Composition Functions

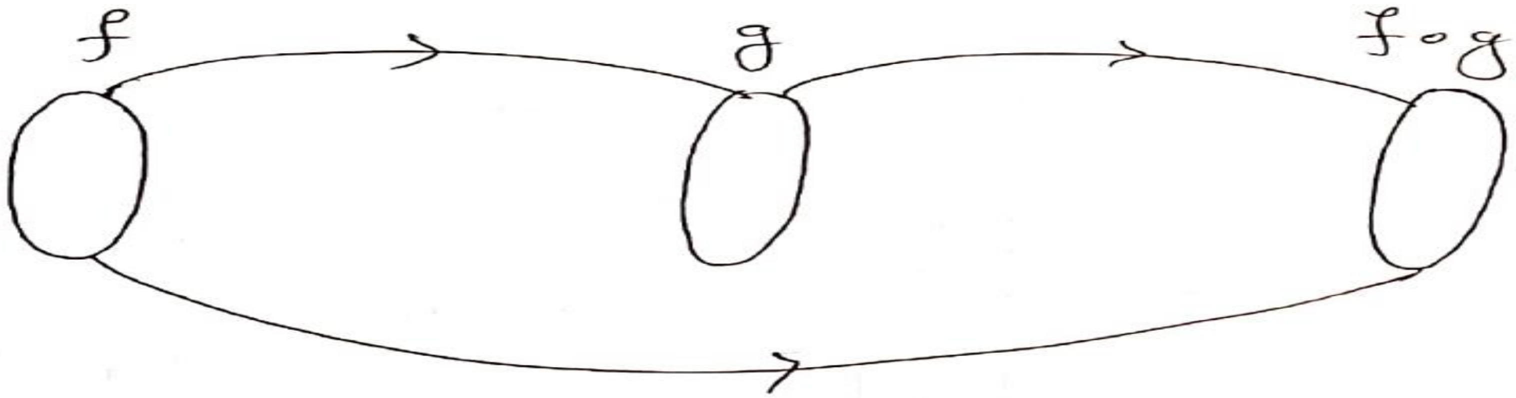
Define $f : \mathrm{R} \to \mathrm{R}$ by $f(x) = x^2$ and $g : \mathrm{R} \to \mathrm{R}$ by $g(x) = x^3$. The composition $f \circ g : \mathrm{R} \to \mathrm{R}$ is the transformation defined by the rule

$$f \circ g(x) = f(g(x)) = f(x^3) = (x^3)^2 = x^6.$$

For instance, $f \circ g(-2) = f(-8) = 64$.

$$f \circ g(-2) = f(g(-2)) = g[(-2)^3] = g(-8) = (-8)^2$$

$$\boxed{f \circ g(-2) = 64}$$

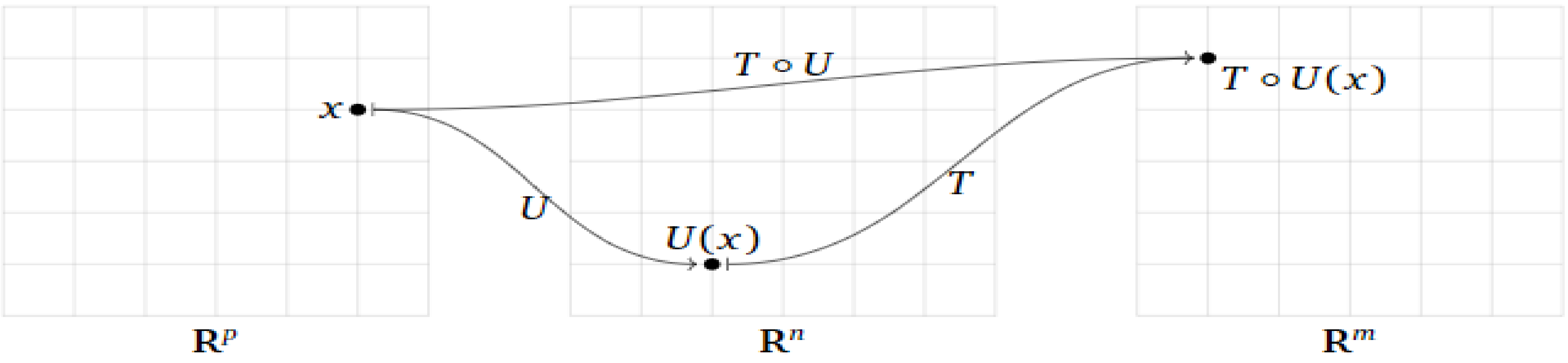


Composition of linear transformations

Definition. Let $T: \mathbf{R}^n \to \mathbf{R}^m$ and $U: \mathbf{R}^p \to \mathbf{R}^n$ be transformations. Their *composition* is the transformation $T \circ U: \mathbf{R}^p \to \mathbf{R}^m$ defined by

$$(T \circ U)(x) = T(U(x)).$$

Composing two transformations means chaining them together: $T \circ U$ is the transformation that first applies U, then applies T (note the order of operations). More precisely, to evaluate $T \circ U$ on an input vector x, first you evaluate $U(x)$, then you take this output vector of U and use it as an input vector of T: that is, $(T \circ U)(x) = T(U(x))$. Of course, this only makes sense when the outputs of U are valid inputs of T, that is, when the range of U is contained in the domain of T.



Properties of composition. Let S, T, U be transformations and let c be a scalar. Suppose that $T : \mathrm{R}^n \to \mathrm{R}^m$, and that in each of the following identities, the domains and the codomains are compatible when necessary for the composition to be defined. The following properties are easily verified:

$$S \circ (T + U) = S \circ T + S \circ U \qquad (S + T) \circ U = S \circ U + T \circ U$$

$$c(T \circ U) = (cT) \circ U \qquad c(T \circ U) = T \circ (cU) \quad \text{if } T \text{ is linear}$$

$$S \circ (T \circ U) = (S \circ T) \circ U$$

Composition of transformations is *not* commutative in general. That is, in general, $T \circ U \neq U \circ T$, even when both compositions are defined.

Theorem. *Let $T : \mathrm{R}^n \to \mathrm{R}^m$ and $U : \mathrm{R}^p \to \mathrm{R}^n$ be linear transformations, and let A and B be their standard matrices, respectively, so A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $T \circ U : \mathrm{R}^p \to \mathrm{R}^m$ is a linear transformation, and its standard matrix is the product AB.*

Products and compositions.

The matrix of the composition of two linear transformations is the product of the matrices of the transformations.

Standard coordinate vectors.

The ***standard coordinate vectors in*** $\mathbf{R}^n$ are the n vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

The ith entry of e_i is equal to 1, and the other entries are zero.

From now on, for the rest of the book, we will use the symbols $e_1, e_2, \ldots$ to denote the standard coordinate vectors.

Matrix transformations are the same as linear transformations.

Composition of transformations is *not* commutative in general. That is, in general, $T \circ U \neq U \circ T$, even when both compositions are defined.

1. Let . $T : R^3 \longrightarrow R^3$ and $U : R^2 \longrightarrow R^3$ are linear trans-formation that can be represented by the matrices

$$T = \begin{bmatrix} 3 & -1 & 5 \\ 0 & 2 & 1 \\ 4 & 0 & -3 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 7 \\ -6 & 1 \\ 1 & -4 \end{bmatrix} \quad \text{respectively}$$

for the transformation T and U just defined, find $(T \circ U) \cdot X = (T \circ U) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then find the matrix of the transformation $T \circ U$

Sol^n Given that $T : R^3 \longrightarrow R^3$ and $U : R^2 \longrightarrow R^3$

$$T = \begin{bmatrix} 3 & -1 & 5 \\ 0 & 2 & 1 \\ 4 & 0 & -3 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 7 \\ -6 & 1 \\ 1 & -4 \end{bmatrix}$$

we have

$$(T \circ U) X = T[U(x)]$$

$$= T\left\{ \begin{bmatrix} 2 & 7 \\ -6 & 1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\}$$

$$= T \begin{bmatrix} 2x_1 + 7x_2 \\ -6x_1 + x_2 \\ x_1 - 4x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & 5 \\ 0 & 2 & 1 \\ 4 & 0 & -3 \end{bmatrix} \begin{bmatrix} 2x_1 + 7x_2 \\ -6x_1 + x_2 \\ x_1 - 4x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 3(2x_1 + 7x_2) - (-6x_1 + x_2) + 5(x_1 - 4x_2) \\ 0(2x_1 + 7x_2) + 2(-6x_1 + x_2) + 1(x_1 - 4x_2) \\ 4(2x_1 + 7x_2) + 0(-6x_1 + x_2) - 3(x_1 - 4x_2) \end{bmatrix}$$

$$= \begin{bmatrix} 6x_1 + \cancel{21x_2} + 6x_1 - \cancel{x_2} + 5x_1 - \cancel{20x_2} \\ 0 - 12x_1 + 2x_2 + x_1 - 4x_2 \\ 8x_1 + 28x_2 + 0 - 3x_1 + 12x_2 \end{bmatrix}$$

$$(T \circ U)x = \begin{bmatrix} 17x_1 + 0x_2 \\ -11x_1 - 2x_2 \\ 5x_1 + 40x_2 \end{bmatrix}$$

matrix of the linear transformation of $T \circ U$.

i.e., $$T \circ U = \begin{bmatrix} 17 & 0 \\ -11 & -2 \\ 5 & 40 \end{bmatrix}$$

2. Let $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ and $U: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1^2 \\ x_1 x_3 \end{bmatrix} \; ; \; U\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 \\ 2x_2 - x_1 \end{bmatrix}$$

Determine whether each of the compositions $T \circ U$ and $U \circ T$ exists.

Sol^n Given that $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ and $U: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1^2 \\ x_1 \, x_3 \end{bmatrix} \; ; \; U\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 \\ 2x_2 - x_1 \end{bmatrix}$$

Since the domain of T is R^3 and the range of U is a subset of R^2.

$\Rightarrow$ the composition T∘U does not exist.

The range of T falls within the domain of U, So the composition U∘T does exist.

Its equation is found by

$$(U \circ T)(x) = U[T(x)]$$

$$= U\left(T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$$

$$= U \begin{bmatrix} x_1^2 \\ x_2 x_3 \end{bmatrix}$$

$$(U \circ T)(x) = \begin{bmatrix} x_1^2 + 3x_2 x_3 \\ 2x_2 x_3 - x_1^2 \end{bmatrix}$$

3. Let $T: R^3 \longrightarrow R^2$ and $U: R^2 \longrightarrow R^3$ are the linear transformation that can be represented by the matrices $T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ respectively for the transformation T and U defined $(T \circ U)X = (T \circ U)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then find the matrix of the transformation $T \circ U$.

Sol: By data $T: R^3 \longrightarrow R^2$ and $U: R^2 \longrightarrow R^3$

$$T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Now we have

$$(T \circ U)(x) = T[U(x)]$$

$$= T\left(U\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

$$= T\left(U\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

$$= T\begin{bmatrix} 1x_1 + 0x_2 \\ 0x_1 + 1x_2 \\ 1x_1 + 0x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1x_1 + 0x_2 \\ 0x_1 + 1x_2 \\ 1x_1 + 0x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1(x_1 + 0x_2) + 1(0x_1 + 1x_2) + 0(1x_1 + 0x_2) \\ 0(x_1 + 0x_2) + 1(0x_1 + 1x_2) + 1(1x_1 + 0x_2) \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + 0x_2 + 0x_1 + x_2 + 0x_1 + 0x_2 \\ 0x_1 + 0x_2 + 0x_1 + 1x_2 + 1x_1 + 0x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + x_2 \\ 1x_2 + x_1 \end{bmatrix}$$

$$(T \circ U)(x) = \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 \end{bmatrix}$$

The matrix of linear transformation of

$$T \circ U = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

4. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ and $U : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_1 \\ -3x_2 \end{bmatrix}, \quad U\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5x_1 - x_2 \\ x_1 + 4x_2 \end{bmatrix}$$

Determine the composition of $T \circ U$ and also to find the matrix of the composition of $T \circ U$

Sol: Given that $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ and $U : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

So that $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_1 \\ -3x_2 \end{bmatrix}$; $U\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5x_1 - x_2 \\ x_1 + 4x_2 \end{bmatrix}$

Now ...

