# UNIT-2

## Linear Equations

→ Solution of a system of linear Equations

1. Gauss - Elimination
2. Gauss - Seidel
3. Gauss - Jordan

### Gauss - Elimination

Consider the system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$Ax = B$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

$$[A:B] = \left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right].$$

Only Row operations are performed to make lower triangular elements as zero.

→

$$2x + y + 4z = 12$$
$$4x + 11y - z = 33$$
$$8x - 3y + 2z = 20.$$

$$[A:B] = \left[\begin{array}{ccc:c} 2 & 1 & 4 & 12 \\ 4 & 11 & -1 & 33 \\ 8 & -3 & 2 & 20 \end{array}\right].$$

$R_2 = R_2 - 2R_1$

$R_3 = R_3 - 4R_1$

$$= \left[\begin{array}{ccc:c} 2 & 1 & 4 & 12 \\ 0 & 9 & -9 & 9 \\ 0 & -7 & -14 & -28 \end{array}\right]$$

$R_3 = 9R_3 + 7R_2.$

$9(-14) + 7(-9)$

$$= \left[\begin{array}{ccc:c} 2 & 1 & 4 & 12 \\ 0 & 9 & -9 & 9 \\ 0 & 0 & -189 & -189 \end{array}\right]$$

$$2x + y + 4z = 12$$
$$9y - 9z = 9$$
$$-189z = -189$$
$$\Rightarrow \boxed{z = 1}$$
$$y - z = 1$$
$$\boxed{y = 2}$$

$$2x + 2 + 4 = 12.$$
$$2x = 6$$
$$\boxed{x = 3}$$

→ $5x_1 + x_2 + x_3 + x_4 = 4$ — (i)

$x_1 + 7x_2 + x_3 + x_4 = 12$ (ii)

$x_1 + x_2 + 6x_3 + x_4 = -5$ — (iii)

$x_1 + x_2 + x_3 + 4x_4 = -6$ — (iv)

$$[A:B] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 1 & 1 & 6 & 1 & -5 \\ 1 & 7 & 1 & 1 & 12 \\ 5 & 1 & 1 & 1 & 4 \end{array}\right]$$

$R_4 = R_4 - 5R_1$

$R_3 = R_3 - R_1$

$R_2 = R_2 - R_1$

$$= \left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & 6 & 0 & -3 & 18 \\ 0 & -4 & -4 & -19 & 34 \end{array}\right]$$

$R_2 \leftrightarrow R_4$

$$= \left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 0 & -4 & -4 & -19 & 34 \\ 0 & 6 & 0 & -3 & 18 \\ 0 & 0 & 5 & -3 & 1 \end{array}\right]$$

$R_3 = 4R_3 + 6R_2$

$$= \left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 0 & -4 & -4 & -19 & 34 \\ 0 & 0 & -24 & -126 & 276 \\ 0 & 0 & 5 & -3 & 1 \end{array}\right]$$

$-12 + 6(-19)$

0

$R_3 = R_3/6$

$$= \left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 0 & -4 & -4 & -19 & 34 \\ 0 & 0 & -4 & -21 & 46 \\ 0 & 0 & 5 & -3 & 1 \end{array}\right]$$

1
2
-1
-2

$R_4 = 4R_4 + 5R_3$

$$= \left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 0 & -4 & -4 & -19 & 34 \\ 0 & 0 & -4 & -21 & 46 \\ 0 & 0 & 0 & -117 & 234 \end{array}\right]$$

