Module-5: Complex Differentiation

- **Function of a complex variable**
- **Limit, continuity and differentiability of a function**
- **Analytic functions**
- **Harmonic functions**
- **Procedure of finding derivative of a complex function**
- **Construction of analytic function given real or imaginary part – Milne Thomson's method**

Pre-requisites:

Consider the quadratic equation, $m^2+1=0$. This equation does not possess solution in reals. This difficulty was overcome by introducing the imaginary unit i defined as $i^2=-1$ or $i=\sqrt{-1}$ Thus the system of real number is extended to the system of complex number. Here i is called imaginary number.

- The numbers of the form $z=x+iy$ where x , y are real numbers and $i=\sqrt{-1}$,known as imaginary unit are called complex numbers
- The real number x is called the real part and the real number y is called the imaginary part of the complex number z . These are denoted by Re(z)and Im (z).
  $z=x+iy \Rightarrow \mathrm{Re}(z)=x, \mathrm{Im}(z)=y$
- The complex number is purely real if $y=0$ and it is purely imaginary if real part is 0.
- The set of complex numbers is denoted by C. All real numbers are complex but all complex numbers need not be reals.
- Two complex numbers $z_1=a+ib$ and $z_2=c+id$ are said to be equal if their real and imaginary parts are separately equal, ie $z_1=z_2 \Rightarrow a+ib=c+id \Rightarrow a=c \;\; and \;\; b=d$.
- The sum of two complex numbers $z_1=a+ib$ and $z_2=c+id$ is the complex number $(a+c)+i(b+d)$
- The product of two complex numbers $z_1=a+ib$ and $z_2=c+id$ is the complex number $z_1z_2=(ac-bd)+i(ad+bc)$
- If $z=x+iy$ is any complex number, then the complex number $\bar{z}=x-iy$ is called the conjugate of the complex number z. Further $z\bar{z}=(x+iy)(x-iy)=x^2+y^2$.
- A complex number and its conjugate satisfy the following properties:

i) $\overline{z_1 z_2} = \overline{z_1} \bullet \overline{z_2}$ ii) $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$

iii) $z + \bar{z} = 2\,\mathrm{Re}(z)$. iv) $z - \bar{z} = 2i\,\mathrm{Im}(z)$.

- $e^{i\theta} = \cos\theta + i\sin\theta$
- $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}, \ \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}.$
- $\cos(i\theta) = \cosh\theta, \ \sin(i\theta) = i\sinh\theta.$
- If n is any rational number then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$

4.1 Complex plane:

The complex numbers can be represented by means of points on the plane. The plan on which the representation is carried out is called complex **plane** or **Argand plane.** We can represent the complex number of the form $z = x + iy$ as a point $P(x, y)$ as shown in the Cartesian plane, called complex plane or $z-$plane. The x-axis is called the real axis and y-axis the imaginary axis. If θ is the angle made by OP with the x-axis and OP=r, then we have

$$x = r\cos\theta, y = r\sin\theta \Rightarrow |z| = \sqrt{x^2 + y^2} = r \ \ and \ \theta = \tan^{-1}\left(\frac{y}{x}\right) = amp\, z.$$

Here r is called radius vector or the modulus of the complex number and θ is called amplitude of z or the argument of z denoted as amp(z) or arg(z).

- $z\bar{z} = x^2 + y^2 = |z|^2 \ and \ |z| = \sqrt{z\bar{z}}$
- $|z_1 \bullet z_2| = |z_1| \bullet |z_2|, \ \dfrac{|z_1|}{|z_2|} = \left|\dfrac{z_1}{z_2}\right|.$
- $amp(z_1 z_2) = amp(z_1) + amp(z_2), \ amp\left(\dfrac{z_1}{z_2}\right) = amp(z_1) - amp(z_2).$
- Complex number in i) Cartesian form: $z = x + iy$
  ii) Polar form: $z = r(\cos\theta + i\sin\theta)$
  iii) Exponential form: $z = re^{i\theta}$.

4.2 Function of a complex variable:

Let D be the set of complex numbers and z and w be two complex variables. The complex variable w is said to be a function of the complex variable z, if to every value of z in the domain D, there correspond one or more values of w. This is denoted by $w = f(z)$. Ex: $w = e^z, w = \dfrac{1}{z}, w = \sin z \ etc.$ are functions of complex variable.

4.3 Limit, continuity and differentiability of a complex function:

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4.3.1 Limit of a complex function:

A function $w = f(z)$ is said to tend to limit l as z tends to z_0 if for every $\varepsilon > 0$, there exists as $\delta > 0$ such that $|f(z) - l| < \varepsilon$ *for all* $|z - z_0| < \delta$.
The above definition implies that the value of $f(z)$ can be made arbitrarily close to l for all z in the neighborhood of z_0, except perhaps at $z = z_0$. Symbolically, we can write $\lim_{z \to z_0} f(z) = l$.

4.3.2 Continuity of a function of a complex variable:

A function $f(z)$ of complex variable z,is said to be continuous at a point z_0 ,if

i. $f(z)$ is defined at z $= z_0$.
ii. $\lim_{z \to z_0} f(z)$ exists.
iii. $\lim_{z \to z_0} f(z) = f(z_0)$.

Note that the sum, difference and product of continuous functions of a complex variable are continuous. And the function $f(z)$ is said to be continuous in a domain D if it is continuous at every point in D.

4.3.3 Differentiability of a complex function:

A function $f(z)$ is said to be differentiable at a point z $= z_0$ if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists. This limit is called the derivative of $f(z)$ at z $= z_0$ and is denoted by $f'(z_0)$. Let $z = z_0 + \delta z$,
$f'(z_0) = \lim_{z \to z_0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z}$.

Note:

Neighbourhood of a complex number z_0:

Let r be a finite number, neighbourhood of a complex number z_0 is defined as $|z - z_0| = r$ or $|z - z_0| < r$

Consider $z = x + iy \therefore z_0 = x_0 + iy_0$

$$|z - z_0| = |x + iy - (x_0 + iy_0)| = r \Rightarrow \sqrt{(x - x_0)^2 + (y - y_0)^2} = r \Rightarrow (x - x_0)^2 + (y - y_0)^2 = r$$

Hence neighbourhood of any point z_0 in the complex plane is the circle centered at z_0 and having radius r and $|z - z_0| < r$ represents only the interior points of the circle centered at z_0 and radius r.

4.3.4 Analytic function:

A function $f(z)$ of a complex variable is said to be analytic at a point z_0 if it is differentiable at z_0 and also at every point of some neighbourhood of z_0. A function $f(z)$ is said to be analytic in a domain D, if it is analytic at every point in D. It is also called as **regular** function or **holomorphic** function.

4.4 Cauchy-Riemann Equations:

Theorem:

A necessary condition that $f(z)=u(x,y)+iv(x,y)$ be analytic in a domain D is that the first order derivatives of u and v with respect to x and y must exist and satisfy the equations $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$. These equations are called Cauchy-Riemann equations in Cartesian form.

Proof:

Given that $f(z)$ is analytic in a domain D. Therefore $f(z)$ has a derivative

$$f'(z)=\lim_{\delta z\to 0}\frac{f(z+\delta z)-f(z)}{\delta z}$$

We have $f(z)=u(x,y)+iv(x,y)$

$$\therefore f(z+\delta z)=u(x+\delta x, y+\delta y)+iv(x+\delta x, y+\delta y).$$

$$\delta z=\delta x+i\delta y$$

$$\therefore f'(z)=\lim_{\delta z\to 0}\frac{[u(x+\delta x, y+\delta y)+iv(x+\delta x, y+\delta y)]-[u(x,y)+iv(x,y)]}{\delta z}$$

$$f'(z)=\lim_{\delta z\to 0}\frac{u(x+\delta x, y+\delta y)-u(x,y)}{\delta z}+i\lim_{\delta z\to 0}\frac{v(x+\delta x, y+\delta y)-v(x,y)}{\delta z} \qquad (1)$$

Case 1: Let δz be purely real. Thus $\delta y=0, \delta z=\delta x$.

$$f'(z)=\lim_{\delta x\to 0}\frac{u(x+\delta x, y+\delta y)-u(x,y)}{\delta x}+i\lim_{\delta z\to 0}\frac{v(x+\delta x, y+\delta y)-v(x,y)}{\delta x}$$

$$f'(z)=\lim_{\delta z\to 0}\frac{u(x+\delta x, y)-u(x,y)}{\delta x}+i\lim_{\delta z\to 0}\frac{v(x+\delta x, y)-v(x,y)}{\delta x}$$

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} \qquad (2)$$

Case 2: Let δz be purely imaginary. Thus $\delta x=0, \delta z=i\delta y$

$$\therefore f'(z)=\lim_{\delta y\to 0}\frac{u(x+\delta x, y+\delta y)-u(x,y)}{i\delta y}+i\lim_{\delta y\to 0}\frac{v(x+\delta x, y+\delta y)-v(x,y)}{i\delta y}$$

$$\Rightarrow f'(z)=\lim_{\delta y\to 0}\frac{u(x, y+\delta y)-u(x,y)}{i\delta y}+i\lim_{\delta y\to 0}\frac{v(x, y+\delta y)-v(x,y)}{i\delta y}$$

$$\Rightarrow f'(z)=\frac{1}{i}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

$$\Rightarrow f'(z)=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y} \qquad (3)$$

Since $f'(z)$ is unique, the right sides of (2) *and* (3) are equal.

$$\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

Equating real and imaginary parts we get,

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

Note:

i) $f(z)\ analytic \Rightarrow v_x$ ii) $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$. need not imply $f(z)$ analytic.

Sufficient Conditions:

The sufficient conditions for the function $f(z)=u+iv$, to be analytic at any point in the domain are

i) u,v and their partial derivatives $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous at that point.

ii) C-R equations are satisfied at that point.

4.5 Cauchy-Riemann equations in polar form:

Statement:

A necessary condition that $f(z)=u(r,\theta)+iv(r,\theta)$ be analytic in a domain D is that the first order derivatives of u and v w.r.t $r\ and\ \theta$ must exist and must satisfy the equations

$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$. These equations are called C-R equations in polar form.

Proof:

Given that $f(z)$ is analytic in the domain D.

$\therefore f'(z)=\lim\limits_{\delta z\to 0}\frac{f(z+\delta z)-f(z)}{\delta z}$ exists.