Now we have $(T \circ U)(x) = T(U(x))$

$$= T\left(U\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

$$= T\begin{bmatrix} 5x_1 - x_2 \\ x_1 + 4x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 5x_1 - x_2 + x_1 + 4x_2 \\ 2(5x_1 - x_2) \\ -3(x_1 + 4x_2 \end{bmatrix}$$

$$(T \circ U)(x) = \begin{bmatrix} 6x_1 + 3x_2 \\ 10x_1 - 2x_2 \\ -3x_1 - 12x_2 \end{bmatrix}$$

The matrix of the composition of $T \circ U$

i.e., $T \circ U = \begin{bmatrix} 6 & 3 \\ 10 & -2 \\ -3 & -12 \end{bmatrix}$

5. Define $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ and $U : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by

$$T = \begin{bmatrix} 1 & 0 & +1 \\ 0 & -1 & 0 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ -1 & 3 \end{bmatrix}$$ then

find the composition of the $T \circ U : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

Sol: By data $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ and $U : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$T = \begin{bmatrix} 1 & 0 & +1 \\ 0 & -1 & 0 \end{bmatrix} ; \quad U = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ -1 & 3 \end{bmatrix}$$

Now we have

$$(T \circ U)(x) = T\left(U\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} 1 & 2 \\ 0 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

$$= T\begin{bmatrix} x_1 + 2x_2 \\ 0x_1 - x_2 \\ -x_1 + 3x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & +1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 + 2x_2 \\ 0x_1 - x_2 \\ -x_1 + 3x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1(x_1+2x_2)+0(0x_1-x_2)+1(-x_1+3x_2) \\ 0(x_1+2x_2)-1(0x_1-x_2)+0(-x_1+3x_2) \end{bmatrix}$$

$$= \begin{bmatrix} x_1+2x_2+0-x_1+3x_2 \\ 0-0x_1+x_2+0 \end{bmatrix}$$

$$(ToU)(x) = \begin{bmatrix} 0x_1+5x_2 \\ 0x_1+1x_2 \end{bmatrix}$$

The matrix of the composition of ToU is

i.e., $ToU = \begin{bmatrix} 0 & 5 \\ 0 & 1 \end{bmatrix}$

Singular and Non-Singular linear transformation

Let U and V be 2 vector spaces over F, a linear transformation $T: U \rightarrow V$ is said to be non-singular if $N(T) = 0$ Nullity

otherwise T is singular

Invertible linear transformation :→

Let U and V be any two vector spaces over F. A linear transformation $T: U \longrightarrow V$ is said to be invertible if there exists a linear transformation $S: V \longrightarrow U$ such that

$$S \circ T = T \circ S = I$$

Here S is called inverse of T and is denoted by T^{-1}

1. Show that the mapping $T : V_3(\mathbb{R}) \longrightarrow V_3(\mathbb{R})$ defined by $T(e_1) = e_1 + e_2$, $T(e_2) = e_2 + e_3$, $T(e_3) = e_1 + e_2 + e_3$ is non-singular and find its inverse.

Sol: Given that $T : V_3(\mathbb{R}) \longrightarrow V_3(\mathbb{R})$

$$T(e_1) = e_1 + e_2, \quad T(e_2) = e_2 + e_3 \text{ and } T(e_3) = e_1 + e_2 + e_3$$

we have

$$T(e_1) = (1,0,0) + (0,1,0)$$

$$T(e_1) = (1,1,0)$$

$$T(e_2) = (0,1,0) + (0,0,1)$$

$$T(e_2) = (0,1,1)$$

$$T(e_3) = (1,0,0) + (0,1,0) + (0,0,1)$$

$$T(e_3) = (1,1,1) \qquad \therefore \; A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow |A| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1(0) - 1(-1) + 0(-1)$$

$$= 1 \neq 0$$

$$\Rightarrow |A| = 1 \neq 0 \qquad \boxed{\therefore \; T \text{ is non-singular}}$$

$\boxed{\text{OR}}$

$$T(x, y, z) = xT(e_1) + yT(e_2) + zT(e_3)$$

$$= (x, x, 0) + (0, y, y) + (z, z, z)$$

$$\boxed{T(x, y, z) = (x+z,\ x+y+z,\ y+z)}$$

Let $(x, y, z) \in N(T) \implies T(x, y, z) = 0$

i.e., $(x+z,\ x+y+z,\ y+z) = (0, 0, 0)$

$x+z=0$; $x+y+z=0$; $y+z=0$

$$\Rightarrow \boxed{x=0,\ y=0,\ z=0}$$

$$(x, y, z) = (0, 0, 0)$$

$\therefore$ T is non-singular

Also to find Inverse of T = ?

Let $T(x, y, z)$ be the pre-image of (r, s, t)

then $T(x, y, z) = (r, s, t)$

$$\Rightarrow (x+z,\ x+y+z,\ y+z) = (r, s, t)$$

$$\Rightarrow x + z = r \rightarrow (1)$$

$$x + y + z = s \rightarrow (2)$$

$$y + z = t \rightarrow (3)$$

from (1) & (2)

$$\begin{array}{r} \cancel{x} + 0y + \cancel{z} = r \\ + \cancel{x} + y + \cancel{z} = s \\ (-) \quad (-) \quad (-) \quad (-) \\ \hline -y = r - s \end{array}$$

$$\Rightarrow \boxed{y = s - r}$$

from (3) $z = t - y$

$= t - (s - r)$

$\boxed{z = t - s + r}$

from (1) $x = r - z$

$x = r - (t - s + r)$

$= \cancel{r} - t + s - \cancel{r}$

$\boxed{x = s - t}$

$\boxed{x = s - t}$, $\boxed{y = s - r}$, $\boxed{z = t - s + r}$

$\therefore T^{-1}(r, s, t) = (x, y, z)$

$$\boxed{T^{-1}(r, s, t) = (s - t, s - r, t - s + r)}$$

2. Let T be the operator on $\mathbb{R}^3$ defined by $T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$ Show that T is invertible and find a formula for T^{-1}

Sol → By data $T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$

Let $(x, y, z) \in N(T)$

$$\Rightarrow T(x, y, z) = 0$$

$$(2x, 4x - y, 2x + 3y - z) = (0, 0, 0)$$

$2x = 0 \qquad 4x - y = 0 \qquad 2x + 3y - z = 0$

$\Rightarrow \boxed{x = 0} \qquad \boxed{y = 0} \qquad \boxed{z = 0}$

$N(T) = (0, 0, 0)$

$\therefore$ T is non-singular and hence invertible

Let (x, y, z) be the pre-image of (r, s, t) then $T(x, y, z) = (r, s, t)$

$\Rightarrow (2x, 4x - y, 2x + 3y - z) = (r, s, t)$

$2x = r \qquad 4x - y = s \qquad 2x + 3y - z = t$

$\boxed{x = \frac{r}{2}}$; $\boxed{y = 2r - s}$ $\quad r + 6r - 3s - z = t$

$\boxed{z = 7r - 3s - t}$

$\therefore T^{-1}(r, s, t) = (x, y, z)$

$$T^{-1}(r, s, t) = \left(\frac{r}{2},\ 2r - s,\ 7r - 3s - t\right)$$

3. Let $V_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ If

$b = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 6 \end{bmatrix}$. Express 'b' as a linear combination of vectors (v_1, v_2, v_3)

Let us consider the linear combination

$$b = x_1 v_1 + x_2 v_2 + x_3 v_3 \longrightarrow (1)$$

$$(-1, 0, 3, 6) = x_1 (1, 0, 1, 0) + x_2 (2, 3, 0, 0) + x_3 (0, 1, 0, 1)$$

$$\Rightarrow (-1, 0, 3, 6) = (x_1 + 2x_2,\ 3x_2 + x_3,\ x_1,\ x_3)$$

Now $x_1 + 2x_2 = -1 \Rightarrow 2x_2 = -1 - x_1$

$3x_2 + x_3 = 0$

$2x_2 = -1 - 3$

$\boxed{x_1 = 3}$

$2x_2 = -4$

$\boxed{x_3 = 6}$

$\boxed{x_2 = -2}$

OR

The augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 3 & 1 & 0 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$R_3 \rightarrow R_3 - R_1$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 3 & 1 & 0 \\ 0 & -2 & 0 & 4 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & -2 & 0 & 4 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$R_1 \rightarrow R_1 + R_2$

$R_3 \rightarrow 2R_3 + 3R_2$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & -2 & 0 & 4 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$R_4 \rightarrow 2R_4 - R_3$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & -2 & 0 & 4 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 \rightarrow \frac{-R_2}{2}$$

$$R_3 \rightarrow \frac{R_3}{2}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \boxed{x_1 = 3}$ $\boxed{x_2 = -2}$

$\boxed{x_3 = 6}$

Eqⁿ (1) becomes

$$\boxed{b = 3v_1 - 2v_2 + 6v_3}$$

4. Test for linear independence, we can write the corresponding matrix in Echelon form. So are the following vectors linearly independent or dependent?

(a) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} \right\}$

Sol Given that

$$x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 v_1 + x_2 v_2 + x_3 v_3 = (0, 0, 0) \longrightarrow (1)$$

The

The augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ -2 & 0 & -1 & 0 \\ 0 & 8 & 5 & 0 \end{array}\right]$$

$R_2 \longrightarrow R_2 + 2R_1$

$$= \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & 8 & 5 & 0 \end{array}\right]$$

$R_3 \rightarrow R_3 - R_2$

$$= \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 \longrightarrow \dfrac{R_2}{8}$

$$= \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 1 & 5/8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 \rightarrow R_1 - 4R_2$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 5/8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore \; x_1 + 0x_2 + \frac{1}{2}x_3 = 0$$

$$0x_1 + 1x_2 + \frac{5}{8}x_3 = 0$$

$\Longrightarrow$ Non - trivial Solution

$\therefore$ Linearly dependent

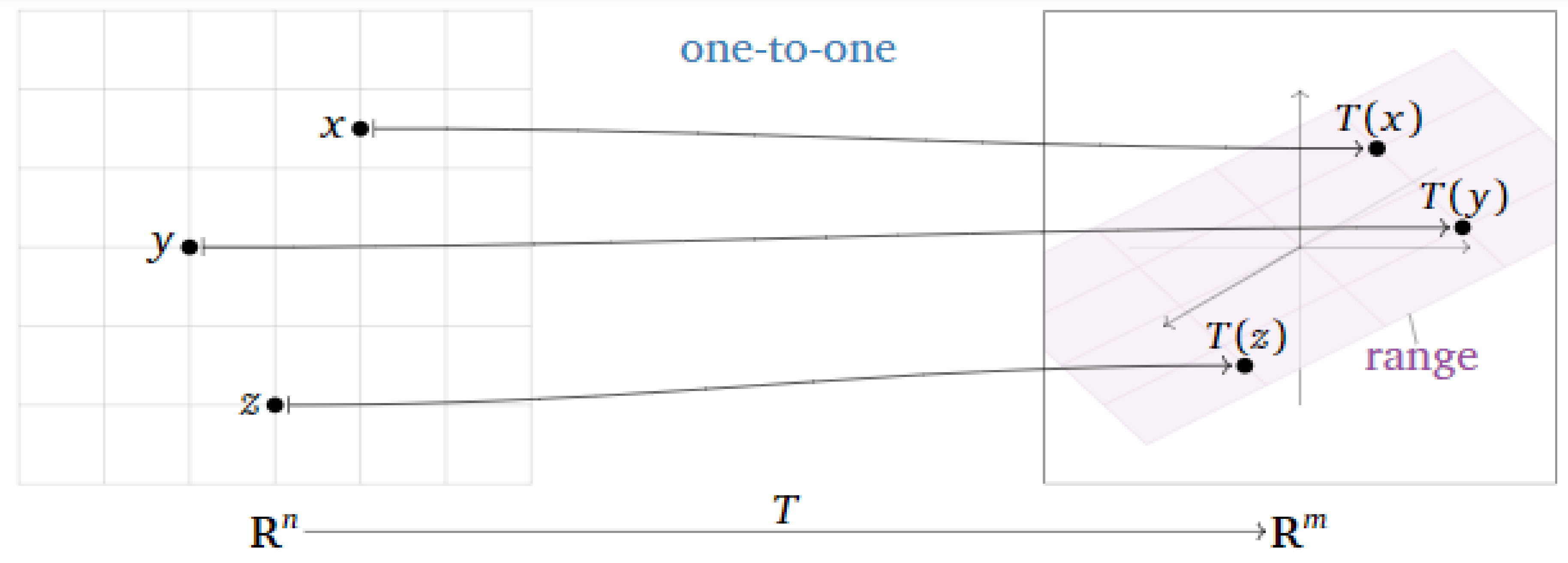
Injective and Surjective Transformations

