MODULE- V

Orthogonality and least Squares

Inner product space :→

If u, v are vectors in $\mathbb{R}^n$, then $u^T.v$ is called the inner product of u and v.

we denote it as $u^T.v$ which is a scalar

Eg :→ 1. Let $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$ and $v = \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix} \in \mathbb{R}^3$ then

$$u^T . v = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix}$$

$$= 5 + 14 + 24$$

$$\boxed{u^T . v = 43}$$

product of two vector is a Scalar

Note :→

1. For any two vectors u, v then $u.v = v.u$

i.e., $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$ and $v = \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix} \in \mathbb{R}^3$

$$u^T . v = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix} = 5 + 14 + 24 = 43$$

$$v^T . u = \begin{bmatrix} 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 5 + 14 + 24 = 43$$

$$\Rightarrow \quad u.v = v.u$$

2. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ [Two - Dimensional space]

$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ [3 - D. space]

$\mathbb{R}^4 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ [4 - D. Space]

$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}$ [n - D Space]

Norm :→

Denoted by " $\| \ \|$ "

Length of the vector

Let u be any n-dimensional vector then it is given by $\|u\| = \sqrt{u.u}$

$= $ Distance from origin $(0, 0, 0 \dots 0)$

where $u = u_1, u_2, u_3 \text{----} u_n$

The length of vector v is a non-negative scalar defined by $\|v\| = \sqrt{v.v}$

$\boxed{\text{OR}}$

The distance from the origin

$$\|v\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

Note :→ Suppose "c" is any scalar and "u" is any vector then $\|c.u\| = |c| \, \|u\|$

Unit Vector :->

A vector whose length is one

If "v" is a non-zero vector it can be converted into a unit vector by using the formula

$$\text{Unit vector } (\hat{n}) = \frac{v}{\|v\|}$$

The process of creating unit vector from any non-zero vector is called Normalization and the unit vector is in the same direction as the given non-zero vector.

Inner product :→

The Inner product (or) dot product of $\mathbb{R}^n$ is denoted (.) and defined by

$$u \cdot v = u^T \cdot v = [u_1, u_2 \ldots\ldots] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix}$$

$$= u_1 v_1 + u_2 v_2 + \ldots\ldots$$

The inner product $\langle -, - \rangle$ satisfies the following properties

1. Linearity : $\langle au + bv, w \rangle = a(u, w) + b(v, w)$

2. Symmetric property $(u, v) = (v, w)$

3. positive definite property

For any $u \in V$, $(u, v) \geq 0$ and $(u, v) = 0$ if and only if $u = 0$

Inner product help use to known the geometric concepts like length of vectors, angle b/w vectors, orthogonality etc

Length :→

The length (or) norm of vector $v \in \mathbb{R}^n$ denoted by $\|v\|$ is defined by $\|v\| = \sqrt{v.v.}$

$$= \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Unit vector :→

A vector with length is 1 is called a unit vector if $v = 0$ then the vector $u = \frac{1}{\|v\|} v$

$$= \frac{1}{\sqrt{v.v}} . v$$ is normalization of v

distance b/w u and v is given by $\| u - v \|$

Orthogonality :→

If two vectors $u, v \in \mathbb{R}^n$ are orthogonal if
$u \cdot v = 0$

Orthogonal Sets :→

A set of vectors $\{u_1, u_2, u_3 \ldots u_k\}$ in $\mathbb{R}^n$ is an orthogonal set if each pair of distinct vectors from the set is orthogonal
i.e., $u_i \cdot u_j = 0$ whenever $i \neq j$

An orthogonal basis for a Subspace W is a basis for W that is also an orthogonal set.

An orthonormal basis for a Subspace W is an orthogo-nal basis for W where each vector has length 1.

Probleme :→

1. compute the length of vector $v = (1, -2, 2, 0)$ and also find an unit vectore and length of unit vector.

Sol: Given that $v = (1, -2, 2, 0)$

Length of vector $= \|v\|$

$$= \sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2}$$

$$= \sqrt{1^2 + (-2)^2 + 2^2 + 0^2}$$

$$= \sqrt{1 + 4 + 4 + 0}$$

$$= \sqrt{9}$$

$$\boxed{\|v\| = 3}$$

Unit vector $(\hat{n}) = \dfrac{v}{\|v\|}$

$$\boxed{\hat{n} = \frac{1}{3}(1, -2, 2, 0)}$$

$u =$ Unit vector $\hat{n} = \left(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}, 0\right)$

Length of an unit vector

$$\|u\| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + 0^2}$$

$$= \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}}$$

$$= \sqrt{\frac{9}{9}} = \sqrt{1}$$

$$\boxed{\|u\| = 1}$$

$\Rightarrow$ u is an unit vector and having a length is one

2. compute the length of vector $v = (2, -1, 1, 2)$ and also find an unit vector

Sol Given that $v = (2, -1, 1, 2)$

Length of vector $= \|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2}$

$= \sqrt{2^2 + (-1)^2 + (1)^2 + (2)^2}$

$= \sqrt{4+1+1+4}$

$\boxed{\|v\| = \sqrt{10}}$

Unit vector $\hat{n} = u = \frac{v}{\|v\|}$

$= \frac{1}{\sqrt{10}}(2, -1, 1, 2)$

$$\boxed{u = \left(\frac{2}{\sqrt{10}}, \frac{-1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right)}$$

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3. Compute the distance between the vectors $u = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Sol: Given that $u = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Distance b/w u and v = $\|u - v\|$

$$= \sqrt{(u-v)(u-v)}$$

$$= \sqrt{(\|u-v\|)^2}$$

$$= \|u-v\|$$

Let $u - v = \begin{bmatrix} 7 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$u - v = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\|u-v\| = \sqrt{(4)^2 + (-1)^2}$$

$$= \sqrt{16+1}$$

$$\boxed{\|u-v\| = \sqrt{17}}$$

4. Show that $\{u_1, u_2\}$ is an orthogonal set where $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$. Express $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as linear combination of vector S.

Sol: Given that $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

To prove that $\{u_1, u_2\}$ are orthogonal to each other

i.e., $u_1^T \cdot u_2 = 0$ $\quad u_2^T \cdot u_1 = 0$

$$u_1^T \cdot u_2 = \begin{bmatrix} 2 & 5 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = -4 + 5 - 1 = 5 - 5 = 0$$

$$u_2^T \cdot u_1 = \begin{bmatrix} -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} = -4 + 5 - 1 = 5 - 5 = 0$$

u_1 and u_2 are orthogonal to each other

$\{u_1, u_2\}$ are orthogonal set.

We know that

$$y = c_1 u_1 + c_2 u_2$$

$$c_1 = \frac{y'u_1}{u_1 \cdot u_1} = \frac{y'u_1}{\|u_1\|^2} = \frac{1}{\left(\sqrt{2^2+5^2+(-1)^2}\right)^2} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$$

$$= \frac{1}{30}\left[2+10-3\right]$$

$$c_1 = \frac{9}{30} = \frac{3}{10} \Rightarrow \boxed{c_1 = \frac{3}{10}}$$

$$c_2 = \frac{y'u_2}{u_2 \cdot u_2} = \frac{y'u_2}{\|u_2\|^2} = \frac{1}{\left(\sqrt{(-2)^2+1^2+1^2}\right)^2} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{6}\left[-2+2+3\right] = \frac{3}{6} = \frac{1}{2}$$

$$\boxed{c_2 = \frac{1}{2}}$$

$\therefore y = c_1 u_1 + c_2 u_2$

$$\boxed{y = \frac{3}{10} u_1 + \frac{1}{2} u_2}$$ as linear combination of vector s

5. Show that $\{u_1, u_2, u_3\}$ is an orthogonal set where (8)

$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ $u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ and $u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$. Express

$y = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ as linear combination of vector s

Sol^n Given that $u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ and $u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$

To prove that u_1, u_2, u_3 are orthogonal to each other.

i.e., $u_1^T . u_2 = 0$, $u_2^T . u_3 = 0$ and $\underline{u_3^T . u_2 = 0}$

$$u_1^T . u_2 = \begin{bmatrix} 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = -3+2+1 = 3-3 = 0$$

$$u_2^T . u_3 = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1/2 \\ -2 \\ +7/2 \end{bmatrix} = \frac{1}{2} - 4 + \frac{7}{2} = \frac{8}{2} - 4 = 4-4 = 0$$

$$u_3^T \cdot u_1 = \begin{bmatrix} -\frac{1}{2} & -2 & \frac{7}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = -\frac{3}{2} - 2 + \frac{7}{2} = \frac{4}{2} - 2 = 2 - 2 = 0$$

$\therefore \{u_1, u_2, u_3\}$ is an orthogonal set.

