# Module - IV

## Linear Transformation

**Definition of Linear Transformation :→**

Let $U$ and $V$ be two vectors spaces over the field $F$. The mapping $T : U \longrightarrow V$ is called as linear transformation. if

(i) $T(\alpha + \beta) = T(\alpha) + T(\beta) \quad \forall \ \alpha, \beta \in U$

(ii) $T(c\alpha) = cT(\alpha) \quad \forall \ c \in F, \ \alpha \in U$

Theorem 1:→

A mapping $T: U \to V$ is a linear transformation iff $T(c_1\alpha + c_2\beta) = c_1 T(\alpha) + c_2 T(\beta)$ $\forall$ $c_1, c_2 \in F$ and $\alpha, \beta \in U$

proof:→ Let $T$ is linear then by the definition,

$$T(c_1\alpha + c_2\beta) = T(c_1\alpha) + T(c_2\beta)$$

$$\boxed{T(c_1\alpha + c_2\beta) = c_1 T(\alpha) + c_2 T(\beta)}$$

Conversely

If $T(c_1\alpha + c_2\beta) = c_1T(\alpha) + c_2T(\beta)$ $\rightarrow$ (i)

Now particularly if $c_1 = c_2 = 1$ then,

eq^n (i) becomes

$$T(\alpha + \beta) = T(\alpha) + T(\beta)$$

Further if $c_1 = c$ and $c_2 = 0$ then

eq^n (i) becomes $T(c\alpha) = cT(\alpha)$

This proves $T$ is linear

Hence the proof

**Theorem 2** :→ If $T: U \longrightarrow V$ is a linear mapping then

i) $T(0) = 0'$ where $0$ and $0'$ be zero vectors in $U$ and $V$ respectively

ii) $T(-\alpha) = -T(\alpha) \quad \forall \alpha \in U$

iii) $T(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n) = c_1T(\alpha_1) + c_2T(\alpha_2) + \cdots + c_n T(\alpha_n)$

where $c_1, c_2, \ldots, c_n \in F$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in U$.

**proof** :→ (i) consider $T(\alpha + 0) = T(\alpha) + T(0)$

$$\cancel{T(\alpha)} + 0' = \cancel{T(\alpha)} + T(0)$$

$$\boxed{0' = T(0)} \quad [\text{By left cancellation law}]$$

(ii) consider $T(\alpha + (-\alpha)) = T(\alpha) + T(-\alpha)$

$$\Rightarrow 0' = T(\alpha) + T(-\alpha)$$

$\Rightarrow T(-\alpha)$ is the inverse of $T(\alpha)$

i.e., $\boxed{T(-\alpha) = -T(\alpha)}$

(iii) To prove the (iii) result we use mathematical induction [Step 1 : Basic step
Step 2 : Induction step]

Given statement

$P(n)$: $T(c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + \dots + c_n\alpha_n) = c_1T(\alpha_1) + c_2T(\alpha_2) + \dots + c_nT(\alpha_n)$

Basic step :-

check the result for $n=1$ i.e., Basic step

$$T(c_1\alpha_1) = c_1 T(\alpha_1)$$

The result is true for $n=1$

Induction step :

Assume the result is true for $k$ i.e., $n=k$

$$T(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_k\alpha_k) = c_1 T(\alpha_1) + c_2 T(\alpha_2) + \cdots + c_k T(\alpha_k)$$

Now we have to prove this result is true for $k+1$

i.e., $n=k+1$

... for $k+1$

$$T\{c_1\alpha_1 + c_2\alpha_2 + \cdots + c_k\alpha_k) + (c_{k+1}\alpha_{k+1})\}$$

$$= T(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_k\alpha_k) + T(c_{k+1}\alpha_{k+1})$$

$$= c_1T(\alpha_1) + c_2T(\alpha_2) + \cdots + c_kT(\alpha_k) + c_{k+1}T(\alpha_{k+1})$$