1. Injective Transformation [one-to-one Transformation]

A transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is one-to-one if, for every vector b in $\mathbb{R}^m$, the equation $T(x) = b$ has at most one solution x in $\mathbb{R}$

Here are some equivalent ways of saying that T is one-to-one

* For every vector b in $\mathbb{R}^m$, the equation $T(x) = b$ has zero or one solution x in $\mathbb{R}^n$.
* Different inputs of T have different outputs.
* If $T(u) = T(v)$ then $u = v$

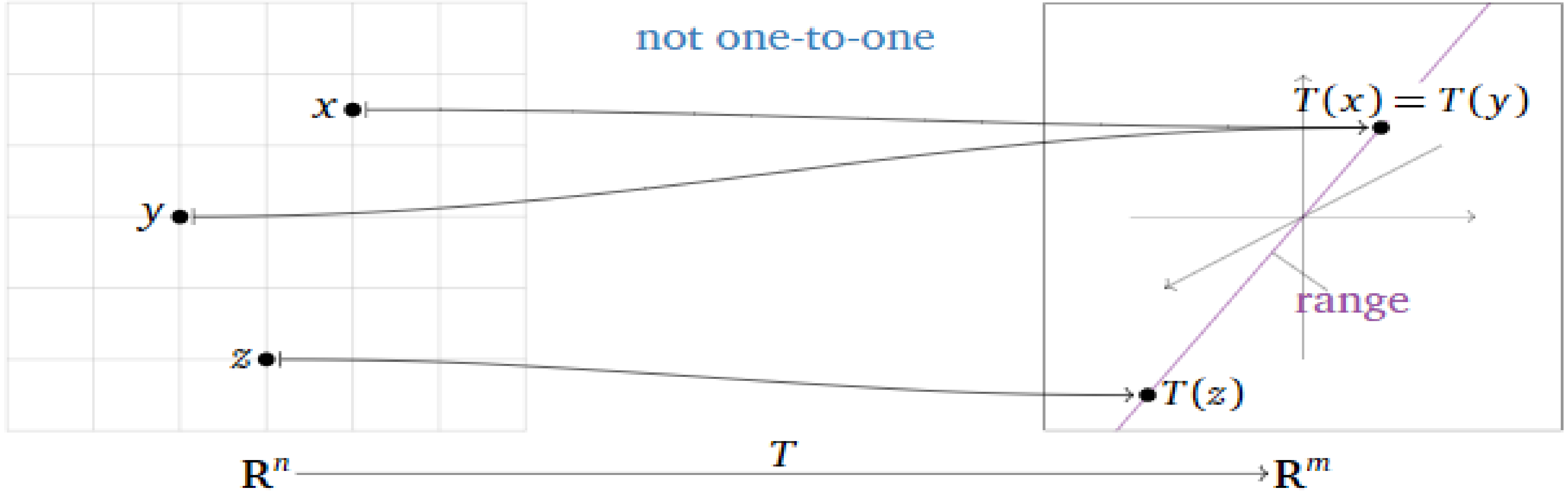


Here some equivalent ways of saying that T is not one-to-one.

* There exists some vector b in R^m such that the equation $T(x) = b$ has more than one solution x in R

* There are two different inputs of T with the same output

* There exist vectors u, v such that $u \neq v$ but $T(u) = T(v)$.



Theorem (One-to-one matrix transformations). *Let A be an $m \times n$ matrix, and let $T(x) = Ax$ be the associated matrix transformation. The following statements are equivalent:*

1. *T is one-to-one.*
2. *For every b in R^m, the equation $T(x) = b$ has at most one solution.*
3. *For every b in R^m, the equation $Ax = b$ has a unique solution or is inconsistent.*
4. *$Ax = 0$ has only the trivial solution.*
5. *The columns of A are linearly independent.*
6. *A has a pivot in every column.*
7. *The range of T has dimension n.*

1. Let A be the matrix $A = \begin{bmatrix} 1 & -3 & 4 \\ 3 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix}$,

and define $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $T(X) = AX$. Is T one to one?

Sol^n By data

$$A = \begin{bmatrix} 1 & -3 & 4 \\ 3 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

$$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \quad \text{and} \quad T(X) = AX$$

The reduce row echelon form of augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 3 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

$$R_2 \leftrightarrow R_3$$

$$= \left[\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 \longrightarrow R_2 - 2R_1$$

$$R_3 \longrightarrow R_3 - 3R_1$$

$$= \left[\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 0 & 7 & -7 & 0 \\ 0 & 9 & -12 & 0 \end{array}\right]$$

$$R_2 \longrightarrow \frac{1}{7} R_2$$

$$R_3 \longrightarrow \frac{1}{3} R_3$$

$$= \left[\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right]$$

$$R_1 \longrightarrow R_1 + 3R_2$$

$$R_3 \longrightarrow R_3 - 3R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 7 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -7 & 0 \end{array}\right]$$

$R_1 \longrightarrow R_1 + R_3$

$R_2 \longrightarrow 7R_2 + R_3$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -7 & 0 \end{array}\right]$$

$R_1 \longrightarrow R_1 + R_3$

$R_2 \longrightarrow 7R_2 + R_3$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -7 & 0 \end{array}\right]$$

$R_3 \longrightarrow \frac{-1}{7} R_3$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\Rightarrow$ Linearly independent

$\Rightarrow$ only trivial solution

Hence A has a pivot in every column.

So T is one-to-one

2. Let A be the matrix $A = \begin{bmatrix} 1 & -3 & 4 \\ 3 & 0 & 3 \\ 2 & 1 & 1 \end{bmatrix}$

and define $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $T(x) = Ax$. Is T is one-to one? If Not, find two different vectors u, v such that $T(u) = T(v)$.

Sol: Given that $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$T(x) = Ax$$

where $A = \begin{bmatrix} 1 & -3 & 4 \\ 3 & 0 & 3 \\ 2 & 1 & 1 \end{bmatrix}$

The reduce row echelon form of augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 3 & 0 & 3 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

$$= \left[\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 0 & 9 & -9 & 0 \\ 0 & 7 & -7 & 0 \end{array}\right]$$

$$R_2 \rightarrow \frac{1}{9} R_2$$

$$R_3 \rightarrow \frac{1}{7} R_3$$

$$= \left[\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$R_1 \rightarrow R_1 + 3R_2$$

$$R_3 \rightarrow R_3 - R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 7 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\hookrightarrow$ Free variable

$\Rightarrow$ Infinity no. of Solution [not just trivial]

$\Rightarrow$ Non-trivial Solⁿ

$\Rightarrow$ Linearly dependent columns

There is not a pivot in every column, so T is not one-to one

we have $Ax = 0$ has non trivial solutions.

If v is a non-trivial (i.e., non zero) solution of $Av = 0$, then $T(v) = Av$

$$\Rightarrow T(v) = 0$$

$$\Rightarrow T(v) = A0$$

$$\boxed{T(v) = T(0)}$$

So o and v are different vectors with the same output. In order to find a non-trivial solution, we find the parametric form of the solutions of $Ax = 0$ using reduced matrix above.

$$x + 0y + 7z = 0 \Rightarrow \quad x + 7z = 0 \rightarrow (1)$$

$$y - z = 0 \qquad\qquad y - z = 0 \rightarrow (2)$$

The free variables are z.

put $z = k$

from ② $y - z = 0$; from ① $x + 7z = 0$

$y = z$ $\qquad$ $x = -7z$

$\boxed{y = k}$ $\qquad$ $\boxed{x = -7k}$

$\therefore \boxed{(x, y, z) = (-7k, k, k)}$

If $k = 1$ then $\boxed{(x, y, z) = (-7, 1, 1)}$

-7

$T(x) = Ax$

$$T\begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 4 \\ 3 & 0 & 3 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}$$

$$T\begin{bmatrix}-7\\1\\1\end{bmatrix} = \begin{bmatrix}-6\\-18\\-12\end{bmatrix}$$

$$\Rightarrow \quad T\begin{bmatrix}-7\\1\\1\end{bmatrix} = T\begin{bmatrix}-6\\-18\\-12\end{bmatrix}$$

$\#$) Here $u = (-7, 1, 1)$ and $v = (-6, -18, -12)$

3. Let A be the matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix}$

and define $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by $T(x) = Ax$. Is T one-to-one? If not, find two different vectors u, v such that $T(u) = T(v)$

Sol^n By data $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ and $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix}$

$T(x) = Ax$

The reduce row echelon form of A

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ -2 & 2 & -4 & 0 \end{array}\right]$$

$R_2 \longrightarrow R_2 + 2R_1$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Free variable

There is not a pivot in column,

So T is not one-to-one

Therefore, $Ax = 0$ has nontrivial solutions.

If v is a nontrivial (i.e., non zero) solution of $Av = 0$

then

$$T(v) = Av$$

$$\Rightarrow T(v) = 0$$

$$\Rightarrow T(v) = A0$$

$$\Rightarrow \boxed{T(v) = T(0)}$$

So u & v are different vectors with same output.

In order to find a non-trivial solution, we find the parametric form of the solution of $Ax = 0$ using reduced matrix above

$$x - y + 2z = 0$$

$$\Rightarrow x = y - 2z \rightarrow (1)$$

The free variables are y & z.

Taking $y = k_1$ & $z = k_2$ gives the non-trivial soln, then $x = k_1 - 2k_2$

$\therefore (x, y, z) = (k_1 - 2k_2, k_1, k_2)$

If $k_1 = 1$ & $k_2 = 0$

$(x, y, z) = (1, 1, 0)$

$T(x) = AX$

$$T\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$T\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$\Rightarrow T\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = T\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

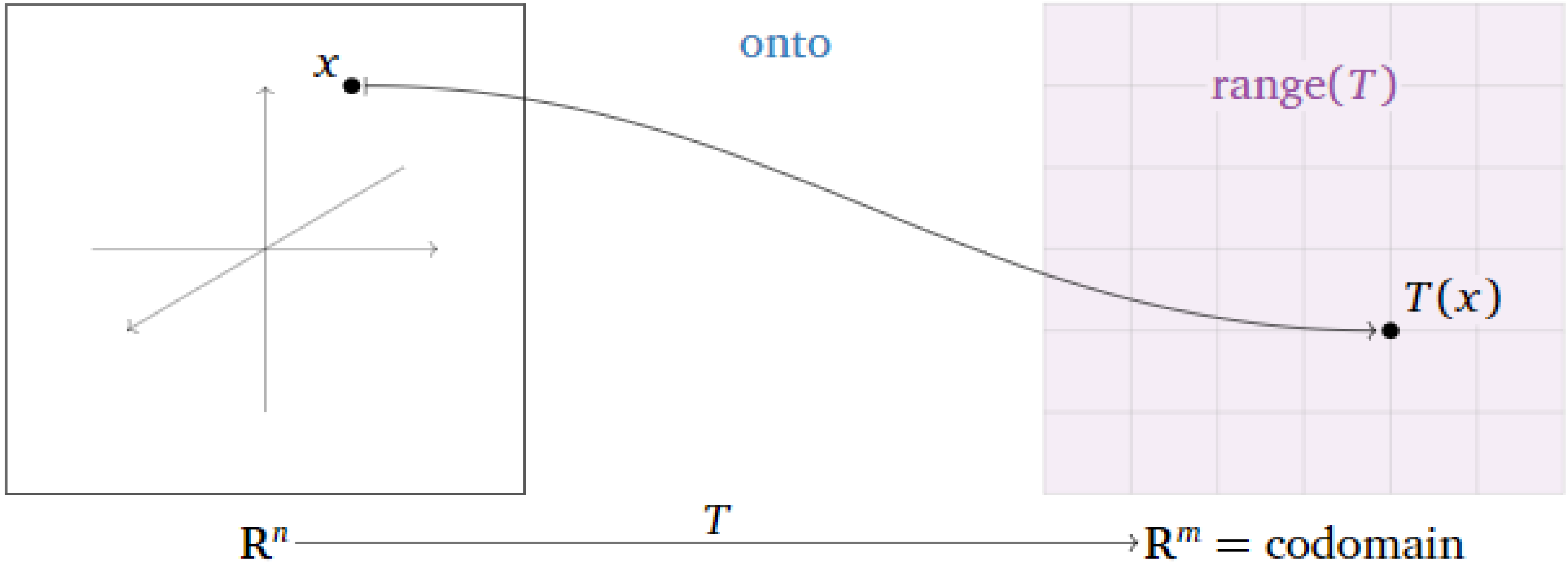
$u = (1, 1, 0)$ and $v = (0, 0, 0)$

Surjective Transformation [onto Transformation] :→

A transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is onto if, for every vector b in $\mathbb{R}^m$, the equation $T(x) = b$ has at least one solution x in $\mathbb{R}^n$.