W.K.T.

$$y = c_1 u_1 + c_2 u_2 + c_3 u_3$$

where

$$c_1 = \frac{y^{1} \cdot u_1}{u_1 \cdot u_1} = \frac{y^{1} \cdot u_1}{\|u_1\|^2} = \frac{1}{\left(\sqrt{3^2 + 1^2 + 1^2}\right)^2} \begin{bmatrix} 6 & -1 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(\sqrt{11})^2} (18 + 1 - 8) = \frac{11}{11} = 1$$

$$\boxed{c_1 = 1}$$

$$c_2 = \frac{y' \cdot u_2}{u_2 \cdot u_2} = \frac{y' u_2}{\|u_2\|^2} = \frac{1}{\left(\sqrt{(-1)^2 + 2^2 + 1^2}\right)^2} \begin{bmatrix} 6 & 1 & -8 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$c_2 = \frac{1}{6}[-6+2-8] = \frac{-12}{6} = -2$$

$$\Rightarrow \boxed{c_2 = -2}$$

$$c_3 = \frac{y' \cdot u_3}{u_3 \cdot u_3} = \frac{y' \cdot u_2}{\|u_3\|^2} = \frac{1}{\left(\sqrt{\left(-\frac{1}{2}\right)^2 + (-2)^2 + \left(\frac{7}{2}\right)^2}\right)^2} \begin{bmatrix} 6 & +1 & -8 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ -2 \\ \frac{7}{2} \end{bmatrix}$$

$$= \frac{1}{\frac{1}{4} + 4 + \frac{49}{4}} \left[-\frac{\overset{3}{\cancel{6}}}{\cancel{2}} - 2 - \frac{\overset{4}{\cancel{8}} \times 7}{\cancel{2}}\right] = \frac{2}{\cancel{33}} \left(-\cancel{33}\right)$$

$$\Rightarrow \boxed{c_3 = -2}$$

$\therefore\ y = c_1 u_1 + c_2 u_2 + c_3 u_3$

$\boxed{y = 1u_1 - 2u_2 - 2u_3}$ as linear combination of vector S

5. Show that $\{u_1, u_2, u_3\}$ is an orthogonal set where $u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ $u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ and $u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$. Express $y = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ as linear combination of vectors

(8)

Sol^n Given that $u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ and $u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$

To prove that u_1, u_2, u_3 are orthogonal to each other.

i.e., $u_1^T . u_2 = 0$, $u_2^T . u_3 = 0$ and $u_3^T . u_2 = 0$

$$u_1^T . u_2 = \begin{bmatrix} 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = -3+2+1 = 3-3 = 0$$

$$u_2^T . u_3 = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1/2 \\ -2 \\ +7/2 \end{bmatrix} = \frac{1}{2} - 4 + \frac{7}{2} = \frac{8}{2} - 4 = 4-4 = 0$$

$$u_3^T \cdot u_1 = \begin{bmatrix} -\frac{1}{2} & -2 & \frac{7}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = -\frac{3}{2} - 2 + \frac{7}{2} = \frac{4}{2} - 2 = 2 - 2 = 0$$

$\therefore \{u_1, u_2, u_3\}$ is an orthogonal set.

W.K.T.

$$y = c_1 u_1 + c_2 u_2 + c_3 u_3$$

where

$$c_1 = \frac{y' \cdot u_1}{u_1 \cdot u_1} = \frac{y' \cdot u_1}{\|u_1\|^2} = \frac{1}{\left(\sqrt{3^2 + 1^2 + 1^2}\right)^2} \begin{bmatrix} 6 & -1 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(\sqrt{11})^2} (18 + 1 - 8) = \frac{11}{11} = 1$$

$$\boxed{c_1 = 1}$$

$$c_2 = \frac{y' \cdot u_2}{u_2 \cdot u_2} = \frac{y' u_2}{\|u_2\|^2} = \frac{1}{\left(\sqrt{(-1)^2 + 2^2 + 1^2}\right)^2} \begin{bmatrix} 6 & 1 & -8 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$c_2 = \frac{1}{6}\left[-6 + 2 - 8\right] = \frac{-12}{6} = -2$$

$$\Rightarrow \boxed{c_2 = -2}$$

$$c_3 = \frac{y' \cdot u_3}{u_3 \cdot u_3} = \frac{y' \cdot u_2}{\|u_3\|^2} = \frac{1}{\left(\sqrt{\left(-\frac{1}{2}\right)^2 + (-2)^2 + \left(\frac{7}{2}\right)^2}\right)^2} \begin{bmatrix} 6 & +1 & -8 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ -2 \\ \frac{7}{2} \end{bmatrix}$$

$$= \frac{1}{\frac{1}{4} + 4 + \frac{49}{\cancel{4}}} \left[-\frac{\overset{3}{\cancel{6}}}{\cancel{2}} - 2 - \frac{\overset{4}{\cancel{8}} \times 7}{\cancel{2}}\right] = \frac{2}{\cancel{33}} \left(-\cancel{33}\right)$$

$$\Rightarrow \boxed{c_3 = -2}$$

$\therefore \quad y = c_1 u_1 + c_2 u_2 + c_3 u_3$

$\boxed{y = 1u_1 - 2u_2 - 2u_3}$ as linear combination of vector S

Orthogonal basis:

An orthogonal basis for a subspace W is a basis for W that is also an orthogonal Set. An orthonormal basis for a Subspace W is an orthogonal basis for W where each vector has length 1.

Orthogonal projection :→

When the vector space has an inner product and is complete. The concept of orthogonality can be use (or) It is a projection for which the range and null space are orthogonal subspace.

6. If $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$, $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Find the orthogonal projection component also write y as sum of two vectors.

Soln Given $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$, $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

we have $\hat{y} = \left[\dfrac{y' u_1}{u_1 . u_1}\right] u_1 = \left[\dfrac{y' u_1}{\|u_1\|^2}\right] u_1$

and component 'z' of y

$z = y - \hat{y}$ and $\boxed{y = z + \hat{y}}$ → To verify

Let $\hat{y} = \left[\dfrac{y' u_1}{\|u_1\|^2}\right] u_1 = \left[\dfrac{\begin{bmatrix} 7 & 6 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix}}{\left(\sqrt{4^2 + 2^2}\right)^2}\right]\begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$$\Rightarrow \hat{y} = \left[\frac{28 + 12}{\left(\sqrt{20}\right)^2}\right]\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{40}{20}\begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\Longrightarrow \hat{y} = 2\begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\Longrightarrow \hat{y} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

Now $$z = y - \hat{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$y = z + \hat{y} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

7. If $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$ and $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$. Find the orthogonal projection component also write y as sum of two vectors.

