Eigen Values and Eigen Vector

Date: 16/03/2020

1. Solve the Eigen Values and Eigen vector for the matrix

$$A = \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$$

Sol^n we have $|A - \lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & 3 \\ -2 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow (-1-\lambda)(4-\lambda) + 6 = 0$

$\Rightarrow \lambda^2 - 3\lambda + 2 = 0$

$\Rightarrow \lambda^2 - 2\lambda - \lambda + 2 = 0$

$\Rightarrow \lambda(\lambda-2) - 1(\lambda-2) = 0$

$\Rightarrow (\lambda-1)(\lambda-2) = 0$

$\Rightarrow \lambda - 2 = 0 \;|\; \lambda - 1 = 0$

$\boxed{\lambda = 2}$ $\quad$ $\boxed{\lambda = 1}$

The Eigen values $\boxed{\lambda = 1, 2}$

Now $[A - \lambda I][X] = [0]$

$$\left.\begin{aligned} (-1-\lambda)x + 3y &= 0 \\ -2x + (4-\lambda)y &= 0 \end{aligned}\right\} \rightarrow (I)$$

Case (i) :-> If $\lambda = 1$ in (I)

$-2x + 3y = 0 \rightarrow (1)$

$-2x + 3y = 0 \rightarrow (2)$

from (1) $\quad -2x + 3y = 0$

$2x = 3y$

$\Rightarrow \frac{x}{3} = \frac{y}{2} = k$ (say)

$X_1 = [3k, 2k]^T$ (or)

$X_1 = [3, 2]^T$ if $\lambda = 1$

Case (ii) If $\lambda = 2$ in (I)

$-3x + 3y = 0 \rightarrow (3)$

$-2x + 2y = 0 \rightarrow (4)$

from (3) $\quad -3x + 3y = 0$

$\Rightarrow 3x = 3y \Rightarrow \frac{x}{1} = \frac{y}{1}$

$\Rightarrow \frac{x}{1} = \frac{y}{1} = k$ (Say)

$\therefore X_2 = [k, k]^T$ if $\lambda = 2$

(or)

$X_2 = [1, 1]^T$ if $\lambda = 2$

$\therefore \lambda_1 = 1$ & $X_1 = [3, 2]^T$

$\lambda_2 = 2$ & $X_2 = [1, 1]^T$

2. Find all the Eigen values and Eigen vector for the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Sol: we have $|A - \lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

Now $\lambda^3 - (\Sigma d)\lambda^2 + (\Sigma md)\lambda - |A| = 0 \rightarrow (i)$

$\Sigma d = 18$, $\Sigma md = 45$, $|A| = 0$

where

Σd = Sum of the diagonal elements of A.

Σmd = Sum of the minores of the diagonal elements of A

$|A|$ = Determinent of A

Eqn (i) $\Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\Rightarrow \lambda(\lambda^2 - 18\lambda + 45) = 0$

$\Rightarrow \lambda = 0$ & $\lambda^2 - 18\lambda + 45 = 0$

$\lambda^2 - 15\lambda - 3\lambda + 45 = 0$

$\Rightarrow \lambda(\lambda - 15) - 3(\lambda - 15) = 0$

$(\lambda - 3)(\lambda - 15) = 0$

$\lambda - 3 = 0$ & $\lambda - 15 = 0$

$\boxed{\lambda = 3}$ $\boxed{\lambda = 15}$

The Eigen values $\boxed{\lambda = 0, 3, 15}$

we have $[A - \lambda I][X] = 0$

$$\left.\begin{aligned} (8-\lambda)x - 6y + 2z &= 0 \\ -6x + (7-\lambda)y - 4z &= 0 \\ 2x - 4y + (3-\lambda)z &= 0 \end{aligned}\right\} \rightarrow (I)$$

Case (i) If $\lambda = 0$ in (I)

$8x - 6y + 2z = 0 \rightarrow (1)$

$-6x + 7y - 4z = 0 \rightarrow (2)$

$2x - 4y + 3z = 0 \rightarrow (3)$

from (1) & (2) by applying rule of cross multiplication, we have

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\frac{x}{24-14} = \frac{-y}{-32+12} = \frac{z}{56-36}$$

$$\Rightarrow \frac{x}{10} = \frac{y}{20} = \frac{z}{20} \quad [\div \text{ by } 10]$$

$$\Rightarrow \frac{x}{1} = \frac{y}{2} = \frac{z}{2} = k \text{ (say)}$$

$X_1 = [1k \quad 2k \quad 2k]$

(or) if $\lambda = 0$

$X_1 = [1 \quad 2 \quad 2]$

Case (ii) If $\lambda = 3$ in (I)

$5x - 6y + 2z = 0 \rightarrow (4)$

$-6x + 4y - 4z = 0 \rightarrow (5)$

$2x - 4y + 0z = 0 \rightarrow (6)$

from (4) & (5) by applying rule of cross multiplication, we have

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 5 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}$$

$$\frac{x}{24-8} = \frac{-y}{-20+12} = \frac{z}{20-36}$$

$$\frac{x}{16} = \frac{\cancel{-}y}{\cancel{-}8} = \frac{z}{-16}$$

$\div$ by 8

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2} = k \text{ (Say)}$$

$X_2 = [2k \quad 1k \quad -2k]^T$

or

$X_2 = [2 \quad 1 \quad -2]^T$ if $\lambda = 3$

Case (iii) If $\lambda = 15$ in (I)

$-7x - 6y + 2z = 0 \rightarrow (7)$

$-6x - 8y - 4z = 0 \rightarrow (8)$

$2x - 4y - 12z = 0 \rightarrow (9)$

from (7) & (8) by applying rule of cross multiplication on both sides.