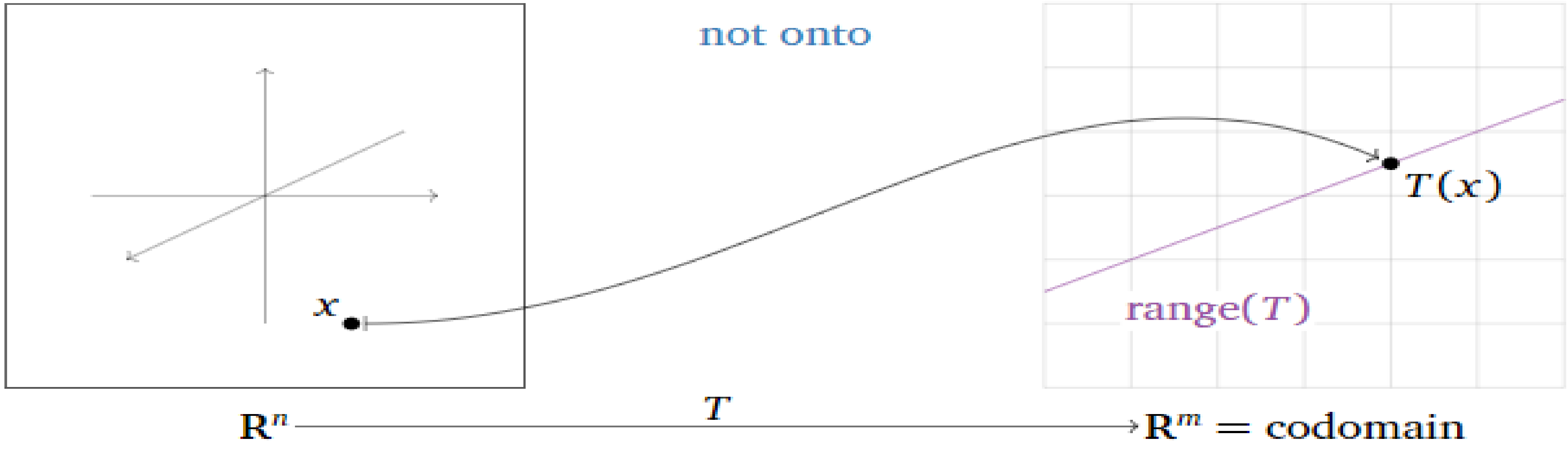
→ Here are some equivalent ways of saying that T is onto

* The range of T is equal to the codomain of T
* Every vector in the codomain is the output of some input vector.



→ Here are some equivalent ways of saying that T not onto

* The range of T is smaller than the codomain of T
* There exists a vector b in $\mathbb{R}^m$ such that the equation $T(x) = b$ does not have a solution
* There is a vector in the codomain that is not the output of any input vector.



Theorem (Onto matrix transformations). *Let A be an $m \times n$ matrix, and let $T(x) = Ax$ be the associated matrix transformation. The following statements are equivalent:*

1. *T is onto.*
2. *$T(x) = b$ has at least one solution for every b in R^m.*
3. *$Ax = b$ is consistent for every b in R^m.*
4. *The columns of A span R^m.*
5. *A has a pivot in every row.*
6. *The range of T has dimension m.*

1. Let A be the matrix $A = \begin{bmatrix} 1 & -3 & 4 \\ 3 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ and

defined $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $T(x) = Ax$. Is T onto?

Sol^n By defn $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

and $T(x) = Ax$ and $A = \begin{bmatrix} 1 & -3 & 4 \\ 3 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix}$

The reduced row ~~ech~~ echelon form of A is

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 3 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

$$R_2 \longleftrightarrow R_3$$

$$= \left[\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 \longrightarrow R_2 - 2R_1$$

$$R_2 \longrightarrow R_2 - 2R_1$$

$$R_3 \longrightarrow R_3 - 3R_1$$

$$= \left[\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 0 & 7 & -7 & 0 \\ 0 & 9 & -12 & 0 \end{array}\right]$$

$$R_2 \longrightarrow \frac{1}{7} R_2$$

$$R_3 \longrightarrow \frac{1}{3} R_3$$

$$= \left[\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right]$$

$R_1 \longrightarrow R_1 + 3R_2$

$R_3 \longrightarrow R_3 - 3R_2$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 7 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -7 & 0 \end{array}\right]$$

$R_1 \longrightarrow R_1 + R_3$

$R_2 \longrightarrow 7R_2 + R_3$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -7 & 0 \end{array}\right]$$

$R_3 \longrightarrow \frac{-1}{7} R_3$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Hence A has a pivot in every row, So T is onto

$\Rightarrow$ To Span $\mathbb{R}^3$, need 3 linearly independent vectors.

[Here $T: \mathbb{R}^3 \to \mathbb{R}^3$ onto if columns of A span $\mathbb{R}^3$]

$$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \right\}$$ are independent

$\Rightarrow$ T is onto

(2) Let A be the matrix $A = \begin{bmatrix} 1 & -3 & 4 \\ 3 & 0 & 3 \\ 2 & 1 & 1 \end{bmatrix}$

and define $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $T(x) = Ax$. Is T onto?

Sol^n Given that $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$T(x) = Ax$ & $A = \begin{bmatrix} 1 & -3 & 4 \\ 3 & 0 & 3 \\ 2 & 1 & 1 \end{bmatrix}$

The red...

The reduce row echelon form of augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 3 & 0 & 3 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

$R_2 \longrightarrow R_2 - 3R_1$

$R_3 \longrightarrow R_3 - 2R_1$

$$= \left[\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 0 & 9 & -9 & 0 \\ 0 & 7 & -7 & 0 \end{array}\right]$$

$R_2 \rightarrow \frac{1}{9} R_2$

$R_3 \longrightarrow \frac{1}{7} R_3$

$$= \left[\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$R_1 \longrightarrow R_1 + 3R_2$

$R_3 \longrightarrow R_3 - R_2$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 7 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \text{Free Variable}$$

There is not a pivot in every row, so T is not onto.

The range of T is the column space of A, which is equal to

$$\text{Span}\left\{\begin{bmatrix}1\\3\\2\end{bmatrix}, \begin{bmatrix}-3\\0\\1\end{bmatrix}, \begin{bmatrix}4\\3\\1\end{bmatrix}\right\} \text{ are not}$$

independent

T is not onto.

3. Let A be the matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix}$

and define $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by $T(x) = Ax$. Is T onto?

Sol^n Given that $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$T(x) = Ax$ and $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix}$

The reduce echelon form of A

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ -2 & 2 & -4 & 0 \end{array}\right]$$

$R_2 \longrightarrow R_2 + 2R_1$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Free variable

There is not a pivot in every row,

So T is not onto.

The range of T is the column space of A.

which is equal to

$$\text{Span}\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix} \right\}$$ are ~~ind~~

not independent.

T is not onto.

Rank and Nullity of a linear transformation :→

$\boxed{\text{Rank}} + \boxed{\text{Nullity}} = \boxed{\text{Dimension}}$

Rank ↓ No. of non-zero rows

Nullity ↓ No. of zero rows

Dimension ↓ No. of rows

Rank of a linear transformation :→

Let T be a linear transformation from a vector Space U to a vector Space V, $T : U \longrightarrow V$

Let T be a linear transformation from a vector Space U to a vector Space V, $T : U \longrightarrow V$

The null space of T is denoted by $N(T)$ and is defined by $N(T) = \{ \alpha \in U / T(\alpha) \}$

where '0' is the zero element of V.

Null space of a linear transformation is also kernel of a linear transformation.

Dimension of Null Space is called "Nullity"

Range space of a linear transformation :-

Let $T : U \longrightarrow V$ over F. Range space of T is denoted by $R(T)$ and defined as

$$R(T) = \{T(\alpha) \mid \alpha \in V\}$$

Dimension of range space is called 'Rank'

1. Find the range space and null space rank of transformation $T : V_2(R) \longrightarrow V_2(R)$ is defined by $T(x, y) = (x+y, y)$

Sol

Sol: Given that $T : V_2(R) \longrightarrow V_2(R)$

$$T(x, y) = (x+y, y)$$

Let e_1 and e_2 be the standard basis vectors in $V_2(R)$

$$T(e_1) = T(1,0) = (1,0)$$

$$T(e_2) = T(0,1) = (1,1)$$

Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $\quad R_2 \rightarrow R_2 - R_1$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

rank = 2
Nullity = 0 } dimension = 2

$\therefore$ Dimension = rank + nullity

$\Rightarrow$ Dimension = 2 + 0

$\Rightarrow$ $\boxed{\text{Dimension} = 2}$

2. If $T : R^3 \longrightarrow R^3$ defined by $T(x, y, z) = (x+y, x-y, 2x+z)$
Find $R(T)$, $N(T)$

Sol Given that $T : R^3 \longrightarrow R^3$

$$T(x, y, z) = (x+y, x-y, 2x+z)$$

Let e_1, e_2 and e_3 be the standard basis vector in R^3

$T(e_1) = T(1,0,0) = (1,1,2)$

$T(e_2) = T(0,1,0) = (1,-1,0)$

$T(e_3) = T(0,0,1) = (0,0,1)$

Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_2 \to R_2 - R_1$

$= \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

rank $= 3$, Nullity $= 0$

$R(T) = 3$; $N(T) = 0$

$\therefore$ Dimension $=$ rank $+$ Nullity

$= 3 + 0$

$\boxed{\text{Dim.} = 3}$

Rank Nullity theorem :→

If $T : U \longrightarrow V$ be a linear transformation and U be a finite dimension vector space then

$$\text{Dim}(R(T)) + \text{Dim}(N(T)) = \text{Dimension of domain},$$

3. If $T : R^3 \longrightarrow R^3$ defined by $T(x, y, z) = (x+y, x-y, 2x+z)$ find $R(T)$, $N(T)$ and verify rank-nullity theorem

Sol^n Given that $T : R^3 \longrightarrow R^3$

$$T(x, y, z) = (x+y, x-y, 2x+z)$$

The co-efficient matrix $= \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_2 \longrightarrow R_2 - R_1$

