FACULTY OF ENGINEERING & TECHNOLOGY
Global Campus
Jakkasandra Post, Kanakapura Taluk, Ramanagara District - Pin Code: 562 112

DEPARTMENT OF MATHEMATICS

Mathematics -1

Calculus, Differential Equations and Linear Algebra (Common to all)

Subject Code	**:**	**18BS3MA01**	**Total Contact Hours**	**:**	**45+15**
Credits	**:**	**04**	**Hours per week**	**:**	**04+01**

Course Outcomes

The objective of this course is to familiarize the prospective engineers with techniques in calculus, multivariate analysis and linear algebra. It aims to equip the students with standard concepts and tools at an intermediate to advanced level that will serve them well towards tackling more advanced level of mathematics and applications that they would find useful in their disciplines.

At the end of the course students will be able to learn

CO1 : The differential and integral calculus and basics of improper integrals

CO2 : The Rolle's and Mean value theorems which is applicable to analyze Engineering problems

CO3 : The functions of several variables that are essential in most branches of engineering.

CO4 : The effective mathematical tools for the solutions of differential equations that model physical processes.

CO5 : The essential tool of matrices and linear algebra in a comprehensive manner.

Module 1: Calculus: **(8 lectures+3Tutorials)**

Reduction formulae (without proof), Evaluation of definite and improper integrals; Beta and Gamma functions and their properties. Tracing of standard curves: Strophoid, Leminscate, Cardioid, and Astroid. Applications of definite integrals to evaluate surface areas and volumes of revolutions.

Module 2: Calculus: **(8 lectures+3Tutorials)**

Rolle's Theorem, Mean value theorems, Taylor's and MacLaurin's series for exponential, trigonometric and logarithm functions; indeterminate forms and L'Hospital's rule

Module 3: Multivariable Calculus (Differentiation): **(9 lectures+3Tutorials)**

Partial derivatives, total derivative; Jacobians, Maxima & Minima; Method of Lagrange multipliers; Directional derivatives Gradient, Curl and Divergence.

Module 4: Ordinary differential equations: **(10 lectures+3Tutorials)**

Exact, Linear and Bernoulli's differential equations, higher order linear differential equations with constant coefficients, method of variation of parameters, Cauchy and Legendre's differential equation; Power series solution of differential equation

Module 5: Matrices **(10 lectures+3Tutorials)**

Echelon form, rank of a matrix, System of linear equations: Consistency, solution by Gauss elimination and Gauss-Siedel methods, Eigenvalues and eigenvectors; Diagonalization of matrices. Conversion of an n^{th} order differential equation to a system of first order linear differential equations, Solution of system of linear differential equations by diagonalization method and discuss the stability of the system.

Self-Study : Evolutes and Involutes, Limit, Continuity, Orthogonal Transformation.
Assignment : Eigen values, Eigen vectors, solution of the system of differential equations using MATLAB.
NOTE : No questions will be asked from self-study and assignment section in the exam

REFERENCES

1. B.S. Grewal; Higher Engineering Mathematics, Khanna Publishers, 41st Edition, 2011.
2. Erwin Kreyszig; Advanced Engineering Mathematics, 9th Edition, 2012.
3. B V Ramana; Higher Engineering Mathematics, 10th Reprint Edition, 2010.
4. Dennis G Zill & Michael R Cullen; Advanced Engineering Mathematics, Second Edition; Jones & Barlett Publishers; 2000.

Module-1
CALCULUS

- Reduction formulae (without proof)
- Evaluation of definite and improper integrals.
- Beta and Gamma functions and their properties.
- Tracing of standard curves: Strophoid, Leminscate, cardioid, and Astroid.
- Applications of definite integrals to evaluate surface areas and volumes of revolutions

Introduction

Reduction formulae are basically a recurrence relation which reduces integral of functions of higher degree in the form $\int [f(x)]^n dx$, $\int [f(x)]^m [g(x)]^n dx$ (where m and n are non-negative integrals) to lower degree. The successive application of the recurrence relation finally ends up with a function of degree 0 or 1 so that we can easily complete the integration process.

We discuss three standard reduction formulae in the form of definite integrals & the evaluation of them with standard limits of integration.

Reduction formulae for integration of $\sin^n x$ **,** $\cos^n x$ **,** $\tan^n x$ **,** $\cot^n x$ **,** $\sec^n x$ **and** $\cos ec^n x$ **.**
(Without derivations, where $'n'$ **being positive integer)**

Definition

A formula which expresses (or reduces) the integral of the n^th^ indexed function in terms of that of $(n-1)^{\text{th}}$ indexed or lower indexed function is called a reduction formula.

The two results are very useful in the evaluation of reduction formulae.

Integration by parts

If u and v are functions of x then $\int uvdx = u\int vdx - \int\left(\int vdx\right)u'dx$, where, u' represents differentiation of the function u .

Bernoulli's Formula:

If u and v are functions of x then $\int uvdx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + ...$ where, u' , u'' and u''' …are derivatives of the function u , v_1, v_2 and v_3 … are integrals of the function v .

Reduction formula for $\int \sin^n x\ dx$ **where** n **is a positive integer**

Consider $I_n = \int \sin^n x\, dx$ which can also be written as

$$I_n = \int \sin^{(n-1)} x \cdot \sin x\, dx \qquad \dots (1)$$

Integrating by parts (1) we get

$$I_n = \sin^{n-1} x(-\cos x) - \int (-\cos x)(n-1)\sin^{n-2} x \cos x dx$$

$$\mathrm{I}_n = -\sin^{(n-1)} x \cos x + (n-1)\int \sin^{(n-2)} x \cos^2 x\, dx$$

$$\mathrm{I}_n = -\sin^{(n-1)} x\ \cos x + (n-1)\int \sin^{(n-2)} x(1-\sin^2 x)\ dx$$

$$I_n = -\sin^{(n-1)} x\cos x + (n-1)\int (\sin^{(n-2)} x - \sin^n x)\, dx$$
$$I_n = -\sin^{(n-1)} x\cos x + (n-1)\int \sin^{(n-2)} x dx - (n-1)\int \sin^n x dx$$
$$I_n = -\sin^{(n-1)} x\cos x + (n-1)I_{n-2} - (n-2)I_n$$
$$I_n + (n-1)\,I_n = -\sin^{(n-1)} x\cos x + (n-1)I_{n-2}$$
$$nI_n = -\sin^{(n-1)} x\cos x + (n-1)I_{n-2}$$
$$I_n = \frac{-\sin^{(n-1)} x\cos x}{n} + \frac{(n-1)}{n} I_{n-2} \qquad \text{Where } I_n = \int \sin^n x\, dx \text{ and } I_{n-2} = \int \sin^{n-2} x\, dx$$

Applying the limits of integration i.e., from 0 to $\frac{\pi}{2}$ for the above reduction formula we get

$$I_n = \int_0^{\pi/2} \sin^n x dx = \left.\frac{-\sin^{n-1} x\cos x}{n}\right|_0^{\pi/2} + I_{n-2}$$
$$I_n = \frac{n-1}{n} I_{n-2}$$

Consider $$I_n = \frac{n-1}{n} I_{n-2} \qquad \ldots (2)$$

Replace n by $(n-2)$ in equation (2)

$$I_{n\text{-}2} = \frac{n\text{-}3}{n\text{-}2} I_{n\text{-}4} \qquad \ldots (3)$$

Replace n by $(n-2)$ in equation (3)

$$I_{n\text{-}4} = \frac{n\text{-}5}{n\text{-}4} I_{n\text{-}6}$$

We finally have

$$I_n = \begin{cases} \frac{n\text{-}1}{n}\frac{n\text{-}3}{n\text{-}2}\frac{n\text{-}5}{n\text{-}4}\ldots\frac{1}{2} I_0 & n \text{ is even} \\ \frac{n\text{-}1}{n}\frac{n\text{-}3}{n\text{-}2}\frac{n\text{-}5}{n\text{-}4}\ldots\frac{2}{3} I_1 & n \text{ is odd} \end{cases} \quad \text{Where, } I_0 = \int_0^{\pi/2} \sin^0 x\, dx = \frac{\pi}{2} \text{ and } I_1 = \int_0^{\pi/2} \sin x\, dx = 1$$

$$I_n = \begin{cases} \frac{n\text{-}1}{n}\frac{n\text{-}3}{n\text{-}2}\frac{n\text{-}5}{n\text{-}4}\ldots\frac{1}{2}\frac{\pi}{2} & n \text{ is even} \\ \frac{n\text{-}1}{n}\frac{n\text{-}3}{n\text{-}2}\frac{n\text{-}5}{n\text{-}4}\ldots\frac{2}{3}1 & n \text{ is odd} \end{cases}$$

Reduction formula for $\int \cos^n x\, dx$ where n is a positive integer

Consider $$I_n = \int \cos^n x\, dx \qquad \ldots (1)$$

$$I_n = \int \cos^{n-1} x\cos x\, dx$$

Integrating the above equation by parts we get

$$I_n = \cos^{n-1} x\sin x - \int (\sin x)(n-1)\cos^{n-2} x(-\sin x)dx$$
$$I = \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\, dx$$
$$I = \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x dx - (n-1)\int \cos^n x\, dx$$
$$I = \cos^{n-1} x\sin x + (n-1)\,I_{n-2} - (n-1)\,I_n$$
$$I_n + (n-1)I_n = \cos^{n-1} x\sin x + (n-1)\,I_{n-2}$$
$$nI_n = \cos^{n-1} x\sin x + (n-1)I_{n-2}$$
$$I_n = \frac{\cos^{n-1} x\sin x}{n} + \frac{(n-1)}{n} I_{n-2} \qquad \ldots (2)$$

Next,

Let $I_n = \int_0^{\pi/2} \cos^n x \ dx$

From (2), $I_n = \left(\frac{\cos^{n-1} x \sin x}{n} \right)_0^{\pi/2} + \frac{n-1}{n} I_{n-2}$

But $\cos(\pi/2) = 0 = \sin 0$

Thus $I_n = \frac{n-1}{n} I_{n-2}$

Note : $\int_0^{\pi/2} \cos^n x\, dx = \int_0^{\pi/2} \cos^n \left(\frac{\pi}{2} - x \right) dx = \int_0^{\pi/2} \sin^n x dx$

$$\int_0^{\pi/2} \cos^n x = \begin{cases} \frac{n-1}{n} . \frac{n-3}{n-2} . \frac{n-5}{n-4} .... \frac{2}{3}, If\ is\ n\ odd \\ \frac{n-1}{n} . \frac{n-3}{n-2} . \frac{n-5}{n-4} .... \frac{1}{2} . \frac{\pi}{2} If\ is\ n \text{ even} \end{cases}$$

Problems

1. **Evaluate** $\int \sin^5 x\, dx$

Solution: $\int \sin^5 x\, dx = \frac{-\sin^4 x \cos x}{5} + \frac{4}{5} \int \sin^3 x dx$

$\int \sin^5 x\, dx = \frac{-\sin^4 x \cos x}{5} - \frac{4}{15} \sin^2 x \cos x - \frac{8}{15} \cos x + c$

2. **Evaluate** $\int_0^{\pi/2} \sin^7 x\, dx$

Solution: $\int_0^{\pi/2} \sin^7 x\, dx = \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 1$

$\int_0^{\pi/2} \sin^7 x\, dx = \frac{16}{35}$

3. **Evaluate** $\int_0^{\pi/6} \sin^6 3x\, dx$

Solution: $I = \int_0^{\pi/6} \sin^6 3x\, dx$

Put $3x = t, dx = \frac{1}{3} dt$, when $x = 0, t = 0$

when $x = \frac{\pi}{6},\ t = \frac{\pi}{2}$

$I = \int_0^{\pi/2} \sin^6 t \times \frac{1}{3} dt$

$I = \frac{1}{3} \int_0^{\pi/2} \sin^6 t dt = \frac{1}{3} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$

$I = \frac{5\pi}{96}$

4. **Evaluate** $\int_0^\infty \frac{1}{(1+x^2)^n}\,dx$

Solution: $I=\int_0^\infty \frac{1}{(1+x^2)^n}\,dx$

Put $x = tan\theta$, $dx = sec^2\theta\, d\theta$ when $x=0$ $\theta=0$

when $x\to\infty$ $\theta\to\frac{\pi}{2}$

$$I=\int_0^{\pi/2}\frac{1}{(1+\tan^2\theta)^n}\sec^2\theta\, d\theta$$

$$I=\int_0^{\pi/2}\cos^{2n-2}\theta\, d\theta$$

$$\int_0^\infty \frac{1}{(1+x^2)^n}\,dx=\frac{2n-3}{2n-2}\times\frac{2n-5}{2n-4}\times\frac{2n-7}{2n-6}\cdots\frac{1}{2}\times\frac{\pi}{2}$$

5. **Evaluate** $\int_0^3 \sqrt{\frac{x^3}{3-x}}\,dx$

Solution: $I=\int_0^3 \sqrt{\frac{x^3}{3-x}}\,dx$

Put $x=3sin^2\theta$, $dx=6\,sin\theta\, cos\theta\, d\theta$ when $x=0$ $\theta=0$

when $x=3$ $\theta=\frac{\pi}{2}$

$$I=\int_0^{\pi/2}\sqrt{\frac{27\sin^6\theta}{3-3\sin^2\theta}}\,6\sin\theta\cos\theta\, d\theta$$

$$I=18\int_0^{\pi/2}\sin^4\theta d\theta=18\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}$$

$$\int_0^3 \sqrt{\frac{x^3}{3-x}}\,dx=\frac{27\pi}{8}$$

6. **Evaluate** $\int_0^\infty \frac{x^2}{\sqrt{(1+x^6)^7}}\,dx$

Solution: $I=\int_0^\infty \frac{x^2}{\sqrt{(1+x^6)^7}}\,dx$

Put $x^3=tan\theta$, $3x^2dx=sec^2\theta d\theta$ when $x=0$ $\theta=0$

when $x\to\infty$ $\theta\to\frac{\pi}{2}$

$$I=\frac{1}{3}\int_0^{\pi/2}\frac{1}{\sec^7\theta}\sec^2\theta\, d\theta$$

$$I=\frac{1}{3}\int_0^{\pi/2}\cos^5\theta d\theta=\frac{1}{3}\times\frac{4}{5}\times\frac{2}{3}\times 1$$

$$\int_0^\infty \frac{x^2}{\sqrt{\left(1+x^6\right)^7}}\,dx=\frac{8}{135}$$

Reduction formula for $\int \tan^n x\,dx$ where n is a +ve integer > 1.

