A Primer on Graph Theory – Preliminaries

Rishi Nair*
University of Toronto Schools, Toronto, ON, Canada M5S 2R7

1 Introduction

Graph theory is a area of mathematics that deals with graphs, a mathematical structure. Graph theory was discovered by Leonhard Euler in the 18th century. He noted the interesting structure in the paths connecting the seven bridges of Königsberg as seen in figure 1, a landmark in the Kingdom of Prussia (now located in Germany). Euler wondered if he could travel along the path, starting anywhere you want, so that he crosses every bridge (as denoted by a black bar) exactly once. By trial and error, we can come to the conclusion that this is not possible. However, what will happen if we remove a bridge, or add a bridge, or change the shape of the path?

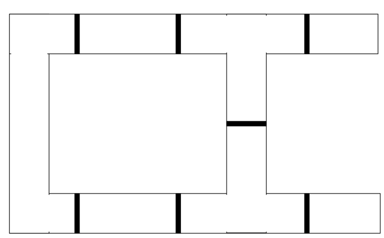

Figure 1: Euler Graph Theory Problem: Seven Bridges of Königsberg (1736).

In order to graphically represent this problem, we need to consider two components: The "land masses" and the "bridges". We can convert figure 1 into a so-called mathematical graph as shown in figure 2 where the "land masses" are vertices and the "bridges" are edges.

*Grade 12 student, email: *nairi@utschools.ca*

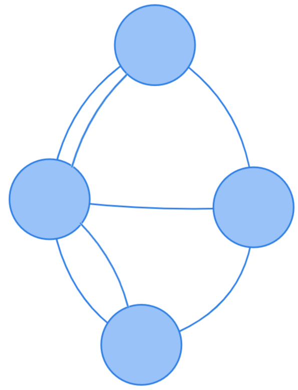

Figure 2: Graph representation of the Seven Bridges of Königsberg.

Graph theory has several applications in fields such as computer science, physics and biochemistry. One such application is in mapping traffic and finding routes that enable drivers to avoid getting stuck in traffic jams. Streets as seen in figure 3 can be represented as a graph data structure where the roads are edges and intersections are vertices. Numerical values that represent traffic density can be assigned to edges. When creating a route from one location (represented by vertices) to another, we want to travel through roads (edges) that have minimal traffic density to minimize the duration of the trip.

Another application of Graph Theory is in drug discovery. Molecules such as the ones in figure 4 can be represented as mathematical graphs, where the atoms are represented by vertices and their bonds represented by edges. Numerical values can be assigned to vertices and edges to represent the elements in the molecule and the type/strength of the bonds. Using machine learning models on such graphs, we can predict certain properties of molecules such as their effectiveness against certain pathogens.

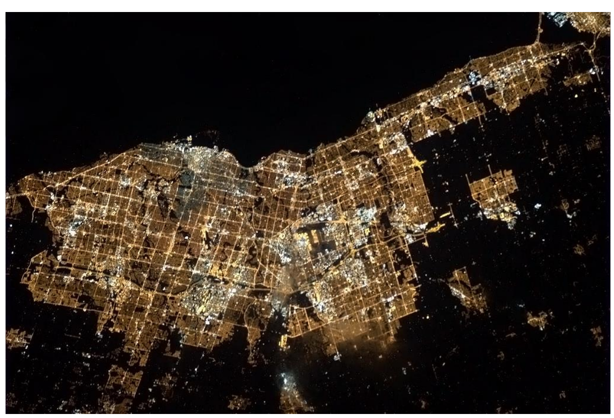

Figure 3: An aerial view of downtown Toronto taken by Chris Hatfield from the International Space Station[4].

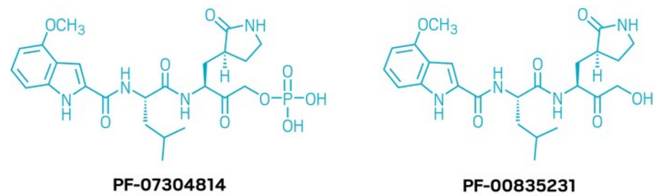

Figure 4: Molecular structures of some COVID-19 antiviral drugs[5].

In this blog post we will cover some basic definitions in graph theory and two interesting theorems on Eulerian trails and circuits.