$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

$R_2 \longrightarrow -\frac{1}{2} R_2$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

rank = 3, Nullity = 0

Dimension = rank + Nullity

= 3 + 0

$\boxed{\text{Dim.} = 3}$

Range space $R(T) = \text{Span}\{(1,1,2), (0,1,1), (0,0,1)\}$

$= \{x(1,1,2) + y(0,1,1) + z(0,0,1) \mid x, y, z \in R\}$

$= \{(x, x, 2x), (0, y, y), (0, 0, z) \mid x, y, z \in R\}$

$= \{(x, x, 2x), (0, y, y), (0, 0, z) \mid x, y, z \in R\}$

$= \{(x, x+y, 2x+y+z) / x, y, z \in R\}$

Let $T(x, y, z) \in N(T)$

i.e., $T(x, y, z) = 0$

$(x+y, x-y, 2x+y+z) = (0, 0, 0)$

$\Rightarrow$ $2x = 0$ $\quad x = y$

$\boxed{x = 0}$ $\quad \boxed{y = 0}$ $\quad \boxed{z = 0}$

$\boxed{N(T) = (0, 0, 0)}$

Rank nullity theorem is verified

4. Find the range space and null space of the transfo-rmation defined by $y' = x$ and $x' = y$ and verify RNT

Sol: Let $T(x, y) = (x', y')$

$$T(x, y) = (y, x)$$

Let e_1 and e_2 be the standard basis vectors in $R_2(V)$

$T(e_1) = T(1, 0) = (0, 1)$

$T(e_2) = T(0, 1) = (1, 0)$

$\Rightarrow$ $T(e_1) = (0, 1)$

$T(e_2) = (1, 0)$

The co-efficient matrix $= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\boxed{\text{rank} = 2}$; $\boxed{\text{Nullity} = 0}$

$R_2 \longleftrightarrow R_1$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Range Space $= \text{Span}\{(1,0);(0,1)\}$

$= \{x(1,0) + y(0,1) \mid x, y \in \mathbb{R}\}$

$= \{(x,0) + (0,y) \mid x, y \in \mathbb{R}\}$

$= \{(x,y) \mid x, y \in \mathbb{R}\}$

Let $T(x, y) \in N(T)$

$$\Rightarrow T(x, y) = 0$$

$$\Rightarrow (y, x) = (0, 0)$$

$$\Rightarrow \boxed{x = 0}, \boxed{y = 0}$$

$$\therefore N(T) = (0, 0)$$

$\therefore$ Rank Nullity theorem is verified

5. Find the rank and nullity of $T : R^3 \longrightarrow R^3$ given by $T(e_1) = e_1 - e_2$, $T(e_2) = 2e_1 + e_2$ and $T(e_3) = e_1 + e_2 + e_3$

$e_1 + e_2 + e_3$

Sol: Given that $T : R^3 \longrightarrow R^3$

$T(e_1) = e_1 - e_2$; $T(e_2) = 2e_1 + e_2$ and $T(e_3) = e_1 + e_2 + e_3$

we have

$$T(e_1) = (1, 0, 0) - (0, 1, 0)$$

$$T(e_1) = (1, -1, 0)$$

$$T(e_2) = (2, 0, 0) + (0, 1, 0)$$

$$T(e_2) = (2, 1, 0)$$

$$T(e_3) = (1, 0, 0) + (0, 1, 0) + (0, 0, 1)$$

$$T(e_3) = (1, 1, 1)$$

The co-efficient matrix = $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 0 \\ 0 & 2 & 1 \end{bmatrix}$.

$R_3 \to 3R_3 - 2R_2$

$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\quad R_2 \to \frac{1}{3} R_2$

$R_3 \to \frac{1}{3} R_3$

$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore$ Rank = 3 ; Nullity = 0

6. If $T: V_3(R) \longrightarrow V_2(R)$ defined by $T(e_1) = (2,1)$, $T(e_2) = (0,1)$, $T(e_3) = (1,1)$, find $R(T)$, $N(T)$ and verify rank - nullity theorem.

Soln By data $T: V_3(R) \longrightarrow V_2(R)$

$T(e_1) = (2,1)$; $T(e_2) = (0,1)$ and $T(e_3) = (1,1)$

The co-efficient matrix $= \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad R_3 \rightarrow 2R_3 - R_1$

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad R_1 \to \frac{1}{2} R_1$$

$$A = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ rank $= 2$

Nullity $= 1$

Dim (domain) $= 3$

Range Space $= \text{Span}\{(2,1), (0,1)\}$

$= \{x(2,1) + y(0,1)\}$

$= \{(2x, x), (0, y) \mid x, y \in \mathbb{R}\}$

$\boxed{\text{Range space} = \{(2x, x+y) \mid x, y \in \mathbb{R}\}}$

Let $T(x, y, z) \in N(T)$

$\Rightarrow T(x, y, z) = 0$

$T(x, y, z) = x(2,1) + y(0,1) + z(1,1)$

$T(x, y, z) = (2x + z, x + y + z)$

$\Rightarrow (2x + z, x + y + z) = 0$

$\Rightarrow$ $2x + z = 0$ $\quad$ $x + y + z = 0$

$\boxed{z = -2x}$ $\quad$ $x + y - 2x = 0$

$y - x = 0$

$\boxed{y = x}$

$\boxed{\therefore x = y = t \;;\; z = -2t}$

$\therefore N(T) = (t, t, -2t)$ where $t \in R$

7. If $T : V_3(R) \longrightarrow V_4(R)$ defined by $T(e_1) = (0,1,0,2)$, $T(e_2) = (0,1,1,0)$ and $T(e_3) = (0,1,-1,4)$. Find $R(T)$ $N(T)$ and verify Rank-nullity theorem.

Sol^n By data $T : V_3(R) \longrightarrow V_4(R)$

$T(e_1) = (0,1,0,2)$; $T(e_2) = (0,1,1,0)$; $T(e_3) = (0,1,-1,4)$

The co-efficient matrix $= \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 4 \end{bmatrix}$

Let $A = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 4 \end{bmatrix}$ $\quad R_2 \longrightarrow R_2 - R_1$

$\quad R_3 \longrightarrow R_3 - R_1$

$$A = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank $= 2$; Nullity $= 1$ $\therefore$ dim(domain) $= 3$

Range Space $=$ Span$\{(0,1,0,2), (0,0,1,-2)\}$

$= \{x(0,1,0,2) + y(0,0,1,-2)\}$

$$= \{(0, x, 0, 2x) + (0, 0, y, -2y)\}$$

Range space $= \{(0, x, y, 2x - 2y) \mid x, y \in R\}$

Let $T(x, y, z) \in N(T)$

$$\Rightarrow T(x, y, z) = 0$$

Now consider

$$T(x, y, z) = x(0, 1, 0, 2) + y(0, 1, 1, 0) + z(0, 1, -1, 4)$$

$$T(x, y, z) = (0, x + y + z, y - z, 2x + 4z)$$

$$\Rightarrow (0, x + y + z, y - z, 2x + 4z) = (0, 0, 0)$$

$x + y + z = 0$; $y - z = 0$ $2x + 4z = 0$

$-2z + y + z = 0$

$\boxed{y - z = 0}$ $\boxed{y = z}$ $2x = -4z$

$\boxed{x = -2z}$

$y = z = t$; $x = -2t$

$\boxed{N(T) = (-2t, t, t)}$ where $t \in R$

8. Find range, rank and Nullity of the transformation whose matrix $\begin{bmatrix} 1 & 3 & 2 \\ -1 & 7 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

Sol: Given that $\begin{bmatrix} 1 & 3 & 2 \\ -1 & 7 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

consider co-efficient matrix $= \begin{bmatrix} 1 & -1 & 1 \\ 3 & 7 & 0 \\ 2 & 2 & 1 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 10 & -3 \\ 0 & 4 & -1 \end{bmatrix} \qquad R_3 \longrightarrow 10R_3 - 4R_2$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 10 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

Rank $= 3$; Nullity $= 0$

dim(domain) = rank + Nullity

$= 3 + 0$

$$\boxed{\text{dim(domain)} = 3}$$

Range Space $= \text{Span}\{(1,-1,1), (0,10,-3), (0,0,1)\}$

$$= \{x(1,-1,1) + y(0,10,-3) + z(0,0,1) \mid x, y, z \in \mathbb{R}\}$$

$$R(T) = \{(x, 10y-x, x-3y+z) \mid x, y, z \in \mathbb{R}\}$$

Now

$$T(x,y,z) = xT(e_1) + yT(e_2) + zT(e_3)$$

$$= x(1,-1,1) + y(3,7,0) + z(2,2,1)$$

$$= (x,-x,x) + (3y,7y,0) + (2z,2z,z)$$

$$T(x,y,z) = (x+3y+2z, -x+7y+2z, x+z)$$

Let $T(x, y, z) \in N(T)$

$$\Rightarrow T(x, y, z) = 0$$

$$(x+3y+2z, -x+7y+2z, x+z) = (0, 0, 0)$$

$$x+3y+2z=0 \; ; \; -x+7y+2z=0 \; ; \; x+z=0$$

$$\begin{bmatrix} 1 & 3 & 2 \\ -1 & 7 & 2 \\ 1 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} R_2 \longrightarrow R_2 + R_1 \\ R_3 \longrightarrow R_3 - R_1 \end{array}$$

$$= \begin{bmatrix} 1 & 3 & 2 \\ 0 & 10 & 4 \\ 0 & -3 & -1 \end{bmatrix} \quad R_3 \longrightarrow 10R_3 + 3R_2$$

$$= \begin{bmatrix} 1 & 3 & 2 \\ 0 & 10 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

$z = 0, \quad y = 0, \quad x = 0$

$\therefore \; N(T) = \underline{\underline{(0, 0, 0)}}$

Isomorphism of a linear transformation

Let V and W be a vector spaces over F. A linear trans-formation $T : V \to W$ is called an isomorphism if T is one to one and onto.

If there is an isomorphism from V to W then V and W are called isomorphic.

An invertible linear transformation is called an isomorphism. we say the linear space V and W are isomorphic if there is an isomorphism from V to W.

Examples :→

1. V is Isomorphic to itself because the identity map is an isomorphism from V onto itself.

2. Every 'n' dimensional vector space over R is isomorphic to R^n

Propertiies of isomorphism

1. If $T : V \longrightarrow W$ is an isomorphism then $T^{-1} : W \longrightarrow V$ is also an isomorphism.

2. For any linear transformation $T : V \longrightarrow W$, $T(0) = 0$, Further if T is an isomorphism then $T(v) = 0$ $\Rightarrow v = 0$.

3. If $T : V \to W$ is an isomorphism and $S = \{v_1, v_2, \ldots, v_k\}$ is a linearly independent set then $T(S) = \{T(v_1), T(v_2), \ldots, T(v_k)\}$ is linearly independent set.

Theorem :→ Two finite dimensional vector spaces over the same field F are isomorphic if and only if they have same dimension.

Eg :→ 1. $T : \mathbb{R}^4 \longrightarrow M_2(\mathbb{R})$ T is linear, one-to-one and onto

$$T\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\Rightarrow \mathbb{R}^4 \cong M_2(\mathbb{R})$

2. $T : M_2(\mathbb{R}) \longrightarrow \mathbb{R}_3[x]$

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ax^3 + bx^2 + cx + d$$

T is linear, one-to-one and onto

$\therefore M_2(\mathbb{R}) \cong \mathbb{R}_3[x]$

3. There is no isomorphism b/w $\mathbb{R}^3$ and $M_2(\mathbb{R})$

Since $\dim \mathbb{R}^3 = 3 \neq 4 = \dim M_2(\mathbb{R})$

1. consider the transformation $T\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

from $\mathbb{R}^4$ to $\mathbb{R}^{2\times 2}$. we are told T is linear transformation. Show that T is invertible

Sol^n The most direct way to show that a function is invertible is to find its inverse.

we can see that

$$T^{-1}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

The linear spaces $\mathbb{R}^4$ and $\mathbb{R}^{2\times 2}$ have essentially the same structure. we say that the linear spaces $\mathbb{R}^4$ and $\mathbb{R}^{2\times 2}$ are isomorphic.

2. Show that the transformation $T(A) = S^{-1}AS$ from $\mathbb{R}^{2\times2}$ to $\mathbb{R}^{2\times2}$ is an isomorphism.

Sol^n To show that T is linear transformation, and that T is invertible.

