summation Block



(4) Unit Delay Block



To Find H(z) :

Let $y(n) = x(n-1)$

By ZT,

$$Y(z) = z^{-1} X(z)$$

$$\frac{Y(z)}{X(z)} = z^{-1}$$

$$H(z) = z^{-1}$$

[I] Direct form Realization

(a) Direct-Form-I

Consider a DT System, $H(z) = \dfrac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$

Always must be 1

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

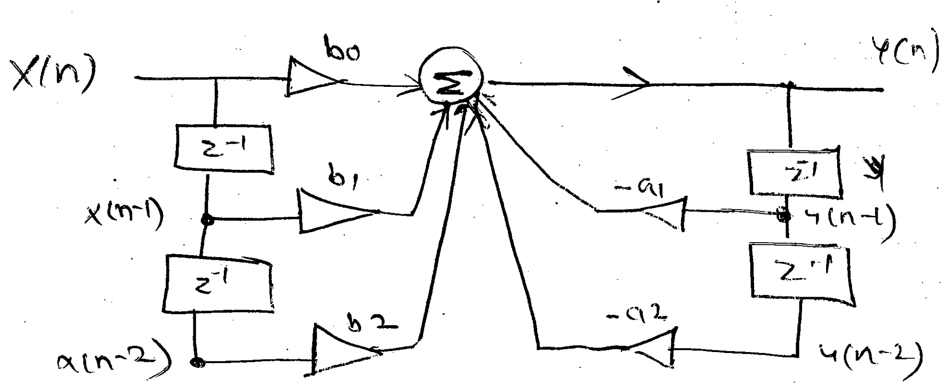
Cross Multiply,

$$Y(z) + a_1 Y(z) z^{-1} + a_2 Y(z) z^{-2} = b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z)$$

By IZT,

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)$$

$$y[n] = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) - a_1 y(n-1) - a_2 y(n-2)$$



The DF-I structure has the following properties:

1. It can be regarded as a two-zero filter section followed by a two-pole filter section.

2. In most fixed-point arithmetic schemes (such as two's complement), there is no possibility of internal filter overflow. That is, since there is fundamentally only one summation point in the filter, and since fixed-point overflow naturally ``wraps around'' from the largest positive to the largest negative number and vice versa, then as long as the final result y[n] is ``in range'', overflow is avoided, even when there is overflow of intermediate results in the sum (see below for an example). This is an important, valuable, and unusual property of the DF-I filter structure.

Two's Complement Wrap-Around

In this section, we give an example showing how *temporary* overflow in two's complement fixed-point causes no ill effects.

In 3-bit signed fixed-point arithmetic, the available numbers are as shown in Table-1

Table-1 : Three-bit two's-complement binary fixed-point numbers.

Decimal	Binary
-4	100
-3	101
-2	110
-1	111
0	000
1	001
2	010
3	011

Let's perform the sum $3+3-4=2$, which gives a temporary overflow ($3+3=6$, which wraps around to -2), but a final result (2) which is in the allowed range $[-4,3]$:

$$\begin{aligned} 011+011 &= 110 \quad (3+3=-2 \pmod 8) \\ 110+100 &= 010 \quad (-2-4=2 \pmod 8) \end{aligned}$$

Now let's do $1+3-2=2$ in three-bit two's complement:

$$\begin{aligned} 001+011 &= 100 \quad (1+3=-4 \pmod 8) \\ 100+110 &= 010 \quad (-4-2=2 \pmod 8) \end{aligned}$$

Thus, in both cases, the intermediate result overflows, but the final result is correct. Another way to state what happened is that a *positive* wrap-around in the first addition is canceled by a *negative* wrap-around in the second addition.

3. Number of Delay blocks are more than necessary. As a result, the DF-I structure is not canonical with respect to delay.

4. In all direct-form filter structures, poles and zeros can be very sensitive to round-off errors in the filter coefficients. This is usually not a problem for a simple second-order section (i.e two pole system), but it can become a problem for higher order direct-form filters. Lower sensitivity is obtained using series low-order sections (*e.g.*, second order), or by using ladder or lattice filter structures.

It is a very useful property of the direct-form I implementation that it cannot overflow internally in two's complement fixed-point arithmetic: As long as the output signal is in range, the filter will be free of numerical overflow. Most IIR filter implementations do not have this property. While DF-I is immune to internal overflow, it should not be concluded that it is always the best choice of implementation. Other forms to consider include parallel and series second-order sections and normalized ladder forms.

(b) Direct-Form-II [also called as canonical form]

The DF-I structure can be seen as an FIR filter followed by an all-pole filter in series. Since LTI filters in series *commute*, we may reverse this ordering, and implement an all-pole filter followed by an FIR filter in series. In other words, the zeros may come first, followed by the poles. When this is done, it is easy to see that the delay elements in the two filter sections contain the same numbers. As a result, a single delay line can be *shared* between the all-pole and all-zero (FIR) sections.

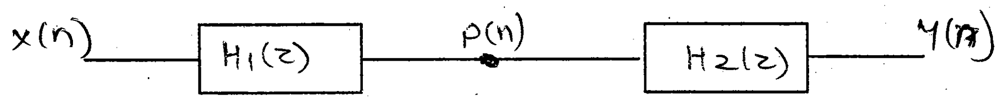
This new combined structure is called ``direct form II''.

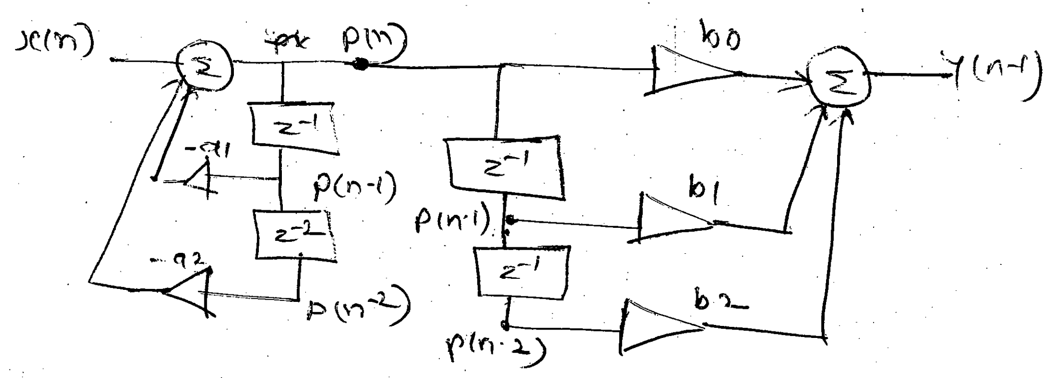
Direct-Form-II

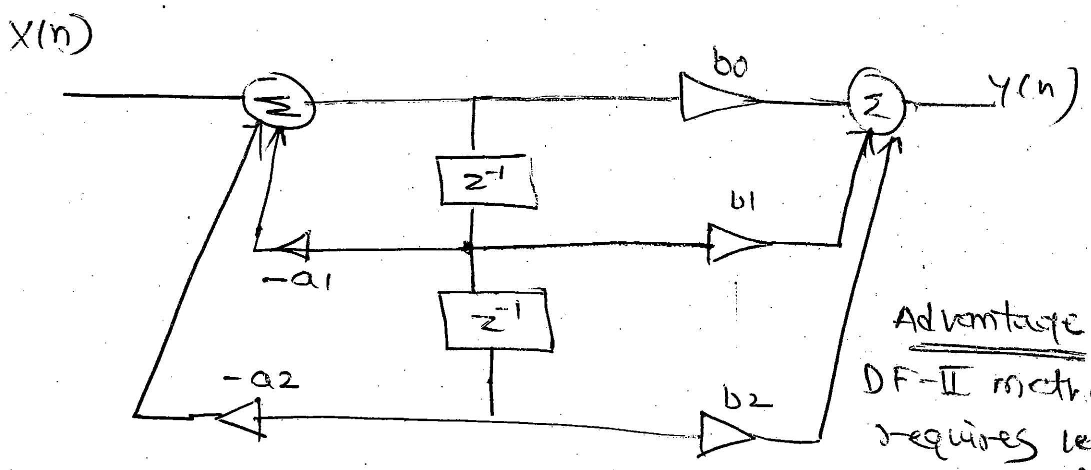
Consider a DT System, $H(z) = \dfrac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$

$$H(z) = \underbrace{\left(\frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}}\right)}_{\text{All pole system}} \underbrace{\left(\frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1}\right)}_{\text{All zero system}}$$

Let $H(z) = H_1(z) \; H_2(z)$







Advantage –
DF-II method
requires less
number of
Delay blocks.

In summary, the DF-II structure has the following properties:

1. It can be regarded as a two-pole filter section followed by a two-zero filter section.

2. It is canonical with respect to delay. This happens because delay elements associated with the two-pole and two-zero sections are shared.

3. Internal overflow can occur at the delay-line input (output of the leftmost summer in Figure-2, unlike in the DF-I implementation.

4. As is the case with all direct-form filter structures, the poles and zeros are sensitive to round-off errors in the coefficients a_i and b_i, especially for high transfer-function orders. Lower sensitivity is obtained using series low-order sections (*e.g.*, second order), or by using ladder or lattice filter structures.

More about Potential Internal Overflow of DF-II

Since the poles come first in the DF-II realization of an IIR filter, the signal entering the state delay-line (see Fig-2) typically requires a larger dynamic range than the output signal $y(n)$. In other words, it is common for the feedback portion of a DF-II IIR filter to provide a large signal *boost* which is then compensated by *attenuation* in the feed forward portion (the zeros). As a result, if the input dynamic range is to remain unrestricted, the two delay elements may need to be implemented with high-order *guard bits* to accommodate an extended dynamic range. If the number of bits in the delay elements is doubled (which still does not guarantee impossibility of internal overflow), the benefit of halving the number of delays relative to the DF-I structure may largely canceled.

In other words, the DF-II structure, which is canonical with respect to delay, may require just as much or more memory as the DF-I structure, even though the DF-I uses twice as many addressable delay elements for the filter state memory.

Q(32) Show cascade and parallel realization of the following casual LTI systems.

$$H(z) = \frac{(z+0.3)\,(z^2-1)}{(z+0.4)\,(z+0.5)\,(z-0.5)}$$

Solution (a) : Cascade Realization

$$H(z) = \frac{(z+0.3)\,(z^2-1)}{(z+0.4)\,(z+0.5)\,(z-0.5)}$$

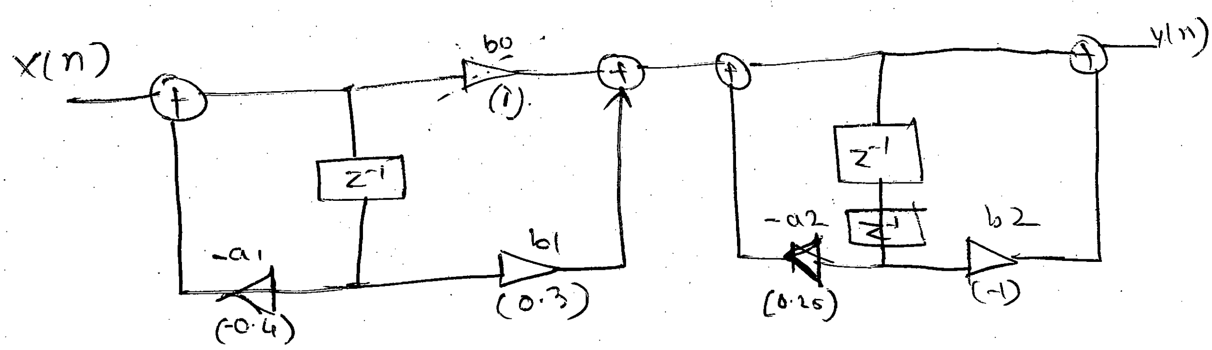
$$= \left[\frac{z+0.3}{z+0.4}\right]\left[\frac{z^2-1}{z^2-0.25}\right]$$

$$H(z) = H_1(z)\ H_2(z)$$

$$H_1(z) = \frac{z+0.3}{z+0.4} = \frac{1+0.3z^{-1}}{1+0.4z^{-1}} = \frac{b_0+b_1z^{-1}}{1+a_1z^{-1}}$$

$$H_2(z) = \frac{z^2-1}{z^2-0.25} = \frac{1-z^{-2}}{1-0.25z^{-2}} = \frac{b_0+b_2z^{-1}}{1+a_2z^{-2}}$$

x(n) —— [$H_1(z)$] —— [$H_2(z)$] —— y(n)



(b) Parallel realization,

$$H(z) = \frac{(z + 0.3)(z^2 - 1)}{(z + 0.4)(z + 0.5)(z - 0.5)}$$

$$\frac{H(z)}{z} = \frac{(z + 0.3)(z^2 - 1)}{z(z + 0.4)(z + 0.5)(z - 0.5)}$$

$$\frac{H(z)}{z} = \frac{A}{z} + \frac{B}{z + 0.4} + \frac{C}{z + 0.5} + \frac{D}{z - 0.5}$$

Where A =   C =

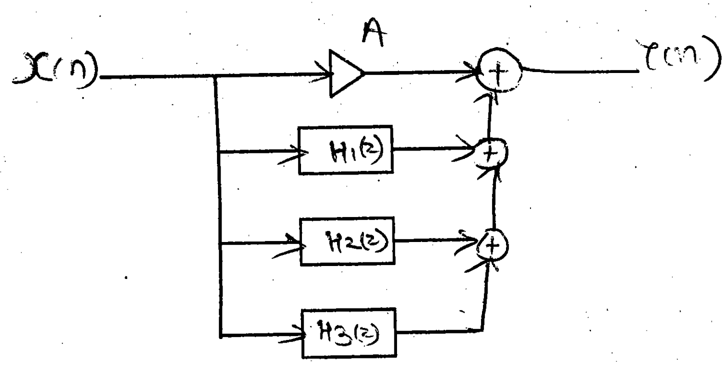
B =   D =

By substituting we get,

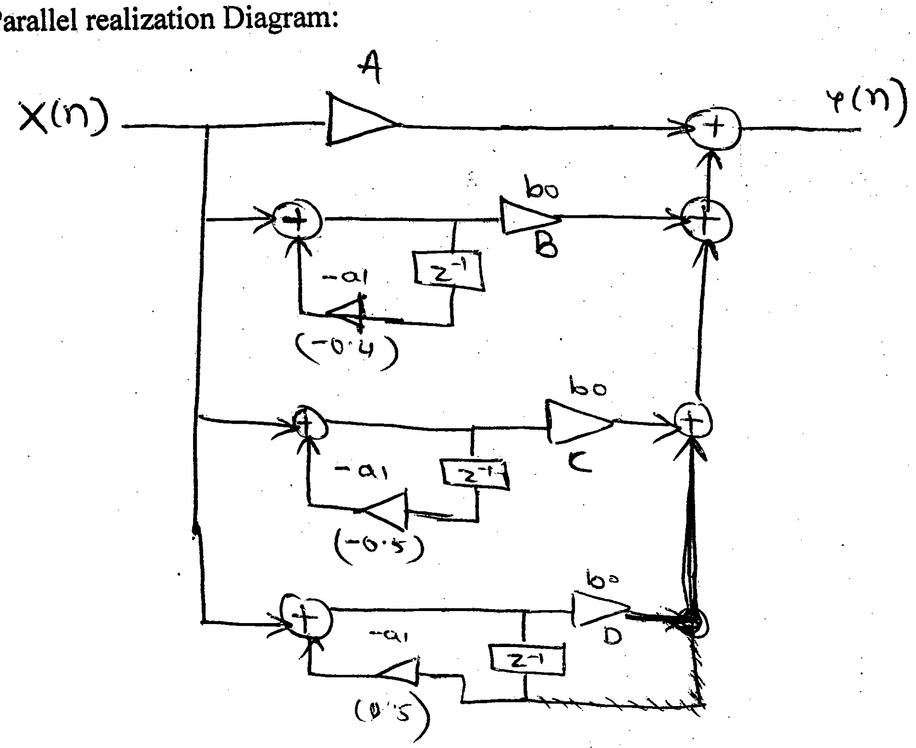
$$H(z) = A + \frac{Bz}{z + 0.4} + \frac{Cz}{z + 0.5} + \frac{Dz}{z - 0.5}$$

$$H(z) = A + \frac{B}{1 + 0.4z^{-1}} + \frac{C}{1 + 0.5z^{-1}} + \frac{D}{z - 0.5z^{-1}}$$

Let H(z) = A +



Parallel realization Diagram:



Q(33) Show cascade and parallel realization of the following casual LTI systems.

$$H(z) = \frac{10\,z\left(z-\frac{1}{2}\right)\left(z-\frac{2}{3}\right)(z+2)}{\left(z-\frac{3}{4}\right)\left(z-\frac{1}{8}\right)\left(z-\frac{1}{2}-j\frac{1}{2}\right)\left(z-\frac{1}{2}+j\frac{1}{2}\right)}$$ **(IMPORTANT)**

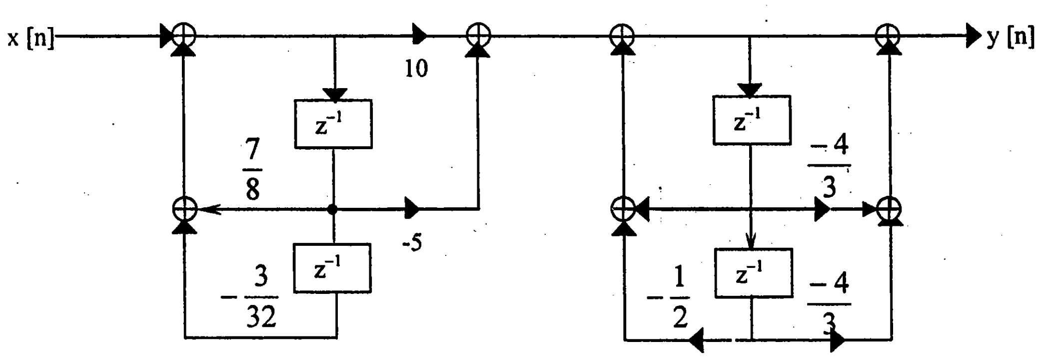
Solution:

a) Cascade Realization

$$H(z) = \left[\frac{10z\left(z-\frac{1}{2}\right)}{z^2-\frac{7}{8}z+\frac{3}{32}}\right]\left[\frac{\left(z-\frac{2}{3}\right)(z+2)}{z^2-z+0\cdot 5}\right]$$

$$H(z) = \left[\frac{10z^2-5z}{z^2-\frac{7}{8}z+\frac{3}{32}}\right]\left[\frac{z^2-\frac{4}{3}z-\frac{4}{3}}{z^2-z+0\cdot 5}\right]$$

Let H (z) = $H_1(z)$ $H_2(z)$



b) Parallel realization,

$$\frac{H(z)}{z} = \frac{10\left(z-\frac{1}{2}\right)\left(2-\frac{2}{3}\right)(z+2)}{\left(z-\frac{3}{4}\right)\left(z-\frac{1}{8}\right)\left(z-\frac{1}{2}-j\frac{1}{2}\right)\left(z-\frac{1}{2}+j\frac{1}{2}\right)}$$

$$H(z) = A\left[\frac{z}{z-\frac{3}{4}}\right] + B\left[\frac{z}{z-\frac{1}{8}}\right] + C\left[\frac{z}{z-p_1}\right] + D\left[\frac{z}{z-p_2}\right]$$

Where

$p_1 = \frac{1}{2} + j\frac{1}{2}$ $\qquad$ $p_2 = \frac{1}{2} - j\frac{1}{2}$

A = 2·93 $\qquad$ B = 17·68

C = 12·25 – j 14·57 $\qquad$ D = 12·25 + j 14·57

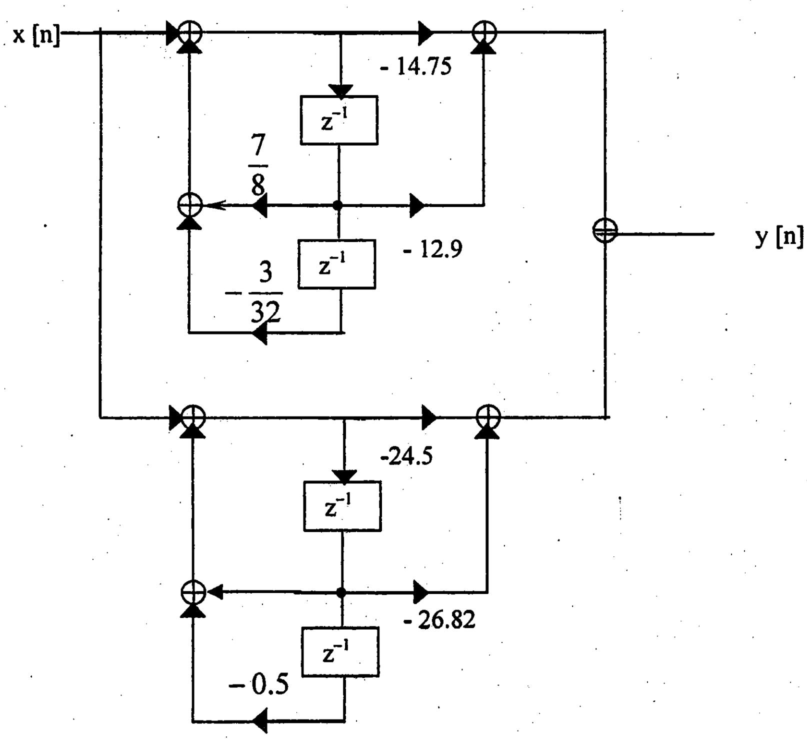
By substituting and simplifying we get,

$$H(z) = \left[\frac{-14\cdot 75z^2-12\cdot 90z}{z^2-\frac{7}{8}z+\frac{3}{32}}\right] + \left[\frac{24\cdot 50z^2+26\cdot 82\,z}{z^2-z+0\cdot 5}\right]$$

$$H(z) = \left[\frac{-14\cdot57 - 12\cdot90z^{-1}}{1 - \frac{7}{8}z^{-1} + \frac{3}{32}z^{-2}}\right] + \left[\frac{-24\cdot50 + 26\cdot82z^{-1}}{1 - z^{-1} + 0\cdot5z^{-2}}\right]$$

Let $H(z) = H_1(z) + H_2(z)$

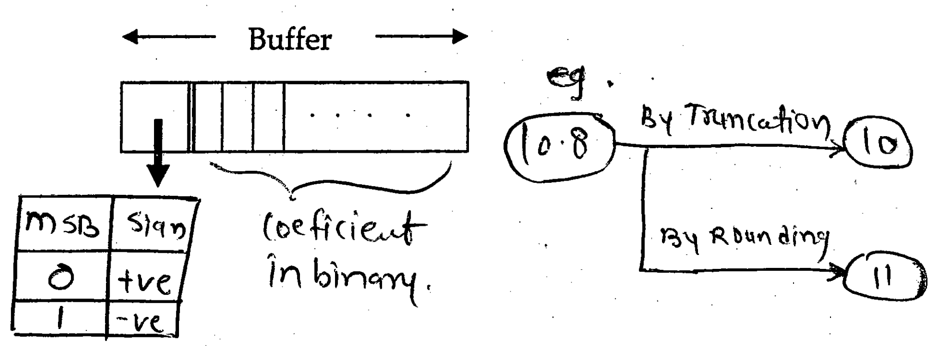
Parallel realization Diagram:



> Quantization by Truncation

Coefficients are stored in Buffer

For example :



✱✱✱✱ **Q(34)** A D T system has Transfer Function $H(z) = \dfrac{1}{1-0.8\,z^{-1} + 0.12\,z^{-2}}$

A four bit processor is used in which MSB represents sign bit and remaining three bits store quantized coefficients.

a) What is the effect of quantization on pole location if direct form is used for realization?

b) What is the effect of quantization on pole location if cascade form is used for realization?

c) In which case (direct / cascade form) the shift from the actual pole locations due to quantization is less.

Solution :

Given, $H(z) = \dfrac{1}{1-0.8\,z^{-1}+0.12\,z^{-2}} = \dfrac{z^2}{z^2-0.8z+0.12} = \dfrac{z^2}{(z-0.6)(z-0.2)}$

POLES : $P_1 = 0.6$ **and** $P_2 = 0.2$

(a) Direct form Realization

i.e. Single system Realization

$H(z) = \dfrac{1}{1-0.8\,z^{-1}+0.12\,z^{-2}} = \dfrac{1}{1+a_1z^{-1}+a_2z^{-2}}$ where $a_1 =$ $a_2 =$

(i) For $\boxed{a_1 = -0.8,}$ $-a_1 = +0.8$

By D2B,

0.8×2	$= \boxed{1}.6$	1
0.6×2	$= \boxed{1}.2$	1
0.2×2	$= \boxed{0}.4$	0

Coeff Stored in Buffer is ,

0	1	1	0

Now, Quantized coefficient

$-a'_1 = (+)\left(2^{-1}+2^{-2}+0\right)$

$= \frac{1}{2}+\frac{1}{4}$

$-a'_1 = 0.75$

$\therefore \boxed{a_1 = -0.75}$

(ii) For $\boxed{a_2 = 0.12,}$ $-a_2 = -0.12$

By D2B,

0.12×2	$= \boxed{0}.24$	0
0.24×2	$= \boxed{0}.48$	0
0.48×2	$= \boxed{0}.96$	0

Coeff Stored in Buffer is ,

1	0	0	0

Now, Quantized coefficient

$-a'_2 = (-)(0)$

$-a'_2 = 0$

$\boxed{a_2 = 0}$

(iii) The modified T.F. is given by,

$$H'(z) = \frac{1}{1+a'_1z^{-1}+a'_2z^{-2}} = \frac{1}{1-0.75\,z^{-1}+0} = \frac{Z}{Z-0.75}$$

Modifed POLES : $P'_1 = 0.75$ $P'_2 =$ Does not exist

(b) Cascade form Realization

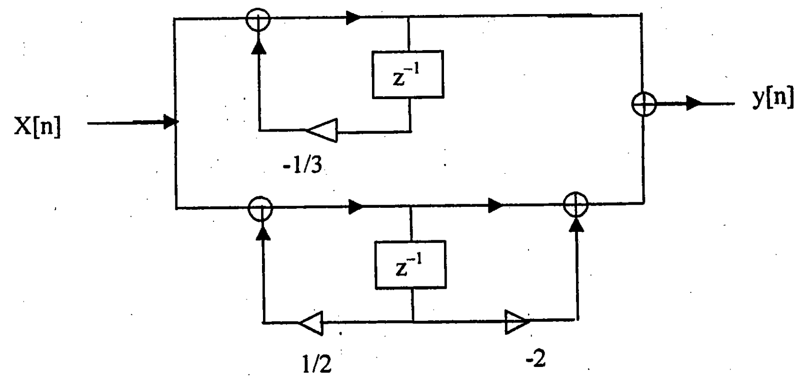
Given, $H(z) = \dfrac{1}{1-0.8\,z^{-1}+0.12\,z^{-2}} = \dfrac{z^2}{z^2-0.8z+0.12} = \dfrac{z^2}{(z-0.6)(z-0.2)}$

$$H(z) = \left[\frac{z}{z-0.6}\right]\left[\frac{z}{z-0.2}\right]$$

Let $H(z) = H_1(z)\; H_2(z)$

Q(35) Determine the transposed structure of the system shown below and verify that both the original and transposed system have the same system function.

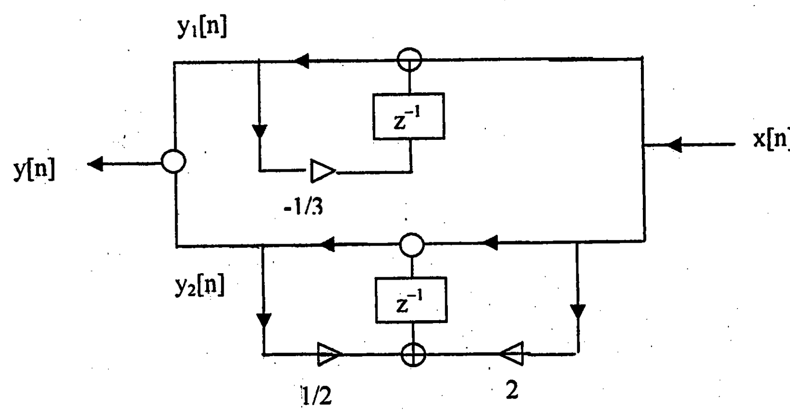
(IMP)



Solution :

(i) System function for original form. $H(z) = \dfrac{1}{1+\frac{1}{3}z^{-1}} + \dfrac{1+2z^{-1}}{1-\frac{1}{2}z^{-1}}$ -------- (eq-1)

(ii) The transposed form is given below.



(iii) System function for transposed form

$$y_1[n] = x[n] - \frac{1}{3} y_1[n-1]$$

$$\therefore\ y_1[n] + \frac{1}{3} y_1[n-1] = x[n]$$

$$\therefore\ Y_1(z)\left\{1+\frac{1}{3}z^{-1}\right\} = X(z) \Rightarrow H_1(z) = \frac{1}{1+\frac{1}{3}z^{-1}}$$

$$y_2[n] = 2x[n-1] + x[n] + \frac{1}{2} y_2[n-1]$$

$$y_2[n] - \frac{1}{2} y_2[n-1] = x[n] + 2x[n-1]$$

$$\therefore\ Y_2(z)\left\{1-\frac{1}{2}z^{-1}\right\} = X(z)\left\{1+2z^{-1}\right\}$$

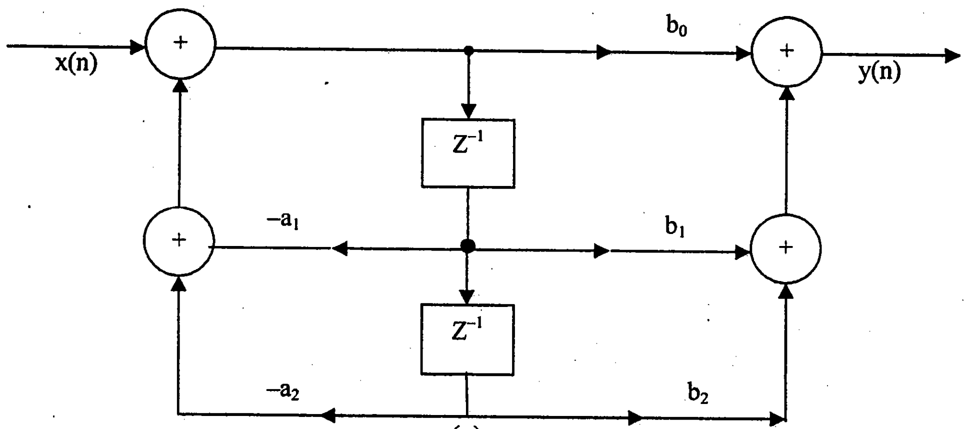
$$\therefore\ H_2(z) = \frac{Y_2(z)}{X(z)} = \frac{1+2z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$H(z) = H_1(z) + H_2(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1+2z^{-1}}{1-\frac{1}{2}z^{-1}}$$ ----------------- (eq-2)

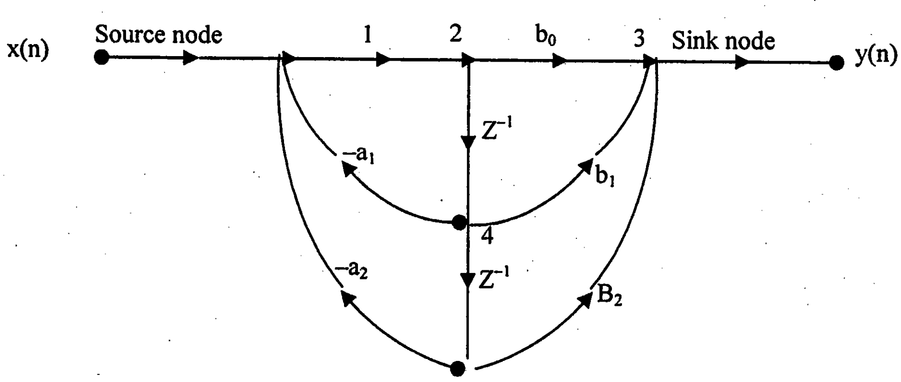
(i) From (eq-1) and (eq-2) , Both the system functions are exactly same. Hence it is verified that original and transposed system have the same system function.

A signal flow graph is basically a set of directed branches that connect at nodes.
This signal out of a branch is equal to the branch gain times the signal into the branch.
Furthermore, the signal at a node of a flow graph is equal to the sum of the signals from all branches connecting to the node.

Ex:



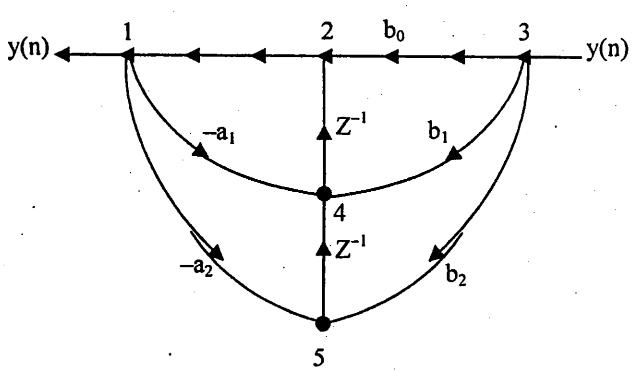
(a)

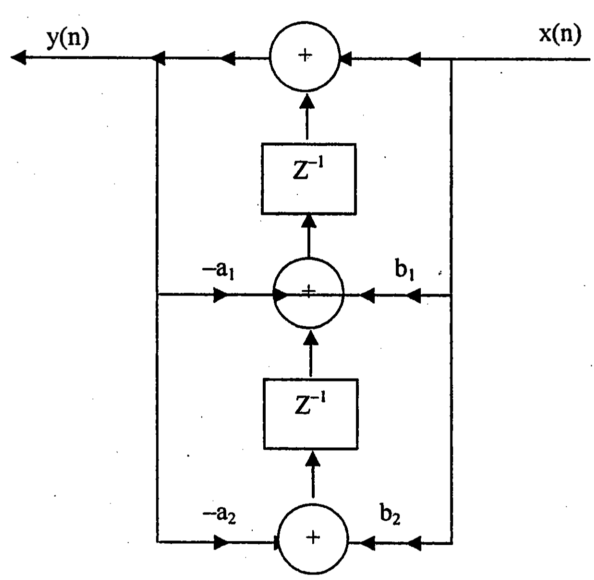


Transposition or flowgraph reversal theorem :

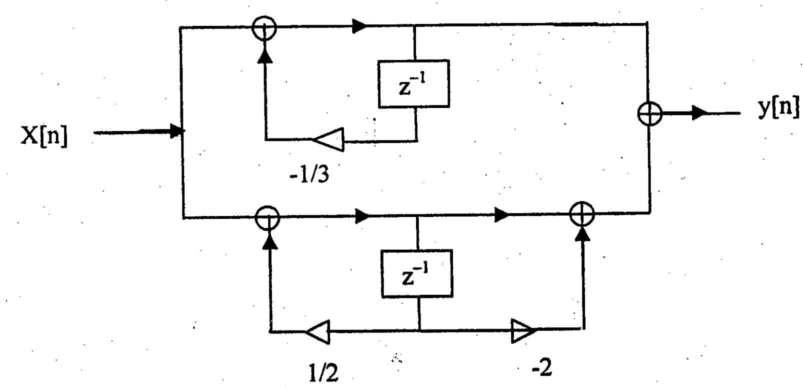
This theorem states that if we reverse the directions of all branch transmittances and interchange the input and output in the flowgraph, the system function remains unchanged.