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -7 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}$$

$$\frac{x}{24+16} = \frac{-y}{28+12} = \frac{z}{56-36}$$

$$\frac{x}{40} = \frac{-y}{40} = \frac{z}{20}$$

$\div$ by 20

$$\frac{x}{2} = \frac{y}{-2} = \frac{z}{1} = k \text{ (Say)}$$

$X_3 = [2k \quad -2k \quad 1k]^T$

or

$X_3 = [2 \quad -2 \quad 1]^T$ if $\lambda = 15$

Convert an nth order differential Equation to a system of equations :→

An nth order differential equation

$$y^{(n)} = F(t, y, y', y'' \ldots\ldots y^{(n-1)}) \longrightarrow ①$$

can be reduced to a system of 'n' first order differential equation i.e

$$\begin{array}{lllll}
x_1 = y & \xrightarrow{\text{Differentiation}} & x_1' = y' & \xrightarrow{\text{Name as}} & x_2 \\
x_2 = y' & \longrightarrow & x_2' = y'' & \longrightarrow & x_3 \\
x_3 = y'' & \longrightarrow & x_3' = y''' & \longrightarrow & x_4 \\
x_4 = y''' & \longrightarrow & x_4' = y^{IV} & \longrightarrow & x_5 \\
x_5 = y^{IV} & \longrightarrow & x_5' = y^{V} & \longrightarrow & x_6 \\
x_6 = y^{V} & \longrightarrow & x_6' = y^{VI} & \longrightarrow & x_7 \\
\equiv \quad \equiv & & \equiv \quad \equiv & & \equiv \\
x_n = y^{(n+1)} & \longrightarrow & x_n' = y^{n} & \longrightarrow & x_{n+1}
\end{array}$$

Sub. in ① we get

$$x_1' = x_2$$
$$x_2' = x_3$$
$$x_3' = x_4$$
$$x_4' = x_5$$
$$\equiv \quad \equiv$$
$$x_n' = x_{n+1}$$

1. Convert the following second order differential equations as a system of first order linear differential equation and represent in the form matrix notation

$$y'' + 3ay' - 4a^2 y = 0$$

Sol: Given that $y'' + 3ay' - 4a^2 y = 0 \longrightarrow ①$

Here $y \rightarrow$ dependent variable and $t \rightarrow$ Independent variable

P.T.O

The Given D.E is Second order D.E

$\therefore$ choose two variables x_1 and x_2

Now

$$x_1 = y \xrightarrow{\text{Diff.}} x_1' = y' \xrightarrow{\text{Name as}} x_2$$

$$x_2 = y' \longrightarrow x_2' = y''$$

Eqn (1) becomes

$$y'' + 3ay' - 4a^2 y = 0$$

$$\Rightarrow y'' = 4a^2 y - 3ay'$$

$$\Rightarrow x_2' = 4a^2 x_1 - 3a x_2$$

Hence System of first order D.E is

$$x_1' = x_2 \qquad \boxed{\text{OR}} \qquad x_1' = 0.x_1 + 1.x_2$$

$$x_2' = 4a^2 x_1 - 3a x_2 \qquad\qquad x_2' = 4a^2 x_1 - 3a x_2$$

In matrix form

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4a^2 & -3a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Diff.

$$\boxed{X' = AX} \longrightarrow \text{Homogeneous equation}$$

where $X' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$; $A = \begin{bmatrix} 0 & 1 \\ 4a^2 & -3a \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

2. Write the following 4th order differential Equation as a system of first order differential equations

$$y^{iv} + 3y'' - \sin t\, y' + 8y = t^2$$

Sol: Given that $y^{iv} + 3y'' - \sin t\, y' + 8y = t^2 \longrightarrow (1)$

Here $y \rightarrow$ dependent Variable and $t \rightarrow$ Independent variable

The given differential equation is 4th order D.E

$\therefore$ choose 4 variables x_1, x_2, x_3 and x_4

Put

$x_1 = y \xrightarrow{\text{Diff.}} x_1' = y' \xrightarrow{\text{Name as}} x_2$

$x_2 = y' \longrightarrow x_2' = y'' \longrightarrow x_3$

$x_3 = y'' \longrightarrow x_3' = y''' \longrightarrow x_4$

$x_4 = y''' \longrightarrow x_4' = y^{IV}$

Eqⁿ (1) becomes $y^{IV} + 3y'' - \sin t\, y' + 8y = t^2$

$$\Rightarrow y^{IV} = -8y + \sin t\, y' - 3y'' + t^2$$

$$\therefore \quad x_4' = -8x_1 + \sin t\, x_2 - 3x_3 + t^2$$

Hence System of first order D.E is

$$x_1' = x_2$$
$$x_2' = x_3$$
$$x_3' = x_4$$
$$x_4' = -8x_1 + \sin t\, x_2 - 3x_3 + t^2$$