Let's think about the linearity of T first:

$$T(M+N) = S^{-1}(M+N)S$$

$$= S^{-1}(MS+NS)$$

$$= S^{-1}MS + S^{-1}NS$$

$$\Rightarrow \boxed{T(M+N) = S^{-1}MS + S^{-1}NS}$$

Next $T(KA) = S^{-1}(KA)S$

$= K(S^{-1}AS)$

$\Rightarrow \boxed{T(KA) = K(S^{-1}AS)}$

The inverse transformation is

$$\boxed{T^{-1}(B) = SBS^{-1}}$$

Properties of isomorphism

1. If T is an isomorphism, then so is T^{-1}

2. If V and W are isomorphism and $\dim(V) = n$, then $\dim(W) = n$.

Singular and Non-Singular linear transformation

Let U and V be 2 vector spaces over F, a linear transformation $T : U \rightarrow V$ is said to be non-singular if $N(T) = 0$

Nullity

otherwise T is singular

Invertible linear transformation :→

Let U and V be any two vector spaces over F. A linear transformation $T : U \longrightarrow V$ is said to be invertible if there exists a linear transformation $S : V \longrightarrow U$ such that

$$S o T = T o S = I$$

Here S is called inverse of T and is denoted by T^{-1}

1. Show that the mapping $T: V_3(\mathbb{R}) \longrightarrow V_3(\mathbb{R})$ defined by $T(e_1) = e_1 + e_2$, $T(e_2) = e_2 + e_3$, $T(e_3) = e_1 + e_2 + e_3$ is non-singular and find its inverse.

Sol: Given that $T: V_3(\mathbb{R}) \longrightarrow V_3(\mathbb{R})$

$$T(e_1) = e_1 + e_2, \quad T(e_2) = e_2 + e_3 \text{ and } T(e_3) = e_1 + e_2 + e_3$$

we have

$$T(e_1) = (1,0,0) + (0,1,0)$$

$$T(e_1) = (1,1,0)$$

$$T(e_2) = (0,1,0) + (0,0,1)$$

$$T(e_2) = (0,1,1)$$

$$T(e_3) = (1,0,0) + (0,1,0) + (0,0,1)$$

$$T(e_3) = (1,1,1) \qquad \therefore A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow |A| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1(0) - 1(-1) + 0(-1)$$

$$= 1 \neq 0$$

$$\Rightarrow |A| = 1 \neq 0 \qquad \boxed{\therefore T \text{ is non-singular}}$$

$\boxed{\text{OR}}$

$$T(x, y, z) = xT(e_1) + yT(e_2) + zT(e_3)$$

$$= (x, x, 0) + (0, y, y) + (z, z, z)$$

$$\boxed{T(x, y, z) = (x+z,\ x+y+z,\ y+z)}$$

Let $(x, y, z) \in N(T) \implies T(x, y, z) = 0$

i.e., $(x+z,\ x+y+z,\ y+z) = (0, 0, 0)$

$x+z=0$; $x+y+z=0$; $y+z=0$

$\Rightarrow \boxed{x=0,\ y=0,\ z=0}$

$(x, y, z) = (0, 0, 0)$

$\therefore$ T is non-singular

Also to find Inverse of T = ?

Let $T(x, y, z)$ be the pre-image of (r, s, t)

then $T(x, y, z) = (r, s, t)$

$\Rightarrow (x+z,\ x+y+z,\ y+z) = (r, s, t)$

$$\Rightarrow x + z = r \rightarrow (1)$$

$$x + y + z = s \rightarrow (2)$$

$$y + z = t \rightarrow (3)$$

from (1) & (2)

$$\begin{array}{r} \cancel{x} + 0y + \cancel{z} = r \\ + \cancel{x} + y + \cancel{z} = s \\ (-) \quad (-) \quad (-) \quad (-) \\ \hline -y = r - s \end{array}$$

$$\Rightarrow \boxed{y = s - r}$$

from ③ $z = t - y$

$= t - (s - r)$

$\boxed{z = t - s + r}$

from ① $x = r - z$

$x = r - (t - s + r)$

$= \cancel{r} - t + s - \cancel{r}$

$\boxed{x = s - t}$

$\boxed{x = s - t}$, $\boxed{y = s - r}$, $\boxed{z = t - s + r}$

$\therefore T^{-1}(r, s, t) = (x, y, z)$

$$\boxed{T^{-1}(r, s, t) = (s - t,\ s - r,\ t - s + r)}$$

2. Let T be the operator on $\mathbb{R}^3$ defined by $T(x, y, z) = (2x, 4x-y, 2x+3y-z)$ Show that T is invertible and find a formula for T^{-1}

Sol $\rightarrow$ By data $T(x, y, z) = (2x, 4x-y, 2x+3y-z)$

Let $(x, y, z) \in N(T)$

$$\Rightarrow T(x, y, z) = 0$$

$$(2x, 4x-y, 2x+3y-z) = (0, 0, 0)$$

$2x = 0$ $\quad$ $4x - y = 0$ $\quad$ $2x + 3y - z = 0$

$\Rightarrow \boxed{x = 0}$ $\quad$ $\boxed{y = 0}$ $\quad$ $\boxed{z = 0}$

$N(T) = (0, 0, 0)$

$\therefore$ T is non-Singular and hence invertible

Let (x, y, z) be the pre-image of (r, s, t) then $T(x, y, z) = (r, s, t)$

$\Rightarrow (2x, 4x - y, 2x + 3y - z) = (r, s, t)$

$2x = r$ $\quad$ $4x - y = s$ $\quad$ $2x + 3y - z = t$

$\boxed{x = \frac{r}{2}}$; $\boxed{y = 2r - s}$ $\quad r + 6r - 3s - z = t$

$\boxed{z = 7r - 3s - t}$

$\therefore T^{-1}(r, s, t) = (x, y, z)$

$$T^{-1}(r, s, t) = \left(\frac{r}{2},\ 2r - s,\ 7r - 3s - t\right)$$

3. Show that the operator T on $\mathbb{R}^3$ defined by $T(x, y, z) = (3x - 2y, 2y, x+y+z)$ is invertible. Find the formula for T^{-1}.

Sol^n Let $(x, y, z) \in N(T)$

$$T(x, y, z) = 0$$

$$(3x - 2y, 2y, x+y+z) = (0, 0, 0)$$

$$3x - 2y = 0 ; \quad 2y = 0 \quad x+y+z = 0$$

$$\Rightarrow \quad x = y = z = 0$$

$\therefore$ T is non-singular and invertible

Let (x, y, z) be pre-image of (r, s, t) then

$$T(x, y, z) = (r, s, t)$$

$$(3x - 2y,\ 2y,\ x + y + z) = (r, s, t)$$

$3x - 2y = r$ $\qquad$ $2y = s$ $\qquad$ $x + y + z = t$

$3x = r + s$ $\qquad$ $\boxed{y = \frac{s}{2}}$ $\qquad$ $z = t - x - y$

$\boxed{x = \frac{r+s}{3}}$ $\qquad$ $= t - \left(\frac{r+s}{2}\right) - \frac{s}{3}$

$$z = t - \frac{s}{2} - \frac{(r+s)}{3}$$

$$= t - \frac{s}{2} - \frac{r}{3} - \frac{s}{3}$$

$$= \frac{6t - 3s - 2r - 2s}{6}$$

$$\boxed{z = \frac{6t - 5s - 2r}{6}}$$

$$T^{-1}(r, s, t) = \left[\frac{r+s}{3}, \frac{s}{2}, \frac{6t - 5s - 2r}{6}\right]$$

4. Find the matrix of Linear transformation $T: V_2(R) \longrightarrow V_3(R)$ defined by $T(x, y) = (x+y, x, 3x-y)$ relative to the basis $B_1 = \{(1,1), (3,1)\}$ and $B_2 = \{(1,1,1), (1,1,0), (1,0,0)\}$

Sol: Given that $T: V_2(R) \longrightarrow V_3(R)$

$$T(x, y) = (x+y, x, 3x-y)$$

$$B_1 = \{(1,1), (3,1)\} \text{ and}$$

$$B_2 = \{(1,1,1), (1,1,0), (1,0,0)\}$$

Now $T(1,1) = (2,1,2)$

Let $(2,1,2) = c_1(1,1,1) + c_2(1,1,0) + c_3(1,0,0)$

$(2,1,2) = (c_1+c_2+c_3,\ c_1+c_2,\ c_1)$

$\Rightarrow c_1+c_2+c_3 = 2 \quad ; \quad c_1+c_2 = 1 \quad ; \quad \boxed{c_1 = 2}$

$\Rightarrow c_3 = 2 - c_1 - c_2 \qquad\qquad c_2 = 1 - c_1$

$= 2 - 2 - (-1) \qquad\qquad = 1 - 2$

$\boxed{c_2 = -1}$

$= \cancel{2} - \cancel{2} + 1$

$\boxed{c_3 = 1} \qquad \therefore (c_1, c_2, c_3) = (2, -1, 1)$

Also $\quad T(x, y) = (x+y,\ x,\ 3x-y)$

$\Rightarrow T(3,1) = (4, 3, 8)$

Let $(4,3,8) = c_1'(1,1,1) + c_2'(1,1,0) + c_3'(1,0,0)$

$\Rightarrow (4,3,8) = (c_1' + c_2' + c_3',\ c_1' + c_2',\ c_1')$

$\Rightarrow c_1' + c_2' + c_3' = 4 \quad ; \quad c_1' + c_2' = 3 \quad ; \quad \boxed{c_1' = 8}$

$\Rightarrow c_3' = 4 - c_2' - c_1'$

$c_2' = 3 - c_1'$

$= 4 - (-5) - 8$

$c_2' = 3 - 8$

$= 4 + 5 - 8$

$\boxed{c_2' = -5}$

$\boxed{c_3' = 1}$

$\therefore (c_1', c_2', c_3') = (8, -5, 1)$

5. Find the matrix of linear transformation $T : R^2 \longrightarrow R^2$ defined by $T(x, y) = (2x - 5y, 3x + y)$ relative to the basis $\{(2,1), (3,2)\}$

Sol: By data

$$T : R^2 \longrightarrow R^2$$

$$T(x, y) = (2x - 5y, 3x + y) \longrightarrow (1)$$

$$T(2,1) = (-1, 7) \; ; \; T(3,2) = (-4, 11)$$

Let $(-1, 7) = c_1(2,1) + c_2(3,2)$

$$\Longrightarrow (-1, 7) = (2c_1 + 3c_2, \; c_1 + 2c_2)$$

$\Rightarrow 2c_1 + 3c_2 = -1 \rightarrow (2)$ $\quad c_1 + 2c_2 = 7 \longrightarrow (3)$

Now $\quad \cancel{2c_1} + 3c_2 = -1$

$\Rightarrow \underset{(-)}{+}\cancel{2c_1} \underset{(-)}{+} 4c_2 = \underset{(-)}{+}14$

$-c_2 = -15$

$\boxed{c_2 = 15}$

x'ly by 2 in (3)

$2c_1 + 4c_2 = 14$

from (2)

$2c_1 = -1 - 3c_2$

$\Rightarrow 2c_1 = -1 - 3(15)$

$\Rightarrow 2c_1 = -1 - 45$

$2c_1 = -46 \Rightarrow c_1 = \frac{-46}{2}$

$\boxed{c_1 = -23}$

$\therefore (c_1, c_2) = (-23, 15)$

IIIly $T(3,2) = (-4, 11)$

Let $(-4, 11) = c_1'(2,1) + c_2'(3,2)$

$\Rightarrow (-4, 11) = (2c_1' + 3c_2', c_1' + 2c_2')$

$\Rightarrow 2c_1' + 3c_2' = -4 \rightarrow (4)$

$c_1' + 2c_2' = 11 \rightarrow (5)$

xly by 2 in (3)

$2c_1' + 4c_2' = 22$

Now

$$\begin{array}{r} \cancel{2c_1'} + 3c_2' = -4 \\ \cancel{2c_1'} + 4c_2' = +22 \\ (-) \quad (-) \quad (-) \\ \hline -c_2' = -26 \end{array}$$

$$\boxed{c_2' = 26}$$

from (4) $2c_1' = -4 - 3c_2'$

$\Rightarrow 2c_1' = -4 - 3(26)$

$\Rightarrow 2c_1' = -4 - 78$

$\Rightarrow 2c_1' = -82$

$c_1' = \frac{-82}{2}$

$$\boxed{c_1' = -41}$$

$$\boxed{\therefore (c_1', c_2') = (-41, 26)}$$

Matrix of LT $= \begin{bmatrix} -23 & -41 \\ 15 & 26 \end{bmatrix}$

Linear combinations :→

Let a vector v is a linear combination of vectors $v_1, v_2, v_3 \cdots v_n$ if

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n$$

$\alpha_i, \forall i$ are Scalar

A Set of vectors $\{v_1, v_2 \cdots v_k\}$ is said to be linearly independent if the only scalars $\alpha_1, \alpha_2 \cdots \alpha_k$ satisfying