Sol: Given $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$ and $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

we have

$$\hat{y} = \left[\frac{y'u_1}{\|u_1\|^2}\right] u_1 + \left[\frac{y'u_2}{\|u_2\|^2}\right] u_2$$

$$\hat{y} = \left(\frac{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}}{\left(\sqrt{2^2+5^2+(-1)^2}\right)^2}\right)\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \left(\frac{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}{\sqrt{(-2)^2+1^2+1^2}}\right)\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\hat{y} = \left(\frac{2+10-3}{4+25+1}\right)\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \left(\frac{-2+2+3}{4+1+1}\right)\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\hat{y} = \frac{\overset{3}{\cancel{9}}}{\underset{10}{\cancel{30}}}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{\overset{1}{\cancel{3}}}{\underset{2}{\cancel{6}}}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\hat{y} = \begin{bmatrix} 6/10 \\ 15/10 \\ -3/10 \end{bmatrix} + \begin{bmatrix} -1 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} \frac{6}{10} - 1 \\ \frac{15}{10} + \frac{1}{2} \\ -\frac{3}{10} + \frac{1}{2} \end{bmatrix}$$

$$\hat{y} = \begin{bmatrix} -4/10 \\ 20/10 \\ 2/10 \end{bmatrix} \Rightarrow \hat{y} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}$$

Now

$$z = y - \hat{y}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} = \begin{bmatrix} 1 + 2/5 \\ 2 - 2 \\ 3 - 1/5 \end{bmatrix}$$

$$\boxed{z = \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix}}$$

$$y = z + \hat{y} = \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix} + \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} = \begin{bmatrix} 7/5 - 2/5 \\ 0 + 2 \\ 14/5 + 1/5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\boxed{\therefore y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

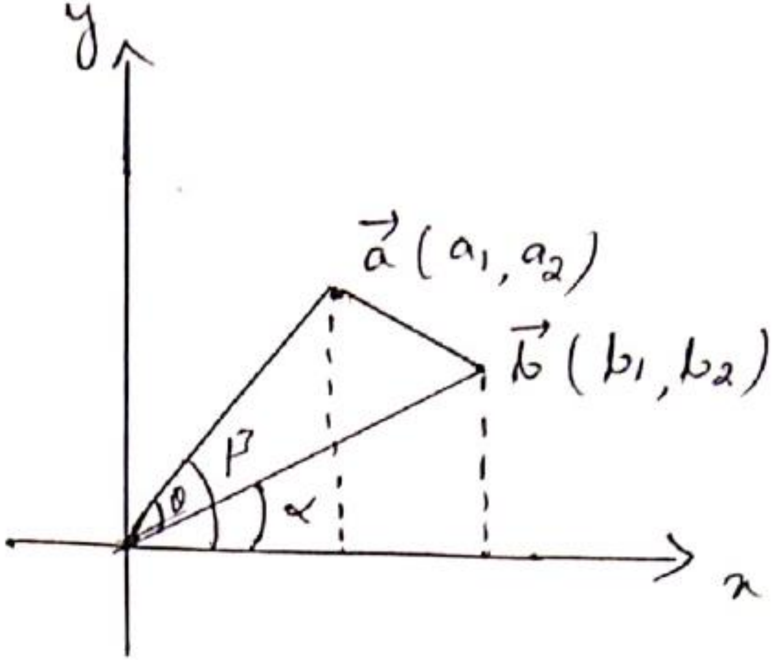
Cosine and projections onto lines :-

Cosines :→

$$\cos\theta = \cos(\beta - \alpha)$$

$$= \cos\beta\cos\alpha + \sin\beta\sin\alpha$$

$$= \frac{a_1}{|\vec{a}|}\frac{b_1}{|\vec{b}|} + \frac{a_2}{|\vec{a}|}\frac{b_2}{|\vec{b}|}$$

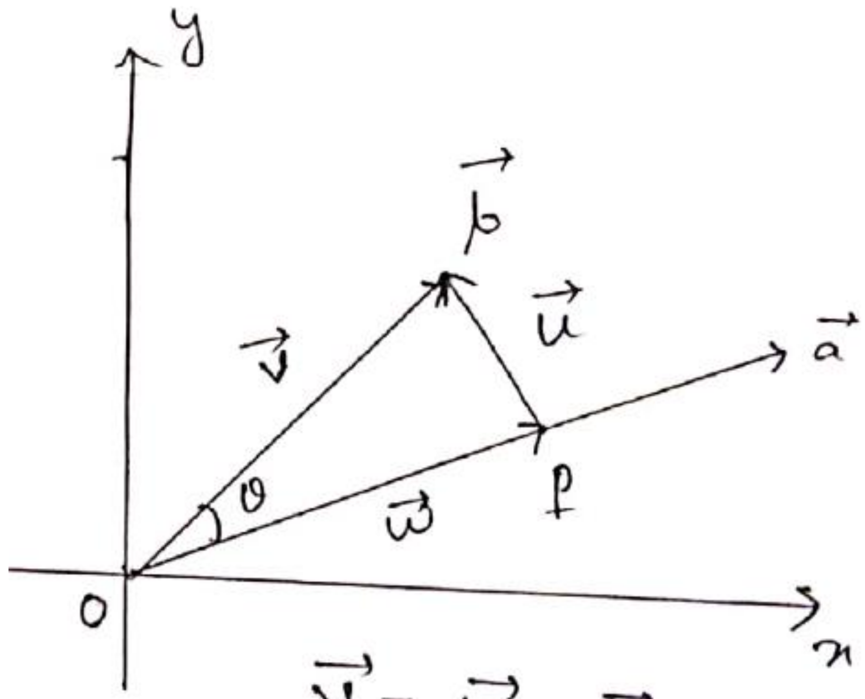


$$\cos\theta = \frac{a_1 b_1 + a_2 b_2}{|\vec{a}|\,|\vec{b}|}$$

$$\Rightarrow \boxed{\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}}$$

projection onto lines :-

The projection of the vector $\vec{b}$ onto the line in the direction of $\vec{a}$ is $(op) = \vec{b} \cdot \frac{\vec{a}}{|\vec{a}|}$



The line connectivity from $\vec{b}$ to point p is $\perp^{r}$ to $\vec{a}$ and is denoted by vector $\vec{u}$

$$\vec{v} = \vec{u} + \vec{w}$$

$$\vec{b} = \vec{v}$$

$$op = \vec{w}$$

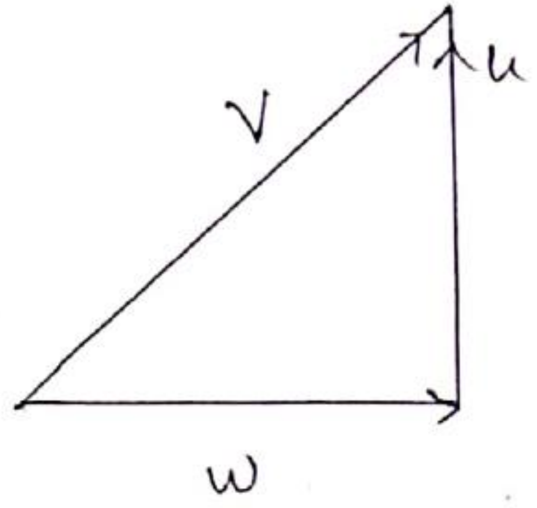
i.e., $\vec{u} \cdot \vec{a} = 0$

$$(\vec{v} - \vec{w}) \cdot \vec{a} = 0$$

Note :- Eventhrough $\vec{a}$ and $\vec{b}$ are not orthogonal, $\vec{u}$ automatically brings the orthogonality concept.

projections :-

If W is a subspace of $\mathbb{R}^n$ with orthonormal basis $\{w_1, w_2, \ldots, w_m\}$ and v is any vector in $\mathbb{R}^n$, then there exist unique vectors w in W and u in $W^\perp$ such that $v = w + u$



$$v = w + u$$

Moreover $w = (v.w_1)w_1 + (v.w_2)w_2 + \cdots + (v.w_m)w_m$

which is called as the orthogonal projection of v and w and is denoted by

$$\operatorname{proj}_w v = w = (v.w_1)w_1 + (v.w_2)w_2 + \cdots + (v.w_m)w_m$$

Note :→ 1. If $w_1, w_2, \ldots, w_m$ are orthogonal vectors then

$$\operatorname{proj}_w v = \left(\frac{v.w_1}{w_1.w_1}\right)w_1 + \left(\frac{v.w_2}{w_2.w_2}\right)w_2 + \cdots + \left(\frac{v.w_m}{w_m.w_m}\right)w_m$$

$(\ldots_m . w_m)$

orthogonal vectors:

If $w_1, w_2 \ldots, w_m$ are orthogonal vectors then all the vectors are mutually prependicular to each other

orthogonal vectors:

If $w_1, w_2 \ldots, w_m$ are orthonormal vector then all the vectors mutually prependicular to each other as well as each vector is a unit vector.