And we have $f(z)=u(r,\theta)+iv(r,\theta)$ and $f(z+\delta z)=u(r+\delta r,\theta+\delta\theta)+iv(r+\delta r,\theta+\delta\theta)$

$$\therefore f'(z)=\lim_{\delta z\to 0}\frac{[u(r+\delta r,\theta+\delta\theta)+iv(r+\delta r,\theta+\delta\theta)]-[u(r,\theta)+iv(r,\theta)]}{\delta z}$$

$$f'(z)=\lim_{\delta z\to 0}\frac{[u(r+\delta r,\theta+\delta\theta)-u(r,\theta)]-i[v(r+\delta r,\theta+\delta\theta)-v(r,\theta)]}{\delta z} \quad (1)$$

We have $z=re^{i\theta}$

$$\delta z=\frac{\delta z}{\delta r}\delta r+\frac{\delta z}{\delta\theta}\delta\theta$$

$$\delta z=e^{i\theta}\delta r+ire^{i\theta}\delta\theta$$

Equation (1) becomes $f'(z)=\lim\limits_{\delta z\to 0}\frac{[u(r+\delta r,\theta+\delta\theta)-u(r,\theta)]}{\delta z}+i\lim\limits_{\delta z\to 0}\frac{[v(r+\delta r,\theta+\delta\theta)-v(r,\theta)]}{\delta z}$ (2)

Case 1: When $\delta z \to 0, \delta r \to 0$ and put $\delta\theta = 0$
Equation (1) becomes,

$$f'(z) = \lim_{\delta r \to 0} \frac{[u(r+\delta r,\theta) - u(r,\theta)]}{e^{i\theta}\delta r} + i \lim_{\delta z \to 0} \frac{[v(r+\delta r,\theta+\delta\theta) - v(r,\theta)]}{e^{i\theta}\delta r}$$

$$f'(z) = e^{-i\theta}\frac{\partial u}{\partial r} + ie^{-i\theta}\frac{\partial v}{\partial r} \qquad (3)$$

Case 2: When $\delta z \to 0, \delta\theta \to 0$ and put $\delta r = 0$
Equation (1) becomes,

$$f'(z) = \lim_{\delta\theta \to 0} \frac{[u(r,\theta+\delta\theta) - u(r,\theta)]}{ire^{i\theta}\delta\theta} + i \lim_{\delta\theta \to 0} \frac{[v(r,\theta+\delta\theta) - v(r,\theta)]}{ire^{i\theta}\delta\theta}$$

$$f'(z) = \frac{-ie^{-i\theta}}{r}\frac{\partial u}{\partial\theta} + \frac{e^{-i\theta}}{r}\frac{\partial v}{\partial r} \qquad (4)$$

Comparing (3) *and* (4),

$$\frac{1}{r}\frac{\partial v}{\partial\theta} = \frac{\partial u}{\partial r}, \; r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial\theta}$$

4.6 Harmonic Functions: (Laplace equation)

The equation $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ is called Laplace equation in two dimensions.

The Laplace equation is of great practical importance and occurs frequently in the study of fluid flow, heat conduction, gravitation and electrostatic
Any function $\phi(x,y)$ which possesses continuous partial derivatives of the first and second order and satisfy the Laplace equation $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ is called a **harmonic function.**

4.6.1 Theorem: (Cartesian form)

Statement:

The real and imaginary parts of an analytic function $f(z) = u(x,y) + iv(x,y)$ are harmonic functions.

Proof:

Given $f(z) = u(x,y) + iv(x,y)$ is analytic function.
We are supposed to prove that u and v are harmonic functions.
Since $f(z)$ is analytic, C-R equations are satisfied.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad (1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \qquad (2)$$

Differentiate (1) w.r.t x and (2) w.r.t y, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 v}{\partial x \partial y} = 0 \quad \Rightarrow \text{u is harmonic function.}$$

Differentiate (1) w.r.t y and (2) w.r.t x and adding the equations we get,

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Note:

Harmonic function in polar form:

A function $\phi(r,\theta)$ is said to be harmonic if it satisfies $\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0.$

4.6.2 Theorem: (Polar form)

Statement:

The real and imaginary parts of an analytic function $f(z) = u(r,\theta) + iv(r,\theta)$ are harmonic functions.

Proof:

Given $f(z) = u(r,\theta) + iv(r,\theta)$ is analytic.
We are supposed to prove that u and v are harmonic.
Since $f(z)$ is analytic C-R equations are satisfied.

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad (1)$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}. \qquad (2)$$

Differentiate (1) w.r.t r and (2) w.r.t θ we get,

$$r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} = \frac{\partial^2 v}{\partial r \partial \theta}$$

$$\Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta} \qquad (3)$$

$$-\frac{\partial^2 u}{\partial \theta^2} = r\frac{\partial^2 v}{\partial \theta \partial r}$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r} \qquad (4)$$

Adding (3) and (4) we get,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \Rightarrow u \text{ is harmonic.}$$

Differentiate (1) w.r.t θ and (2) w.r.t r and adding we get,

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}=0. \Rightarrow v \text{ is harmonic.}$$

4.6.3 Theorem:

Statement:

If $f(z)=u(x,y)+iv(x,y)$ is analytic function in the domain D of the complex plane, then $u=c_1$ and $v=c_2$ where c_1 and c_2 are constants, represent orthogonal family of curves.

Proof:

Given $f(z)=u+iv$ is analytic function

And $u(x,y)=c_1$ (1)

$v(x,y)=c_2$ (2)

Differentiate (1) w.r.t x we get,

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\left(\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}\right)=m_1$$

Again differentiate (2) w.r.t y we get,

$$\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\left(\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}\right)=m_2$$

Since $f(z)=u(x,y)+iv(x,y)$ is analytic function, we have C-R equations: $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$.

$$\therefore m_1 m_2=\left(\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}\right)\left(\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}\right)=\left(\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}\right)\left(\frac{-\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}}\right)=-1$$

This shows that the curves $u(x,y)=c_1$ and $v(x,y)=c_2$ cut orthogonally.

Note:

If $f(z)=u+iv$ is analytic function then v is an harmonic conjugate of u and vice-versa.

Procedure of finding derivative of a complex function:

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1. Given $f(z)$, replace z by $x+iy \text{ or } re^{i\theta}$. And find the real part of $f(z)$ i.e. u and imaginary part v.
2. Verify whether $f(z)$ is analytic or not. If so, find $f'(z)$ using the relevant formula

$$f'(z)=u_x+iv_x \text{ or } f'(z)=e^{-i\theta}(u_r+iv_r) \quad \text{where } u_x=\frac{\partial u}{\partial x},\ v_x=\frac{\partial v}{\partial x}, u_r=\frac{\partial u}{\partial r} \text{and } v_r=\frac{\partial v}{\partial r}.$$

Problems:

1. Find the complex derivative of $f(z)=z$

Solution:

Given $f(z)=z$

$\Rightarrow f(z)=x+iy$

$\Rightarrow u=x, v=y$

Consider $u_x=1,\ v_x=0,\ v_y=1$ and $u_y=0$

Clearly $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=1$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.=0$. C-R equations are satisfied

And all the partial derivatives are continuous. $\therefore f(z)$ is analytic.

We have $f'(z)=u_x+iv_x$

$$f'(z)=1+i0=1 \quad .$$

2. Find $f'(z)$ if $f(z)=e^z$

Solution:

Given $f(z)=e^z$

$\Rightarrow f(z)=e^{x+iy}=e^x e^{iy}=e^x(\cos y+i\sin y)$

$\Rightarrow u=e^x\cos y, v=e^x\sin y$

Now $u_x=\mathrm{e}^x\cos y, u_y=-\mathrm{e}^x\sin y, v_x=\mathrm{e}^x\sin y$ *and* $v_y=\mathrm{e}^x\cos y$.

Clearly $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=\mathrm{e}^x\cos y$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$.C-R equations are satisfied.

And all the parial derivatives are continuous. $\therefore f(z)$ is analytic.

We have $f'(z)=u_x+iv_x$

$$f'(z)=\mathrm{e}^x\cos y+i\ \mathrm{e}^x\sin y=e^x(e^{iy})=e^z \quad .$$

3. Find $f'(z)$ if $f(z)=e^z+z$.

Solution:

Given $f(z) = e^z + z$.

$\Rightarrow f(z) = e^{x+iy} + (x+iy) = e^x e^{iy} + (x+iy)$

$= e^x(\cos y + i\sin y) + x + iy$

$\Rightarrow u = e^x \cos y + x, v = e^x \sin y + y$

$Now\ u_x = e^x \cos y + 1, u_y = -e^x \sin y$

$v_x = e^x \sin y\ and\ v_y = e^x \cos y + 1.$

Clearly $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^x \cos y + 1$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. C-R equations are satisfied

And all the partial derivatives are continuous. $\therefore f(z)$ is analytic.

We have $f'(z) = u_x + iv_x$

$f'(z) = e^x \cos y + 1 + i(e^x \sin y)$

$f'(z) = e^z + 1$.

4. Find $f'(z)$ if $f(z) = \cos z$.

Solution:

Given $f(z) = \cos z$.

$\Rightarrow f(z) = \cos(x+iy) = \cos x \cos(iy) - \sin x \sin(iy)$

$= \cos x \cosh y - i \sin x \sinh y$

$\Rightarrow u = \cos x \cosh y, v = -\sin x \sinh y$

$Now\ u_x = -\sin x \cosh y, u_y = \cos x \sinh y, v_x = -\cos x \sinh y\ and\ v_y = -\sin x \cosh y$

Clearly $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. C-R equations are satisfied

And all the partial derivatives are continuous. $\therefore f(z)$ is analytic.

We have $f'(z) = u_x + iv_x$

$f'(z) = -\sin x \cosh y - i \cos x \sinh y$

$f'(z) = -\sin x \cos(iy) - \cos x \sin(iy) = -\sin(x+iy) = -\sin z.$

5. Find $f'(z)$ if $f(z) = \cosh z$.

Solution:

Given $f(z) = \cosh z$

$\Rightarrow f(z) = \cos(iz) = \cos i(x+iy) = \cos(xi - y)$

$\Rightarrow \cos(ix)\cos y + \sin(xi)\sin y = \cosh x \cos y + i \sinh x \sin y$

$\Rightarrow u = \cosh x \cos y, v = \sinh x \sin y$

$Now\ u_x = \sinh x \cos y, u_y = -\cosh x \sin y,.$

$v_x = \cosh x \sin y\ and\ v_y = \sinh x \cos y$

Clearly $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$. C-R equations are satisfied

And all the partial derivatives are continuous. $\therefore f(z)$ is analytic.