2 Preliminaries

In this section, we present some definitions with examples.

Definition 1. A **graph** G is a triple $(V, E, f : E \to \binom{V}{2})$. We call elements of V the vertices of G, and we call the elements of E the edges of G.

The graph in figure 5 has a set of vertices $V = \{A, B, C, D, E\}$ and a set of edges $E = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$. The function $f : E \to \binom{V}{2}$ tells us which edge connects which two vertices. $f : b_1 \to \{A, B\}$, $f : b_2 \to \{B, C\}$, $f : b_3 \to \{C, C\}$, $f : b_4 \to \{C, D\}$, $f : b_5 \to \{A, D\}$, $f : b_6 \to \{E, D\}$, $f : b_7 \to \{E, A\}$.

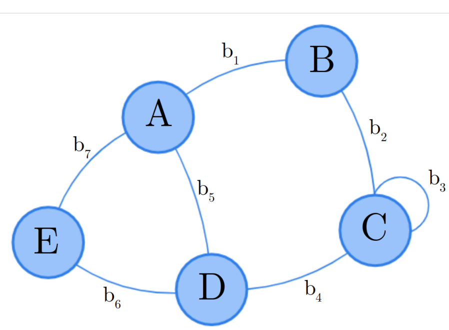

Figure 5: An example of a graph.

Definition 2. For an edge $e \in E$, the elements of $f(e)$ are called the **ends** of e. If $f(e) = \{u, v\}$, then the ends of e are u and v, and e joins them.

In figure 5, vertices A and B are the ends of edge b_1 and b_1 joins A and B.

Definition 3. We say $u, v \in V$ are **adjacent** if there is an edge that joins them.

In figure 5, vertices E and D are adjacent because edge b_6 joins them.

Definition 4. If a vertex $v \in V$ is an end of $e \in E$, we say v and e are **incident**. $e_1, e_2 \in E$ are **incident** if they have a common end.

In figure 5, b_4 and C are incident because C is an end of b_4. The edges b_5 and b_7 are incident because they both have vertex A as an end.

Definition 5. The **order** of a graph G is the number of vertices in G. The **size** of G is the number of edges.

The order of the graph in figure 5 is 5 and its size is 7.

Definition 6. An **empty graph** is a graph whose set of edges $E = \emptyset$.

The graph in figure 6 is an empty graph of order 5. This means the graph has 5 vertices and no edges.

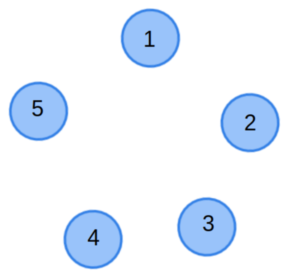

Figure 6: An empty graph.

Definition 7. A graph G is **simple** if it does not have loops or multiple edges between the same 2 vertices.

The graph in figure 7 is simple but the graph in figure 5 is not simple because it has a loop b_3 which connects vertex C to itself.

Definition 8. A **complete graph** is a simple graph with one edge connecting every pair of vertices. A complete graph with n vertices can be written as K_n. The set of vertices of a complete graph K_n is $V = \{1, 2, 3, ..., n\}$ and its set of edges is $E = \binom{V}{2}$

The graph in figure 7 is a complete graph and can be expressed as K_5 because it has 5 vertices. It has a set of vertices $V = \{1, 2, 3, 4, 5\}$ and set of edges $E = \binom{V}{2}$.

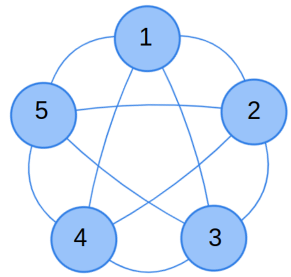

Figure 7: A complete graph.

Definition 9. A **path graph** is a graph whose vertices are connected by edges to form a "line." A path graph with n vertices can be written as P_n. The length of a path graph is the number of edges. The set of vertices in a path graph of length k is $V = \{1, 2, 3, ..., k, k+1\}$ and its edges are $\{\{1, 2\}, \{2, 3\}, ..., \{k, k+1\}\}$.

The graph in figure 8 is a path graph with a length of 4 and can be expressed as P_5. It has a set of vertices $V = \{1, 2, 3, 4, 5\}$ and set of edges $E = \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}$.