The resulting structure is called a transposed structure signal flowgraph of transposed structure is shown in the following fig.





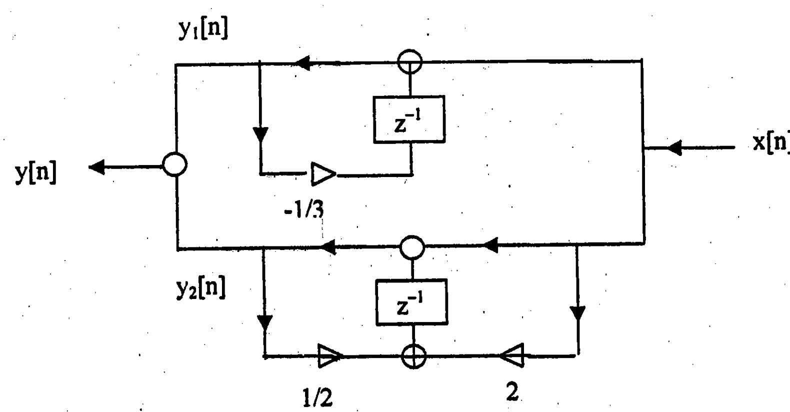
Q(36) Determine the transposed structure of the system shown below and verify that both the original and transposed system have the same system function.



Solution :

(i) System function for original form. $H(z) = \dfrac{1}{1+\frac{1}{3}z^{-1}} + \dfrac{1+2z^{-1}}{1-\frac{1}{2}z^{-1}}$ -------- (eq-1)

(ii) The transposed form is given below.



(ii) System function for transposed form

$$y_1[n] = x[n] - \frac{1}{3} y_1[n-1]$$

$$\therefore\ y_1[n] + \frac{1}{3} y_1[n-1] = x[n]$$

$$\therefore\ Y_1(z)\left\{1+\frac{1}{3}z^{-1}\right\} = X(z) \Rightarrow H_1(z) = \frac{1}{1+\frac{1}{3}z^{-1}}$$

$$y_2[n] = 2x[n-1] + x[n] + \frac{1}{2}y_2[n-1]$$

$$y_2[n] - \frac{1}{2}y_2[n-1] = x[n] + 2x[n-1]$$

$$\therefore\ Y_2(z)\left\{1-\frac{1}{2}z^{-1}\right\} = X(z)\left\{1+2z^{-1}\right\}$$

$$\therefore\ H_2(z) = \frac{Y_2(z)}{X(z)} = \frac{1+2z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$H(z) = H_1(z) + H_2(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1+2z^{-1}}{1-\frac{1}{2}z^{-1}}$$ ------------------ (eq-2)

(ii) From (eq-1) and (eq-2), Both the system functions are exactly same. Hence it is verified that original and transposed system have the same system function.

V POLE-ZERO DIAGRAM OF DIGITAL FILTERS

Q(37) Plot pole-zero diagram of the following systems.

a) $H(z) = 1 - 81\,z^{-4}$
b) $H(z) = 1 + 81\,z^{-4}$
c) $H(z) = 4 - 16\,z^{-4}$
d) $H(z) = 4 + 16\,z^{-4}$
e) $h[n] = (2)^n \cos(n\,\pi/4)\; u[n]$
f) $h[n] = (½)^n$ for $0 \le n \le 7$
g) $h[n] = \{ \underset{\uparrow}{1}, 2, 4, 8, 16, 32 \}$

Solution :

(a) $H(z) = 1 - 81z^{-4}$

$$= 1 - \frac{81}{z^{+4}}$$

$$\therefore H(z) = \frac{z^4 - 81}{z^4}$$

Poles : $z^4 = 0$ $\therefore$

system has 4 poles at origin

ZEROS : $Z^4 - 81 = 0$;

$Z^4 = 3^4 (1)$

$Z^4 = 3^4 e^{j2\pi k}$

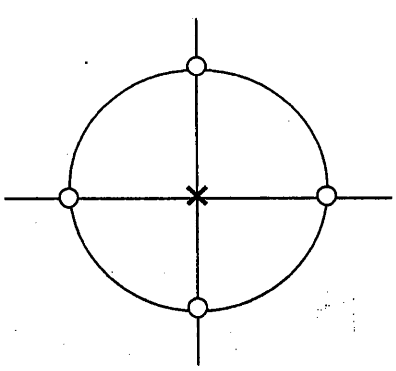
$\therefore Z_k = 3e^{j\pi k/2}$ $k = 0, 1, 2, 3$

$k = 0, \;\; \therefore Z_o = 3,$

$k = 1, \;\; \therefore Z_1 = 3 \angle \frac{\pi}{2},$

$k = 2, \;\; \therefore Z_2 = 3 \angle \pi,$

$k = 3, \;\; \therefore Z_3 = 3 \angle \frac{3\pi}{2}$



(f) $h[n] = (0.5)^n$; $0 \le n \le 7$

$$H(z) = \sum_{n=0}^{7} (0.5)^n z^{-n}$$

$$= \sum_{n=0}^{7} (0.5\, z^{-1})^n$$

$$= \frac{1 - (0.5z^{-1})^8}{1 - 0.5z^{-1}}$$

$$= \frac{z^8 - (0.5)^8}{z^7 (z - 0.5)}$$

ZEROS : $Z^8 - (0.5)^8 = 0$

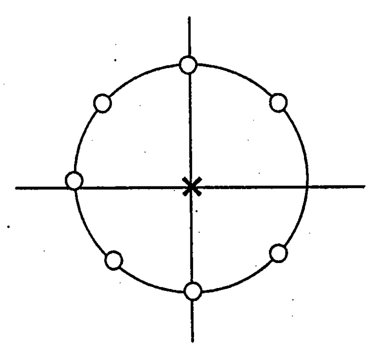
$Z^8 = (0.5)^8$

$Z^8 = (0.5)^8 e^{j2\pi k}$

$Z^8 = (0.5)^8 e^{j\pi k}$, $k = 0, 1, \ldots 7$

$k = 0$, $Z_o = 0.5$
$k = 1$, $Z_1 = 0.5 \angle \pi/4$
$k = 2$, $Z_2 = 0.5 \angle \pi/2$
$k = 3$, $Z_3 = 0.5 \angle 3\pi/4$
$k = 4$, $Z_4 = 0.5 \angle \pi$
$k = 5$, $Z_5 = 0.5 \angle 0\pi/4$
$k = 6$, $Z_6 = 0.5 \angle 3\pi/2$
$k = 7$, $Z_7 = 0.5 \angle 7\pi/4$

The ZERO at $Z_o = 0.5$ cancels the pole at 0.5



VI STABILITY OF DIGITAL FILTERS :-

I) Time Domain Test : Bounded Input Bounded output (**BIBO**) Test :-

A linear Time Invariant system is BIBO stable if and only if, $\underbrace{\sum_{-\infty}^{\infty} |h[n]| < \infty}_{finite}$

Now, $H(z) = \sum_{n=-\infty}^{\infty} h[n]\ z^{-n}$

$$|H(z)| = \sum_{-\infty}^{\infty} \left| h[n]\ z^{-n} \right| = \sum_{-\infty}^{\infty} |h[n]|\, |z^{-n}|$$

When evaluated on the unit circle ie $|z| = 1$

$$|H(z)| = \sum_{-\infty}^{\infty} |h[n]| < \infty \text{ for stable system.}$$

That means H (z) must contain the unit circle within Roc. Therefore a LTI system is BIBO stable if and only if the ROC of the system includes the unit circle.

Proof of stability property

Let h [n] be the impulse response of filter,
Then y [n] = x [n] * h [n]
y [n] = $\sum$ x [m] h[n – m]

$$|y[n]| = \sum_{-\infty}^{\infty} |x[m]|\, |h[n-m]|$$

If Input is Bounded , ie $\sum_{-\infty}^{\infty} |x[m]| \le Mx$. [Let Mx be any finite constant]

$$|y[n]| = \sum_{n=-\infty}^{\infty} |h[n-m]| . Mx$$

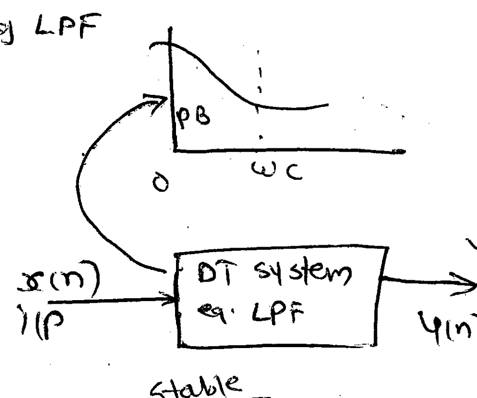
If $\sum_{n=-\infty}^{\infty} |h[n-m]| < \infty$

Then $|y[n]| < \infty$ ie Bounded output

That means, if $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

Then the system produces bounded output for bounded input.

eg LPF



$y(n) = x(n) * h(n)$

$y(n) = \sum_{-\infty}^{\infty} x(m)\, h(n-m) \underbrace{< \infty}_{\text{finite.}}$

Bounded I/P $\sum_{-\infty}^{\infty} |x(n)| < \infty$

Bounded O/P $\sum_{-\infty}^{\infty} |y(n)| < \infty$

V Imp

II) Transform Domain Test :

If ROC includes unit circle then system is stable.

For causal and stable system, all the poles must lie inside the unit circle. $|\text{pole}| < 1$

For anti-causal and stable system, all the poles must lie outside the unit circle. $|\text{pole}| > 1$

Classification of DT

- can be Practically Implemented → **Causal System**
  - If output of the system depends on **Past/Present Input/Output** values, Then system is Causal System.
  - $h(n) \leftarrow$ Causal signal.
- cannot be implemented practically. → **Non-Causal System**
  - Anti Causal system: $h(n) \leftarrow$ Anti Causal Signal
  - Bothsided system: $h(n) \leftarrow$ Both sided signal.

Case -1 When h[n] is Causal Signal, System is Causal System

Eg $h[n] = (a)^n\, u[n]$

By ZT, $H(z) = \dfrac{z}{z-a}$

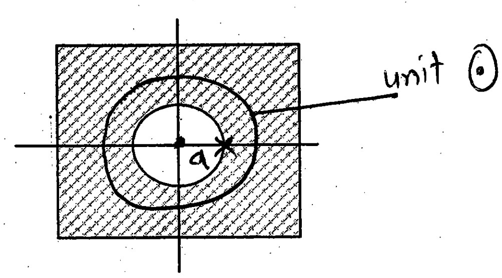
POLE : $\boxed{Z = a}$

By BIBO,

$$\sum_{-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} |a^n|$$

$$= \underbrace{\left[\frac{1}{1-a}\right]}_{\text{finite}} \ast \text{ provided } |a| < 1 \text{ i.e. } |\text{pole}| < 1$$

= System is stable.



NOTE :

For causal and stable system all the POLES must lie INSIDE the unit circle.

i.e. $|\text{POLE}| < 1$

Case -2 When h[n] is Anti-Causal Signal, System is Anti-Causal System

Eg $\quad h[n] = (a)^n\, u[-n-1]$

By ZT, $H(z) = \dfrac{-z}{z-a} \quad |z| \le |a|$

POLE : $z = a$

By BIBO,

$$\sum_{-\infty}^{\infty} \left| h[n] \right| = \sum_{-1}^{-\infty} \left| a^n \right|$$

Put $n = -m$

$$= \sum_{1}^{\infty} \left| a^{-m} \right|$$

$$= \sum_{1}^{\infty} \left| (a^{-1})^m \right|$$

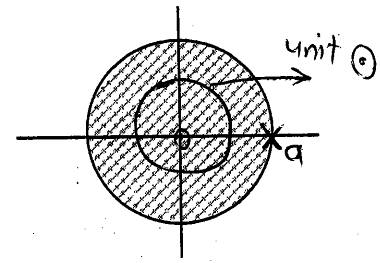
$$= \frac{a^{-1}}{1-a^{-1}} \quad \text{provided } |a^{-1}| < 1$$

$1/|a| < 1$

$\therefore |a| > 1$

$\therefore$ System is stable

$\therefore |\text{Pole}| > 1$



NOTE :

For ~~Anti-causal~~ and stable system all the POLES must lie OUTSIDE the unit circle.

i.e. $\quad |\,\text{POLE}\,| > 1$

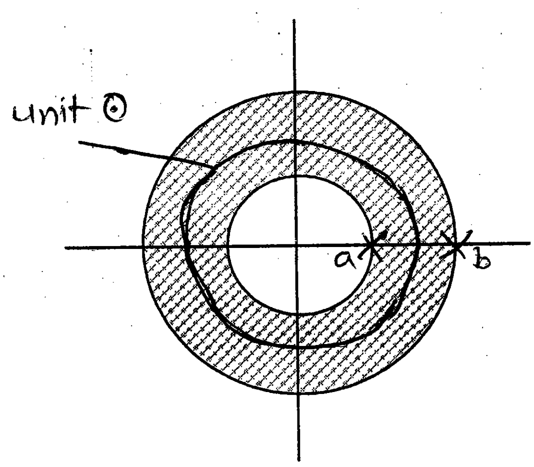
Case -3 When h[n] is Both-sided Signal, System is Both-sided System

eg $\quad h[n] = a^n u(n) + b^n u(-n-1)$

By ZT, $H(z) = \dfrac{z}{z-a} + \dfrac{-z}{z-b}$

$\boxed{|b| > |z| > |a|}$ $\quad$ $\boxed{|z| > |a|}$ $\quad$ $\boxed{|z| < |b|}$

Bothsided & Stable $\quad$ causal If $|a| < 1$ then stable $\quad$ Anti-causal If $|b| > 1$ then stable.



Q(38) Determine whether the following causal systems are stable or not

a) $H(z) = \dfrac{z}{(z+½)(z-¼)}$ $\quad$ b) $H(z) = \dfrac{z(z-2)}{(z-½)(z-2)}$ $\quad$ c) $H(z) = \dfrac{z}{z^2 - 2z + 1}$

d) $H(z) = \dfrac{z}{(z-0.5)(z-1)}$ $\quad$ e) $H(z) = \dfrac{z}{(z-2.5)(z-1)}$ $\quad$ f) $H(z) = \dfrac{z^8 - 3^8}{z^7(z-3)}$ $\quad$ g)

$H(z) = \dfrac{z(z-1)}{(z-½)(z-0.6-j0.6)(z-0.6+j0.6)}$

Solution :

(a) $H(z) = \dfrac{z}{(z+½)(z-¼)}$ (i) POLES : $P_1 = -1/2 \quad P_2 = 1/4$. (ii) ROC : $\lvert z\rvert > 1/2$ All poles are inside unit ⊙ ∴ system is stable.	(b) $H(z) = \dfrac{z\,\cancel{(z-2)}}{(z-½)\,\cancel{(z-2)}}$ (i) POLES : $P_1 = 1/2$ ~~$P_2 = 2$~~ (ii) ROC : $\lvert z\rvert > \lvert 1/2\rvert$ Pole is inside unit ⊙ System is stable.
(c) $H(z) = \dfrac{z}{z^2-2z+1} = \dfrac{z}{(z-1)^2}$ (i) POLES : $P_1 = 1,\ P_2 = 1$ (ii) ROC : $\lvert z\rvert > 1$ Poles on unit ⊙ ∴ system is Marginally stable.	(d) $H(z) = \dfrac{z}{(z-0.5)(z-1)}$ (i) POLES : $P_1 = 0.5 \quad P_2 = 1$ (ii) ROC : marginally stable.
(e) $H(z) = \dfrac{z}{(z-2.5)(z-1)}$ (i) POLES : $P_1 = 2.5 \quad P_2 = 1$ (ii) ROC : $2.5 > \lvert z\rvert > \lvert 1\rvert$ Not stable	(g) $H(z) = \dfrac{z(z-1)}{(z-½)(z-0.6-j0.6)(z-0.6+j0.6)}$ (i) POLES : $p_1 = ½$ $p_2 = 0.6 + j\,0.6 \quad \lvert P_2\rvert = \sqrt{.72}$ $p_3 = 0.6 - j\,0.6 \quad P_3 = \sqrt{0.72}$ (ii) ROC $\lvert z\rvert > \sqrt{.72}$ All POLES are INSIDE the unit Circle Therefore System is STABLE

Q(39) Given $H(z) = \dfrac{1}{(1+0.5\,z^{-1})\,(1-2\,z^{-1})}$

a) State all possible convergence regions.

b) What is the convergence condition if the sequence h[n] is causal? Whether such system be stable? Why or Why not? Find h[n] in this case.

c) If h[n] is both sided sequence, state the condition of convergence and find h[n] in this case. State whether the system is stable.

Solution :

$$H(z) = \frac{z^2}{(z+0.5)(z-2)}$$

$$\therefore\ H(z) = \frac{1}{5}\left(\frac{z}{z+\frac{1}{2}}\right) + \frac{4}{5}\left(\frac{z}{z-2}\right)$$

(a) All possible convergence regions are : $|z| > 2$, $\quad |z| < 0.5$, $\quad 0.5 < |z| < 2$

(b) If h[n] is causal ⇒ system is causal ∴ convergence condition is $|z| > 2$.

$$h[n] = \frac{1}{5}\left(\frac{-1}{2}\right)^n u[n] + \frac{4}{5}\,(2)^n\,u[n]$$

(c) If h[n] is both sided sequence, ROC is $0.5 < |z| < 2$

$$h[n] = +\frac{1}{5}\left(\frac{-1}{2}\right)^n u[-n-1] - \frac{4}{5}(2)^n u[-n-1]$$

Q(40) Given $H(z) = \dfrac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}}$

Specify the ROC and determine h[n] for the specified conditions.
a) The system is stable b) The system is causal. c) The system is anti-causal.

Solution :

$$H(z) = \frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}} = \frac{3z^2-4z}{z^2-3.5z+1.5} \qquad \therefore H(z) = \frac{z(3z-4)}{(z-\frac{1}{2})(z-3)}$$

$$\therefore \left(\frac{H(z)}{Z}\right) = \frac{3z-4}{(z-\frac{1}{2})(z-3)} = \frac{A}{z-\frac{1}{2}} + \frac{B}{Z-3} \quad \text{where } A = 1 \text{ and } B = 2$$

$$\therefore H(z) = \left(\frac{z}{z-\frac{1}{2}}\right) + 2\left(\frac{z}{z-3}\right)$$

i) For stable system, ROC should include unit circle i.e. ROC : $\frac{1}{2} < |z| < 3$

$$h[n] = \left(\frac{1}{2}\right)^n u[n] - 2(3)^n u[-n-1]$$

ii) For causal system, ROC : $|z| > 3$ $\qquad h[n] = \left(\frac{1}{2}\right)^n u[n] + 2(3)^n u[n]$

iii) For anticausal system, ROC : $|z| < ½$ $\qquad h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - 2(3)^n u[-n-1]$

Q(41) A certain DT system is stable and has transfer function as given below, find the impulse response of the system. $H(z) = \dfrac{z^3}{(z - 0.2)(z - 0.5)(z - 2)}$

Solution : $\dfrac{H(z)}{Z} = \dfrac{Z^2}{(Z-0.2)(Z-0.5)(Z-2)}$

BY PFE,

$$\therefore H(z) = 0.24\underbrace{\left(\frac{Z}{Z-0.2}\right)}_{\text{Causal}} - 1.5\underbrace{\left(\frac{Z}{Z-0.5}\right)}_{\text{Causal}} + 1.48\underbrace{\left(\frac{Z}{Z-2}\right)}_{\text{AntiCausal}}$$

$\boxed{2 > |z| > 0.5}$

By iZT,

$h(n) = 0.24(0.2)^n u(n) - 1.5(0.5)^n u(n) - 1.48(2)^n u(-n-1)$

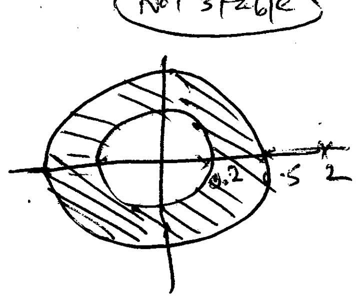
(Bothsided)

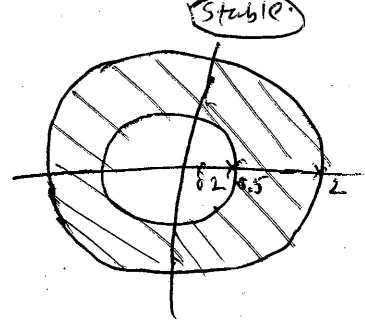
Ref Poles $P_1 = 0.2$ $P_2 = 0.5$ $P_3 = 2$

i) Causal $|z| > 2$ Not stable

ii) Anticausal $|z| < 0.2$ Not stable

iii) Bothsided (a) $\boxed{0.5 > |z| > 0.2}$ Not stable (b) $\boxed{2 > |z| > 0.5}$ Stable





VII THE FREQUENCY RESPONSE OF THE SYSTEM.

Definition. The *frequency response* of an LTI filter is defined as the spectrum of the output signal divided by the spectrum of the input signal.

The frequency Response of the system is determined by giving the sinusoidal input to the system. For sinusoidal input, output signal is also sinusoidal of the same frequency as the input signal. Thus in passing through the system the input signal is subjected only to an **amplitude scaling and a phase shift.**

The ratio of the output signal amplitude to the input signal amplitude is the amplitude response at the input frequency.

The output phase minus the input phase is the phase response is the phase response at the input frequency.

Consider a sampled sinusoidal sequence, $x[n] = A\ e^{j(nw+\theta)}$ as the input to the DT system.

The response of the system y[n] with impulse h[n] for input sequence x[n] is given by,

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

$$= \sum h[m]\, A.e^{j(wn+\theta)}$$

$$y[n] = A.\ e^{j(wn+\theta)} \sum_{-\infty}^{\infty} h[m]e^{-jmw}$$

$$= x[n].\sum_{-\infty}^{\infty} h[m]e^{-jmw}$$

Let $y[n] = x[n] . G(w)$ Where $G(w) = \sum_{-\infty}^{\infty} h[m]e^{-jmw}$

The output sequence y[n] is the input sequence x[n] multiplied by a complex weighting factor G(w) that completely depends on input frequency w.

The multiplier G (w) is called the frequency response of the system.

Since y[n] is also a sampled sinusoidal, we then have,

$$y[n] = B\ e^{j(wn+\phi)} = x[n]\ G(w)$$

$$B\ e^{j(wn+\phi)} = A\ e^{j(wn+\theta)}.G(w)$$

$$G(w) = \frac{B e^{j(wn+\phi)}}{A\ e^{j(wn+\theta)}} = \frac{B}{A}\ e^{j(\phi-\theta)}.$$

$$|G(W)| = \frac{B}{A} = \frac{\textit{Magnitude of Numerator}}{\textit{Magnitude of Denominator}}$$

$$\angle G(W) = \phi - \theta = \text{. Angle of Numerator Angle of Denominator}$$

The frequency response of the system can be obtained Analytically OR Graphically.

(I) Analytical Method :

The frequency response can be obtained from its Z – Transform. By substituting $Z = e^{jw}$.

$$G(W) = H(z)\Big|_{z=e^{jwt}} = H(e^{jwT})$$

i) $\text{Magnitude Response} = \dfrac{\text{Magnitude of numerator}}{\text{Magnitude of Denominator}}$

Where $\text{Magnitude} = \sqrt{(\text{Real})^2 + (\text{Imaginary})^2}$

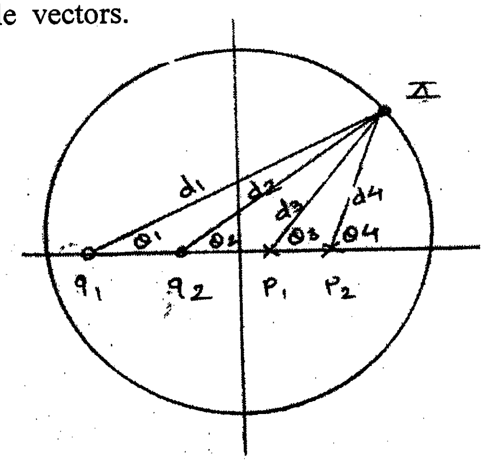
ii) Phase Response = Angle of Numerator – Angle of denominator

$$\text{Where angle} = \begin{cases} \tan^{-1}\left(\dfrac{\text{Imaginary}}{\text{Real}}\right) & \text{When Real} > 0 \\ 180 + \tan^{-1}\left(\dfrac{\text{Imaginary}}{\text{Real}}\right) & \text{When Real} < 0 \end{cases}$$

II) Graphical Method :

In Graphical method, the frequency response at a given frequency w is determined by the ratio of the product of the zero vectors with the product of pole vectors.

$$\text{Magnitude Response} = \frac{\text{Product of distance from zeros}}{\text{Product of distance from poles}}$$



Example :

$$\text{Consider } H(z) = \frac{(z-q_1)(z+q_2)}{(z-p_1)(z-p_2)}$$

$$\text{Magnitude Response} = \frac{d_1 . d_2}{d_3 . d_4}$$

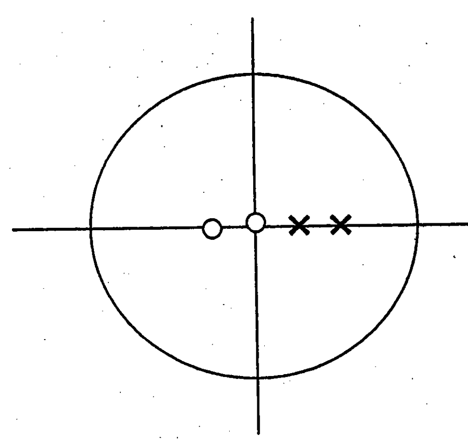
Phase Response = Summation of angles from ZEROS – Summation of angles from POLES.

$$\text{Phase Response} = (\theta_1 + \theta_2) - (\theta_3 + \theta_4)$$

Q(42) Given $H(z) = \dfrac{z(z+0.2)}{(z-0.5)(z-0.3)}$. Find Magnitude Response and Phase Response at w = 0 and w=π using graphical method.

Solution:

Given $H(z) = \dfrac{z(z+0.2)}{(z-0.5)(z-0.3)}$



POLES : $P_1 = 0.5$ and $P_2 = 0.3$
ZEROS : $Z_1 = 0$ and $Z_2 = -0.2$

(i) At $w = 0$

$$H(w) = \frac{d_1\, d_2}{d_3\, d_4} = \frac{(1)\,(1.2)}{(0.5)(0.7)} = \frac{1.2}{0.35} = \boxed{\quad}$$

(ii) At $w = \pi$

$$H(w) = \frac{d_1\, d_2}{d_3\, d_4} = \frac{(1)\,(0.8)}{(1.5)(1.3)} = \frac{0.8}{1.95} = \boxed{\quad}$$

Q(43) Given h[n] = $(0.5)^n$ u[n]. Plot Magnitude spectrum and Phase spectrum of the Low Pass Filter.

Solution:

Given $h[n] = (0.5)^n u[n]$. Causal infinite length.

(Infinite Impulse Response)

By ZT,

$$H(\ ) = \frac{z}{z-0.5}$$

Put $z = e^{j\omega}$

$$H(e^{jw}) = \frac{e^{jw}}{e^{jw}-0.5}$$

Put $e^{j\omega} = \cos(\omega) + j\sin(\omega)$

$$H(w) = \frac{\cos(\omega) + j\sin(\omega)}{[\cos(\omega) - 0.5] + j\sin(\omega)}.$$

To evaluate set calculator in complex mode, radians mode.

$$\left((\cos A + \mathbf{i} \sin A) \div ((\cos A - 0.5) + \mathbf{i} \sin A) \right) r \angle \Theta$$

Then CALC for w = 0 , 0.2π , 0.4π , 0.6π , 0.8π , π

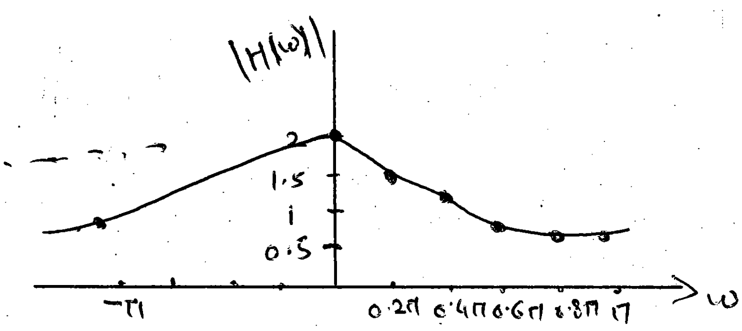
	Freq.	Magnitude	Phase
1	0	2	0
2	0.2π	1.5	−0.46
3	0.4π	1.03	−0.51
4	0.6π	0.8	−0.39
5	0.8π	0.68	−0.2
6	π	0.66	0

NOTE:

(i) For IIR filter
Take Step Size = 0.2π

(ii) For FIR filter
Take Step Size = 0.1π

(a) Magnitude spectrum



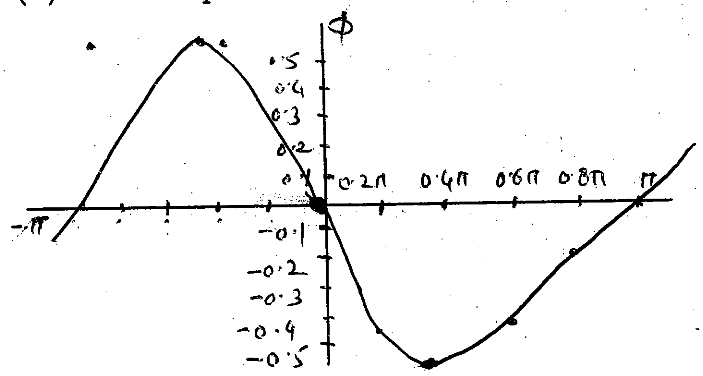
Magnitude Spectrum is :

(i) Continuous function of w

(ii) Symmetric about w = 0

(iii) Periodic with Period = 2π

(b) Phase spectrum



Phase Spectrum is :

(i) Continuous function of w

(ii) Anti-Symmetric about w = 0

(iii) Periodic with Period = 2π

Q(44) Given $H(z) = \dfrac{0.7z^2 - 0.252}{z^2 - 0.1\ z - 0.72}$. Find Magnitude Response and Phase Response of the filter.

Solution :

Given $H(z) = \dfrac{0.7z^2 - 0.252}{z^2 - 0.1\ z - 0.72}$

Put $z = e^{jw}$

$$H(e^{jw}) = \frac{0.7e^{j2w} - 0.252}{e^{j2w} - 0.1\ e^{jw} - 0.72}$$

$$H(e^{jw}) = \frac{0.7(\cos(2w) + j\sin(2w)] - 0.252}{[\cos(2w) + j\sin(2w)] - 0.1\ [\cos(w) + j\sin(w)] - 0.72}$$

$$H(e^{jw}) = \frac{[0.7(\cos(2w) - 0.252] + j\,[0.7\sin(2w)]}{[\cos(2w) - 0.1\cos(w) - 0.72] + j[\sin(2w) - j\ 0.1\sin(2w)]}$$

$$\underbrace{|H(e^{jw})|}_{\text{Magnitude Response}} = \frac{\sqrt{[0.7\cos(2w) - 0.252]^2 + [0.7\sin(2w)]^2}}{\sqrt{[\cos(2w) - 0.1\cos(w) - 0.72]^2 + j\,[\sin(2w) - 0.1\sin(w)]^2}}$$

$$\underbrace{\underline{|H(e^{jw})}}_{\text{Phaze response}} = \tan^{-1}\left[\frac{0.7\sin(2w)}{0.7\cos(2w) - 0.252}\right] - \tan^{-1}\left[\frac{\sin(2w) - 0.1\sin(w)}{\cos(2w) - 0.1\cos(w) - 0.72}\right]$$

Q(45) Let h[n] = { 0.2, 0.4, 0.5, 0.9 }. What is the magnitude response of the system at w = 0 and at w = π using Analytical and Graphical Method.

Q(46) Given $H(e^{jw}) = e^{-j3w}\,[\,2 + 1.8\cos(3w) + 1.2\cos(2w) + 0.5\cos(w)\,]$. Find h[n].

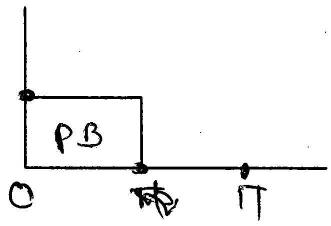
Q(47) Find the squared magnitude response of the IIR filter which has two poles at $P_{1,2} = (0.8)\ e^{\pm j\pi/4}$ and two zeros at the origin. Plot it wrt frequency and identify the filter type.

Q(48) Given h[n] = { 1, 2, 2, 1 }. Plot the magnitude and phase response of the system.