$\cancel{x + y + z +}$

$x_1 + x_2 + x_3 + 4x_4 = -6$

$-4x_2 - 4x_3 - 19x_4 = 34$

$-4x_3 - 21x_4 = 46$

$-117x_4 = 234$

$\boxed{x_4 = -2}$

$-4x_2 + 4 - 19(-2) = 34$

$-4x_2 = -8$

$\boxed{x_2 = 2}$

$-4x_3 - 42 = 46$

$-4x_3 = 4$

$\boxed{x_3 = -1}$

$\cancel{x + 2 - 1 - 8 = -6}$

$x_1 + 2 - 1 - 8 = -6$

$\boxed{x_1 = 1}$

## Gauss - Seidel

$\rightarrow$ $10x + y + z = 12$ $\quad x = \frac{1}{20}(17 - y + 2z)$

$x + 10y + z = 12$

$x + y + 10z = 12.$ $\quad y = \frac{1}{20}(-18 - 3x + 3z)$

$x_1 = \frac{1}{10}(12 - y_0 - z_0)$ $\quad z = \frac{1}{20}(25 - 2x + 3y)$

$y_1 = \frac{1}{10}(12 - x_1 - z_1)$

$z_1 = \frac{1}{10}(12 - x_1 - y_1)$

$x_0 = 0, y_0 = 0, z_0 = 0$

$x_1 = \frac{1}{10}(12 - 0 - 0) = 1.2$

$y_1 = \frac{1}{10}(12 - 0 - 0) = 1.2$

$z_1 = \frac{1}{10}(12 - 0 - 0) = 1.2$

$(x_1, y_1, z_1) = (1.2, 1.2, 1.2)$

$x_2 = \frac{1}{10}(12 - 1.2 - 1.2) = 0.96$

$y_2 = \frac{1}{10}(12 - 1.2 - 1.2) = 0.96$

$z_2 = \frac{1}{10}(12 - 1.2 - 1.2) = 0.96$

$(x_2, y_2, z_2) = (0.96, 0.96, 0.96)$

$x_3 = \frac{1}{10}(12 - 0.96 - 0.96) = 1.008$

$y_3 = 1.008$

$z_3 = 1.008.$

$(x_3, y_3, z_3) = (1.008, 1.008, 1.008)$

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## Gauss Seidel

$\rightarrow$ $10x + y + z = 12$

$x + 10y + z = 12$

$x + y + 10z = 12$

$x = \frac{1}{10}(12 - y - z)$

$y = \frac{1}{10}(12 - x - z)$

$z = \frac{1}{10}(12 - x - y)$

$x_1 = \frac{1}{10}(12 - 0 - 0) = 1.2.$

$y_1 = \frac{1}{10}(12 - 1.2 - 0) = 1.08$

$z_1 = \frac{1}{10}(12 - 1.2 - 1.08) = 0.972.$

$(1.2, 1.08, 0.972)$

$x_2 = \frac{1}{10}(12 - 1.08 - 0.972)$

$= 0.9948$

$y_2 = \frac{1}{10}(12 - 0.9948 - 0.972)$

$= 1.00332$

$z_2 = \frac{1}{10}(12 - 0.9948 - 1.00332)$

$= 1.000188$

$(0.9948,\ 1.00332,\ 1.000188)$

$x_3 = \frac{1}{10}(12 - 1.00332 - 1.000188)$

$= 0.9996492$

$y_3 = \frac{1}{10}(12 - 0.9996492 - 1.000188)$

$= 1.00001628$

$z_3 = \frac{1}{10}(12 - 0.9996492 - 1.00001628)$

$= 1.000033452.$

→ $x + y + 54z = 110$

$27x + 6y - z = 85$

$6x + 15y + 2z = 72.$

$x = \frac{1}{27}(85 - 6y + z)$

$y = \frac{1}{15}(72 - 6x - 2z)$

$z = \frac{1}{54}(110 - x - y)$

$(x_0, y_0, z_0) = (0, 0, 0)$

$x_1 = \frac{1}{27}(85) = 3.148$

$y_1 = \frac{1}{15}(72 - 18.888 - 0) = 3.5408$

$z_1 = \frac{1}{54}(110 - 3.148 - 3.5408) = 6.88741$

$(3.148,\ 3.5408,\ 6.8874)$

$x_2 = \frac{1}{27}(85 - 21.2448 + 6.8874) = 2.6163$

$y_2 = \frac{1}{15}(72 - 15.6978 - 13.7748) = 2.83516$

$z_2 = \frac{1}{54}(110 - 2.6163 - 2.83516) = 1.93608$

$(2.6163,\ 2.83516,\ 1.93608)$

$x_3 = \frac{1}{27}(85 - 15.6978 + 1.93608) = 2.6384$

$y_3 = \frac{1}{15}(72 - 15.8304 - 3.87216) = 3.486496.$

$x + y + 54z = 110$

$8x + 6y - z = 85$

$18x + 15y + 2z = 72$

→ $20x + y - 2z = 17$

$3x + 20y - z = -18$

$2x - 3y + 20z = 25$

→ $5x + 2y + z = 12$

$x + 4y + 2z = 15$

$x + 2y + 5z = 20$

with $(1, 0, 3)$

Gauss Jordan Method

$a_{11}x + a_{12}y + a_{13}z = c_1$

$a_{21}x + a_{22}y + a_{23}z = c_2$

$a_{31}x + a_{32}y + a_{33}z = c_3$

$$[A:B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & : c_1 \\ a_{21} & a_{22} & a_{23} & : c_2 \\ a_{31} & a_{32} & a_{33} & : c_3 \end{bmatrix}$$

① $10x + y + z = 12$

$x + 10y + z = 12$

$x + y + 10z = 12.$

$$[A:B] = \begin{bmatrix} 10 & 1 & 1 & : 12 \\ 1 & 10 & 1 & : 12 \\ 1 & 1 & 10 & : 12 \end{bmatrix}.$$