We have $f'(z)=u_x+iv_x$

$$f'(z)=\sinh x\cos y+i\cosh x\sin y$$
$$f'(z)=\cos y\sin(ix)+i\cos(ix)\sin y$$
$$f'(z)=\frac{\sin(ix)}{i}\cos y+i\cos(ix)\sin y$$
$$=-i[\sin(ix)\cos y+\cos(ix)\sin y]=-i\sin[i(x+iy)]$$
$$=-i\sin(iz)=-i^2\sinh z=\sinh z\,.$$

6. Find $f'(z)$ if $f(z)=\log z$.

Solution:

Given $f(z)=\log z$

$$\Rightarrow f(z)=\log\left(re^{i\theta}\right)=\log r+i\theta$$
$$\Rightarrow u=\log r, v=\theta$$
$$Now\ u_r=\frac{1}{r}, u_\theta=0, v_r=0\ and\ v_\theta=1.$$
$$Consider\ ru_r=1=v_\theta\ \ and\ \ rv_r=0=u_\theta$$
$$\therefore ru_r=v_\theta\ \ and\ \ rv_r=u_\theta.\text{C - R equations are satisfied}$$

And the partial derivatives are continuous. $\therefore f(z)$ is analytic.

$$f'(z)=e^{-i\theta}\frac{\partial u}{\partial r}+ie^{-i\theta}\frac{\partial v}{\partial r}$$
$$f'(z)=u_re^{-i\theta}+ie^{-i\theta}v_r=e^{-i\theta}(\frac{1}{r}+0)=\frac{1}{re^{i\theta}}=\frac{1}{z}.$$

7. Find the derivative of complex valued function $f(z)=z^n$

Solution:

$Given\ f(z)=z^n$

$$\Rightarrow (re^{i\theta})^n=r^ne^{in\theta}=r^n\left(\cos n\theta+i\sin n\theta\right).$$
$$\therefore u=r^n\cos n\theta, v=r^n\sin n\theta$$
$$u_r=nr^{n-1}\cos n\theta, u_\theta=-nr^n\sin n\theta, v_r=nr^{n-1}\sin n\theta, v_\theta=nr^n\cos n\theta.$$
$$Clearly\ \ ru_r=v_\theta\ \ and\ \ rv_r=-u_\theta$$

C-R equations are satisfied and partial derivatives are continuous.

Thus $f(z)$ is analytic.

$$f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}(nr^{n-1}\cos n\theta + inr^{n-1}\cos n\theta) = nr^{n-1}.e^{-i\theta}e^{in\theta}$$
$$= nr^n \frac{1}{re^{i\theta}} e^{in\theta} = nz^{n-1}$$

4.7 Construction of analytic function using Milne's Thomson's method:

This method can be applied to construct the analytic function when only real or imaginary part is given.

4.7.1 Procedure:

1. If u or v of analytic function is given in Cartesian form:

Suppose the real part u is given,
Since the function is analytic we have $f'(z) = u_x + iv_x$.

By C-R equations we know that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. $\therefore f'(z) = u_x - iu_y$

Then put x=z and y=0 in the above equation. After substitution, integrate w.r.t z, so that the analytic function $f(z)$ is obtained. The procedure is vice-versa if v is given.

2. If u or v of analytic function is given in polar form:

Suppose the real part u is given,
Since the function is analytic we have $f'(z) = e^{-i\theta}(u_r + iv_r)$.

By C-R equations we have, $ru_r = v_\theta$ *and* $rv_r = -u_\theta \therefore f'(z) = e^{-i\theta}(u_r - i\frac{1}{r}u_\theta)$.

Then put r=z and θ=0 so that the function $f'(z)$ will be in terms of z. Now on integration, we get the required analytic function.

4.7.2 Problems:

1. Find the analytic function whose real part is $x^3 - 3xy^2$.

Solution:

Let $f(z) = u + iv$ be the required analytic function.
Given $u = x^3 - 3xy^2$.

$$\Rightarrow \frac{\partial u}{\partial x} = 3x^2 + 3y^2, \frac{\partial u}{\partial y} = -6xy.$$

We know $f(z)$ is analytic $\Rightarrow f'(z)$ exists.

And $f'(z) = u_x + iv_x$

$\Rightarrow f'(z) = u_x - iu_y = (3x^2 + 3y^2) + 6ixy.$

Apply Milne's Thomson's method, Put $x = z$ and $y = 0$ in the above equation,

We get, $f'(z) = 3z^2$

On integration, we get $f(z) = z^3 + c$ which is required analytic function.

2. Find the analytic function whose imaginary part is $v = e^x(x\sin y + y\cos y)$

Solution:

Let $f(z) = u + iv$ be the required analytic function.

Given $v = e^x(x\sin y + y\cos y)$

$$\Rightarrow \frac{\partial v}{\partial x} = e^x(x\sin y + y\cos y) + e^x(\sin y), \frac{\partial v}{\partial y} = e^x(x\cos y + \cos y - y\sin y).$$

We know $f(z)$ is analytic $\Rightarrow f'(z)$ exists.

And $f'(z) = u_x + iv_x$

$\Rightarrow f'(z) = v_y + iv_x = e^x(x\cos y + \cos y - y\sin y) + i\left[e^x(x\sin y + y\cos y) + e^x(\sin y)\right]$

Apply Milne's Thomson's method, Put $x = z$ and $y = 0$ in the above equation,

We get, $f'(z) = e^z(z+1) + i(0)$

On integration, we get $f(z) = ze^z + c$ which is required analytic function.

3. Find the analytic function whose real part is $\left(r + \frac{1}{r}\right)\cos\theta.$

Solution:

Let $f(z) = u + iv$ be the required analytic function.

Given $u = \left(r + \frac{1}{r}\right)\cos\theta.$

$$\Rightarrow \frac{\partial u}{\partial r} = \left(1 - \frac{1}{r^2}\right)\cos\theta, \frac{\partial u}{\partial \theta} = -\left(r + \frac{1}{r}\right)\sin\theta.$$

We know $f(z)$ is analytic $\Rightarrow f'(z)$ exists.

And $f'(z) = e^{-i\theta}(u_r + iv_r)$

$$\Rightarrow f'(z) = e^{-i\theta}(u_r - i\frac{1}{r}u_\theta). = e^{-i\theta}\left[\left(1 - \frac{1}{r^2}\right)\cos\theta + i\left(r + \frac{1}{r}\right)\sin\theta\right]$$

Apply Milne's Thomson's method, Put $r = z$ and $\theta = 0$ in the above equation,

We get, $f'(z) = \left(1 - \frac{1}{z^2}\right)$

On integration, we get $f(z) = z + \frac{1}{z} + c$ which is required analytic function.

4. Find the analytic function whose imaginary part is $r\sin\theta + \frac{\cos\theta}{r}.$

Solution:

Let $f(z)=u+iv$ be the required analytic function.

Given $v=r\sin\theta+\dfrac{\cos\theta}{r}$.

$$\Rightarrow \frac{\partial v}{\partial r}=\sin\theta-\frac{\cos\theta}{r^2}, \frac{\partial v}{\partial \theta}=r\cos\theta-\frac{\sin\theta}{r}.$$

We know $f(z)$ is analytic $\Rightarrow f'(z)$ exists.

And $f'(z)=e^{-i\theta}(u_r+iv_r)$

$$\Rightarrow f'(z)=e^{-i\theta}(\frac{1}{r}v_\theta+iv_r).=e^{-i\theta}\left[\frac{1}{r}\left(r\cos\theta-\frac{\sin\theta}{r}\right)+i\left(\sin\theta-\frac{\cos\theta}{r^2}\right)\right]$$

Apply Milne's Thomson's method, Put $r=z$ and θ=0 in the above equation,

We get, $f'(z)=\left(1-i\dfrac{1}{z^2}\right)$

On integration, we get $f(z)=z+\dfrac{i}{z}+c$ which is required analytic function.

5. Find the analytic function $f(z)=u+iv$ given $u-v=e^x(\cos y-\sin y)$.

Solution:

Given: $u-v=e^x(\cos y-\sin y)$. (1)

Differentiate (1) partially w.r.t x, $u_x-v_x=e^x(\cos y-\sin y)$ (2)

Differentiate (1) partially w.r.t y, $u_y-v_y=e^x(-\sin y-\cos y)$ (3)

Since the function is analytic, we have $u_y=-v_x, u_x=v_y$

$\therefore$ (3) becomes, $-v_x-u_x=-e^x(\sin y+\cos y)\Rightarrow u_x+v_x=e^x(\sin y+\cos y)$

Add (2) and the above equations we get, $u_x=e^x\cos y\Rightarrow v_x=e^x\sin y$

$f'(z)=u_x+iv_x=e^x(\cos y+i\sin y)=e^z$

6. Find the analytic function $f(z)=u+iv$ given $2u+v=e^x(\cos y-\sin y)$.

Solution:

Given: $2u+v=e^x(\cos y-\sin y)$. (1)

Differentiate (1) partially w.r.t x, $2u_x+v_x=e^x(\cos y-\sin y)$ (2)

Differentiate (1) partially w.r.t y, $2u_y+v_y=-e^x(\sin y+\cos y)$ (3)

Since the function is analytic, we have $u_y=-v_x, u_x=v_y$

$\therefore$ (3) becomes, $-2v_x+u_x=-e^x(\sin y+\cos y)$

multiply the above equation by 2,we get $-4v_x+2u_x=-2e^x(\sin y+\cos y)$

Solve (2) and the above equation we get, $v_x = \frac{1}{5}e^x[3\cos y + \sin y]$

Substitute in (2) we get, $u_x = \frac{1}{5}e^x(\cos y - 3\sin y)$

$f'(z) = u_x + iv_x = \frac{1}{5}e^x(\cos y - 3\sin y) + i\frac{1}{5}e^x(3\cos y + \sin y)$

Put $x = z, y = 0$ we get, $f'(z) = \frac{1}{5}e^z(1+3i)$

$f(z) = \frac{1}{5}e^z(1+3i) + c$ is required analytic function

7. Find the analytic function $f(z) = u + iv$ given $u + v = x^3 - y^3 + 3x^2y - 3xy^2$

Solution:

Given: $u + v = x^3 - y^3 + 3x^2y - 3xy^2$ (1)
Differentiate (1) partially w.r.t x, $u_x + v_x = 3x^2 + 6xy - 3y^2$ (2)
Differentiate (1) partially w.r.t y, $u_y + v_y = -3y^2 + 3x^2 - 6xy$ (3)
Since the function is analytic, we have $u_y = -v_x, u_x = v_y$
$\therefore$ (3) becomes, $u_x - v_x = 3x^2 - 3y^2 - 6xy$ (4)
Add (2) and the above equation we get, $u_x = 3(x^2 - y^2)$
Substitute in equation (2), we get $v_x = 6xy$
$f'(z) = u_x + iv_x = 3(x^2 - y^2) + i6xy$
Put $x = z, y = 0$
$f'(z) = 3z^2 \Rightarrow f(z) = z^3 + c$ is the required analytic function .

8. Find the analytic function $f(z) = u + iv$ given $u + v = \frac{1}{r^2}(\cos 2\theta - \sin 2\theta)$

Solution:

Given: $u + v = \frac{1}{r^2}(\cos 2\theta - \sin 2\theta)$ (1)

Differentiate (1) partially w.r.t r, $u_r + v_r = -\frac{2}{r^3}(\cos 2\theta - \sin 2\theta)$ (2)

Differentiate (1) partially w.r.t θ, $u_\theta + v_\theta = \frac{1}{r^2}(-2\sin 2\theta - 2\cos 2\theta)$ (3)

Since the function is analytic, $ru_r = v_\theta$ *and* $rv_r = -u_\theta$

$\therefore$ (3) becomes, $v_r - u_r = \frac{2}{r^3}(\sin 2\theta + \cos 2\theta)$

Add (2) and the above equation we get, $v_r = \frac{2}{r^3}\sin 2\theta$

Substitute in equation (2), we get $u_r = -\frac{2}{r^3}\cos 2\theta$

$$f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(-\frac{2}{r^3}\cos 2\theta + i\frac{2}{r^3}\sin 2\theta\right) = \frac{-2}{r^3 e^{3i\theta}}$$

Put $r = z, \theta = 0$ we get, $f'(z) = -\frac{2}{z^3}$

On integration, $f(z) = \frac{1}{z^2} + c$ is the required analytic function.

