FM122 – Mathematics of Uncertainty

Mini Project 1 – Misadventures of Squeaky the Squirrel

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Section 2

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Question 1: Consider an island defined by a one-dimensional grid of length 100 units and the initial position of the squirrel is 40 units from the left (tagged 0), create a simulation of the squirrel hooping on the island over time.

Solution: Code Snippet

```
start_pos= input("Starting Position?")
grid_length= input("Grid length?")
curr_pos = start_pos;
num_hops = 1;
while (curr_pos > 0 && curr_pos < grid_length)
        toss = rand(1);
    if (toss < 0.5)
        curr_pos = curr_pos - 1;
    elseif (toss >= 0.5)
        curr_pos = curr_pos + 1;
    end
    stem(curr_pos,0);
    xlim([0,grid_length]);
    F=getframe;
    movie(F);
    num_hops= num_hops+1


end
```

Question 2: Does the squirrel eventually fall of and die or does she just bounce on and off on the island in a never ending fashion? Play your simulation and justify your answer.

Solution: Code Snippet –

```
start_pos= 5
grid_length= 10
curr_pos = start_pos;
num_hops = 1;
while (curr_pos > 0 && curr_pos < grid_length)
        toss = rand(1);
    if (toss < 0.5)
        curr_pos = curr_pos - 1;
    elseif (toss >= 0.5)
        curr_pos = curr_pos + 1;
    end
    num_hops= num_hops+1
```

Results –

Trial Number	Fell?	Number of Hops (n)
1	Yes	32
2	Yes	58
3	Yes	10

4	Yes	64
5	Yes	6
6	Yes	38
7	Yes	16
8	Yes	10
9	Yes	24
10	Yes	34
11	Yes	12
12	Yes	104
13	Yes	8
14	Yes	26
15	Yes	24
16	Yes	62
17	Yes	14
18	Yes	8
19	Yes	20
20	Yes	52
21	Yes	62
22	Yes	28
23	Yes	24
24	Yes	14
25	Yes	46
26	Yes	52
27	Yes	10
28	Yes	8
29	Yes	6
30	Yes	10

Based on these results, we can reasonably infer that Squeaky will fall on the 31st simulation as well, as there hasn't been a single instance in the previous 30 simulations that Squeaky did not fall (assuming results from running the simulation 30 times is a reasonable data set to infer from).

Question 3: Does your answer above depend on the size of the island or the initial position of the squirrel? Repeat your experiment with different grid size and initial position to justify your answer.

Solution:

Case 1 – Fixed Grid Length, Varying Starting Position (Grid Length - 15)
Code Snippet –

Function:

```
function [num_hops] = Lab2_LifeExpectancy(grid_length, start_pos)
    curr_pos=start_pos;
    num_hops=1;
    while (curr_pos >0 && curr_pos<grid_length)
      toss=rand(1);
      if (toss < 0.5)
        curr_pos = curr_pos- 1;
      elseif (toss >= 0.5)
        curr_pos = curr_pos + 1;
      end

      num_hops=num_hops+1;
    end
    num_hops
```

Function Call:

```
life_expectancy=0;
    for l=1:30
    grid_length=15;
    start_pos=1; %vary for different starting positions
    [num_hops] = Lab2_LifeExpectancy(grid_length, start_pos);
    life_expectancy = life_expectancy+num_hops;
    end
    avg=life_expectancy/30
```

Results –

Starting Position	**Average Life Expectancy (Average of 30 iterations)**
1	18.50
2	30.66
3	32.00
4	42.26
5	57.80
6	67.16
7	69.06
8	52.53
9	43.46
10	42.20
11	41.10
12	37.33
13	30.73
14	15.63

Number of hops vs Starting Position –

X-Axis = Starting Position.
Y-Axis = Average number of hops (for 30 iterations) before Squeaky fell.