② $10x + y + z = 12$

$2x + 10y + z = 13$

$x + y + 5z = 7$

$$[A:B] = \begin{bmatrix} 10 & 1 & 1 & : 12 \\ 2 & 10 & 1 & : 13 \\ 1 & 1 & 5 & : 7 \end{bmatrix}$$

$R_2 = 5R_2 - R_1$ $\quad$ $R_3 = 10R_3 - R_1$

$$= \begin{bmatrix} 10 & 1 & 1 & : 12 \\ 0 & 49 & 4 & : 53 \\ 0 & 9 & 49 & : 58 \end{bmatrix}.$$

$R_1 = 49R_1 - R_2$ $\quad$ $R_3 = 49R_3 - 9R_2$

$$= \begin{bmatrix} 490 & 0 & 45 & : 535 \\ 0 & 49 & 4 & : 53 \\ 0 & 0 & 2365 & : 2365 \end{bmatrix} = \begin{bmatrix} 490 & 0 & 45 & : 535 \\ 0 & 49 & 4 & : 53 \\ 0 & 0 & 1 & : 1 \end{bmatrix}$$

$R_3 = R_3/2365$

$R_1 = R_1 - 45R_3$ $\quad$ $R_2 = R_2 - 4R_3$

$$= \begin{bmatrix} 490 & 0 & 0 & : 490 \\ 0 & 49 & 0 & : 49 \\ 0 & 0 & 1 & : 1 \end{bmatrix}$$

$R_1 = R_1/490$

$R_2 = R_2/49$

$$= \begin{bmatrix} 1 & 0 & 0 & : 1 \\ 0 & 1 & 0 & : 1 \\ 0 & 0 & 1 & : 1 \end{bmatrix}$$

$\Rightarrow x = 1, y = 1, z = 1$

(3) $x+y+z=6$,
$x-y+2z=5$
$3x+y+z=8$

$$[A:B]=\left[\begin{array}{ccc:c}1&1&1&6\\1&-1&2&5\\3&1&1&8\end{array}\right]$$

$R_2=R_2-R_1$ $\quad R_3=R_3-3R_1$

$$=\left[\begin{array}{ccc:c}1&1&1&6\\0&-2&1&-1\\0&-2&-2&-10\end{array}\right] \quad R_3=\frac{R_3}{-2}=\left[\begin{array}{ccc:c}1&1&1&6\\0&-2&1&-1\\0&1&1&5\end{array}\right]$$

$R_1=2R_1+R_2$ $\quad R_3=2R_3+R_2$.

$$=\left[\begin{array}{ccc:c}2&0&3&11\\0&-2&1&-1\\0&0&3&9\end{array}\right] \quad \begin{array}{l}R_1=R_1-R_3\\R_2=3R_2-R_3\end{array}$$

$$=\left[\begin{array}{ccc:c}2&0&0&2\\0&-6&0&-12\\0&0&3&9\end{array}\right] \quad \begin{array}{l}R_1=R_1/2\\R_2=R_2/-6\\R_3=R_3/3\end{array}$$

$$=\left[\begin{array}{ccc:c}1&0&0&1\\0&1&0&2\\0&0&1&3\end{array}\right] \quad \begin{array}{l}x=1\\y=2\\z=3\end{array}$$

## Homogenous & Non-homogenous System of LE

Non-homogenous (consistency method)

$\rho(A)=\rho(A:B)$ $\quad$ n = no of unknown values

⇒ finite solutions : $\rho(A)=\rho(A:B)=n$

⇒ Infinite solution :- $[\rho(A)=\rho(A:B)]<n$

$\rho(A)\neq\rho(A:B)$ Inconsistent (No solution)

1) $x+y+z=6$
$x-y+2z=5$
$3x+y+z=8$

$$[A:B]=\left[\begin{array}{ccc:c}1&1&1&6\\1&-1&2&5\\3&1&1&8\end{array}\right] \quad \begin{array}{l}R_2=R_2-R_1\\R_3=R_3-3R_1\end{array}$$

$$=\left[\begin{array}{ccc:c}1&1&1&6\\0&-2&1&-1\\0&-2&-2&-10\end{array}\right] \quad R_3=R_3-R_2$$

$$=\left[\begin{array}{ccc:c}1&1&1&6\\0&-2&1&-1\\0&0&-3&-9\end{array}\right] \quad \begin{array}{l}\rho(A)=3\\\rho(A:B)=3\end{array} \quad n=3$$

$\rho(A)=\rho(A:B)=n=3$

finite solution.