Problems :-

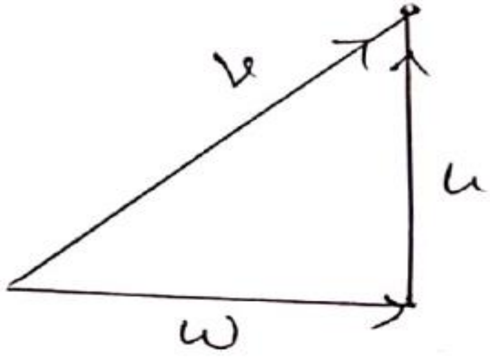
1. Let W be the 3-D Subspace of $\mathbb{R}^3$ with ortho-normal basis (w_1, w_2) where $w_1 = \left(\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}\right)$ and $w_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ using standard inner product of $\mathbb{R}^3$. Find the orthogonal projection of $v = (2, 1, 3)$ on W and the vector u that is orthogonal to every vector in W.

Sol^n By data

$$w_1 = \left(\frac{2}{3}, \frac{-1}{3}, -\frac{2}{3}\right) ; w_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$v = (2, 1, 3)$$

We have



$$u = v - w$$

$$w = (v\cdot w_1)\,w_1 + (v\cdot w_2)\,w_2$$

$$\Rightarrow w = \left[(2,1,3)\cdot\left(\frac{2}{3}, \frac{-1}{3}, -\frac{2}{3}\right)\right]w_1 + \left[(2,1,3)\cdot\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\right]w_2$$

$$\Rightarrow w = \left(\frac{4}{3} - \frac{1}{3} - \frac{2}{1}\right)w_1 + \left(\frac{2}{\sqrt{2}} + 0 + \frac{3}{\sqrt{2}}\right)w_2$$

$$\Rightarrow w = \left(\frac{4-1-6}{3}\right)w_1 + \left(\frac{5}{\sqrt{2}}\right)w_2$$

$$\Rightarrow w = -w_1 + \frac{5}{\sqrt{2}}\,w_2$$

$$= -\left(\frac{2}{3}, \frac{-1}{3}, -\frac{2}{3}\right) + \frac{5}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$= \left(-\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right) + \left(\frac{5}{2}, 0, \frac{5}{2}\right)$$

$$= \left(-\frac{2}{3} + \frac{5}{2}, \frac{1}{3} + 0, \frac{2}{3} + \frac{5}{2}\right)$$

$$= \left(\frac{-4+15}{6}, \frac{1}{3}, \frac{4+15}{6}\right)$$

$$\boxed{w = \left(\frac{11}{6}, \frac{2}{6}, \frac{19}{6}\right)}$$

Now

$$u = v - w$$

$$= (2, 1, 3) - \left(\frac{11}{6}, \frac{2}{6}, \frac{19}{6}\right)$$

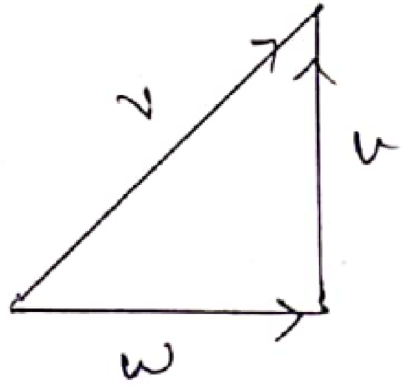
$$= \left(2 - \frac{11}{6}, 1 - \frac{2}{6}, \frac{3}{1} - \frac{19}{6}\right)$$

$$= \left(\frac{12-11}{6}, \frac{6-2}{6}, \frac{18-19}{6}\right)$$

$$\boxed{u = \left(\frac{1}{6}, \frac{4}{6}, \frac{-1}{6}\right)}$$

2. Let W be the subspace of $\mathbb{R}^3$ defined $w_1 = \left(\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}\right)$ and $w_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ and let $v = (1, 1, 0)$. Find the distance v to W

Sol: Given that



$$w_1 = \left(\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}\right), \quad w_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$v = (1, 1, 0)$$

Now $w = (v \cdot w_1) w_1 + (v \cdot w_2) w_2$

$$= \left[(1,1,0) \cdot \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)\right] w_1 + \left[(1,1,0) \cdot \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\right] w_2$$

$$= \left(\frac{2}{3} - \frac{1}{3}\right) w_1 + \frac{1}{\sqrt{2}} w_2$$

$$= \frac{1}{3} w_1 + \frac{1}{\sqrt{2}} w_2$$

$$= \frac{1}{3}\left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$= \left(\frac{2}{9}, -\frac{1}{9}, -\frac{2}{9}\right) + \left(\frac{1}{2}, 0, \frac{1}{2}\right)$$

$$= \left(\frac{2}{9} + \frac{1}{2}, -\frac{1}{9} + 0, -\frac{2}{9} + \frac{1}{2}\right)$$

$$= \left(\frac{4+9}{18}, -\frac{1}{9}, \frac{-4+9}{18}\right)$$

$$\boxed{w = \left(\frac{13}{18}, -\frac{1}{9}, \frac{5}{18}\right)} \Rightarrow \boxed{w = \left(\frac{13}{18}, -\frac{2}{18}, \frac{5}{18}\right)}$$

Now $u = v - w$

$$= (1,1,0) - \left(\frac{13}{18}, \frac{-2}{18}, \frac{5}{18}\right)$$

$$= \left(1 - \frac{13}{18}, 1 + \frac{2}{18}, 0 - \frac{5}{18}\right)$$

$$= \left(\frac{18-13}{18}, \frac{18+2}{18}, \frac{-5}{18}\right)$$

$$\boxed{u = \left(\frac{5}{18}, \frac{20}{18}, \frac{-5}{18}\right)}$$

$$\boxed{\|u\| = \frac{15\sqrt{2}}{18}}$$

$$\|u\| = \sqrt{\frac{25}{18^2} + \left(\frac{20}{18}\right)^2 + \frac{25}{18^2}}$$

$$= \sqrt{\frac{25}{18^2} + \frac{400}{18^2} + \frac{25}{18^2}} = \frac{\sqrt{450}}{18} = \frac{\sqrt{225 \times 2}}{18}$$

3. Let w be the subspace of R^4, $w_1 = (0,0,1,0)$, $w_2 = \left(\frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}}\right)$ and $w_3 = \left(\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right)$ and $v = (1, 2, -1, 0)$. Find the distance from v to w

Sol: Given that $w_1 = (0, 0, 1, 0)$, $w_2 = \left(\frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}}\right)$
$w_3 = \left(\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right)$, $v = (1, 2, -1, 0)$

Now $w = (v.w_1)w_1 + (v.w_2)w_2 + (v.w_3)w_3$ (17)

$= \{(1, 2, -1, 0).(0, 0, 1, 0)\}w_1 + \{(1, 2, -1, 0).\left(\frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}}\right)\}$
$w_2 + \{(1, 2, -1, 0).\left(\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right)\} w_3$

