# Understanding Quantum Entanglement through Bell's Inequalities

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# 1 Abstract

This report examines the EPR paradox's propositions and Bell's Inequalities; On the eve of the Nobel Prize being won for the same, we aim to explore Bell's Inequalities using the Copenhagen Interpretation of Quantum Mechanics and to incorporate IBM's Qiskit modules to try and come to an experimental verification for Quantum Entanglement through the violation of CHSH Inequality, thereby verifying Bell's Theorem.

# 2 Acknowledgement

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# 3 Introduction

'God does not play dice' - an infamous statement by Albert Einstein now echoes in all physicist's and mathematicians' minds. Around 1935, Einstein authored a paper aptly titled 'Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?'. While the report presented an objective criticism against the fundamental nature of quantum mechanics, essentially, the EPR paper presented a straightforward conclusion to satisfy the principles of quantum mechanics. The idea of local reality for classical quantum mechanics.

While the paper was well-received and initially considered as the final verdict against Quantum Mechanics, Niels Bohr, an early advocate of Quantum Mechanics, provided a rather complicated criticism of EPR, which we will examine shortly.

While the debate between the two giants persisted, the fallout from these intellectual ideals created the ideas of modern quantum physics, posing fundamental concepts on quantum entanglement.

Expanding on the fundamental nature of Quantum Entanglement, the construct assumed by ERP [Einstein, Rosen, and Podolsky] would imply that quantum mechanics has a relatively local realistic explanation that guarantees the values of spins of any two entangled charges at any point in time or space. And while most tests do ensure the present existence of Quantum Mechanics and all the properties associated with it, the local realism argument has yet to be dissipated.

However, in 1964 John Stewart Bell assuming locality as a construct in Quantum mechanics, disproved the EPR's assumption of local reality. Yet the experimental verification of the same was undermined. This present paper wishes to follow an approach involving entangled particles; however, it actively incorporates modern technology to develop an intuitive solution toward disproving Bell's Inequalities.

## 3.1 Laying the Groundwork: Copenhagen Interpretation of Quantum Mechanics

The significance of the Copenhagen Interpretation is imperative for developing a concise model of Quantum Mechanics. A culmination of the efforts of Bohr and Heisenberg, the Copenhagen Interpretation lists three fundamental principles:

- Quantum mechanics describes individual systems.
- Quantum mechanical probabilities are primary, i.e., they cannot be derived from a deterministic theory.
- The world must be divided into two parts. The object under study must be described quantum mechanically. The remaining part, which includes the apparatus, is classical. The cut between the system and the apparatus can be made arbitrarily.
- *The observation process is irreversible.*
- *Complementary properties cannot be measured simultaneously.*

The above two italicized principles are fundamental to understanding the interpretation of Quantum Mechanics. They would be two principles that we would apply to approach a solution to our problem statement.

On closer examination of the principles, a pivotal observation can be derived. The interpretation denies the possibility of measuring the atomic scale of the scales of measurement apparatus used in the measurement itself. The debate would thus further spiral into the idea that an experiment would be deemed complete if every single experimental arrangement were accounted for. Thus birthing the convoluted consequences of completeness and contextuality in Quantum Mechanics.

Apart from the convoluted consequences birthed out of the Copenhagen Interpretation, another important assumption formulated from it would be the *Complementarity principle.* Fundamentally distinguishing Classical Mechanics from Quantum Mechanics, the Complementarity Principle essentially argues the following,

'*All particles of atomic size or smaller can exhibit particle or wave-like properties but not both*'

While the principle births various interpretations, Heisenberg extended the complementarity principle to the atomic properties of spin, angular momentum, and charge. However, the significant debate revolved around the consequences of completeness and contextuality instead of Complementarity.

## 3.2 Completeness and Continuity: EPR's bespoke assumptions

The exploration of Completeness and Contextuality properties resulted as a consequence of the EPR. To understand our report further, let's go through a quick dive into EPR.