$x+y+z=6$

$-2y+z=-1$

$-3z=-9 \Rightarrow \boxed{z=3}$

$-2y+3=-1$

$-2y=-4 \Rightarrow \boxed{y=2}$

$x+2+3=6$

$\boxed{x=1}$

2) $5x+3y+7z=5$

$3x+26y+2z=9$

$7x+2y+10z=5$

$$[A:B] = \left[\begin{array}{ccc:c} 5 & 3 & 7 & 5 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right]$$

$R_2 = 5R_2 - 3R_1$

$R_3 = 5R_3 - 7R_1$

$$= \left[\begin{array}{ccc:c} 5 & 3 & 7 & 5 \\ 0 & 121 & -11 & 30 \\ 0 & -11 & 1 & -10 \end{array}\right]$$

$R_3 = 11R_3 + R_2$

$$= \left[\begin{array}{ccc:c} 5 & 3 & 7 & 5 \\ 0 & 121 & -11 & 30 \\ 0 & 0 & 0 & -80 \end{array}\right]$$

$\rho(A) \neq \rho(A:B)$

Inconsistent.

→ Investigate the values of $\lambda$ & $\mu$ s.t. system of eq's

(1) may have unique sol'n

(2) $\infty$ sol'n

(3) no sol'n.

$x+y+z=6$

$x+2y+3z=10$

$x+2y+\lambda z=\mu$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$R_2 = R_2 - R_1$ ; $R_3 = R_3 - R_1$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$R_3 = R_3 - R_2$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

(1) unique sol'n.

$\lambda - 3 \neq 0$

$\lambda \neq 3$

~~$\mu - 10 \neq 0$~~

~~$\mu \neq 10$.~~

$\mu$ can be any number.

$\mu \in (-\infty, \infty)$.

(2) $\lambda = 3$, $\mu = 10$.

$\rho(A) = \rho(A:B) = 2 < 3$.

(3) $\lambda = 3$, $\mu \neq 10$.

$\rho(A) \neq \rho(A:B)$

→ $x+2y+3z=14$

$4x+5y+7z=35$

$3x+3y+4z=21$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 4 & 5 & 7 & 35 \\ 3 & 3 & 4 & 21 \end{array}\right]$$

$R_2 = R_2 - 4R_1 \qquad R_3 = R_3 - 3R_1$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & -3 & -5 & -21 \\ 0 & -3 & -5 & -21 \end{array}\right]$$

$R_3 = R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & -3 & -5 & -21 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = \rho(A:B) = 2 < 3$

∴ infinite sol

$x+2y+3z=14$

$-3y-5z=-21$

$\boxed{z=K}$

$3y = 21-5z$

$y = \frac{1}{3}(21-5K)$ $\qquad \boxed{7-5/3 K = y}$

$x + \frac{2}{3}(21-5K) + 3K = 14$

$x = 14 - \frac{10}{3}K + 3K = 14$

$x = \frac{10}{3}K - 3K$

$\boxed{x = 1/3 K}$

Homogenous Method

Trivial solⁿ $[x=y=z=0]$

Non Trivial solⁿ / Infinite solⁿ $[\rho(A) \text{ or } \rho(A:B)] < n$

→ Solve the system of eqs $x+3y-2z=0$
$2x-y+4z=0$
$x-11y+14z=0$

$AX = 0$

$$\left[\begin{array}{ccc} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{array}\right] \left[\begin{array}{c} x \\ y \\ z \end{array}\right] = 0$$

$$A = \left[\begin{array}{ccc} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{array}\right]$$

$R_3 = R_3 - R_1 \qquad R_2 = R_2 - 2R_1$

$$A = \left[\begin{array}{ccc} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{array}\right]$$

$R_3 = R_3 - 2R_2$

$$A = \left[\begin{array}{ccc} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = 2 < 3$

non-trivial solⁿ

$x+3y-2z=0$

$-7y+8z=0$

$\boxed{z=K}$

$\boxed{y = \frac{8K}{7}}$

$x + \frac{24}{7}K - 2K = 0$

$\boxed{x = -10/7 K}$

$\rightarrow$ $3x - y - z + 2u = 0$

$3x + y - z - 2u = 0$

$12x + 3y - 4z - 6u = 0$ $\quad AX = 0$

$$\begin{bmatrix} 3 & -1 & -1 & 2 \\ 3 & 1 & -1 & -2 \\ 12 & 3 & -4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 = R_3 - 4R_1$

$R_2 = R_2 - R_1$

$$= \begin{bmatrix} 3 & -1 & -1 & 2 \\ 0 & 2 & 0 & -4 \\ 0 & 7 & 0 & -14 \end{bmatrix} \quad R_2 = R_2/2, \; R_3 = R_3/7$$

$$= \begin{bmatrix} 3 & -1 & -1 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & -2 \end{bmatrix} \quad R_3 = R_3 - R_1$$

$$= \begin{bmatrix} 3 & -1 & -1 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \rho(A) = 2 < 4$$