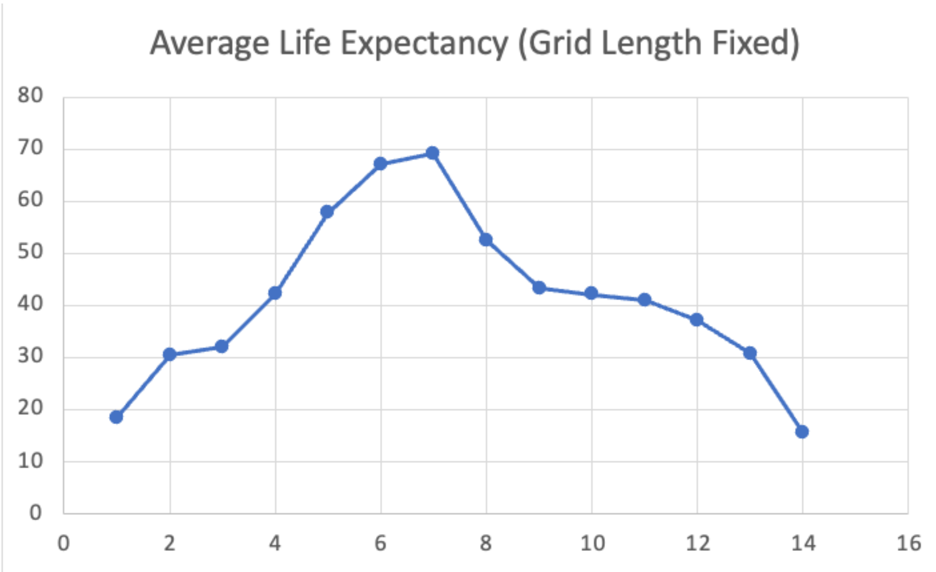

Figure 1: Number of hops vs Starting Position

Conclusion: From the experiment, it is clearly visible that life expectancy of Squeaky is less when it starts at a position closer to the edge of the island. On the contrary, life expectancy is observed to be higher when it starts closer to the middle.

Case 2 – Fixed Starting Position, Varying Grid Length (Starting Position – 7 from left)

Code Snippet –

Function:

```
function [num_hops] = Lab2_LifeExpectancy(grid_length, start_pos)
    curr_pos=start_pos;
    num_hops=1;
    while (curr_pos >0 && curr_pos<grid_length)
      toss=rand(1);
      if (toss < 0.5)
        curr_pos = curr_pos- 1;
      elseif (toss >= 0.5)
        curr_pos = curr_pos + 1;
      end

      num_hops=num_hops+1;
    end
    num_hops
```

Function Call:

```
life_expectancy=0;
    for l=1:30
    grid_length=15; %vary for different grid lengths
    start_pos=7;
    [num_hops] = Lab2_LifeExpectancy(grid_length, start_pos);
    life_expectancy = life_expectancy+num_hops;
    end
    avg=life_expectancy/30
```

Result –

Grid Length	Average Life Expectancy (Average of 30 iterations)
15	043.83
16	057.13
17	069.83
18	096.27
19	102.63
20	129.13
21	117.70
22	106.40
23	131.40
24	110.20
25	119.30
26	111.46
27	126.36
28	145.06
29	184.80

Number of Hops vs Grid Length –

X-Axis = Grid Length.
Y-Axis =Average Number of hops (average of 30 iterations) before Squeaky fell.