Let us start building on a local-realistic world according to Einstein's view. Let us break down the idea of local reality. Locality can be defined as the fundamental phenomenon that an event in some place would not have a consequence in another location before a signal from the origin reaches the other location. Secondly, realism argues that every property of a given physical system exists before they are measured.

*If the property of an object can be measured, then the given object under discussion would have to be real.*

This proposes certain objective formulations from EPR:

- Quantum Bits in opposite states given as $\frac{1}{\sqrt{2}}|00\rangle$ and $\frac{1}{\sqrt{2}}|10\rangle$ will have definite values, but can only be learned after observation.
- When observing the first quantum bit, we find that there is a probability of $\frac{1}{2}$ from this, we can infer with certainty that the value of the second bit would be $\frac{1}{2}$, as the values are related due to their local-reality properties.

From these pivotal assumptions, the EPR paper was able to conclude the inconsistencies of Quantum Mechanics, with the primary being the disregard for, as Einstein would say, infamously 'Spooky Action at a Distance', with the claim of entangled particles being able to stay in a juxtaposition of multiple states being disposed of, and instead a rather classical interpretation being presented of trying to provide an answer. Utilizing the fundamental notion of local realism, ERP presented the possibility of quantum particles all having definite pre-determined states to which they already belong.

With a seeming lack of a befitting reply, the EPR paradox was finally given a befitting answer by John C Bell, which we will elaborate on further.

## 3.3 Mathematical operators, spin operators, and states

The SG (Stern-Gerlach) experiment suggests that the spin states of the electron can be described using two basis. The first corresponds to an electron with $S_z = \frac{\hbar}{2}$. The $z$ label indicates the component of the spin, and 2 the + the fact that the component of spin is positive. This state is also called 'spin up' along $z$. The second state corresponds to an electron with $S_z = \frac{\hbar}{2}$, that is a 'spin down' along z. Mathematically, we have an operator $S_z$ for which the above states are eigenstates with opposite eigenvalues:

$$\hat{S}_z |z;+\rangle = +\frac{\hbar}{2} |z;+\rangle$$

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If we have two basis states, then the state space of electron spin is a two-dimensional complex vector space. Each vector in this vector space represents a possible electron spin state. We are not discussing other degrees of freedom of the electron, such as its position, momentum, or energy. The general vector in the two-dimensional space is an arbitrary linear combination of the basis states and thus takes the form :

$$|\psi\rangle = c_1 |z;+\rangle + c_2 |z;-\rangle$$

with $c_1, c_2 \in C$

It is customary to call the state $|z;+\rangle$ the *first* basis state and it is denoted by $|1\rangle$. The state $|z;-\rangle$ is termed as the *second* basis state and is denoted by $|2\rangle$. . States are vectors in some vector space. A vector is explicitly represented as a column vector with two components in a two-dimensional vector space. The first basis vector can be represented as $\begin{pmatrix}1\\0\end{pmatrix}$ and the second basis vector can be represented as $\begin{pmatrix}0\\1\end{pmatrix}$. Thus we have the following names for states and their concrete representation as column vectors,

$$|z:+\rangle = |1\rangle \longleftrightarrow \begin{pmatrix}1\\0\end{pmatrix}$$

$$|z:-\rangle = |2\rangle \longleftrightarrow \begin{pmatrix}0\\1\end{pmatrix}$$

Using these options the state in prior expressions takes the possible forms

$$|\psi\rangle = c_1 |z;+\rangle + c_2 |z;-\rangle = c_1 |1\rangle + c_2 |2\rangle \longleftrightarrow c_1 \begin{pmatrix}1\\0\end{pmatrix} + c_2 \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}c_1\\c_2\end{pmatrix}$$