Non-Trivial sol$^n$

$3x - y - z + 2u = 0$

$y - 2u = 0$ $\quad \boxed{u = k_1}$ $\boxed{z = k_2}$

$\boxed{y = 2k_1}$

$3x - 2k_1 - k_2 + 2k_1 = 0$

$\boxed{x = \frac{k_2}{3}}$

$\rightarrow$ $x + 2y - 4z = 0$

$3x - y + 2z = 0$

$5x - 3y + z = 0$

$$A = \begin{bmatrix} 1 & 2 & -4 \\ 3 & -1 & 2 \\ 5 & -3 & 1 \end{bmatrix}$$

$R_3 = R_3 - 5R_1$

$R_2 = R_2 - 3R_1$

$$A = \begin{bmatrix} 1 & 2 & -4 \\ 0 & -7 & 14 \\ 0 & -13 & 21 \end{bmatrix} \quad R_2 = R_2/7$$

$R_2 =$

$$A = \begin{bmatrix} 1 & 2 & -4 \\ 0 & -1 & 2 \\ 0 & -13 & 21 \end{bmatrix}$$

$R_3 = R_3 - 13R_2$

$$A = \begin{bmatrix} 1 & 2 & -4 \\ 0 & -1 & 2 \\ 0 & 0 & -5 \end{bmatrix} \quad \rho(A) = 3$$

Trivial sol$^n$

$\Rightarrow x = y = z = 0$

Row reduction & echelon form of L.E

Working process:

Given system of L.E write an argumented matrix reduce argumented into echelon form then matrix form into eq form & solve for unknowns.

→ Solve $x+y+z=6$, $x-y+2z=5$, $3x+y+z=8$ by row reduction & echelon form.

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \end{array}\right]$$

$R_3 = R_3 - 3R_1$

$R_2 = R_2 - R_1$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \end{array}\right]$$

$R_3 = R_3 - R_2$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -3 & -9 \end{array}\right]$$

$R_2 = R_2/-2$

$R_3 = R_3/-3$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1/2 & +1/2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$x+y+z=6$

$y-\frac{1}{2}z=\frac{1}{2}$ ⇒ $y-3/2=1/2$

$\boxed{z=3}$

$y=4/2$

$\boxed{y=2}$

$x+2+3=6$

$\boxed{x=1}$

→ $4x-5y+z=3$

$2x+3y-z=3$

$3x-y+2z=5$

$x+2y-5z=9$

$$[A:B] = \left[\begin{array}{ccc|c} 4 & -5 & 1 & 3 \\ 2 & 3 & -1 & 3 \\ 3 & -1 & 2 & 5 \\ 1 & 2 & -5 & 9 \end{array}\right]$$

$R_1 \leftrightarrow R_4$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -5 & 9 \\ 2 & 3 & -1 & 3 \\ 3 & -1 & 2 & 5 \\ 4 & -5 & 1 & 3 \end{array}\right]$$

$R_2 = R_2 - 2R_1$

$R_3 = R_3 - 3R_1$

$R_4 = R_4 - 4R_1$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -5 & 9 \\ 0 & -1 & 9 & -15 \\ 0 & -7 & 17 & -22 \\ 0 & -13 & 21 & -33 \end{array}\right]$$

$R_3 = R_3 - 7R_2$

$R_4 = R_4 - 13R_2$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -5 & 9 \\ 0 & -1 & 9 & -15 \\ 0 & 0 & -46 & 83 \\ 0 & 0 & -96 & +162 \end{array}\right]$$

$R_4 = 46R_4 - 96R_3$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -5 & 9 \\ 0 & -1 & 9 & 15 \\ 0 & 0 & -46 & 83 \\ 0 & 0 & 0 & -516 \end{array}\right]$$

→ $2x+y+z=10$

$3x+2y+3z=18$

$x+4y+9z=16$

$$(A:B) = \left[\begin{array}{ccc:c} 1 & 4 & 9 & 16 \\ 3 & 2 & 3 & 18 \\ 2 & 1 & 1 & 10 \end{array}\right]$$

$R_2 = R_2 - 3R_1$

$R_3 = R_3 - 2R_1$

$$= \left[\begin{array}{ccc:c} 1 & 4 & 9 & 16 \\ 0 & -10 & -24 & -30 \\ 0 & -7 & -17 & -22 \end{array}\right]$$

$R_3 = 10R_3 - 7R_2$

$$= \left[\begin{array}{ccc:c} 1 & 4 & 9 & 16 \\ 0 & -10 & -24 & -30 \\ 0 & 0 & -2 & -10 \end{array}\right]$$

$R_2 = R_2/-10$

$R_3 = R_3/-2$

$$= \left[\begin{array}{ccc:c} 1 & 4 & 9 & 16 \\ 0 & 1 & \frac{24}{10} & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$x+4y+9z=16$

$y+\frac{12}{5}z=3$

$\boxed{z=5}$

$y+\frac{12}{5}(5)=3$

$\boxed{y=-9}$

$x+12+45=16$

$\boxed{x=-41}$

$x-36+45=16$

$\boxed{x=7}$

→ Reduce the matrix in row reduction & Echelon form and hence find the rank.