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_k v_k = 0$$

or

$$\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0.$$

otherwise linearly dependent

Consider a vectore

$$v_1 = \{ b_{11}, b_{12} \ldots b_{1n} \}$$

$$v_2 = \{ b_{21}, b_{22} \ldots b_{2n} \}$$

$$\vdots$$

$$v_n = \{ b_{n1}, b_{n2} \ldots b_{nn} \}$$

then

$$|A| = \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix}$$

$\Rightarrow$ If $|A| = 0 \rightarrow$ Linearly dependent

$\Rightarrow$ If $|A| \neq 0 \rightarrow$ Linearly independent

Eg:- 1. Express the vector $b = \begin{bmatrix} 2 \\ 13 \\ 6 \end{bmatrix}$ as a linear combination of vectors $v_1 = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$v_3 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

Solⁿ we need to find numbers x_1, x_2, x_3 satisfying

$$b = x_1 v_1 + x_2 v_2 + x_3 v_3 \longrightarrow (1)$$

$$\longrightarrow ①$$

$$(2, 13, 6) = x_1(1, 5, -1) + x_2(1, 2, 1) + x_3(1, 4, 3)$$

$$(2, 13, 6) = (x_1 + x_2 + x_3,\ 5x_1 + 2x_2 + 4x_3,\ -x_1 + x_2 + 3x_3)$$

Now

$$x_1 + x_2 + x_3 = 2$$

$$5x_1 + 2x_2 + 4x_3 = 13$$

$$-x_1 + x_2 + 3x_3 = 6$$

Solving $\boxed{x_1 = 1}$ $\boxed{x_2 = -2}$ $\boxed{x_3 = 3}$

Eqn ① becomes

$$\boxed{b = v_1 - 2v_2 + 3v_3}$$

$\boxed{OR}$

we apply elementary row operation and obtain matrix reduced to row echelon form as following

The augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 5 & 2 & 4 & 13 \\ -1 & 1 & 3 & 6 \end{array}\right] \qquad = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -3 & -1 & 3 \\ 0 & 2 & 4 & 8 \end{array}\right]$$

$R_2 \longrightarrow R_2 - 5R_1$

$R_3 \longrightarrow R_3 + R_1$

$R_3 \rightarrow \frac{R_3}{2}$

$R_2 \longrightarrow -R_2$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 3 & 1 & -3 \\ 0 & 1 & 2 & 4 \end{array}\right]$$

$$R_2 \longleftrightarrow R_3$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 1 & -3 \end{array}\right]$$

$$R_1 \longrightarrow R_1 - R_2$$

$$R_3 \longrightarrow R_3 - 3R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -5 & -15 \end{array}\right]$$

$$R_3 \longrightarrow -\frac{1}{5} R_3$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$R_1 \longrightarrow R_1 + R_3$$

$$R_2 \longrightarrow R_2 - 2R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

we have

$\boxed{x_1 = 1}$, $\boxed{x_2 = -2}$ $\boxed{x_3 = 3}$

Eqn (1) becomes

$$\boxed{b = 1v_1 - 2v_2 + 3v_3}$$

(4

2. Find the value of h for which the following set of vectors is linearly independent

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} h \\ 1 \\ -h \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 2h \\ 3h+1 \end{bmatrix}$$

Soln Let us consider the linear combination

$$x_1 v_1 + x_2 v_2 + x v_3 = 0$$

This is homogenous system has only zero solution $x_1 = x_2 = x_3 = 0$ then the vectors v_1, v_2, v_3 are linearly independent.

Now we reduce the Augmented matrix for the system follows

$$\left[\begin{array}{ccc|c} 1 & h & 1 & 0 \\ 0 & 1 & 2h & 0 \\ 0 & -h & 3h+1 & 0 \end{array}\right]$$

$$R_3 \longrightarrow R_3 + hR_2$$

$$= \left[\begin{array}{ccc|c} 1 & h & 1 & 0 \\ 0 & 1 & 2h & 0 \\ 0 & 0 & 2h^2+3h+1 & 0 \end{array}\right]$$

From this we see that the homogenous system has only the zero solution iff

$$2h^2 + 3h + 1 \neq 0$$

$$\Rightarrow \boxed{h \neq -\frac{1}{2}, -1}$$

i.e., 'h' Expect $-1/2, -1$, Vectors v_1, v_2, v_3 are linearly independent

3. Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ If

$b = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 6 \end{bmatrix}$. Express 'b' as a linear combination of vectors (v_1, v_2, v_3)

Let us consider the linear combination

$$b = x_1 v_1 + x_2 v_2 + x_3 v_3 \longrightarrow (1)$$

$$(-1, 0, 3, 6) = x_1(1, 0, 1, 0) + x_2(2, 3, 0, 0) + x_3(0, 1, 0, 1)$$

$\Rightarrow (-1, 0, 3, 6) = (x_1 + 2x_2 , \quad 3x_2 + x_3 , \quad x_1 , \quad x_3)$

Now $\quad x_1 + 2x_2 = -1 \quad \Rightarrow 2x_2 = -1 - x_1$

$3x_2 + x_3 = 0 \qquad\qquad 2x_2 = -1 - 3$

$\boxed{x_1 = 3} \qquad\qquad 2x_2 = -4$

$\boxed{x_3 = 6} \qquad\qquad \boxed{x_2 = -2}$

OR

The augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 3 & 1 & 0 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$R_3 \to R_3 - R_1$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 3 & 1 & 0 \\ 0 & -2 & 0 & 4 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & -2 & 0 & 4 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$R_1 \to R_1 + R_2$

$R_3 \to 2R_3 + 3R_2$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & -2 & 0 & 4 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$R_4 \to 2R_4 - R_3$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & -2 & 0 & 4 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 \longrightarrow \frac{R_2}{2}$

$R_3 \longrightarrow \frac{R_3}{2}$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \boxed{x_1 = 3} \quad \boxed{x_2 = -2}$

$\boxed{x_3 = 6}$

Eqⁿ (1) becomes

$$\boxed{b = 3v_1 - 2v_2 + 6v_3}$$

4. Test for linear independence, we can write the corresponding matrix in Echelon form. So are the following vectors linearly independent or dependent?

(a) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} \right\}$

Sol: Given that

$$x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies x_1 v_1 + x_2 v_2 + x_3 v_3 = (0, 0, 0) \longrightarrow (1)$$

The augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ -2 & 0 & -1 & 0 \\ 0 & 8 & 5 & 0 \end{array}\right]$$

$$R_2 \longrightarrow R_2 + 2R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & 8 & 5 & 0 \end{array}\right]$$

$$R_3 \rightarrow R_3 - R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 \longrightarrow \frac{R_2}{8}$$

$$= \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 1 & 5/8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 \rightarrow R_1 - 4R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 5/8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore \; x_1 + 0x_2 + \frac{1}{2}x_3 = 0$$

$$0x_1 + 1x_2 + \frac{5}{8}x_3 = 0$$

$\Longrightarrow$ Non - trivial Solution

$\therefore$ Linearly dependent

Matrix Transformation :→

It is a function $f : R^n \longrightarrow R^m$ defined by $f(u) = Au$ where A is $(m \times n)$ matrix and u is n-vector then the matrix product Au is a m-vector.

The vector $f(u)$ in R^m is called the image of u, and the set of all images of the vectors in R^n is called the range of f.

Eg :→ 1. If f be the matrix transformation defined by

$$f(u) = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} u.$$

The image of $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is

$$f\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\} = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 - 4 \\ 6 - 1 \end{bmatrix}$$

$$f\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

2. Let A be a 2×3 matrix Say $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 2 \end{bmatrix}$

Now $$Ax = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$Ax = \begin{bmatrix} x - z \\ 3x + y + 2z \end{bmatrix}$$

$$= (x - z, 3x + y + 2z)$$

$f : \mathbb{R}^3 \to \mathbb{R}^2$ is a function, where $f(x) = Ax$

x is an n dimensional vector.

1. Let the function $f(x, y) = (2x+y, y, x-3y)$, which is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$.

The matrix A associated with f will be a 3×2 matrix which will write as $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

we need A to satisfy $f(x) = Ax$ where $x = (x, y)$,

Sol The easiest way to find A is the following

If we let $x = (1, 0)$ then $f(x) = Ax$ is the first column of A

$$f(x, y) = (2x+y,\ y,\ x-3y)$$

$$f(1,0) = (2, 0, 1) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Similarly

If $x = (0,1)$ then $f(x) = Ax$ is the second column of A

$$f(x, y) = (2x+y,\ y,\ x-3y)$$

$$f(0,1) = (1, 1, -3) = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

putting these together, we see that the linear transformation $f(x)$ is associated with the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & -3 \end{bmatrix}$$

2. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

i) write an expression for function f

ii) Find $f(1,0)$ and $f(0,1)$

iii) Find all points (x, y) such that $f(x,y) = (1,0)$

Sol^n (i) Given that $A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

we have $f(x,y) = Ax$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$f(x,y) = \begin{bmatrix} x + 2y \\ 3x + 7y \end{bmatrix}$$

$$\Rightarrow \quad f(x,y) = (x+2y,\ 3x+7y) \rightarrow (i)$$

(ii) from (i) $f(x,y) = (x+2y,\ 3x+7y)$

$$f(1,0) = (1,3)$$

$$f(0,1) = (2,7)$$

(iii) By data $f(x, y) = (1, 0)$

from (1), we have

$$f(x, y) = (x + 2y, 3x + 7y)$$

$$\Rightarrow (1, 0) = (x + 2y, 3x + 7y)$$

$\therefore$ $x + 2y = 1 \rightarrow (1)$ x'ly by 3 $\Rightarrow$ $\cancel{3x} + 6y = 3$

$3x + 7y = 0 \rightarrow (2)$ $\quad \underset{(-)}{+}\cancel{3x} \underset{(-)}{+} 7y = 0$

$-y = 3$

$\boxed{y = -3}$

from (1)

$x = 1 - 2y$

$x = 1 - 2(-3) = 1 + 6$

$\boxed{x = 7}$

$\therefore$ $\boxed{x = 7}$ $\boxed{y = -3}$

Matrix Representation of Linear Transformation :-

Let $T : U \longrightarrow V$ is a linear transformation, U with dimension m, V with dimension n,

$$\beta_1 = \{\alpha_1, \alpha_2, \ldots \alpha_m\}$$

$\beta_2 = \{\beta_1, \beta_2 \ldots \beta_n\}$ be ordered bases of U and V and Transformation is defined by

$$T(a_j) = a_{1j}\beta_1 + a_{2j}\beta_2 + a_{3j}\beta_3 + \cdots \quad ; j = 1, 2 \ldots n$$

then $T(a_j)$ written as column vectors

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

3. Let $T : R^4 \longrightarrow R^3$ be a linear transformation defined by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 3x_3 - x_4 \\ 3x_1 + 5x_2 + 8x_3 - 2x_4 \\ x_1 + x_2 + 2x_3 \end{bmatrix}$$

Find a matrix A such that $T(X) = AX$

Sol: Let e_1, e_2, e_3 and e_4 be the Standard basis vectors in $\mathbb{R}^4$

By data $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 3x_3 - x_4 \\ 3x_1 + 5x_2 + 8x_3 - 2x_4 \\ x_1 + x_2 + 2x_3 \end{bmatrix}$

$T(e_1) = T\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ $\quad T(e_2) = T\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$

$T(e_3) = T\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix}$ $\quad T(e_4) = T\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$

$$\therefore A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & 5 & 8 & -2 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