(or)

$$x_1' = 0.x_1 + 1.x_2 + 0.x_3 + 0.x_4 + 0.t^2$$
$$x_2' = 0.x_1 + 0.x_2 + 1.x_3 + 0.x_4 + 0.t^2$$
$$x_3' = 0.x_1 + 0.x_2 + 0.x_3 + 1.x_4 + 0.t^2$$
$$x_4' = -8x_1 + \sin t\, x_2 - 3x_3 + 0.x_4 + t^2$$

In matrix form

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & \sin t & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ t^2 \end{bmatrix}$$

$X' = AX + F \rightarrow$ Non-homogeneous Diff. equation

where
$$X' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & \sin t & -3 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}; \quad F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ t^2 \end{bmatrix}$$

Home work

1. Convert the differential equation $y'' + 5y' + 6y = 0$ into System of linear differential equation.
2. Convert the differential equation $y''' + 2y'' + y' + 2y = 0$ into System of linear differential equation.

Scanned with CamScanner

Solution system of linear differential equation by diagonalization method :→

Homogenous :→

Let $x' = Ax \rightarrow (1)$

Assume solution be as $x = Py$

Diff

$x' = Py'$

Eq^n (1) becomes

$$Py' = APy$$

$$y' = (P^{-1}AP)y$$

$$y' = Dy$$

where $\boxed{D = P^{-1}AP}$

$$\therefore y' = Dy$$

In 2x2 matrix

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Solve D.E by Variable Seperable method [VSM]

$$y_1' = \lambda_1 y_1 \longrightarrow y_1 = c_1 e^{\lambda_1 t}$$

$$y_2' = \lambda_2 y_2 \longrightarrow y_2 = c_2 e^{\lambda_2 t}$$

Now Solution of System i.e., given by

$$X = PY$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$x_1 = a_1 y_1 + b_1 y_2$$

$$x_2 = a_2 y_1 + b_2 y_2$$

$$X = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} y_1 + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} y_2$$

$$X = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} c_1 e^{\lambda_1 t} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} c_2 e^{\lambda_2 t}$$

In general $[n \times n]$

$$\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \cdots & \lambda_n \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Solve D.E by variable seperable method [VSM]

$$y_1' = \lambda_1 y_1 \xrightarrow{\text{VSM}} y_1 = c_1 e^{\lambda_1 t}$$

$$y_2' = \lambda_2 y_2 \longrightarrow y_2 = c_2 e^{\lambda_2 t}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$y_n' = \lambda_n y_2 \longrightarrow y_n = c_n e^{\lambda_n t}$$

Now Solution of System i.e., given by

$$X = PY$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & & \vdots \\ z_1 & z_2 & \cdots & z_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$x_1 = a_1 y_1 + a_2 y_2 + \cdots + a_n y_n$$
$$x_2 = b_1 y_1 + b_2 y_2 + \cdots + b_n y_n$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$x_n = z_1 y_1 + z_2 y_2 + \cdots + z_n y_n$$

$$X = \begin{bmatrix} a_1 \\ b_1 \\ \vdots \\ z_1 \end{bmatrix} y_1 + \begin{bmatrix} a_2 \\ b_2 \\ \vdots \\ z_2 \end{bmatrix} y_2 + \cdots + \begin{bmatrix} a_n \\ b_n \\ \vdots \\ z_n \end{bmatrix} y_n$$

where $y_1 = c_1 e^{\lambda_1 t}$, $y_2 = c_2 e^{\lambda_2 t}$ ... $y_n = c_n e^{\lambda_n t}$

Eigen value and Stability of the System :→

The table below given ~~as~~ a complete overview of the stability corresponding to each type of eigen value

Eigen values	Stability
All real and positive	Unstable
All real and negative	Stable
mixed positive & negative real	Unstable
$a+ib$	Unstable
$-a+ib$	Stable
$0+bi$	Unstable
Repeated values	Depends upon orthogonality of eigen vectors

1. Solve the system $x' = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix} x$ by diagona-lization and hence discuss the stability of the System

Sol: Given that $x' = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix} x$

$\Rightarrow x' = Ax \longrightarrow (1)$ where $A = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix}$

we have $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -5-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = 0$

$\Rightarrow (-5-\lambda)(-2-\lambda) - 4 = 0 \Rightarrow 10 + 5\lambda + 2\lambda + \lambda^2 - 4 = 0$

$\Rightarrow \lambda^2 + 7\lambda + 6 = 0$

Solving $\boxed{\lambda = -1, -6} \longrightarrow$ Eigen values

Now consider $[A - \lambda I][X] = [0]$

$\Rightarrow \left.\begin{aligned} (-5-\lambda)x + y &= 0 \\ 4x + (-2-\lambda)y &= 0 \end{aligned}\right\} \longrightarrow (I)$

case (i) If $\lambda = -1$ in (I)

$-4x + y = 0 \longrightarrow (1)$

$4x - y = 0 \longrightarrow (2)$

from (1) $-4x + y = 0$

$-4x = -y$

$\frac{x}{1} = \frac{y}{4} = k$ (Say)

$x_1 = [1k, 4k]^T$

(or)

$\boxed{x_1 = [1, 4]^T \text{ if } \lambda = -1}$

case (ii) If $\lambda = -6$ in (I)