Consider $I_n = \int \tan^n x\,dx$

$$I = \int \tan^{n-2} x \tan^2 x\,dx$$

$$I = \int \tan^{n-2} x(\sec^2 x - 1)\,dx$$

$$I = \int (\tan^{n-2} x \sec^2 x - \tan^{n-2} x)\,dx$$

$$I = \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx$$

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2} \qquad \left[\because \int [f(x)]^n f^{'}(x) dx = \frac{[f(x)]^{n-1}}{n+1}\right]$$

Applying the limits of integration i.e., from 0 to $\frac{\pi}{4}$ to the above equation we get

$$I_n = \frac{1}{n-1} - I_{n-2}$$

Reduction formula for $\int \cot^n x$, where n is a +ve integer > 1

Consider $I_n = \int \cot^n x\,dx$

$$I_n = \int \cot^{n-2} x \cot^2 x dx$$

$$I_n = \int \cot^{n-2} x (\operatorname{cosec}^2 x - 1)\,dx$$

$$I_n = \int (\cot^{n-2} x \operatorname{cosec}^2 x - \cot^{n-2} x)\,dx$$

$$I_n = \int \cot^{n-2} x \operatorname{cosec}^2 x dx - \int \cot^{n-2} x\,dx$$

$$I_n = \frac{-\cot^{n-1} x}{n-1} - I_{n-2}$$

Applying the limits of integration to the above equation from $\frac{\pi}{4}$ to $\frac{\pi}{2}$ we get

$$I_n = \int_{\pi/4}^{\pi/2} \cot^n x\,dx = \left.\frac{-\cot^{n-1} x}{n-1}\right|_{\pi/4}^{\pi/2} - \left\{ I_{n-2} = \int_{\pi/4}^{\pi/2} \cot^{n-2} x\,dx \right\}$$

$$I_n = \frac{1}{n-1} - I_{n-2}$$

Reduction formula for $\int \sec^n x\,dx$ where n is a +ve integer > 1

Consider $I_n = \int \sec^n x\,dx$

$$I_n = \int \sec^{n-2} x \sec^2 x\,dx$$

Integrating by parts we get

$$I_n = \sec^{n-2} x \tan x - \int \tan x (n-2) \sec^{n-3} x \sec x \tan x\,dx$$

$$I_n = \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x\,dx$$

$$I_n = \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1)\,dx$$

$$I_n = \sec^{n-2} x \tan x - (n-2) \int (\sec^n x - \sec^{n-2} x)\,dx$$

$$I_n = \sec^{n-2} x \tan x - (n-2) \int \sec^n x dx + (n-2) \int \sec^{n-2} x\,dx$$

$$I_n = \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2}$$
$$[1+n-2]I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}$$
$$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

Applying limits to the above equation from 0 to $\frac{\pi}{4}$ we get

$$I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

Reduction formula for $\int \operatorname{cosec}^n x\, dx$, where n is a +ve integer > 1

Consider $I_n = \int \operatorname{cosec}^n x\, dx$

$$I_n = \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x dx$$

Integrating by parts we get

$$I_n = \frac{-\operatorname{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

Applying the limits from $\frac{\pi}{4}$ to $\frac{\pi}{2}$ to the above equation, we get

$$I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

Problems

7. **Evaluate** $\int \tan^6 x\, dx$

Solution: We have $I_n = \int \tan^n x\, dx = \dfrac{\tan^{n-1} x}{n-1} - I_{n-2}$

$$I_6 = \int \tan^6 x\, dx$$

$$I_6 = \int \tan^6 x dx = \frac{\tan^5 x}{5} - I_4 = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + I_2$$

$$\int \tan^6 x\, dx = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} - \tan x + x + C$$

8. **Evaluate** $\int_0^a x^5 (2a^2 - x^2)^{-3}\, dx$

Solution: Let $I = \int_0^a x^5 (2a^2 - x^2)^{-3} dx$

Put $x = \sqrt{2} a \sin\theta$ $dx = \sqrt{2} a \cos\theta\, d\theta$ when $x = 0$ $\theta = 0$

when $x = a$ $\theta = \frac{\pi}{4}$

$$I = \int_0^a x^5 (2a^2 - x^2)^{-3} dx$$

$$I = \int_0^a \frac{x^5}{\left(2a^2 - x^2\right)^3} dx$$

$$I=\int_0^{\pi/4}\frac{(\sqrt{2}a)^5\sin^5\theta}{(2a^2-2a^2\sin^2\theta)^3}(\sqrt{2}a\cos\theta d\theta)$$

$$I=\frac{2^3a^6}{2^3a^6}\int_0^{\pi/4}\frac{\sin^5\theta\cos\theta}{(1-\sin^2\theta)^3}d\theta$$

$$I=\int_0^{\pi/4}\frac{\sin^5\theta\cos\theta}{\cos^6\theta}d\theta=\int_0^{\pi/4}\tan^5\theta d\theta$$

$$=\frac{1}{4}-I_3=\frac{1}{4}-\left(\frac{1}{2}-I_1\right)=-\frac{1}{4}+(\log\sec\theta)_0^{\frac{\pi}{4}}\quad=\log\sqrt{2}-\frac{1}{4}$$

$$\int_0^a x^5(2a^2-x^2)^{-3}\,dx=\frac{1}{2}\left(\log 2-\frac{1}{2}\right)$$

9. Show that $\displaystyle\int_{\pi/4}^{\pi/2}\cot^4 xdx=\frac{\pi}{4}-\frac{2}{3}$

Solution: For $n=4$

$$I_4=\int_{\pi/4}^{\pi/2}\cot^4 x\,dx=-\frac{2}{3}+I_0$$

$$I_4=-\frac{2}{3}+\int_{\pi/4}^{\pi/2}dx$$

$$\int_{\pi/4}^{\pi/2}\cot^4 x\,dx=\frac{\pi}{4}-\frac{2}{3}$$

10. Evaluate $\displaystyle\int_0^a(a^2+x^2)^{5/2}dx$

Solution: $\displaystyle I=\int_0^a(a^2+x^2)^{5/2}dx$

Put $x=a\tan\theta$, $dx=a\sec^2\theta\ d\theta$, when $x=0$ $\theta=0$

when $x=a$ $\theta=\frac{\pi}{4}$

$$I=\int_0^{\pi/4}(a^2+a^2\tan^2\theta)^{5/2}(a\sec^2\theta)d\theta$$

$$I=a^6\int_0^{\pi/4}(1+\tan^2\theta)^{5/2}\sec^2\theta d\theta$$

$$I=a^6\int_0^{\pi/4}\sec^7\theta d\theta$$

$I=a^6I_7$ where $\displaystyle I_n=\int_0^{\pi/4}\sec^n\theta\,d\theta$

$$I=a^6\left[\frac{(\sqrt{2})^5}{6}+\frac{5}{6}I_5\right]$$

$$\int_0^a(a^2+x^2)^{5/2}dx=a^6\left[\frac{(\sqrt{2})^5}{6}+\frac{5}{6}\left\{\frac{7}{4\sqrt{2}}+\frac{3}{8}\log|\sqrt{2}+1|\right\}\right]$$

11. Evaluate $\int_0^a x^2(a^2+x^2)^{3/2}\,dx$

Solution: $I=\int_0^a x^2(a^2+x^2)^{3/2}dx$

Put $x=a\tan\theta$, we get, when $x=0 \quad \theta=0$

when $x=a \quad \theta=\dfrac{\pi}{4}$

$$I=\int_0^{\pi/4}(a^2\tan^2\theta)(a^2+a^2\tan\theta)^{3/2}a\sec^2\theta d\theta$$

$$I=a^6\int_0^{\pi/4}(\sec^2\theta-1)\sec^3\theta\sec^2\theta d\theta=a^6(I_7-I_5) \qquad \text{where } \mathrm{I_n}=\int_0^{\pi/4}\sec^{\mathrm{n}}\theta\,\mathrm{d}\theta$$

$$I=a^6\left[\left\{\frac{\sqrt{2}}{6}+\frac{5}{6}I_5\right\}-I_5\right]=a^6\left[\frac{4\sqrt{2}}{6}-\frac{1}{6}I_5\right]$$

$$\int_0^a x^2(a^2+x^2)^{3/2}dx=\frac{a^6}{6}\left[4\sqrt{2}-\left\{\frac{7}{4\sqrt{2}}+\frac{3}{8}\log(\sqrt{2}+1)\right\}\right]$$

Evaluation of $\sin^m x\cos^n x$ **integrals**

Let $I_{m,n}=\int\sin^m x\cos^n x dx$

$$I_{m,n}=\int\sin^{m-1}x\cos^n x\sin x\,dx$$

Integrating the above equation by parts we get

i.e. $I_{m,n}=\int\sin^{m-1}x\cos^n x\sin xdx$

$$I_{m,n}=\sin^{m-1}x\left[\frac{-\cos^{n+1}x}{n+1}\right]-\int\left(-\frac{\cos^{n+1}x}{n+1}\right)(m-1)\sin^{(m-2)}x\cos x\,dx$$

$$I_{m,n}=\frac{-\sin^{m-1}x\cos^{n+1}x}{n+1}+\frac{m-1}{n+1}\int\sin^{m-2}x\cos^{n+2}x\,dx$$

$$I_{m,n}=\frac{-\sin^{m-1}x\cos^{n+1}x}{n+1}+\frac{m-1}{n+1}\int\sin^{m-2}x\cos^n x\cos^2 x\,dx$$

$$I_{m,n}=\frac{-\sin^{m-1}x\cos^{n+1}x}{n+1}+\frac{m-1}{n+1}\int\sin^{m-2}x\cos^n x(1-\sin^2 x)\,dx$$

$$I_{m,n}=\frac{-\sin^{m-1}x\cos^{n+1}x}{n+1}+\frac{m-1}{n+1}\int(\sin^{m-2}x\cos^n x-\sin^m x\cos^n x)dx$$

$$I_{m,n}=\frac{-\sin^{m-1}x\cos^{n+1}x}{n+1}+\frac{m-1}{n+1}\int\sin^{m-2}x\cos^n xdx-\frac{m-1}{n+1}\int\sin^m x\cos^n x\,dx$$

$$I_{m,n}=-\frac{\sin^{m-1}x\cos^{n+1}x}{n+1}+\frac{m-1}{n+1}I_{m-2,n}-\frac{m-1}{n+1}I_{m,n}$$

$$\left[1+\frac{m-1}{n+1}\right]I_{m,n}=\frac{-\sin^{m-1}x\cos^{n+1}x}{n+1}+\frac{m-1}{n+1}I_{m-2,n}$$

$$I_{m,n}=\frac{-\sin^{m-1}x\cos^{n-1}x}{m+n}+\frac{m-1}{m+n}I_{m-2,n}$$

Applying limits to the above equation from 0 to $\frac{\pi}{2}$, we get

$$I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\,dx = \left.\frac{-\sin^{m-1} x \cos^{n+1} x}{m+n}\right|_0^{\pi/2} + \frac{m-1}{m+n} I_{m-2,n}$$

$$I_{m,n} = \frac{m-1}{m+1} I_{m-2,n} \qquad \ldots (1)$$

which is the reduction formula by reducing the power of $\sin x$ function.

Note : If we reduce the power of $\cos x$ function then we will get the reduction formula as

$$I_{m,n} = \frac{m-1}{m+n} I_{m,n}$$

Replace m by $(m-2)$ in equation (1) we get

$$I_{m-2,n} = \frac{m-3}{m+n-2} I_{m-4,n} \qquad \ldots (2)$$

Replace m by $(m-4)$ in equation (1) we get

$$I_{m-4,n} = \frac{m-5}{m+n-4} I_{m-6,n} \qquad \ldots (3)$$

If m is even then by putting $m=2$ in equation (1)

$$I_{2,n} = \frac{1}{2+n} I_{0,n}$$

$$I_{0,n} = \int_0^{\pi/2} \cos^n x\,dx$$

$$= \begin{cases} \frac{n-1}{n} \times \frac{n-3}{n-2} \ldots \frac{1}{2} \times \frac{\pi}{2} & n \text{ is even} \\ \frac{n-1}{n} \times \frac{n-3}{n-2} \ldots \frac{2}{3} \times 1 & n \text{ is odd} \end{cases}$$

If n is odd then by putting $m=3$ in equation (1)

$$I_{3,n} = \frac{2}{3+n} I_{1,n}$$

$$I_{1,n} = \int_0^{\pi/2} \sin x \cos^n x dx = \left.\frac{-\cos^{n+1}}{n+1}\right|_0^{\pi/2}$$

$$= \frac{1}{n+1}$$

Summarizing the above cases in (1), we get

$$I_{m,n} = \begin{cases} \frac{m-1}{m+1} \times \frac{m-3}{m+n-2} \times \frac{m-5}{m+n-4} \cdots \frac{1}{2+n} \times \begin{cases} \frac{n-1}{n} \times \frac{n-3}{n-2} \ldots \frac{1}{2} \times \frac{\pi}{2} & \text{if } m \text{ is even } n \text{ is even} \\ \frac{n-1}{n} \times \frac{n-3}{n-2} \ldots \frac{2}{3} \times 1 & \text{if } m \text{ is even } n \text{ is odd} \end{cases} \\ \frac{m-1}{m+n} \times \frac{m-3}{m+n-2} \times \frac{m-5}{m+n-4} \cdots \frac{2}{3+n} \times \frac{1}{n+1}, \begin{matrix} \text{if } m \text{ odd } n \text{ is odd} \\ \text{if } m \text{ is odd } n \text{ is even} \end{matrix} \end{cases}$$