9. If $\phi + i\psi$ represents the complex potential of an electrostatic field where $\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$. Find the complex potential as a function of complex variable z and determine ϕ.

Solution:

Given $\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$.

Differentiate partially w.r.t x we get, $\psi_x = 2x + \frac{-x^2+y^2}{(x^2+y^2)^2}$

Again differentiate partially w.r.t y we get, $\psi_y = -2y + \frac{-2xy}{x^2+y^2}$.

We have $f'(z) = \phi_x + i\psi_x$. But $\phi_x = \psi_y$.

$\therefore f'(z) = \phi_x + i\psi_x$. Put x=z,y=0

$$f'(z) = 0 + i\left(2z - \frac{1}{z^2}\right)$$

On integration, $f(z) = i\left(z^2 + \frac{1}{z}\right) + c$ is required complex potential.

To find ϕ

$$f(z) = \phi + i\psi = i\left[(x+iy)^2 + \frac{1}{x+iy}\right] + c = \left(-2xy + \frac{y}{x^2+y^2}\right) + i\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right) + c.$$

Equating real part, $\phi = \left(-2xy + \frac{y}{x^2+y^2}\right)$

10. If $f(z)$ is analytic function of z then show that $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}\right]|ref(z)|^2 = 2|f'(z)|^2$.

Solution:

Let $f(z)=u+iv$ and $\text{Re}\, f(z)=u$ and $\therefore |ref(z)|^2=u^2$

L.H.S= $\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial x^2}\right]|ref(z)|^2=\frac{\partial^2(u^2)}{\partial x^2}+\frac{\partial^2(u^2)}{\partial y^2}=\frac{\partial}{\partial x}\left[2u\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial y}\left[2u\frac{\partial u}{\partial y}\right]$

$=2(u_x)^2+2u_{xx}+2(u_y)^2+2u_{yy}=2\left[u_x^{\,2}+u_y^{\,2}\right]+2u(0) \qquad \because u_{xx}+u_{yy}=0$

$=2|f'(z)|^2$=R.H.S

11. If $f(z)$ is analytic function of z then show that $\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial x^2}\right]|f(z)|^2=4|f'(z)|^2$.

Solution:

Let $f(z)=u+iv$ be analytic. And $|f(z)|=\sqrt{u^2+v^2}$

$\therefore |f(z)|^2=u^2+v^2$, let $\phi=u^2+v^2$

To prove that: $\phi_{xx}+\phi_{yy}=4(u_x^{\,2}+v_x^{\,2})$

Proof: Consider $\phi=u^2+v^2$

Differentiate partially w.r.t x twice,

$\phi_x=2uu_x+2vv_x$; $\phi_{xx}=2\left[uu_{xx}+u_x^{\,2}+vv_{xx}+v_x^{\,2}\right]$

Similarly $\phi_{yy}=2\left[uu_{yy}+u_y^{\,2}+vv_{yy}+v_y^{\,2}\right]$

Consider $\phi_{xx}+\phi_{yy}=2\left[u(u_{xx}+u_{yy})+u(u_{xx}+u_{yy})+u_x^{\,2}+u_y^{\,2}+v_x^{\,2}+v_y^{\,2}\right]=4(u_x^{\,2}+v_x^{\,2})$.

12. If $f(z)$ is regular function of z then show that

$$\left[\frac{\partial}{\partial x}|f(z)|\right]^2+\left[\frac{\partial}{\partial y}|f(z)|\right]^2=|f'(z)|^2$$

Solution:

Let $f(z)=u+iv$ be analytic. And $|f(z)|=\sqrt{u^2+v^2}$

Let $|f(z)|=\sqrt{u^2+v^2}=\phi$

We have to prove $\phi_x^{\,2}+\phi_y^{\,2}=|f'(z)|^2$

Proof: Consider $\phi^2=u^2+v^2$

Differentiate $2\phi\phi_x=2uu_x+2vv_x \Rightarrow \phi\phi_x=uu_x+vv_x$

Similarly $\phi\phi_y=2uu_y+2vv_y$

Squaring and adding the above equations we get,

$$\phi^2\left(\phi_x^{\,2}+\phi_y^{\,2}\right)=\left(uu_x+vv_x\right)^2+\left(uu_y+vv_y\right)^2=\left(u^2+v^2\right)\left(u_x^{\,2}+v_x^{\,2}\right)$$
$$\Rightarrow\left(\phi_x^{\,2}+\phi_y^{\,2}\right)=\left(u_x^{\,2}+v_x^{\,2}\right)=\left|f'(z)\right|^2$$

4.8 CONFORMAL MAPPINGS.

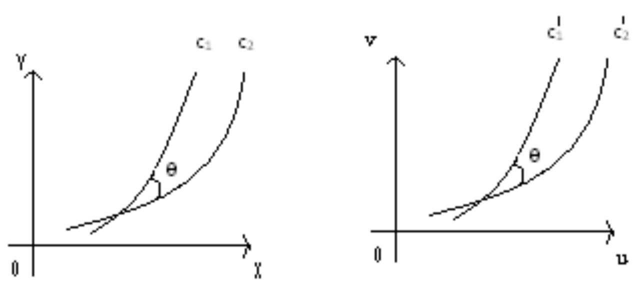
Consider a complex valued function

$$w = f(z) = u + iv. \quad (\text{Put} \quad z = x + iy.)$$
$$f(x+iy) = u(x, y) + iv(x, y).$$

Let the complex quantities $z = z(x, y), w = w(u, v)$ be represented in two separate planes namely the z-plane and the w-plane respectively.
A point (x, y) in the z-plane corresponds to the point (u, v) in the w-plane. If a set of points (x, y) traces a curve c in the z-plane and the corresponding points (u, v) traces a curve c' under the transformation $w = f(z)$
The corresponding sets of points in the two planes are called images of each other.

4.8.1 CONFORMAL TRANSFORMATON.

Let the two curves C_1 and C_2 in the z-plane interest at the point P and the corresponding curves C_1' and C_2' c'2 in the w-plane intersect at P'.



If the angle of intersection of the curves at P is the same as the angle of intersection of the curves at P' in magnitude sense then the transformation is said to be conformal. i.e. $\theta = \theta'$ ..

Note: If $w = f(z)$ is an analytic function of z in a region of z-plane then $w = f(z)$ is conformal at all points of region where $f'(z) \neq 0$.

4.8.2 DISCUSSION OF CONFORMAL TRANSFORMATIONS.

1. Discuss the transformation $w = e^z$. (General)

Consider $w = e^z$

$$u + iv = e^{x+iy}$$
$$= e^{x} e^{iy}$$
$$= e^{x} (\cos y + i \sin y)$$
$$u = e^{x} \cos y, v = e^{x} \cos y \quad \text{----------(1)}$$

We shall find the image in w-plane corresponding to the straight lines parallel to the co-ordinate axes in the z-plane.

i.e. $x = a$ constant (say a) and $y = b$ constant (say b). --------(2)

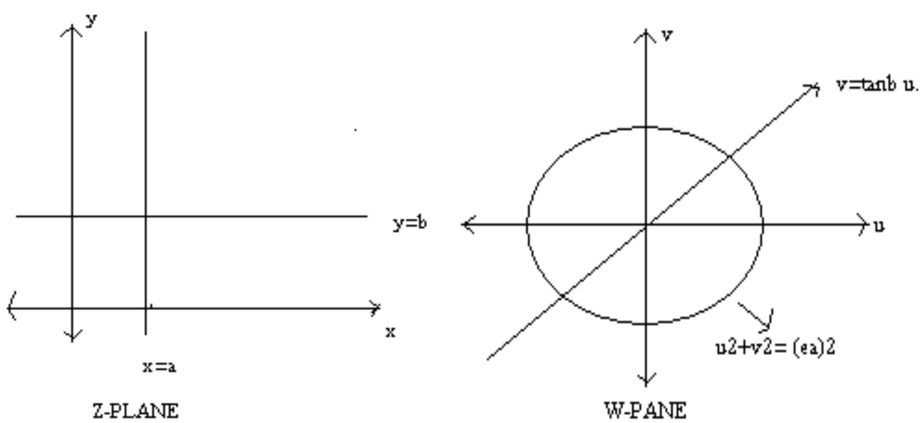
Eliminate x and y separately from (1)

Squaring and adding we get $u^2 + v^2 = e^{2x}$ ------------- (3)

Squaring and dividing we get $\frac{v}{u} = \tan y$ ------------ (4)

Use (2) in (3) & (4) we will get,

$u^2 + v^2 = (e^a)^2$ *and* $v = \tan bu$. These two equations represent a circle with centre at the origin and radius e^a also a straight line passing through the origin with the slope $\tan b$ respectively. This is shown in the figure as fallows.



CONCLUSION:

The straight lines parallel to y-axis (x=a) and x-axis (y=b) in the z-plane maps on to a circle $u^2 + v^2 = (e^a)^2$ and a straight line $(v = \tan bu)$ in the w-plane.

2. Discuss the transformation $w = z^2$ (general).

Solution:

Given $w = z^2$

Let $z = x + iy$ and $w = u + iv$ then

$$= (x + iy)^2$$

$= x^2 - y^2 + i2xy$

Implies that $u = x^2 - y^2$ and $v = 2xy$ -----------(1)

We shall find the image in w-plane corresponding to the straight lines parallel to the co-ordinate axes in the z-plane.

i.e $x = a$ constant (say a) and $y = b$ constant (say b).-------(2)

Case 1: Put $x = a$ in set of equations (1) and eliminate.

$u = a^2 - y^2$, $v = 2ay$ then

$$u = a^2 - \frac{v^2}{4a^2}$$

$v^2 = 4a^2(u - a^2)$ ---------------(3)

This equation represents a parabola in w-plane symmetrically about real axis with its vertex $(a^2, 0)$ and focus at the origin.

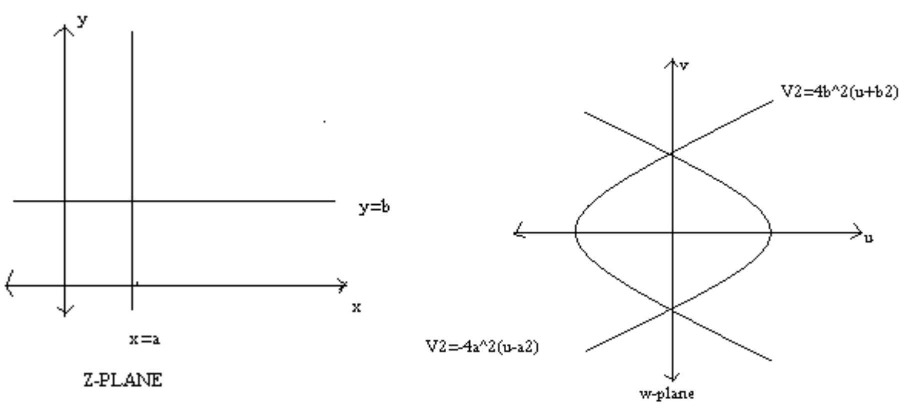
Case 2: Put $y = b$ in set of equations (1) and eliminate x and y.