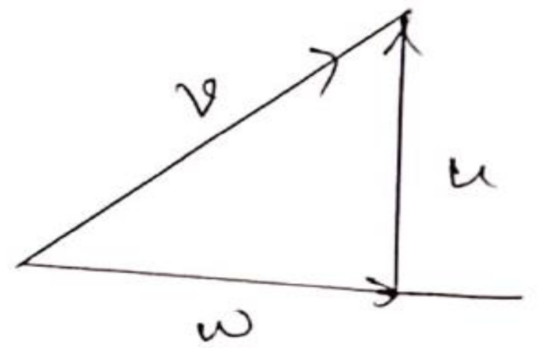
$$= -1\,w_1 + \frac{1}{\sqrt{2}} w_2 + \frac{1}{\sqrt{2}} w_3$$

$$= -1(0,0,+1,0) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right)$$

$$= (0,0,-1,0) + \left(\frac{1}{2}, 0, 0, -\frac{1}{2}\right) + \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$$

$$\boxed{w = (1, 0, -1, 0)}$$

Now $u = v - w$



$$= (1, 2, -1, 0) - (1, 0, -1, 0)$$

$$u = (0, 2, 0, 0)$$

$$\|u\| = \sqrt{0^2 + 2^2 + 0^2 + 0^2}$$

$$= \sqrt{4}$$

$$\boxed{\|u\| = 2}$$

4. The distance from a point y in $\mathbb{R}^n$ to a subspace W is defined as the distance from y to the nearest point in W. Find distance from y to $W = \text{Span}\{u_1, u_2\}$ where $y = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}$, $u_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$

and $u_2 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$

Sol: Given that $y = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}$, $u_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

we have

$$u = \left[\frac{y' \cdot u_1}{u_1 \cdot u_1}\right] u_1 + \left[\frac{y' u_2}{u_2 \cdot u_2}\right] u_2$$

$$\Rightarrow u = \left\{ \frac{[-1, -5 \quad 10] \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}}{\left(\sqrt{5^2 + (-2)^2 + 1^2}\right)^2} \right\} \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + \left\{ \frac{[-1 \quad -5 \quad 10] \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}{\sqrt{1^2 + 2^2 + (-1)^2}} \right\} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow u = \frac{\cancel{15}}{\cancel{30}_{2}}\begin{bmatrix}5\\-2\\1\end{bmatrix} + \left[-\frac{\cancel{21}^{7}}{\cancel{6}_{2}}\right]\begin{bmatrix}1\\2\\-1\end{bmatrix}$$

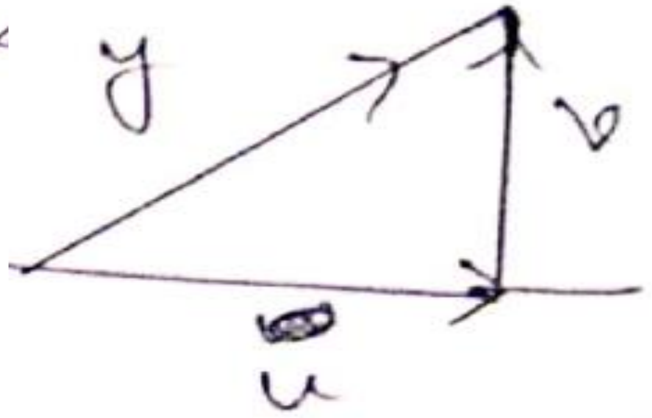
$$\Rightarrow u = \frac{1}{2}\begin{bmatrix}5\\-2\\1\end{bmatrix} + \left(-\frac{7}{2}\right)\begin{bmatrix}1\\2\\-1\end{bmatrix}$$

$$\Rightarrow u = \begin{bmatrix}5/2\\-1\\1/2\end{bmatrix} + \begin{bmatrix}-7/2\\-7\\7/2\end{bmatrix} = \begin{bmatrix}\frac{5}{2}-\frac{7}{2}\\-1-7\\\frac{1}{2}+\frac{7}{2}\end{bmatrix} = \begin{bmatrix}-2/2\\-8\\8/2\end{bmatrix}$$

$$\Rightarrow \boxed{u = \begin{bmatrix}-1\\-8\\4\end{bmatrix}}$$

Now $v = y - u$

$$= \begin{bmatrix}-1\\-5\\10\end{bmatrix} - \begin{bmatrix}-1\\-8\\4\end{bmatrix} = \begin{bmatrix}-1+1\\-5+8\\10-4\end{bmatrix}$$



$$v = \begin{bmatrix}0\\3\\6\end{bmatrix}$$

$$\|v\| = \sqrt{0+9+36}$$

$$\|v\| = \sqrt{35} = 3\sqrt{5}$$

$$\boxed{\|v\| = 3\sqrt{5}}$$

Least Squares [curve fitting]

Working Rule :→

Curve : $y = a_2x^2 + a_1x + a_0$

Step 1 :→ From $B = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, $A = \begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ \vdots & \vdots & \vdots \\ x_n^2 & x_n & 1 \end{bmatrix}$ and $X = \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix}$

Step 2 :→ Solve the normal System

$A^TAX = A^TB$ for finding x by Gauss-Jordan reduction.

<u>curve</u> :→ $y = a_1 x + a_0$

<u>Step 1</u> :→

$$b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}, \quad X = \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$$

<u>Step 2</u> :→ Solve the normal system

$$A^T A X = A^T B$$

for finding X by Gauss - Jordan reduction.

1. **Find the Least square solutions to the system $Ax = b$ where**

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \; b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

Sol: Given that $Ax = b$;

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}_{3\times 2}, \quad b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}_{3\times 1}$$

order of $A = 3\times 2$
order of $b = 3\times 1$
order of $x = 2\times 1$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

W.K.T. $Ax = b$

The normal equation is given by

$$A^T A x = A^T b$$

$$\Rightarrow \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 16+0+1 & 0+0+1 \\ 0+0+1 & 0+4+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8+0+11 \\ 0+0+11 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

$\Rightarrow \quad 17x_1 + x_2 = 19 \rightarrow (1)$

$x_1 + 5x_2 = 11 \longrightarrow (2) \quad \times^{h}$ by 17

$\Rightarrow \quad \cancel{17x_1} + x_2 = 19$

$\overset{+}{\underset{(-)}{}}\cancel{17x_1} \underset{(-)}{+} 85x_2 = \underset{(-)}{+}187$

$\cancel{+}84x_2 = \cancel{+}168$

$x_2 = \dfrac{168}{84} \Rightarrow \boxed{x_2 = 2}$

from (2)

$x_1 + 5x_2 = 11$

$\Rightarrow x_1 = 11 - 5x_2$

$\Rightarrow x_1 = 11 - 5(2)$

$\Rightarrow x_1 = 11 - 10$

$\boxed{x_1 = 1}$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \therefore y = a_1 x + a_0$$

$$\boxed{y = 1x + 2}$$

2. Find the Least square solutions to the system $Ax = b$ where

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix}$$

Sol^n Given that

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}_{6\times 4} \qquad b = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix}$$

we have

$$Ax = b$$

order of $A = 6\times 4$

order of $b = 6\times 1$

order of $x = 4\times 1$

The Normal equation is given by

$$A^T A x = A^T b$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 2 \\ 6 \end{bmatrix}$$

$$6x_1 + 2x_2 + 2x_3 + 2x_4 = 4$$

$$2x_1 + 2x_2 + 0x_3 + 0x_4 = -4$$

$$2x_1 + 0x_2 + 2x_3 + 0x_4 = 2$$

$$2x_1 + 0x_2 + 0x_3 + 2x_4 = 6$$

Use Gauss elimination method to find x_1, x_2, x_3 and x_4

$$[A : B] = \left[\begin{array}{cccc|c} 6 & 2 & 2 & 2 & 4 \\ 2 & 2 & 0 & 0 & -4 \\ 2 & 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 2 & 6 \end{array}\right]$$

$R_2 \to 3R_2 - R_1$

$R_3 \to 3R_3 - R_1$

$R_4 \to 3R_4 - R_1$

$$= \left[\begin{array}{cccc|c} 6 & 2 & 2 & 2 & 4 \\ 0 & 4 & -2 & -2 & -16 \\ 0 & -2 & 4 & -2 & 2 \\ 0 & -2 & 2 & 4 & 14 \end{array}\right]$$

$R_3 \to 2R_3 + R_2$

$R_4 \to 2R_4 + R_2$

$$= \left[\begin{array}{cccc|c} 6 & 2 & 2 & 2 & 4 \\ 0 & 4 & -2 & -2 & -16 \\ 0 & 0 & 6 & -6 & -12 \\ 0 & 0 & -6 & 6 & 12 \end{array}\right]$$

$R_4 \to R_4 + R_3$

$$[A:B] = \left[\begin{array}{cccc:c} 6 & 2 & 2 & 2 & 4 \\ 0 & 4 & -2 & -2 & -16 \\ 0 & 0 & 6 & -6 & -12 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = \rho[A:B] = r = 3$