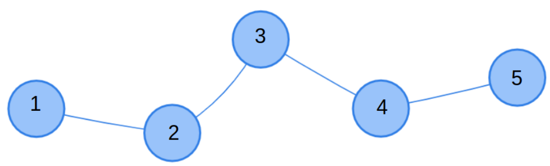

Figure 8: A path graph

Definition 10. A **cycle graph** is a graph whose vertices are connected by edges to form a "closed loop." The length of a cycle graph is the number of edges in the graph which is equal to the number of vertices in the graph. A cycle graph with n vertices can be written as C_n. The set of vertices in a path graph of length n is $V = \{1, 2, 3, ..., n\}$ and its edges are $\{\{1, 2\}, \{2, 3\}, ..., \{n-1, n\}, \{n, 1\}\}$.

The graph in figure 9 is a cycle graph with a length of 5 and can be expressed as C_5. It has a set of vertices $V = \{1, 2, 3, 4, 5\}$ and set of edges $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\}$.

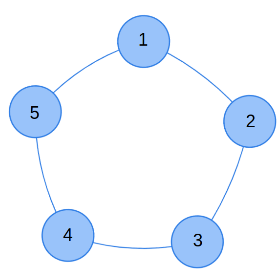

Figure 9: A cycle graph.

Definition 11. A **wheel graph** can be constructed from a cycle graph by adding another vertex that is adjacent to all the vertices in the cycle. A wheel graph with n vertices can be written as W_n. The set of vertices of a wheel graph W_n is $V = \{1, 2, 3, ..., n\}$ and its edges are $\{\{1, 2\}, \{2, 3\}, ..., \{n - 2, n - 1\}, \{n - 1, 1\}, \{n, 1\}, \{n, 2\}, ..., \{n, n - 1\}\}$.

Figure 10 shows a wheel graph W_6, which has a set of vertices $V = \{1, 2, 3, 4, 5, 6\}$ and set of edges $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}, \{6, 1\}, \{6, 2\}, \{6, 3\}, \{6, 4\}, \{6, 5\}\}$.

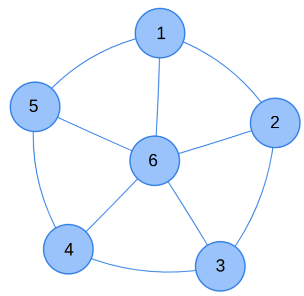

Figure 10: A wheel graph.

Definition 12. A **star graph** can be constructed from an empty graph by adding one vertex and edges so that the added vertex is adjacent to all the other vertices. A star graph with n vertices can be written as S_n. The set of vertices of a wheel graph W_n is $V = \{1, 2, 3, ..., n\}$ and its edges are $\{\{n, 1\}, \{n, 2\}, ..., \{n, n - 1\}\}$

Figure 11 depicts a star graph S_6, which has a set of vertices $V = \{1, 2, 3, 4, 5, 6\}$ and set of edges $E = \{\{6, 1\}, \{6, 2\}, \{6, 3\}, \{6, 4\}, \{6, 5\}\}$.

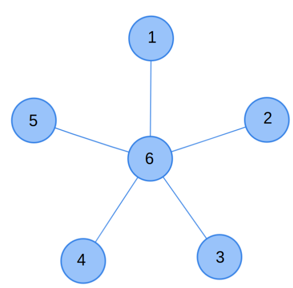

Figure 11: A star graph.

Definition 13. Two simple graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are **isomorphic** if there exists a function $\phi : V_1 \to V_2$ such that:

1) ϕ is bijective, meanings its both adjective and subjective (in simple terms, ϕ is just a relabeling of the vertices).

1.1) $\phi : A \to B$ is **adjective** if for all $a_1, a_2 \in A$ then if $a_1 \neq a_2$, $\phi(a_1) \neq \phi(a_2)$.

1.2) For $u, v \in V_1$, u and v are **subjective** in G_1 if and only if $\phi(u)$ and $\phi(v)$ are adjacent in G_2

2) For $u, v \in V_1$, u and v are adjacent in G_1 if and only if $\phi(u)$ and $\phi(v)$ are adjacent in G_2.

If G_1 is isometric to G_2, we write $G_1 \cong G_2$.

The two graphs, known as Petersen graphs, as shown in figure 12, are isometric.