Evaluate the following integrals

12. Evaluate $\int_0^{\pi/6} \cos^4 3x \sin^3 6x\,dx$

Solution: $I = \int_0^{\pi/6} \cos^4 3x \sin^3 6x dx$

Put $3x = t$, we get when $x=0$ $t=0$

when $x = \frac{\pi}{6}$ $t = \frac{\pi}{2}$

$$I = \int_0^{\pi/2} \cos^4 t \sin^3 2t\,(1/3\,dt)$$

$$I=\frac{1}{3}\int_0^{\pi/2}(\cos^4 t)(2\sin t\cos t)^3 dt$$

$$I=\frac{8}{3}\int_0^{\pi/2}\sin^3 t\cos^7 t dt=\frac{8}{3}.\frac{2}{10}.\frac{1}{8}$$

$$\int_0^{\pi/6}\cos^4 3x\sin^3 6x\ dx=\frac{1}{15}$$

13. **Evaluate** $\int_0^{\pi}(\sin^2 x)(1-\cos x)^3 dx$

Solution: $I=\int_0^{\pi}(\sin^2 x)(1-\cos x)^3 dx$

Put $x=2t$, we get when $x=0$ $t=0$

when $x=\pi$ $t=\frac{\pi}{2}$

$$I=2\int_0^{\pi/2}(\sin^2 2t)(1-\cos 2t)^3 dt$$

$$I=64\int_0^{\pi/2}\sin^8 t\cos^2 t dt$$

$$I=64.\frac{7\times5\times3\times1}{10\times8\times6\times4}.\frac{1}{2}.\frac{\pi}{4}$$

$$\int_0^{\pi}\left(\sin^2 x\right)(1-\cos x)^3 dx=\frac{7\pi}{16}$$

14. **Evaluate** $\int_0^{\infty}\frac{x^5}{(1+x^2)^6}dx$

Solution: Put $x=\tan\theta$ when $x=0$ $\theta=0$

when $x\to\infty$ $\theta=\frac{\pi}{2}$

$$\int_0^{\infty}\frac{x^5}{(1+x^2)^6}dx=\int_0^{\pi/2}\frac{\tan^5\theta}{\sec^{12}\theta}\sec^2\theta d\theta=\int_0^{\pi/2}\sin^5\theta\cos^5\theta d\theta=\frac{4.2}{10.8.6}=\frac{1}{60}$$

15. **Evaluate** $\int_0^{\infty}\frac{x^6}{(1+x^2)^{9/2}}dx$

Solution: Put $x=\tan\theta$ when $x=0$ $\theta=0$

when $x\to\infty$ $\theta=\frac{\pi}{2}$

$$\int_0^{\infty}\frac{x^6}{(1+x^2)^{9/2}}dx=\int_0^{\pi/2}\frac{\tan^6\theta}{\sec^9\theta}\sec^2\theta d\theta=\int_0^{\pi/2}\sin^6\theta\cos\theta d\theta=\frac{1}{7}$$

16. **If** n **is a positive integer, show that** $\int_0^{2a}x^n\sqrt{2ax-x^2}dx=\pi a^2\left(\frac{a}{2}\right)^n.\frac{(2n+1)!}{(n+2)!n!}$**, hence evaluate** $\int_0^{2a}x^3\sqrt{2ax-x^2}dx$

Solution: $I=\int_0^{2a}x^n\sqrt{2ax-x^2}dx$

$$I=\int_0^{2a} x^n\sqrt{a^2-(a-x)^2}\,dx$$

Put $x=2a\sin^2\theta$, $dx=4a\sin\theta\cos\theta\,d\theta$, θ varies from 0 to $\pi/2$

$$\mathrm{I}=\int_0^{\pi/2} 2^{\mathrm{n}}\mathrm{a}^{\mathrm{n}}\sin^{2\mathrm{n}}\theta\sqrt{4\mathrm{a}^2\sin^2\theta-4\mathrm{a}^2\sin^4\theta}\;4\mathrm{a}\sin\theta\cos\theta\,\mathrm{d}\theta$$

$$I=(2a)^{n+2}\int_0^{\pi/2} 2\sin^{2n+2}\theta\cos^2\theta\,d\theta$$

$$I=(2a)^{n+2}\times 2\times\frac{(2n+1)(2n-1)...1}{(2n+4)(2n+2)...2}\cdot\frac{\pi}{2}$$

$$I=(2a)^{n+2}\pi\frac{(2n+1)(2n-1)...1}{2^{n+2}(n+2)(n+1)...1}$$

$$I=a^{n+2}\pi\times\frac{(2n+1)(2n)(2n-1)(2n-2)...2.1}{(n+2)!(2n)(2n-2)...2}$$

$$I=\frac{a^{n+2}}{2^n n!}\frac{(2n+1)!}{(n+2)!}\pi$$

Next, for $n=3$, $\int_0^{2a} x^3\sqrt{2ax-x^2}\,dx=\frac{7!}{5!*3!*2!}\pi a^5=\frac{7\pi a^5}{8}$

Beta and Gamma functions, properties of Beta and Gamma functions.

Gamma Function

The integral $\int_0^{\infty} x^{n-1}e^{-x}$ … (1) is called a gamma function $\Gamma(n)$.

Properties

Property 1 : $\Gamma(1)=1$

Proof : By definition

$$\Gamma(1)=\int_0^{\infty} x^0e^{-x}dx=-e^{-x}\Big]_0^{\infty}=1$$

$$\Gamma(1)=1$$

Property 2 : **Recurrence formula** $\Gamma(n+1)=n\Gamma(n)$

Proof : By definition

$$\Gamma(n+1)=\int_0^{\infty} x^ne^{-x}\,dx=x^n\frac{e^{-x}}{-1}\Bigg]_0^{\infty}-\int_0^{\infty}\frac{e^{-x}}{-1}nx^{n-1}dx=0+n\int_0^{\infty} x^{n-1}e^{-x}dx$$

$$\Gamma(n+1)=n\Gamma(n)$$

Case (i) If n is a positive integer

$$\Gamma(n+1)=n\Gamma(n)=n(n-1)\Gamma(n-1)=n(n-1)(n-2)\Gamma(n-2)$$

$$\Gamma(n+1)=n(n-1)(n-2)...21\,\Gamma(1)\qquad\text{Since }\Gamma(1)=1$$

$$\Gamma(n+1)=n!$$

Case (ii) If n is a negative real number then $\Gamma(n)=\frac{\Gamma(n+1)}{n}$.

Case (iii) If $n=0$ or n is negative integral value then $\Gamma(n)$ is not defined.

Properties

Property 1 : **Show that** $\beta(m,n) = \beta(m,n)$

Proof : $\beta(m,n) = \int_0^1 x^{m-1}\ (1-x)^{n-1}\ dx$

$$= \int_0^1 (1-x)^{m-1}\ \{1-(1-x)\}^{n-1}\, dx\text{, by using } \int_0^a f(x)\,dx = \int_0^a f(a-x)\ dx$$

$$= \int_0^1 (1-x)^{m-1}\, x^{n-1}\, dx$$

$$\beta(m,n) = \beta(n,m)$$

Property 2 : $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\ d\theta$

Proof : We have $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

Put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

When $x = 0$, $\theta = 0$ and $x = 1$, $\theta = \dfrac{\pi}{2}$

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}\ (1-\sin^2\theta)^{n-1}.2\sin\theta\ \cos\theta\ d\theta$$

$$\beta(m,n) = \int_0^{\pi/2} \sin^{2m-2}\theta\ \cos^{2n-2}\theta\ 2\sin\theta\ \cos\theta\ d\theta$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\, d\theta$$

Note : $\int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$ letting $2m-1 = p, 2n-1 = q$

Property 3 : $\beta(m,n+1) + \beta(m+1,n) = \beta(m,n)$

Proof : LHS $= \beta(m,n+1) + \beta(m+1,n)$

$$= \int_0^1 x^{m-1}(1-x)^n\, dx + \int_0^1 x^m (1-x)^{n-1} dx$$

$$= \int_0^1 \left\{x^{m-1}(1-x)^n + x^n(1-x)^{n-1}\right\} dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}\{(1-x)+x\}dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}dx = \beta(m,n)$$

Property 4 : $\dfrac{\beta(m,n+1)}{n} = \dfrac{\beta(m+1,n)}{m} = \dfrac{\beta(m,n)}{m+n}$

Proof : By definition

$$\beta(m,n+1)=\int_0^1 x^{m-1}(1-x)^n\,dx$$

Integrating by parts

$$\beta(m,n+1)=(1-x)^n\,\frac{x^m}{m}\Bigg]_0^1-\int_0^1\frac{x^m}{m}n(-x)^{n-1}(-1)dx$$

$$\beta(m,n+1)=\frac{n}{m}\beta(m+1,n)$$

$$\therefore\ \frac{\beta(m,n+1)}{n}=\frac{\beta(m+1,n)}{m}$$

we know that, if $\frac{a}{b}=\frac{c}{d}$ then $\frac{a}{b}=\frac{c}{d}=\frac{a+c}{b+d}$

$$\therefore\ \frac{\beta(m,n+1)}{n}=\frac{\beta(m+1,n)}{m}=\frac{\beta(m+1,n)+\beta(m,n+1)}{m+n}=\frac{\beta(m,n)}{m+n}$$

Property 5 : $\therefore\ \beta(m,n)=\int_0^\infty\frac{x^{m-1}}{(1+x)^{m+n}}dx$

Proof : We have

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx\quad put\ x=\frac{t}{1+t}\ \Rightarrow t=\frac{x}{1-x}\Rightarrow dx=\frac{1}{(1+t)^2}\,dt$$

when $x=0$ $\quad t\to 0$

$x=1$ $\quad t\to\infty$

$$\therefore\ \beta(m,n)=\int_0^\infty\left(\frac{t}{1+t}\right)^{m-1}\left(1-\frac{t}{1+t}\right)^{n-1}.\frac{1}{(1+t)^2}\,dt$$

$$=\int_0^\infty\frac{t^{m-1}}{(1+t)^{m-1}}.\frac{1}{(1+t)^{n-1}}.\frac{1}{(1+t)^2}\,dt=\int_0^\infty\frac{t^{m-1}}{(1+t)^{m+1}}dt$$

$$\therefore\ \beta(m,n)=\int_0^\infty\frac{x^{m-1}}{(1+x)^{m+n}}dx$$

Relation between Beta and gamma function

$$By\ definition\ \Gamma(n)=\int_0^\infty x^{n-1}e^{-t}\,dt\qquad put\ t=x^2\Rightarrow dt=2x\,dx$$

$$=\int_0^\infty(x^2)^{n-1}\,e^{-x^2}\,2x\,dx$$

$$\Gamma(n)=2\int_0^\infty x^{2n-1}e^{-x^2}dx$$

Consider Γ(n) Γ(m) $=\left(2\int_0^\infty x^{2n-1}e^{-x^2}dx\right)\left(2\int_0^\infty y^{2m-1}e^{-g^2}dy\right)$

$$=4\int_{y=0}^\infty\int_{x=0}^\infty x^{2n-1}y^{2m-1}e^{-(x^2+y^2)}dx\,dy$$

Transforming this to polar coordinates using $x=r\cos\theta, y=r\sin\theta\ dxdy=rdrd\theta$ and θ varies 0 from $\frac{\pi}{2}$ and r varies 0 from ∞.

$$\Gamma(m)(n)=4\int_{r=0}^\infty\int_{\theta=0}^{\pi/2}e^{-r^2}\,(r\cos\theta)^{2n-1}\,(r\sin\theta)^{2m-1}r\,d\theta\,dr$$

$$= 4\int_{r=0}^{\infty} e^{-r^2} r^{2(n+m)-1}\left\{\int_{\theta=0}^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta\right\}dr$$

$$= 4\int_{r=0}^{\infty} e^{r^2} r^{2(n+m)-1} dr.\frac{1}{2}B(m+n)$$

$$\Gamma(m).\Gamma(n) = 2\beta(m,n)\int_{r=0}^{\infty} e^{-r^2} r^{2(m+n)-1} dr$$

$$\Gamma(m).\Gamma(n) = \beta(m,n)\left\{2\int_{0}^{\infty} e^{-r^2} r^{2(n+m)-1} dr\right\}$$

$$= \beta(m,n)\,\Gamma(n+m) \qquad \text{Using } \dots (1)$$

$$\therefore \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Some Useful results

Result 1 : $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof : We have $\beta\left(\frac{1}{2},\frac{1}{2}\right) = \dfrac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}$

$$2\int_{0}^{\pi/2}\sin^{2(1/2)-1}\theta.\cos^{2(1/2)-1}\theta\, \mathrm{d}\theta = \frac{\left\{\Gamma\left(\frac{1}{2}\right)^2\right\}^2}{\Gamma(1)}$$

$$2\int_{0}^{\pi/2}\mathrm{d}\theta = \left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 \quad \Rightarrow 2\frac{\pi}{2} = \left\{\Gamma\left(\frac{1}{2}\right)\right\}^2$$

$$\therefore \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Result 2 : **Duplication formula** $\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \dfrac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$

Proof : *We have* $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+m)} \therefore \beta(m,m) = \dfrac{\{\Gamma(m)\}^2}{\Gamma(2m)}$ ...(1)

$$\beta(m,m) = 2\int_{0}^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$$

$$\beta(m,m) = 2\int_{0}^{\pi/2}\{\sin\theta\cos\theta\}^{2m-1}d\theta$$

$$\beta(m,m) = 2\int_{0}^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^{2m-1}d\theta$$

$$\beta(m,m) = \frac{1}{2^{2m-2}}\int_{0}^{\pi/2}(\sin 2\theta)^{2m-1}d\theta$$

Put $2\theta = \varphi$

$\Rightarrow 2d\theta = d\varphi$

when $\theta = 0$, $\varphi = 0$ and $\theta = \dfrac{\pi}{2}, \varphi = \pi$

$$\beta(m,m)=\frac{1}{2^{2m-2}}-\int_0^{\pi}(\sin\varphi)^{2m-1}\frac{d\varphi}{2}$$

$$\beta(m,m)=\frac{1}{2^{2m-2}}2\int_0^{\pi/2}(\sin\varphi)^{2m-1}.d\varphi.\frac{1}{2}$$

$$\beta(m,m)=\frac{2}{2^{2m-2}}\int_0^{\pi/2}\sin^{2m-1}\varphi\cos^0\varphi\,d\varphi$$

$$\beta(m,m)=\frac{1}{2^{2m-2}}\frac{1}{2}\beta\left(m,\frac{1}{2}\right)$$

$$\beta\,(m,m)\;=\;\frac{1}{2^{m-1}}\,\frac{\Gamma(m)\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)} \qquad ...(2)$$