There is solution

i) Unique Solution $r = n$ Here $r \neq n$

$3 \neq 4$

ii) Infinite many Solution

$r < n \Rightarrow 3 < 4$

Let $n - r = 4 - 3 = 1$

choose $\boxed{x_4 = k_1}$

$$6x_1 + 2x_2 + 2x_3 + 2x_4 = 4 \longrightarrow (1)$$

$$4x_2 - 2x_3 - 2x_4 = -16 \longrightarrow (2)$$

$$6x_3 - 6x_4 = -12 \longrightarrow (3)$$

from (3)

$$6x_3 = -12 + 6x_4$$

$$6x_3 = \cancel{k_1} -12 + 6k_1$$

$$\cancel{6}x_3 = \cancel{6}(k_1 - 2)$$

$$\boxed{x_3 = k_1 - 2.}$$

from (2)

$$4x_2 = -16 + 2x_3 + 2x_4$$

$$= -16 + 2(k_1 - 2) + 2k_1$$

$$= -16 + 2k_1 - 4 + 2k_1$$

$$4x_2 = 4k_1 - 20$$

$$\cancel{4}x_2 = \cancel{4}(k_1 - 5)$$

$$\boxed{x_2 = k_1 - 5}$$

$4x_2$

$\cancel{4}x$

from (1)

$$6x_1 = 4 - 2x_2 - 2x_3 - 2x_4$$

$$= 4 - 2(k_1 - 5) - 2(k_1 - 2) - 2k_1$$

$$= 4 - 2k_1 + 10 - 2k_1 + 4 - 2k_1$$

$$6x_1 = 18 - 6k \implies \cancel{6}x_1 = \cancel{6}(3 - k_1)$$

$$\implies \boxed{x_1 = 3 - k_1}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3-K_1 \\ K_1-5 \\ K_1-2 \\ K_1 \end{bmatrix}$$

put $K=0$

$$X = \begin{bmatrix} 3 \\ -5 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}$$

$\therefore y = a_3 x^3 + a_2 x^2 + a_1 x + a_0$

$$\boxed{y = 3x^3 - 5x^2 - 2x + 0}$$

3. Find the least square solutions of $Ax = b$ for

$$A = \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix}$$

Sol^n Given that

$$A = \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix}$$

order of $A = 4 \times 2$
order of $b = 4 \times 1$
order of $x = 2 \times 1$

we have

$$Ax = b$$

The Normal equation is given by

$$A^T A x = A^T b$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ -6 & -2 & 1 & 7 \end{bmatrix} \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -6 & -2 & 1 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1+1+1+1 & -6-2+1+7 \\ -6-2+1+7 & 36+4+1+49 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1+2+1+6 \\ 6-4+1+42 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 0 \\ 0 & 90 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 45 \end{bmatrix}$$

$$4x_1 + 0x_2 = 8 \quad \Rightarrow \quad x_1 = 8/4$$

$$0x_1 + 90x_2 = 45 \qquad \boxed{x_1 = 2}$$

$$90x_2 = 45$$

$$x_2 = \frac{45}{90}$$

$$\boxed{x_2 = \frac{1}{2}}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$$

$$y = a_1 x + a_0$$

$$\boxed{y = 2x + \frac{1}{2}}$$

4. In the manufacturer of product X, the amount of the compound beta present in the product is controlled by the amount of ingredient alpha used in the process. In manufacturing the gallon of X, the amount of alpha used, and the amount of beta present are recorded. The following data were obtained

Alpha used(x) (Ounces/ gallon)	3	4	5	6	7	8	9	10	11	12
Beta Present(y) (Ounces/ gallon)	4.5	5.5	5.7	6.6	7.0	7.7	8.5	8.7	9.5	9.7

a) Find an equation of the least square line for the given data.

b) Use the equation obtained to predict the number of ounces of beta present in a gallon of product X if 30 ounces of alpha are used per gallon.

Sol Let $b = \begin{bmatrix} 4.5 \\ 5.5 \\ 5.7 \\ 6.6 \\ 7.0 \\ 7.7 \\ 8.5 \\ 8.7 \\ 9.5 \\ 9.7 \end{bmatrix}$ $A = \begin{bmatrix} 3 & 1 \\ 4 & 1 \\ 5 & 1 \\ 6 & 1 \\ 7 & 1 \\ 8 & 1 \\ 9 & 1 \\ 10 & 1 \\ 11 & 1 \\ 12 & 1 \end{bmatrix}$ $x = \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$

$y = a_1 x + a_0$

we have $Ax = b$

order of $A = 10 \times 1$

order of $b = 10 \times 2$

order of $x = 2 \times 1$

The normal equation is given by

$$A^T A x = A^T b$$

$$\Rightarrow \begin{bmatrix} 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 1 \\ 5 & 1 \\ 6 & 1 \\ 7 & 1 \\ 8 & 1 \\ 9 & 1 \\ 10 & 1 \\ 11 & 1 \\ 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4.5 \\ 5.5 \\ 5.7 \\ 6.6 \\ 7.0 \\ 7.7 \\ 8.5 \\ 8.7 \\ 9.5 \\ 9.7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 645 & 75 \\ 75 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 598.6 \\ 73.4 \end{bmatrix}$$

$$645x_1 + 75x_2 = 598.6 \rightarrow ①$$

$$75x_1 + 10x_2 = 73.4 \rightarrow ② \quad \times^{ly} \text{ by } 7.5$$

$$645x_1 + \cancel{75x_2} = 598.6$$

$$\underline{\mp 562.5x_1 \mp \cancel{75x_2} = \mp 550.5}$$

$$82.5x_1 = 48.1$$

$$\boxed{x_1 = 0.5830}$$

from ② $\quad 75x_1 + 10x_2 = 73.4$

$$10x_2 = 73.4 - 75x_1$$

$$= 73.4 - 75(0.5830)$$

$$= 73.4 - 43.725$$

$$10x_2 = 29.675$$

$$x_2 = \frac{29.675}{10}$$

$$\boxed{x_2 = 2.9675}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 0.5830 \\ 2.9675 \end{bmatrix}$$

$$\boxed{y = (0.5830)x + 2.9675}$$

Also to find y at $x = 30$

$$y = (0.5830)(30) + 2.9675$$

$$= 17.49 + 2.9675$$

$$\boxed{y = 20.4575}$$

5. The following data shows atmospheric pollutants y_i (relative to an EPA standard) at half-hour intervals t_i.

t_i	1	1.5	2	2.5	3	3.5	4	4.5	5
y_i	-0.15	0.24	0.68	1.04	1.21	1.15	0.86	0.41	-0.08

A plot of these data points suggests that a quadratic polynomial $y = a_2t^2 + a_1t + a_0$. Find the least squares quadratic fit for the data

SoS

order of $A = 9 \times 3$

order of $b = 9 \times 1$

order of $T = 3 \times 1$

$$X = \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix}.$$

Let

$$b = \begin{bmatrix} -0.15 \\ 0.24 \\ 0.68 \\ 1.04 \\ 1.21 \\ 1.15 \\ 0.86 \\ 0.41 \\ -0.08 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ 2.25 & 1.5 & 1 \\ 4 & 2 & 1 \\ 6.25 & 2.5 & 1 \\ 9 & 3 & 1 \\ 12.25 & 3.5 & 1 \\ 16 & 4 & 1 \\ 20.25 & 4.5 & 1 \\ 25 & 5 & 1 \end{bmatrix}$$