$x + y = 0 \longrightarrow (3)$

$4x + 4y = 0 \longrightarrow (4)$

from (3) $x + y = 0$

$x = -y$

$\Rightarrow \frac{x}{-1} = \frac{y}{1} = k$ (Say)

$x_2 = [-1k, 1k]^T$

(or)

$\boxed{x_2 = [-1, 1]^T \text{ if } \lambda = -6}$

The modal matrix (or) spectral matrix $P = [x_1, x_2]$

$P = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}$

To find $p^{-1} = ?$

we have $p^{-1} = \frac{adj\,p}{|P|}$; $|P| = 1+4$

$\boxed{|P| = 5}$

$adj\,p = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}$

$\therefore\ p^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}$

$p^{-1} = \begin{bmatrix} 1/5 & 1/5 \\ -4/5 & 1/5 \end{bmatrix}$ or $P = \begin{bmatrix} 0.2 & 0.2 \\ -0.8 & 0.2 \end{bmatrix}$

Now to find $D = p^{-1}Ap$

$D = \begin{bmatrix} 1/5 & 1/5 \\ -4/5 & 1/5 \end{bmatrix}\begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}$

$D = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix}$

we have $x' = Ax \rightarrow (1)$ [from given data]

put $x = py \rightarrow (2)$ [Assume Soln]

Diff.

$x' = py'$

Eqn (1) becomes

$\cancel{x'} = py' = Apy$

$y' = (p^{-1}Ap)y$

$y' = Dy$

$\Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$y_1' = -y_1 + 0y_2$

$y_2' = 0y_1 - 6y_2$

$y_1' = -y_1$ $\qquad\qquad$ $y_2' = -6y_2$

$\frac{dy_1}{dt} = -y_1$ [SVM] $\qquad\qquad$ $\frac{dy_2}{dt} = -6y_2$

$\frac{dy_1}{y_1} = -dt$ $\qquad\qquad$ $\frac{dy_2}{y_2} = -6dt$

on Integrating on both sides we get

$\int \frac{dy_1}{y_1} = -\int dt$ $\qquad\qquad$ $\int \frac{dy_2}{y_2} = -6\int dt$

$\log y_1 = -t + c_1$ $\qquad\qquad$ $\log y_2 = -6t + c_2$

$y_1 = e^{-t+c_1}$ $\qquad\qquad$ $y_2 = e^{-6t+c_2}$

$= e^{-t} . e^{c_1}$ $\qquad\qquad$ $y_2 = e^{-6t} . e^{c_2}$

$\boxed{y_1 = C_1 e^{-t}}$ $\qquad\qquad$ $\boxed{y_2 = C_2 e^{-6t}}$

where $e^{c_1} = C_1$ $\qquad\qquad$ where $e^{c_2} = C_2$

Let $X = PY$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{-6t} \end{bmatrix}$$

$$x_1 = c_1 e^{-t} - c_2 e^{-6t}$$

$$x_2 = 4c_1 e^{-t} + c_2 e^{-6t}$$

$$X = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}$$

∵ The Eigen values are real and distinct and negative [real and negative]

∴ The System is stable

2. Solve $x' = \begin{bmatrix} -2 & -1 & 8 \\ 0 & -3 & 8 \\ 0 & -4 & 9 \end{bmatrix} x$ by diagonalization

Sol^n Given that $x' = \begin{bmatrix} -2 & -1 & 8 \\ 0 & -3 & 8 \\ 0 & -4 & 9 \end{bmatrix} x$

$\Rightarrow x' = Ax \rightarrow (1)$

where $A = \begin{bmatrix} -2 & -1 & 8 \\ 0 & -3 & 8 \\ 0 & -4 & 9 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} -2-\lambda & -1 & 8 \\ 0 & -3-\lambda & 8 \\ 0 & -4 & 9-\lambda \end{vmatrix} = 0$$

Now $\lambda^3 - (\Sigma d)\lambda^2 + (\Sigma md)\lambda - |A| = 0$

where $\boxed{\Sigma d = +4}$ $\boxed{\Sigma md = -7}$ $\boxed{|A| = -10}$

$\therefore \lambda^3 - 4\lambda^2 - 7\lambda + 10 = 0$

Solving $\boxed{\lambda = -2, 1, 5} \rightarrow$ Eigen values

consider $[A - \lambda I][X] = 0$

$$\left.\begin{aligned} (-2-\lambda)x - y + 8z &= 0 \\ 0x + (-3-\lambda)y + 8z &= 0 \\ 0x - 4y + (9-\lambda)z &= 0 \end{aligned}\right\} \rightarrow (I)$$

case (i) If $\lambda = -2$ in (I)