From (1) and (2)

$$\frac{\{\Gamma(m)\}^2}{\Gamma(2m)}=\frac{1}{2^{2m-1}}\,\frac{\Gamma(m)\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$$

17. **Prove that** $\Gamma(n)=2a^n\int_0^{\infty}t^{2n-1}e^{-at^2}dt$

Solution : By definition $\Gamma(n)=\int_0^{\infty}e^{-x}x^{n-1}dx$

Put $x=at^2$

$\Rightarrow dx=2atdt$

$$\Gamma(n)=\int_0^{\infty}e^{-at^2}(at^2)^{n-1}(2atdt)$$

$$\Gamma(n)=2a^n\int_0^{\infty}t^{2n-1}e^{-at^2}dt$$

18. **Evaluate** $\int_0^{\infty}x^4e^{-x^2}dx$

Solution : $I=\int_0^{\infty}x^4\;e^{-x^2}\;dx$

Put $t=x^2$, $\quad dx=\frac{dt}{2\sqrt{t}}$

$$=\int_0^{\infty}t^2e^{-t}\frac{dt}{2\sqrt{t}}$$

$$=\frac{1}{2}\int_0^{\infty}t^{3/2}\;e^{-t}\;dt$$

$$=\frac{1}{2}\int_0^{\infty}t^{5/2-1}e^{-t}dt=\frac{1}{2}\Gamma\left(\frac{5}{2}\right)=\frac{1}{2}\Gamma\left(\frac{3}{2}+1\right)=\frac{1}{2}\frac{3}{2}\Gamma\left(\frac{3}{2}\right)$$

$$=\frac{1}{2}.\frac{3}{2}.\Gamma\left(\frac{3}{2}\right) \quad =\frac{1}{2}.\frac{3}{2}.\Gamma\left(\frac{1}{2}+1\right)=\frac{1}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$=\frac{3}{8}\sqrt{\pi}$$

19. Evaluate $\int_0^{\infty}\frac{dx}{3^{4x^2}}$

Solution: $I=\int_0^{\infty}\frac{1}{3^{4x^2}}dx$

Put $3=e^m, 4x^2=t$

$$x=\frac{\sqrt{t}}{2}\Rightarrow dx=\frac{1}{4}t^{-1/2}dt$$

$$=\int_0^{\infty}\frac{\frac{1}{4}t^{-1/2}\ dt}{e^{mt}}$$

$$=\frac{1}{4}\int_0^{\infty}t^{-1/2}\ e^{mt}\ dt$$

Put $mt=u$

$mdt = du$

$$=\frac{1}{4}\cdot\int_0^{\infty}\left(\frac{u}{m}\right)^{-1/2}e^{-u}\ \frac{du}{m}=\frac{1}{4\sqrt{m}}\int_0^{\infty}u^{\frac{1}{2}-1}e^{-u}\ du$$

$$=\frac{1}{4\sqrt{m}}\ \Gamma\left(\frac{1}{2}\right)$$

$$=\frac{1}{4\sqrt{\log 3}}\sqrt{\pi}$$

20. Evaluate $\int_0^{1}(x\log x)^4 dx$

Solution: $I=\int_0^{1}(x\ \log x)^4\ dx$

$\log x=-t$

$\Rightarrow x=e^{-t}$

$dx=-e^{-t}dt$

$x=0\Rightarrow t\to\infty$ and $x=1\Rightarrow t=0$

$$=\int_{\infty}^{0}e^{-4t}\left(-t\right)^4 e^{-t}dt$$

$$=\int_0^{\infty}t^4e^{-5t}dt$$

Put $5t=u\Rightarrow dt=\frac{du}{5}$

$$=\frac{1}{5}\int_0^{\infty}\left(\frac{u}{5}\right)^4\ e^{-u}\ du=\frac{1}{5^5}\int_0^{\infty}u^{5-1}e^{-u}\ du$$

$$=\frac{1}{5^5}\Gamma(5)$$

$$=\frac{4!}{5^5}$$

21. **Evaluate**: i) $\Gamma\left(\frac{9}{2}\right)$ ii) $\Gamma\left(-\frac{7}{2}\right)$

Solution: i) $\Gamma\left(\frac{9}{2}\right)=\Gamma\left(\frac{7}{2}+1\right)$

$$=\frac{7}{2}\Gamma\left(\frac{7}{2}\right)=\frac{7}{2}\Gamma\left(\frac{5}{2}+1\right)=\frac{7}{2}.\frac{5}{2}\Gamma\left(\frac{5}{2}\right)=\frac{7}{2}.\frac{5}{2}.\frac{3}{2}\Gamma\left(\frac{3}{2}\right)=\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{3}{2}\Gamma\left(\frac{1}{2}\right)$$

$$=\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}=\frac{105}{16}\sqrt{\pi}$$

ii) $\Gamma\left(-\frac{7}{2}\right)=\dfrac{\Gamma\left(-\frac{7}{2}+1\right)}{-\frac{7}{2}}=-\frac{2}{7}\Gamma\left(-\frac{5}{2}\right)=-\frac{2}{7}.\dfrac{\Gamma\left(-\frac{5}{2}+1\right)}{-\frac{5}{2}}$

$$=\frac{4}{35}\Gamma\left(-\frac{3}{2}\right)=\frac{4}{35}\frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}}=\frac{-8}{105}\Gamma\left(-\frac{1}{2}\right)$$

$$=\frac{-8}{105}\frac{\Gamma\left(\frac{-1}{2}+1\right)}{\frac{-1}{2}+1}$$

$$=\frac{-16}{105}\Gamma\left(\frac{1}{2}\right)$$

$$=\frac{-16}{105}\sqrt{\pi}$$

22. **Evaluate** $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta.$

Solution: $I=\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta$

$$=\int_0^{\pi/2}\sin^{1/2}\theta\,\cos^{-1/2}\theta\,d\theta$$

$$=\frac{1}{2}\beta\left(\frac{1/2+1}{2},\frac{-1/2+1}{2}\right)$$

Use $\Gamma(n)\Gamma(1-n)=\dfrac{\pi}{\sin n\pi}$

$$=\frac{1}{2}\beta\left(\frac{3}{4},\frac{1}{4}\right)$$

$$=\frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}$$

$$=\frac{1}{2}\frac{\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)}{1}$$

$$=\frac{1}{2}\frac{\pi}{\sin\left(\frac{\pi}{4}\right)}$$

$$=\frac{\pi}{\sqrt{2}}$$

23. **Evaluate** $\int_0^\infty \sqrt{y}e^{-y^2}dy\int_0^\infty \frac{e^{-y^2}}{\sqrt{y}}dy$

Solution:

$$I_1 = \int_0^\infty y^{\frac{1}{2}}e^{-y^2}dy$$

$$y^2 = u \Rightarrow y = u^{\frac{1}{2}}$$

$$\Rightarrow dy = \frac{1}{2}u^{\frac{1}{2}-1}du$$

$$= \int_0^\infty u^{\frac{1}{4}}e^{-u} \cdot \frac{1}{2} u^{\frac{1}{2}-1} du$$

$$= \frac{1}{2}\int_0^\infty u^{\frac{3}{4}-1}e^{-u} du$$

$$= \frac{1}{2}\Gamma\left(\frac{3}{4}\right)$$

$$I_2 = \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}} dy$$

Put $y^2 = u$

$$\Rightarrow y = u^{\frac{1}{2}}$$

$$\Rightarrow dy = \frac{1}{2}u^{\frac{1}{2}-1}du$$

$$= \int_0^\infty e^{-u}u^{\frac{-1}{4}} \cdot \frac{1}{2}u^{\frac{1}{2}-1}$$

$$= \frac{1}{2}\int_0^\infty e^{-u}u^{\frac{1}{4}-1}du$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)$$

$$I_1 . I_2 = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\frac{1}{2}\Gamma\left(\frac{1}{4}\right)$$

$$= \frac{1}{4}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{4}.\frac{\pi}{\sin\frac{\pi}{4}}$$

$$= \frac{\pi}{4\ \frac{1}{\sqrt{2}}}$$

$$= \frac{\pi}{2\sqrt{2}}$$

24. **Prove that** $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$

Solution: We have $\beta(m,n) = \int_0^\infty x^{m-1}(1-x)^{n-1}dx$

Put $x=\dfrac{1}{1+y}$

$\Rightarrow dx=\dfrac{-1}{(1+y)^2}\ dy$

Now $y=\dfrac{1}{x}-1\Rightarrow y=\dfrac{1-x}{x}$

When $x=0\Rightarrow y\to\infty$ and $x=1\Rightarrow y=0$

$$\beta(m,n)=\int_{\infty}^{0}\left(\frac{1}{1+y}\right)^{m-1}\left(1-\frac{1}{1+y}\right)^{n-1}\frac{-1}{(1+y)^2}dy$$

$$=\int_0^{\infty}\frac{y^{n-1}}{(1+y)^{m+n}}\ dy$$

$$\beta(m,n)=\int_0^{\infty}\frac{x^{n-1}}{(1+x)^{m+n}}\ dx \qquad \text{Since } \beta(m,n)=\beta(n,m)$$

$$=\int_0^{\infty}\frac{x^{m-1}}{(1+x)^{m+n}}\ dx$$

25. Evaluate $\int_0^2 x(8-x^3)^{1/3}dx$

Solution: $I=\int_0^2 x(8-x^3)^{1/3}\ dx$

Put $x^3=8t$

$\Rightarrow x=2t^{1/3}$

$\Rightarrow dt=2\dfrac{1}{3}t^{\frac{1}{3}-1}dt$

When $x=0\Rightarrow t=0$ and $x=2\Rightarrow t=1$

$$=\int_0^1 2t^{1/3}(8-8t)^{\frac{1}{3}}\,2\frac{1}{5}t^{\frac{1}{3}-1}dt$$

$$=\frac{4}{3}\int_0^1 t^{\frac{2}{3}-1}.2(1-t)^{\frac{1}{3}}\ dt$$

$$=\frac{8}{3}\int_0^1 t^{\frac{2}{3}-1}(1-t)^{\frac{4}{3}-1}\ dt$$

$$=\frac{8}{3}\,B\left(\frac{2}{3},\frac{4}{3}\right)=\frac{8}{3}\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma(2)}=\frac{8}{3}\frac{\Gamma\left(\frac{2}{3}\right)\frac{1}{3}\Gamma\left(\frac{1}{3}\right)}{1}$$

$$=\frac{8}{9}\,\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)=\frac{8}{9}\frac{\pi}{\sin\frac{\pi}{3}} \qquad \text{Use } \Gamma(n)\Gamma(1-n)=\frac{\pi}{\sin n\pi}$$

$$=\frac{8}{9}.\frac{\pi}{\frac{\sqrt{3}}{2}}=\frac{16\pi}{9\sqrt{3}}$$

26. Show that $\int_a^b (x-a)^m(b-x)^n dx=(b-a)^{m+n+1}\beta(m+1,n+1)$ **where 'a' and 'b' are unequal constants.**

Solution: Let $I=\int_a^b (x-a)^m(b-a)^n dx$

Put $x-a=t(b-a)$

$$x = a \Rightarrow t = 0 \text{ and } x = b \Rightarrow t = 1$$
$$\therefore I = \int_0^1 t^m (b-a)^m \{b-a-t(b-a)\}^n (b-a)dt$$
$$= \int_0^1 t^m \ (b-a)^m \ (b-a)^n \{1-t\}^n \ (b-a)\,dt$$
$$= (b-a)^{m+n+1} \int_0^1 t^m (1-t)^n dt$$
$$= (b-a)^{m+n+1} \beta(m+1, n+1).$$

Tracing of standard curves

Introduction

In many engineering applications we need the length of a curve or something distributed along the length of a curve. Area of some region or something over the region bounded by closed curve, volume of solids formed by the revolution of the area bounded by a curve about a line something distributed in that volume, surfaces formed by the revolution of the arc of a curve about the line.

Procedure for tracing Cartesian Curves

Let $f(x, y) = 0$ represent the equation of the given Cartesian curve

Symmetry of the curve

A curve is symmetrical about x-axis, if $f(x, -y) = f(x, y)$ or y occurs only in even powers
A curve is symmetrical about y-axis, if $f(-x, y) = f(x, y)$ or x occurs only in even powers
A curve is symmetrical about the origin if $f(-x, -y) = f(x, y)$ or both x and y occurs in even powers.
A curve is symmetrical about the line $y = x$ if on interchanging x and y, the equation of the curve remains unchanged.

Origin

If $x = 0, y = 0$ satisfy the equation of the curve, then the curve passes through origin
If it does, then find the equation of the tangents at the origin by equating the lowest degree terms in the equation to 0

Asymptotes

See if the curve has only asymptote parallel to axes.

Asymptotes Parallel to y axis

Equate the co-efficient of highest degree term in y to 0 to obtain asymptotes parallel to y-axis.

Asymptotes Parallel to x axis

Equate the co-efficient of highest degree term in x to 0 to obtain asymptotes parallel to x-axis.

Oblique Asymptotes

Asymptotes not parallel to the axes, called oblique asymptotes, are determined by putting $y = mx + c$ in the equation of the curve and finding both possible values of m and c by equating the coefficients of the first two highest powers of x to 0.

Points of Intersection

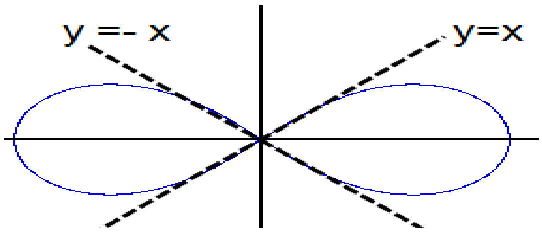
The points of intersection of the curve with the x-axis and the y _axis are found by putting $x = 0$ and $y = 0$ respectively and solving the equation of the curve.