The state $|z;+\rangle$ entering the second machine must have zero overlaps with $|z,-\rangle$ since no such down spins emerge. Moreover, the overlap of $|z;+\rangle$ with itself must be one, as all states emerge from the second machine top output. We thus write

$$\langle z;-|z;+\rangle = 0, \langle z;+|z;+\rangle = 1$$

and similarly, we expect

$$\langle z;+|z;-\rangle = 0, \langle z;-|z;-\rangle = 1$$

Using the notation where the basis states are labeled as $|1\rangle$ and $|2\rangle$ we have the simpler form that summarizes the four equations above:

$$\langle i|j\rangle = \delta_(ij)i, j = 1, 2$$

We have not yet made precise what we mean by the 'bras' so let us do so briefly. We define the basis 'bras' as the row vectors obtained by transposition and complex conjugation:

$$\langle 1| \longleftrightarrow, \langle 2| \longleftrightarrow (0,1)$$

Given states $\langle \alpha|$ and $\langle \beta|$

$$\langle \alpha| = \alpha_1 \langle 1| + \alpha_2 \langle 2| \longleftrightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$\langle \beta| = \beta_1 \langle 1| + \beta_2 \langle 2| \longleftrightarrow \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

we associate

$$\langle \alpha| \equiv \alpha_1^* \langle 1| + \alpha_2^* \langle 2| \longleftrightarrow (\alpha_1^*, \alpha_2^*)$$

and the 'bra-ket' inner product is defined as the 'obvious' matrix product of the row vector and column vector representatives:

$$\langle \alpha|\beta \rangle \equiv (\alpha_1^*, \alpha_2^*). \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \alpha_1^* \beta_1 + \alpha_2^* \beta_2.$$

When we represent the states as two-component column vectors the operators that act on the statesto give new states can be represented as two-by-two matrices. We can thus represent the operator $\hat{S}_z$ as a 2 × 2 matrix which we claim takes the form:

$$\boxed{\hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

## 4 Bell's Inequalities, a befitting response to EPR.

Bell had a relatively concise response to this which we will elaborate on below, assuming:

| | | | | |
|---|---|---|---|---|
| $E_1$ | $Z^+$ | $X^+$ | $Q^+$ | |
| $E_2$ | $Z^+$ | $X^+$ | $Q^-$ | |
| $E_3$ | $Z^+$ | $X^-$ | $Q^+$ | |
| $E_4$ | $Z^+$ | $X^-$ | $Q^-$ | *All possible events for Alice's particle 1* |
| $E_5$ | $Z^-$ | $X^+$ | $Q^+$ | |
| $E_6$ | $Z^-$ | $X^+$ | $Q^-$ | |
| $E_7$ | $Z^-$ | $X^-$ | $Q^+$ | |
| $E_8$ | $Z^-$ | $X^-$ | $Q^-$ | |

Let's take the universe of EPR: Hidden Variable (local) -
Linear momentum ($\rho$) and angular momentum is conserved.

P($Z^+$, $X^+$) = If Alice measures + in Z - axis, probability that Bob measures + in X-axis.

For Bob to measure X+, Alice should have X−. So Similarly for

$$P(Z^+, X^+) = \frac{E_3 + E_4}{8} = \frac{2}{8} = \frac{1}{4}$$

$$P(Z^+, Q^+) = \frac{E_2 + E_4}{8} = \frac{1}{4}$$

Similarly,

$$P(Q^+, Z^+) = \frac{E_3 + E_7}{8} = \frac{1}{4}$$

Now, using the insights John Bell had:

total number of events = 8

$$\begin{aligned} 8 \times P(Z^+, X^+) &\leq (P(Z^+, Q^+) + P(Q^+, X^+)) \times 8 \\ P(Z^+, X^+) &\leq (P(Z^+, Q^+) + P(Q^+, X^+)) \end{aligned}$$

This is Bell's Inequality:

$$\begin{aligned} \frac{E_3 + E_4}{8} &\leq \frac{E_2 + E_4}{8} + \frac{E_3 + E_7}{8} \\ E_3 + E_4 &\leq E_2 + E_4 + E_3 + E_7 \end{aligned}$$

Since, here all these values are positive, these inequalities have to be true for any hidden variable theory to be true.