$0x - y + 8z = 0 \rightarrow (1)$

$0x - y + 8z = 0 \rightarrow (2)$

$0x - 4y + 11z = 0 \rightarrow (3)$

from 2 & 3 by applying rule of cross multiplication, we have

$$\Rightarrow \frac{x}{-11+32} = \frac{-y}{0-0} = \frac{z}{0-0}$$

$$\Rightarrow \frac{x}{21} = \frac{y}{0} = \frac{z}{0}$$

x/ly by 21

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{0} = k \text{ (say)}$$

$X_1 = [1k, 0, 0]^T$

(or) $\boxed{X_1 = [1, 0, 0]^T \text{ if } \lambda = -2}$

case ii

Case (iii) :- If $\lambda = 5$ in (I)

$$-7x - y + 8z = 0 \rightarrow (7)$$

$$0x - 8y + 8z = 0 \rightarrow (8)$$

$$0x - 4y + 4z = 0 \rightarrow (9)$$

From (7) & (8) by applying RCM, we have

$$\Rightarrow \frac{x}{56} = \frac{-y}{-56-0} = \frac{z}{56-0}$$

$$\Rightarrow \frac{x}{56} = \frac{\cancel{-}y}{\cancel{-}56} = \frac{z}{56}$$

x/ly by 56

$$\Rightarrow \frac{x}{1} = \frac{y}{1} = \frac{z}{1} = k \text{ (say)}$$

$X_3 = [1k, 1k, 1k]^T$ if $\lambda = 5$

(or) $\boxed{X_3 = [1, 1, 1]^T \text{ if } \lambda = 5}$

Case (ii) :- If $\lambda = 1$ in (I)

$$-3x - y + 8z = 0 \rightarrow (4)$$

$$0x - 4y + 8z = 0 \rightarrow (5)$$

$$0x - 4y + 8z = 0 \rightarrow (6)$$

from (4) & (5) by applying RCM, we have

$$\Rightarrow \frac{x}{24} = \frac{-y}{-24-0} = \frac{z}{12-0}$$

$$\Rightarrow \frac{x}{24} = \frac{\cancel{-}y}{\cancel{-}24} = \frac{z}{12}$$

x/ly by 12

$$\Rightarrow \frac{x}{2} = \frac{y}{2} = \frac{z}{1} = k \text{ (say)}$$

$\Rightarrow X_2 = [2k, 2k, 1k]$

(or) $\boxed{X_2 = [2, 2, 1]^T \text{ if } \lambda = 1}$

The

The modal matrix $P = [X_1 \ X_2 \ X_3]$

$$P = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

To find $P^{-1} = ?$; $P^{-1} = \dfrac{adj\,P}{|P|}$; $\boxed{|P| = 1}$

$$P^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Also to find :- Diag $= P^{-1}AP$

$$D = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -2 & -1 & 8 \\ 0 & -3 & 8 \\ 0 & -4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Eqⁿ (1) becomes $X' = AX$ [Given data]

put $X = PY \rightarrow (2)$ [Assume Solⁿ]

Diff

$X' = PY'$

Eqⁿ (2) becomes $PY' = APY$

$\Rightarrow Y' = (P^{-1}AP)Y$

$Y' = DY$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$y_1' = -2y_1 + 0y_2 + 0y_3$$

$$y_2' = 0y_1 + 1y_2 + 0y_3$$

$$y_3' = 0y_1 + 0y_2 + 5y_3$$

$$y_1' = -2y_1$$

$$\frac{dy_1}{dt} = -2y_1$$

$$\frac{dy_1}{y_1} = -2dt \quad \boxed{\text{SVM}}$$

$$y_2' = 1y_2$$

$$\frac{dy_2}{dt} = 1y_2$$

$$\frac{dy_2}{y_2} = dt \quad \boxed{\text{SVM}}$$

$$y_3' = 5y_3$$

$$\frac{dy_3}{dt} = 5y_3$$

$$\frac{dy_3}{y_3} = 5dt$$

Integrating on both sides we get

$$\int \frac{dy_1}{y_1} = -2\int dt$$

$$\log y_1 = -2t + c_1$$

$$y_1 = e^{-2t+c_1}$$

$$y_1 = e^{-2t} \cdot e^{c_1}$$

$$\boxed{y_1 = C_1 e^{-2t}}$$

where $C_1 = e^{c_1}$

$$\int \frac{dy_2}{y_2} = \int dt$$

$$\log y_2 = t + c_2$$

$$y_2 = e^{t+c_2}$$

$$y_2 = e^{t} \cdot e^{c_2}$$

$$\boxed{y_2 = C_2 e^{t}}$$

where $C_2 = e^{c_2}$

$$\int \frac{dy_3}{y_3} = 5\int dt$$

$$\log y_3 = 5t + c_3$$

$$y_3 = e^{5t+c_3}$$

$$y_3 = e^{5t} \cdot e^{c_3}$$

$$\boxed{y_3 = C_3 e^{5t}}$$

where $C_3 = e^{c_3}$

Now we have $X = PY$ [from (2)]

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 = 1y_1 + 2y_2 + 1y_3$$

$$x_2 = 0y_1 + 2y_2 + 1y_3$$

$$x_3 = 0y_1 + 1y_2 + 1y_3$$

$x_1 = 1y_1 + 2y_2 + 1y_3$

$x_2 = 0y_1 + 2y_2 + 1y_3$

$x_3 = 0y_1 + 1y_2 + 1y_3$

$$X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} y_1 + \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} y_2 + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} y_3$$

$$X = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{5t}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-2t} \\ c_2 e^{t} \\ c_3 e^{5t} \end{bmatrix}$$