Find the points where the tangent is parallel or perpendicular to the x-axis
(i.e., the points where $\frac{dy}{dx} = 0$ or ∞)

Find the region of existence of the curve by considering all the above points.

27. Trace the curve Leminscate given by the equation $(x^2 + y^2)^2 = a^2(x^2 - y^2)$
Solution:

Both x and y occur only in even powers, therefore the curve is symmetrical about both the coordinate axes
The point $(0,0)$ satisfies the equation and hence the curve passes through the origin
The tangents at the origin are found by rewriting the given equation in the form $x^4 + y^4 + 2x^2y^2 + a^2\left(y^2 - x^2\right) = 0$ and equating the lowest degree term in this equation to zero. This gives us the equation $y^2 - x^2 = 0 \Rightarrow y = \pm x$, hence $y = x$ and $y = -x$ are the tangents at the origin
For $y = 0$, the equation gives $x = 0, \pm a$, hence the points of intersection are $(\pm a, 0)$
Observe that at the points $(\pm a,\ 0)$ $y' \to \infty$, therefore, at the points $(\pm a,\ 0)$, the tangents are parallel to the $y-$axis.
The curve lies in the region for which $x^2 \le a^2$ i.e. between $x = -a$ and $x = a$.



Procedure for tracing Parametric Curves

Let the equation of the curve to be traced is given by the equation $x = f(t)$ and $y = \phi(t)$
Symmetry

A curve is symmetrical about the x-axis if on replacing t by $-t$, $f(t)$ remains unchanged and $\phi(t)$ changes to $-\phi(t)$

A curve is symmetrical about the $y-$axis if on replacing t by $-t$, $f(t)$ changes to $-f(t)$ and $\phi(t)$ remains unchanged

A curve is symmetrical in the opposite quadrant if on replacing t by $-t$, both $f(t)$ and remains unchanged.

Limits

Find the greatest and the least values of and y so as to determine the strips parallel to the axes within or outside which the curve lies.

Asymptote

If the value of a parameter makes either x or y infinite, then that value of t gives asymptotes parallel to either the x or the y-axis respectively

Special Point

Determine the points where the curve crosses the axes. The points of intersection of the curve with the x-axis are given by the roots of $\phi(t)=0$, while those with the $y-$axis are given by the roots of $f(t)=0$ Giving t a series of values, plot the corresponding values of x and y, noting whether x and y increase or decrease for the intermediate values of t. For this purpose, we consider the signs of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ for different values of t.

Determine the points where the tangent is parallel or perpendicular to the x-axis (i.e., where $\frac{dy}{dx}=0 \; or \to \infty$)

When x and $y-$ are periodic functions of t with a common period, we need to study the curve only for one period as the other values of t will repeat the same curve.

28. Trace the curve Strophoid.

Solution:

The equation contains only even powers of y. Thus the curve is symmetrical about x-axis.

The curve passes through the origin. Equating the lowest degree terms to zero, we have th equation of tangents at the origin as $y^2=x^2$ **(i.e.,** $y=\pm x$ **).**The two tangents are real and district. Thus the origin is a node.

When $y=0$, we have $x=0$ and $x=-a$.

The curve meet the x-axis only at $(0,0)$ and $(0,-a)$.

Further the curve meets the y-axis only at $(0,0)$.

Equating the highest degree term in y, we get the asymptote parallel to y-axis as $a-x=0$ (i.e., $x=a$).

The curve has no other asymptotes.

Now, $y^2(a-x)=x^2(a+x) \Rightarrow y=\pm\sqrt{\frac{a+x}{a-x}}$

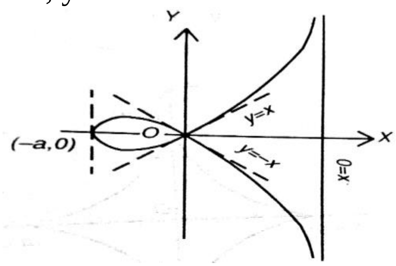
Taking +ve sign, we have $y=+\sqrt{\frac{a+x}{a-x}}$

When $x>a$ or $x<-a$, y is imaginary.

$\Rightarrow$ The curve does not lie beyond the lines.

i.e., the curve lies between the lines $x=-a$ and $x=a$.

Further as x increases from –a to -0.6a,y increases & as x increases form -0.6a to a, y decreases & tends to ∞.



Hence the shape of the curve is as shown in the figure.

29. Trace the curve Astroid given by the equation $x^{2/3}+y^{2/3}=a^{2/3}$ also given by the parametric equations in the form $x=a\cos^3 t, y=a\sin^3 t$

Solution:

The curve is symmetrical about both the coordinates axes and also about the lines $y=\pm x$
The curve does not pass through origin and hence no tangents at the origin
For $t=0$ and $t=\pi$, we have $x=\pm a$ and $y=0$ therefore $(\pm a,0)$ are the points of intersection.

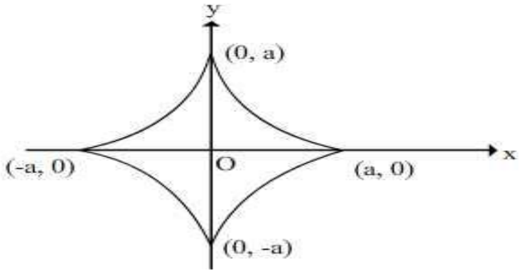
Similarly, for $t=\pi/2$ and $t=3\pi/2$, we have $x=0$ and $y=\pm a$, therefore $(0,\pm a)$ are the points of intersection on the axes of reference

Since $|x|=a|\cos t|^3\le a$ and $|y|=a|\sin t|^3\le a$, the entire curve lies in the region for which $-a\le x\le a, -a\le y\le a$

We have $y' = \dfrac{dy}{dx}=\dfrac{dy/dt}{d/dt}=\dfrac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t}=-\tan t$

At the points $(\pm a,\ 0)$, we have $y'=0$, therefore $x-$axis is the tangent at the points $(\pm a,\ 0)$ and similarly, $y-$axis is the tangent the points $(0,\pm a)$.

In the first quadrant (where $x>0, y>0$) we have $\tan t>0$, therefore $y'<0$ and hence the curve is decreasing in this quadrant.



Procedure for tracing Polar Curves

Let $f(r,\theta)=0$ represent the equation of the given Cartesian curve

Symmetry

A curve is symmetrical about the initial line, if the equation remains unchanged when θ is changed to $-\theta$
A curve is symmetrical about the line through the pole perpendicular to the initial line iff (or $\operatorname{cosec}\theta$) occur in its equation. (i.e., it remains unchanged when θ is changed to $\pi-\theta$)
A curve is symmetrical about the pole if only even powers of r occur in the equation(i.e., it remains unchanged when r is changed to $-r$)

Limits

Determine numerically the greatest value of r so as to notice whether the curve lies within a circle or not
Determine the region in which no portion of the curve lies by finding those values of θ for which r is imaginary

Points of Intersection

Giving successive values to θ, find the corresponding values of r

Determine the points where the tangent coincides with the radius vector or is perpendicular to it. (i.e., the point where $\tan\theta = r\dfrac{d\theta}{dr}=\theta$ or ∞).

30. Trace the curve Cardioid given by the equation $r = a(1+\cos\theta)$.

Solution:

When θ is changed to $-\theta$, the equation remains unaltered, therefore the curve is symmetrical about the initial line

Since $|\cos\theta| \le 1,$ we have $r \le 2a$, therefore the entire curve lies within the circle centered at the pole with $r = 2a$ as the radius.

When $\theta = \pi$, $r = 0$, therefore the curve passes through the pole and the tangent thereat is the line $\theta = \pi$

Here $\frac{dr}{d\theta} = -a\sin\theta$, so that $\tan\phi = r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta} = -\cot\left(\frac{\theta}{2}\right) = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right),$

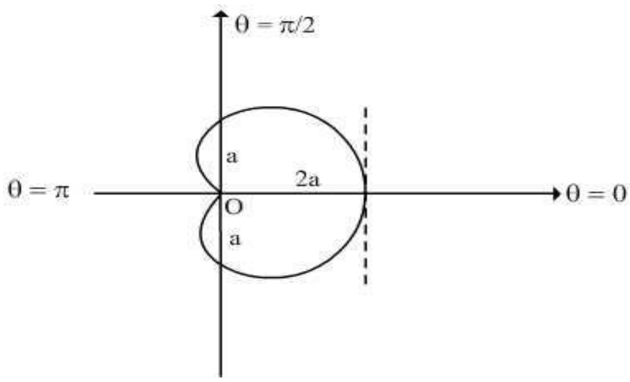
$\Rightarrow \phi = \frac{\pi}{2}$ at $\theta = 0$, therefore at the point $\theta = 0$(for which $r = 2a$) on the curve, the tangent is perpendicular to the radius vector.

$r = a$ When $\theta = \pi/2$, therefore the curves cuts the line $\theta = \pi/2$ at $(a, \pi/2)$.

As θ increases from 0 to π, then r decreases from $2a$ to a and as θ increases from $\pi/2$ to π, further decreases from a to 0.

As set of corresponding values of θ and r on the curve are shown below:

θ	0	$\pi/2$	$2\pi/3$	π	$3\pi/2$	2π
r	$2a$	a	$a/2$	0	a	$2a$



31. Trace the curve Leminscate of Bernoulli $r^2 = a^2\cos 2\theta$.

Solution:

The equation remains same θ changes to $-\theta$. Thus the curve is symmetrical about the initial line.

The equation remains same θ changes to $\pi-\theta$. Therefore the curve is symmetrical about the line $\theta = \frac{\pi}{2}$.(i.e., y-axis).

Further the equation remains same when r is changed to –r. Thus the curve is symmetrical about the pole.

If $r = 0$, then $cos2\theta = 0 \Rightarrow 2\theta = \pm\frac{\pi}{2} \Rightarrow \theta = \pm\frac{\pi}{4}$.

Thus the curve passes through the pole. the tangents to the curve at the pole are $\theta = \pm\frac{\pi}{4}$.

When $\theta = 0$, we have $r^2 = a^2 \Rightarrow r = \pm a$

the curve meets the initial line at $(a, 0)$ and $(-a, 0)$.

The curve has no asymptotes, because there is no finite value of θ for which .

Now, $r^2 = a^2\cos 2\theta \quad \Rightarrow \quad r\frac{dr}{d\theta} = -a^2\sin 2\theta$

$$\Rightarrow \quad r\frac{dr}{d\theta} = -\cot 2\theta$$

$$\Rightarrow \qquad \tan\varphi = \tan\left(\frac{\pi}{2} + 2\theta\right)$$

$$\Rightarrow \qquad \varphi = \frac{\pi}{2} + 2\theta$$

When $\theta = 0$, we have $\varphi = \frac{\pi}{2}$

the tangents at $(\pm a, 0)$ are perpendicular to the initial line.

Now $r^2 = a^2 \cos 2\theta \;\; \Rightarrow \;\; r = a\sqrt{\cos 2\theta}$

Here we have taken the positive square root.

Now, $\frac{dr}{d\theta} = -\frac{a\sin 2\theta}{\sqrt{\cos 2\theta}}$

If $0 < \theta < \frac{\pi}{4}$, then $\frac{dr}{d\theta}$ will be negative .

$\Rightarrow$ r decreases in this region .

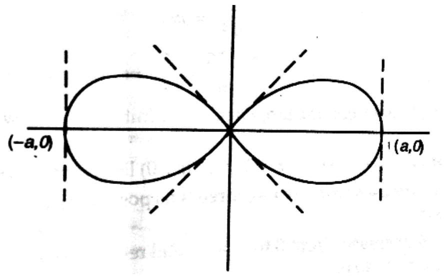
If $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, then r will be imaginary and therefore no portion of the curve lies between the lines $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$.

If $\frac{3\pi}{4} < \theta < \pi$, then $\frac{dr}{d\theta}$ will be positive.

$\Rightarrow$ r increases in this region .

As the curve is symmetrical about the initial line, the shape of the curve below the initial line can be traced by symmetry.

The shape of the curve is as shown in the figure.

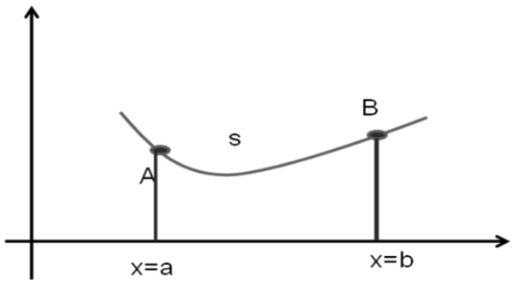


Applications: Area under a plane curve, length of a plane curve, illustrative examples on volume of revolution and surface area of revolution by a given curve (without proof).

Expressions for Derivatives of Arc Length

Let AB be the curve $y = f(x)$ between the points A and B where $x = a$ and $x = b$.

Let $P(x, y)$ be any point on the curve and arc AP = s so that it is a function of x, then



The length of the arc of the curve $y = f(x)$ between the points where $x = a$ and $x = b$ is $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

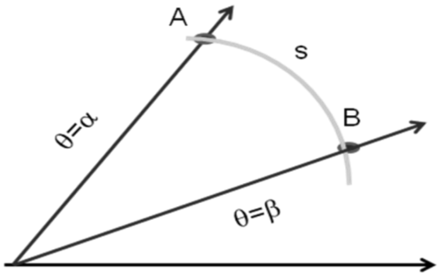
$$\therefore \int_a^b \frac{ds}{dx}.dx = |S|_{x=a}^{x=b} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Hence AB = $\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$

The length of the arc of the curve $x = f(y)$ between the points where $y = a$ and $y = b$ is $\int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$.

The length of the arc of the curve $x = f(t)$, $y = \phi(t)$ between the points at $t = a$ & $t = b$ is $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$

The length of the arc of the curve $r = f(\theta)$ between $\theta = \alpha$ and $\theta = \beta$ is $\int_{\theta=\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$.