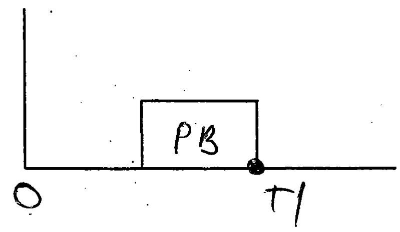
Viva

★ Ideal Characteristics of Digital Filters

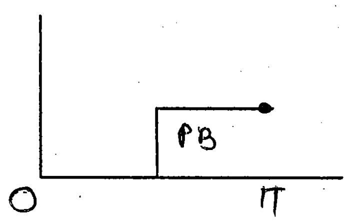
(iii) Low Pass Filter



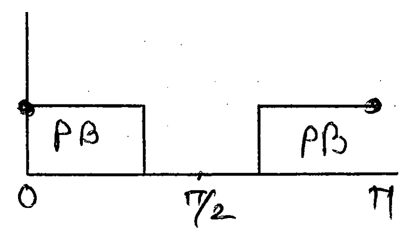
(iii) Band Pass Filter



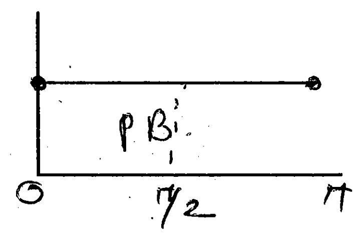
(iv) High Pass Filter



(iv) Band Stop Filter (ie Band Reject Filter)



(v) All Pass Filter



Q(49) Identify the following filters based on pass-band.

a) $H[z] = \dfrac{z}{z-0.5}$ b) $H(z) = 1 - z^{-2}$ c) $H[z] = \dfrac{Z+2}{Z+0.5}$ d) $H[z] = \dfrac{3+2z^{-1}+z^{-2}}{1+2z^{-1}+3z^{-2}}$

Solution :

a) $H[z] = \dfrac{z}{z-0.5}$

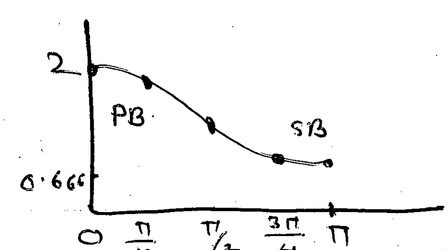
(i) At $w = 0$, $z = 1$

$H(w) = \dfrac{1}{1-0.5} = \boxed{2}$

(ii) At $w = \pi$, $z = -1$

$H(w) = \dfrac{-1}{-1-0.5} = 0.666$

Magnitude Spectrum



∴ Filter is low pass filter

b) $H(z) = 1 - z^{-2} = \dfrac{z^2 - 1}{z^2}$

(i) At $w = 0$ $z = 1$

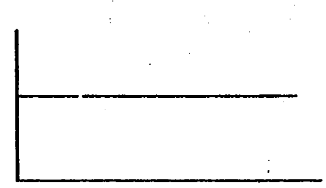
$H(w) = 0$

(ii) At $w = \pi$ $z = -1$

$H(w) = 0$

(iii) At $w = \pi/2$ $z = j$

$$H(w) = \frac{j^2 - 1}{j^2} = \frac{-1-1}{-1} = 2$$



To Find Steady State Response (SSR)

(i) Find the input signal frequencies

$W = \{0, \pi/5, \frac{4\pi}{5}\}$

(ii) Find the Frequency Response

Now, $H(z) = \frac{z}{z - 0.5}$ Put $z = e^{jw} = \cos w + j \sin w$

$$H(w) = \frac{\cos(\omega) + j\sin(\omega)}{(\cos(\omega) - 0.5) + j\sin(\omega)}$$

(iii) Find magnitude and Phase value for every input signal frequency

1) At $w = 0$ $\quad H(w) = 2 \angle 0$

2) At $w = \frac{\pi}{5}$ $\quad H(w) = 1.5 \angle -0.46$

3) At $w = \frac{4\pi}{5}$ $\quad H(w) = 0.68 \angle -0.20$.

(iv) Find Steady state Response (SSR)

The SSR of the system to the input $x[n] = 10 - 5\sin\left(\frac{\pi n}{5}\right) + 20\cos\left(\frac{4\pi n}{5}\right)$ is given by, $y[n] = 10(2) - 5(1.5)\sin\left(\frac{n\pi}{5} - 0.46\right)$

$$+ 20(0.68)\cos\left(\frac{n4\pi}{5} - 0.20\right)$$

$$y(n) = 20 - 7.5\sin\left(\frac{n\pi}{5} - 0.46\right) + 13.6\cos\left(n\frac{4\pi}{5} - 0.20\right)$$

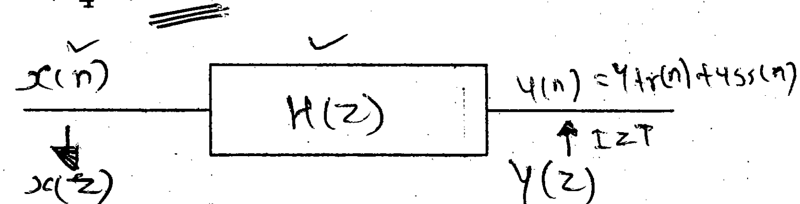
Ans

Amp Freq. Phase → Amp(magn) (Freq) (Phase + Phase)

> Case-2 : If the exponential or sinusoidal signal is applied to the system at some finite time instant, say at n = 0, the response of the system consists of two terms, the **Transient Response** and the **Steady State Response.**
>
> The transient response decays toward zero as n extends to ∞.
>
> The output of the system is given by $y[n] = y_{tr}[n] + y_{ss}[n]$

Q(51) Given $H(z) = \frac{z}{z - 0.5}$. Find the response of the system to the input $x[n] = 10\cos\left(\frac{n\pi}{4}\right) u[n]$.

Solution :



Here sinusoidal I/P signal is applied at n = 0.

(i) For $x[n] = 10\cos\left(n\frac{\pi}{4}\right)u(n)$

By ZT,

$$X(z) = 10\left[\frac{z^2 - z\cos(\frac{\pi}{4})}{z^2 - 2z\cos\left(\frac{\pi}{4}\right)+1}\right] = 10\left[\frac{z^2 - 0.707\,z}{z^2 - 1.414\,z + 1}\right]$$

(ii) Let $Y(z) = H(z)\,X(z)$

$$Y(z) = \frac{z}{z-0.5}\,\frac{10\,z\,(z-0.707)}{z^2 - 1.44z + 1}$$

$$Y(z) = \frac{10\,z^2(z-0.707)}{(z-0.5)\,(z-p_1)\,(z-p_2)} \quad \text{Where} \quad P_1 = 0.707 + j0.707 = e^{j\frac{\pi}{4}}$$

$$P_2 = 0.707 - j0.707 = e^{-j\frac{\pi}{4}}$$

By PFE,

$$\frac{Y(z)}{z} = \frac{A}{z-0.5} + \frac{B}{z-P_1} + \frac{C}{z-P_3}$$

where $A = -1.035$ $\quad B = 6.78\,e^{-j28.68}$ and $\quad C = 6.78\,e^{j28.68}$

$$Y(z) = A\left[\frac{z}{z-0.5}\right] + B\left[\frac{z}{z-P_1}\right] + C\left[\frac{z}{z-P_3}\right]$$

By iZT,

$$y[n] = A\,(0.5)^n u[n] + B\,(P_1)^n u[n] + C\,(P_3)^n u[n].$$

$$y[n] = -1.035\,(0.5)^n u[n] + 6.78\,e^{-j28.67}\left(e^{j\frac{\pi}{4}}\right)^n u[n] + 6.78\,e^{j28.68}\left(e^{j\frac{\pi}{4}}\right)^n u[n]$$

ANS : $\mathbf{y[n] = -1.035\,(0.5)^n\,u[n] + 13.56}\cos\left[n\frac{\pi}{4} - 28.68^0\right]u[n]$

Q(52) Given $H(z) = \frac{z}{z-0.5}$ Find the steady state and transient responses of the system to the input $x[n] = 10\cos\left(\frac{n\pi}{4}\right)$ $-\infty < n < \infty$

Q(53) Given $H(z) = \frac{z}{z-0.5}$ Find the steady state and transient responses of the system to the input $x[n] = 10\cos\left(\frac{n\pi}{4}\right)\,u[n]$. Same as Q. 51

Q(54) Given $h[n] = \{\underset{\uparrow}{1}, 0.5, 0, -0.5, -1\}$. Find the response of the system to the input $x[n] = (\frac{1}{4})^n \cos\left(\frac{n\pi}{2}\right)$

Q(55) Given $h[n] = \{\underset{\uparrow}{1}, 0.5, 0, -0.5, -1\}$. Find the response of the system to the input $x[n] = (\frac{1}{4})^n \cos\left(\frac{n\pi}{2}\right)\,u[n]$.

3.8 Linear Invariant System as Frequency Selective Filters

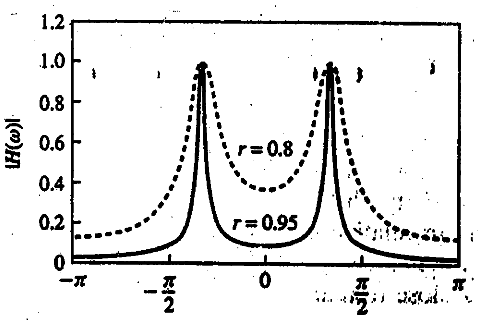
[1]. DIGITAL RESONATOR

A digital resonator is a special two pole Band-pass filter with the pair of complex conjugate poles located near the unit circle as shown below. Digital resonator is a narrowband band-pass filter. The name resonator refers to the fact that the filter has a large magnitude response (ie it resonates) in the vicinity of the pole location. The angular position determines the resonant frequency of the filter. Digital resonators are useful for bandpass filtering, speech generation etc.

Design of digital resonator : To have a peak at $\omega = \omega_o$ select the complex conjugate poles at $p_1 = r\,e^{j\omega o}$ and $p_2 = r\,e^{-j\omega o}$ and zeros at $Z_1 = 1$ and $Z_2 = -1$

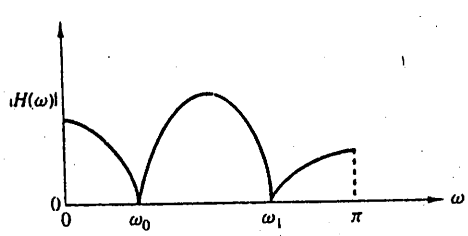
The system function and the magnitude response is given by,

$$H(z) = \frac{G(z-1)(z+1)}{(z-re^{j\omega_o})(z-re^{-j\omega_o})} = \frac{G(z-1)(z+1)}{z^2 - 2\,r\,z\cos(\omega_o) + r^2}$$



[2]. NOTCH FILTER

A notch filter is a filter that contains one or more deep notches or ideally a perfect nulls in the frequency response characteristics. The following figure illustrates the frequency characteristics of a notch filter with nulls at frequencies ω_o and ω_1.

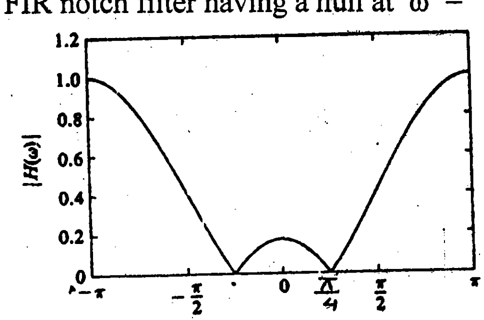


Notch filters are useful in many applications where the specific frequency components must be eliminated. For example, instrumentation and recording systems require that the power line frequency of 60 Hz and its harmonics be eliminated.

To create a null in the frequency response of a filter at a frequency ω_o, we simply introduce a pair of complex-conjugate zeros on the unit circle at an angle ω_o.
That is $Z_1 = e^{j\omega o}$ $Z_2 = e^{-j\omega o}$

Thus the system function of FIR notch filter is simply, $H(z) = \dfrac{G\,(z-e^{j\omega_o})(z-e^{-j\omega_o})}{z^2}$

Magnitude response of FIR notch filter having a null at $\omega = \dfrac{n\,\pi}{4}$



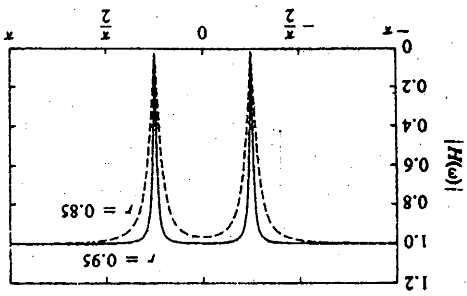
The problem with FIR notch filter is that the other frequency components around the desired NULL are severely attenuated.
Suppose a pair of complex conjugate poles at at $p_1 = r\,e^{j\omega o}$ and $p_2 = r\,e^{-j\omega o}$

The effect of pole is to introduce a resonance in the vicinity of the null that reduces the bandwidth of the null.

The system function of the resulting filter is

$$H(z) = \frac{G\,(z-e^{j\omega_o})(z-e^{-j\omega_o})}{(z-re^{j\omega_o})(z-re^{-j\omega_o})} = \frac{z^2 - 2\,z\cos(\omega_o) + 1}{z^2 - 2\,r\,z\cos(\omega_o) + r^2}$$

The magnitude response of the filter with $\omega_0 = \frac{\pi}{4}$ and r = 0.85.



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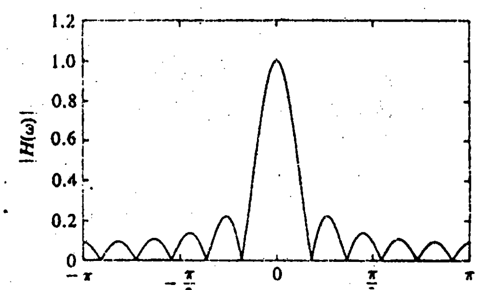
[3]. COMB FILTER

Comb filter is a notch filter in which the nulls occur periodically across the frequency band. Comb filter has applications in a wide range of practical systems such as in the rejection of power line harmonics.

Comb filter has zeros on the unit circle at, $z = e^{\hat{j}\frac{2\pi k}{M+1}}$ k=1,2.. M

Ex. Magnitude response of comb filter with M = 10.



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[4]. ALL PASS FILTER

An all pass filter is defined as a system that has a constant magnitude response for all frequencies, that is $|H(\omega)| = 1$ for $0 \le \omega \le \pi$

The simplest example of an all pass filter is a pure delay system with system function H(z) $= z^{-k}$. All pass filter passes all signals without modification. It simply adds delay of k samples.

All pass filter is also described by the system function

$$H(z) = \frac{a_N + a_{N-1}\,z^{-1} + a_{N-2}\,z^{-2} + \dots + a_1\,z^{-N+1} + z^{N}}{1 + a_1\,z^{-1} + a_2\,z^{-2} + \dots + a_N\,z^{N}}$$ where all coefficients are real.

If we define polynomial A(z) as $A(z) = \sum_{k=0}^{N} a_k\,z^{-k}$ with $a_o = 1$ Then H(z) can be expressed as

$$A(z) = z^{-N}\,\frac{A(z^{-1})}{A(z)}$$

Since $|H(w)|^2 = H(z)\ H(z^{-1})|_{z=e^{jw}} = 1$, the system is an all pass filter.

All pass filter finds application as Phase Equalizer. When placed in cascade with a system that has an undesired phase response, a phase equalizer is designed to compensate for the poor phase characteristics of the system to produce an overall Linear phase Response.

In All Pass filter POLES and ZEROS are always in reciprocal order.

---

[6]. DIGITAL SINUSOIDAL OSCILLATOR

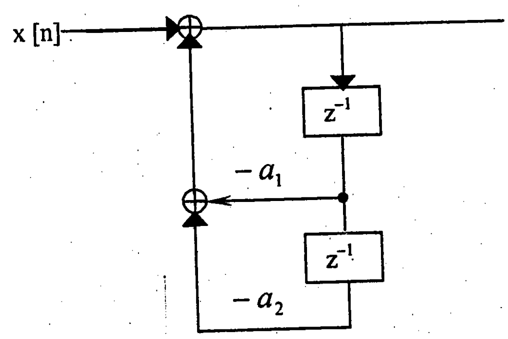
A Digital sinusoidal oscillator can be viewed as a limiting form of a two pole resonator for which the complex conjugate poles lie on the unit circle.
Consider the complex poles at $P_1 = r\,e^{j\omega o}$ and $P_2 = r\,e^{-j\omega o}$

System function $H(z) = \dfrac{b_o\, z}{z^2 - 2\, r\, z \cos(\omega_o) + r^2}$

If $b_o =$ is set to A $\sin(w_o)$ Then $h[n] = A \sin(n\, w_o)\, u[n]$

Thus the impulse response of the second order system with complex conjugate poles on the unit circle is is a sinusoid and the system is called a Digital Sinusoidal generator.
A digital sinusoidal oscillator is a basic component of a digital systhesizer.

Block Diagram Representation :



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[7]. INVERTIBILITY OF LINEAR TIME INVARIANT SYSTEM

A system is said to be invertible if there is a one to one correspondence between its input and output signals. The inverse system with input y[n] and output x[n] is denoted by 'T^{-1}.

Then, $w[n] = h_I[n] * h[n] * x[n] = x[n]$

This implies that, $h[n] * h_I[n] = \delta[n]$

By ZT, $H(z)\;\; H_I(z) = 1$

$$H_I(z) = \frac{1}{H(z)}$$

If H(z) has a rational system function, $H(z) = \dfrac{B(z)}{A(z)}$

Then Inverse system $H_I(z) = \dfrac{A(z)}{B(z)}$

Thus the zeros of H(z) become the poles of the inverse system and vice versa.
Furthermore, if H(z) is FIR system then $H_I(z)$ is an all pole system or if H(z) is an all pole system then $H_I(z)$ is FIR system.

EX-1 Determine the inverse of the system with impulse response $h[n] = \left(\frac{1}{2}\right)^n u[n]$.

Solution : . $h[n] = (0.5)^n u[n]$

By ZT $\quad H(z) = \dfrac{z}{z - 0.5} \quad$ Let $\quad H_I(z) = \dfrac{1}{H(z)} = 1 - 0.5z^{-1}$

By iZT, $h_I[n] = \delta[n] - \frac{1}{2}\delta[n-1]$.

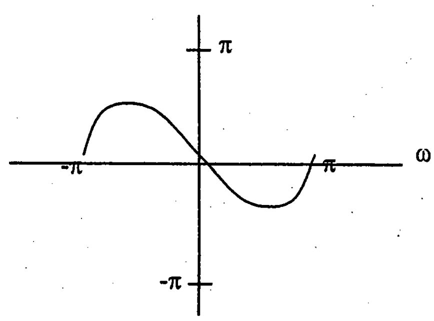
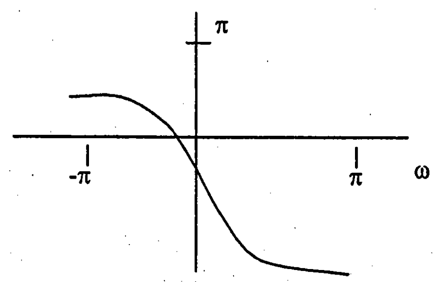
Ex-2 Determine the inverse of the system with impulse response $h[n] = \delta[n] - \frac{1}{2}\delta[n-1]$.

Solution. : By ZT, $\quad H(z) = 1 - \frac{1}{2}z^{-1},\ |z| > 0$

$$H_I(z) = \frac{1}{H(z)} = \frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{z}{z - \frac{1}{2}} \qquad |z| > \frac{1}{2}$$

[8] MINIMUM PHASE, MAXIMUM PHASE AND MIXED PHASE SYSTEM

Consider two FIR Systems characterized by system functions,

(I)		(II)	
	$H_1(z) = 1+\frac{1}{2}z^{-1} = = \left(\frac{z+\frac{1}{2}}{z}\right)$ By iZT $h_1[n] = \{1, \frac{1}{2}\}$		$H_2(z) = \frac{1}{2} + z^{-1} = \frac{1}{2}\left(\frac{z+2}{z}\right)$ By iZT, $h_2[n] = \{1/2, 1\}$
i)	Zero: $z_1 = -½$	i)	Zero: $z_1 = -2$
ii)	Magnitude Response : $H_1(z) = \left(\frac{z+\frac{1}{2}}{z}\right) = \left(\frac{e^{jw}+\frac{1}{2}}{e^{jw}}\right)$ $H_1(z) = \left(\frac{\cos(w)+j\sin(w)+\frac{1}{2}}{\cos(w)+j\sin(w)}\right)$ $\lvert H_1(w)\rvert = \sqrt{\frac{5}{4}+\cos(w)}$	ii)	Magnitude Response : $H_2(z) = \frac{1}{2}\left(\frac{z+2}{z}\right) = \frac{1}{2}\left(\frac{e^{jw}+2}{e^{jw}}\right)$ $H_2(z) = \frac{1}{2}\left(\frac{cow(w)+j\sin(w)+2}{cow(w)+j\sin(w)}\right)$ $\lvert H_2(w)\rvert = \sqrt{\frac{5}{4}+\cos(w)}$
iii)	Phase Response: $\theta_1(w) = \tan^{-1}\left(\frac{\sin(w)}{\frac{1}{2}+\cos(w)}\right) - w$ 	iii)	Phase Response : $\theta_2(w) = \tan^{-1}\left[\frac{\sin(w)}{1+\frac{1}{2}\cos(w)}\right] - w$ 

For $H_1(w)$ with zero inside the unit circle, the net phase change $\theta_1(\pi) - \theta_1(0) = 0$ i.e. minimum phase. For $H_2(w)$ with zero outside the unit circle, the net phase change, $\theta_2(\pi) - \theta_2(0) = \pi$ i.e. max. phase. Therefore, $H_1(z)$ is minimum phase system and $H_2(z)$ is maximum phase system.

FIR system with M zeros, $H(w) = b_0 (1-z_1 e^{-jw}) (1-z_2 e^{-jw})$ -------------- $(1 - Z_m e^{-jw})$

When all zeros are inside the unit circle, each term in the product corresponds to a real valued zero, will undergo a net phase change of zero between $\omega = 0$ and $\omega = \pi$. Also each pair of complex conjugate factors in H(w) will undergo a net phase change of zero.

Therefore, $\angle$ **H(π) – $\angle$ H(0) = 0** Hence, the system is called a **Minimum Phase** System.

On the other hand, when all zeros are outside the unit circle, a real valued zero will contribute a net change of 'π' radians as the freq. varies from $\omega = 0$ to $w = \pi$ and each pair of complex conjugate zero will contribute a net change of 2π radians.

Therefore, $\angle$ **H(π) – $\angle$ H(0) = M. π** Hence the system is called **Maximum Phase** System.
Which is the Largest possible phase change for FIR system with M zeros. Hence the system is called Maximum Phase System.

FIR system with M zeros, if some of the zeros are inside the unit circle and remaining zeros outside the unit circle, then the system is called mixed phase system or non-minimum phase system.

Minimum Phase characteristic implies a min. delay function while a max. phase characteristic impulses that the delay characteristic is also maximum.

Q(56) Determine the zeros and indicate whether the system is Min. Phase, Max. Phase or Mixed Phase.

(a) $H_1(z) = 6 + z^{-1} - z^{-2}$
(b) $H_2(z) = 1 - z^{-1} - 6z^{-2}$
(c) $H_3(z) = 1\ 5/2\ z^{-1} - 3/2\ z^{-2}$
(d) $H_4(z) = 1 + 5/2\ z^{-1} - 2/3\ z^{-2}$

Solution :

(a) $H_1(z) \Rightarrow Z_1, Z_2 :\ -½, 1/3$: Min. Phase.
(b) $H_2(z) \Rightarrow Z_{1,2} = -2, 3$: Max. Phase.
(c) $H_3(z) \Rightarrow Z_{1,2} = -1/2, 3$: Mixed Phase
(d) $H_4(z) \Rightarrow Z_{1,2} = -2, 1/3$: Mixed Phase.

Q(57) Determine whether the following systems are of Minimum Phase, Miximum Phase or Mixed Phase type.

(a) $H_1(z) = z^2 + 2z - 8$ (b) $H_2(z) = 3z^2 + \frac{1}{2}z - \frac{1}{2}$

(c) $H_3(z) = 1 + 2z^{-1} + \frac{3}{4}z^{-2}$ (d) $H_4(z) = 1 + \frac{7}{8}z^{-1} + \frac{2}{6}z^{-2}$

Solution:

(a) $H_1(z) = z^2 + 2z - 8 = (z+4)(z-2)$
Hence, the zeros are at $z = -4$ and $z = 2$.
As both the zeros are outside the unit circle, this system is Maximum Phase System.

(b) $H_2(z) = 3z^2 + \frac{1}{2}z - \frac{1}{2} = (3z-1)\left(z + \frac{1}{2}\right)$

Hence the zeros are at $z = \frac{1}{3}$ and $z = -\frac{1}{2}$.
For this system, both the zeros are inside the unit circle, hence it is Minimum Phase System.

(c) $H_3(z) = 1 + 2z^{-1} + \frac{3}{4}z^{-2} = (z + \frac{3}{2})(z + \frac{1}{2})$

Hence, the zeros are at $z = -\frac{3}{2}$ and $= -\frac{1}{2}$.
For this system one zero is inside the unit circle and one is outside the unit circle, hence the system is Mixed Phase System.

(d) $H_4(z) = 1 + \frac{7}{6}z^{-1} + \frac{2}{6}z^{-2} = (z + \frac{2}{3})\ (z + \frac{1}{2})$

Hence, the zeros are at $z = -\frac{2}{3}$ and $z = -\frac{1}{2}$.
i.e. both the zeros are inside the unit circle, hence the system is Minimum Phase.

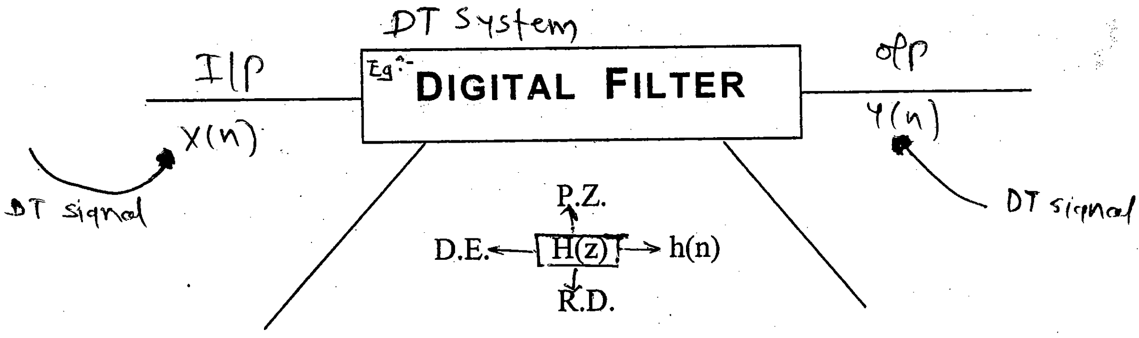
****

DIGITAL FILTERS

Priority 🔒 : **Definitely Everything** √ : **You must do this**

$: **You should not leave this** 🕒 : **If possible, do it**

	EXAM	IT	ELX	COMP	EXTC	INSTRU
1	May-2004	10	10	--	--	--
2	Dec-2004	08	08	38	54	60
3	May-2005	34	43	20	70	50
4	Dec-2005	32	28	28	50	64
5	May-2006	15	00	20	47	10
6	Dec -2006	43		25	50	50
7	May-2007	40	07	40		
8	Dec -2007	48	05	45	54	
9	May-2008	24	05	32		
AVERAGE						



I I R Filter

e.g. $h[n] = (1/2)^n u(n)$

Causal signal
Infinite length
Stable.

F I R Filter

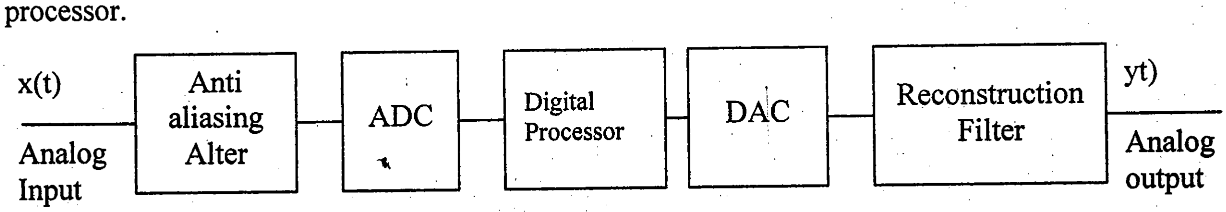
e.g. $h[n] = \{1, 2, 3, 4\}$

Causal signal
Finite length
Stable.

Q(1) What is digital filter ?

Digital filter is a Discrete Time System which produces a discrete time output sequence y[n] for the discrete time input sequence x[n]. Digital filter is nothing but mathematical algorithm implemented in hardware or software.

Real time digital filter consist of processing of real time signal using digital device called digital processor.



As shown in figure, analog input signal is band limited using anti-aliasing filter which is then sampled and DT signal thus obtained is converted into digital signal using ADC. Digital processor, perform the operation depending upon the algorithm programmed in digital processor.

The output of the digital processor is converted into analog signal using DAC. Reconstruction filter is used to obtain the corresponding analog signal from the output DT signal.

Q(2) Explain the advantage and disadvantages of discrete time filter over the analog filter.

	Parameter	Analog filter	Digital Filter
1	Input/output signal	Analog	Digital(discrete time sequences)
2	Composition	Lumped elements such as R.L. and C or analog /C_s	Software + digital hardware
3	Filter representation	In terms of system components	By difference equation
4	Flexibility	Not flexible	Highly flexible
5	Portability	Not easily portable	Portable
6	Design objective and result	Specifications to values of R.L and C components	Specifications to difference equation
7	Environmental effects	Negligible effect of environmental parameters	Negligible effect of environmental parameters
8	Interference notes and other effects	Maximum effect	Minimum/negligible effect
9	Storage/maintenance failure	Difficult storage and maintenance and higher failure rate	Easier storage and maintenance and reduced failure rate

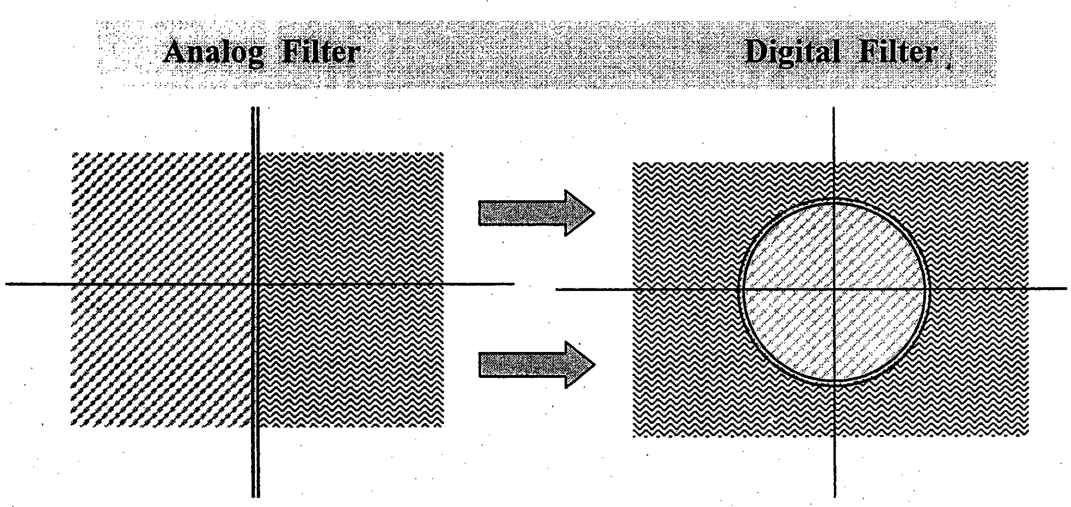
---

Q(3) What is the requirement of design of digital IIR filter ?

Digital I I R filters are designed from Analog filters. The designed filter must be causal and stable. An analog filter H(s) is stable if all the poles lie in the left half of s-plane.

To obtain stable digital filter from stable analog filter, the filter design technique should have the following properties.

1. The jΩ axis in the s-plane should map onto the unit circle in the z-plane. This gives a direct relationship between the two frequency variables in the two domains.
2. The left-half plane of the s-plane should map into the inside of the unit circle in the z-plane to convert a stable analog filter into a stable digital filter.



Q(4) What are the Advantages of digital filters-?

The following list gives some of the main advantages of digital over analog filters.

1. **A digital filter is *programmable*,** i.e. its operation is determined by a program stored in the processor's memory. This means the digital filter can easily be changed without affecting the circuitry (hardware). An analog filter can only be changed by redesigning the filter circuit. (i.e. **Flexibility in parameter setting**)
2. **Digital filters are easily *designed, tested* and *implemented*** on a general-purpose computer or workstation.

3. The characteristics of analog filter circuits (particularly those containing active components) are subject to drift and are dependent on temperature. Digital filters do not suffer from these problems, and so are **extremely *stable*** with respect both to time and temperature.
4. Unlike their analog counterparts, **digital filters can handle *low frequency signals* accurately**. As the speed of DSP technology continues to increase, digital filters are being applied to high frequency signals in the RF (radio frequency) domain, which in the past was the exclusive preserve of analog technology.
5. **Digital filters are very much more *versatile* in their ability to process signals in a variety of ways**; this includes the ability of some types of digital filter to adapt to changes in the characteristics of the signal.

Q(5) What are Advantages of FIR Filters-?

1) **They can easily be designed to be "linear phase"** (and usually are). Put simply, linear-phase filters delay the input signal, but don't distort its phase.
2) **They are simple to implement**. On most DSP microprocessors, the FIR calculation can be done by looping a single instruction.
3) **They are suited to multi-rate applications**. By multi-rate, we mean either "decimation" (reducing the sampling rate), "interpolation" (increasing the sampling rate), or both. Whether decimating or interpolating, the use of FIR filters allows some of the calculations to be omitted, thus providing an important computational efficiency. In contrast, if IIR filters are used, each output must be individually calculated, even if it that output is discarded (so the feedback will be incorporated into the filter).
4) **They have desirable numeric properties**. In practice, all DSP filters must be implemented using "finite-precision" arithmetic, that is, a limited number of bits. The use of finite-precision arithmetic in IIR filters can cause significant problems due to the use of feedback, but FIR filters have no feedback, so they can usually be implemented using fewer bits, and the designer has fewer practical problems to solve related to non-ideal arithmetic.
5) **They can be implemented using fractional arithmetic**. Unlike IIR filters, it is always possible to implement a FIR filter using coefficients with magnitude of less than 1.0. (The overall gain of the FIR filter can be adjusted at its output, if desired.) This is an important consideration when using fixed-point DSP's, because it makes the implementation much simpler.

Q(6) What are the *disadvantages* of FIR Filters (compared to IIR filters)?

Compared to IIR filters, FIR filters sometimes have the disadvantage that they require more memory and/or calculation to achieve a given filter response characteristic.

Q(7) What are the advantages of IIR filters (compared to FIR filters)?
IIR filters can achieve a given filtering characteristic using less memory and calculations than a similar FIR filter.

Q(8) What are the disadvantages of IIR filters (compared to FIR filters)?

1) They are more susceptible to problems of finite-length arithmetic, such as noise generated by calculations, and limit cycles. (This is a direct consequence of feedback: when the output isn't computed perfectly and is fed back, the imperfection can compound.)
2) They are harder (slower) to implement using fixed-point arithmetic.
3) They don't offer the computational advantages of FIR filters for multirate (decimation and interpolation) applications.

Q(9) The physically realizable and stable IIR filter cannot have a linear phase. Justify.

➡ The physically realizable and stable IIR filter cannot have a linear phase. For a filter to have a linear phase, the condition is $h[n] = h[N-1-n]$ and the filter would have a mirror image pole outside the unit circle for every pole inside the unit circle. This results in an unstable filter. As a result, a causal and stable IIR filter cannot have a linear phase.