$u = x^2 - b^2$; $v = 2bx$ *then*

$$u = \frac{v^2}{4b^2} - b^2$$

$v^2 = 4b^2(u + b^2)$ ----------(4)

This equation represents a parabola in w-plane symmetrically about real axis with its vertex $(-b^2, 0)$ and focus at the origin. The figures are shown as follows.



CONCLUSION:

The lines x=a and y=b in the z-plane maps on to the parabolas $v^2 = -4a^2(u - a^2)$, $v^2 = 4b^2(u + b^2)$ in the w-plane.

3. Discuss the transformation $w = z + \frac{1}{z}$

Solution:

Let $z = re^{i\theta}$ then $w = u + iv$

$$u + iv = re^{i\theta} + \frac{1}{r}e^{-i\theta}$$

$$u + iv = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

$$\Rightarrow u = \left(r + \frac{1}{r}\right)\cos\theta \quad and \quad v = \left(r - \frac{1}{r}\right)\sin\theta \qquad (1)$$

Eliminating r and θ from (1) we obtain

To eliminate θ let us put (1) in the form

$$\frac{u}{r + \frac{1}{r}} = \cos\theta \quad (2) \quad and \quad \frac{v}{r - \frac{1}{r}} = \sin\theta \quad (3)$$

Squaring and adding (2) and (3) we get

$$\frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left(r - \frac{1}{r}\right)^2} = 1 \qquad (4)$$

To eliminate r , let us put (1) in the form

$$\frac{u}{\cos\theta} = \left(r + \frac{1}{r}\right) \quad (5) \; and \quad \frac{v}{\sin\theta} = \left(r - \frac{1}{r}\right) \quad (6)$$

Squaring and subtracting (5) and (6), we get

$$\frac{u^2}{(2\cos\theta)^2} - \frac{v^2}{(2\sin\theta)^2} = 1$$

Consider
$$z = re^{i\theta} \qquad |z| = r$$
$$ampz = \theta$$
$$|z| = r = \sqrt{x^2 + y^2} \qquad or \; x^2 + y^2 = r^2$$

This represent a circle with center origin and radius r in the z – plane, when r is constant.

$$amp\, z = \theta \quad \Rightarrow \tan^{-1}(y/x) = \theta$$
$$\frac{y}{x} = \tan\theta$$

Represents a straight line in z –plane when θ is constant.

We shall discuss image in w – plane corresponding to r = constant circle and θ= constant (straight line) in z – plane.

Case 1: Let r = constant.

(2) becomes $\frac{u^2}{a^2}+\frac{v^2}{b^2}=1 \quad where\, a=r+\frac{1}{r}\, and\, b=r-\frac{1}{r}$

This equation represents an ellipse in w – plane with foci $(\pm\sqrt{a^2-b^2},0)$ i.e. $(\pm 2k,0)$ $(\pm 2,0)$

$$\sqrt{a^2-b^2}=\sqrt{\left(r+\frac{1}{r}\right)^2-\left(r-\frac{1}{r}\right)^2}=\sqrt{4k^2}\sum \pm 2k=\pm 2$$

We conclude that $|z|=r=cons\tan t$ in z – plane maps onto ellipse w- plane with foci ($\pm 2,0)$

But $|u|=2|\cos\theta|\le 2$

Image consists of segments of real axis from -2 to 2 or $-2\le u\le 2$

Problems:

1. Find the image in w-plane corresponding to the straight lines $x=a, x=b, y=b\, and\, y=d$ **under the transformation** $w=z^2$ **. Indicate the region with sketches.**

Solution:

Consider $w=e^z$

$$u+iv=e^{x+iy}$$
$$= e^x e^{iy}$$
$$= e^x(\cos y+i\sin y)$$

$u=e^x\cos y, v=e^x\cos y$ ----------(1)

We shall find the region in the w-plane corresponding to the straight lines parallel to the co-ordinate axes in the z-plane.

i.e. $x=a,b$ constants and $y=c,d$ constants.

Eliminate x and y separately from (1)

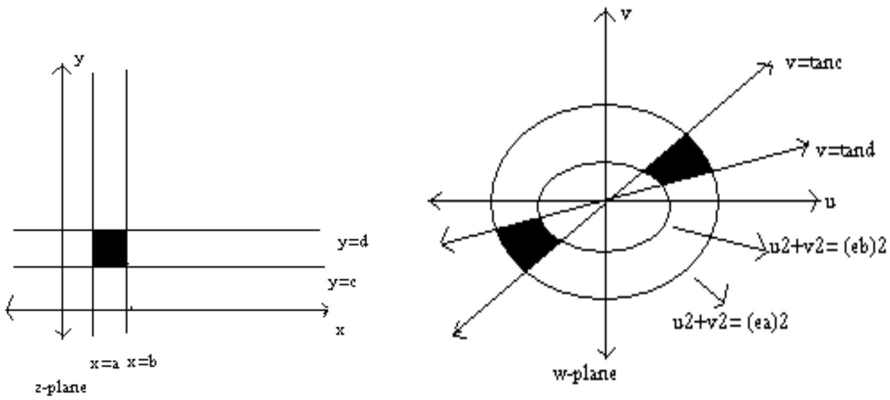
Squaring and adding we get $u^2+v^2=e^{2x}$ (2)

Squaring and dividing we get $\frac{v}{u}=\tan y$ (3)

Use (1) in (2) & (3) we will get,

$u^2+v^2=(e^a)^2,\ u^2+v^2=(e^b)^2, v=\tan cu\ \&\ v=\tan du$

These equations represent a circle with centre at the origin, radius e^a & e^b also a straight lines passing through the origin with the slope $\tan c$ & $\tan d$.
The shaded part in the above figure is required region, which are shown in the figure as fallows.



2. Discuss the transformation $w = e^z$, and show that it transforms the region between the
real axis and a line parallel to real axis $y = \frac{\pi}{2}$ into the upper of the w-plane.

Solution:

Consider $w = e^z$

$$u + iv = e^{x+iy}$$

$$= e^x e^{iy}$$

$$= e^x(\cos y + i \sin y)$$

$u = e^x \cos y, v = e^x \cos y$ ----------(1)

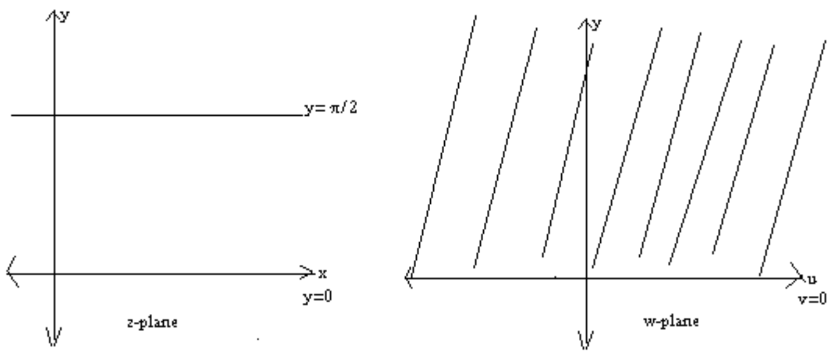
We shall find the region in the w-plane corresponding to the straight lines parallel to the real axis and $y = \frac{\pi}{2}$ in the z-plane.

i.e. $y = 0$ & $y = \frac{\pi}{2}$ (2)

Eliminate y separately from (1)

Squaring and dividing we get $\frac{v}{u} = \tan y$ (3)

Use (2) in (3) we get, v=0 and v=∞.
The above equation v=0 represent a straight line which is a real axis in the w-plane and v=∞ is a line parallel to real axis which is above at infinite distance and is shown below.



CONCLUSION:

The lines $y = 0$ and $y = \frac{\pi}{2}$ in the z-plane maps on to the lines v=0 and v=∞ in the w-plane.

3. Find the image in w-plane corresponding to the straight lines x=a, x=c, y=b &y=d under the transformation $w = z^2$. Indicate the region with sketches.

Solution:

Given $w = z^2$

Let $z = x + iy$ and $w = u + iv$ then

$$= (x + iy)^2$$

$$= x^2 - y^2 + i2xy$$

Implies that $u = x^2 - y^2$ and $v = 2xy$ -----------(1)

We shall find the image in w-plane corresponding to the straight lines parallel to the co-ordinate axes in the z-plane.

i.e. x=a constant (say a & c) and y=a constant (say b and d).-------- (2)

Case: 1 Put x=a in set of equations (1) and eliminate.

$u = a^2 - y^2;\ v = 2ay$ then

$$u = a^2 - \frac{v^2}{4a^2}$$

$v^2 = -4a^2(u - a^2)$ -------------(3)

Similarly, if we put x=c then it gives

$v^2 = -4c^2(u - c^2)$ ----------------(4)

These equations represent a parabolas in w-plane symmetrically about real axis with its vertices (a^2, 0) and

(c^2, 0) and focus at the origin.

Case2: Put y=b in set of equations (1) and eliminate x and y.

$u = x^2 - b^2;\ v = 2bx$ then

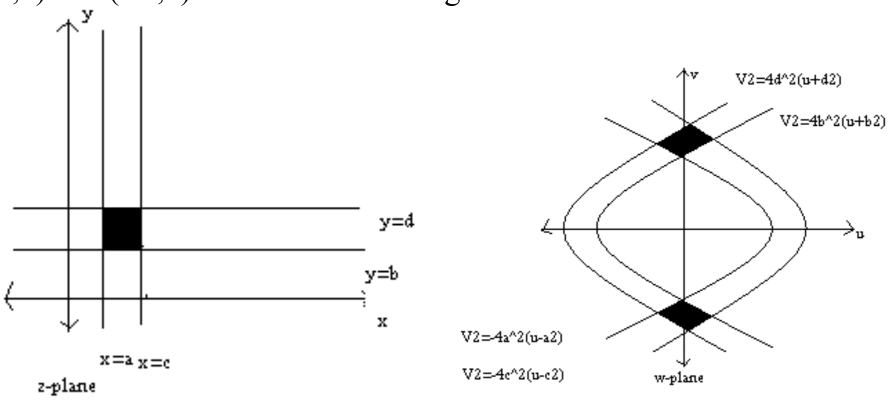
$u = \frac{v^2}{4b^2} - b^2$

$v^2 = 4b^2(u + b^2)$ -----------------(5)

Similarly, if we put y=d then it gives

$v^2 = 4d^2(u - d^2)$ -----------------(6)

These equations represent a parabolas in w-plane symmetrically about real axis with its vertices ($-b^2$,0) and ($-d^2$,0) and focus at the origin. The sketches are shown as follows.



The shaded parts in the above figure is the required region.