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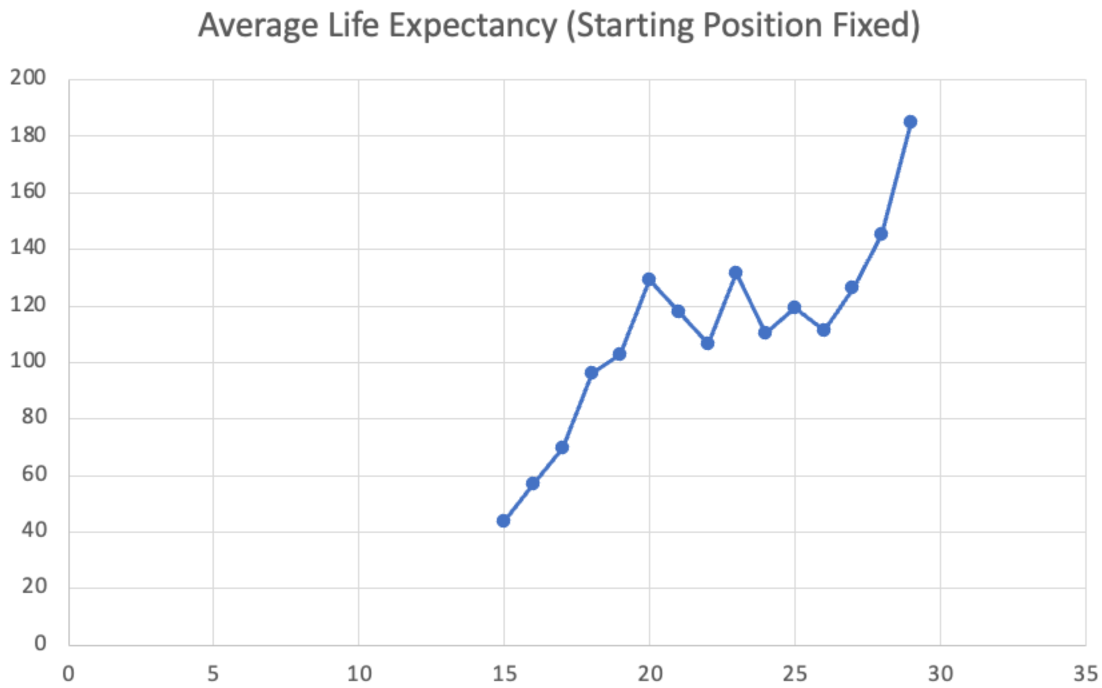

Figure 2: Number of Hops vs Grid Length

Conclusion: From the experiments, increase in the grid length leading to increase in the life expectancy can be observed as a **rough** trend.

Question 4: What is the life expectancy of the squirrel? Does your answer tally with the theoretically predicted expected number of steps or is there a discrepancy? Explain why?

Solution:

a. First, we theoretically derive the life expectancy of Squeaky using the *Law of Total Expectation.*

Let m be the starting (current) position (X_0) of Squeaky at the beginning of her walk. Let E be the event where Squeaky's first hop is to the left. We condition our expectation over event E and initial condition $X_0 = m$.

Let N be the number of hops after which Squeaky falls off the island.

Let E_m represent the expected life span of Squeaky when $X_0 = m$.

Hence,

$$E_m = E(N|E \ \& \ X_0 = m)$$

$$\Longrightarrow E_m = E(N|E \ \& \ X_0 = m)P(E|X_0 = m) + E\left(N \,\middle|\, E^c \& X_0 = m\right)P(E^c|X_0 = m)$$

$$\Rightarrow E_m = E(N|X_1 = m-1)\frac{1}{2} + E\big(N\,\big|\,X_1 = m+1\big)\frac{1}{2}$$

$$\Rightarrow E_m = E(N|X_1 = m-1)\frac{1}{2} + E\big(N\,\big|\,X_1 = m+1\big)\frac{1}{2}$$

$$\Rightarrow E_m = \frac{1}{2}\big(1 + E(N \mid X_0 = m-1)\big) + \frac{1}{2}\big(1 + E(N \mid X_0 = m+1)\big)$$

$$\Rightarrow E_m = 1 + \frac{1}{2}E_{m-1} + \frac{1}{2}E_{m+1}$$

This is a non-homogenous recurrence relation.