4. Find the matrix for $T : V_2 \to V_3$ is a linear transformation such that $T(x_1, x_2) = \{x_1 + x_2,\ 2x_1 - x_2,\ 7x_2\}$

Sol^n Given that

$$\beta_1 = \{(1,0), (0,1)\}$$

$$\beta_2 = \{(1,0,0), (0,1,0), (0,0,1)\}$$

$T(x$

$$T(x_1, x_2) = \{x_1 + x_2,\ 2x_1 - x_2,\ 7x_2\}$$

$$T(1,0) = \{1, 2, 0\} \quad \text{and} \quad T(0,1) = \{1, -1, 7\}$$

$$T(a_j) = a_{1j}\beta_1 + a_{2j}\beta_2 + a_{3j}\beta_3 + \dots$$

$$= a_{11}\beta_1 + a_{21}\beta_2 + a_{31}\beta_3 = \overset{a_{11}}{1}(1,0,0) + \overset{a_{21}}{2}(0,1,0) + \overset{a_{31}}{0}(0,0,1),$$

$$= a_{12}\beta_1 + a_{22}\beta_2 + a_{32}\beta_3 = \overset{a_{12}}{1}(1,0,0) \overset{a_{22}}{-1}(0,1,0) + \overset{a_{32}}{7}(0,0,1)$$

$$\text{Matrix } A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 7 \end{bmatrix}$$

5. Find the matrix for $T : V_2 \longrightarrow V_2$ is a linear transformation such that $T(x_1, x_2) = (x_1, -x_2)$ $\beta_1 \{(1,0), (0,1)\}$ and $\beta_2 = \{(1,1), (1,-1)\}$

Sol^n Given data

$$T : V_2 \longrightarrow V_2$$

$$T(x_1, x_2) = (x_1, -x_2)$$

$$T(1,0) = (1,0)$$

and $$T(0,1) = (0,-1)$$

Now $$T(a_j) = a_{1j}\beta_1 + a_{2j}\beta_2 + a_{3j}\beta_3 + \cdots$$

$$(1,0) = a_{11}\beta_1 + a_{21}\beta_2 = \overset{a_{11}}{\frac{1}{2}}(1,1) + \overset{a_{21}}{\frac{1}{2}}(1,-1)$$

$$(0,-1) = a_{12}\beta_1 + a_{22}\beta_2 = \overset{a_{12}}{-\frac{1}{2}}(1,1) + \overset{a_{22}}{\frac{1}{2}}(1,-1)$$

$$\text{matrix } A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

6. Find the matrix of linear transformation $T : V_2(R) \longrightarrow V_2(R)$ defined by $T(x, y) = (x, -y)$

Sol^n Given that

$$T : V_2(R) \longrightarrow V_2(R)$$

$$T(x, y) = (x, -y)$$

Let e_1 and e_2 be the Standard basis vectors in $V_2(R)$

$$T(e_1) = T(1,0) = (1,0)$$

$$T(e_2) = T(0,1) = (0,-1)$$

$$\beta_1 = \{(1,0),(0,1)\}$$

$$\beta_2 = \{(1,0),(0,1)\}$$

$$\therefore T(a_j) = a_{1j}\beta_1 + a_{2j}\beta_2$$

$$(1,0) = a_{11}\beta_1 + a_{21}\beta_2 = \overset{a_{11}}{1}(1,0) + \overset{a_{21}}{0}(0,1)$$

$$(0,1) = a_{12}\beta_1 + a_{22}\beta_2 = \overset{a_{12}}{0}(1,0) + \overset{a_{22}}{(-1)}(0,1)$$

Matrix of LT is $= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

7. Find the matrix of Linear Transformation $T : V_3(R) \longrightarrow V_2(R)$ defined by $T(x, y, z) = (x+y, y+z)$

Sol: Given that $T : V_3(R) \longrightarrow V_2(R)$

$$T(x, y, z) = (x+y, y+z)$$

Let $B_1 = \{(1,0,0), (0,1,0), (0,0,1)\}$ and

$B_2 = \{(1,0), (0,1)\}$

Now e_1, e_2 & e_3 be the standard basis vectors in $V_3(R)$

$\therefore T(x, y, z) = (x+y, y+z) \longrightarrow (1)$

$T(e_1) = T(1,0,0) = (1,0)$

$$T(e_2) = T(0,1,0) = (1,1)$$

$$T(e_3) = T(0,0,1) = (0,1)$$

$$\therefore \; T(a_j) = a_{1j}\beta_1 + a_{2j}\beta_2$$

$$(1,0) = a_{11}\beta_1 + a_{21}\beta_2 = \overset{a_{11}}{1}(1,0) + \overset{a_{21}}{0}(0,1)$$

$$(1,1) = a_{12}\beta_1 + a_{22}\beta_2 = \overset{a_{12}}{1}(1,0) + \overset{a_{22}}{1}(0,1)$$

$$(0,1) = a_{13}\beta_1 + a_{23}\beta_2 = \overset{a_{13}}{0}(1,0) + \overset{a_{23}}{1}(0,1)$$

matrix of LT is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

8. Find the matrix of LT $T: V_3(R) \longrightarrow V_2(R)$ with respect to standard basis defined by

$$T(x, y, z) = (z - 2y, x + 2y - z)$$

Sol: Given that

$$T: V_3(R) \longrightarrow V_2(R)$$

$$T(x, y, z) = (z - 2y, x + 2y - z)$$

Let e_1, e_2 and e_3 be the standard basis vector in $V_3(R)$ and

$$B_1 = \{(1,0,0), (0,1,0), (0,0,1)\}$$

$$B_2 = \{(1,0), (0,1)\}$$

$$T(e_1) = T(1,0,0) = (0,1)$$
$$T(e_2) = T(0,1,0) = (-2,2)$$
$$T(e_3) = T(0,0,1) = (1,-1)$$

$$\therefore T(a_j) = a_{1j}\beta_1 + a_{2j}\beta_2$$

$$(0,1) = a_{11}\beta_1 + a_{21}\beta_2 = \overset{a_{11}}{0}(1,0) + \overset{a_{21}}{1}(0,1)$$
$$(-2,2) = a_{12}\beta_1 + a_{22}\beta_2 = \overset{a_{12}}{-2}(1,0) + \overset{a_{22}}{2}(0,1)$$
$$(1,-1) = a_{13}\beta_1 + a_{23}\beta_2 = \overset{a_{13}}{1}(1,0) \overset{a_{23}}{-1}(0,1)$$

$\therefore$ matrix of LT is $\begin{bmatrix} 0 & -2 & 1 \\ 1 & 2 & -1 \end{bmatrix}$

Note :→ 1. We know that $e_1 = (1,0)$ & $e_2 = (0,1)$

① consider $(x, y) = c_1(1,0) + c_2(0,1)$

$$(x, y) = x(1,0) + y(0,1)$$

$$(x, y) = (x, 0) + (0, y)$$

$$(x, y) = x(1,0) + y(0,1)$$

$$(x, y) = x(e_1) + y(e_2)$$

Next

$$T(x, y) = T(xe_1, ye_2)$$

$$= T(xe_1) + T(ye_2)$$

$$\boxed{T(x, y) = x\,T(e_1) + y\,T(e_2)}$$

Similarly ② $T(x, y, z) = x\,T(e_1) + y\,T(e_2) + z\,T(e_3)$

1. ...

1. Find the linear transformation and also find the matrix of a linear transformation. $T: R^2 \longrightarrow R^3$ such that $T(1,1) = (0,1,2)$, $T(-1,1) = (2,1,0)$

Sol^n Given that $T: R^2 \longrightarrow R^3$

$T(1,1) = (0,1,2)$ and $T(-1,1) = (2,1,0)$

we have $T(1,1) = T(1,0) + T(0,1)$

$T(1,1) = T(e_1) + T(e_2) = (0,1,2) \longrightarrow (1)$

and $T(-1,1) = -T(1,0) + T(0,1)$

$T(-1,1) = -T(e_1) + T(e_2) = (2,1,0) \longrightarrow (2)$

from (1) & (2)

$$\cancel{T(e_1)} + T(e_2) = (0,1,2)$$
$$-\cancel{T(e_1)} + T(e_2) = (2,1,0)$$

$$2T(e_2) = (2,2,2)$$

$$\boxed{T(e_2) = (1,1,1)}$$

from (1) $T(e_1) = (0,1,2) - T(e_2)$

$$= (0,1,2) - (1,1,1) = (-1,0,1)$$

$$\boxed{T(e_1) = (-1,0,1)}$$

W.K.T. $T(x,y) = xT(e_1) + yT(e_2)$

$$= x(-1,0,1) + y(1,1,1)$$

$$= (-x, 0, x) + (y, y, y)$$

$$\boxed{T(x,y) = (-x+y,\ y,\ x+y)} \rightarrow LT$$

Matrix of $LT = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

2. Find the matrix of linear transformation $T: V_2(R) \longrightarrow V_3(R)$ defined by $T(-1,1)=(-1,0,2)$, $T(2,1) = (1,2,1)$ with respect to standard basis.

Sol: Given that $T: V_2(R) \longrightarrow V_3(R)$

$T(-1,1) = (-1,0,2)$ and $T(2,1) = (1,2,1)$

we have $T(-1,1) = -T(1,0) + T(0,1)$

$T(-1,1) = -T(e_1) + T(e_2) = (-1,0,2) \rightarrow (1)$

IIIly

$T(2,1) = 2T(1,0) + T(0,1)$

$T(2,1) = 2T(e_1) + T(e_2) = (1,2,1) \rightarrow (2)$

from (1) & (2) we have

$$-T(e_1) + T(e_2) = (-1, 0, 2)$$
$$\underset{(-)}{\overset{+}{2}}T(e_1) \underset{(-)}{+} T(e_2) = \underset{(-)}{+}(1, 2, 1)$$

$$-3T(e_1) = (-2, -2, 1)$$

$$\Rightarrow \boxed{T(e_1) = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)}$$

from (1)

$$T(e_2) = (-1, 0, 2) + T(e_1)$$
$$= (-1, 0, 2) + \left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right)$$

$$\boxed{T(e_2) = \left(-\frac{1}{3}, \frac{2}{3}, \frac{5}{3}\right)}$$

$\therefore$ The matrix of LT $= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{5}{3} \end{bmatrix}$

3. Find the matrix of linear transformation $T : V_2(\mathbb{R}) \longrightarrow V_2(\mathbb{R})$ such that $T(1,2) = (3,0)$ and $T(2,1) = (1,2)$ with respect to the standard basis

Soln By data $\quad T : V_2(\mathbb{R}) \longrightarrow V_2(\mathbb{R})$

$T(1,2) = (3,0)$ and $T(2,1) = (1,2)$

we have $T(1,2) = T(1,0) + 2T(0,1)$

$T(1,2) = T(e_1) + 2T(e_2) = (3,0) \rightarrow (1)$

and $T(2,1) = 2T(1,0) + T(0,1)$

$T(2,1) = 2T(e_1) + T(e_2) = (1,2) \longrightarrow (2)$

from (1) & (2) $T(e_1) + 2T(e_2) = (3,0) \rightarrow (1)$ x'ly by 2

$2T(e_1) + T(e_2) = (1,2) \rightarrow (2)$

x'ly by 2 in (1) we get

$$\cancel{2T(e_1)} + 4T(e_2) = (6,0)$$

$$\underset{(-)}{+}\cancel{2T(e_1)} \underset{(-)}{+} T(e_2) = \underset{(-)}{+}(1,2)$$

$$3T(e_2) = (5,-2)$$

$$\boxed{T(e_2) = \left(\frac{5}{3}, \frac{-2}{3}\right)}$$

from (1)

$$T(e_1) = (3,0) - 2T(e_2)$$

$$= (3,0) - 2\left(\frac{5}{3}, -\frac{2}{3}\right)$$

$$= (3,0) - \left(\frac{10}{3}, \frac{-4}{3}\right)$$

$$\boxed{T(e_1) = \left(-\frac{1}{3}, \frac{4}{3}\right)}$$

Matrix of LT $= \begin{bmatrix} -\frac{1}{3} & \frac{5}{3} \\ \frac{4}{3} & \frac{-2}{3} \end{bmatrix}$