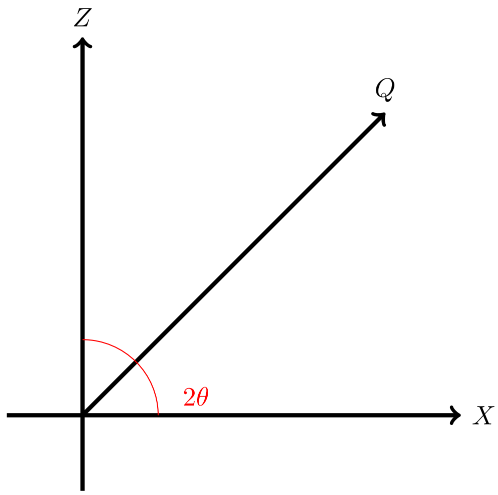
Now, let's consider when Alice measures her particle's spin in the 'Z'-axis and Bob's spin in 'Q'-axis.

Therefore the probability, if the hidden variable theory is true, will be given by:

$$P(Z^+, Q^+) = \frac{1}{4}$$

since 'Q' can have 50% chance of both positive and negative.

But, to our surprise, this is not what happens in quantum mechanics! This is because measurement of particles follows probability laws of the wave function for a particle rotated at $\theta^\circ$ angle.




Now let us take a general case:
i) Alice measure '+' or '-' spins in 'z-axis' and Bob measures '+' and '-' in $\theta$ axis ii) Now, Alice measures at angle $\theta$ axis and Bob measures at angle $2\theta$ axis iii) Lastly, Alice measures in 'z-axis' and Bob measures at $2\theta$

Therefore,
Probability of i) is P(Z,Q)
Probability of ii) is P(Q,X)
Probability of iii) is P(Z,X)

Therefore,

$$P(Z, \bar{Q}) + P(Q, \bar{X}) \geq P(Z, \bar{X})$$
$$P(\uparrow_z, \downarrow_\theta) + P(\uparrow_\theta, \downarrow_{2\ \theta}) \geq P(\uparrow_z, \downarrow_{2\ \theta})$$

Here P(A,B) represents probability of measurements taken by Alice,Bob.

Spin operators:

$\mathrm{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix}$

$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

We are working in the basis of Eigenstates of $S_z$, i.e., with definitive values of $S_z$,

$$\psi(x) = \psi_{\uparrow}(x)\begin{pmatrix}1\\0\end{pmatrix} + \psi_{\downarrow}(x)\begin{pmatrix}0\\1\end{pmatrix}$$

Where we observe that $\begin{pmatrix}1\\0\end{pmatrix}$ represents $\uparrow_z$ *and* $\begin{pmatrix}0\\1\end{pmatrix}$ *indicates* $\downarrow_z$

The above conclusion also helps us understand why $S_z$ is a matrix with diagonal entries as well. Continuing our above discussion, we observe that for eigenvalues of $S_x$.

$$S_x\begin{pmatrix}1\\1\end{pmatrix} = \frac{\hbar}{2}S_x\begin{pmatrix}1\\1\end{pmatrix} = \uparrow_z + \downarrow_z$$

where $\frac{\hbar}{2}$ is the eigenvalue and $\begin{pmatrix}1\\1\end{pmatrix}$ is the eigenvector.