4. **Find the transformation of $x + y = 1$ under $w = z^2$.**

Solution:

Given $w = z^2$

Let $z = x + iy$ and $w = u + iv$ then

$= (x + iy)^2$

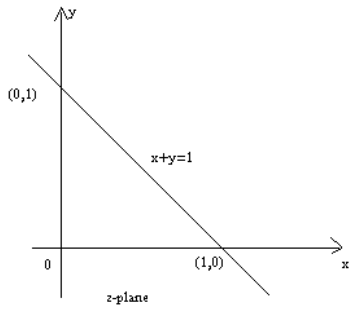
$= x^2 - y^2 + i2xy$

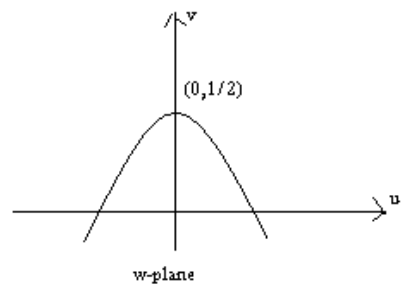
Implies that $u = x^2 - y^2$ and $v = 2xy$ -----------(1)

We shall find the image in w-plane corresponding to the straight line $x + y = 1$ in the z-plane.

i.e. $y = 1 - x$, Then (1) becomes $u = x^2 - (1 - x)^2;\ v = 2x(1 - x)$

$again\ \ 2x = 1 + u\ \ then\ \ v = (1 + u)\left[1 - \left(\frac{1 + u}{2}\right)\right]$ implies that $u^2 = -2(v - \frac{1}{2})$ .This equation represents a parabola in w-plane symmetrically about imaginary axis with its vertex $\left(0, \frac{1}{2}\right)$ and focus at the origin. The figure is shown as follows.





CONCLUSION:

The straight line x+y=1 in z-plane maps on to a parabola with vertex $\left(0,\frac{1}{2}\right)$ and focus at the origin in the w-plane.

5. Find the transformation of $x^2 - y^2 = 1$ under $w = z^2$.

Solution:

Given $w = z^2$

Let $z = x + iy$ and $w = u + iv$ then

$$= (x + iy)^2$$

$$= x^2 - y^2 + i2xy$$

Implies that $u = x^2 - y^2$ and $v = 2xy$ -----------(1)

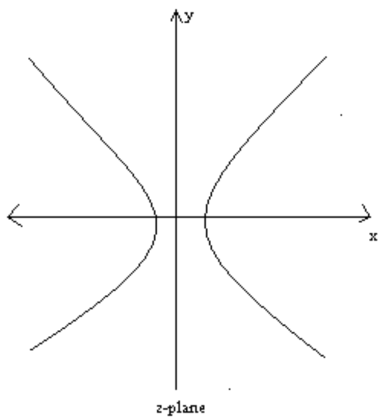
We shall find the image in w-plane corresponding to the hyperbola $x^2 - y^2 = 1$ in the z-plane.

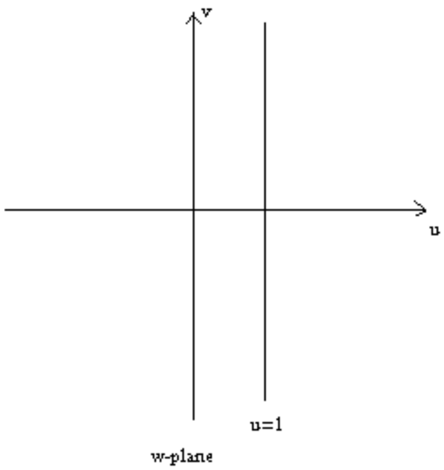
Given $x^2 - y^2 = 1$------------(2)

Then sub (2) in (1) we get $u = 1$

This equation represents a straight line u=1 in w-plane. The figure is shown as follows.

CONCLUSION: The hyperbola $x^2 - y^2 = 1$ in the z-plane maps on to the line u=1 in the w-plane





6. Find he region in the w-plane bounded by the lines x=1, y=1& $x+y=1$ **under the transformation** $w=z^2$**. Indicate the region with sketches.**

Solution:

We shall find the image in w-plane corresponding to the straight line $x+y=1$ in the z-plane.

Given $w=z^2$

Let $z=x+iy$ and $w=u+iv$ then

$$=(x+iy)^2$$
$$=x^2-y^2+i2xy$$

Implies that $u=x^2-y^2$ and $v=2xy$ -----------(1)

Case: 1 Put x=1 in set of equations (1) and eliminate.

$$u=1-y^2;\ v=2y\ \ then$$
$$u=1-\frac{v^2}{4}$$
$$v^2=-4(u-1)\text{----------------------(2)}$$

This equation represents a parabola in w-plane symmetrically about real axis with its vertex (1,0) and focus at the origin.

Case 2: Put y=b in set of equations (1) and eliminate x and y.

$$u=x^2-1;\ v=2x\ \ \text{then}$$
$$u=\frac{v^2}{4}-1$$
$$v^2=4(u+1)\text{-----------------(3)}$$

This equation represents a parabola in w-plane symmetrically about real axis with its vertex (a^2,0) and focus at the origin.

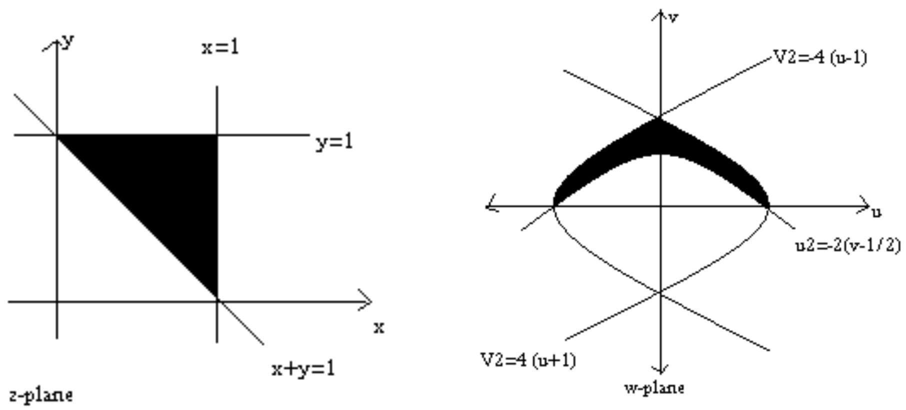
Case 3: $y=1-x$, Then (1) becomes $u=x^2-(1-x)^2;\ v=2x(1-x)$

again $\quad 2x = 1 \text{ then } v = (1+u)\left(1-\left(\frac{1+u}{2}\right)\right)$

implies that $\quad u^2 = -2\left(v - \frac{1}{2}\right)$

This equation represents a parabola in w-plane symmetrically about imaginary axis with its vertex $\left(0, \frac{1}{2}\right)$ and focus at the origin. The figure is shown as follows.

The figures are shown as follows.



Conclusion: The straight lines x=1, y=1& x+y=1 in the z-plane maps on to the parabolas $v^2 = -4(u-1)$, $v^2 = 4(u+1)$ & $u^2 = -2\left(v - \frac{1}{2}\right)$ in the w-plane respectively.

7. Show that the circle $|z - a| = r$ is mapped to a limacon or cardioids under $w = z^2$.

Solution:

Given $|z - a| = r$ represent a circle with radius r and center at (a, 0) in z-plane.

$$z - a = re^{i\theta}$$
$$z = a + re^{i\theta}$$

Then
$$\begin{aligned} w &= z^2 \\ &= \left(a + re^{i\theta}\right)^2 \\ &= a^2 + 2are^{i\theta} + r^2 e^{2i\theta} \end{aligned}$$
$$w - a^2 = 2are^{i\theta} + r^2 e^{2i\theta}$$

Add r^2 on both sides

$$w - (a^2 - r^2) = r^2 + r^2 e^{2i\theta} + 2are^{i\theta}$$
$$w - (a^2 - r^2) = r^2(1 + e^{2i\theta} + 2are^{i\theta})$$
$$w - (a^2 - r^2) = 2re^{i\theta}\left[a + \frac{r}{2}\left(e^{i\theta} + e^{-i\theta}\right)\right]$$
$$w - (a^2 - r^2) = 2re^{i\theta}(a + r\cos\theta)$$

Suppose $w-(a^2-r^2)=\mathrm{Re}^{i\phi}$ then the above equation becomes $\mathrm{Re}^{i\phi}=2re^{i\theta}(a+r\cos\theta)$ so that the pole in w-plane is at the point (a^2-r^2).

$R(\cos\phi+i\sin\phi)=2r(a+r\cos\theta)(\cos\phi+i\sin\phi)$.

$R\cos\phi=2r(a+r\cos\theta)\cos\theta$; -------------(1)

$R\sin\phi=2r(a+r\cos\theta)\sin\theta$; --------------(2)

Squaring and adding the above equations, we get

$R^2=[2r(a+r\cos\theta)]^2$;

$R=2r(a+r\cos\theta)$ -------------------------(3)

Dividing (2) by (1) we get

$\frac{R\sin\phi}{R\cos\phi}=\tan\theta$

Therefore θ=φ.------------------------------ (4)

Using (4) in (3) we get

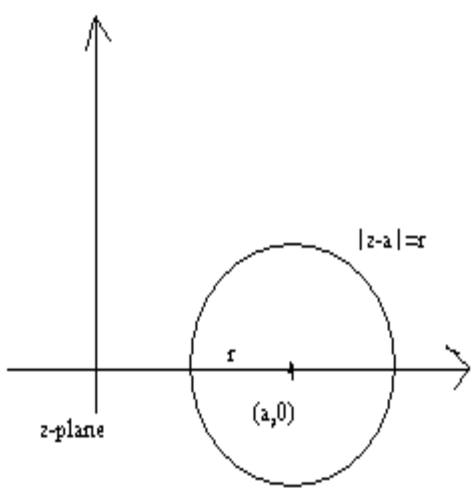
$R=2r(a+r\cos\phi)$ -----------------------------(5)

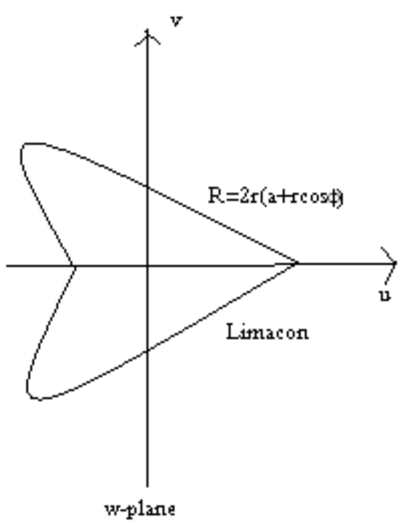
The above equation represents a limacon in the w-plane.

(Where a>r>0)

[Standard form of limacon is given by $r=a+b\cos\theta$]

The figure as follows





Conclusion: The circle with centre a and radius r in the z-plane is mapped onto limacon in w-plane.