Q(10) Compare FIR filters and IIR filters

	FIR filter	IIR filter
1	Provides exact linear phase.	Not linear phase.
2	Provides good stability.	Stability is not guaranteed.
3	Order required is higher.	Order required is lower.
4	Computationally not efficient.	Computationally more efficient.
5	More memory required for the storage of coefficients.	Less memory required fro storage of coefficients.
6	Requires more processing time.	Requires less processing time.
7	Requires M multiplications per output sample	Requires 2M + 1 multiplications per output sample.

➤ IIR FILTER DESIGN

Time Domain Method

(1) Impulse Invariant Method

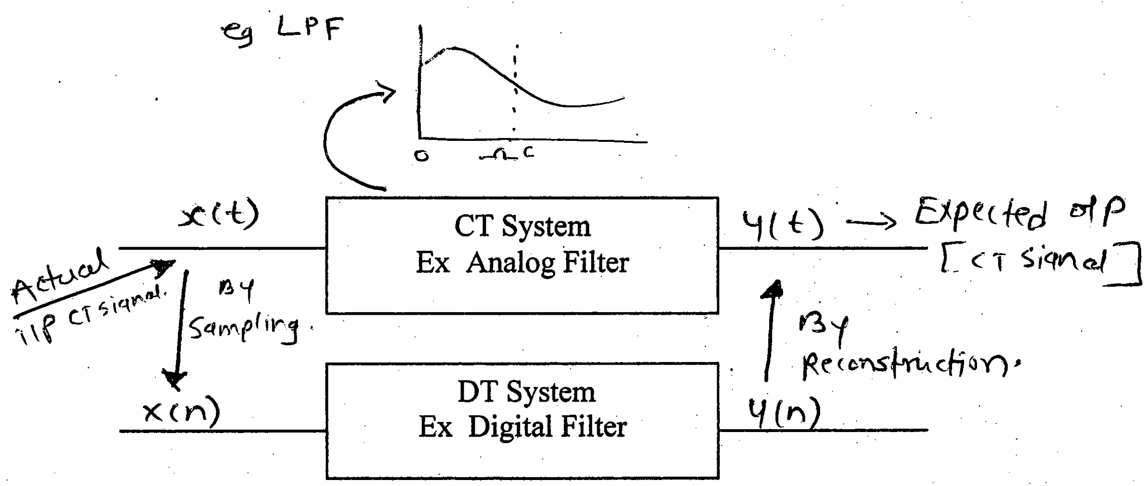
(2) Step Invariant Method

If not mentioned in problem Do by ⟶

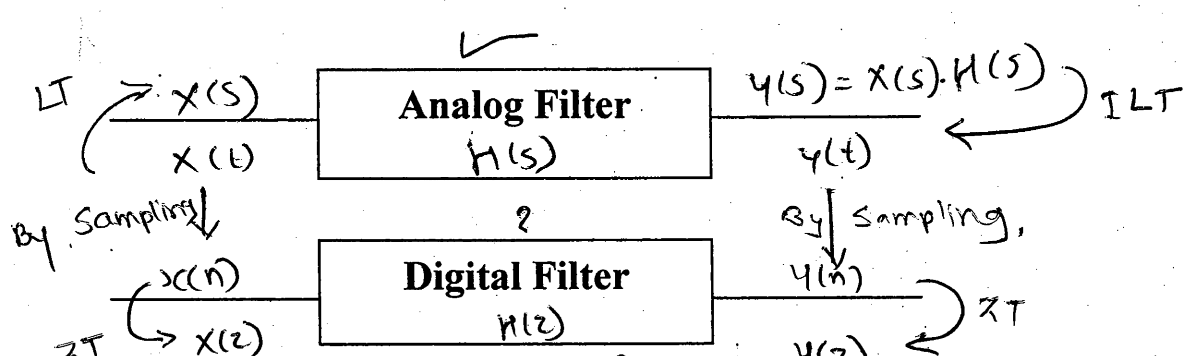
Transform Domain Method

(1) Bilinear transformation (BLT) Method

Note (I) Concept of Design of Digital IIR Filter



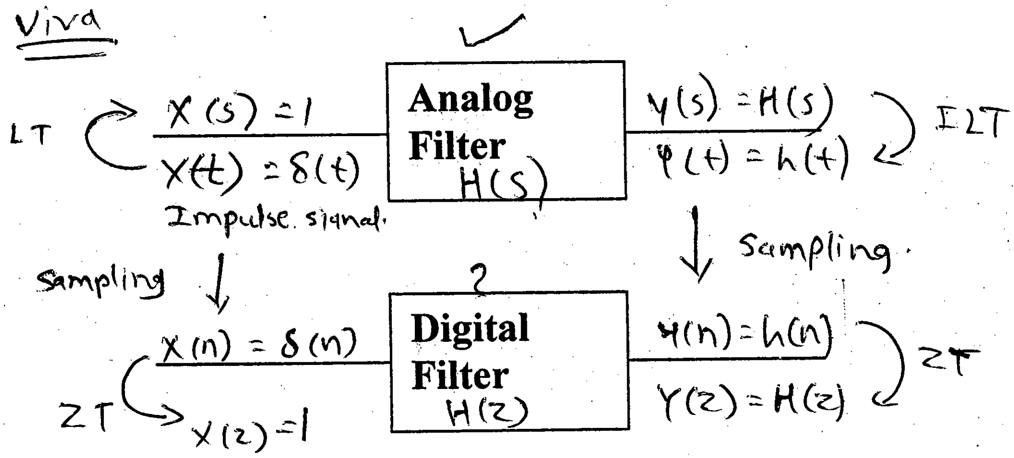
Note (II) Concept of Invariant Method



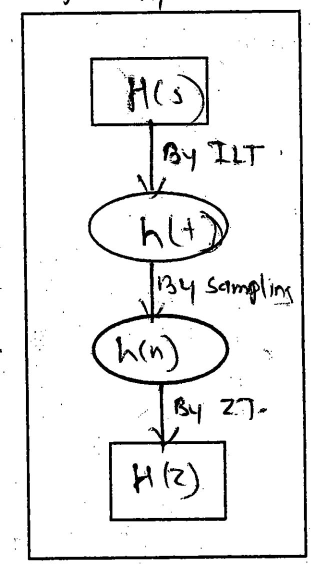
$$H(z) = \frac{Y(z)}{X(z)}$$

❖ Digital I I R Filter design by Impulse Invariant Method

Viva



Theory.



Q(11) Give Invariant Method Design Procedure :-

I. Determine the normalized analog filter Transfer function H(s) that satisfies the specification for the desired filter.

II. Determine the analog filter output by tasking inverse laplace transform of Y(s)

ie y (t) = ILT {Y(s)} where Y(s) = X(s) . H(s) .

III. Sample the output of the analog filter .

$$y[nT] = y(t)\Big|_{t=nT}$$

IV Obtain the Transfer function of the digital filter.

$$Y(z) = ZT\ \{y[nt]\}$$

Then $H(z) = \dfrac{Y(z)}{X(z).}$

For Impulse Invariant Method take x(t) = δ(t)

For Step Invariant Method take x(t) = u(t).

NOTE :

(1) **LT** $\{\ \delta(t)\ \}$ == 1

(2) **LT** $\{\ u(t)\ \}$ == $1/s$

(3) $LT\ \left\{e^{-at}\,u(t)\right\}$ == $1/s+a.$

(4) $LT\ \left\{e^{-at}\cos(bt)\ u(t)\right\}$ == $\dfrac{s+a}{(s+a)^2+b^2}$

(5) $LT\ \left\{e^{-at}Sin(bt)\ u(t)\right\}$ == $\dfrac{b}{(s+a)^2+b^2}$

where 'b' is Analog resonant freq.

5M

Q(12) An analog filter has transfer function $H(s) = \dfrac{1}{(s+1)(s+3)}$ Obtain H(z) using impulse invariant method. Take sampling period Ts = 2 sec.

Solution :

(i) Find h(t)

$$H(s) = \frac{1}{(s+1)(s+3)}$$

By PFE,

$$H(s) = \frac{A}{s+1} + \frac{B}{s+3} \qquad \text{Where } A = 1/2 \quad B = -1/2$$

$$H(s) = \frac{1}{2}\left[\frac{1}{s+1}\right] - \frac{1}{2}\left[\frac{1}{s+3}\right]$$

By ILT,

$$h(t) = \frac{1}{2} e^{-t} u(t) - \frac{1}{2} e^{-3t} u(t)$$

(ii) Find h[n]

Put $t = nT$

$$h(nT) = \frac{1}{2} e^{-nT} u(nT) - \frac{1}{2} e^{-3nT} u(nT)$$

$$h(n) = \frac{1}{2} (e^{-T})^n u(n) - \frac{1}{2} (e^{-3T})^n u(n)$$

(iii) Find H(z)

$$H(z) = \frac{1}{2}\left[\frac{z}{z - e^{-T}}\right] - \frac{1}{2}\left[\frac{z}{z - e^{-3T}}\right] \qquad \text{Put } T = 2 \text{ sec}$$

$$H(z) = \frac{1}{2}\left[\frac{z}{z - 0.135}\right] - \frac{1}{2}\left[\frac{z}{z - 0.027}\right]$$

Q(13) An analog domain filter has a transfer function $H(s) = \dfrac{b}{(s+a)^2 + b^2}$ filter is to be converted into digital filter so that its impulse response characteristics are retained. The sampling frequency is 100 Hz. Find the transfer function H(z).

Q(14) Derive the relationship between Analog Filter POLE and Digital Filter POLE when Impulse Invariant Method is used for filter design.

⟹ Consider Analog Filter, $H(s) = \dfrac{1}{2}\left[\dfrac{1}{s+1}\right] - \dfrac{1}{2}\left[\dfrac{1}{s+3}\right]$

with Poles : $S_1 = -1$ and $S_2 = -3$

By Impulse invariant Method, a Digital filter is given by,

$$H(z) = \frac{1}{2}\left[\frac{z}{z - e^{-T}}\right] - \frac{1}{2}\left[\frac{z}{z - e^{-3T}}\right]$$

with Poles : $Z_1 = e^{-T}$ and $Z_2 = e^{-3T}$

In general, The Relation between Analog Filter Pole and Digital filter Pole is given by,

$$\boxed{Z = e^{ST}}$$

Sampling Time.

Digital Filter pole

Analog Filter Pole.

Q(15) Derive the relation between Analog Filter Frequency and Digital Filter Frequency when Impulse Invariant Method is used for filter design.

The Relation between Analog Filter Pole and Digital filter Pole is given by, $Z = e^{sT}$

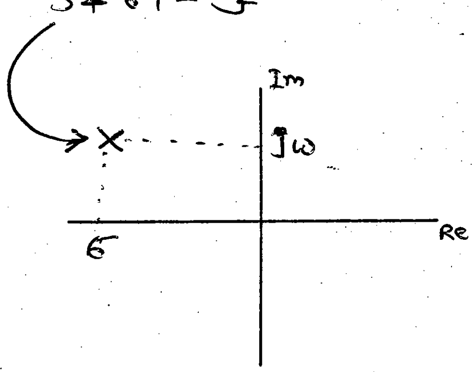
Let (i). $s = \sigma + \Omega j$
where Ω is Analog filter frequency

(ii) $z = r e^{jw}$
where w is Digital filter frequency

Range of Ω is $(-\infty, \infty)$

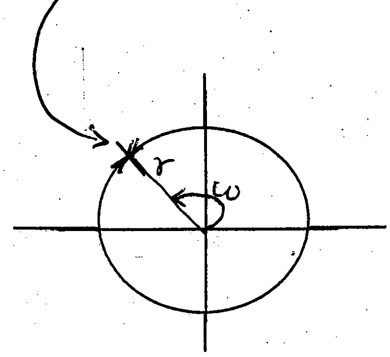
Range of w is $(-\pi$ to $\pi]$

$S = \sigma + \Omega j$



S-plane.

$Z = re^{jw}$



Z-plane.

Now $Z = e^{sT}$

$re^{jw} = e^{(\sigma + \Omega j)T}$

$r\, e^{jw} = e^{\sigma T}\, e^{j\Omega T}$

By equating we get,

(1) $r = e^{\sigma T}$

(2) $w = \Omega T$

i.e. $\boxed{\omega = \frac{\Omega}{Fs}}$

Analog filter freq in rad/sec.

Digital filter freq. in radians

Sampling freq in Hz.

Q(16) Explain the Mapping of points from s-plane to z–plane when Impulse Invariant Method is used for filter design.

Case-I When $\sigma = 0$, $r = 1$

Analog poles which lies on imaginary axis gets mapped onto the unit circle in the z-plane.

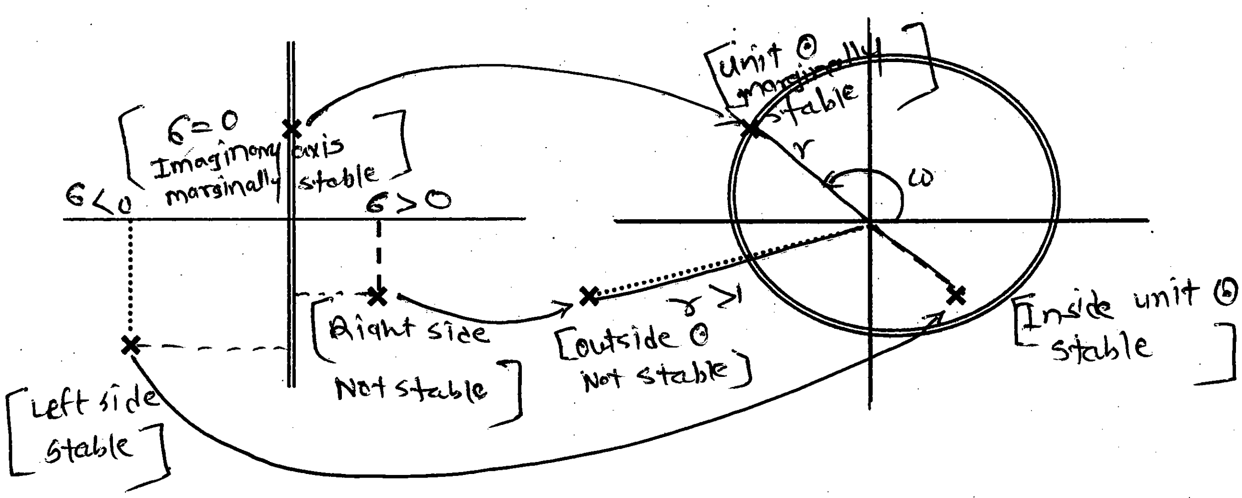
Case-II When $\sigma < 0$, $r < 1$,

Analog poles that lies on LEFT half of s-plane gets mapped INSIDE the unit circle in the z–plane.

Case–III When $\sigma > 0, r > 1$.

Analog poles that lies on RIGHT half of s-plane gets mapped OUTSIDE the unit circle in the z–plane.

ANALOG FILTER → [$s = \sigma + j\Omega$, $z = re^{j\omega}$, $r = e^{\sigma T}$, $\omega = \Omega T$] → DIGITAL FILTER



Q(17) Explain the effect of Aliasing in Impulse Invariant Method.

The relation between analog filter pole and digital filter pole when impulse invariant technique is used for filter design is given by, $Z = e^{ST}$.

However the relation $Z = e^{ST}$ does not describe one to one mapping between s-plane and z-plane.

Case –1, Consider analog pole at $S_1 = \sigma + j\Omega$

$$\text{Then} \quad Z_1 = e^{S_1 T} = e^{(\sigma + j-2)T} = e^{\sigma T} e^{j\Omega T} \quad \text{\_\_\_\_\_\_\_\_ I}$$

Case – 2.

Consider analog pole at $S_2 = \sigma + j\left(\Omega + \frac{2\pi}{T}\right)$

$$\text{Then} \quad Z_2 = e^{S_2 T} = e^{\left[\sigma + j\left(\Omega + \frac{2\pi}{T}\right)\right]T} = e^{\sigma T} e^{j\left(\Omega + \frac{2\pi}{T}\right)T}$$

$$= e^{\sigma T} e^{j\Omega T} e^{j2\pi} \qquad \text{But } e^{j2\pi} = 1$$

$$\therefore \quad = e^{\sigma T} e^{j\Omega T} \quad \text{\_\_\_\_\_\_\_\_ II}$$

From eq^n I and II, Analog poles $S_1 \neq S_2$ But the corresponding digital poles $Z_1 = Z_2$.

That means, all frequencies $\left(\Omega + \frac{2\pi k}{T}\right)$ are mapped to the same point in the Z- plane.

The transformation maps all points in the s plane given by $S = \sigma + j\left(\Omega + \frac{2\pi k}{T}\right)$,

$k = 0, \pm 1, \pm 2, \pm 3....$ on to a single point in the z – plane at $z = e^{ST}$. ie $Z = e^{\sigma T} \; e^{j\Omega T}$

The mapping implies that the interval $\frac{-\pi}{T} \le \Omega \le \frac{\pi}{T}$ maps into the corresponding values of $-\pi \le w \le \pi$ in the digital domain. Further the frequency interval $\frac{\pi}{T} \le \Omega \le \frac{3\pi}{T}$ of s-plane also maps into the interval $-\pi \le w \le \pi$ in the z-plane.

In general, any frequency in the interval $\frac{(2k-1)}{T} \le \Omega \le \frac{(2k+1)\pi}{T}$ will also map into the interval $-\pi \le w \le \pi$ in the z–plane. Thus the mapping from the analog frequency Ω to the digital frequency w is not one to one mapping which reflects the effect of aliasing due to sampling. A one to one mapping is thus possible only if freq. Ω lies in the principle range of $-\frac{\pi}{T} \le \Omega \le \frac{\pi}{T}$.

Q(18) Impulse Invariant method is not suitable for HPF / BPF design. Justify.

In Impulse Invariant method, the relation between analog filter pole and digital filter pole and digital filter pole when impulse invariant technique is used for filter design is given by, $Z = e^{ST}$. However the relation $Z = e^{ST}$ does not describe one to one mapping between s-plane and z-plane. That means, all frequencies $\left(\Omega + \frac{2\pi k}{T}\right)$ are mapped to the same point in the z-plane. The transformation maps all points in the s plane given by

$S = \sigma + j\left(\Omega + \frac{2\pi k}{T}\right)$, $k = 0, \pm 1, \pm 2, \pm 3....$ on to a single point in the z – plane at $z = e^{ST}$.

ie $Z = e^{\sigma T} \; e^{j\Omega T}$

The mapping implies that the interval $\frac{-\pi}{T} \le \Omega \le \frac{\pi}{T}$ maps into the corresponding values of $-\pi \le w \le \pi$ in the digital domain. Further the frequency interval $\frac{\pi}{T} \le \Omega \le \frac{3\pi}{T}$ of s-plane also maps into the interval $-\pi \le w \le \pi$ in the z-plane.

Thus the mapping from the analog frequency Ω to the freq. variable w in the digital domain is many to one. which reflects the effect of aliasing due to sampling. A one to one mapping is thus possible only if freq. Ω lies in the principle range of $-\frac{\pi}{T} \le \Omega \le \frac{\pi}{T}$.

That means if cut off frequency of analog filter Ωc is greater than $\frac{\pi}{T}$. then one to one mapping from analog filter frequency to digital filter frequency is not possible.

Therefore the filter such as HPF or BPF with cut off frequency of analog filter Ωc greater than $\frac{\pi}{T}$. can not be designed using impulse invariant method.

Q(19) Determine H(z) by using impulse invariance technique for the analog system function

$$H(s) = \frac{1}{(S+0.5)\,(S^2+0.5S+2)} \quad \textbf{(IMP)}$$

By Impulse Invariance, $H(s) \rightarrow h(t) \rightarrow h(n) \rightarrow H(z)$

(i) $H(s) = \dfrac{1}{(S+0.5)\,(S^2+0.5\,S+2)}$

By PFE, $H(s) = \dfrac{A}{S+0.5} + \dfrac{BS+C}{S^2+0.5\,S+2}$

Where $\boxed{A = 0.5}$

$$H(s) = \frac{0.5}{S+0.5} + \frac{BS+C}{S^2+0.5\,S+2}$$

$$\frac{1}{(S+0.5)\,(S^2+0.5\,S+2)} = \frac{0.5\,(S^2+0.5S+2)+(S+0.5)\,(BS+C)}{(S+0.5)\,(S^2+0.5\,S+2)}$$

$$= \frac{0.5\,S^2+0.25\,S+1+BS^2+SC+0.5\,BS+0.5\,C}{(S+0.5)\,(S^2+0.5\,S+2)}$$

Equating Numerators

$1 = S^2\,(0.5+B) + S\,(0.25+C+0.5B) + (1+0.5C)$

$\therefore\ 0.5 + B = 0$ $\qquad$ $1 + 0.5\,C = 1$

$\boxed{B = -0.5}$ $\qquad$ $\boxed{C = 0}$

By substituting,

$$H(s) = \frac{0.5}{S+0.5} + \frac{-0.5\,S}{S^2+0.5\,S+2}$$

$$= \frac{0.5}{S+0.5} + \frac{-0.5\,[(S+0.25)-0.25]}{(S+0.0.25)^2+(1.3919)^2}$$

$$H(s) = \frac{0.5}{S+0.5} - 0.5\left[\frac{S+0.25}{(S+0.25)^2+(1.3919)^2}\right] + 0.0898\left[\frac{1.3919}{(S+0.25)^2+(1.3919)^2}\right]$$

By I.L.T.,

$$h(t) = 0.5\,e^{-0.5t}\,u(t) - 0.5e^{-0.25t}\cos(1.3919t)\,u(t) + 0.0898\,e^{-0.25t}\sin(1.3919t)\,u(t)$$

(ii) Put t = nT $\qquad$ Let T = 1 sec $\therefore$ t = n

$$h(n) = 0.5\,e^{-0.5n}\,u(n) - 0.5\,e^{-0.25n}\cos(1.3919\,n)\,u(n) + 0.0898\,e^{-0.25}\sin(1.3919\,n)\,u(n)$$

(iii) By ZT,

$$H(z) = \frac{0.5z}{z-e^{-0.5}} - 0.5\left[\frac{z^2-e^{-0.25}z\cos(1.3919)}{z^2-2(e^{-0.25})z\cos(1.3919)+e^{-0.5}}\right] + 0.0898\left[\frac{e^{-0.25}z\sin(1.3919)}{z^2-2(e^{-0.25})z\cos(1.3919)+e^{-0.5}}\right]$$

Put $e^{-0.5} = 0.6065$ $\qquad$ $\cos(1.3919) = 0.1779$

$e^{-0.25} = 0.7788$ $\qquad$ $\sin(1.3919) = 0.9840$

ANS : $H(z) = \dfrac{0.5\,z}{Z-0.6065} - 0.5\dfrac{z^2+0.1385\,z}{z^2-0.2770\,z+0.6065} + 0.0898\left[\dfrac{0.7614\,z}{Z^2-0.2770\,7+0.6065}\right]$

> **BILINEAR TRANSFORMATION (BLT) METHOD**
>
> **Bilinear Transformation is a mapping of points from s-plane to corresponding points in the z-plane. The BLT transforms, the entire $j\Omega$ axis in the s-plane into one revolution of the unit circle in the z-plane ie. only once and therefore avoids the aliasing of frequency components**

***
IMP **Q(20) Derive BLT equation**
***

In IIR filter, ~~order~~ order = No. of poles

Consider the first order low pass analog filter with,

$$H(s) = \frac{1}{s+1} \quad \text{\_\_\_\_\_\_(i)}$$

$$\frac{Y(s)}{X(s)} = \frac{1}{s+1}$$

$$SY(s) + Y(s) = X(s)$$

By Inverse LT,

$$y'(t) + y(t) = x(t)$$

$$\therefore\ y'(t) = -y(t) + x(t)$$

put $t = nT$,

$$y'[nT] = -y[nT] + x[nT] \quad \text{\_\_\_\_\_\_(ii)}$$

By integral calculus,

$$y(t) = \int_{to}^{t} y^1(t)\,dt + y(t.)$$

Let $t = nT$

$to = (n-1)T$

By substituting,

$$y(nT) = \int_{(n-1)T}^{nt} y^1(t)\,dt + y[(n-1)T]$$

Applying trapezoidal integral approximation rule,

$$y[nT] = \frac{T}{2}\{y'[nT] + y'[(n-1)T\} + y[(n-1)T] \quad \text{\_\_\_\_\_\_(iii)}$$

From eq^n (ii),

$$y'[nT] = -y[nT] + x[nT]$$

$$y'[(n-1)T] = -y[(n-1)T] + x[(n-1)T]$$

By substituting y'[nT] and y'[(n-1)T] in eq^n (iii) we get,

$$y[nT] = \frac{T}{2}\{(-y[nT] + x[nT]) + (-y[(n-1)T + x[(n-1)T)\} + y[(n-1)T]$$

i.e. $$y[n] = \frac{T}{2}\{(-y[n] + x[n]) + (-y[(n-1) + x[(n-1))\} + y[(n-1)]$$

By ZT

$$Y[z] = \frac{T}{2}\{(-Y[z] + X[z]) + (-z^{-1}Y[z] + z^{-1}X[z])\} + z^{-1}Y[z]$$

$$Y(z) = Y(z)\left\{-\frac{T}{2} - z^{-1}\frac{T}{2} + z^{-1}\right\} = X(z)\left\{\frac{T}{2} + z^{-1}\frac{T}{2}\right\}$$

$$Y(z)\left\{1 + \frac{T}{2} + z^{-1}\frac{T}{2} - z^{-1}\right\} = X(z)\left\{\frac{T}{2} + z^{-1}\frac{T}{2}\right\}$$

$$Y(z)\left\{(1 - z^{-1}) + \frac{T}{2}(1 + z^{-1})\right\} = X(z)\left\{\frac{T}{2}(1 + z^{-1})\right\}$$

$$\frac{Y(z)}{X(z)} = \frac{\frac{T}{2}(1+z^{-1})}{(1-z^{-1})+\frac{T}{2}(1+z^{-1})} = \frac{1}{\frac{(1-z^{-1})}{\frac{T}{2}(1+z^{-1})}+1}$$

$$H(z) = \frac{1}{\frac{2}{T}\frac{(z-1)}{(z+1)}+1} \text{ ----------------(iv)}$$

Digital filter   Analog filter.

By comparing H(s) and H(z) eq^n (i) & eq^n (iv) we get, $H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{(Z-1)}{(z+1)}}$

The Bilinear Transformation is characterized by $s = \frac{2}{T}\frac{(Z-1)}{(Z+1)}$

Q(21) Explain Mapping of points from s-plane to z–plane when BLT Method is used for filter design.

⇨ BLT Transformation is characterized by, $s = \frac{2}{T}\frac{(z-1)}{(z+1)}$ And so $\therefore\ Z = \frac{2+ST}{2-ST}$

Put $z = r\,e^{jw}$ and $s = \sigma + j\Omega$

$$r\,e^{jw} = \frac{2+(\sigma+j\Omega)T}{2-(\sigma+j\Omega)T} = \frac{(2+\sigma T)+j\,\Omega T}{(2-\sigma T)-j\Omega T}$$

$$r\,e^{jw} = \frac{\left(\frac{2}{T}+\sigma\right)+j\Omega}{\left(\frac{2}{T}-\sigma\right)-j\Omega}$$

$$|z| = r = \frac{\sqrt{\left(\frac{2}{T}+\sigma\right)^2+\Omega^2}}{\sqrt{\left(\frac{2}{T}-\sigma\right)^2+\Omega^2}} \quad \text{and} \quad \angle Z = w = \tan^{-1}\left(\frac{\Omega}{\frac{2}{T}+\sigma}\right) - \tan^{-1}\left(\frac{-\Omega}{\frac{2}{T}-\sigma}\right)$$

Case-I When $\sigma = 0$, $r = 1$

Analog poles which lies on imaginary axis gets mapped onto the unit circle in the z- plane.

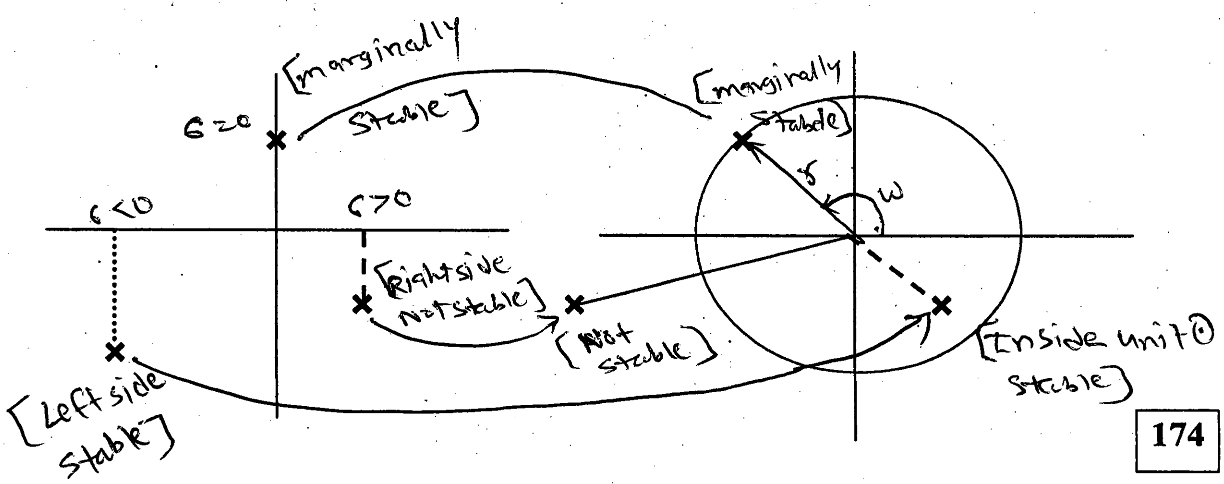
Case-II When $\sigma < 0$, $r < 1$,

Analog poles that lies on LEFT half of s-plane gets mapped INSIDE the unit circle in the z–plane.

Case–III When $\sigma > 0, r > 1$.

Analog poles that lies on RIGHT half of s-plane gets mapped OUTSIDE the unit circle in the z – plane.

ANALOG FILTER **DIGITAL FILTER**



3m **Q(22)** An analog filter has transfer function $H(s) = \dfrac{1}{(s+1)(s+3)}$ Obtain H(z) using BLT. Take sampling period Ts = 2 sec.

By BLT,

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\frac{(z-1)}{(z+1)}}$$

Put T = 2sec

$$H(z) = H(s)\Big|_{s = \frac{(z-1)}{(z+1)}}$$

$$H(z) = \frac{1}{s^2+4s+3}\Big|_{s=\frac{z-1}{z+1}}$$

$$= \frac{1}{\left[\frac{z-1}{z+1}\right]^2 + 4\left[\frac{z-1}{z+1}\right] + 3} = \frac{(z+1)^2}{(z-1)^2 + 4(z-1)(z+1) + 3(z+1)^2}$$

simplify further

Q(23) What is the relationship between Analog Filter Pole and Digital Filter Pole when BLT Method is used for filter design ?

$S = \dfrac{2}{T}\dfrac{(Z-1)}{(Z+1)}$ where S is analog filter pole and Z is digital filter pole.

Q(24) What is the relationship between Analog Filter Frequency and Digital Filter Frequency when BLT Method is used for filter design ?

$$S = \frac{2}{T}\frac{(z-1)}{(z+1)} \quad \text{—— (1)}$$

Let $s = \sigma + j\Omega$

$z = r e^{j\omega}$

when $\sigma = 0$, $r = 1$

$s = j\Omega$, $z = e^{j\omega}$

Put in eqⁿ (1)

$$j\Omega = \frac{2}{T}\frac{(e^{j\omega}-1)}{(e^{j\omega}+1)}$$

Multiply & Divide by $e^{j\omega/2}$

$$\therefore j\Omega = \frac{2}{T}\frac{(e^{j\omega/2} - e^{-j\omega/2})}{(e^{j\omega/2} + e^{-j\omega/2})}\frac{\cancel{e^{j\omega/2}}}{\cancel{e^{j\omega/2}}}$$

$$\therefore \cancel{j}\Omega = \frac{2}{T}\frac{\sin(j\omega/2)}{\cos(j\omega/2)}\frac{\cancel{2}\cancel{j}}{\cancel{2}}$$

$$\therefore \boxed{\Omega = \frac{2}{T}\tan(j\omega/2)}$$

Analog Filter freq. in rad/sec

Sampling Time in sec

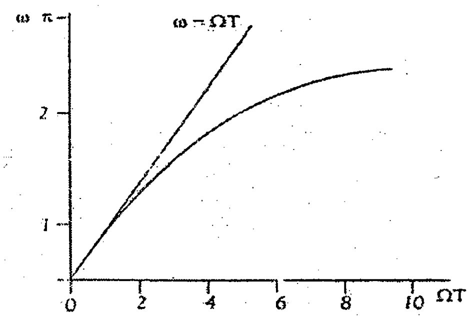
Digital filter freq in radian.

Q(25) Explain frequency warping in BLT method

The relation between Analog Filter frequency Ω and Digital filter Frequency ω is given by, $\Omega = \frac{2}{T}\tan\left(\frac{w}{2}\right)$

For small value of θ
$\tan(\theta) = \theta$

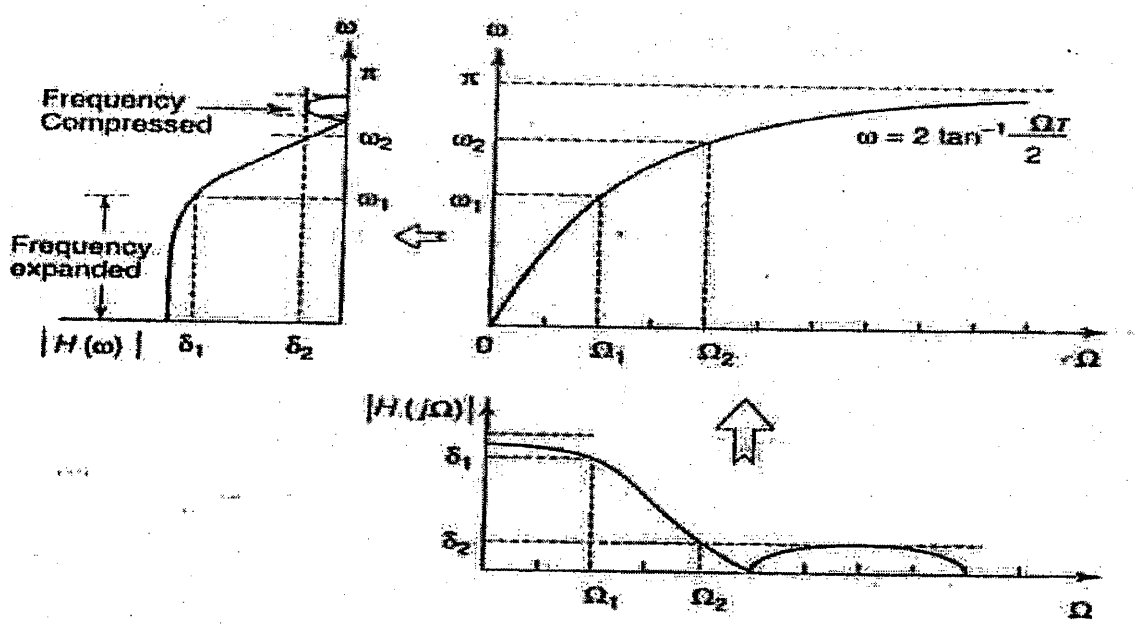
(i) For small value of w ; $\Omega = \frac{2}{T}\left(\frac{w}{2}\right) = \frac{w}{T}$



$\therefore$ w = Ω T
For low frequencies the relationship between Ω and w is linear, as a result, the digital filter have the same amplitude response as the analog filter

.(ii) For large value of w ; $\Omega = \frac{2}{T}\tan\left(\frac{w}{2}\right)$

The influence of warping effect on the amplitude response is shown below. If the analog filter has number of pass-bands centered at regular intervals, the derived digital filter will have same number of pass-bands, but the center frequencies and bandwidth of higher frequency pass-band will tend to reduce disproportionately.