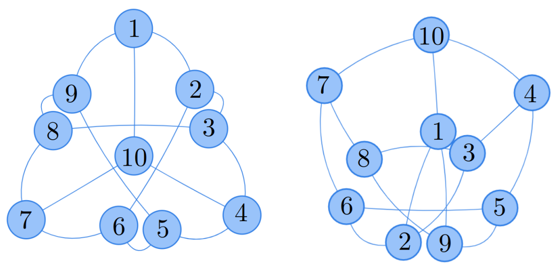

Figure 12: The Petersen graph.

Definition 14. A graph $H = (V(H), E(H), f_H)$ is a (subgraph) of a graph $G = (V(G), E(G), f_G)$ if:

1) $V(H) \subseteq V(G)$

2) $E(H) \subseteq E(G)$

3) For $e \in E(H) : f_H(e) = f_G(e)$

We can construct subgraphs from a graph by either removing a set of vertices, removing a set of edges, or deleting a subgraph (deleting both vertices and edges) from the graph.

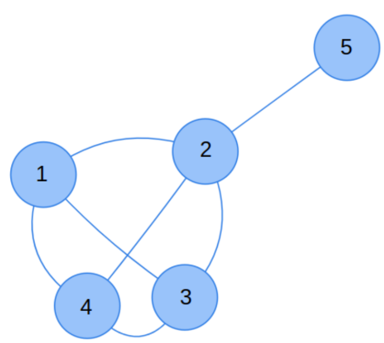

Figure 13: Graph G.

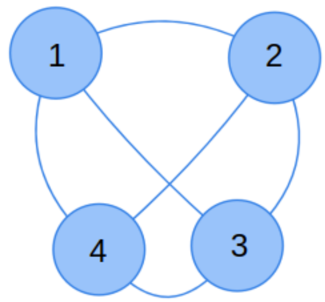

Figure 14: A subgraph of G.

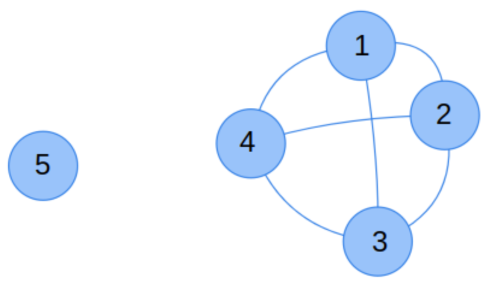

Figure 15: A subgraph of G.

Definition 15. *Deleting a set of vertices $S \subseteq V(G)$:*

$G - S = V(G - S) = V(G) \setminus S$

$E(G - S) = E(G) \setminus \{e \in E(G) | e \text{ is incident with somes} \in S\}$

$F_a - s = f_a|_{E(G-S)}$

Figure 14 shows a subgraph of the graph in figure 13 obtained by deleting the vertex labelled "5."

Definition 16. *Deleting a set of edges $S \subseteq E(G)$:*

$G - S = (V(G), E(G) \setminus S$

$f_{G-S} = f|_{E(G)-S}$

Figure 15 shows a subgraph of the graph in figure 13 obtained by deleting the edge that connects the two vertices labelled "4" and "5."

Definition 17. A subgraph $H \subseteq G$ is a **spanning subgraph** if $V(H) = V(G)$. A subgraph $H \subseteq G$ is an **induced subgraph** if for all $u, v \in V(H)$, every edge joining u, v in G is also an edge of H. We write $H = G(V(H))$. An induced subgraph that is an empty graph is called an **independent set**. A subgraph is called a **clique** if it is a complete graph.

The subgraph in figure 15 is a spanning subgraph of the graph in figure 5 because their set of vertices are equal to each other. The subgraph in figure 14 is a clique and is also an induced subgraph of the graph in figure 13.

In graph theory, we ask questions such as “How do different properties of graphs interact with each other?” and “How do local properties influence larger, global properties and vice versa?”

3 Eulerian trails and circuits

Definition 18. A **walk** in a graph is a sequence $v_1, e_1, v_2, e_2, ..., v_k, e_k, v_{k+1}$ where e_i joins v_i and v_{i+1} for $i = 1, ..., k$. This is a walk of length k. A walk is a **trail** if each edge appears at most once. An **Eulerian trail** is a trail in which which each edge appears exactly once. A walk is **closed** if $v_k = v_1$ and a closed trail is called a **circuit**. A closed Eulerian trail is called a **Eulerian Circuit**.