8. If $w=z^2$ sketch the family of the curves u=a and v=b. show that the two family of curves intersect orthogonally.

Solution:

Given $w=z^2$

Let $z = x + iy$ and $w = u + iv$ then

$$= (x+iy)^2$$
$$= x^2 - y^2 + i2xy$$

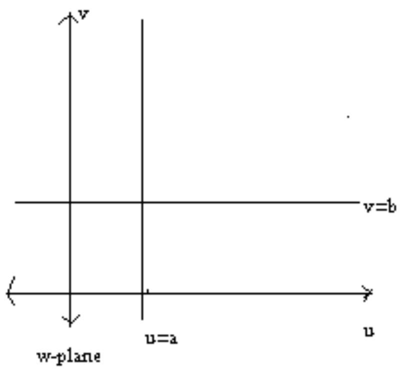
Implies that $u = x^2 - y^2$ and $v = 2xy$ -----------(1)

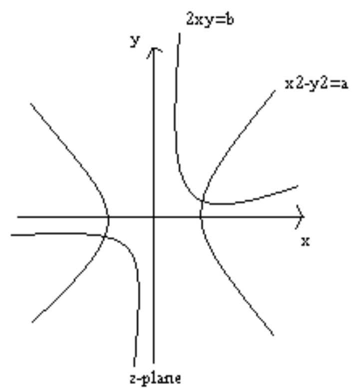
Given that u=a and v=b, these equations represent the straight line in the w-plane parallel to the coordinate axes (2)

Use (2) in (1) we get

That implies x^2-y^2=a and 2xy=b (3)

These equations represent rectangular hyperbolas in the z-plane. The figures are shown as follows.





Conclusion: The straight lines u=a and v=b in the w-plane maps on to rectangular hyperbolas x^2-y^2=a and 2xy=b in the z-plane respectively.

9. Find the image of circles $|z| = 1 \& |z| = 2$ (equivalently $x^2 + y^2 = 1 \ \& \ x^2 + y^2 = 2$) under $w = z + \frac{1}{z}$.

Solution:

Let $z = re^{i\theta}$ then $w = u + iv$

$$u + iv = re^{i\theta} + \frac{1}{r}e^{-i\theta}$$

$$u + iv = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

$$\Rightarrow u = \left(r + \frac{1}{r}\right)\cos\theta \ \ and \ \ v = \left(r - \frac{1}{r}\right)\sin\theta \qquad (1)$$

Eliminating r and θ from (1) we obtain

To eliminate θ let us put (1) in the form

$$\frac{u}{r+\frac{1}{r}}=\cos\theta \quad (2) \quad and \quad \frac{v}{r-\frac{1}{r}}=\sin\theta \quad (3)$$

Squaring and adding (2) and (3) we get

$$\frac{u^2}{\left(r+\frac{1}{r}\right)^2}+\frac{v^2}{\left(r-\frac{1}{r}\right)^2}=1 \qquad (4) \quad r\neq 1$$

Case: 1 Given $|z|=1$

From (1) $u=2\cos\theta$ & $v=0$.

In w-plane $v=0$ represent u-axis and $|u|=2|\cos\theta|\leq 2$ or $-2\leq u\leq 2$.

Therefore $|z|=1$ maps on to the segment of real axis from -2 to 2 in w-plane.

Let θ=constant

(3) is of the form $\frac{u^2}{A^2}-\frac{v^2}{B^2}=1,\ where\ A=2\cos\theta,\ B=2\sin\theta.$

This equation represents a hyperbola in w-plane with foci $\left(\pm\sqrt{A^2+B^2},0\right)=(\pm 2,0)$

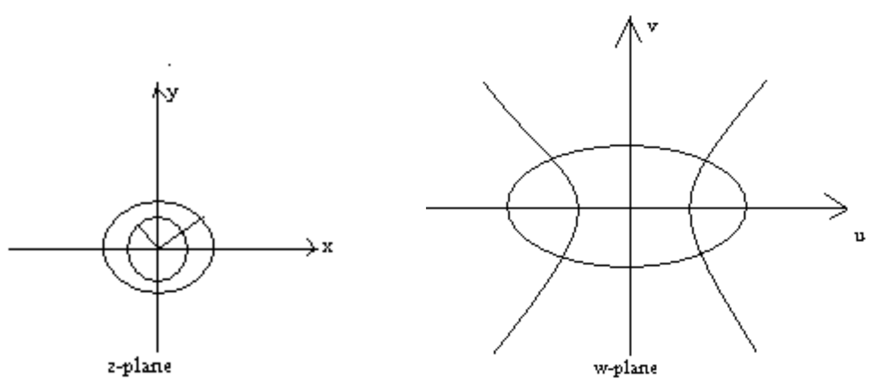
Case: 2 $|z|=2\ \ or\, r=2.$

$$\frac{u^2}{\left(\frac{5}{2}\right)^2}+\frac{v^2}{\left(\frac{3}{2}\right)^2}=1$$

This equation represents an ellipse in w-plane.

Conclusion: The circle maps on to ellipse in w-plane.

The figures as shown follows.



Since both conics (ellipse and hyperbola) have same foci hence they called confocal conics.

4.9 BILINEAR TRANSFORMATION

The transformation $w = \dfrac{az+b}{cz+d}$ *where* a, b, c & d are complex constants such that $ad - bc \neq 0$ is called a bilinear transformation.

Note:

1. The condition $ad - bc \neq 0$ ensures that the conformal mapping property of the bilinear transformation.

$$w = \frac{az+b}{cz+d} \quad \frac{dw}{dz} = \frac{(cz+d)-(az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2}$$

Implies that $ad - bc \neq 0 \Rightarrow \dfrac{dw}{dz} \neq 0$

Therefore, the bilinear transformation (B.T) is conformal.

2. B.T is also called Mobius transformation.

3. If z_1, z_2, z_3 & z_4 are four distinct points then the ratio $\dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$ is called as the cross ratio of these points and it is denoted by (z_1, z_2, z_3, z_4).

4. A B.T preserves cross ratio of four points z_1, z_2, z_3, z_4 be four distinct points in the z-plane. Let w_1, w_2, w_3, w_4 be corresponding images of points in w-plane.

Then $\dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = \dfrac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)}$

Theorem 1: The bilinear transformation preserves the cross ratio. i.e., the cross ratio of 4 points remains invariant under a bilinear transformation.

Proof:

Let z_1, z_2, z_3, z_4 be four distinct points in the z-plane and w_1, w_2, w_3, w_4 be corresponding images of points in w-plane under the bilinear transformation ; $ad - bc \neq 0$.

We have to prove that $w_1, w_2, w_3, w_4 = z_1, z_2, z_3, z_4$.

i.e $\dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = \dfrac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)}$

$$w_1 - w_2 = \frac{az_1 + b}{cz_1 + d} - \frac{az_2 + b}{cz_2 + d} = \frac{(az_1 + b)(cz_2 + d) - (az_2 + b)(cz_1 + d)}{(cz_2 + d)(cz_1 + d)}$$

$$= \frac{ad(z_1 - z_2) - bc(z_1 - z_2)}{(cz_2 + d)(cz_1 + d)} = \frac{(ad - bc)(z_1 - z_2)}{(cz_2 + d)(cz_1 + d)}$$

Similarly $w_2 - w_3 = \dfrac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}$

$$w_3 - w_4 = \frac{(ad-bc)(z_3-z_4)}{(cz_3+d)(cz_4+d)}$$

$$w_4 - w_1 = \frac{(ad-bc)(z_4-z_1)}{(cz_4+d)(cz_1+d)}$$

Consider $\dfrac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \dfrac{\dfrac{(ad-bc)(z_1-z_2)}{(cz_2+d)(cz_1+d)} \times \dfrac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}}{\dfrac{(ad-bc)(z_3-z_4)}{(cz_3+d)(cz_4+d)} \times \dfrac{(ad-bc)(z_4-z_1)}{(cz_4+d)(cz_1+d)}} = \dfrac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$

Hence bilinear transformation preserves the cross ratio.

Theorem 2: A bilinear transformation transforms circles into circles or straight lines.

Proof :

The equation of the circle with z_1 *and* z_2 as inverse points is given by proof of this result

$$\left|\frac{z-z_1}{z-z_2}\right| = K \quad where\ K \neq 1$$

Let $w = \dfrac{az+b}{cz+d}$ be the bilinear transformation.

$$w(cz+d) = az+b$$

$$w(cz-a) = b - wd$$

$$\therefore z = \frac{dw-b}{a-cw}$$

The equation of circle becomes

$$\left|\frac{\dfrac{dw-b}{a-cw} - z_1}{\dfrac{dw-b}{a-cw} - z_2}\right| = K \Rightarrow \left|\frac{w(d+cz_1)-(az_1+b)}{w(d+cz_2)-(az_2+b)}\right| = K \Rightarrow \left|\frac{(d+cz_1)}{(d+cz_2)}\right| \left|\frac{w - \dfrac{(az_1+b)}{(cz_1+d)}}{w - \dfrac{(az_2+b)}{(cz_2+d)}}\right| = K$$

$$i.e \left|\frac{w-p}{w-q}\right| = K' \quad where\ K' = K\left|\frac{(cz_2+d)}{(cz_1+d)}\right|$$

$$p = \frac{(az_1+b)}{(cz_1+d)}, \quad q = \frac{(az_2+b)}{(cz_2+d)}$$

If $K^{'} \neq 1$ then $\left|\frac{w-p}{w-q}\right| = K^{'}$ represents circle in w-plane with p and q as inverse points.

If $K^{'} = 1$ then $\left|\frac{w-p}{w-q}\right| = K^{'}$ represents a straight line.

Hence the bilinear transformation transforms the given circle into circles or a straight line.

Note: Invariant Points
If the point Z maps onto itself i.e. $w=z$ under the BLT then the point is said to be invariant.