Q(26) A digital filter is required to have a cut off frequency 100 Hz and sampling frequency of 1000 Hz..What is the analog domain cut off frequency. Assume that Bilinear Transformation Technique is used.

ANS :

Sol^n

To find Ω_c

$$\Omega_c = \frac{2}{T}\tan\left(\frac{w_c}{2}\right)$$

when (i) $T = \frac{1}{f_s} = \frac{1}{1000}$

ii) $w_c = 2\pi f_c$

$= 2\pi\left[\frac{100\ Hz}{1000\ Hz}\right]$

$= 2\pi\,[0.1]$

ii) $= 0.2\pi$

$$\Omega_c = \frac{2}{1/1000}\tan\left(\frac{0.2\pi}{2}\right)$$

$$\boxed{\Omega_c = 649.84\ r/s}$$

Ans

Note –

i) Range of w is $[-\pi, \pi]$

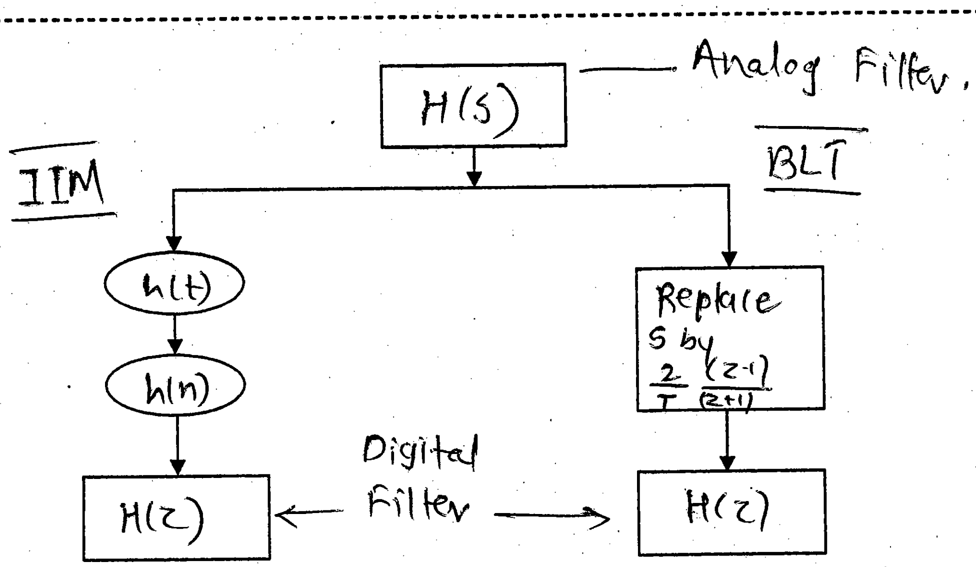
ii) Range of f is $(-1/2, 1/2]$

$f = \frac{F}{F_s}$

↓

Digital freq, No unit.

NOTE :



Q(27) Convert the analog filter with system function $H(s) = \dfrac{s+0.1}{(s+0.1)^2+16}$ into a digital IIR filter by means of a bilinear transformation. The digital filter is to resonate at freq. $w_r = \pi/2$.

ANS : Now, $H(s) = \dfrac{s+0.1}{(s+0.1)^2+16}$

Let $H(s) = \dfrac{s+a}{(s+a)^2+b^2}$ where b is analog resonant frequency.

By comparing we get, analog resonant frequency $b = 4$. **Let $\Omega r = b = 4$**

Digital resonant frequency $w_r = \pi/2$

Now, $\Omega r = \dfrac{2}{T}\tan\left(\dfrac{w_r}{2}\right)$

$4 = \dfrac{2}{T}\tan\left(\dfrac{\pi/2}{2}\right)$ By solving we get $T = 0.5$ sec

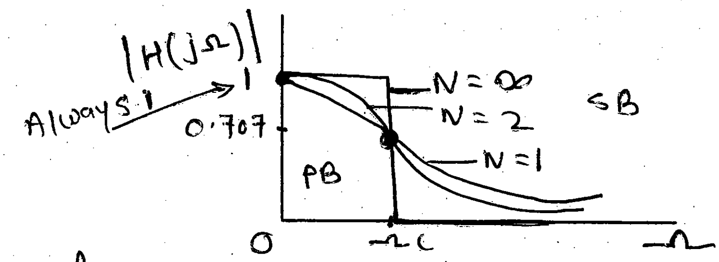
By BLT, Digital Filter is given by, $H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{(z-1)}{(z+1)}}$ Put $T = 0.5$ sec.

$$H(z) = \frac{0.128 + 0.006\ z^{-1} - 0.122\ z^{-2}}{1 + 0.0006\ z^{-1} - 0.975\ z^{-2}}$$

➤ Analog Butterworth LPF

(1) Magnitude Response

$$|H(j\Omega)| = \frac{1}{\sqrt{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}}$$

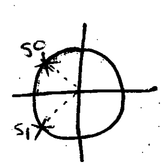


where N is order of filter i.e. No. of poles.

(2) Analog Butterworth LPF POLES and ZEROS

(i) ZEROS : Analog Butterworth LPF has NO ZEROS.

(ii) POLES : $S_K = \Omega c\; e^{j\pi\left(\frac{N+1+2k}{2N}\right)}$ ---------------- 1

Eg-1 LPF N = 1 Ωc = 1 rad/sec Poles : $S_K = \Omega c\, e^{j\pi\left(\frac{N+1+2k}{2N}\right)}$ $S_K = e^{j\pi\left(\frac{2+2k}{2}\right)}$ -- k=0, $S_0 = e^{j\pi} = -1$	**To find H(s) :** $H(s) = \frac{1}{(s-s_0)}$ $\hat{H}(s) = \frac{1}{(s+1)}$ Normalized LPF i.e. $\Omega c = 1$ r/sec.
Eg-2 LPF N = 2 Ωc = 1 rad/sec Poles : $S_K = \Omega c\, e^{j\pi\left(\frac{N+1+2k}{2N}\right)}$ $S_K = e^{j\pi\left(\frac{3+2k}{4}\right)}$ -- complex conjugate. k=0, $S_0 = e^{j\frac{3\pi}{4}}$ k=1, $S_1 = e^{j\frac{5\pi}{4}}$ $= e^{-j\frac{3\pi}{4}}$ 	**To find H(s) :** $H(s) = \frac{1}{(s-s_0)(s-s_1)}$ $H(s) = \frac{1}{\left(s-e^{j\frac{3\pi}{4}}\right)\left(s-e^{-j\frac{3\pi}{4}}\right)}$ $\hat{H}(s) = \frac{1}{s^2+\sqrt{2}s+1}$
Eg-3 LPF N = 3 Ωc = 1 rad/sec Poles : $S_K = \Omega c\, e^{j\pi\left(\frac{N+1+2k}{2N}\right)}$ $S_K = e^{j\pi\left(\frac{4+2k}{6}\right)}$ -- k=0, $S_0 = e^{j2\pi/3}$ k=1, $S_1 = e^{j\pi} = -1$ k=2, $S_2 = e^{-j\frac{2\pi}{3}}$	**To find H(s) :** $H(s) = \frac{1}{(s-s_0)(s-s_1)(s-s_2)}$ ********* $H(s) = \frac{1}{\left(s-e^{j2\pi/3}\right)(s+1)\left(s-e^{-j\frac{2\pi}{3}}\right)}$ $\hat{H}(s) = \frac{1}{s^3+2s^2+2s+1}$

➤ Analog Butterworth HPF

Transfer Function of Normalized Analog Butterworth HPF is given by,

$$H_{HPF}(s) = H_{LPF}(s)\Big|_{S=\frac{1}{s}}$$

Order	Normalized Analog Butterworth LPF	Normalized Analog Butterworth HPF
N = 1	$\hat{H}(s) = \dfrac{1}{s+1}$	$H(s) = \dfrac{s}{s+1}$
N = 2	$\hat{H}(s) = \dfrac{1}{s^2+\sqrt{2}s+1}$	$H(s) = \dfrac{s^2}{s^2+\sqrt{2}s+1}$
N = 3	$\hat{H}(s) = \dfrac{1}{s^3+2s^2+2s+1}$	$H(s) = \dfrac{s^3}{s^3+2s^2+2s+1}$

Not for comp.

Another method to find Normalized H(s) for LPF:

$$\overline{H}(s) = \frac{1}{a_0 + a_1 S^1 + \ldots\ldots + ..a_N\ S^N} \quad \text{where } a_0 = a_N == 1$$

And $$a_k = \left[\frac{\cos\left\{\dfrac{(k-1)\pi}{2N}\right\}}{\sin\left(\dfrac{k\pi}{2N}\right)}\right] a_{k-1}$$ where N is the filter order.

~~Not~~ for comp

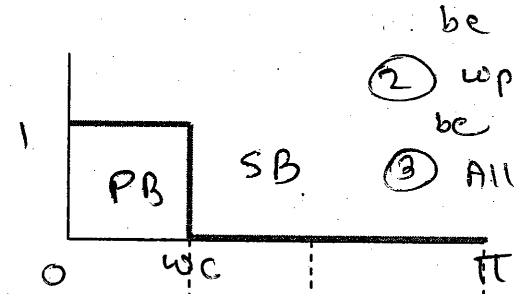
NOTE : Digital Butterworth LPF

① AP & As must be in dB

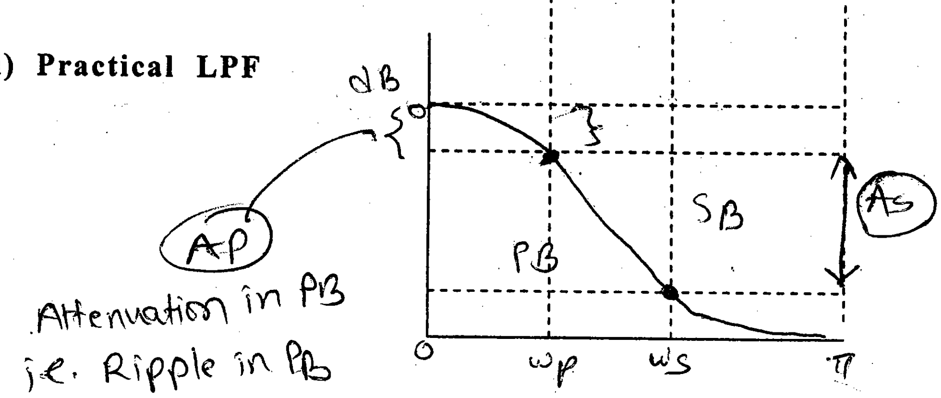
② ωp & ωs must be in radian

③ All parameters must be always +ve.

(i) Ideal LPF



(ii) Practical LPF



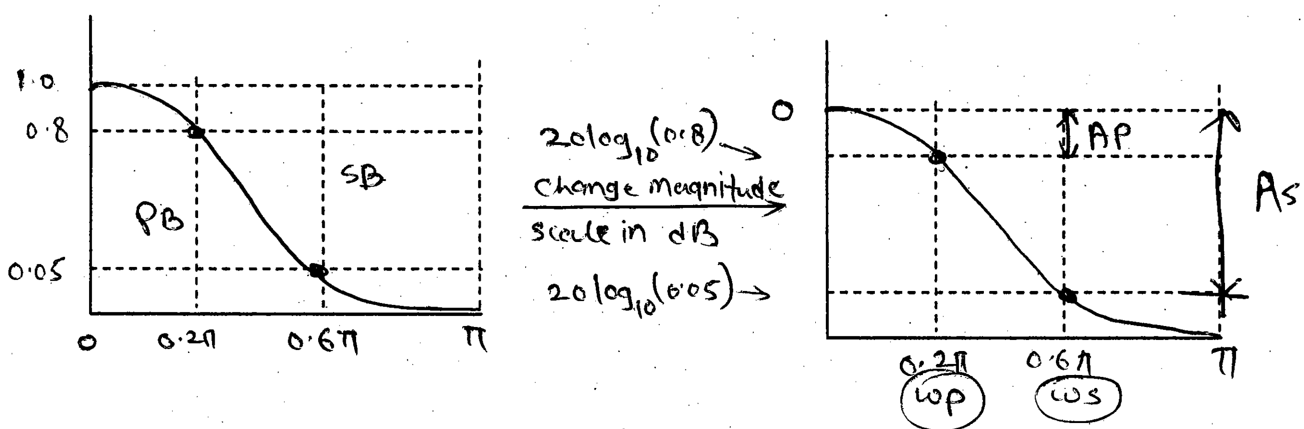
Q(28) Given,

$$0.8 \le |H(e^{jw})| \le 1.0 \quad \text{for} \quad 0 \le w \le 0.2\pi$$

$$|H(e^{jw})| \le 0.05 \quad \text{for} \quad 0.6\pi \le w \le \pi$$

Fs = 1 KHz. Design a Digital Butterworth filter.

Solution :



(i) $A_p = 0 - 20\log_{10}(0.8) = \boxed{1.93\,dB}$

(ii) $A_s = 0 - 20\log_{10}(0.05) \approx \boxed{26.02}$

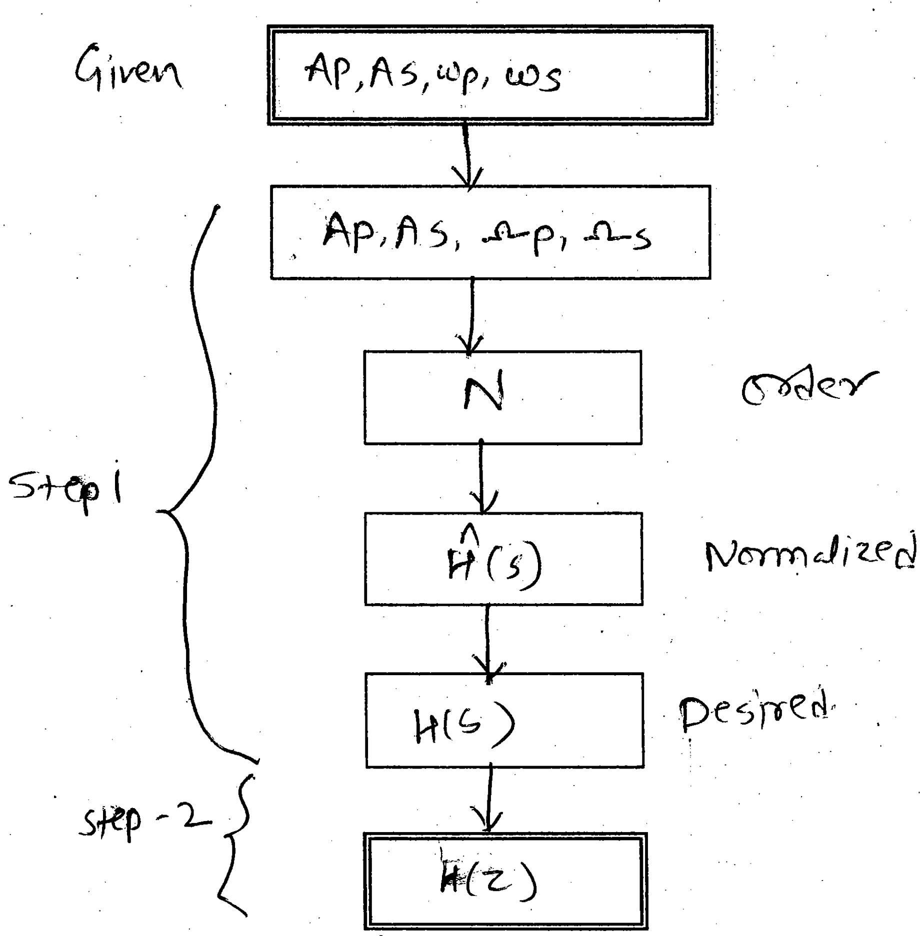
(iii) $w_p = 0.2\pi$

(iv) $w_s = 0.6\pi$

(v) $F_s = 1$ KHz.

vi) LPF

---------------------------------- * ALGORITHM * ----------------------------------



STEP – 1 Design Analog Butterworth L P F

(1) Calculate $\Omega p, \Omega s$

(i) $\Omega p = \frac{2}{T} \tan\left(\frac{\omega p}{2}\right)$

Put $T = \frac{1}{Fs} = \frac{1}{1000}$

$\omega p = 0.2\pi$

$\Omega p = \frac{2}{1/1000} \tan\left(\frac{0.2\pi}{2}\right)$

$\therefore \boxed{\Omega p = 649.84 \ r/sec}$

$\Omega s = \frac{2}{T}\left(\frac{\omega s}{2}\right)$

Put $T = 1/Fs = 1/1000$

$\omega s = 0.6\pi$

$\Omega s = \frac{2}{1/1000} \tan\left(\frac{0.6\pi}{2}\right)$

$\therefore \boxed{\Omega s = 2752.76 \ r/sec}$

---

(2) Calculate Filter order N

$$\underline{N}_{\underline{LPF}} = \frac{\log\left[\frac{10^{As/10}-1}{10^{Ap/10}-1}\right]^{1/2}}{\log\left[\frac{\Omega s}{\Omega p}\right]} \quad \text{——— (2)}$$

$= 2.72$

$\therefore \boxed{N \cong 3}$

---

(3) Calculate Normalized LPF

LPF N = 3 Ωc = 1 rad/sec

Poles : $S_K = \Omega c \, e^{j\pi\left(\frac{N+1+2k}{2N}\right)}$

$S_K = e^{j\pi\left(\frac{4+2k}{6}\right)}$

---

k=0, $S_0 = e^{j2\pi/3}$

k=1, $S_1 = -1$

k=1, $S_2 = e^{-j2\pi/3}$

To find H(s) :

$$H(s) = \frac{1}{(s-s_0)(s-s_1)(s-s_2)} == \frac{1}{s^3+2s^2+2s+1}$$

(4) Calculate Desired LPF

By De-Normalization,

$$H(s) = \hat{H}(s)\Big|_{s=\frac{s}{\Omega_c}} \quad \text{when } \Omega_c = \frac{\Omega_P}{\left(10^{A_P/10}-1\right)^{1/2N}} \quad -③$$

$$H(s) = \frac{1}{S^3+2S^2+2S+1}\Bigg|_{S=\frac{S}{715.86}} \qquad \therefore \boxed{\Omega_c = 715.86 \text{ r/sec}}$$

$$H(s) = \frac{1}{\left[\frac{s}{715.86}\right]^3 + 2\left[\frac{s}{715.86}\right]^2 + 2\left[\frac{s}{715.86}\right] + 1}$$

$$H(s) = \frac{367061696}{s^3 + 1432s^2 + 1025312s + 367061696}$$

STEP – 2 Design Digital Butterworth LPF

By BLT, Digital Filter is given by,

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{(z-1)}{(z+1)}} \qquad \text{put } T = \frac{1}{1000}$$

$$H(z) = \frac{367061696}{\left[\frac{2000(z-1)}{z+1}\right]^3 + 1432\left[\frac{2000(z-1)}{z+1}\right]^2 + 1025312\left[\frac{2000(z-1)}{z+1}\right] + 367061696}$$

=

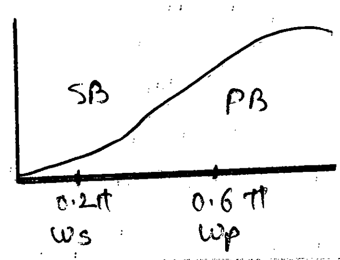
Q(29) Given, $0.8 \le |H(e^{jw})| \le 1.0$ for $0 \le F \le 1$ kHz

$|H(e^{jw})| \le 0.2$ for $3\text{ kHz} \le F \le 5$ kHz

Fs = 10 KHz. Design a Digital Butterworth filter.

Q(30) Given $A_p = 1.93$ dB $A_s = 13.97$ dB $w_s = 0.2\pi$ $w_p = 0.6\pi$
Design a Digital Butterworth filter.

Solution :



$0.2\pi < 0.6\pi$
$\omega_s < \omega_p$
SB PB
High pass filter.

STEP = 1 Design Analog Butterworth

(1) Calculate Ω_p, Ω_s

i) $\Omega_p = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right)$

Assume $T = 1$ sec

$\Omega_p = 2\tan\left(\frac{0.6\pi}{2}\right)$

$\therefore \boxed{\Omega_p = 2.752 \text{ r/sec}}$

(ii) $\Omega_s = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right)$

$= 2\tan\left(\frac{0.2\pi}{2}\right)$

$\boxed{\Omega_s = 0.649 \text{ r/sec}}$

(2) Calculate

$$N_{HPF} = \frac{\log\left[\frac{10^{A_s/10}-1}{10^{A_p/10}-1}\right]^{1/2}}{\log\left[\frac{\Omega_p}{\Omega_s}\right]}$$

$= 1.29$

Let, $\boxed{N = 2}$

(3) Calculate Normalized HPF

HPF $N = 2$ $\Omega c = 1$ rad/sec

Poles : $S_K = \Omega c\, e^{j\pi\left(\frac{N+1+2k}{2N}\right)}$

$S_K = e^{j\pi\left(\frac{3+2k}{4}\right)}$

$k=0,\ S_0 = e^{j\pi \times \frac{3}{4}}$

$k=1,\ S_1 = e^{-j\pi \times \frac{3}{4}}$

Now, $H(s) = \frac{s^2}{(s-s_0)(s-s_1)}$ $H(s) = \frac{s^2}{s^2+\sqrt{2}s+1}$

(4) Calculate Desired HPF

By De-normalization,

$$\mathbf{H(s)} = \tilde{H}(s)\Big|_{S=\frac{S}{\Omega_c}} \qquad \textbf{where } \Omega_c = \frac{\Omega_p}{\left(10^{A_P/10}-1\right)^{1/2N}}$$

$$= 3.18 \text{ r/sec}$$

$$H(s) = \frac{s^2}{s^2+\sqrt{2}\,s+1}\Bigg|_{S=\frac{S}{3.18}}$$

$$H(s) = \frac{\left[\frac{s}{3.18}\right]^2}{\left[\frac{s}{3.18}\right]^2+\sqrt{2}\left[\frac{s}{3.18}\right]+1}$$

$$H(s) = \frac{S^2}{S^2+4.496\ S\ +10.106}$$

STEP – 2 Design Digital Butterworth H P F

By BLT, Digital Filter is given by,

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{(z-1)}{(z+1)}} \qquad \text{Put } T=1 \text{ sec.}$$

$$H(z) = \frac{\left[\frac{2(z-1)}{z+1}\right]^2}{\left[\frac{2(z-1)}{z+1}\right]^2+4.496\left[\frac{2(z-1)}{z+1}\right]+10.106}$$

$$= \frac{\frac{4(z-1)^2}{(z+1)^2}}{\frac{4(z-1)^2}{(z+1)^2}+\cancel{4.496}\ 8.992\frac{(z-1)}{(z+1)}+10.106}$$

$4.496 \times 2 = 8.992$

$=$

Q(31) Design a second order digital Butterworth high pass filter using BLT method with cutoff frequency ω_c = **0.643 π**

Solution :

STEP – 1 Design Analog Butterworth HPF

(1) No AP, As, ωp, ωs is given No need to calculate,
In this eq, order of filter is given,
i.e N = 2

---

(2) Calculate Normalized HPF

HPF N = 2 Ωc = 1 rad/sec

Poles : $S_K = \Omega c \; e^{j\pi\left(\frac{N+1+2k}{2N}\right)}$

$S_K = e^{j\pi\left(\frac{3+2k}{4}\right)}$

---

k=0, $S_0 = e^{j\frac{3\pi}{4}}$

k=1, $S_1 = e^{-j\frac{3\pi}{4}}$

Now, $H(s) = \dfrac{s^2}{(s-s_0)(s-s_1)}$

$$H(s) = \frac{s^2}{s^2+\sqrt{2}s+1}$$

---

(3) Calculate Desired HPF

By De-normalization,

0.643π

$$\mathbf{H(s)} = \tilde{H}(s)\Big|_{S=\frac{S}{\Omega_c}} \qquad \textbf{where } \Omega_c = \frac{2}{T}\tan\left(\frac{\omega_c}{2}\right)$$

$= 3.18$ r/sec.

$$H(s) = \frac{s^2}{s^2+\sqrt{2}\,s+1}\Bigg|_{S=\frac{S}{3.18}}$$

$$H(s) = \frac{\left[\frac{s}{3.18}\right]^2}{\left[\frac{s}{3.18}\right]^2+\sqrt{2}\left[\frac{s}{3.18}\right]+1}$$

$$H(s) = \frac{S^2}{S^2+4.496\;S\;+10.106}$$

STEP – 2 Design Digital Butterworth H P F

By BLT, Digital Filter is given by,

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{(z-1)}{(z+1)}} \qquad \textbf{Put T = 1 sec.}$$

$$H(z) = \frac{\cancel{\left[3.348\right]^2}\left[\frac{2(z-1)}{z+1}\right]^2}{\left[\frac{2(z-1)}{z+1}\right]^2 + 4.496\left[\frac{2(z-1)}{z+1}\right] + 10.106}.$$

Q(32) A Digital Butterworth is required to meet the following specifications

Pass band ripple	$\leq$ 1 dB	
Pass band edge	= 4 KHz	
Stop band attenuation	$\geq$ 40 dB	
Stop band edge	= 6 KHz	
Sampling rate	= 24 KHz	

Find the filter order and cutoff freq. if
a) Impulse Invariant Method is used
b) BLT technique is used.

ANS : a) $\Omega p = 25132.74$ $\Omega s = 37699.11$ $N = 13.02 \cong 14$ $\Omega c = 26375.32$ $w_c = 1.09$
b) $\Omega p = 27712.8$ $\Omega s = 48000$ $N = 9.61 \cong 10$ $\Omega c = 29649.7$ $w_c = 1.106$

Q(33) A Digital IIR Low-Pass Filter required to meet the following specifications:

Passband ripple	: $\leq$ 0.5 dB	Stopband attenuation	: $\leq$ 40 dB
Passband edge	: = 1.2 kHz	Stopband edge	: = 2.0 kHz
Sample rate	: = 8.0 kHz		

Determine the required filter order for (A) A Digital Butterworth Filter
(B) A Digital Chebyshev Filter

Solution : Ap = 0.5 dB As = 40 dB $w_p = 0.3\,\pi$ $w_p = 0.5\,\pi$ **LPF**

(a) A Digital Butterworth Filter

$$N = \frac{\log\left[\dfrac{10^{As/10}-1}{10^{Ap/10}-1}\right]^{1/2}}{\log\left[\dfrac{\Omega s}{\Omega p}\right]}$$

$N = 8.38 \cong 9$

(b) A digital Chebyshev filter

$$N = \frac{\cosh^{-1}\left[\dfrac{10^{As/10}-1}{10^{Ap/10}-1}\right]^{1/2}}{\cosh^{-1}\left[\dfrac{\Omega s}{\Omega p}\right]}$$

N $= 4.09 \cong$ **5**

Q(34) The cut off frequency of a LPF is required to be 100 Hz . Sampling frequency is 1 KHz Design a second order Butterworth filter using :-
a) Bilinear Transformation Method b) Impulse Invariant Method

---

Q(35) Design second order Butterworth low pass filter using impulse invariance technique. The cut off frequency required to be 50 Hz and sampling frequency 500 samples/second. Show filter realization in appropriate form.

Q(36) A third order analog Butterworth filter with cut off frequency = 1 rad/sec, has transfer function $H(s) = \dfrac{1}{S^3 + 2s^2 + 2s + 1}$ Obtain equivalent digital filter Transfer function if sampling frequency is 4 times the cut off frequency of analog filter.

Q(37) Design a first order high pass DT Butterworth filter whose cutoff frequency is 1 kHz at the sampling rate of 10^4 sample/sec.

$\Longrightarrow$

HINT : 1. Transfer function for 1st order H.P Filter : $H(s) = \dfrac{s}{s+1}$

2. Digital Cut-off frequency $\omega_C = 2\pi f_C = 0.2\pi$ rad

3. Prewarp frequency

$$\Omega_C = \frac{2}{T_s}\tan\left(\frac{\omega_C}{2}\right) = \frac{2}{T_s}\tan\left(\frac{0.2\pi}{2}\right) = 6498.39 \text{ rad/sec}$$

4. By denormalization $H_{HPF}(s) = H_{LPF}(s)\big|_{s=\frac{S}{\Omega_C}=\frac{S}{6498.39}}$

5. By BLT transformation, $H(z) = H(s)\big|_{s=\frac{2(z-1)}{T(z+1)}}$

Q(38) Find the attenuation at the frequency 800Hz of a 4^{th} order Butterworth filter whose 1 dB pass band edge is located at 25 00 Hz.

⟹

Butterworth LPF

Stop band freq : $\Omega s = 2\pi(8000)$ rad/sec   Ap = 1 dB

Pass band freq : $\Omega p = 2\pi(2500)$ rad/sec   As = ?

Order N = 4

$$N = \frac{\log\left[\frac{10^{As/10}-1}{10^{Ap/10}-1}\right]^{1/2}}{\log\left[\frac{\Omega s}{\Omega p}\right]} \quad \text{(LPF)}$$

By solving we get, **As =**

Q(39) Design a first order low pass DT Butterworth filter whose cutoff frequency is 1 kHz at the $Fs = 10^4$ Hz using Impulse Invariant Method

---

Q(40) Given $H(s) = \frac{1}{S^2+S+1}$ describes the transfer function of a LPF with a passband of 1 rad/sec. Using Frequency transformations find the transfer function of the following filters.

⟹

(a) A LPF with passband freq = 10 rad/sec

Solution : $H_{LPF}(s) = \hat{H}(s)\Big|_{S=\frac{S}{10}}$

$$= \frac{100}{S^2+10S+100}$$

(b)A HPF with a cutoff freq of 10 rad/sec

Solution :

$$\hat{H}_{HPF}(S) = \bar{\hat{H}}_{LPF}(S)\Big|_{S=\frac{1}{S}} = \frac{S^2}{S^2+S+1}$$

$$\hat{H}_{HPF}(S) = \bar{\hat{H}}_{HPF}(S)\Big|_{S=\frac{S}{10}}$$

$$H_{HPF}(S) = \frac{[S/10]^2}{[S/10]^2 + [S/10] + 1} = \frac{S^2}{S^2+10S+100}$$

(c) A BPF with a pass band of 10 rad/sec and a centre freq of 100 rad/sec

Solution : $H_{BPF}(s) = H_{LPF}(s)\Big|_{S=\frac{S^2+\Omega_o^2}{SB}}$

where $B = 10$ and $\Omega_o = 100$

$$H_{BPF}(s) = \frac{1}{S^2+S+1}\Big|_{S=\frac{S^2+10000}{10S}=\frac{S^2+10^4}{10S}}$$

$$= \frac{1}{\left[\frac{S^2+10^4}{10S}\right]^2 + \left[\frac{S^2+10^4}{10S}\right] + 1}$$

$$H_{BPF}(s) = \frac{100\,S^2}{S^4+10\,S^3+20100S^2+10^5\,S+10^8}$$

(d)A Band stop filter with a stopband of 2 rads/sec and a centre freq of 10 rads/sec

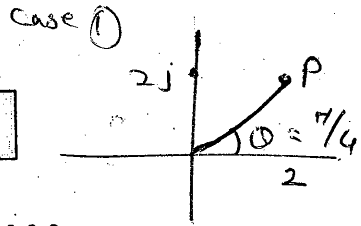
Solution :

$$H_{BSF}(s) = \hat{H}_{LPF}(s)\Big|_{S=\frac{SB}{S^2+\Omega_o^2}}$$

where $B = 2$ and $\Omega_o = 10$

$$H_{BSF}(s) = \frac{1}{S^2+S+1}\Big|_{S=\frac{10S}{S^2+10000}=\frac{10\,S}{S^2+10^2}}$$

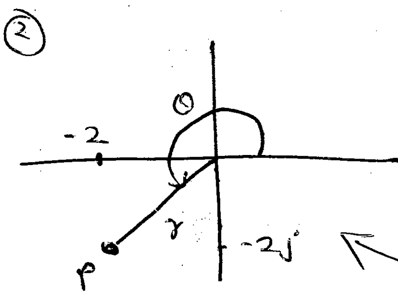
Case ①



DIGITAL F I R FILTERS

➤ Characteristics of FIR filters : •••••

Case ②



[1]. FIR filter is always STABLE.

Consider FIR filter with $h[n] = \{ \underset{\uparrow}{1}, 2, 3, 4 \}$

By ZT, $H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$

multiply & Divide by z^3

$$H(z) = \frac{z^3 + 2z^2 + 3z + 4}{z^3}$$

POLES : $P_1 = P_2 = P_3 = 0$

☞ In FIR filter POLES are always only at origin.

☞ For causal and stable filter all the poles must lie INSIDE the unit circle. i.e. | pole | ≤ 1.

∴ Therefore FIR filters are always STABLE.

---

[2]. FIR filter can have Linear Phase Response why linear phase ? Distortionlessness

Consider FIR filter with $h[n] = \{ \underset{\uparrow}{3}, 2, 1, 2, 3 \}$

By ZT, $H(z) = 3 + 2z^{-1} + z^{-2} + 2z^{-3} + 3z^{-4}$

Put $z = e^{jw}$

$\therefore H(e^{jw}) = 3 + 2e^{-jw} + e^{-j2w} + 2e^{-j3w} + 3e^{-j4w}$

$$H(e^{jw}) = e^{-j\left(\frac{N-1}{2}\right)\omega}\left[\;\_\_\_\_\;\right]$$

only when h(n) is symm or Antisymm.

$$H(e^{jw}) = e^{-j2w}\left[3e^{j2w} + 2e^{jw} + 1 + 2e^{-jw} + 3e^{-j2w}\right]$$

$$H(e^{jw}) = e^{-j2w}\left[3(e^{j2w} + e^{-j2w}) + 2(e^{jw} + e^{-jw}) + 1\right]$$

$$H(e^{jw}) = e^{-j2w}\left[6\cos(2\omega) + 4\cos(\omega) + 1\right]$$

$$\boxed{H(w) = e^{-j2\omega}\left[6\cos(2\omega) + 4\cos(\omega) + 1\right]}$$

freq. response i.e. DTFT.