Definition 19. The **degree** of a vertex v, denoted $deg_G(V)$, in a graph G is the number of edges incident with V that are not loops plus two times the number of loops incident with V.

Theorem 1. *If a graph has a Eulerian circuit, then every vertex has an even degree.*

Proof. Let G be a connected graph (a graph in which there is a walk that includes all vertices) with vertices $V(G) = \{v_1, v_2, v_3, ..., v_n\}$ and edges $E(G) = \{e_1, e_2, e_3, ..., e_n\}$. Assume that G has a Eulerian circuit $v_1, e_1, v_2, e_2, ..., v_n, e_n, v_1$. Since the walk is closed, (because it starts and ends at v_1), we leave all vertices other than v_1 as many times as we enter it. Since the walk is Eulerian, all the edges in $E(G)$ are traversed in the walk and since graph G is connected, we enter/leave each vertex at least once. Since we leave and enter each vertex the same number of time without traversing an edge more or less than one, there are an even number of edges incident to each vertex other than v_1. If we enter a vertex k times, there are $2k$ edges attached to the vertex. This means that the degree of an vertex that is not v_1 is equal to $deg(v_i) = 2k$. Therefore all vertices other than v_1 have an even degree.

At the end of the walk we will traverse an edge e_n which is incident to v_1 to reach v_1. Ignoring the first edge e_1 and last edge e_n traversed, we enter and leave vertex v_1 the same number of times, meaning that an even number of edges are incident to v_1. Let j denote the number of times we exit vertex v_1 during the walk. Since the walk is Eulerian, $v_1 \neq e_n$ and there are two additional edges incident to v_1 which are traversed at the start and end of the walk. This means that $deg(v_1) = 2j + 2$. Therefore v_1 has an even degree. □

Theorem 2. *If a graph has a non-closed Eulerian trail, then it has exactly two verticies with odd degrees.*

Proof. Let G be a connected graph with an open Eulerian trail that starts and ends on vertices $v_1, v_n \in V(G)$, respectively, with $v_1 \neq v_n$. Let H be a graph such that G is a subgraph of H. The set of vertices of H is $V(G) = \{V(G), v_{n+1}\}$, with the vertex v_{n+1} adjacent to v_1 and v_n. A Eulerian circuit exists on graph H by taking the Eulerian trail from v_1 to v_n on graph G and then going to vertex $v_n + 1$ and back to vertex v_1. Since we start at v_1 and end at v_1, this is a Eulerian circuit. From Theorem 1, all vertices in graph H have an even degree. Let $\deg(v_1) = 2k$ and $\deg(v_n) = 2j$. If we subtract $v_n + 1$ from H to get G, we remove two edges from H. Vertices v_1 and v_n both have one less incident edge. This means that in G, $\mathrm{def}(v_1) = 2k - 1$ and $\deg(v_n) = 2j - 1$. Since the other vertices in graph H have even degrees, v_1 and v_n are the only vertices with odd degrees in G. □

4 Closing Remarks

In the next part of this series, we will go over some theoretical results on Hamiltonian walks and we will also look into tree graphs. Please see "References and Further Reading" for some additional resources on graph theory.

5 References and Further Reading

[1] R.J.Trudeau, Introduction to Graph Theory, Dover Publications, 1993. https://books.google.gg/books?id=eRLE

[2] R.J.Wilson, Introduction to Graph Theory 4th Edition, Addison Wesley Longman Limited, 1996. https://www.maths.ed.ac.uk/~v1ranick/papers/wilsongraph.pdf

[3] M. Singh, 10 Graph Theory Applications in Real Life, Number Dyslexia, 2022. https://numberdyslexia.com/graph-theory-applications-in-real-life/

[4] S. Kupferman, A Collection of Chris Hadfield's Pictures of Toronto from Space, 2013. https://torontoist.com/2013/02/a-collection-of-chris-hadfields-pictures-of-toronto-from-space/

[5] B. Halford, Pfizer's novel COVID-19 antiviral heads to clinical trials, 2020. https://cen.acs.org/pharmaceuticals/d discovery/Pfizers-novel-COVID-19-antiviral/98/web/2020/09