$x_1 = c_1 e^{-2t} + 2c_2 e^{t} + c_3 e^{5t}$

$x_2 = 0c_1 e^{-2t} + 2c_2 e^{t} + c_3 e^{5t}$

$x_3 = 0c_1 e^{-2t} + c_2 e^{t} + c_3 e^{5t}$

$$X = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{5t}$$

Solve the system of linear differential equation by diagonalization method

2. Non - Homogeneous :→

Let $x' = Ax + F \rightarrow (1)$

Assume solution be as

$$x = Py$$

Diff. wrt 't'

$$x' = Py'$$

Eq^n (1) becomes

$$Py' = APy + F$$

$$\Rightarrow y' = P^{-1}[APy + F]$$

$$\Rightarrow y' = (P^{-1}AP)y + P^{-1}F$$

$$\Rightarrow y' = Dy + P^{-1}F$$

where $D = P^{-1}AP$

In 2x2 matrix :→

$$P = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

$$y' = Dy + P^{-1}F$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} a_1' & b_1' \\ a_2' & b_2' \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$y_1' = \lambda_1 y_1 + a_1' f_1 + b_1' f_2$$

$$y_2' = \lambda_2 y_2 + a_2' f_1 + b_2' f_2$$

Solving using linear D.E method

$\dfrac{dy}{dt} + Py = Q$; $I.F = e^{\int P dt}$; y_1 and y_2

Sol^n is $y\,(I.F) = \int Q\,(I.F)\,dt + c$

Now Solution of System is given by

$$X = PY$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$x_1 = a_1 y_1 + b_1 y_2$$

$$x_2 = a_2 y_1 + b_2 y_2$$

$$X = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} y_1 + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} y_2$$

In general [$n \times n$ matrix]

$$Y' = DY + P^{-1}F$$

$$\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & & \vdots \\ z_1 & z_2 & \cdots & z_n \end{bmatrix} \begin{bmatrix} f_{t_1} \\ f_{t_2} \\ \vdots \\ f_{t_n} \end{bmatrix}$$

$$y_1' = \lambda_1 y_1 + A_1 f_{t_1} + \cdots$$

$$y_2' = \lambda_2 y_2 + B_1 f_{t_1} + \cdots$$

$$\vdots$$

$$y_n' = \lambda_n y_n + Z_1 f_{t_1} + \cdots$$

Solve by using Linear D.E method

$$\frac{dy}{dt} + Py = Q \; ; \; I.F = e^{\int P dt}$$

$$\therefore \; y(I.F) = \int Q(I.F)\, dt + c$$

$y_1 =$ ____

$y_2 =$ ____

$\vdots$

$y_n =$ ____

Now Solution of System is given by

$$X = PY$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ = & = & & = \\ z_1 & z_2 & \cdots & z_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 = a_1 y_1 + a_2 y_2 + \cdots + a_n y_n$$

$$x_2 = b_1 y_1 + b_2 y_2 + \cdots + b_n y_n$$

$$\equiv$$

$$x_n = z_1 y_1 + z_2 y_2 + \cdots + z_n y_n$$

$$X = \begin{bmatrix} a_1 \\ b_1 \\ \equiv \\ z_1 \end{bmatrix} y_1 + \begin{bmatrix} a_2 \\ b_2 \\ \equiv \\ z_2 \end{bmatrix} y_2 + \cdots + \begin{bmatrix} a_n \\ b_n \\ \equiv \\ z_n \end{bmatrix} y_n$$

1. Solve $X' = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} X + \begin{bmatrix} 3e^t \\ e^t \end{bmatrix}$ by diagonalization and hence discuss the stability of the system.

Sol^n Given that $X' = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} X + \begin{bmatrix} 3e^t \\ e^t \end{bmatrix}$

$$\Rightarrow X' = AX + F \rightarrow (1)$$

where $A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$; $X' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$; $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ &

$$F = \begin{bmatrix} 3e^t \\ e^t \end{bmatrix}$$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)(1-\lambda) - 4 = 0$$

$$\Rightarrow \cancel{4} - 4\lambda - \lambda + \lambda^2 - \cancel{4} = 0 \Rightarrow \lambda^2 - 5\lambda = 0$$

$$\Rightarrow \lambda(\lambda - 5) \Rightarrow \boxed{\lambda = 0} \text{ \& } \lambda - 5 = 0 \Rightarrow \boxed{\lambda = 5}$$

$\boxed{\lambda = 0, 5} \rightarrow$ Eigen values

Now consider $[A - \lambda I][X] = 0$

$$\left.\begin{aligned}(4-\lambda)x + 2y &= 0\\ 2x + (1-\lambda)y &= 0\end{aligned}\right\} \rightarrow (I)$$

Case (i) :→ If $\lambda = 0$ in (I)

$4x + 2y = 0 \rightarrow (1)$

$2x + y = 0 \rightarrow (2)$

from (1) $4x + 2y = 0$

$\Rightarrow 4x = -2y$

$\Rightarrow 2x = -y$

$\Rightarrow \frac{x}{1} = \frac{-y}{2} = k$ (Say)

$\Rightarrow X_1 = [k, -2k]^T$

(or) $\boxed{X_1 = [1, -2]^T \text{ if } \lambda = 0}$

Case (ii) :→ If $\lambda = 5$ in (I)

$-x + 2y = 0 \rightarrow (3)$

$2x - 4y = 0 \rightarrow (4)$

from (3) $-x + 2y = 0$

$\Rightarrow x = 2y$

$\Rightarrow \frac{x}{2} = \frac{y}{1} = k$ (Say)