Phase Response $\phi(\omega)$

↳ Real part of $H(\omega)$ i.e. $H_r(\omega)$

(i) Magnitude Response $|H(w)| = |6\cos(2w) + 4\cos(w) + 1|$

(ii) Phase Response $\phi(w) = e^{-j2w} = e^{-j\phi}$

(iii) Generalized Phase $\phi = \begin{cases} -2\omega & \text{if } H_r(\omega) \geq 0 \\ -2\omega + \pi & \text{if } H_r(\omega) < 0 \end{cases}$

$a + jb = re^{j\theta}$

where $r = \sqrt{a^2 + b^2}$

$\theta = \begin{cases} \tan^{-1}(b/a) & \text{if } a > 0 \\ \tan^{-1}(b/a) + \pi & \text{if } a < 0 \end{cases}$

Case ①
$P = 2 + 2j$
Let $P = re^{j\theta}$
where $r = 2\sqrt{2}$
$\theta = \pi/4$

Case ②
$P = -2 - 2j$
Let, $P = re^{j\theta}$
where $r = 2\sqrt{2}$
$\theta = \frac{\pi}{4} + \pi$

$\Phi = -2\omega + \pi$ if $H_r(\omega) < 0$

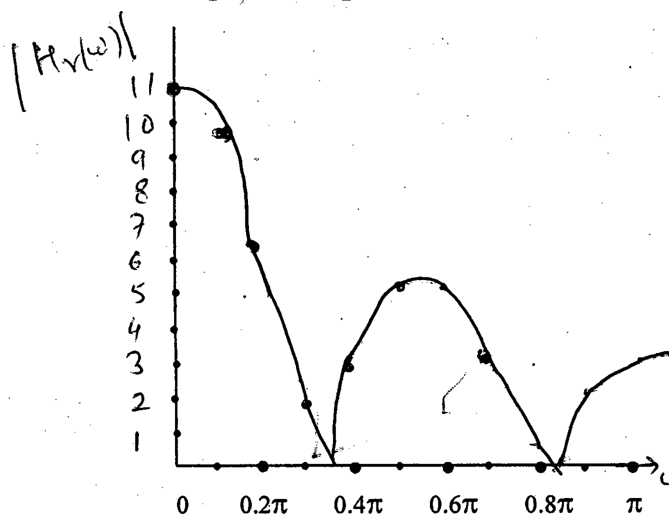
Sr No.	Freq. W	$H_r(w)$	Phase Response
1	0	**11.0**	0
2	0.1 π	**9.66**	−0.2π
3	0.2 π	**6.09**	−0.4π
4	0.3 π	**1.50**	−0.6π
5	0.4 π	**− 2.62**	−0.8π + π = 0.2π
6	0.5 π	**− 5.00**	−π + π = 0
7	0.6 π	**− 5.09**	−1.2π + π = −0.2π
8	0.7 π	**− 3.20**	−1.4π + π = −0.4π
9	0.8 π	**− 0.38**	−1.6π + π = −0.6π
10	0.9 π	**2.05**	−1.8π + (2π) = 0.2π
11	π	**3.00**	−2π + (2π) = 0

NOTE :

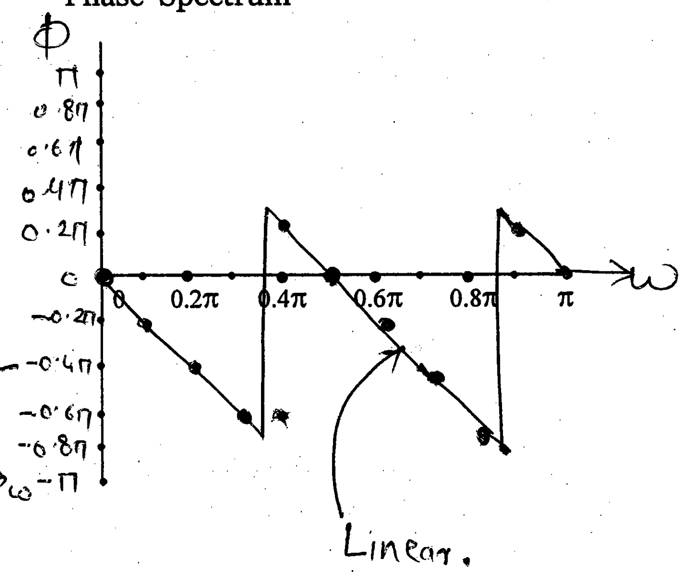
(i) Range of w is (−π to π]

(i) Range of ∅ is (−π to π]

Magnitude Spectrum



Phase Spectrum



$$\phi \;=\; -2 \; w$$

Linearly Varying with Frequency Phase **i.e. Linear Phase**

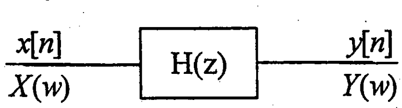
constant

Frequency

Q(41) Explain the concept of Linear Phase and its importance.

Consider a LPF with frequency response $H(e^{jw})$ given by

$$H(e^{jw}) = \begin{cases} e^{-jw\alpha} & |w| \le w_c \\ 0 & wc < w \le \pi \end{cases}$$



Let X(w) = DTFT { X[n] } ,

The FT of y[n] is then given by

$$Y(w) = X(w) \,.\, H(w)$$

$$Y(w) = X(w) \,.\, e^{-jw\alpha}$$

By iDTFT, $\boxed{y[n] = x[n - \alpha] \leftarrow \text{o/p of filter}}$

Conclusion :

(1) The phase response of the filter can be linear or non linear.

(2) If the phase response of the filter is linear then the output of the filter is same as original input delayed by α.
The linear Phase does not alter the shape of the original signal. In this example, the phase response is , $\phi(w) = -\alpha.w$ where α is any constant.

(3) If the phase response of the filter is non linear the output is distorted version of input x[n].

I. If the Phase Response is Linear the output of the Filter during pass-band is delayed input.

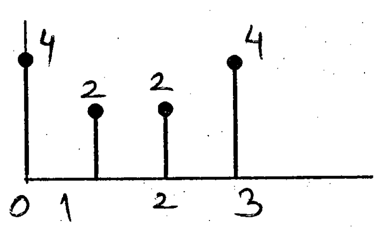
II. If the phase Response is non Linear the output of the filter during pass-band is distorted one

The linear Phase characteristic is important when the phase distortion is not tolerable.

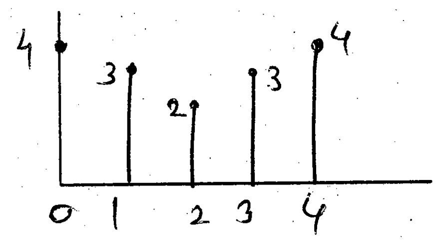
FIR Filter can be designed with linear phase characteristic. In application like data transmission, speech processing etc phase distortion can not be tolerated and here linear phase characteristic of FIR filter is useful.

Note : (I) When h[n] is <u>symmetric</u> i.e. $\boxed{h[n] = h[N-1-n]}$ Causal & symm.

Eg h[n] N = 4 (EVEN)



h[n] N = 5 (ODD)



(i) Phase Response : $\phi(w) = e^{-j\left(\frac{N-1}{2}\right)\omega}$

(iii) Generalized phase $\phi = \begin{cases} -\left(\frac{N-1}{2}\right)\omega & \text{if } H_r(\omega) > 0 \\ -\left(\frac{N-1}{2}\right)\omega + \pi & \text{if } H_r(\omega) < 0 \end{cases}$

phase

$$\phi = -\left(\frac{N-1}{2}\right) w$$

Linear Phase

constant

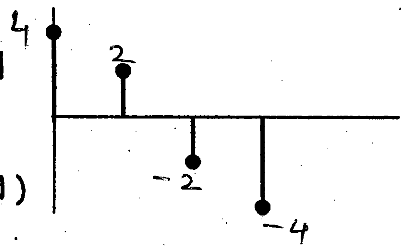
Frequency

Note : (II) When h[n] is <u>Anti-symmetric</u> i.e. $\boxed{h[n] = -h[N-1-n]}$ Causal & anti symm.

Eg h[n]

N = 4

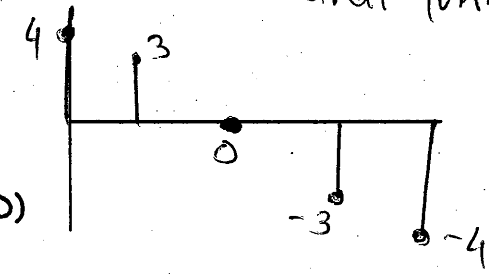
(EVEN)



h[n]

N = 5

(ODD)



(i) Phase Response : $\phi(w) = e^{j\left(\frac{\pi}{2} - \left(\frac{N-1}{2}\right)\omega\right)}$

(ii) Generalized phase $\phi = \begin{cases} \frac{\pi}{2} - \left(\frac{N-1}{2}\right)\omega & \text{if } Hr(\omega) > 0 \\ \frac{\pi}{2} - \left(\frac{N-1}{2}\right)\omega + \pi & \text{if } Hr(\omega) < 0 \end{cases}$

$$\boxed{\phi = \frac{\pi}{2} - \left(\frac{N-1}{2}\right) w}$$

Linear Phase

constant

Frequency

Note : (III) For Linear Phase FIR filter,

(1) **Order** = N-1 where N is length of h(n)

(2) **Number of POLES** ≤ N-1

(3) **Max number of POLES** = N-1

(4) **Max number of ZEROS** = N-1

Q(42) Given $Hd(e^{jw}) = e^{-j2w}$ What is the order of the filter ?

Solution :

$Hd(e^{j\omega}) = 1 \cdot e^{-j2\omega}$

freq Response $|Hd(e^{j\omega})|$ $\phi(\omega)$

Phase Response $\phi(\omega) = e^{-j2\omega}$

Phase $\phi = -2\omega$ — freq.

Linear phase FIR const

for symm. h(n)

$\phi = -\left(\frac{N-1}{2}\right)\omega = -2\omega$

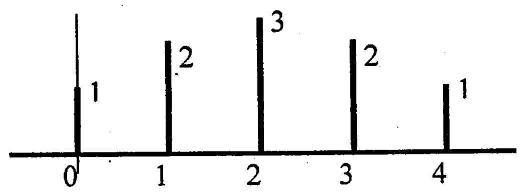
$-\frac{N-1}{2} = -2$

$N-1 = 4$

$\boxed{\text{order} = 4}$ Ans.

Q(43) Find frequency response, phase delay and group delay of Linear Phase FIR Filter with symmetric h[n] and N odd (i.e Type-I Filter)

Consider Type-I Linear Phase FIR Filter with symmetric h[n] and N odd.



$h[n] = h[n-1-n]$
Consider h[n] with N = 5

$h[n] = \{ h[o],\ h[1],\ h[2], h[3], h[4] \}$

By ZT, $H(z) = \sum_{n=0}^{N-1} h[n]\, z^{-n} = \sum_{n=0}^{4} h[n]\, z^{-n}$

$H(z) = h[0] + h[1]\, z^{-1} + h[2]\, z^{-2} + h[3]\, z^{-3} + h[4]\, z^{-4}$

Put $z = e^{jw}$

$H(e^{jw}) = h[0] + h[1]\, e^{-jw} + h[2]\, e^{-j2w} + h[3]\, e^{-j3w} + h[4]\, e^{-j4w}$

For Symmetric $\begin{matrix} h[o] = h[4] \\ h[1] = h[3] \end{matrix}$ and

Substituting, $H(e^{jw}) = h[0] + h[1]\, e^{-jw} + h[2]\, e^{-j2w} + h[3]\, e^{-j3w} + h[4]\, e^{-j4w}$

$$= e^{-j2w}[h[0]e^{j2w} + h[1]\, e^{jw} + h[2] + h[1]\, e^{-jw} + h[0]\, e^{-2w}]$$

$$= e^{-j2w}[h[0](e^{j2w} + e^{-j2w}) + h[2] + h[1](e^{jw} + e^{-jw})]$$

$$H(e^{jw}) = e^{-j2w}[h[2] + 2h[1]\cos(w) + 2h[0]\cos(2w)]$$

In general

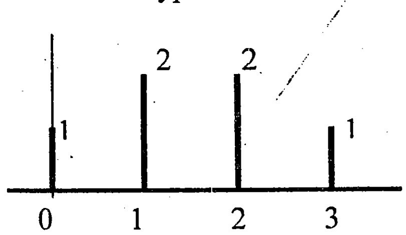
$$\underbrace{H(e^{jw})}_{\text{Frequency Response}} = \underbrace{e^{-jw\alpha}}_{\text{phase Response}} \underbrace{\left[h[\alpha] + \sum_{n=0}^{\frac{N-3}{2}} 2h[n]\cos(\alpha - n) \right]}_{\text{Magnitude Response}}$$

☞ Phase delay $= \dfrac{N-1}{2}$

☞ Group delay $= \left(\dfrac{N-1}{2}\right)$.

Q(44) Find frequency response, phase delay and group delay of Linear Phase FIR Filter with symmetric h[n] and N Even (i.e Type-II Filter)

Consider Type-II Linear Phase FIR Filter with symmetric h[n] and N Even.



Ex. $h[n] = h[N-1-N]$.
Consider h[n] with N = 4
$h[n] = \{ h[o], h[1], h[2], h[3] \}$
↑

By ZT, $H(z) = \sum_{n=0}^{N-1} h[n] z^{-n} = \sum_{n=0}^{3} h[n] z^{-n}$

$$H(z) = h[0] + h[1]\, z^{-1} + h[2]\, z^{-2} + h[3]\, z^{-3}$$

put $z = e^{jw}$ $\quad H(e^{jw}) = h[0] + h[1]e^{-jw} + h[2]\, e^{-j2w} + h[3]\, e^{-j3w}$

$$= e^{-j\frac{3}{2}w}\left[h[0]\, e^{j\frac{3}{2}w} + h[1]\, e^{j\frac{1}{2}w} + h[z]\, e^{-j\frac{1}{2}w} + h[3]\, e^{-j\frac{3}{2}w}\right]$$

For symmetric h[n]

h[n] = h [N − 1 − N]. For N = 4

ie h[0] = h[3]

h[1] = h[2]

$$H(e^{jw}) = e^{-j\frac{3}{2}w}\left[h[0]\, e^{j\frac{3}{2}w} + h[1]\, e^{j\frac{1}{2}w} + h[1]\, e^{-j\frac{1}{2}w} + h[0]\, e^{-j\frac{3}{2}w}\right]$$

$$= e^{-j\frac{3}{2}w}\left[h[0]\left(e^{j\frac{3}{2}w} + e^{-j\frac{3}{2}w}\right) + h[1]\left(e^{+j\frac{1}{2}w} + e^{-j\frac{1}{2}w}\right)\right]$$

$$H(e^{jw}) = e^{-j\frac{3}{2}w}\left[2\ h(0)\ \cos\left(\frac{3}{2}w\right) + 2h[1]\ \cos\left(\frac{1}{2}w\right)\right]$$

$$H(z) = e^{-j\frac{3}{2}w}\left[2\ h(0)\ \cos\left(\frac{3}{2}w\right) + 2h[1]\ \cos\left(\frac{1}{2}w\right)\right]$$

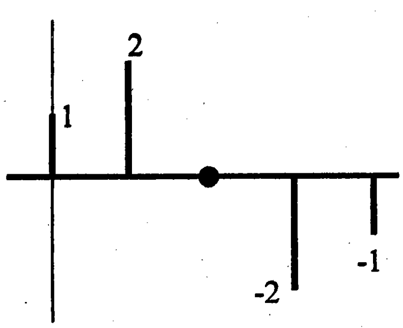
In General, $\underbrace{H(e^{jw})}_{\text{Frequency Response}} = \underbrace{e^{-jw\alpha}}_{\text{phase Response}} \sum_{n=0}^{N-1} 2\ \underbrace{h[n]}_{\text{Magnitude Response}}\ \cos(\alpha - n)\ \ where\ \ \alpha = \frac{N-1}{2}$

- Phase delay = $\frac{N-1}{2}$
- Group delay = $\left(\frac{N-1}{2}\right)$.

Q(45) Find frequency response, phase delay and group delay of Type-III Linear Phase FIR Filter with Anti-symmetric h[n] and N odd (i.e Type-III Filter)

Consider Type-III Linear Phase FIR Filter with Anti-symmetric h[n] and N odd.

Ex.



h[n] = − h [N − 1 − N]

Consider h[n] with N = 5

h[n] = { h[0] , h[1], h[2], h[3], h[4] }

By ZT, $H(z) = \sum_{n=0}^{N-1} h[n]z^{-n} = \sum_{n-0}^{4} h[n]z^{-n}$

$$H(z) = h[0] + h[1]\, z^{-1} + h[2]\, z^{-2} + h[3]\, z^{-3} + h[4]\, z^{-4}$$

For antisymmetric h[n], h[n] = − h[N − 1 − n]

ie h[0] = − h[4]

h[1] = − h[3] and

h[2] = 0

By substituting,

$$H(z) = h[0] + h[1]\, z^{-1} + 0 - h[2]\, z^{-3} - h[0]\, z^{-4}$$

Put $z = e^{jw}$ $\quad H(e^{jw}) = h[0] + h[1]\, e^{-jw} - h[1]\, e^{-j3w} - h[0]\, e^{-j4w}$

$$= e^{-j2w}[h[0]e^{j2w} + h[1]\, e^{jw} - h[1]e^{-jw} - h[0]e^{-j2w}]$$

$$= e^{-j2w}[h[0](e^{j2w} + e^{-j2w}) - h[1](e^{jw} - e^{-jw}]$$

$$= e^{-j2w}\ [2j\ h[0]\sin(2w) + 2j\ h[1]\sin(w)]$$

$$H(e^{jw}) = e^{j\left[\frac{\pi}{2}-\left(\frac{N-1}{2}\right)w\right]}\ [\,2[h]\sin(2w) + 2h[1]\sin(w)]$$

In general,

$$\underset{\substack{Frequency\\Response}}{H(e^{jw})} = \underset{\substack{Phase\\Response}}{e^{j\left[\frac{\pi}{2}-\left(\frac{N-1}{2}\right)w\right]}} \left(\sum_{n=0}^{\frac{N-3}{2}} 2\ h[n]\ \sin(\alpha - n)\right)$$

$$\uparrow \quad \alpha = \frac{N-1}{2}$$

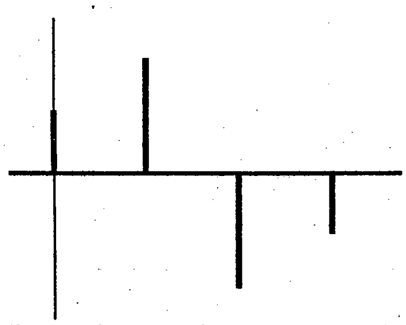
magnitude Response

$\rightarrow$ Phase delay $= \frac{\pi}{2} - \left(\frac{N-1}{2}\right)w$

$\rightarrow$ Group delay : $\frac{N-1}{2}$

Q(46) Find frequency response, phase delay and group delay of Linear Phase FIR Filter with Anti-symmetric h[n] and N Even (i.e Type-IV Filter)

$\Rightarrow$ Consider Type-IV Linear Phase FIR Filter with Anti-symmetric h[n] and N EVEN.



$h[n] = -h[N-1-n]$

Consider h[n] with N = 4

$h[n] = \{\, h[0], h[1], h[2], h[3] \,\}$

By ZT,

$$H(z) = \sum_{n=o}^{N-1} h[n]z^{-n} = \sum_{n=o}^{3} h[n]z^{-n}$$

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3}$$

For antisymmtric h[n],

$$h[n] \ = -h[N-1-n]$$
$$\text{i.e } h[0] = -h[3]$$
$$h[1] \ = -h[2]$$

Substituting,

$$H[z] = h[0] + h[1]z^{-1} - h[1]\ z^{-2} - h[0]\ z^{-3}$$

Put $z = e^{jw}$,

$$H(e^{jw}) = h[0] + h[1]z^{-jw} - h[1]z^{-j2w} - h[0]z^{-j3w}$$

$$= e^{-j\frac{3}{2}w}\left[h[0]\,e^{j\frac{3}{2}w} + h[1]\,e^{j\frac{3}{2}w} - h[1]\,e^{-j\frac{1}{2}w} - h[0]\,e^{-j\frac{3}{2}w}\right]$$

$$= e^{-j\frac{3}{2}w}\left[h[0]\left(e^{j\frac{3}{2}w} - e^{-j\frac{3}{2}w}\right) - h[1]\left(e^{j\frac{1}{2}w} - e^{-j\frac{1}{2}w}\right)\right]$$

$$= e^{-j\frac{3}{2}w}\left[2j\ h[0]\sin\left(\frac{3}{2}w\right) + 2j\ h[1]\ \sin\left(\frac{1}{2}w\right)\right]$$

$$H(e^{jw}) = e^{-j\frac{3}{2}w}\ j\left[2\ h[0]\sin\left(\frac{3}{2}w\right) + 2\ h[1]\ \sin\left(\frac{1}{2}w\right)\right]$$

Put $e^{j\frac{\pi}{2}} = j$

$$H(e^{jw}) = e^{-j\frac{3}{2}w}\ e^{j\frac{\pi}{2}}\left[2\ h[0]\sin\left(\frac{3}{2}w\right) + 2\ h[1]\ \sin\left(\frac{1}{2}w\right)\right]$$

In general, $H(e^{jw}) = e^{j\left[\frac{\pi}{2}-\left(\frac{N-1}{2}\right)w\right]} \left[\sum_{n-0}^{\frac{N}{2}-1} 2\, h[n]\, \sin(\alpha - n)\right]$

$\uparrow$ *Frequency* Response $\quad$ $\uparrow$ *Phase* Response $\quad$ $\uparrow$ *Magnitude* Response $\quad \alpha = \frac{N-1}{2}$

→ Phase delay = $\frac{\pi}{2} - \left(\frac{N-1}{2}\right)w$

→ Group delay : $\frac{N-1}{2}$

Q(47) Show that if Z_1 is ZERO of the filter then $\frac{1}{z_1}$ is also a ZERO of Linear Phase filter. ✱✱✱ Imp

Consider a Linear Phase FIR filter,

By ZT, $\quad H(z) = \sum h[n]\ z^{-n}$

(i) For symmetric h[n] = h[N-1-n]

$$H(z) = \sum_{n=0}^{N-1} h[N-1-n]\ z^{-n}$$

Put N – 1 - n = m
∴m = N - 1 - n

$$H(z) = \sum_{m=0}^{N-1} h[m]\ z^{-(N-1-m)}$$

$$H(z) = \sum_{m=0}^{N-1} h[m]\ z^{-(N-1)}\ z^{m}$$

$$H(z) = z^{-(N-1)} \sum_{m=0}^{N-1} h[m]\ (Z^{-1})^{-m}$$

$$H(z) = z^{-(N-1)}\ H(z^{-1})$$

$$H(z) = z^{-(N-1)}\ H\left(\frac{1}{Z}\right)$$

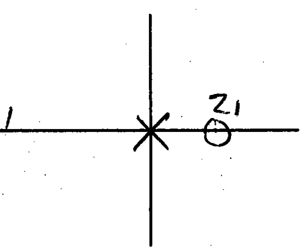
If Z_1 is ZERO of the filter,
Then

$$H(z)\big|_{z=Z_1} = 0$$

$$\therefore H(z_1) = z_1^{-(N-1)}\ H\left(\frac{1}{z_1}\right) = 0$$

But $z_1 \neq 0$

In FIR filter poles, are always only at origin.



$\therefore H\left(\frac{1}{z_1}\right) = 0.$

i.e. $H(z)\big|_{z=\frac{1}{z_1}} = 0.$

Therefore, if Z_1 is ZERO of the filter, Then $\frac{1}{z_1}$ is also a ZERO of the filter.

NOTE :

(1) For **Linear Phase** FIR filter h[n] must be either **Symmetric** OR **Antisymmetric.**

(2) When h[n] is either Symmetric OR Antisymmetric, **ZEROS** of the filter are always in **Reciprocal order**.

i.e. If Z_1 is ZERO of the filter, Then $\frac{1}{z_1}$ is also a ZERO of the filter.

(3) If ZEROS of the filter are in reciprocal order, then filter is Linear Phase FIR filter.

****

Q(48) A fourth order FIR filter has following two pairs of complex conjugate zeros. Z_1 , $Z_2 = e^{\pm j\pi/2}$ and Z_3 , $Z_4 = 2e^{\pm j\pi/3}$ State giving reasons whether the filter has Linear Phase property.

Solution :

For, order = 4

$z_1 = e^{j\pi/2}$, $\frac{1}{z_1} = e^{-j\pi/2}$ exists.

$z_2 = e^{-j\pi/2}$, $\frac{1}{z_2} = e^{j\pi/2}$ exists

$z_3 = 2e^{+j\pi/3}$, $\frac{1}{z_3} = \frac{1}{2}e^{-j\pi/3}$ does not exits

$z_4 = 2e^{-j\pi/3}$, $\frac{1}{z_4} = \frac{1}{2}e^{+j\pi/3}$ does not exists.

As, zeros of filter are NOT in reciprocal order the filter is NOT linear phase FIR filter

*---

Q(49) Zeros of fourth order FIR filter are given. $Z_1 = e^{j\pi/2}$, $Z_2 = e^{-j\pi/2}$ and $Z_3 = 2e^{-j\pi/3}$ State giving reasons whether the filter has Linear Phase property.

ANS : Order = 4

$z_1 = e^{j\pi/2}$

$z_2 = e^{-j\pi/2}$

Complex conjugate $z_3 = 2e^{-j\pi/3}$

Same as 48

Q(50) A fourth order FIR filter has following transfer function. State whether the filter is of Linear Phase type. Justify your answer.

$$H[z] = (1 - e^{-j\pi/2} z^{-1})(1 - e^{j\pi/2} z^{-1})(1 - 0.5 z^{-1})(1 - 2 z^{-1})$$

ANS :

➢ Position of definite ZEROS of a Linear Phase FIR filter.

Case-1 : When h[n] is Symmetric with N even

$$H(z) = z^{-(N-1)} \quad H\left(\frac{1}{Z}\right)$$

$$H(z) = z^{-ODD} \quad H\left(\frac{1}{Z}\right)$$

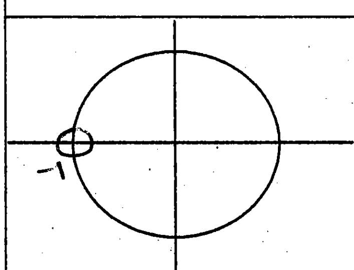
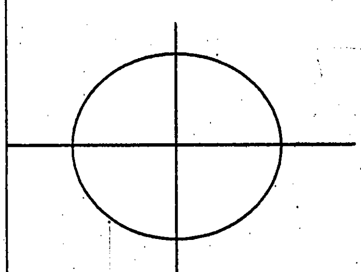
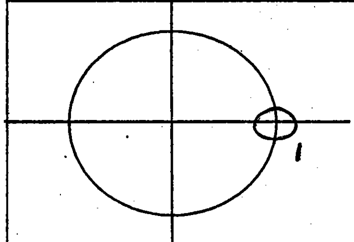
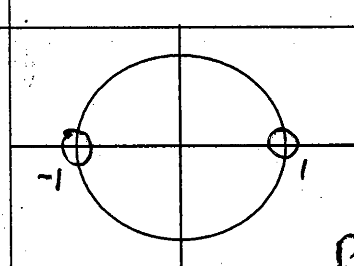
(1) At $Z=-1$ $w=\pi$	(2) At $Z=1$ $w=0$
$H(-1) = (-1)^{-ODD} \; H(-1)$ $H(-1) = (-1) \; H(-1)$ $H(-1) = -H(-1)$ So, $H(-1) = 0$ $H(z)\big\vert_{z=-1} = 0$ That means there exists definite ZERO at $z=-1$	$H(1) = (1)^{-ODD} \; H(1)$ $H(1) = (1) \; H(1)$ $H(1) = H(1)$ No definite ZERO exists at $z = 1$

Case-2 : When h[n] is symmetric with N odd ***************

Case-3 : When h[n] is Anti-Symmetric with N even *************

Case-4 : When h[n] is Anti-Symmetric with N odd ******************

➢ Conclusion :

I. When h[n] is Symmetric	
Type -2 filter : **N EVEN**	Type -1 filter : **N ODD**
 LPF ✓ HPF ✗	 LPF ✓ HPF ✓ BPF ✓
II. When h[n] is Anti-symmetric	
Type -4 filter : **N EVEN**	Type -3 filter : **N ODD**
 LPF ✗ HPF ✓	 LPF ✗ HPF ✗ BPF ✓

NOTE :- If no. of poles & zero's are not given, consider order no.

Q(55) One of the zeros of a third order Linear Phase filter lies at z = ½. Find transfer function and Impulse response of the filter.

Solution :

Linear Phase Filter

Order = 3

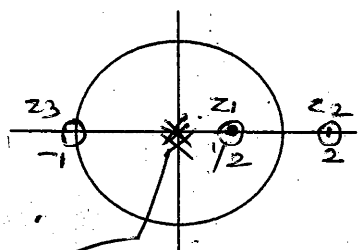
$N - 1 = 3 \qquad Z_1 = \frac{1}{2}$

$N = 4$ (Even)

Let No of POLES = 3

Let No of ZEROS = 3

For symmetric h(n) with N even there exists definite zero at $Z = -1$



3 Poles are at origin

(i) To find H(z)

$$H(z) = \frac{G(z-z_1)(z-z_2)(z-z_3)}{z^3}$$

where G is gain of filter.

Let $G = 1$,

$$H(z) = \frac{(z-\frac{1}{2})(z-2)(z+1)}{z^3}$$

ii) To find H(n) $= \frac{[z^2-(2.5)z+1](z+1)}{z^3}$

$$= \frac{z^2 - 1.5z^2 - 1.5z + 1}{z^3}$$

by power series expansion (pg no. 108)

$$H(z) = 1 - 1.5z^{-1} - 1.5z^{-2} + z^{-3}$$

By IZT,

$$h(n) = \{\underset{\uparrow}{1}, -1.5, -1.5, 1\}$$

Q(56) One of the zeros of a third order Linear Phase High Pass Filter lies at z = ½. Find transfer function and Impulse response of the filter.

Solution :

Linear Phase Filter

Order = 3

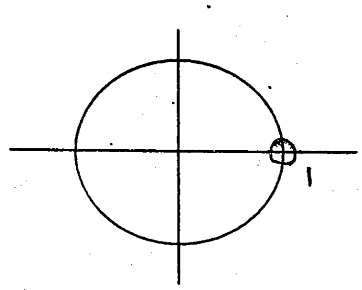
$N - 1 = 3 \qquad Z_1 = \frac{1}{2}$

$N = 4$

Let No of POLES = 3

Let No of ZEROS = 3

For Antisymm. h(n) with N even, there exists definite zero at $z = 1$

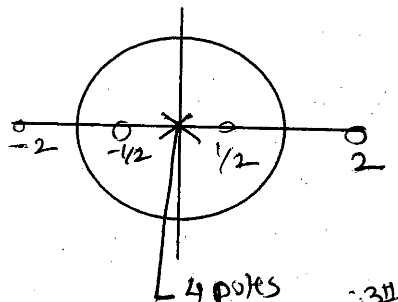


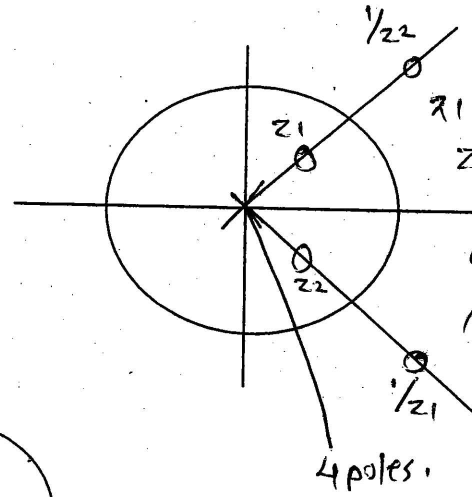
Q(57) Show three possible POLE ZERO pattern of 4^{th} order Linear Phase FIR filter with (a) symmetric h[n] and (b) Anti-symmetric h[n]

Solution : Linear Phase FIR filter

Order = 4	Let No of POLES = 4
$N-1=4$	Let No of ZEROS = 4
$N=5$	

(a) Symmetric h[n]



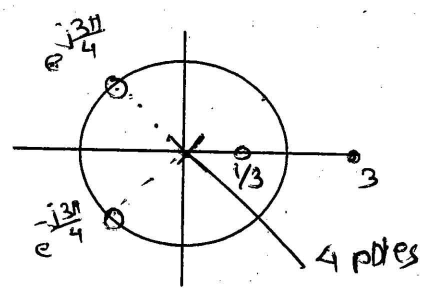


$z_1 = \frac{1}{2} e^{j\pi/4}$

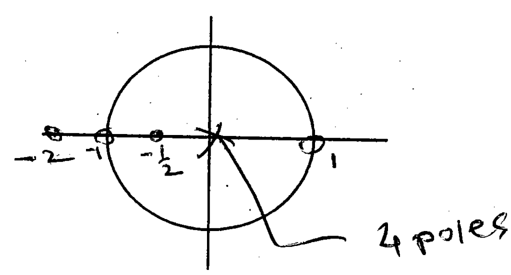
$z_2 = \frac{1}{2} e^{-j\pi/4}$

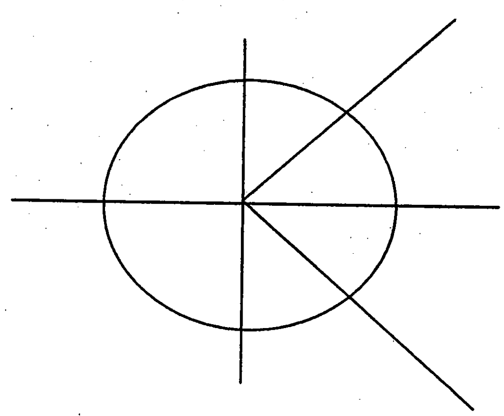
$\frac{1}{z_1} = 2 e^{-j\pi/4}$

$\frac{1}{z_2} = 2 e^{j\pi/4}$

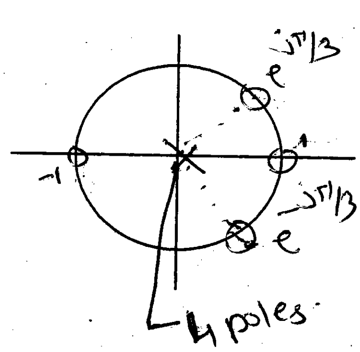


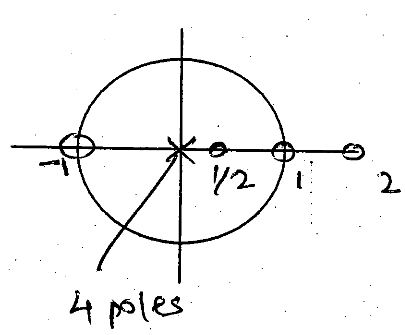
(a) Anti-symmetric h[n] with N odd has definite zeros at z = 1 and z = -1

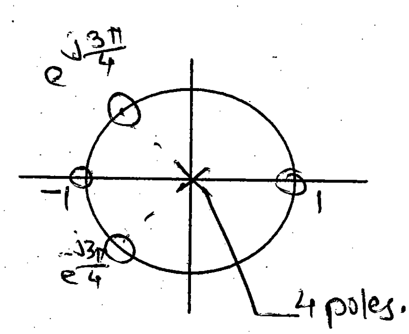




Not possible.