Length of Arc

Problems

32. Find the entire length of the asteroid $x^{2/3} + y^{2/3} = a^{2/3}$ **.**

Solution:

We have observed that the curve is symmetrical about both the axes. For the branch of the curve in the first quadrant, x varies from 0 to a. The total length s of the curve is 4 times the length of this branch.

$$\therefore s = 4\int_0^a \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{1/2} dx$$

Let $x^{2/3} + y^{2/3} = a^{2/3} \Rightarrow \frac{dy}{dx} = \frac{x^{-1/3}}{y^{-1/3}} = -\left(\frac{y}{x}\right)^{1/3}$

$$\therefore 1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{y^{2/3}}{x^{2/3}} = \frac{x^{2/3}+y^{2/3}}{x^{2/3}} = \frac{a^{2/3}}{x^{2/3}}$$

Hence we get $s = 4\int_0^a \frac{a^{1/3}}{x^{1/3}} dx = 6a$.

33. Find the perimeter of the loop of the curve $3ay^2 = x^2(a-x), a > 0$

Solution:

Since that the curve is symmetrical about the $x-$axis and the loop of the curve is bounded between $x = 0$ and $x = a$.

$\therefore$ The length of the loop is twice the length of its upper part.

$\therefore$ The required length is $s = 2\int_0^a \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{1/2} dx,\ y > 0$

Let $3ay^2 = x^2(a-x), a>0 \Rightarrow 6ay\frac{dy}{dx} = 2ax - 3x^2$,

$$\left(\frac{dy}{dx}\right)^2 = \frac{(2ax-3x^2)^2}{36a^2y^2} = \frac{x^2(2a-3x)^2}{12a[x^2(a-x)]} = \frac{(2a-3x)^2}{12a(a-x)}$$

$$1+\left(\frac{dy}{dx}\right)^2 = \frac{(4a-3x)^2}{12a(a-x)}$$

$$s = 2\int_0^a \frac{4a-3x}{\sqrt{12}\sqrt{a}\sqrt{a-x}}dx = \frac{1}{\sqrt{3a}}\int_0^a \frac{a+3(a-x)}{\sqrt{a-x}}\,dx$$

$$= \frac{1}{\sqrt{3a}}\left\{\int_0^a \frac{a}{\sqrt{a-x}}\,dx + 3\int_0^a \sqrt{a-x}\,dx\right\}$$

$$= \frac{1}{\sqrt{3a}}\left[-a.\frac{(a-x)^{1/2}}{(1/2)} - 3\frac{(a-x)^{3/2}}{(3/2)}\right]_0^a$$

$$s = \frac{4}{\sqrt{3}}a$$

34. Obtain the total length of the cardioid $r = a(1+\cos\theta)$. Show that the arc of the upper half is bisected at $\theta = \pi/3$.

Solution:

We note that the curve is symmetrical about the initial line $(\theta = 0)$ and for the upper half of the curve, θ varies from 0 to π

$\therefore$ the length of the upper half of the curve is

$$s = \int_0^\pi \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{1/2} d\theta$$

$$= \int_0^\pi \{a^2(1+\cos\theta)^2 + a^2\sin^2\theta\}^{1/2} d\theta,$$

$$= \int_0^\pi a\{1+\cos^2\theta + 2\cos\theta + \sin^2\theta\}^{1/2} d\theta$$

$$= 2a\int_0^\pi \cos(\theta/2)\,d\theta = 4a\sin\frac{\pi}{2} = 4a$$

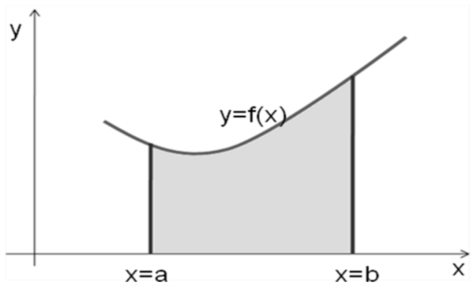
Therefore the total length of the given cardioid is $8a$. Now the length of the arc between $\theta = 0$ and $\theta = \pi/3$ is

$$s_1 = \int_0^{\pi/3} \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{1/2} d\theta = 2a\int_0^{\pi/3}\cos(\theta/2)\,d\theta = 2a.$$

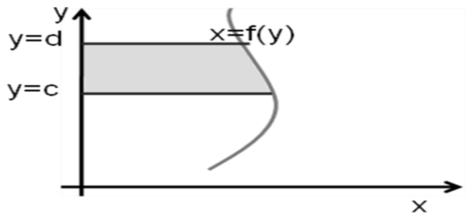
We observe that the point for which is $\theta = \pi/3$ bisects the length of the upper half of the curve.

Area

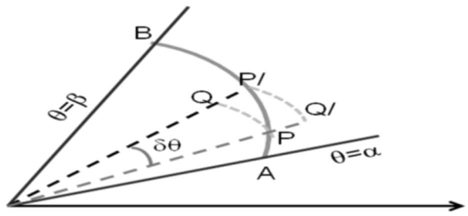
Area bounded by the curve $y = f(x)$, the axis and the ordinates $x = a$, $x = b$ is given by $A = \int_{x=a}^{x=b} y\,dx$.



Interchanging x and y in the above formula, we see that the area bounded by the curve $x = f(y)$, the $y-$axis and the abscissa $y = c$ and $y = d$ is given by $A = \int_c^d x dy$.



Area bounded by the polar curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$ is given by $A = \frac{1}{2}\int_{\theta=\beta}^{\theta=\beta} r^2\, d\theta$



Problems

35. Find the area enclosed by the curve $a^2x^2 = y^3(2a - y), a > 0$**.**

Solution:

The line of symmetry is the $y-$axis

$\therefore$ The required area A is twice the area bounded by the curve to the right of the y-axis between $y = 0$ and $y = 2a$.

$$A = 2\int_{y=0}^{y=2a} x dy = 2\int_0^{2a} \frac{y^{3/2}\sqrt{2a-y}}{a} dy$$

Put $y = a\sin^2\theta$

$$A = \frac{2}{a}\int_0^{\pi/2} (2a\sin^2\theta)^{3/2} \sqrt{2a - 2a\sin^2\theta}\,(4a\sin\theta\cos\theta)\, d\theta,$$

$$= \frac{2}{a}.(2a)^{3/2}\sqrt{2a}.4a = \int_0^{\pi/2} \sin^4\theta\cos^2\theta d\theta$$

$$= 32a^2.\frac{3\times 1}{6\times 4}.\frac{1}{2}.\frac{\pi}{2} = \pi a^2.$$

36. Find the area bounded by the Astroid $x^{2/3} + y^{2/3} = a^{2/3}$**.**

Solution:

This curve is already familiar. We observe that the area bounded by the curve is 4 times the area bounded by the arc of the curve in the first quadrant and the $x-$axis between $x = 0$ and $x = a$. Thus, the required area is $A = 4\int_0^a y dx$

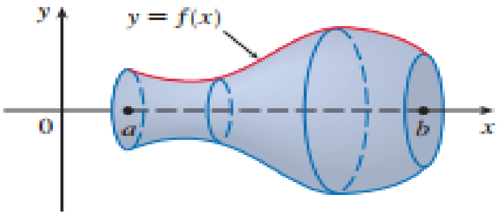
Since $x = a\cos^3\theta$, we have $dx = -3a\cos^2\theta\sin\theta\, d\theta$ and $\theta = \pi/2$ for $x = 0$ and $\theta = 0$ for $x = a$.

$$A = 4\int_{\pi/2}^{0} \left(a\sin^3\theta\right)\left(-3a\cos^2\theta\sin\theta\, d\theta\right) = 12a^2\int_0^{\pi/2} \sin^4\theta\cos^2\theta\, d\theta$$

$$= 12a^2.\frac{3.1}{6.4.2}.\frac{\pi}{2} = \frac{3}{8}\pi a^2.$$

Surface Area of Solid of Revolution

Consider the revolution of an arc of a curve $y = f(x)$ about the $x-$axis between the ordinates $x = a$ and $x = b$.



The Surface area of the solid so generated is called the volume of revolution generated by the curve. This volume is given by the formula

$$S = \int_{x=a}^{x=b} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

If the axis of revolution is the $y-$axis, then the surface area is given by the formula

$$S = \int_{y=c}^{y=d} 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$ Where $y = c$ and $y = d$ are appropriate limits of integration

In the case of polar curves, the volume generated by revolving the curve about the initial line $\theta = 0$ $x-$axis) is given by

$$S = \int_{\theta=\alpha}^{\theta=\beta} 2\pi r \sin\theta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

If the line of revolution is $\theta = \frac{\pi}{2}$ ($y-$axis), then the volume generated by revolving the curve is given by

$$S = \int_{\theta=\alpha}^{\theta=\beta} 2\pi r \cos\theta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Problems

37. Determine the surface area of the solid obtained by rotating $y = \sqrt{9-x^2}, -2 \le x \le 2$ about the x-axis.

Solution: The formula that we'll be using here is, $S = \int_{x=a}^{x=b} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

Here $\frac{dy}{dx} = \frac{-x}{\sqrt{9-x^2}}$

$$\Rightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{x^2}{9-x^2}} = \frac{3}{\sqrt{9-x^2}}$$

$$\therefore S = \int_{x=-2}^{x=2} 2\pi\sqrt{9-x^2}\,\frac{3}{\sqrt{9-x^2}}dx$$

$$= \int_{x=-2}^{x=2} 6\pi dx = 24\pi$$

38. Determine the surface area of the solid obtained by rotating the following parametric curve about the x-axis $x=\cos^3\theta, y=\sin^3\theta, 0\le\theta\le\frac{\pi}{2}$.

Solution:

We'll first need the derivatives of the parametric equations.

$$\frac{dx}{dt}=-3\cos^2\theta\sin\theta, \frac{dy}{dt}=3\sin^2\theta\cos\theta$$

But $S=\int_{\theta=\alpha}^{\theta=\beta}2\pi x\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$

$$S=\int_{\theta=0}^{\theta=\pi/2}2\pi\cos^3\theta\sqrt{(-3\cos^2\theta\sin\theta)^2+(3\sin^2\theta\cos\theta)^2}\,d\theta$$

$$=\int_{\theta=0}^{\theta=\pi/2}2\pi\cos^3\theta(3\cos\theta\sin\theta)d\theta$$

$$=6\pi\int_{\theta=0}^{\theta=\pi/2}\sin^4\theta\cos\theta\,d\theta$$

$$=6\pi/5$$

39. Find the area of the solid generated by the rotating of the loop of the curve $r^2=a^2\cos2\theta$ **about the initial line.**

Solution:

The equation of the curve is $r^2=a^2\cos2\theta$

There are two loops of the curve. Differentiating the above equation we get

$$2r\frac{dr}{d\theta}=-2a^2\sin2\theta\Rightarrow\frac{dr}{d\theta}=\frac{-a^2\sin2\theta}{r}$$

$$S=2\int_{\theta=0}^{\theta=\pi/4}\left(2\pi a\sqrt{\cos2\theta}\sin\theta\sqrt{a^2\cos2\theta+\frac{a^2\sin^2 2\theta}{\cos2\theta}}\right)d\theta$$

$$=4\pi a^2\int_{\theta=0}^{\theta=\pi/4}(\sin\theta)d\theta=4\pi a^2(-\cos\theta)_0^{\pi/4}=4\pi a^2\left(1-\frac{1}{\sqrt2}\right).$$

40. Find the area of the solid formed by the revolution of the cardioid $r=a(1+\cos\theta)$ **of about the initial line.**

Solution:

The equation of the curve is $r=a(1+\cos\theta)$

The curve is symmetric about the initial line. The upper portion of the curve θ varies from 0 to π

$$\frac{dr}{d\theta}=-a\sin\theta$$

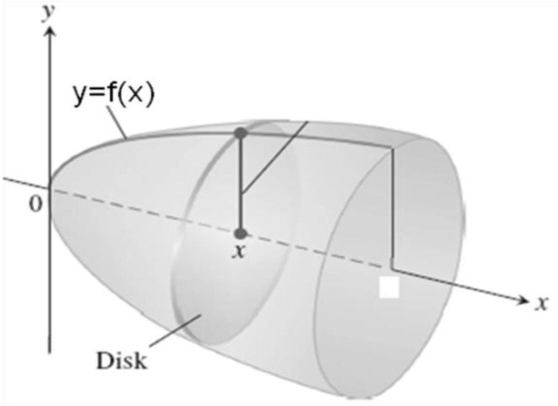
$$S=\int_{\theta=0}^{\theta=\pi}\left(2\pi a(1+\cos\theta)\sin\theta\sqrt{a^2(1+\cos\theta)^2+a^2\sin^2\theta}\right)d\theta$$

$$=2\pi a^2\int_{\theta=0}^{\theta=\pi}\left(2\cos^2(\theta/2)2\sin(\theta/2)\cos(\theta/2)\right)2\cos(\theta/2)d\theta$$

$$=16\pi a^2\int_{\theta=0}^{\theta=\pi}\left(\cos^4(\theta/2)\sin(\theta/2)\right)d\theta=\frac{32\pi a^2}{5}$$

Volume of Solid of Revolution

Consider the revolution of an arc of a curve $y = f(x)$ about the $x-$axis between the ordinates $x = a$ and $x = b$.



The volume of the solid so generated is called the volume of revolution generated by the curve. This volume is given by the formula

$$V = \int_{x=a}^{x=b} \pi y^2 dx$$

If the axis of revolution is the $y-$axis, then the volume is given by the formula

$$V = \int_{y=c}^{y=d} \pi x^2 dy$$

Where $y = c$ and $y = d$ are appropriate limits of integration

In the case of polar curves, the volume generated by revolving the curve about the initial line $\theta = 0$ ($x-$axis) is given by

$$V = \int \frac{2}{3}\pi r^3 \sin\theta \, d\theta$$

If the line of revolution is $\theta = \frac{\pi}{2}$ ($y-$axis), then the volume generated by revolving the curve is given by

$$V = \int \frac{2}{3}\pi r^3 \cos\theta \, d\theta$$

In the above two formulae, appropriate limits for integration with respect to θ have to be taken.