$2x+y+z=10$

$3x+2y+3z=18$

$x+4y+9z=16$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 4 & 9 & 16 \\ 3 & 2 & 3 & 18 \\ 2 & 1 & 1 & 10 \end{array}\right]$$

$$= \left[\begin{array}{ccc:c} 1 & 4 & 9 & 16 \\ 0 & 1 & \frac{24}{10} & 3 \\ 0 & 0 & 1 & 5 \end{array}\right] \quad \rho(A)=3$$

## Matrix Equations

$AX=B$

$X=A^{-1}B \quad , |A| \neq 0$

① $x+2y=2$

$2x+3y=3$

$$A=\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \quad X=\begin{bmatrix} x \\ y \end{bmatrix} \quad B=\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$AX=B.$

$$A^{-1} = \frac{\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}}{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

$$X = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3(2)+2(3) \\ 2(2)-(1)(3) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

2) $x - y + 2z = 7$

$3x + 4y - 5z = -5$

$2x - y + 3z = 12$

$A \quad B$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 7/4 & 1/4 & -3/4 \\ -19/4 & -1/4 & 11/4 \\ -11/4 & -1/4 & 7/4 \end{bmatrix}$$

$$\cancel{X = \begin{bmatrix} 7/4 & 1/4 & -3/4 \\ -19/4 & -1/4 & \end{bmatrix}}$$

$$X = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7(7)+1(-5)-3(12)}{4} \\ \frac{-19(7)-1(-5)+11(12)}{4} \\ \frac{-11(7)-1(-5)+7(12)}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

3) $x - y - 2z = 3$

$2x + y + z = 5$

$4x - y - 2z = 1$

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 2 & 1 & 1 \\ 4 & -1 & -2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1/3 & 0 & 1/3 \\ 8/3 & 2 & -5/3 \\ -2 & -1 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} -1/3 & 0 & 1/3 \\ 8/3 & 2 & -5/3 \\ -2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 + 0 + 1/3 \\ 8 + 10 - 5/3 \\ -6 - 5 + 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 49/3 \\ -10 \end{bmatrix}$$

4) $x + y + z = 9$

$2x + y - z = 0$

$2x + 5y + 7z = 52.$

$$A^{-1} = \begin{bmatrix} 3 & -1/2 & -1/2 \\ -4 & 5/4 & 3/4 \\ 2 & -3/4 & -1/4 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 & -1/2 & -1/2 \\ -4 & 5/4 & 3/4 \\ 2 & -3/4 & -1/4 \end{bmatrix}\begin{bmatrix} 9 \\ 0 \\ 52 \end{bmatrix}$$

$$= \begin{bmatrix} 9(3) - 1/2(52) \\ -4(9) + 3/4(52) \\ 2(9) - 1/4(52) \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

5) $x+2y+z=8$

$2x+3y+4z=20$

$4x+3y+2z=16$

$$A^{-1} = \begin{bmatrix} -1/2 & -1/12 & 5/12 \\ 1 & -1/6 & -1/6 \\ -1/2 & 5/12 & -1/12 \end{bmatrix}$$

$$X = \begin{bmatrix} -1/2 & -1/12 & 5/12 \\ 1 & -1/6 & -1/6 \\ -1/2 & 5/12 & -1/12 \end{bmatrix} \begin{bmatrix} 8 \\ 20 \\ 16 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2(8) - 1/12(20) + 5/12(16) \\ 1(8) - 1/6(20) - 1/6(16) \\ -1/2(8) + 5/12(20) - 1/12(16) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

6) $x+y+z=6$

$x-y+2z=5$

$3x+y+z=8$

$$A^{-1} = \begin{bmatrix} -1/2 & 0 & 1/2 \\ 5/6 & -1/3 & -1/6 \\ 2/3 & 1/3 & -1/3 \end{bmatrix}$$

$$X = \begin{bmatrix} -1/2(6) + 0 + 8(1/2) \\ 5/6(6) - 1/3(5) - 1/6(8) \\ 2/3(6) + 1/3(5) - 1/3(8) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