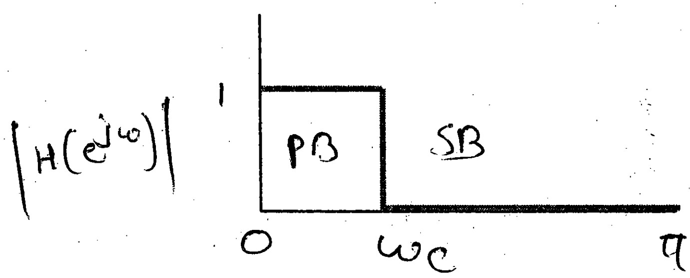






NOTE :

(I) Linear Phase LPF Design



$$H(e^{jw}) = \begin{cases} 1 & 0 \le w \le w_c \\ 0 & Otherwise \end{cases}$$

For LPF,

$H(e^{jw}) \neq 0$ *for* $0 \le w \le w_c$

$H(e^{jw}) \neq 0$ *at* $w=0$

i.e. $H(e^{jw})\Big|_{w=0} \neq 0$ *for LPF*

Put $z = e^{jw}$

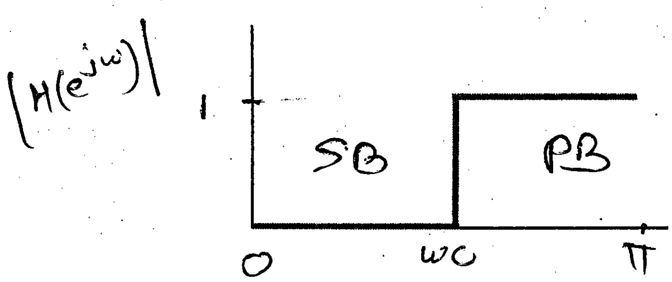
At $w = 0$ $z = 1$

$H(z)\big|_{z=1} \neq 0$ *for LPF*

For Anti-symmetric h[n], there exists definite ZERO at z = 1. i.e. $H(z)\big|_{z=1} = 0$

That means Anti-symmetric h[n] can-not be used for LPF design.

(II) Linear Phase HPF Design



$$H(e^{jw}) = \begin{cases} 1 & w_c \le w \le \pi \\ 0 & Otherwise \end{cases}$$

For HPF,

$H(e^{jw}) \neq 0$ *for* $w_c \le w \le \pi$

$H(e^{jw}) \neq 0$ *at* $w=\pi$

i.e. $H(e^{jw})\Big|_{w=\pi} \neq 0$ *for HPF*

Put $z = e^{jw}$

At $w = \pi$ $z = -1$

$H(z)\big|_{z=-1} \neq 0$ *for HPF*

For symmetric h[n] with N even and Anti-symmetric h[n] with N odd, there exists definite ZERO at z = –1. i.e. $H(z)\big|_{z=-1} = 0$

That means symmetric h[n] with N even and Anti-symmetric h[n] with N odd can-not be used for HPF design.

Q(58) One of the zeros of a causal linear phase FIR filter is at 0.5 ∠60. Show the locations of the other zeros and hence find transfer function and impulse response of the filter.

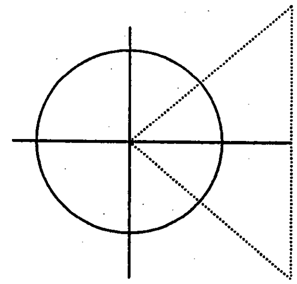
Solution : Zero of a linear phase filter occur at reciprocal location

$$z_0 = 0.5\ \angle 60^0 \quad \text{or} \quad 0.5\, e^{\frac{\pi}{3}}$$

$$z_0^* = 0.5\ \angle -60^0 \quad \text{or} \quad 0.5\, e^{-\frac{\pi}{3}}$$

$$\frac{1}{z_0} = 2 \angle -60 \quad \text{or} \quad 2e^{-\frac{\pi}{3}}$$

$$\frac{1}{z_0^*} = 2\ \angle 60^0 \quad \text{or} \quad 2e^{\frac{\pi}{3}}$$



To find H[z] :

$$H(z) = \frac{\left(z-\frac{1}{2}e^{\frac{\pi}{3}}\right)\left(z-\frac{1}{2}e^{-\frac{\pi}{3}}\right)(z-2\ e^{\frac{\pi}{3}})\ (z-2e^{-\frac{\pi}{3}})}{z^4}$$

$$= \frac{\left[z^2-\frac{1}{2}\ (e^{\frac{\pi}{3}}+e^{-\frac{\pi}{3}})\ 2+\frac{1}{4}\right]\left[z^2-2\ (e^{\frac{\pi}{3}}+e^{-\frac{\pi}{3}})\ z+4\right]}{z^4}$$

$$= \frac{\left[z^2-\cos\frac{\pi}{3}z+\frac{1}{4}\right]\left[z^2-4\ \cos\frac{\pi}{3}z+4\right]}{z^4}$$

$$= \frac{\left[z^2-\frac{1}{2}z+\frac{1}{4}\right][z^2-2z+4]}{z^4}$$

$$H(z) = \frac{z^4-2.5z^3+5.25\ z^2-2.5\ z+1}{z^4}$$

$$H(z) = 1-2.5\ z^{-1}+5.25\ z^{-2}-2.5\ z^{-3}+z^{-4}$$

By I ZT, $h(n) = \left\{ \underset{\uparrow}{1}\ -2.5,\ 5.25,\ -2.5, 1 \right\}$

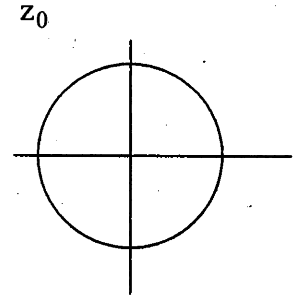
Q(59) One of the zeros of a third order causal linear phase High pass FIR filter lies at $z=\frac{1}{2}$ Find the location of the other zeros and hence find the transfer function and impulse response of the filter.

Solution : $N-1=3 \quad \therefore N=4$ (Even)

For HPF, consider anti-symmetric h[n] with N even. For Anti symmetric h[n] with N even there exists definite zero at z = 1.

Zero of a linear phase filter occur at reciprocal location $z_0=\frac{1}{2}.\ \therefore=\frac{1}{z_0}=2$.

$$H(z) = \frac{\left(z-\frac{1}{2}\right)(z-2)\ (z-1)}{z^3} = \frac{\left(z^2-\frac{5}{2}z+1\right)\ (z-1)}{z^3}$$



$$= \frac{z^3-\frac{7}{2}z^2+\frac{7}{2}z-1}{z^3}$$

$$= 1-\frac{7}{2}z^{-1}+\frac{7}{2}z^{-2}-z^{-3}$$

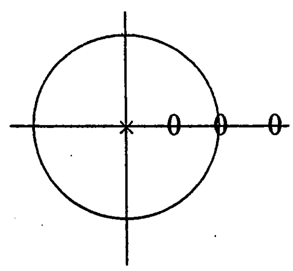
ANS : $h(n) = \left\{ \underset{\uparrow}{1}\ \frac{-7}{2},\ \frac{7}{2},\ -1 \right\}$

Q(60) An antisymmetric filter has one zero at $z = \frac{1}{2}$. What is the minimum order of this filter? Justify your answer.

Solution :

Zeros of a linear phase filter occur at reciprocal location $z_0 = \frac{1}{2}$. $\therefore = \frac{1}{z_0} = 2$.

For antisymmetric h[n] there exists definite zero at z = 1.



Total No of zeros = 3

$\therefore$ order = 3

i.e. N-1 = 3

so N = 4 (Even)

Minimum order is 3.

Q(61) Draw pole zero location of third order linear phase LPF & HPF.

Solution : Linear Phase FIR filter

Order = 3

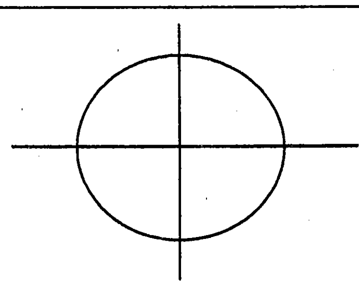
N – 1 = 3

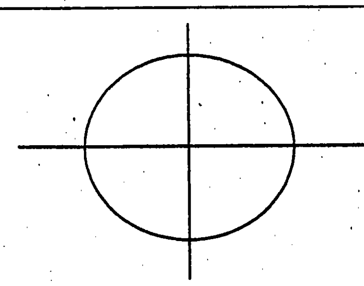
N = 4 (EVEN)

Let no of POLES = 3

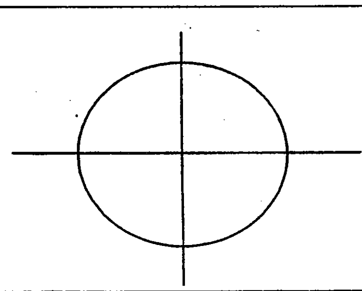
No of ZEROS = 3

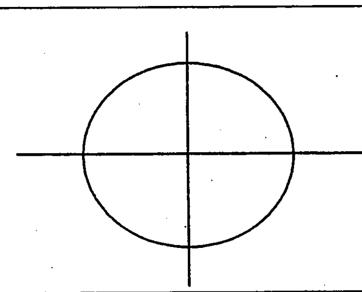
I. Linear Phase LPF with Symmetric h[n] and N EVEN
(Definite ZERO at z = – 1)





II. Linear Phase HPF with Anti-Symmetric h[n] and N EVEN.
(Definite ZERO at z = 1)





Q(62) Plot magnitude and phase response of second order antisymmetric, linear phase Band-Pass FIR Filter.

Q(63) A fourth order antisymmetric, linear phase filter has zero at 0.5. What is the response of filter to the input $x[n] = (\frac{1}{2})^n \cos(n\frac{\pi}{3})\, u[n]$.

Q(64) Explain FIR Filter design Using Window Function (Note on windowing method)

FIR Filter is designed by truncating the infinite samples of IIR Filter. Let hd[n] denote the impulse response sequence of IIR Filter. To have the linear phase response, the impulse response must be symmetric or antisymmetric about some point in time. To simplify the analysis we consider hd[n] to be symmetric about n = 0.

The N point impulse response of FIR can be obtained as, $h[n] = \begin{cases} hd[n] & 0 \le n \le N-1 \\ O & otherwise \end{cases}$ To truncate the samples of hd[n], window function is used.The impulse response of FIR Filter is then given by, **h[n] = hd[n] . w[n]** where w[n] is a window function and is symm about n=0 to maintain the symmetry that is present in hd[n].

steps Algo

(1) $\boxed{Hd(\omega)}$ Desired freq Respons

(2) $\boxed{hd(n)}$

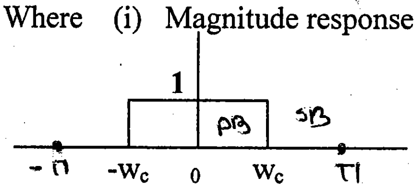
(3) $\boxed{h(n)}$

Ex. Linear Phase LPF design

Step 1 : Find Hd(w)

Let $\boxed{Hd(w) = | Hd(w) | \; \phi(w)}$

Where (i) Magnitude response :



$$H_d(\omega) = \begin{cases} 1 & for \; -w_c \le |\omega| \le w_c \\ 0 & otherwise \end{cases}$$

(ii). Phase Response : $\phi(w) = e^{j\phi}$

For Linear Phase LPF with symmetric h[n] $\phi = -\left(\frac{N-1}{2}\right) w = -\alpha w$

$$\therefore \phi(w) = e^{-j\alpha w}$$

By substituting, $\mathbf{H_d(\omega)} = \begin{cases} e^{-j\alpha w} & -w_c \le |\omega| \le w_c \\ 0 & otherwise \end{cases}$

Step 2 : Find hd[n]

By Inverse DTFT, $\boxed{h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(w)e^{jnw}\, dw}$

$$= \frac{1}{2\pi}\int_{-w_c}^{w_c} (e^{-j\alpha w})e^{jnw}\, dw$$

$$= \frac{1}{2\pi}\int_{-w_c}^{w_c} e^{j(n-\alpha)w}\, dw$$

$$= \frac{1}{2\pi}\left[\frac{e^{j(n-\alpha)w}}{(n-\alpha)j}\right]_{-w_c}^{w_c}$$

$$= \frac{1}{\pi(n-\alpha)}\left[\frac{e^{j(n-\alpha)w_c} - e^{-j(n-\alpha)w_c}}{2j}\right]$$

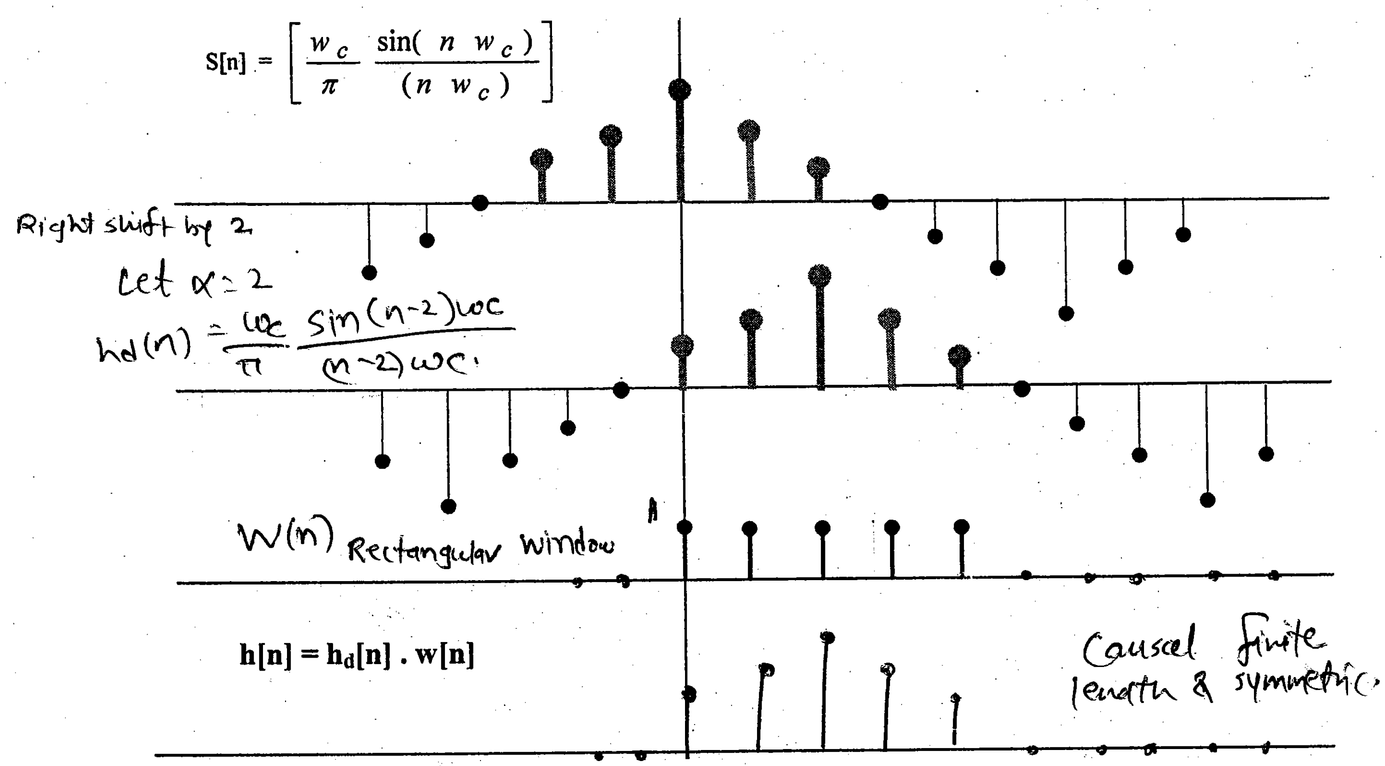
$$= \frac{1}{\pi(n-\alpha)}\sin(n-\alpha)w_c \times \frac{(n-\alpha)w_c}{(n-\alpha)w_c}$$

$$\boxed{h_d[n] = \frac{w_c}{\pi}\,\frac{\sin(n-\alpha)w_c}{(n-\alpha)w_c}}$$

L'Hospital Rule $\lim_{w \to 0} \frac{\sin(w)}{w} = 1$

Step 3. Find h[n]

Linear phase FIR filter with impulse response h[n] is given by, $h[n] = h_d[n] \; w[n]$ where w[n] is window function



Q(65) Design 6^{th} order Linear Phase Low Pass FIR filter with $w_c = \frac{\pi}{2}$ using following window function
$$w[n] = \begin{cases} 0.42 - 0.5 \cos\left(\frac{2\pi n}{N-1}\right) + 0.08 \cos\left(\frac{4\pi n}{N-1}\right) & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

Solution : Linear Phase FIR filter

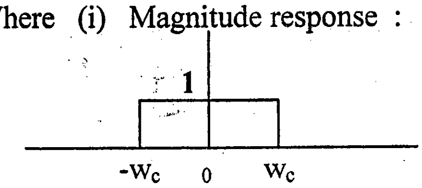
Order = 6
$N - 1 = 6$
$N = 7$

$$\alpha = \frac{N-1}{2} = \underline{\underline{3}}$$

Step 1 : Find Hd(w)

Let $Hd(w) = |Hd(w)| \; \phi(w)$

Where (i) Magnitude response :



$$H_d(\omega) = \begin{cases} 1 & \text{for } -w_c \le |\omega| \le w_c \\ 0 & \text{otherwise} \end{cases}$$

(ii) Phase Response : $\phi(w) = e^{j\phi}$

For Linear Phase LPF with symmetric h[n] $\phi = -\left(\frac{N-1}{2}\right) w = -\alpha w$

$$\therefore \phi(w) = e^{-j\alpha w}$$

By substituting, $H_d(w) = \begin{cases} e^{-j\alpha w} & -w_c \le |\omega| \le w_c \\ 0 & otherwise \end{cases}$

where $\alpha = 3$ and $w_c = \frac{\pi}{2}$

Step 2 : Find hd[n]

By Inverse DTFT, $h_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(w) e^{jnw}\, dw$

$$= \frac{1}{2\pi} \int_{-w_c}^{w_c} (e^{-j\alpha w}) e^{jnw}\, dw$$

$$= \frac{1}{2\pi} \int_{-w_c}^{w_c} e^{j(n-\alpha)w}\, dw$$

$$= \frac{1}{2\pi} \left[\frac{e^{j(n-\alpha)w}}{(n-\alpha)j} \right]_{-w_c}^{w_c}$$

$$= \frac{1}{\pi(n-\alpha)} \left[\frac{e^{j(n-\alpha)w_c} - e^{-j(n-\alpha)w_c}}{2j} \right]$$

$$= \frac{1}{\pi(n-\alpha)} \sin(n-\alpha)w_c \times \frac{(n-\alpha)w_c}{(n-\alpha)w_c}$$

$$\boxed{h_d[n] = \frac{w_c}{\pi} \frac{\sin(n-\alpha)w_c}{(n-\alpha)w_c}} \quad \text{where } \alpha = 3 \text{ and } w_c = \frac{\pi}{2}$$

Step 3. Find h[n]

Linear phase FIR filter with impulse response h[n] is given by,

h[n] = h_d[n] w[n]

$$h[n] = \left[\frac{1}{2} \frac{\sin(n-3)\frac{\pi}{2}}{(n-3)\frac{\pi}{2}} \right] \left[0.42 - 0.5 \cos\left(\frac{2\pi n}{6}\right) + 0.08 \cos\left(\frac{4\pi n}{6}\right) \right]$$

$$h[n] = \begin{bmatrix} -0.106 & n=0 \\ 0 & n=1 \\ 0.318 & n=2 \\ 0.5 & n=3 \\ 0.318 & n=4 \\ 0 & n=5 \\ -0.106 & n=6 \end{bmatrix} \begin{bmatrix} 0 \\ 0.13 \\ 0.63 \\ 1.00 \\ 0.63 \\ 0.13 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.2 \\ 0.5 \\ 0.2 \\ 0 \\ 0 \end{bmatrix}$$

Ans

Q(66) Design 6^{th} order Linear Phase High Pass FIR filter with $w_c = \frac{\pi}{2}$ using following window function. $w[n] = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right) & 0 \le n \le N-1 \\ 0 & otherwise \end{cases}$

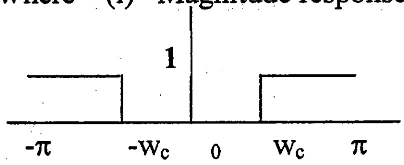
Solution : Linear Phase FIR filter

Order = 6
N – 1 = 6
N = 7

Step 1 : Find Hd(w)

Let $Hd(w) = |Hd(w)|\ \phi(w)$

Where (i) Magnitude response :



$$\mathbf{H_d(\omega)} = \begin{cases} 1 & for\ -\pi < \omega \le -w_c \\ 0 & for\ -w_c \le \omega \le w_c \\ 1 & for\ w_c \le \omega \le \pi \end{cases}$$

(ii.) Phase Response : $\phi(w) = e^{j\phi}$

For Linear Phase HPF with symmetric h[n] $\phi = -\left(\frac{N-1}{2}\right) w = -\alpha w$

$$\therefore \phi(w) = e^{-j\alpha w}$$

By substituting, $\mathbf{H_d(\omega)} = \begin{cases} e^{-j\alpha w} & for\ -\pi < \omega \le -w_c \\ 0 & for\ -w_c \le \omega \le w_c \\ e^{-j\alpha w} & for\ w_c \le \omega \le \pi \end{cases}$

where $\alpha = 3$ and $w_c = \frac{\pi}{2}$

Step 2 : Find hd[n]

By Inverse DTFT, $h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(w)e^{jnw}\,dw$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-w_c} e^{-j\alpha w} e^{jnw}\,dw + \int_{w_c}^{\pi} e^{-j\alpha w} e^{jnw}\cdot dw\right]$$

$$= \frac{1}{2\pi}\left[\left\{\frac{e^{j(n-\alpha)w}}{(n-\alpha)j}\right\}_{-\pi}^{-w_c} + \left\{\frac{e^{j(n-\alpha)w}}{(n-\alpha)j}\right\}_{w_c}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j(n-\alpha)w_c} - e^{-j(n-\alpha)\pi}}{(n-\alpha)j} + \frac{e^{j(n-\alpha)\pi} - e^{j(n-\alpha)w_c}}{(n-\alpha)j}\right]$$

$$= \frac{1}{\pi(n-\alpha)}\left[\left(\frac{e^{j(n-\alpha)\pi} - e^{-j(n-\alpha)\pi}}{2j}\right) - \left(\frac{e^{j(n-\alpha)w_c} - e^{-j(n-\alpha)w_c}}{2j}\right)\right]$$

$$h[n] = \frac{1}{\pi(n-\alpha)}[\sin(n-\alpha)\pi - \sin(n-\alpha)w_c]$$

$$\boxed{\therefore\ h[n] = \frac{\sin(n-\alpha)\pi}{(n-\alpha)\pi} - \frac{w_c}{\pi}\,\frac{\sin(n-\alpha)w_c}{(n-\alpha)w_c}}$$

where $\alpha = 3$ and $w_c = \frac{\pi}{2}$

Step 3. Find h[n]

Linear phase FIR filter with impulse response h[n] is given by,

$$h[n] = h_d[n]\ \ w[n]$$

$$h[n] = \left[\frac{\sin(n-3)\pi}{(n-3)\pi} - \frac{1}{2}\,\frac{\sin(n-3)\frac{\pi}{2}}{(n-3)\frac{\pi}{2}} \right] \left[0.54 - 0.46\ \cos\left(\frac{2\pi n}{6}\right) \right]$$

$$h[n] = \left[\quad\quad\quad \right] \left[\quad\quad\quad \right] = \left[\quad\quad\quad \right]$$

Q(67) Design 5^{th} order Linear Phase Low Pass FIR filter with $w_c = \frac{\pi}{3}$ using hamming window function.

ANS : h[n] = { 0.0051, 0.0844, 0.2900, 0.2900, 0.0844, 0.0051 }

Q(68) Design 5^{th} order Linear Phase High Pass FIR filter with $w_c = \frac{\pi}{3}$ using hamming window function.

ANS : h[n] = { 0.0051 , - 0.1689 , 0.2900, - 0.2900, 0.1689, - 0.0051 }

Q(69) A filter is required to be designed with the following frequency response

$$Hd(e^{jw}) = \begin{cases} 2\ e^{-j2w} & \frac{-\pi}{4} \le w \le \frac{\pi}{4} \\ 0 & \text{Otherwise} \end{cases}$$

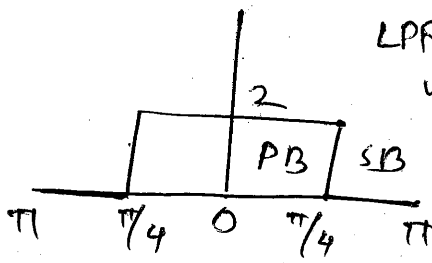
a) Determine the filter coefficients if rectangular window function is used for design
b) Determine the frequency response $H(e^{jw})$ of the designed filter.
c) Show realization of the filter using minimum number of multipliers.
d) Find the response of the filter to the input

i) $x[n] = (\frac{1}{2})^n \cos(n\frac{\pi}{3})\, u[n]$.   ii) $x[n] = (\frac{1}{2})^n \cos(n\frac{\pi}{3})$.

Solution : $H(w) = Hd(e^{jw}) = \begin{cases} 2\ e^{-j2w} & \frac{-\pi}{4} \le w \le \frac{\pi}{4} \\ 0 & \text{Otherwise} \end{cases}$

Freq. Response $|Hd(\omega)|$ $\phi(\omega)$

(i) Magnitude response



LPF with $w_c = \frac{\pi}{4}$,

(ii) Phase $\varnothing = -2\,w$

Linear Phase. constant freq,

for symm. h(n),

$\phi = -\left(\frac{N-1}{2}\right) w = -2w$

$\frac{N-1}{2} = 2$

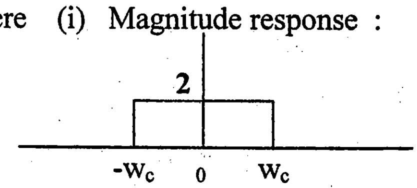
$N-1 = 4 \quad \therefore N=5, \boxed{\alpha = 2}$

(a) To find filter coefficients

Step 1 : Find Hd(w)

Let $Hd(w) = |Hd(w)|\ \phi(w)$

Where (i) Magnitude response :



$$H_d(\omega) = \begin{cases} 2 & for\ -w_c \le |\omega| \le w_c \\ 0 & otherwise \end{cases}$$

(ii) Phase Response : $\phi(w) = e^{j\phi}$

For Linear Phase LPF with symmetric h[n] $\phi = -\left(\frac{N-1}{2}\right) w = -\alpha w$

$\therefore \phi(w) = e^{-j\alpha w}$

By substituting, $H_d(\omega) = \begin{cases} 2\,e^{-j\alpha w} & -w_c \le |\omega| \le w_c \\ 0 & otherwise \end{cases}$

where $\alpha = 2$ and $w_c = \frac{\pi}{4}$

Step 2 : Find hd[n]

By Inverse DTFT, $h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(w)e^{jnw}\,dw$

****

$$\boxed{h_d[n] = \frac{2\,w_c}{\pi}\ \frac{\sin(\ n-\alpha\ )w_c}{(n-\alpha\ )w_c}}$$ where $\alpha = 2$ and $w_c = \frac{\pi}{4}$

Step 3. Find h[n]

Linear phase FIR filter with impulse response h[n] is given by,

$h[n] = h_d[n]\ w[n]$ where $w(n) = 1$ for rectangular function

$$h[n] = \left[\frac{1}{2}\ \frac{\sin(\ n-2)\frac{\pi}{4}}{(n-2)\frac{\pi}{4}}\right]$$

Ans : h[n] = { 0.318, 0.45, 0.5, 0.45, 0.318 } for n ≥ 0

Solution : (b) To find Frequency Response

Now, $h[n] = \{ 0.318, 0.45, 0.5, 0.45, 0.318 \}$

Let $h[n] = \{ h_o, h_1, h_2, h_1, h_o \}$

By ZT, $H(z) = h_o + h_1 z^{-1} + h_2 z^{-2} + h_1 z^{-3} + h_o z^{-4}$

Put $z = e^{jw}$

$$H(e^{jw}) = h_0 + h_1 e^{-jw} + h_2 e^{-j2w} + h_1 e^{-j3w} + h_0 e^{-j4w}$$

Substituting,
$$= e^{-j2w} [h_0 e^{j2w} + h_1 e^{jw} + h_2 + h_1 e^{-jw} + h_0 e^{-2w}]$$

$$= e^{-j2w} [h_0 (e^{j2w} + e^{-j2w}) + h_2 + h_1 (e^{jw} + e^{-jw})]$$

$$H(e^{jw}) = e^{-j2w} [2 h_0 \cos(2w) + 2 h_1 \cos(w) + h_2]$$

$$H(e^{jw}) = e^{-j2w} [2 (0.318) \cos(2w) + 2 (0.45) \cos(w) + (0.5)]$$

$$H(e^{jw}) = e^{-j2w} [0.636 \cos(2w) + 0.9 \cos(w) + 0.5)] \quad \textbf{ANS}$$

Solution : (c) To find Realization diagram

Now, $h[n] = \{ 0.318, 0.45, 0.5, 0.45, 0.318 \}$

Let $h[n] = \{ h_o, h_1, h_2, h_1, h_o \}$

By ZT, $H(z) = h_o + h_1 z^{-1} + h_2 z^{-2} + h_1 z^{-3} + h_o z^{-4}$

$$\frac{Y(z)}{X(z)} = h_o + h_1 z^{-1} + h_2 z^{-2} + h_1 z^{-3} + h_o z^{-4}$$

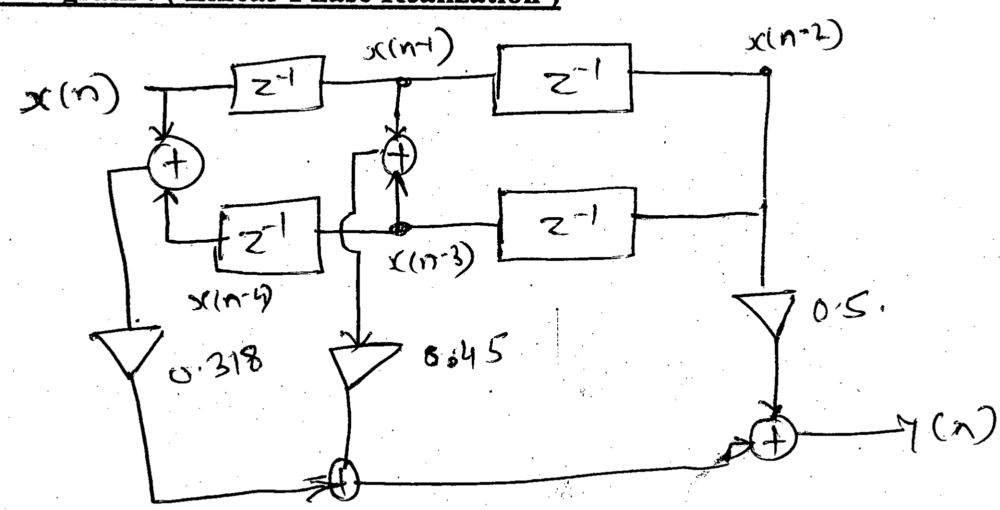
$$Y(z) = h_o X(z) + h_1 z^{-1} X(z) + h_2 z^{-2} X(z) + h_1 z^{-3} (z) + h_o z^{-4} X(z)$$

By IZT, $y[n] = h_o x[n] + h_1 x[n-1] + h_2 x[n-2] + h_1 x[n-3] + h_o x[n-4]$

$$y[n] = h_o (x[n] + x[n-4]) + h_1 (x[n-1] + x[n-3]) + h_2 x[n-2]$$

$$y[n] = \mathbf{0.318} (x[n] + x[n-4]) + \mathbf{0.45} (x[n-1] + x[n-3]) + \mathbf{0.5} x[n-2]$$

<u>Realization Diagram : (Linear Phase Realization)</u>



[1] Linear Phase Realization Method requires Less no of Multipliers.

[2] In IIR filter output depends on output values.

e.g. $y[n] = x[n] + x[n-1] + y[n] + y[n-1]$.

Therefore **IIR filters** are also called as **Recursive** Filters.

[3] In FIR filter output depends only on input values. It doesn't depend on output values.
e.g. $y[n] = x[n] + x[n-1]$

Therefore **FIR filters** are also called as **Non-Recursive** Filters.