Problems

41. Determine the volume of the solid of revolution when the arc of the curve $y = xe^x$ between $x = 0$ and $x = 1$ is revolved about the line $y = 0$ ($x-$axis).

Solution:

The required volume is

$$V = \int_0^1 \pi y^2 dx$$

$$= \pi \int_0^1 x^2 e^{2x} dx)$$

$$= \frac{\pi}{4}(e^2 - 1)$$

42. Find the volume of the spindle-shaped solid generated by revolving the Astroid $x^{2/3}+y^{2/3}=a^{2/3}$ about the x-axis.

Solution:

The required volume is twice the volume of the solid generated by the revolution of the arc of the curve in the first quadrant (for which x varies from 0 to a), about the $x-$axis.

The required volume is $V=2\times\int_0^a \pi y^2 dx$

We have $x=a\cos^3 t, y=a\sin^3 t$ as the parametric equations and $t=\frac{\pi}{2}$ when $x=0 \,\&\, t=0$ when $x=a$, we get

$$V=2\pi\int_{\pi/2}^{0}(a^2\sin^6 t)-(-3a\cos^2 t\sin t\,dt)$$

$$=16\pi a^3\int_0^{\pi/2}\sin^7 t\cos^2 t dt$$

$$=6\pi a^3.\frac{6.4.2}{9.7.5}.\frac{1}{3}=\frac{32}{105}\pi a^3$$

43. Find the volume generated by revolving the Cardioid $r=a\left(1+\cos\theta\right)$ about $\theta=0$.

Solution:

Since the given curve is symmetrical about the initial line $\theta=0$.

The solid obtained by revolving the upper part of the curve about the initial line is the same as the solid obtained by revolving the whole curve.

For the upper part of the given Cardioid, θ varies from 0 to π.

Therefore the required volume is

$$V=\int_0^{\pi}\frac{2}{3}\pi r^3\sin\theta\, d\theta=\frac{2}{3}\pi a^3\int_0^{\pi}(1+\cos\theta)^3\sin\theta\ d\theta$$

$$=\frac{2}{3}\pi a^3\int_0^2 t^3 dt, \quad where\ t=1+\cos\theta$$

$$=\frac{2}{3}\pi a^3.\frac{2^4}{4}=\frac{8}{3}\pi a^3.$$

Question Bank

Sl.No	Problems	Answers
1	Evaluate $\int_0^{\pi} x\sin^6 x\cos^4 x \; dx$	$\frac{3\pi^2}{512}$
2	Evaluate $\int_0^{\pi} x\sin^5 x \; dx$	$\frac{8\pi}{15}$
3	Evaluate $\int_0^{\pi} \frac{\sqrt{1-\cos x}}{1+\cos x}\sin^2 x \; dx$	$\frac{8\sqrt{2}}{3}$
4	Evaluate $\int_0^{1} \frac{x^3}{\left(1+x^2\right)^4} dx$	$\frac{1}{12}$
5	Evaluate $\int_0^{\infty} x^6 e^{-3x} dx$	$\frac{80}{243}$
6	Evaluate $\int_0^{\infty} x^{-7/4} e^{-\sqrt{x}} dx$	$\frac{8\sqrt{\pi}}{3}$
7	Evaluate $\frac{\Gamma(3)\Gamma(5/2)}{\Gamma(11/2)}$	$\frac{16}{315}$
8	Evaluate the following by expressing in terms of beta function $\int_0^{\pi/2} \sin^{1/2} x\cos^{3/2} x \; dx$ $\int_0^{\infty} \frac{x}{1+x^6} dx$ $\int_0^{2} x\left(8-x^3\right)^{1/3} dx$	$\frac{\pi}{4\sqrt{2}}$ $\frac{\pi}{3\sqrt{3}}$ $\frac{16\pi}{9\sqrt{3}}$
9	Trace the curve $xy^2 = a^2(a-x)$	
10	Trace the curve $y^2(a-x) = x^3, \; a>0$	

Module-2
CALCULUS

- Rolle’s Theorem
- Mean Value Theorems (Lagrange's Mean Value theorem & Cauchy's Mean value theorem)
- Taylor’s and MacLaurin’s series for Exponential,Trigonometric and Logarithm functions
- Indeterminate forms and L'Hospital's rule.

Introduction

Closed Interval

An interval of the form $a \le x \le b$, that includes every point between ‘a’ & ‘b’ and also the end points, is called a closed interval and is denoted by $[a,b]$.

Open Interval

An interval of the form $a < x < b$, that includes every point between ‘a’ & ‘b’ but not the end points, is called an open interval and is denoted by (a,b).

Continuity

A real valued function $f(x)$ is said to be continuous at a point x_0, if $\underset{x \to 0}{Lim} f(x) = f(x_0)$. The function $f(x)$ is said to be continuous in an interval if it is continuous at every point in the interval.

Roughly speaking,if we can draw a curve without lifting the pen, then it is a continuous curve otherwise it is discontinuous, having discontinuities at those points at which the curve will have breaks or jumps. We note that all elementary functions such as algebraic, exponential, trigonometric, logarithmic, hyperbolic functions are continuous functions. Also the sum, difference, product of continuous functions is continuous. The quotient of continuous functions is continuous at all those points at which the denominator does not become zero.

Differentiability

A real valued function $f(x)$ is said to be differentiable at point x_0 if $\underset{x \to x_0}{Lim} \frac{f(x) - f(x_0)}{x - x_0}$ exists uniquely and it is denoted by $f'(x_0)$.A real valued function f(x) is said to be differentiable in an interval if it is differentiable at every in the interval or if $\underset{h \to 0}{Lim} \frac{f(x+h) - f(x)}{h}$ exists uniquely.This is denoted by $f'(x)$. We say that either $f'(x)$ exists or $f(x)$ is differentiable.

Geometrically, it means that the curve is a smooth curve. In other words a curve is said to be smooth if there exists a unique tangent to the curve at every point on it. For example a circle is a smooth curve. Triangle, rectangle, square etc are not smooth, since we can draw more number of tangents at every corner point.We note that if a function is differentiable in an interval then it is necessarily continuous in that interval. The converse of this need not be true. That means a function is continuous need not imply that it is differentiable.

Note

In Calculus texts and lecture, Rolle’s Theorem is given first since it’s used as part of the proof for the Mean Value Theorem (MVT). Students can easily remember it, though, as just a special case of the MVT.

Rolle's Theorem and its Geometrical Interpretation

Statement

If a function $f(x)$ is defined in the interval $[a,b]$ such that

i. $f(x)$ is continuous in $[a,b]$

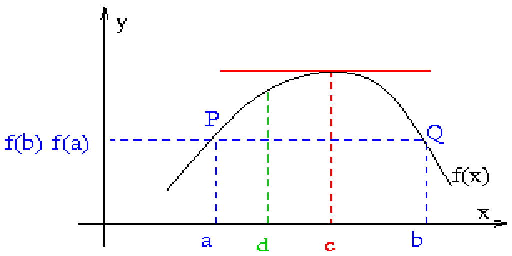
ii. $f(x)$ is differentiable in (a,b)

iii. $f(a)=f(b)$

Then there exists at least one point c in (a,b) that is $a<c<b$ such that $f'(c)=0$.

Geometrical Interpretation

- There are no breaks or gaps in between $'a'$ and $'b'$ for the given curve and including at the end points, hence the function is continuous in the $[a,b]$.
- Since a unique tangent can be drawn at each and every point in the interval except at the end points, the function is differentiable in the (a,b).
- At the end points $'a'$ and $'b'$ they are at the same height from the x-axis.



Conclusion

Therefore there exits at least one point $'c'$ (say) in between $'a'$ and $'b'$ such that the tangent at $'c'$ is parallel to x-axis.

Figure-1

Note: There may be exists more than one point in between (a, b) at which $f'(x)$ vanishes.

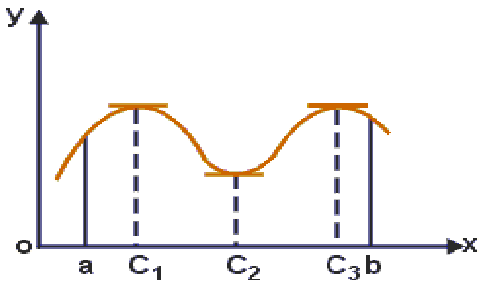


Figure-2

Problems on Rolle's Theorem

1. Verify Rolle's theorem for the function $f(x)=x^3-3x^2-x+3$ in $[1,3]$

Solution : Since $f(x)$ is a polynomial, it is continuous in $[1,3]$

$f'(x)=3x^2-6x-1$ is defined for all x in $(1,3)$ $\therefore$ $f(x)$ is differentiable

$f(1)=f(3)=0$ All three conditions of Rolle's Theorem are satisfied,

Therefore $\exists\, c$ such that $f'(c)=0$

$$\Rightarrow 3c^2-6c-1=0 \Rightarrow c=\frac{6\pm\sqrt{36+12}}{6}=\frac{6\pm4\sqrt{3}}{6}=1\pm\frac{2\sqrt{3}}{3}=1\pm\frac{2}{\sqrt{3}}\approx 2.15\in(1,3)$$

$\therefore$ Rolles' Theorem is verified.

2. Verify Rolles' Theorem for the function $f(x)=\log\left\{\dfrac{(x^2+ab)}{x(a+b)}\right\}$ $in\ [a,b];\ b>a>0.$

Solution : Since $f(x)$ is a standard logarithmic function, it is continuous in $[a,b]$

$f(x)=\log(x^2+ab)-\log(x)-\log(a+b),\Rightarrow f'(x)=\dfrac{2x}{x^2+ab}-\dfrac{1}{x}$ is defined for all in (a,b) $\therefore$ $f(x)$ is differentiable

$f(a)=\log\dfrac{(a^2+ab)}{a^2+ab}=\log 1=0$ and $f(b)=\log\dfrac{(b^2+ab)}{ba+b^2}=\log 1=0\Rightarrow f(a)=f(b)$

All conditions of Rolles' Theorem are satisfied.

$\therefore\ \exists\, c$ such that $\Rightarrow f'(c)=0\ x$

$$\Rightarrow \frac{2c}{c^2+ab}-\frac{1}{c}=0\Rightarrow 2c^2=c^2+ab$$

$$\Rightarrow c=\pm\sqrt{ab}$$

But $c=+\sqrt{ab}\in(a,b)$

Hence Rolles' Theorem is verified.

3. Verify the Rolle's theorem for the function $f(x)=(x-a)^p(x-b)^q$ **in** $[a\ b]$

Solution : Since $f(x)$ is a polynomial, it is continuous in $[a,b]$

$f'(x)=(x-a)^{p-1}(x-b)^{q-1}\left[(q+p)x-(qa+pb)\right]$ is defined for all x in (a,b)

$\therefore$ $f(x)$ Is differentiable, $f(a)=f(b)=0$

Hence all the conditions of Rolle's Theorem are satisfied.

$\therefore\ \exists\, c$ such that $\Rightarrow f'(c)=0$

$$\Rightarrow(c-a)^{p-1}(c-b)^{q-1}\left[(q+p)c-(qa+pb)\right]=0$$

$$\Rightarrow\left[(q+p)c-(qa+pb)\right]=0=0,$$

$$\Rightarrow c=\frac{pb+qa}{p+q}\in(a,b);$$

Thus the Rolle 's Theorem is verified.

4. Verify Rolle's theorem for the function $f(x)=e^x(\sin x-\cos x)$ **in** $\left[\dfrac{\pi}{4},\dfrac{5\pi}{4}\right]$

Solution : Since $f(x)$ is a combination of standard functions, it is continuous in $\left[\dfrac{\pi}{4},\dfrac{5\pi}{4}\right]$

$f'(x)=2e^x\sin x$ is defined for all x in $\left(\dfrac{\pi}{4},\dfrac{5\pi}{4}\right)$ $\therefore$ $f(x)$ is differentiable in $\left(\dfrac{\pi}{4},\dfrac{5\pi}{4}\right)$

$$f\left(\frac{\pi}{4}\right)=e^{\frac{\pi}{4}}\left(\sin\frac{\pi}{4}-\cos\frac{\pi}{4}\right)=e^{\pi/4}\left(\frac{1}{\sqrt2}-\frac{1}{\sqrt2}\right)=0$$

$$f\left(\frac{5\pi}{4}\right)=e^{\frac{5\pi}{4}}\left(\sin\frac{5\pi}{4}-\cos\frac{5\pi}{4}\right)=e^{5\pi/4}\left(-\frac{1}{\sqrt2}+\frac{1}{\sqrt2}\right)=0$$

Therefore $f\left(\dfrac{\pi}{4}\right)=0=f\left(\dfrac{5\pi}{4}\right)$

Hence all the conditions of Rolle's theorem as satisfied.

$\therefore\ \exists\, c$ such that $\Rightarrow f'(c)=0\ \Rightarrow e^c\sin c=0\Rightarrow\sin c=0$

$\Rightarrow c = n\pi$, where $n = 0,1,2,3.....$ $c = \pi \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$;

Thus Rolle's theorem is satisfied.

5. Verify Rolle's theorem for the function $f(x) = \frac{\sin 2x}{e^{2x}}$ **in** $\left[0, \frac{\pi}{2}\right]$

Solution : Since $f(x)$ is a combination of standard functions, it is continuous in $\left[0, \frac{\pi}{2}\right]$

$f'(x) = \frac{e^{2x} 2\cos 2x - \sin 2x\, 2e^{2x}}{\left(e^{2x}\right)^2}$ is defined for all x in $\left(0, \frac{\pi}{2}\right) \therefore f(x)$ is differentiable

$f(0) = f\left(\frac{\pi}{2}\right) = 0$,

Hence all the conditions of Rolle's theorem as satisfied.

$\therefore \exists\, c$ such that $\Rightarrow f'(c) = 0$

$\Rightarrow \frac{2(\cos 2c - \sin 2c)}{\left(e^{2c}\right)} = 0$ $\Rightarrow \cos 2c - \sin 2c = 0 \Rightarrow \cos 2c = \sin 2c$

$\Rightarrow \tan 2c = 1 \Rightarrow c = \frac{\pi}{8}$

Thus Rolle's theorem is satisfied.