7) $4x-5y+z=-3$

$2x+3y-z=5$

$3x-y+2z=5$

$$A^{-1} = \begin{bmatrix} 5/44 & 9/44 & 1/22 \\ -7/44 & 5/44 & 3/22 \\ -1/4 & -1/4 & 1/2 \end{bmatrix}$$

$$X = \begin{bmatrix} 5/44(-3) + 9/44(5) + 1/22(5) \\ -7/44(-3) + 5/44(5) + 3/22(5) \\ -1/4(-3) - 1/4(5) + 1/2(5) \end{bmatrix}$$

$$= \begin{bmatrix} 10/11 \\ 19/11 \\ 2 \end{bmatrix}$$

Application of Markov chain:

Markov: It is a process with finite no. of states (or outcomes or events) in which the probability of being a particular state at step n+1, depends only on the state occupied at step n.

Transitive probability Matrix (TPM):

It consist of a square matrix that gives the probability of different states going from one to another.

A transition matrix of a markov process is called regular if all the entries in some power of P/T are regular.

Probability vector:

A vector $v$ is called a probability vector if each one of its components are non-negative & their sum is unity (1).

→ P.T the markov chain its transitive probability matrix is regular & hence find its unique fixed probability vector

Sol:

$$P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$P^2 = P \times P$

$$= \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/4 & 7/12 & 1/6 \\ 1/4 & 1/3 & 5/12 \end{bmatrix}$$

regular

$v = [x \; y \; z]$

$vP = v$

$$[x \; y \; z] \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [x \; y \; z]$$

$\frac{1}{2}(y+z) = x.$

$\frac{2}{3}x + \frac{1}{2}z = y \Rightarrow \frac{4x+3z}{6} = y$

$\frac{1}{3}x + \frac{1}{2}y = z$

~~$y+z = 2x$~~

$2x - y - z = 0$ — ①

$4x - 6y + 3z = 0$ — ②

$2x + 3y - 6z = 0$ — ③

① − ③

$-y - z - 3y + 6z = 0$

$-4y + 5z = 0$ — ④

② − ③ × 2.

$4x - 6y + 3z - 4x - 6y + 12z = 0$

$-12y + 15z = 0$

① × 2 − ②

$4x - 2y - 2z - 4x + 12y - 6z = 0$

$10y - 8z = 0.$

$5y - 4z = 0.$ — ⑤

$\Rightarrow x = 1/3 \quad y = 10/27$

$z = 8/27.$

$1/3$

$10/27$

$8/27$

→ A student is studing one night he is 70% sure not to study the next night on the other hand if he does not study one night he is 60% sure not to study next night in the longrun how often does he study.

Sol: A = studying  B = Not studying

$$P = \begin{array}{c|cc} & A & B \\ \hline A & 0.3 & 0.7 \\ B & 0.4 & 0.6 \end{array}$$

$vP = v$

$$[x \ \ y] \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = [x \ \ y].$$

$0.3x + 0.4y = x$

$0.7x + 0.6y = y$

$0.7x - 0.4y = 0$

$\cancel{0.7x - 0.4y = 0}$

$x + y = 1$

$0.3(1-y) + 0.4y = 1-y$

$0.3 - 0.3y + 0.4y = 1 - y$

$1.1y = 0.7$

$\boxed{y = 7/11}$

$\boxed{x = 4/11}$

→ Each year a ~~man~~ man trades his car for a new car in 3 brands of the popular company maruti Udoyog limited. if he has a standard each rates for ZEN. if he has a ZEN he trades for esteem. if he has esteem he light trated for new esteem or new or standard. In 1996 he bought his 1st car which was esteem

(i) find the probability that he has

ⓐ 1998 esteem ⓑ 1998 standard ⓒ 1999 ZEN

ⓓ 1999 esteem.

(ii) In the long run how often he will have a esteen.

Sol: x: Std y = Zen z = Esteem

$$P = \begin{array}{c|ccc} & x & y & z \\ x & 0 & 1 & 0 \\ y & 0 & 0 & 1 \\ z & 1/3 & 1/3 & 1/3 \end{array}$$

With 1996 as 1st year, 1998 recorded as 2 years after and 1999 as 3 years after.

$$P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \\ 4/27 & 7/27 & 16/27 \end{bmatrix}$$

(i) ⓐ 4/9 ⓑ 1/9 ⓒ 7/27 ⓓ 16/27

(ii) $vP = v$

$$[x \ \ y \ \ z] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} = [x \ \ y \ \ z]$$

$(\because x+y+z=1)$

$y \ne x$

$z \ne y$

$x + y + z = 3z$

$\Rightarrow$

$x = 1/3$

$y = 1/3$

$z = 1/3$

$\frac{1}{3}z = x \Rightarrow z = 3x$.