Problems:

1. Find the BLT which maps the points z=2, i,-2 to w=1, i,-1. Hence find invariant points.

Solution:

we know that the definition of cross ratio

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} = \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)}$$

Given that $z_1 = z, z_2 = 1, z_3 = i, z_4 = -1$

$$w_1 = z, w_2 = 2, w_3 = i, w_4 = -2$$

$$\frac{(w-2)(i+2)}{(-2-w)(2-i)} = \frac{(z-1)(i+1)}{(-1-z)(i-1)} \Rightarrow \frac{(w-2)}{(w+2)} = \frac{(z-1)(i+1)(2-i)}{(1+z)(i-1)(i+2)}$$

$$\frac{(w-2)}{(w+2)} = \frac{(z-1)(3+i)}{(1+z)(3-i)} \Rightarrow w-2 = w\frac{(z-1)(3+i)}{(1+z)(3-i)} + 2\frac{(z-1)(3+i)}{(1+z)(3-i)}$$

$$w - w\frac{(z-1)(3+i)}{(1+z)(3-i)} = 2\frac{(z-1)(3+i)}{(1+z)(3-i)} + 2 \Rightarrow w\left(\frac{(1+z)(3-i)-(z-1)(3+i)}{(1+z)(3-i)}\right) = 2\left(\frac{(1+z)(3-i)+(z-1)(3+i)}{(1+z)(3-i)}\right)$$

$$2w(3-zi) = 2(6z-2i)$$

$$w = \frac{(6z-2i)}{3-zi} \textit{ is the required transformation.}$$

To find the invariant points put w=z

$$-iz^2 - 3z + 2i = 0$$

$$z = \frac{3 \pm \sqrt{9-8}}{-2i}$$

$$z = 2i, i \text{ are the invariant points.}$$

2. Find the BLT which maps the points $z = 0, i, \infty$ onto $w = 1, -i, -1$ respectively. Hence find invariant points.

Solution:

Given that $z_1 = 0, z_2 = i, z_3 = \infty \quad w_1 = 1, w_2 = -i, w_3 = -1$

we know that the definition of cross ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-1)(-i+1)}{(1+i)(-1-w)} = \frac{(z-0)z_3\left(\frac{z_2}{z_3}-1\right)}{(z_1-i)z_3\left(1-\frac{z}{z_3}\right)} \Rightarrow \frac{(w-1)(-i+1)}{(1+i)(-1-w)} = \frac{z(-1)}{-i(1)} = \frac{z}{i}$$

$$\frac{(w-1)}{(1+w)} = \frac{z(1+i)}{(1-i)} \Rightarrow (w-1) = -z(1+w)$$

$$\therefore \quad w = \frac{1-z}{1+z}$$

To find the invariant points

Put $w=z$ in $w = \frac{1-z}{1+z}$ we get

$$z = \frac{1-z}{1+z} \Rightarrow z^2 + 2z - 1 = 0$$

$z = -1+\sqrt{2}$ *and* $-1-\sqrt{2}$ are the invariant points.

3. Find the BLT which maps the points $z = -1, 1, \infty$ onto $w = -i, -1, i$ respectively.

Solution:

Given that $z_1 = -1, z_2 = 1, z_3 = \infty \quad w_1 = -i, w_2 = -1, w_3 = i$

we know that the definition of cross ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w+i)(-1-i)}{(-i+1)(i-w)} = \frac{(z-z_1)z_3\left(\frac{z_2}{z_3}-1\right)}{(-1-1)z_3\left(1-\frac{z}{z_3}\right)}$$

$$\frac{(w+i)(-1-i)}{(-i+1)(i-w)} = \frac{(z+1)(-1)}{(-2)(1-0)} \Rightarrow \frac{(w+i)(-1-i)}{(-i+1)(i-w)} = \frac{(z+1)}{(2)}$$

$$\frac{(w+i)}{(w-i)} \times i = \frac{(z+1)}{(2)} \Rightarrow \frac{(wi+i^2)}{(w-i)} = \frac{(z+1)}{(2)}$$

$$2(wi-1) = (w-i)(z+1) \Rightarrow w(2i-z-1) = -i(z+1)+2$$

$$w = \frac{-iz-i+2}{-z+2i-1} \Rightarrow w = \frac{-(iz+i-2)}{-(z-2i+1)}$$

$$w = \frac{iz+(i-2)}{z+(1-2i)}$$

4. Find the map of real axis of z-plane in w-plane under the transformation $w = \dfrac{1}{z+i}$.

Solution:

Given $w = \dfrac{1}{z+i}$ and y=0

$$z + i = \frac{1}{w}$$

$$z = \frac{1}{w} - i$$

Then $x + iy = \dfrac{1}{u+iv} - i$

$$= \frac{u - iv}{(u+iv)(u-iv)} - i$$

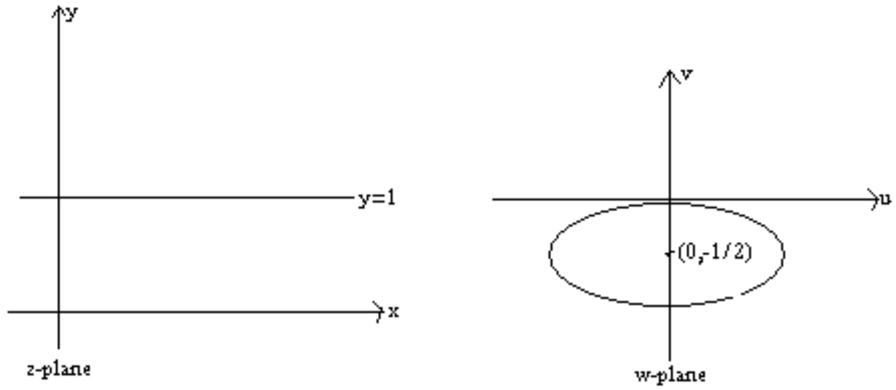
$$= \frac{u}{u^2+v^2} + i\left[\frac{-v}{u^2+v^2} - 1\right]$$

Therefore $x = \dfrac{u}{u^2+v^2}, \quad y = \dfrac{-v}{u^2+v^2} - 1$

We have y=0 , So $u^2 + v^2 + v = 0 \Rightarrow (u-0)^2 + (v+\frac{1}{2})^2 = \left(\frac{1}{2}\right)^2$

This equation represents a circle in w-plane with $\left(0,-\frac{1}{2}\right)$ *and radius* $\frac{1}{2}$.

The figure shown as follows.



Conclusion: The line y=1 in the z-plane maps on to the circle $u^2 + v^2 + v = o$ in the w-plane.

5. Show that the transformation $w = \frac{2z+3}{z-4}$ maps the circle $x^2 + y^2 - 4x = 0$ into a straight line.

Solution:

Given $w = \frac{2z+3}{z-4}$

$$w(z-4) = 2z+3$$
$$wz - w4 = 2z+3$$
$$z(w-2) = 4w+3$$
$$z = \frac{4w+3}{(w-2)}$$

Given $x^2 + y^2 - 4x = 0$

Then we know that $z = x + iy; \bar{z} = x - iy$

Therefore $z\bar{z} = x^2 + y^2; z + \bar{z} = 2x$

(1) Becomes $z\bar{z} - 2(z + \bar{z}) = 0$

$$\left(\frac{4w+3}{w-2}\right)\left(\frac{4\bar{w}+3}{\bar{w}-2}\right) - 2\left[\left(\frac{4w+3}{w-2}\right) + \left(\frac{4\bar{w}+3}{\bar{w}-2}\right)\right] = 0$$
$$(4w+3)(4\bar{w}+3) - 2(4w+3)(\bar{w}-2) - 2(4\bar{w}+3)(w-2) = 0$$
$$16w\bar{w} + 12\bar{w} + 12w + 9 - (8w\bar{w} - 16w + 6\bar{w} - 12) - (8w\bar{w} - 16\bar{w} + 6w - 12) = 0$$
$$22\bar{w} + 22w + 33 = 0$$
$$2(\bar{w} + w) + 33 = 0$$
$$4u + 3 = 0$$

This represents a straight line in w-plane.

EXERCISE:

1. Show that $f(z) = \sin z$ is analytic and hence find $f(z)'$. (Ans: $f(z)' = \cos z$)

2. Show that $f(z) = \sinh z$ is analytic and hence find $f(z)'$. (Ans: coshz)

3. Show that $f(z) = e^x(\sin y + i \sin y)$ is holomorphic and hence find $f(z)'$. (Ans: e^z)

4. Show that $f(z) = z^2 + 2z$ is analytic and hence find $f(z)'$. (Ans : $2z + 2$)

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5. Show that $f(z)=z^2$ is analytic and hence find $f(z)'$. (Ans : $z^2(1+\log z)$)

6. Construct the analytic function whose imaginary part is $-\frac{\sin\theta}{r}$. (Ans: $\frac{1}{r}\cos\theta$). Hence find its real part.

7. Construct the analytic function whose imaginary part is $\left(r-\frac{1}{r}\right)\sin\theta$. (Ans : $u=\left(r+\frac{1}{r}\right)\cos\theta;\quad z+\frac{1}{z}$)

8. Construct the analytic function whose real part is $\frac{x^4-y^4-2x}{x^2+y^2}$. Hence find its imaginary part.

(Ans: $f(z)=z^2-\frac{2}{z}+c;\quad v=\frac{2xy^3+2x^3y+2y}{x^2+y^2}$)

9. Construct the analytic function whose real part is $e^{-x}\{(x^2-y^2)\cos y+2xy\sin y\}$. (Ans: $f(z)=z^2e^{-z}+c$)

10. Construct the analytic function whose real part is $r^2\cos 2\theta$. (Ans: $f(z)=z^2+c$)

11. Show that the function $u=\frac{1}{r}\cos\theta$ is harmonic and find its harmonic conjugate. Also find the analytic function. (Ans: $\frac{-\sin\theta}{r}$; $f(z)=\frac{1}{z}$).

12. Show that the function $v=\cos x\sinh y$ is harmonic and find its harmonic conjugate. Also find the analytic function. (Ans: $f(z)=\sin z,\ u=\sin x\cosh y$)

13. Show that the function $u=(x-1)^3-3xy^2+3y^2$ is harmonic and find its harmonic conjugate. Also find the analytic function. (Ans: $(z-1)^3;\quad v=x^2-y^2+2(2x+y)$)

14. Show that the function $u=\frac{1}{2}\log(x^2+y^2),\ x\neq 0, y\neq 0$ is harmonic and find its harmonic conjugate. Also find the analytic function. (Ans: $\tan^{-1}\left(\frac{y}{x}\right)+c$)

15. Find the analytic function $f(z)=u+iv$ given $u-v=(x-y)+e^x(\cos y+\sin y)$. (Ans: $z+e^z$)

16. Find the analytic function $f(z)=u+iv$ given $u+v=\frac{2\sin 2x}{e^{2y}-e^{-2y}-2\cos 2x}$. (Ans : $\cot z$)

17. Discuss the transformation on $w = e^z$ w.r.t lines represented by the coordinate axes in the z-plane.

18. Under $w = z^2$, find the image of square region bounded by the lines $x = 1, x = 2,\ y = 1, y = 2.$

19. Discuss the transformation $w = z + \frac{1}{z}$ with respect to the curves $r =$ constant ($\neq 0$) $\theta =$ constant ($\neq 0$). Hence find the image of $r = 1$ and $\theta = \pi$ under this transformation.

20. Find the bilinear transformation that map the points $\infty, i, 0$ onto the points $0, i, \infty$ respectively. (Ans: $w = -\frac{1}{z}$)

21. Find the bilinear transformation that map the points $1, i, -1$ onto the points $2, i, -2$ respectively. Hence find the invariant points. (Ans: $w = \frac{(6z - 2i)}{(iz + 3)}$; invatiant points are $2i, i.$)

22. Find the bilinear transformation that map the points $1, i, -1$ onto the points $i, 0, -i$ respectively. Find the image of the region $|z| \leq 1$ under this transformation.(Ans: $w = \frac{i - z}{i + z}$)

23. Find the bilinear transformation that map the points $z = 0, 1, \infty$ onto the points $w = -5, -1, 3$ respectively. Also find the fixed points of this transformation. (Ans: $w = \frac{-3z + 5i}{-iz + 1}$,invaiant points are $i, -5i.$)