$\therefore X_2 = [2k, 1k]^T$

$\boxed{X_2 = [2, 1]^T \text{ if } \lambda = 5}$

The modal matrix $P = [X_1 \ X_2]$

$$P = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \quad ; \ |P| = 1 + 4$$

$\boxed{|P| = 5}$

To find :- $P^{-1} = \frac{\operatorname{adj} P}{|P|}$

$$\therefore P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \quad \boxed{(or)} \quad P^{-1} = \begin{bmatrix} 0.2 & -0.4 \\ 0.4 & 0.2 \end{bmatrix}$$

Also to find :- $D = P^{-1}AP$

$$D = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$

Now Non-Homogeneous

$X' = AX + F \rightarrow (1)$ [Given data]

$\because$ put $X = PY$ [Assume Sol^n] $\rightarrow (2)$

D wrt 't'

$X' = PY'$

Eq (1) becomes

$$PY' = APY + F$$

$$Y' = P^{-1}[APY + F]$$

$$Y' = (P^{-1}AP)Y + P^{-1}F$$

$$Y' = DY + P^{-1}F \quad \text{where } D = P^{-1}AP$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{5} & \frac{-2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 3e^t \\ e^t \end{bmatrix}$$

$$y_1' = 0y_1 + 0y_2 + \frac{3}{5}e^t - \frac{2}{5}e^t$$

$$y_2' = 5y_2 + \frac{6}{5}e^t + \frac{1}{5}e^t$$

$$y_1' = \underbrace{\frac{3}{5}e^t - \frac{2}{5}e^t}_{\text{Same}}$$

$$y_1' = \frac{1}{5}e^t$$

$$\frac{dy_1}{dt} = \frac{1}{5}e^t$$

$\boxed{\text{VSM}}$

$$\frac{dy_1}{1} = \frac{1}{5}e^t dt$$

IOBS

$$\int dy_1 = \frac{1}{5}\int e^t dt$$

$$\boxed{y_1 = \frac{1}{5}e^t + c_1}$$

$$y_2' = 5y_2 + \underbrace{\frac{6}{5}e^t + \frac{1}{5}e^t}_{\text{Same}}$$

$$y_2' = 5y_2 + \frac{7}{5}e^t$$

$$\frac{dy_2}{dt} - 5y_2 = \frac{7}{5}e^t \rightarrow \text{Linear in } y_2$$

$$\frac{dy_2}{dt} + Py_2 = Q \rightarrow \text{linear in } y_2$$

$$I.F = e^{\int P dt} \qquad P = -5,$$

$$= e^{\int(-5)dt} \qquad Q = \frac{7}{5}e^t$$

$$\boxed{I.F = e^{-5t}}$$

$$\therefore y_2 (I.F) = \int Q(I.F)\,dt + c_2$$

$$\Rightarrow y_2 e^{-5t} = \int \frac{7}{5} e^{t} (e^{-5t})\,dt + c_2$$

$$\Rightarrow y_2 e^{-5t} = \frac{7}{5} \int e^{-4t}\,dt + c_2$$

$$\Rightarrow y_2 e^{-5t} = \frac{7}{5}\left(\frac{e^{-4t}}{-4}\right) + c_2$$

$$\Rightarrow y_2 e^{-5t} = -\frac{7}{20} e^{-4t} + c_2$$

$$\boxed{y_2 = -\frac{7}{20} e^{t} + c_2 e^{5t}}$$

Now we have $X = Py$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{5} e^{t} + c_1 \\ -\frac{7}{20} e^{t} + c_2 e^{5t} \end{bmatrix}$$

$$x_1 = \frac{1}{5} e^{t} + c_1 - \frac{14}{20} e^{t} + 2 c_2 e^{5t}$$

$$x_2 = -\frac{2}{5} e^{t} - 2c_1 - \frac{7}{20} e^{t} + c_2 e^{5t}$$

$$x_1 = -\frac{1}{2} e^{t} + c_1 + 2 c_2 e^{5t}$$

$$x_2 = -\frac{15}{20} e^{t} + 2c_1 + c_2 e^{5t}$$

$\therefore$ (i) $\frac{1}{5} - \frac{14}{20}$

$\frac{4-14}{20} = \frac{-10}{20} = -\frac{1}{2}$

$\therefore$ (ii) $-\frac{2}{5} - \frac{7}{20}$

$\frac{-8-7}{20} = \frac{-15}{20}$

$$X = \begin{bmatrix} 1 \\ 2 \end{bmatrix} c_1 + \begin{bmatrix} 2 \\ 1 \end{bmatrix} c_2 e^{5t} - \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} e^{t}$$

$\therefore \lambda = 0, 5$

$\Rightarrow$ All eigen values are real, distinct and positive

$\Rightarrow$ Hence System is unstable

2. Convert the differential Equation $x''+5x'+6x=0$ into system of linear differential equation and hence solve them by diagonalization method. Also discuss the stability of the system