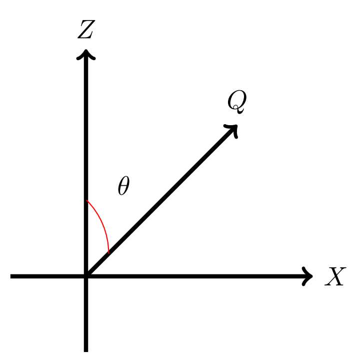
Post normalization, we draw the following inferences:

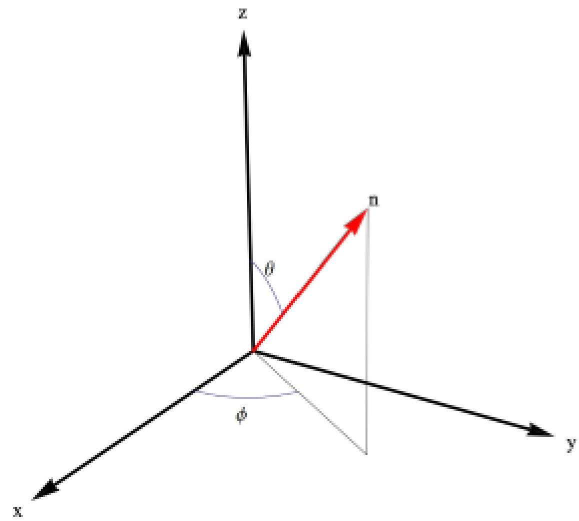
- $\uparrow_x = \frac{1}{\sqrt{2}}(\uparrow_z + \downarrow_z)$
- $\downarrow_x = \frac{1}{\sqrt{2}}(\uparrow_z - \downarrow_z)$

From the above, we can infer that $P(\uparrow_z, \downarrow_z) = (|\dfrac{1}{\sqrt{2}}|)^2 = \frac{1}{2}$

And thus, in our case, considering the operator $S_\theta$

$S_\theta =$






With the Azimuthal angle $\phi = 0$, we observe and '$\theta$' is the angle between $\hat{n}$ of the z-axis, we are able to develop the following conclusions,

$n_x = \cos\phi \sin\theta = \sin\theta$,
$n_y = \sin\phi \sin\theta = 0$
$n_z = \cos\theta$

Therefore,

$$S_{\vec{n}} = \frac{\hbar}{2}(\sigma_1 n_x \hat{+} n_y \sigma_z + n_z \sigma_3)$$
$$= \frac{\hbar}{2}\begin{pmatrix} n_z & n_x - 0i \\ n_x + 0i & -n_z \end{pmatrix}$$

Thus, we obtain

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-\iota\phi} \\ \sin\theta e^{-\iota\phi} & \cos\theta \end{pmatrix}$$

Calculating Eigenvalues and Eigenvectors of the above, we obtain the following :

$$C_1 = \frac{\cos\theta}{2}, C_2 = \frac{\sin\theta}{2} e^{-\iota\phi}$$

Hence,

$$|\vec{n}+\rangle = \cos\frac{\theta}{2}\,|\vec{z}+\rangle + \sin\frac{\theta}{2} e^{-\iota\phi}$$

Substituting from our prior conclusions, we arrive at a final conclusion i.e,

$$\uparrow_\theta = \cos\frac{\theta}{2}\uparrow_z + \sin\frac{\theta}{2}\downarrow$$

$$\downarrow_\theta = \cos\frac{\theta}{2}\downarrow_z + \sin\frac{\theta}{2}\uparrow_z$$

Therefore,

$$P(\uparrow_z, \downarrow_\theta) = || \sin\frac{\theta}{2} ||^2 = \sin^2\frac{\theta}{2} \tag{4}$$

Similarly for

$$P(\uparrow_\theta, \downarrow_{2\ \theta}) = \sin^2\frac{\theta}{2}$$

Similarly for

$$P(\uparrow_z, \downarrow_{2\ \theta}) = \sin^2\theta$$

Arriving at the final equation of :

$$\sin^2\theta \leq \sin^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}$$

So, Quantum mechanics predicts that if we do the EPR experiment using these observables repeatedly and build up statistics, we will find an explicit violation of Bell's Inequality mentioned above.

## 5 Conclusions:

An important inference we observed through the experiment would be a few,

- If certain predictions of quantum mechanics were true, then our world is non-local.
- Bell successfully proved the lack thereof of a given hidden variable theory.

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