On rearranging the terms, we get,

$2E_m - E_{m-1} - E_{m+1} = 2$ …………….. (1)

Characteristic equation for homogenous part,

$-x^2 + 2x - 1 = 0 \Rightarrow x^2 - 2x + 1 = 0$

Finding roots of the characteristic equation,

$x^2 - 2x + 1 = 0 \Rightarrow (x-1)^2 = 0$

$\Rightarrow\ x = 1,1$

Therefore, the homogenous solution is $c_1 1^m + mc_2 1^m = c_1 + mc_2$

As the equation is non-homogenous, we need to find a particular solution. The non-homogenous term is a constant, hence the particular solution has to be a constant, i.e. of the form $E_m = c$. But because the homogenous solution is of the form $c_1 + mc_2$, the particular solution can also be of the forms $E_m = c + dm$ and $E_m = c + dm + em^2$. In this case, $E_m = c$ and $E_m = c + dm$ don't work. Hence, we take $E_m = c + dm + em^2$.

Upon substitution in equation (1), we get the following,

$$2(c + dm + em^2) - (c + d(m-1) + e(m-1)^2) - (c + d(m+1) + e(m+1)^2) = 2$$

$$\Rightarrow 2c + 2dm + 2em^2 - c - dm + d - em^2 + 2em - e - c - dm - d - em^2 - 2em - e = 2$$

$$\Rightarrow\ -2e = 2\ \Rightarrow\ e = -1$$

Since all *c* and *d* terms cancel out, $E_m = c + dm + (-1)m^2 = c + dm - m^2$ is a particular solution for all values of *c* and *d*. If we take $c = d = 0$ for the sake of simplicity, we arrive at $\boldsymbol{E_m = -m^2}$**,** as our particular solution.

Adding the homogenous solution and the particular solution gives us the general form of our solution, as follows,

$E_m = c_1 + mc_2 - m^2$

From the problem, we can infer the following boundary conditions:

$E_0 = 0$ and $E_n = 0$ where n is the grid length (i.e. the horizontal length of the island). On applying these boundary conditions, we get,

$E_0 = c_1 \Rightarrow c_1 = 0$ and $E_n = c_1 + nc_2 - n^2 \Rightarrow 0 = nc_2 - n^2 \Rightarrow c_2 = n$

From these values of c_1 and c_2, we get the final life expectancy, E_m of Squeaky:

$$\boldsymbol{E_m = m(n - m)}$$

Where $\boldsymbol{m}$ is **Squeaky's starting position (current position)** and $\boldsymbol{n}$ is the **grid length (horizontal length of the island.)**

After punching in values for grid length (n) and current position (m) as 15 and 1 - 14 respectively, we get the following values for theoretical life expectancy :

Trial Number	Starting Position	Life Expectancy (n)
1	1	14
2	2	26
3	3	36
4	4	44
5	5	50
6	6	54
7	7	56
8	8	56
9	9	54
10	10	50
11	11	44
12	12	36
13	13	26
14	14	14

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b. Calculating discrepancy between theoretical life expectancy and life expectancy observed experimentally in Question 3, Case 1:

Starting Position	Observed Life Expectancy (o)	Theoretical Life Expectancy (t)	Discrepancy (o-t)
1	18.50	14	04.50
2	30.66	26	04.66
3	32.00	36	-04.00
4	42.26	44	-01.74
5	57.80	50	07.80
6	67.16	54	13.16
7	69.06	56	13.06
8	52.53	56	-03.47
9	43.46	54	-10.54
10	42.20	50	-07.80
11	41.10	44	-02.90
12	37.33	36	01.33
13	30.73	26	04.73
14	15.63	14	01.63

This discrepancy arises because a theoretical formula only gives expected value for a single event, whereas average taken from an iterative computer simulation would give an average value for multiple probabilistic events. Every time the simulation is run, we arrive at a different life expectancy for a given current position. It is the average of these values that we compare with our theoretical average. As we increase the number of iterations for our simulation, we arrive at an average expected value that is closer to the theoretical average.

For life expectancy of Squeaky for any given current position, it can be inferred that the formula will give the same value every time a fixed value of n and m are punched in, whereas a computer simulation will throw out a different value, as the event is probabilistic (and not deterministic).

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