Sol: Given that $x''+5x'+6x=0 \longrightarrow (1)$

The given diff. Eq^n is second order D.E

$\therefore$ choose two variables x_1 and x_2

Let $x_1 = x \xrightarrow{\text{Diff}} x_1' = x' \xrightarrow{\text{Name}} x_2$

$x_2 = x' \longrightarrow x_2' = x''$

Eq^n (1) becomes

$$x''+5x'+6x=0$$

$$\Rightarrow x'' = -6x - 5x'$$

$$\Rightarrow x_2' = -6x_1 - 5x_2$$

Hence system of first order D.E is

$x_1' = x_2$ $\quad\boxed{\text{or}}\quad$ $x_1' = 0.x_1 + 1.x_2$

$x_2' = -6x_1 - 5x_2$ $\qquad\qquad$ $x_2' = -6x_1 - 5x_2$

In matrix form

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$X' = AX \longrightarrow (2)$$

where $X' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$; $A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$

The characteristic equation of A is $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 0-\lambda & 1 \\ -6 & -5-\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(-5-\lambda) + 6 = 0$

$\Rightarrow 5\lambda + \lambda^2 + 6 = 0 \Rightarrow \lambda^2 + 5\lambda + 6 = 0$

$\Rightarrow \lambda^2 + 3\lambda + 2\lambda + 6 = 0 \Rightarrow \lambda(\lambda+3) + 2(\lambda+3) = 0$

$\Rightarrow (\lambda+2)(\lambda+3) = 0$

$\Rightarrow \lambda + 2 = 0$ & $\lambda + 3 = 0$

$\Rightarrow \boxed{\lambda = -2}$ & $\boxed{\lambda = -3}$

$\therefore \boxed{\lambda = -2, -3} \longrightarrow$ Eigen values

Now consider $[A - \lambda I][X] = 0$

$$\left.\begin{aligned} -\lambda x + y &= 0 \\ -6x + (-5-\lambda)y &= 0 \end{aligned}\right\} \longrightarrow (I)$$

Case (i) If $\lambda = -2$ in (I)

$+2x + y = 0 \rightarrow (3)$

$-6x - 3y = 0 \rightarrow (4)$

from (3) $+2x + y = 0$

$2x = -y$

$\Rightarrow \frac{x}{1} = \frac{-y}{2} = k$ (say)

$\Rightarrow X_1 = [1k, -2k]$

(or) $\boxed{X_1 = [1, -2]^T \text{ if } \lambda = -2}$

Case (ii) :→ If $\lambda = -3$ in (I)

$3x + y = 0 \rightarrow (5)$

$-6x - 2y = 0 \rightarrow (6)$

from (5) $3x + y = 0$

$\Rightarrow 3x = -y$

$\Rightarrow \frac{x}{1} = -\frac{y}{3} = K$ (say)

$\Rightarrow X_2 = [1k, -3k]^T$

(or) $\boxed{X_2 = [1, -3]^T \text{ if } \lambda = -3}$

The modal matrix $P = [X_1, X_2]$

$$P = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} ; \quad |P| = -3 + 2$$

$$\boxed{|P| = -1}$$

To find $\bar{p}^{1} = \dfrac{adj\,p}{|P|}$ $\quad\therefore \bar{p}^{1} = \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix}$

Also to find $D = \bar{p}^{1} A p$

$$= \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}$$

$$D = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$$

Eqn (2) becomes

$$x' = Ax$$

put $x = py \rightarrow (7)$ [Assume Soln]

Dwrt 't'

$$x' = py'$$

$$\therefore py' = Apy$$

$$y' = (\bar{p}^{1} A p) y \quad \text{where } \boxed{D = \bar{p}^{1} A p}$$

$$y' = Dy$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$y_1' = -2y_1 + 0y_2$$

$$y_2' = 0y_1 - 5y_2$$

$y_1' = -2y_1$

$\dfrac{dy_1}{dt} = -2y_1$

$\dfrac{dy_1}{y_1} = -2dt$

$\displaystyle\int \frac{dy_1}{y_1} = -\int 2\,dt$

VSM

I.O.B.S

$y_2' = -5y_2$

$\dfrac{dy_2}{dt} = -5y_2$

$\dfrac{dy_2}{y_2} = -5dt$

$\displaystyle\int \frac{dy_2}{y_2} = -5\int dt$

$\Rightarrow \log y_1 = -2t + c_1$

$\Rightarrow y_1 = e^{-2t+c_1}$

$\Rightarrow y_1 = e^{-2t} \cdot e^{c_1}$

$\Rightarrow \boxed{y_1 = C_1 e^{-2t}}$

where $C_1 = e^{c_1}$

$\Rightarrow \log y_2 = -5t + c_2$

$\Rightarrow y_2 = e^{-5t+c_2}$

$\Rightarrow y_2 = e^{-5t} \cdot e^{c_2}$

$\Rightarrow \boxed{y_2 = C_2 e^{-5t}}$

where $C_2 = e^{c_2}$

Now we have $X = PY$ [from (7)]

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} C_1 e^{-2t} \\ C_2 e^{-5t} \end{bmatrix}$$

$$x_1 = c_1 e^{-2t} + c_2 e^{-5t}$$

$$x_3 = -2c_1 e^{-2t} - 3c_2 e^{-5t}$$

$$X = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-5t}$$

Here $\lambda = -2, -3$

$\Rightarrow$ All eigen values are real, distinct & negative

$\Rightarrow$ Hence System is Stable

Assignment - 6

1. Solve $X' = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} X + \begin{bmatrix} 3t \\ e^{-t} \end{bmatrix}$ by diagonalization and hence discuss the stability of the system.

2. Solve $X' = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ by diagonalization and hence discuss the stability of the system.