$y = a_2 t^2 + a_1 t + a_0$ we have $Ax = b$

The Normal equation is given by

$$AA^T x = A^T b$$

$$
\begin{bmatrix} 1 & 2.25 & 4 & 6.25 & 9 & 12.25 & 16 & 20.25 & 25 \\ 1 & 1.5 & 2 & 2.5 & 3 & 3.5 & 4 & 4.5 & 5 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 1 & 1 \\ 2.25 & 1.5 & 1 \\ 4 & 2 & 1 \\ 6.25 & 2.5 & 1 \\ 9 & 3 & 1 \\ 12.25 & 3.5 & 1 \\ 16 & 4 & 1 \\ 20.25 & 4.5 & 1 \\ 25 & 5 & 1 \end{bmatrix}
\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 2.25 & 4 & 6.25 & 9 & 12.25 & 16 & 20.25 & 25 \\ 1 & 1.5 & 2 & 2.5 & 3 & 3.5 & 4 & 4.5 & 5 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}
\begin{bmatrix} -0.15 \\ 0.24 \\ 0.68 \\ 1.04 \\ 1.21 \\ 1.15 \\ 0.86 \\ 0.41 \\ -0.08 \end{bmatrix}
$$

$$\begin{bmatrix} 1583.25 & 378 & 96 \\ 378 & 96 & 27 \\ 96 & 27 & 9 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} 54.6725 \\ 16.7250 \\ 05.3700 \end{bmatrix}$$

$$1583.25\, t_1 + 378 t_2 + 96 t_3 = 54.6725$$

$$378\, t_1 + 96 t_2 + 27 t_3 = 16.7250$$

$$96 t_1 + 27 t_2 + 9 t_3 = 5.3700$$

Solving $a_2 = t_1 = -0.32714$, $a_1 = t_2 = 2.0038$

$a_0 = t_3 = -1.9253$

$$\therefore\ y = a_2 t^2 + a_1 t + a_0$$

$$\boxed{y = (-0.32714)t^2 + 2.0038t - 1.9253}$$

$\boxed{Q12}$

use Gauss elimination method to find t_1, t_2 and t_3

$$[A:B] = \left[\begin{array}{ccc:c} 1583.25 & 378 & 96 & 54.6725 \\ 378 & 96 & 27 & 16.7250 \\ 96 & 27 & 9 & 5.3700 \end{array}\right] \quad R_1 \longleftrightarrow R_3$$

$$= \left[\begin{array}{ccc:c} 96 & 27 & 9 & 5.3700 \\ 378 & 96 & 27 & 16.7250 \\ 1583.75 & 378 & 96 & 54.6725 \end{array}\right]$$

Solving $a_2 = t_1 = -0.32714$, $a_1 = t_2 = 2.0038$

$a_0 = t_3 = -1.9253$

$\therefore y = a_2 t^2 + a_1 t + a_0$

$$\boxed{y = (-0.32714)t^2 + 2.0038t - 1.9253}$$

The Gram–Schmidt orthonormalization Process

It is a process for computing an orthonormal basis $T = \{w_1, w_2, \dots, w_m\}$ for a non-zero Subspace W of R^n with basis $S = \{u_1, u_2, \dots, u_n\}$

Working Rule

Step 1 :→ $v_1 = u_1$

Step 2 :→ compute the vectors $v_2, v_3, \dots, v_m$ successively one at a time by the formula

$$v_i = u_i - \left(\frac{u_i \, v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{u_i \, v_2}{v_2 \cdot v_2}\right) v_2 - \cdots - \left(\frac{u_i \, v_{i-1}}{v_{i-1} \cdot v_{i-1}}\right) v_{i-1}$$

$$v_i = u_i - \left(\frac{u_i' v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{u_i' v_2}{v_2 \cdot v_2}\right) v_2 - \cdots\cdots - \left(\frac{u_i' v_{i-1}}{v_{i-1} \cdot v_{i-1}}\right) v_{i-1}$$

The set of vectors $T^* = \{v_1, v_2, ----, v_m\}$ is an orthogonal set.

Step 3 :→ Let $w_i = \frac{1}{|v_i|} v_i \quad (1 \leq i \leq m)$

then $T = \{w_1, w_2, \dots, w_m\}$ is an orthonormal basis for S.

Remark :

$u \cdot (cv) = 0$ for some $c \in \mathbb{R}$ some times in Gram - schmidt process.

Problems :→

1. consider the basis $S = \{u_1, u_2, u_3\}$ for R^3, where $u_1 = (1,1,1)$, $u_2 = (-1,0,-1)$ and $u_3 = (-1,2,3)$. use the Gram - Schmidt process to transform S to an orthonormal basis for R^3.

Sol^n Given that $u_1 = (1,1,1)$, $u_2 = (-1,0,-1)$ and $u_3 = (-1,2,3)$

Let $v_1 = u_1 = (1,1,1)$

$$v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} \cdot v_1 = u_2 - \left(\frac{u_2 \cdot v_1}{\|v_1\|^2}\right) \cdot v_1$$

$$= (-1,0,-1) - \left[\frac{(-1,0,-1).(1,1,1)}{\left(\sqrt{1^2+1^2+1^2}\right)^2}\right](1,1,1)$$

or

$$= (-1,0,-1) - \left[\frac{(-1,0,-1).(1,1,1)}{(1,1,1).(1,1,1)}\right](1,1,1)$$

$$= (-1,0,-1) - \left[\frac{-1+0-1}{1+1+1}\right](1,1,1)$$

$$= (-1,0,-1) - \left[\frac{-2}{3}\right](1,1,1)$$

$$= (-1,0,-1) + \frac{2}{3}(1,1,1)$$

$$= \left(-1+\frac{2}{3},\ 0+\frac{2}{3},\ -1+\frac{2}{3}\right)$$

$$v_2 = -\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}$$

$$\boxed{v_2 = (-1, 2, -1)}$$ [$\times^{ly}$ by 3 throughout the vector]

$$v_3 = u_3 - \frac{u_3 v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$= (-1, 2, 3) - \left[\frac{(-1, 2, 3) \cdot (1, 1, 1)}{(1, 1, 1) \cdot (1, 1, 1)}\right](1, 1, 1) - \left[\frac{(-1, 2, 3) \cdot (-1, 2, -1)}{(-1, 2, -1) \cdot (-1, 2, -1)}\right](-1, 2, -1)$$

$$= (-1, 2, 3) - \left[\frac{-1+2+3}{1+1+1}\right](1, 1, 1) - \left[\frac{1+4-3}{1+4+1}\right](-1, 2, -1)$$

$$= (-1, 2, 3) - \frac{4}{3}(1, 1, 1) - \frac{2}{6}(-1, 2, -1)$$

$$= \left(-\frac{1}{1} - \frac{4}{3} + \frac{1}{3},\ \frac{2}{1} - \frac{4}{3} - \frac{2}{3},\ \frac{3}{1} - \frac{4}{3} + \frac{1}{3}\right)$$

$$v_3 = (-2, 0, 2)$$

$$\boxed{v_3 = (-1, 0, 1)} \quad \{x^{ly} \text{ by } \tfrac{1}{2} \text{ throughout the vector}\}$$

$T^* = \{(1,1,1), (-1,2,-1), (-1,0,1)\}$ is an orthogonal basis for R^3

Now $w_1 = \frac{1}{|v_1|} v_1 = \frac{1}{\sqrt{1^2+1^2+1^2}}(1,1,1) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$$w_2 = \frac{1}{|v_2|} v_2 = \frac{1}{\sqrt{(-1)^2+2^2+(-1)^2}} (-1, 2, -1)$$

$$\boxed{w_2 = \left(\frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)}$$

$$w_3 = \frac{1}{|v_3|} v_3 = \frac{1}{\sqrt{(-1)^2+0^2+1^2}} (-1, 0, 1)$$

$$\boxed{w_3 = \left(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)}$$

$\therefore T = \{ w_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \; w_2 = \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right),$

$w_3 = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \}$ are the orthonormal basis for $\mathbb{R}^3$.

2. Let W be the subspace of $\mathbb{R}^4$ with basis $S = \{u_1, u_2, u_3\}$ where $u_1 = (1, -2, 0, 1)$, $u_2 = (-1, 0, 0, -1)$ and $u_3 = (1, 1, 0, 0)$. Use Gram-Schmidt process to transformation S to orthonormal basis for W.