Solution : (d) To find Response of the filter

(i) Input : $x[n] = (\frac{1}{2})^n \cos(n\frac{\pi}{3})\, u[n]$

⟹ Input is sinusoidal, Infinite Length and applied to the system at $n = 0$

Impulse Response is : $h[n] = \{ 0.318,\ 0.45,\ 0.5,\ 0.45,\ 0.318 \}$ Finite Length

To find y[n] :

Let $h[n] = \{ ho,\ h_1,\ h_2,\ h_1,\ ho \}$

By ZT, $H(z) = ho + h_1 z^{-1} + h_2 z^{-2} + h_1 z^{-3} + ho\, z^{-4}$

$$\frac{Y(z)}{X(z)} = ho + h_1 z^{-1} + h_2 z^{-2} + h_1 z^{-3} + ho\, z^{-4}$$

cross multiply,

$$Y(z) = ho\, X(z) + h_1 z^{-1} X(z) + h_2 z^{-2} X(z) + h_1 z^{-3} (z) + ho\, z^{-4} X(z)$$

By IZT, $y[n] = ho\, x[n] + h_1 x[n-1] + h_2 x[n-2] + h_1 x[n-3] + ho\, x[n-4]$

$$\text{ANS : } y[n] = 0.318\ (\tfrac{1}{2})^n \cos(n\tfrac{\pi}{3})\ u[n] + \\ 0.45\ (\tfrac{1}{2})^{n-1} \cos(n-1)\tfrac{\pi}{3}\ u[n-1] + \\ 0.50\ (\tfrac{1}{2})^{n-2} \cos(n-2)\tfrac{\pi}{3}\ u[n-2] + \\ 0.45\ (\tfrac{1}{2})^{n-3} \cos(n-3)\tfrac{\pi}{3}\ u[n-3] + \\ 0.318\ (\tfrac{1}{2})^{n-4} \cos(n-4)\tfrac{\pi}{3}\ u[n-4]$$

(ii) Input : $x[n] = (\frac{1}{2})^n \cos(n\frac{\pi}{3})$

⟹ Input is sinusoidal, Infinite Length and applied to the system at $n = -\infty$

The output of the system is given by $y[n] = y_{tr}[n] + y_{ss}[n]$

But $y_{tr}[n] = 0 \quad \therefore\ y[n] = y_{ss}[n]$

To find $y_{ss}[n]$:

Find Input signal frequency components : $w = \{ \frac{\pi}{3} \}$

Find freq response : $H(e^{jw}) = e^{-j2w}[\ 0.636\ \cos(2w) + 0.9\ \cos(w) + 0.5\ )\ ]$

Find magnitude and phase value for every input signal frequency :

Magnitude : M_1 =

Phase : ϕ_1 =

The steady state response of the system is then given by

ANS : y[n] =

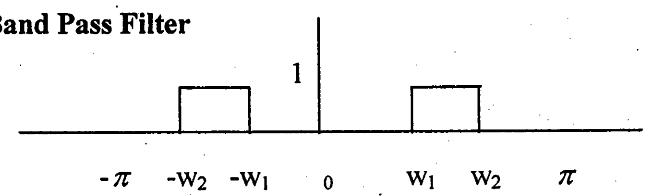
Q(70) Design sixth order linear phase band pass FIR filter with pass-band frequencies $w_1 = 0.25\,\pi$ and $w_2 = 0.6\pi$ using hamming window function.

ANS : h[n] = { 0.0051 , -- 0.1689 , 0.2900, -- 0.2900, 0.1689, -- 0.0051 }

Q(71) Consider the magnitude response of ideal filters as shown below. Find impulse response h[n] in each case

Solution :

a) Band Pass Filter



Solution : By Inverse DTFT,

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(w)\, e^{jnw} dw = \frac{1}{2\pi}\left[\int_{-w_2}^{-w_1} H(w)e^{jnw}\ dw + \int_{w_1}^{w_2} H(w)e^{jnw}\ dw\right]$$

$$= \frac{1}{2\pi}\left[\int_{-w_2}^{-w_1} (1)e^{jnw}\ dw + \int_{w_1}^{w_2} (1)e^{jnw}\ dw\right]$$

$$= \frac{1}{2\pi}\left[\left\{\frac{e^{jnw}}{nj}\right\}_{-w_2}^{-w_1} + \left\{\frac{e^{jnw}}{nj}\right\}_{w_1}^{w_2}\right]$$

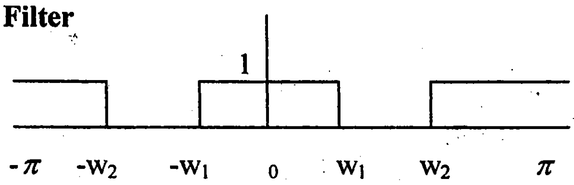
$$= \frac{1}{2\pi}\left[\frac{e^{-jnw_1} - e^{-jnw_2}}{nj} + \frac{e^{jnw_2} - e^{jnw_1}}{nj}\right]$$

$$= \frac{1}{\pi n}\left[\frac{e^{jnw_2} - e^{-jnw_2}}{2j} - \frac{e^{jnw_1} - e^{-jnw_1}}{2j}\right]$$

$$h[n] = \frac{1}{\pi n}[\sin(nw_2) - \sin(nw_1)]$$

$$\boxed{h[n] = \frac{w_2}{\pi}\frac{\sin(nw_2)}{nw_2} - \frac{w_1}{\pi}\frac{\sin(nw_1)}{nw_1}}$$

b) Band Stop Filter



Solution : By Inverse DTFT,

$$h[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} H(w)e^{jnw}dw \quad =\frac{1}{2\pi}\left[\int_{-\pi}^{-w_2} H(w)e^{jnw}\,dw+\int_{-w_1}^{w_1} H(w)e^{jnw}dw+\int_{w_2}^{\pi} H(w)\,e^{jnw}dw\right]$$

$$=\frac{1}{2\pi}\left[\int_{-\pi}^{-w_2}(1)e^{jnw}\;dw+\int_{-w_1}^{w_1}(1)e^{jnw}\;dw+\int_{w_2}^{\pi}(1)e^{jnw}\;dw\right]$$

$$=\frac{1}{2\pi}\left[\left\{\frac{e^{jnw}}{nj}\right\}_{-\pi}^{-w_2}+\left\{\frac{e^{jnw}}{nj}\right\}_{-w_1}^{w_1}+\left\{\frac{e^{jnw}}{nj}\right\}_{w_2}^{\pi}\right]$$

$$=\frac{1}{2\pi}\left[\frac{e^{-jnw_2}-e^{-jn\pi}}{nj}+\frac{e^{jnw_1}-e^{-jnw_1}}{nj}+\frac{e^{jn\pi}-e^{jnw_2}}{nj}\right]$$

$$=\frac{1}{\pi n}\left[\frac{e^{jn\pi}-e^{-jn\pi}}{2j}+\frac{e^{jnw_1}-e^{-jnw_1}}{2j}-\left(\frac{e^{jnw_2}-e^{-jnw_2}}{2j}\right)\right]$$

$$=\frac{1}{\pi n}\left[\sin(n\pi)+\sin(nw_1)-\sin(nw_2)\right]$$

$$\boxed{\therefore\; h[n]=\frac{\sin(n\pi)}{n\pi}+\frac{w_1}{\pi}\,\frac{\sin(nw_1)}{nw_1}-\frac{w_2}{\pi}\,\frac{\sin(nw_2)}{nw_2}}$$

➢ Algorithm To Design Linear Phase F I R Filter Using Window Function

I. Accept the specifications : Ap, As, wp, ws

II. Select the window function such that window function stop-band Attenuation exceeds the specified As. Select Hamming window.

III. Select the number of points in the window to satisfy the transition width for the type of window function. find N.

Transition width Of desired filter $\geq$ Transition width of window function

$$f_2 - f_1 \geq \frac{C}{N}$$

i.e. $N \geq \dfrac{C}{fs - fp}$ for LPF

$N \geq \dfrac{C}{fp - fs}$ for HPF

IV. Select the cut off frequency w_c : Take $w_c = \dfrac{w_p + w_s}{2}$ find wc

V. The impulse response h[n] of desired linear phase FIR is then given by,

$$\mathbf{h[n] = h_d[n].\ w[n]}$$

where

1	**LPF**	$hd[n] = \dfrac{wc}{\pi}\dfrac{\sin(n-\alpha)wc}{(n-\alpha)wc}$
2	**HPF**	$hd[n] = \dfrac{\sin(n-\alpha)\pi}{(n-\alpha)\pi} - \dfrac{wc}{\pi}\dfrac{\sin(n-\alpha)wc}{(n-\alpha)wc}$
3	**BPF**	$hd[n] = \dfrac{w_2}{\pi}\dfrac{\sin(n-\alpha)w_2}{(n-\alpha)w_2} - \dfrac{w_1}{\pi}\dfrac{\sin(n-\alpha)w_1}{(n-\alpha)w_1}$
4	**BRF / BSF**	$hd[n] = \dfrac{\sin(n-\alpha)\pi}{(n-\alpha)\pi} + \dfrac{w_1}{\pi}\dfrac{\sin(n-\alpha)w_1}{(n-\alpha)w_1} - \dfrac{w_2}{\pi}\dfrac{\sin(n-\alpha)w_2}{(n-\alpha)w_2}$

	Window Function	**Main lobe width**	**Transition width C/N**	**As**
1	Rectangular $w[n] = \begin{cases} 1 & 0 \leq n \leq N-1 \\ 0 & otherwise \end{cases}$	$\frac{4\pi}{N}$	**C = 0.92**	**21** ✓
2	Bartlett $w[n] = \begin{cases} 2n/(N-1) & 0 \leq n \leq (N-1)/2 \\ 2-2n/(N-1) & (N-1)/2 \leq N-1 \\ 0 & otherwise \end{cases}$	$\frac{8\pi}{N}$	**C = 2.1**	**25**
3	Hanning $w[n] = \begin{cases} \left\{1-\cos\left(\frac{2\pi n}{N-1}\right)\right\}/2 & 0 \leq n \leq N-1 \\ 0 & otherwise \end{cases}$	$\frac{8\pi}{N}$	**C = 3.21**	**44**
4	Hamming $w[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right) & 0 \leq n \leq N-1 \\ 0 & otherwise \end{cases}$	$\frac{8\pi}{N}$	**C = 3.14**	**53** ✓
5	Blackman $w[n] = \begin{cases} 0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right) & 0 \leq n \leq N-1 \\ 0 & otherwise \end{cases}$	$\frac{12\pi}{N}$	**C = 5.71**	**74**

Q(72) Given Ap = 1 dB As = 40 dB Wp = 0.2 π Ws = 0.8 π

Design a Linear Phase FIR filter using appropriate window function.

Solution :

$W_p = 2\pi f_p = 0.2\pi \quad \therefore f_p = 0.1$

$W_s = 2\pi f_s = 0.8\pi \quad \therefore f_s = 0.4$

Let $f_1 = f_p = 0.1$ and $f_2 = f_s = 0.4$

(i) Select window [Hamming]

For Hamming window, As = 53 db

(ii) Find N

Transition width of filter > Transition width of window.

$$f_2 - f_1 \geq \frac{C}{N}$$

$$N \geq \frac{C}{f_2 - f_1}, \quad N \geq \frac{C = 3.14}{0.4 - 0.1}$$ for Hamming window

$$N \geq \frac{3.14}{0.3} \geq 10.46 = 11$$

(iii) Find Wc

$$w_c = \frac{w_p + w_s}{2} = 0.5\pi \text{ radians.}$$

(I) Find $H_d(w)$

$$H_d(w) = \begin{cases} e^{-j\alpha w} & -w_c \leq |\omega| \leq w_c \\ 0 & otherwise \end{cases}$$

where $\alpha = 5 \quad w_c = 0.5\pi$ radians

(II) Find $h_d[n]$. [Derivation is expected in exam.]

$$hd[n] = \frac{wc}{\pi} \frac{\sin(n-\alpha)wc}{(n-\alpha)wc}$$

(III) Find h[n]

$$h[n] = h_d[n].\ w[n]$$

$$h[n] = \left[\frac{1}{2} \frac{\sin(n-5)\frac{\pi}{2}}{(n-5)\frac{\pi}{2}} \right] \left[0.54 - 0.46 \cos\left(\frac{2\pi n}{10}\right) \right]$$

$$h[n] = \begin{bmatrix} 0.06 \\ 0 \\ -0.106 \\ 0 \\ 0.318 \\ 0.5 \\ 0.318 \\ 0 \\ -0.106 \\ 0 \\ 0.06 \end{bmatrix} \begin{bmatrix} 0.08 \\ 0.16 \\ 0.39 \\ 0.68 \\ 0.91 \\ 1.0 \\ 0.91 \\ 0.68 \\ 0.39 \\ 0.16 \\ 0.08 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

Q(73) Design a FIR linear phase digital filter approximating the ideal frequency response.

$$H_d(\omega) = \begin{cases} 1 & \text{for } |\omega| \le \pi/6 \\ 0 & \text{for } \pi/6 < |\omega| \le \pi \end{cases}$$

Determine the coefficients of a 25 tap filter based on the window method using rectangular window.

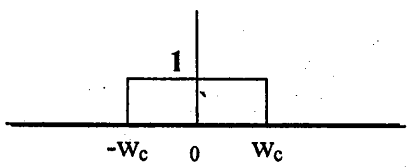
⟹

The ideal filter that we would like to approximate is a low pass filter with a cut off frequency $\omega_p = \pi/6$ and $N = 25$.

Step 1 : Find Hd(w)

Let $Hd(w) = |Hd(w)|\ \phi(w)$

Where (i) Magnitude response :



$$\mathbf{H_d(\omega)} = \begin{cases} 1 & \textit{for } -w_c \le |\omega| \le w_c \\ 0 & \textit{otherwise} \end{cases}$$

(iii) Phase Response : $\phi(w) = e^{j\phi}$

For Linear Phase LPF $\phi = -\dfrac{N-1}{2} = -12w$

$$\therefore \phi(w) = e^{-j12w}$$

By substituting, $\mathbf{H_d(\omega)} = \begin{cases} e^{-j12w} & -w_c \le |\omega| \le w_c \\ 0 & \textit{otherwise} \end{cases}$

Step 2 : Find hd[n]

By Inverse DTFT, $h_d[n] = \sum_{n=0}^{N-1} H_d(w)\ e^{-jn\omega}$ where N = 25

****

$$h_d[n] = \frac{\sin\frac{\pi}{6}(n-12)}{\frac{\pi}{6}(n-12)}$$

Step 3. Find h[n]

Linear phase FIR filter with impulse response h[n] is given by, $h[n] = h_d[n]\ w[n]$ where w[n] is window function
For rectangular window $w[n] = 1$ for $0 < n < N-1$.

h[n] is then given by, $h[n] = \dfrac{\sin\frac{\pi}{6}(n-12)}{\frac{\pi}{6}(n-12)}$ for $0 \le n \le 24$.

---

1) magnitude & phase spectrum
2) Rec to polar.
3) signal

FIR Filter Design by FREQUENCY SAMPLING METHOD

In frequency sampling method, a desired frequency response is sampled at $w = \frac{2\pi k}{N}$ For $k = 0, 1, 2, \ldots\ldots N-1$.

i.e. $H[k] = H_d(w)\Big|_{w=\frac{2\pi k}{N}}$ For $k = 0, 1, 2, \ldots\ldots N-1$

The samples thus obtained are identified as DFT coefficients.

The filter coefficients are then obtained by taking IDFT of this set of samples.

The filter coefficients are given by $h[n] = \frac{1}{N}\sum_{k=0}^{N-1} H[k]\, W_N^{-nk}$

Q(74) Design 6^{th} order Linear Phase $\boxed{\textbf{Low Pass FIR filter}}$ with cut off frequency $w_c = \frac{\pi}{2}$ using frequency sampling method

Algo

(1) $\boxed{H_d(\omega)}$ Desired freq. response (DTFT)

$\downarrow$

(2) $\boxed{H(k)}$ DFT coefie.

$\downarrow$

(3) $\boxed{h(n)}$

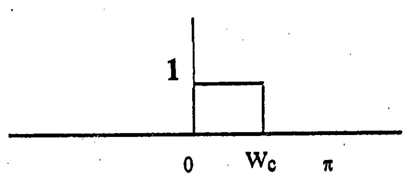
ANS :

Order = 6
N-1 = 6
N = 7

STEP - 1. Find $H_d(w)$

Let $H_d(w) = |H_d(w)|\ \phi(w)$

Where (i) Magnitude response :



$$H_d(\omega) = \begin{cases} 1 & \text{for } 0 \le w \le w_c \\ 0 & \text{otherwise} \end{cases}$$

(ii) Phase Response : $\phi(w) = e^{j\phi}$

For Linear Phase LPF, $\phi = -\frac{N-1}{2} = -3w \quad \therefore \phi(w) = e^{-j3w}$

By substituting, $$H_d(\omega) = \begin{cases} e^{-j3w} & 0 \le w \le w_c \\ 0 & \text{otherwise} \end{cases}$$ **where** $w_c = \frac{\pi}{2}$

$$H_d(\omega) = \begin{cases} e^{-j3w} & 0 \le w \le \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

STEP - 2. Find H[k]

By frequency sampling,

$$H[k] = H_d(w)\Big|_{w=\frac{2\pi k}{N}} \qquad \text{For } k = 0, 1, 2, \ldots\ldots N-1$$

$$H[k] = H_d(w)\Big|_{w=\frac{2\pi k}{7}}$$

$$H[k] = \begin{cases} e^{-j3\left(\frac{2\pi k}{7}\right)} & 0 \le w \le \frac{\pi}{2} \\ 0 & otherwise \end{cases}$$

$$H[k] = \left[\begin{array}{lll} 1 & \longrightarrow k=0 & \omega = 0 \\ e^{-j\frac{6\pi}{7}} & \longrightarrow k=1 & \omega = 2\pi/7 \\ 0 & \longrightarrow k=2 & \omega = \frac{4\pi}{7} \rightarrow \text{pos}, K = N/2 \\ 0 & \longrightarrow k=3 & \omega = 6\pi/7 \\ 0 & \longrightarrow k=4 & \\ e^{+j6\pi/7} & \longrightarrow k=5 & \text{By symm. property of DFT} \\ 0 & \longrightarrow k=6 & \end{array}\right.$$

STEP – 3. Find h[n]

By Inverse DFT,

$$h[n] = \frac{1}{N}\sum_{k=0}^{N-1} H[k]\, W_N^{-nk} \quad \text{where } W_N^1 = e^{\frac{-j2\pi}{N}}$$

$$h[n] = \frac{1}{7}\sum_{k=0}^{6} H[k]\; e^{\frac{j2\pi nk}{7}}$$

$$h[n] = \frac{1}{7}\left\{1 + H[0]\, e^{\frac{j2\pi n}{7}} + H[6]\, e^{\frac{j12\pi n}{7}}\right\} \quad \text{where } e^{\frac{j12n\pi}{7}} = e^{\frac{-j2\pi n}{7}}$$

$$h[n] = \frac{1}{7}\left\{1 + e^{\frac{-j6\pi}{7}}\, e^{\frac{j2\pi n}{7}} + e^{\frac{j6\pi}{7}}\, e^{\frac{-j2\pi n}{7}}\right\}$$

$$h[n] = \frac{1}{7}\left\{1 + e^{\frac{j2\pi(n-3)}{7}} + e^{\frac{-j2\pi(n-3)}{7}}\right\}$$

$$h[n] = \frac{1}{7}\left\{1 + 2\left[\cos\left(\frac{2\pi(n-3)}{7}\right)\right]\right\}$$

h[n] ={ – 0.11465, 0.07910, 0.32094, 0.42860, 0.32094, 0.07910, – 0.11465 } for n $\ge$ 0.

Q(75) Design 6th order **Low Pass FIR filter** with cut off frequency $w_c = \frac{\pi}{2}$ using frequency sampling method

ANS :

h[n] = { 0.4285, 0.3210, 0.0790, - 0.1145, - 0.1145, 0.0790, 0.3210 }

Q(76) Design 6th order **Linear Phase High Pass FIR filter** with cut off frequency $w_c = \frac{\pi}{2}$ using frequency sampling method.

Imp ***

Q(77) Show that FIR filter can also be realized using IIR filters.
(i.e. FREQUENCY SAMPLING REALIZATION)

$\Longrightarrow$ By ZT,

$$H(z) = \sum_{n=0}^{N-1} h[n]\, z^{-n}$$

$$= \sum_{n=0}^{N-1} \left\{ \frac{1}{N} \sum_{k=0}^{N-1} H[k]\, W_N^{-nk} \right\} Z^{-n}$$

$$= \sum_{k=0}^{N-1} H[k] \left\{ \frac{1}{N} \sum_{n=0}^{N-1} W_N^{-nk}\, Z^{-n} \right\}$$

$$= \sum_{k=0}^{N-1} H[k] \left\{ \frac{1}{N} \sum_{n=0}^{N-1} \left(W_N^{-k}\, Z^{-1} \right)^n \right\}$$

$$= \sum_{k=0}^{N-1} H[k].\ \frac{1}{N}.\ \sum_{n=0}^{N-1} \left(e^{\frac{j2\pi k}{N}}\, Z^{-1} \right)^n$$

refer $\sum_{n=0}^{N-1} r^n = \frac{(1-r^N)}{1-r}$.

$$= \frac{1}{N} \sum_{k=0}^{N-1} H[k]\ .\ \frac{1 - \left(e^{\frac{j2\pi k}{N}}\, Z^{-1} \right)^N}{1 - e^{\frac{j2\pi k}{N}}\, Z^{-1}}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} H[k]\ .\ \frac{1 - e^{j2\pi k}\, Z^{-N}}{1 - e^{\frac{j2\pi k}{N}}\, Z^{-1}}$$

But $e^{j2\pi k} = 1$

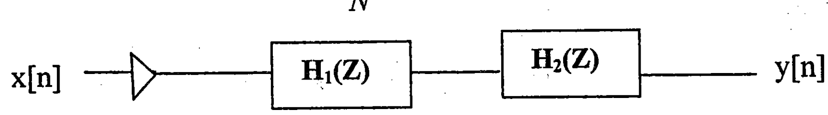
For any integer value of k,
$$H(z) = \frac{1}{N} (1 - z^{-N}) \sum_{k=0}^{N-1} \frac{H[k]}{1 - e^{\frac{j2\pi k}{N}}\, z^{-1}}$$

Let $H(z) = \frac{1}{N} H_1(z) \cdot H_2(z)$

Where $H_1(z) = (1 - z^{-N})$ $\leftarrow$ FIR Filter

And $H_2(z) = \sum_{k=0}^{N-1} \frac{H[k]}{1 - e^{\frac{j2\pi k}{N}}\, z^{-1}}$ $\leftarrow$ Parallel Bank of IIR Filters.

Frequency sampling realization is realization of FIR (i.e. Non recursive) filter using IIR (i.e. Recursive) filters.

> NOTE : Frequency Sampling Realization of FIR Filter (Non recursive Filter)
>
> *Let* $H(z) = \frac{1}{N} H_1(z) \cdot H_2(z)$
>
> 
>
>
> Where $H_1(z) = (1 - z^{-N})$ And $H_2(z) = \sum_{k=0}^{N-1} \frac{H[k]}{1 - e^{\frac{j2\pi k}{N}}\, z^{-1}}$

Q(78) Given $H\left(\frac{2\pi k}{7}\right) = \begin{cases} 1 & k = 0,1 \\ 0 & otherwise \end{cases}$ Show frequency sampling realization diagram of sixth order Linear Phase FIR filter.

Now, $\omega = \frac{2\pi K}{N}$

At, $K = 0$, $\omega = 0$

$K = \frac{N}{2}$, $\omega = \pi$

$\Longrightarrow$

$$H[k] = \begin{bmatrix} 1 & k=0 \\ 1 & k=1 \\ 0 & k=2 \\ 0 & k=3 \\ 0 & k=4 \\ 0 & k=5 \\ 1 & k=6 \end{bmatrix}$$

$\cdot$ Pos, $K = N/2$

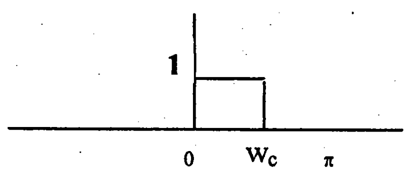
Note:
For Linear phase, phase information must be there. Here, it is missing, so calculate

---

STEP - 1. Find $H_d(w)$

Let $H_d(w) = |H_d(w)|\ \phi(w)$

Where (i) Magnitude response :



$$H_d(\omega) = \begin{cases} 1 & for\ 0 \le w \le w_c \\ 0 & otherwise \end{cases}$$

(ii) Phase Response : $\phi(w) = e^{j\phi}$

For Linear Phase LPF $\phi = -\frac{N-1}{2} = -3w \quad \therefore \phi(w) = e^{-j3w}$

By substituting, $H_d(\omega) = \begin{cases} e^{-j3w} & 0 \le w \le w_c \\ 0 & otherwise \end{cases}$

STEP - 2. Find H[k]

Now, $H_d(\omega) = \begin{cases} e^{-j3w} & 0 \le w \le w_c \\ 0 & otherwise \end{cases}$

Put $w = \frac{2\pi k}{N} = \frac{2\pi k}{7}$

$$H[k] = \begin{cases} e^{-j3\left(\frac{2\pi k}{7}\right)} & 0 \le w \le w_c \\ 0 & otherwise \end{cases}$$

as $r e^{j\phi}$ — Phase
r — magnitude

$$H[k] = \begin{bmatrix} 1 & k=0 \\ e^{-j\frac{6\pi}{7}} & k=1 \\ 0 & k=2 \\ 0 & k=3 \\ 0 & k=4 \\ 0 & k=5 \\ e^{j6\pi/7} & k=6 \end{bmatrix}$$

By Freq Sampling, $H(z) = \frac{1}{N} H_1(z)\ H_2(z)$

Where (i) $N = 7$

(ii) $H_1(z) = 1 - z^{-N} = 1 - z^{-7} = \frac{b_0 + b_7 z^{-7}}{1}$

(iii) $H_2(z) = \sum_{k=0}^{N-1} \frac{H[k]}{1 - e^{\frac{j2\pi k}{N}} z^{-1}}$

Expand for non-zero of H(k) values

$$H_2(z) = \frac{H[0]}{1-z^{-1}} + \frac{H[1]}{1-e^{j\frac{2\pi}{7}} z^{-1}} + \frac{H[6]}{1-e^{j\frac{12\pi}{7}} z^{-1}}$$

* Numerator must be in real **

$$H_2(z) = \frac{1}{1-z^{-1}} + \frac{e^{-j\frac{6\pi}{7}}}{1-e^{j\frac{2\pi}{7}} z^{-1}} + \frac{e^{j\frac{6\pi}{7}}}{1-e^{-j\frac{2\pi}{7}} z^{-1}}$$

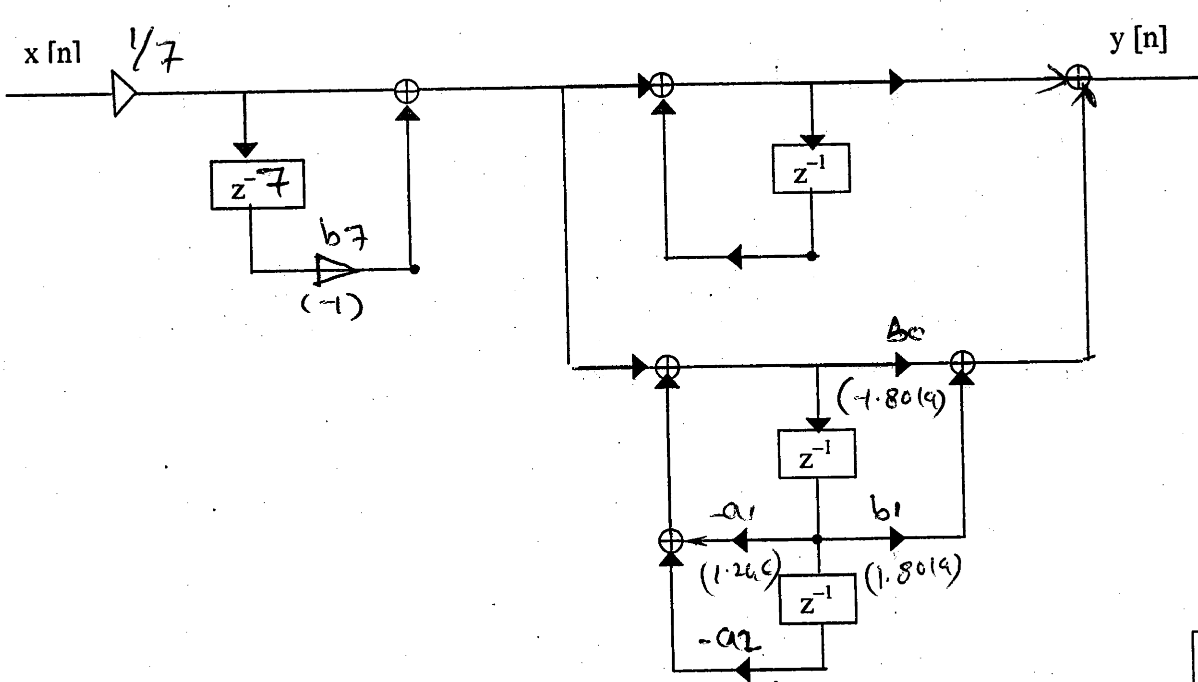
$$H_2(z) = \frac{1}{1-z^{-1}} + \left[\frac{e^{-j\frac{6\pi}{7}}\left(1-e^{-j\frac{2\pi}{7}}z^{-1}\right) + e^{j\frac{6\pi}{7}}\left(1-e^{j\frac{2\pi}{7}}z^{-1}\right)}{\left(1-e^{j\frac{2\pi}{7}}z^{-1}\right)\left(1-e^{-j2\pi/7}z^{-1}\right)}\right]$$

$$H_2(z) = \frac{1}{1-z^{-1}} + \left[\frac{e^{-j\frac{6\pi}{7}} - e^{-j\frac{8\pi}{7}}z^{-1} + e^{j\frac{6\pi}{7}} - e^{j\frac{8\pi}{7}}z^{-1}}{1 - e^{-j\frac{2\pi}{7}}z^{-1} - e^{j\frac{2\pi}{7}}z^{-1} + z^{-2}}\right]$$

$$H_2(z) = \frac{1}{1-z^{-1}} + \left[\frac{\left(e^{j\frac{6\pi}{7}} + e^{-j\frac{6\pi}{7}}\right) - z^{-1}\left(e^{j\frac{8\pi}{7}} + e^{-j\frac{8\pi}{7}}\right)}{1 - z^{-1}\left(e^{j\frac{2\pi}{7}} + e^{-j\frac{2\pi}{7}}\right) + z^{-2}}\right]$$

$$H_2(z) = \frac{1}{1-z^{-1}} + \left[\frac{2\cos\left(\frac{6\pi}{7}\right) - 2z^{-1}\cos(8\pi/7)}{1 - 2z^{-1}\cos\left(\frac{2\pi}{7}\right) + z^{-2}}\right]$$

$$H_2(z) = \frac{1}{1-z^{-1}} + \left[\frac{-1.8019 + 1.8019 z^{-1}}{1 - 1.246 z^{-1} + z^{-2}}\right]$$



Q(79) Given $H(0)=1,\quad H(1)=e^{\frac{-j6\pi}{7}}\quad H(6)=e^{\frac{j6\pi}{7}}$

$H(2)=H(3)=H(4)=H(5)=0.$

Show frequency sampling realization diagram of the above sixth order Linear Phase FIR filter using real coefficients only.

Q(80) Using frequency sampling realization, realize the filter which has following transfer function.

$$H\left(e^{j\frac{2\pi k}{16}}\right)=\begin{cases}1.0 & k=0,1\\ 0.5 & k=2\\ 0 & k=3,4.....7\end{cases}$$

⟹

By Freq Sampling, $\quad H(z)=\frac{1}{N}H_1(z)\,H_2(z)$

Where (i) $N=16$

(ii) $H_1(z)=1-z^{-N}=1-z^{-16}$

(iii) $H_2(z)=\sum_{k=0}^{N-1}\frac{H[k]}{1-e^{\frac{j2\pi k}{N}}z^{-1}}$

$$H_2(z)=\frac{H[0]}{1-z^{-1}}+\frac{H[1]}{1-e^{j\frac{2\pi}{16}}z^{-1}}+\frac{H[2]}{1-e^{j\frac{4\pi}{16}}z^{-1}}+\frac{H[14]}{1-e^{j\frac{28\pi}{16}}z^{-1}}+\frac{H[15]}{1-e^{j\frac{30\pi}{16}}z^{-1}}$$

$$H_2(z)=\frac{1}{1-z^{-1}}+\frac{1}{1-e^{j\frac{\pi}{8}}z^{-1}}+\frac{0.5}{1-e^{j\frac{\pi}{4}}z^{-1}}+\frac{0.5}{1-e^{-j\frac{\pi}{4}}z^{-1}}+\frac{1}{1-e^{-j\frac{\pi}{8}}z^{-1}}$$

$$=\frac{1}{1-z^{-1}}+\left[\frac{1}{1-e^{j\frac{\pi}{8}}z^{-1}}+\frac{1}{1-e^{-j\frac{\pi}{8}}z^{-1}}\right]+\left[\frac{0.5}{1-e^{j\frac{\pi}{4}}z^{-1}}+\frac{0.5}{1-e^{-j\frac{\pi}{4}}z^{-1}}\right]$$

$$=\frac{1}{1-z^{-1}}+\left[\frac{1-e^{-j\frac{\pi}{8}}z^{-1}+1-e^{j\frac{\pi}{8}}z^{-1}}{\left(1-e^{j\frac{\pi}{8}}z^{-1}\right)\left(1-e^{-j\frac{\pi}{8}}z^{-1}\right)}\right]+\left[\frac{0.5-0.5\,e^{-j\frac{\pi}{4}}z^{-1}+0.5-0.5\,e^{j\frac{\pi}{4}}z^{-1}}{\left(1-e^{j\frac{\pi}{4}}z^{-1}\right)\left(1-e^{-j\frac{\pi}{4}}z^{-1}\right)}\right]$$

$$=\frac{1}{1-z^{-1}}+\left[\frac{2-z^{-1}\left(e^{j\frac{\pi}{8}}+e^{-j\frac{\pi}{8}}\right)}{1-z^{-1}\left(e^{j\frac{\pi}{8}}+e^{-j\frac{\pi}{8}}\right)+z^{-2}}\right]+\left[\frac{1-0.5z^{-1}\left(e^{j\frac{\pi}{4}}+e^{-j\frac{\pi}{4}}\right)}{1-z^{-1}\left(e^{j\frac{\pi}{4}}+e^{-j\frac{\pi}{4}}\right)+z^{-2}}\right]$$

$$=\frac{1}{1-z^{-1}}+\left[\frac{2-2z^{-1}\cos(\pi/8)}{1-2\cos\left(\frac{\pi}{8}\right)z^{-1}+z^{-2}}\right]+\left[\frac{1-z^{-1}\cos(\pi/4)}{1-2\cos\left(\frac{\pi}{4}\right)z^{-1}+z^{-2}}\right]$$

$$H_2(z) = \frac{1}{1-z^{-1}} + \left[\frac{2-1.847\,z^{-1}}{1-1.847\,z^{-1}+z^{-2}} \right] + \left[\frac{1-0.707\,z^{-1}}{1-1.414\,z^{-1}+z^{-2}} \right]$$

<u>Realization Diagram</u> : HW