$x + \frac{1}{3}z = y \Rightarrow 3x + z = 3y \Rightarrow 3x + 3x = 3y$

$3y = 6x$

$y = 2x$

$y + \frac{1}{3}z = z$

$\Downarrow$

$\cancel{2x + \frac{1}{3}(3x) = 3x}$

$\cancel{2x + 1 = 3x}$

$\boxed{x = 1}$

$2z = 3(1 - x - z)$

$2z = 3(1 - \frac{z}{3} - z)$

$2z = 3 - z - 3z$

$2z + z + 3z = 3$

$6z = 3$

$\boxed{z = \frac{1}{2}}$

→ A gambler is a member of 2 clubs. He visits either of the club for playing cards. He never visits club A on 2 consecutive days. But he visits club B on a particular day that next day, he is likely to visit club B/ club A. Find the transition matrix & S.T it is regular. Find unit fixed probability vector. If the person has visited club B on monday find the probability that he visits club A on thursday.

Sol:

$$P = \begin{array}{c|c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

$$P^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

∴ it is regular.

$vP = v$

$$(x \ \ y)\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = (x \ \ y)$$

$[\because x + y = 1]$

$\frac{1}{2}y = x$

$x + \frac{1}{2}y = y$

$\frac{1}{2}y = 1 - y$

$\frac{3}{2}y = 1$

$\boxed{y = \frac{2}{3}}$

$\boxed{x = \frac{1}{3}}$

$$P^3 = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{8} & \frac{5}{8} \end{bmatrix}$$

monday - B

the thursday for club A

$\Rightarrow \frac{3}{8}$.

→ A Gamblers luck follows a pa[...] a game the probality that he wins next day 0.6, ho[...] ever if he losses a game the probability of lossing next game is 0.7. there is a even chan[ce] of gambler that wining 1st game if so,

(1) what is the probability of wining the 2nd game

(2) " " " " " " 3rd game

(3) In the long run how often he will win.

Sol:

~~A = luck~~ B = luck.

$$P = \begin{array}{c|cc} & W & L \\ \hline W & 0.6 & 0.4 \\ L & 0.3 & 0.7 \end{array},$$

Initial probability vector

$$p^{(0)} = \left(\tfrac{1}{2} \;\; \tfrac{1}{2}\right)$$

(1) $p^{(1)} = p^{(0)} \cdot P$

$$= \left[\tfrac{1}{2} \;\; \tfrac{1}{2}\right] \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}$$

$$= \left[0.45 \;\; 0.55\right]$$

$$= \left[\tfrac{9}{20} \;\; \tfrac{11}{20}\right].$$

(2) $p^{(2)} = p^{(1)} \cdot P$

$$= \left[\tfrac{9}{20} \;\; \tfrac{11}{20}\right] \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}$$

$$= \left[\tfrac{87}{200} \;\; \tfrac{113}{200}\right]$$

(3) $vP = v$

$$[x \;\; y] \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix} = [x \;\; y]$$

$[\because x + y = 1]$

$0.6x + 0.3y = x$

$0.4x + 0.7y = y$

$\cancel{0.6x + 0.3(1-y) = 1-y}$

$\cancel{0.6x + 0.3 - 0.3y = 1-y}$

$\cancel{0.6x + 0.7y = 0}$

$0.6x + 0.3(1-x) = x$

$0.6x + 0.3 - 0.3x - x = 0$

$-0.7x = -0.3$

$x = 3/7$

$y = 4/7$

→ find the unique fixed probability for the matrix

$$P = \begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix}$$

$vP = v$

$$[x \;\; y \;\; z \;\; w] \begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} = [x \;\; y \;\; z \;\; w]$$

$[\because x+y+z+w=1]$

$$\frac{y}{2}+\frac{z}{2}+\frac{w}{2}=x$$

$$\frac{x}{2}+\frac{z}{2}+\frac{w}{2}=y$$

$$\frac{x}{4}+\frac{y}{4}=z$$

$$\frac{x}{4}+\frac{y}{4}=w.$$

$\Rightarrow$ $y+z+w=2x.$

$1-x=2x$

$3x=1$

$\boxed{x=1/3}$

$x+z+w=2y$

$1-y=2y$

$3y=1$

$\boxed{y=1/3}$

$x+y=4z$

$\frac{1}{3}+\frac{1}{3}=4z$

$z=\frac{\cancel{2}}{3}\times\frac{1}{\cancel{4}_2}$ $\boxed{z=1/6}$

$x+y=4w.$

$\frac{2}{6}=4w$ $\Rightarrow$ $\boxed{w=1/6}$