Sol^n Given that $u_1 = (1, -2, 0, 1)$, $u_2 = (-1, 0, 0, -1)$ and $u_3 = (1, 1, 0, 0)$

Let $v_1 = u_1 = (1, -2, 0, 1)$

$$v_2 = u_2 - \left[\frac{u_2 \cdot v_1}{v_1 \cdot v_1}\right] v_1$$

$$= (-1, 0, 0, -1) - \left[\frac{(-1, 0, 0, -1) \cdot (1, -2, 0, 1)}{(1, -2, 0, 1) \cdot (1, -2, 0, 1)}\right] (1, -2, 0, 1)$$

$$= (-1, 0, 0, -1) + \frac{2}{6}(1, -2, 0, 1)$$

$$v_2 = \left(-1 + \frac{1}{3}, -\frac{2}{3}, 0, \frac{1}{3} - 1\right)$$

$$v_2 = \left(-\frac{2}{3}, -\frac{2}{3}, 0, -\frac{2}{3}\right)$$ [x^{ly} by 3 we get]

$$\boxed{v_2 = (-2, -2, 0, -2)}$$

$$v_3 = u_3 - \left[\frac{u_3 \cdot v_1}{v_1 \cdot v_1}\right] v_1 - \left[\frac{u_3 \cdot v_2}{v_2 \cdot v_2}\right] v_2$$

$$= (1,1,0,0) - \left[\frac{(1,1,0,0)\cdot(1,-2,0,1)}{(1,-2,0,1)\cdot(1,-2,0,1)}\right](1,-2,0,1)$$

$$- \left[\frac{(1,1,0,0)\cdot(-2,-2,0,-2)}{(-2,-2,0,-2)\cdot(-2,-2,0,-2)}\right](2,-2,0,-2)$$

$$= (1,1,0,0) + \frac{1}{6}(1,-2,0,1) + \frac{4}{12}(-2,-2,0,-2)$$

$$= \left(1 + \frac{1}{6} - \frac{2}{3},\ 1 - \frac{2}{6} - \frac{2}{3},\ 0+0+0,\ 0 + \frac{1}{6} - \frac{2}{3}\right)$$

$= \left(\frac{1}{2}, 0, 0, -\frac{1}{2}\right)$ [x^{ly} by 2 we get]

$$\boxed{v_3 = (1, 0, 0, -1)}$$

$T^* = \{(1, -2, 0, 1), (-2, -2, 0, -2), (1, 0, 0, -1)\}$ is the orthogonal vectors of R^4

$$T = \left\{\left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}\right), \left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, 0, \frac{-1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{2}}, 0, 0, \frac{-1}{\sqrt{2}}\right)\right\}$$ are the orthonormal basis of R^4

3. Given that $x_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix}$ is a basis for a subspace of W. Use the Gram - Schmidt process to produce an orthogonal basis for W.

Sol^n Given that $x_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix}$

Let $v_1 = x_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$

$$v_2 = x_2 - \left[\frac{x_2' \cdot v_1}{v_1 \cdot v_1}\right] v_1 = x_2 - \left[\frac{x_2' \cdot v}{\|v_1\|^2}\right] v_1$$

$$= \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix} - \left\{ \frac{\begin{bmatrix} 8 & 5 & -6 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}}{\left[\sqrt{3^2 + 0^2 + (-1)^2}\right]^2} \right\} \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix} - \left[\frac{24 + 0 + 6}{9 + 1} \right] \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix} - \frac{3\cancel{0}}{1\cancel{0}} \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix} - \begin{bmatrix} 9 \\ 0 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 - 9 \\ 5 - 0 \\ -6 + 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} \qquad \boxed{\therefore\ v_2 = \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}}$$

$$T^* = \left\{ \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} \right\}$$ is the orthogonal basis.

$$T = \left\{ \begin{bmatrix} 3/\sqrt{10} \\ 0 \\ -1/\sqrt{10} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{35} \\ 5/\sqrt{35} \\ 3/\sqrt{35} \end{bmatrix} \right\}$$ is the orthonormal basis for W.

(2

4. Use Gram-Schmidt orthonormalization $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ construct the orthogonal basis for w?

Sol: Given that $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

Let $y_1 = v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$y_2 = v_2 - \left[\frac{v_2' y_1}{y_1 \cdot y_1}\right] y_1 = v_2 - \left[\frac{v_2' y_1}{\|y_1\|^2}\right] y_1$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left[\frac{\begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\left(\sqrt{1^2+1^2+1^2+1^2}\right)^2} \right] \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left[\frac{0+1+1+1}{4} \right] \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3/4 \\ 3/4 \\ 3/4 \\ 3/4 \end{bmatrix}$$

$$y_2 = \begin{bmatrix} 0 - 3/4 \\ 1 - 3/4 \\ 1 - 3/4 \\ 1 - 3/4 \end{bmatrix} = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \qquad \boxed{\therefore\ y_2 = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}}$$

$$y_3 = v_3 - \left[\frac{v_3' \, y_1}{\|y_1\|^2}\right] y_1 - \left[\frac{v_3' \, y_2}{\|y_2\|^2}\right] y_2$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \left[\frac{\begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\left[\sqrt{1^2+1^2+1^2+1^2}\right]^2}\right]\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left[\frac{\begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}}{\sqrt{\left(\frac{3}{4}\right)^2+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^2}^{\,2}}\right]\begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

$$y_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{0+0+1+1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

$$y_3 = \begin{bmatrix} 0 - 1/2 + 1/2 \\ 0 - 1/2 - 1/6 \\ 1 - 1/2 - 1/6 \\ 1 - 1/2 - 1/6 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$\boxed{\therefore\ y_3 = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}}$$

orthogonal basis for W is $\{y_1, y_2, y_3\}$

$$= \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}, \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} \right\}$$

5. Use Gram-Schmidt orthonormalization. In $\mathbb{R}^4$ let use find an orthonormal basis for the linear span of three vectors. $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}$

Sol:

Given that

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$

Let $y_1 = v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$y_2 = v_2 - \left[\frac{v_2' y_1}{\|y_1\|^2}\right] y_1$$

$$= \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} - \left[\frac{\begin{bmatrix} -1 & 4 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\left(\sqrt{1^2+1^2+1^2+1^2}\right)^2}\right] \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$y_2 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} - \left[\frac{\begin{bmatrix} -1+4+4-1 \end{bmatrix}}{4}\right] \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$y_2 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 3/2 \\ 3/2 \\ 3/2 \end{bmatrix} = \begin{bmatrix} -1 - 3/2 \\ 4 - 3/2 \\ 4 - 3/2 \\ -1 - 3/2 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{bmatrix}$$

$$y_3 = v_3 - \left[\frac{v_3' y_1}{\|y_1\|^2}\right] y_1 - \left[\frac{v_3' y_2}{\|y_2\|^2}\right] y_2$$

$$y_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} - \left[\frac{\begin{bmatrix} 4 & -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{4}\right] \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} -$$

$$\left[\frac{\begin{bmatrix} 4 & -2 & 2 & 0 \end{bmatrix}}{25} \begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ 5/2 \end{bmatrix}\right] \begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ 5/2 \end{bmatrix}$$

$$y_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} - \left[\frac{4-2+2}{4}\right]\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left[\frac{4\left(-\frac{5}{2}\right) - 2\left(\frac{5}{2}\right) + 2\left(\frac{5}{2}\right)}{25}\right]\begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ 5/2 \end{bmatrix}$$

$$y_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ 5/2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\boxed{y_3 = \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}}$$

$$W = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix} \right\}$$

6. Using Gram-Schmidt orthonormalization. Let $x_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. construct the orthogonal basis for ω

Sol^n Given that $x_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Let $y_1 = x_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ $\boxed{\therefore y_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}}$

$$y_2 = x_2 - \frac{x_2' y_1}{\|y_1\|^2} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \left[\frac{\begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}}{\left(\sqrt{3^2+6^2+0^2}\right)^2} \right] \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \left[\frac{3+12+0}{9+36} \right] \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{\cancel{15}^{1}}{\cancel{45}_{3}} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$y_2 = \begin{bmatrix} 1-1 \\ 2-2 \\ 2-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}. \quad \boxed{\therefore\ y_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}$$

$$W = \{ y_1, y_2 \} = \left\{ \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$$