Thus the result is true for $k+1$ i.e., $n = k+1$

Hence the result is true for $\forall n$

## Probleme

1. Define $T : V_3(R) \longrightarrow V_3(R)$ by $T(x_1, x_2, x_3) = (0, x_2, x_3)$

Show that $T$ is a linear transformation

Sol$^n$ Let $\alpha = (x_1, x_2, x_3)$ and $\beta = (y_1, y_2, y_3) \in V_3(R)$

$$T(\alpha + \beta) = T((x_1, x_2, x_3) + (y_1, y_2, y_3))$$

$$= T(x_1 + y_1, x_2 + y_2, x_3 + y_3)$$

$$= (0, x_2 + y_2, x_3 + y_3)$$

$$= (0, x_2, x_3) + (0, y_2, y_3)$$

$$= T(x_1, x_2, x_3) + T(y_1, y_2, y_3)$$

$$\boxed{T(\alpha+\beta) = T(\alpha)+T(\beta)}$$

Now consider

$$T(c\alpha) = T(c(x_1, x_2, x_3))$$

$$= T(cx_1, cx_2, cx_3)$$

$$= (0, cx_2, cx_3)$$

$$= c(0, x_2, x_3)$$

$$\boxed{T(c\alpha) = cT(\alpha)}$$

Hence 'T' is a linear transformation

2. If $T : V_1(R) \longrightarrow V_3(R)$ is defined by $T(x) = (x, x^2, x^3)$

Verify where T is linear or not

Sol$^n$ Let $\forall\ x, y \in V_1(R)$

$$T(x+y) = [(x+y), (x+y)^2, (x+y)^3] \longrightarrow (1)$$

$$T(x) + T(y) = (x, x^2, x^3) + (y, y^2, y^3)$$

$$T(x) + T(y) = (x+y, x^2+y^2, x^3+y^3) \longrightarrow (2)$$

Compare (1) & (2)

$$T(x+y) \neq T(x) + T(y)$$

$\therefore$ T is not linear

$\left\{ \because (x+y)^2 \neq x^2+y^2 \right.$ and $\left. (x+y)^3 \neq x^3+y^3 \right\}$

3. Find the linear transformation $f : R^2 \to R^2$ such that $f(1,0) = f(1,1)$ and $f(0,1) = (-1,2)$

Sol$^n$

$$(x, y) = x(1,0) + y(0,1)$$

$$f(x,y) = f[x(1,0) + y(0,1)]$$

$$= x f(1,0) + y f(0,1)$$

$$= x(1,1) + y(-1,2)$$

$$= (x,x) + (-y, 2y)$$

$$\boxed{f(x,y) = (x-y, x+2y)}$$

4. Find the linear transformation $f: R^2 \to R^2$ such that $f(1,1) = (0,1)$ and $f(-1,1) = (3,-2)$

Sol$^n$ Consider

$$(x, y) = c_1(1,1) + c_2(-1,1) \longrightarrow (1)$$

$$(x, y) \quad (c_1 - c_2 , c_1 + c_2)$$

Now

$$c_1 - \cancel{c_2} = x$$

$$c_1 + \cancel{c_2} = y$$

Adding

$$2c_1 = x + y$$

$$\boxed{c_1 = \frac{x+y}{2}}$$

and $c_2 = y - c_1$

$$c_2 = y - \left(\frac{x+y}{2}\right)$$

$$\boxed{c_2 = \frac{y-x}{2}}$$

Eq (1) becomes

$$(x, y) = \left(\frac{x+y}{2}\right)(1,1) + \left(\frac{y-x}{2}\right)(-1,1)$$

Applying 'f' on both side we get

$$f(x, y) = f\left[\left(\frac{x+y}{2}\right)(1,1) + \left(\frac{y-x}{2}\right)(-1,1)\right]$$

$$= \left(\frac{x+y}{2}\right) f(1,1) + \left(\frac{y-x}{2}\right) f(-1,1)$$

$$= \left(\frac{x+y}{2}\right)(0,1) + \left(\frac{y-x}{2}\right)(3,2)$$

$$= \left(0, \frac{x+y}{2}\right) + \left(\frac{3}{2}(y-x)\quad \frac{2}{2}(y-x)\right)$$

$$= \left[0 + \frac{3}{2}(y-x),\quad \frac{x+y}{2} + \frac{2}{2}(y-x)\right]$$

$$\boxed{f(x,y) = \left[\frac{3}{2}(y-x),\ \frac{3y-x}{2}\right]}$$

5. Is there a linear map $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ for which $T(2,2) = (4,-6)$ and $T(5,5) = (2,-3)$?

Sol: consider $(5,5) = 5(1,1) = \frac{5}{2}(2,2)$

$$\Rightarrow (5,5) = \frac{5}{2}(2,2)$$

$$\Rightarrow T(5,5) = T\left(\frac{5}{2}(2,2)\right)$$

$$\Rightarrow \quad T(5,5) = \frac{5}{2}\, T(2,2)$$

$$\Rightarrow \quad T(5,5) = \frac{5}{2}(4,-6)$$

$$\Rightarrow \quad T(5,5) = \frac{1}{2}(20,-30)$$

$$\Rightarrow \quad T(5,5) = (10,-15) \rightarrow (1)$$

But $T(5,5) = (2,-3)$ is given $\rightarrow (2)$

From (1) and (2) it is clear

T is not a linear transformation