Lagrange's Mean Value Theorem

Statement

If a function $f(x)$ is defined in the interval $[a, b]$ such that

i. $f(x)$ is continuous in $[a, b]$

ii. $f(x)$ is differentiable in (a, b)

Then there exists at least one point c in (a,b) that is $a < c < b$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

Proof: Let us construct a function $\varphi(x) = f(x) - kx$ ---- (1)

where k is a constant to be choosen suitably later. Since $f(x)$ and x are continuous in $[a, b]$, differentiable in (a,b), and kx is also continous in [a,b], differentiable in (a,b). We can Conclude that $\varphi(x)$ is also continous in [a, b] and differentiable in (a, b).

from (1) we have, $\varphi(a) = f(a) - ka$; $\varphi(b) = f(b) - kb$

$\therefore \varphi(a) = \varphi(b)$ holds good if

$f(a) - ka = f(b) - kb \Rightarrow k = \frac{f(b) - f(a)}{b - a}$ ---- (2)

Hence if k is choosen as given in (2), then $\varphi(x)$ satisfy all the condition of Rolls theorem.

Therefore by Rolles theorem there exist atleast one point c in (a,b) such that $\varphi'(c) = 0$

Differentiating (1) w.r.t x,

we have $\varphi'(x) = f'(x) - k$ and $\varphi'(c) = 0 \Rightarrow f'(c) - k = 0$

i.e. $k = f'(c)$ -------(3),

Equating R.H.S of (2) and (3)

we have $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Geometrical Interpretation of Lagrange's Mean Value Theorem

- There are no breaks or gaps in between 'a' and 'b' for the given curve and including at the end points, hence the given function is continuous in $[a,b]$
- Since a unique tangent can be drawn at each and every point in the interval except at the end points, the function is differentiable in the (a,b)

Conclusion: Therefore there exits at least one point 'c' (say) in between 'a' and 'b' at which the tangent at 'c' is parallel to the chord AB

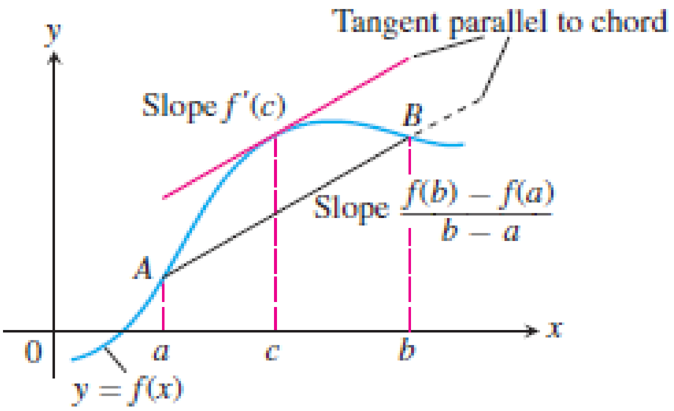


Figure-3

Note:
There may be exists more than one point in between (a, b) at which the tangents are parallel to the chord AB.

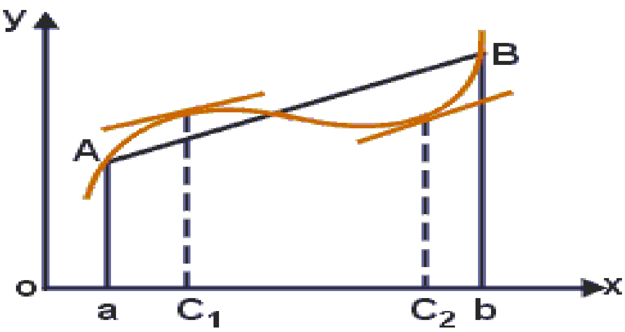


Figure-4

Lagrange's Mean Value Theorem

If a function $f(x)$ is defined in the interval $[a, a+h]$ such that

i. $f(x)$ is continuous in $[a, a+h]$

ii. $f(x)$ is differentiable in $(a, a+h)$

Then there exists at least one number θ, where $0<\theta<1$ such that $c = a + \theta h$ and $f'(a+\theta h) = \frac{f(a+h)-f(a)}{h} \Rightarrow f(a+h) = f(a) + hf'(a+\theta h)$ Where, $\theta = \frac{c-a}{h} \Rightarrow \theta = \frac{c-a}{b-a}$, This leads the generalization of the Lagrange's Mean value as Taylor's series. This is very useful to express any function in terms of approximate polynomial function.

Problems on Lagrange's Mean Value theorem:

6. Verify Lagrange's Mean Value Theorem for the function $f(x) = \ln x$ in $[1, e]$.

Solution: $f(x)$ is continuous in $[1, e]$

$f'(x) = \frac{1}{x}$ is defined $\forall x \in (1, e)$ $f(x)$ is differentiable in $(1, e)$ $\therefore$

All conditions of Lagrange's Mean Value Theorem are satisfied, therefore there exist one point c such that $f'(c) = \frac{f(e)-f(1)}{e-1} \Rightarrow \frac{1}{c} = \frac{1}{e-1} \Rightarrow c = e-1 \in (1\ e)$

Hence Lagrange's Mean Value Theorem is verified.

7. Verify Lagrange's Mean Value Theorem for the function $f(x)=(x-1)(x-2)(x-3)$ in $[0,4]$.

Solution : is continuous as it is an algebraic function in $[0,4]$.

$f(x)=x^3-6x^2+11x-6 \Rightarrow f'(x)=3x^2-12x+11$ is defined $\forall\, x\in(0,4)$

$\therefore\ f(x)$ is differentiable in $(0,4)$, All conditions of Lagrange's Mean Value Theorem are satisfied, therefore there exist on $f(x)$ e point c such that $f'(c)=\dfrac{f(4)-f(0)}{4-0}$

$\Rightarrow 3c^2-12c+11=\dfrac{6-(-6)}{4} \Rightarrow 3c^2-12c+8=0 \Rightarrow c=\dfrac{12\pm\sqrt{48}}{6}$

$\therefore c=3.15$ and 0.85 both $\in(0,4)$

8. Prove that $\dfrac{b-a}{\sqrt{1-a^2}}<\sin^{-1}b-\sin^{-1}a<\dfrac{b-a}{\sqrt{1-b^2}}$ where $a<c<b<1$ using Lagrange's mean value theorem

Solution : $f(x)=\sin^{-1}x \therefore f'(x)=\dfrac{1}{\sqrt{1-x^2}}$, $f(x)$ is continous in [a,b] and differentiable in (a,b).

Applying Lagrange's mean value theorem , for $f(x)$ in $[a,b]$ we get when $a<c<b$

$\dfrac{\sin^{-1}b-\sin^{-1}a}{b-a}=\dfrac{1}{\sqrt{1-c^2}}$ We know that $a<c<b$

$\Rightarrow a^2<c^2<b^2 \Rightarrow -a^2>-c^2>-b^2 \Rightarrow 1-a^2>1-c^2>1-b^2$

$\dfrac{1}{\sqrt{1-a^2}}<\dfrac{1}{\sqrt{1-c^2}}<\dfrac{1}{\sqrt{1-b^2}} \Rightarrow \dfrac{1}{\sqrt{1-a^2}}<\dfrac{\sin^{-1}b-\sin^{-1}a}{b-a}<\dfrac{1}{\sqrt{1-b^2}}$

on multiplying by $(b-a)$ which is positive, we have

$\dfrac{b-a}{\sqrt{1-a^2}}<\sin^{-1}b-\sin^{-1}a<\dfrac{b-a}{\sqrt{1-b^2}}$

9. If $f'(x)>0$ for all the points in $[a,b]$, then prove that $f(x)$ is strictly increasing in the interval.

Solution : Let x_1,x_2 be two numbers such that $a\le x_1<x_2\le b$.

Applying LMVT in $[x_1,x_2]$ $\exists\, c\in(x_1,x_2)$ such that $\dfrac{f(x_2)-f(x_1)}{x_2-x_1}=f'(c)$

We have $f'(x)>0\ \forall\ x\in[a,b] \therefore \dfrac{f(x_2)-f(x_1)}{x_2-x_1}>0 \Rightarrow f(x_2)>f(x_1)$

$\therefore\ f(x)$ is an increasing function.

10. Show that for $x>0$, $\log(1+x)>\dfrac{x}{1+x}$

Solution : Let $f(x)=\log(1+x)-\dfrac{x}{1+x}$

$f'(x)=\dfrac{1}{1+x}-\left\{\dfrac{(1+x).1-x.1}{(1+x)^2}\right\}$

$f'(x)=\dfrac{x}{(1+x)^2}$, clearly $f'(x)>0$ since $x>0$ and also $f(x)$ is continous in [0,x] and differentiable in (0,x). Applying lagrange's mean value theorem for this $f(x)$ in [0,x] we have,

$f(x) = f(0) + (x-0)f'(c)$ but $f(0) = 0$

$\therefore$ $f'(x) = xf'(c)$; $f'(x) > 0 \Rightarrow f'(c) = 0$ and hence $f(x) > 0$

i.e. $\log(1+x) - \dfrac{x}{1+x} > 0$ or $\log(1+x) > \dfrac{x}{1+x}$.

Cauchy's Mean Value Theorem

Statement :

If the functions $f(x)$ and $g(x)$ are defined in the interval $[a,b]$ such that

i. $f(x)$ and $g(x)$ are continuous in $[a,b]$

ii. $f(x)$ and $g(x)$ are differentiable in (a,b)

iii. $g'(x) \neq 0$ in (a,b)

Then there exists at least point c in (a,b) such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$.

Proof : Let us define a function $\varphi(x) = f(x) - kg(x)$ ---(1)

where k is a constant to be choosen suitably later.from the given conditions it is evident that $\varphi(x)$ is also continous in [a,b] and differentiable in (a,b).

from (1) we have, $\varphi(a) = f(a) - kg(a)$; $\varphi(b) = f(b) - kg(b)$

$\therefore$ $\varphi(a) = \varphi(b)$ holds good if $f(a) - kg(a) = f(b) - kg(b) \Rightarrow k = \dfrac{f(b)-f(a)}{g(b)-g(a)}$ ----(2)

Here $g(b) \neq g(a)$, because if $g(b) = g(a)$ then $g(x)$ would satisfy all the conditions of Rolle's theorem and accordingly there must exist atleast one point c in (a,b) such that $g'(c) = 0$.

This contradicts the data that $g'(x) \neq 0$ for all x in (a,b). Hence if k is choosen as given in (2), then $\varphi(x)$ satisfy all the condition of Rolls theorem. Therefore by Rolles theorem there exist atleast one point c in (a,b) such that $\varphi'(c) = 0$

Differentiating (1) w.r.t x, we have $\varphi'(x) = f'(x) - kg'(x)$ and $\varphi'(c) = 0 \Rightarrow f'(c) - kg'(c) = 0$

i.e. $k = \dfrac{f'(c)}{g'(c)}$ -------(3), Equating R.H.S of (2) and (3) we have $f'(c) = \dfrac{f(b)-f(a)}{g(b)-g(a)}$.

Problems on Cauchy Mean Value Theorems

11. Verify Cauchy's Mean Value Theorem for the function $f(x) = e^x; g(x) = e^{-x}$ in $[a,b]$

Solution : $f(x)$ and $g(x)$ are both continuous in $[a,b]$

$f'(x) = e^x$; $g'(x) = -e^{-x}$ are defined $\forall x \in (a,b)$

$\therefore$ $f(x)$ and $g(x)$ are both differentiable in (a,b)

$g'(x) = -e^{-x} \neq 0, \forall x \in (a,b)$

$\exists c$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

$$\Rightarrow \frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}} \quad \Rightarrow e^{2c} = e^{a+b} \Rightarrow 2c = a+b$$

$\Rightarrow c = \frac{a+b}{2} \in (a,\, b)$ Hence the Cauchy's mean value theorem is verified.

12. Verify Cauchy's mean value theorem for the functions $\sqrt{x+9}$ and $\sqrt{x}$ in $[0,16]$

Solution : $f(x)$ and $g(x)$ are both continuous in $[0,16]$

$f'(x) = \frac{1}{2\sqrt{x+9}}$; $g'(x) = \frac{1}{2\sqrt{x}}$ are defined $\forall x \in (0,16)$

$\therefore$ $f(x)$ and $g(x)$ are both differentiable in $(0,16)$

$g'(x) = \frac{1}{2\sqrt{x}} \neq 0, \forall x \in (a,b)$, $\exists c$ such that $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\frac{f(16)-f(0)}{g(16)-g(0)} = \frac{1/2\sqrt{c+9}}{1/2\sqrt{c}} \quad \Rightarrow \frac{\sqrt{25}-\sqrt{9}}{\sqrt{16}-\sqrt{0}} = \frac{1/\sqrt{c+9}}{1/\sqrt{c}}$$

On simplifying we get $c = 3 \in (0,16)$. Thus the theorem is verified.

13. Verify Cauchy's mean value theorem for the functions $f(x)$ and $f'(x)$ in $[1,e]$ where $f(x) = \log x$

Solution : $f(x) = \log x$, Let $g(x) = f'(x) = 1\backslash x$

$f(x)$ and $g(x)$ are both continuous in $[1,e]$

$f'(x) = \frac{1}{x}$; $g'(x) = \frac{-1}{x^2}$ are defined $\forall x \in (1,e)$

$\therefore$ $f(x)$ and are both differentiable in $(1,e)$

$g'(x) = \frac{-1}{x^2} \neq 0 \;\; \forall x \in (1,e)$

$$\exists c \text{ such that } \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)} \Rightarrow \frac{1/c}{-1/c^2} = \frac{f(e)-f(1)}{g(e)-g(1)} \Rightarrow -c = \frac{\log e - \log 1}{(1/e)-1}$$

on simplifying we get $c = 1.6 \in (1,e)$ since $e = 2.7$.

Thus the theorem is verified.

14. Show that the constant c of Cauchy's mean valu A e theorem for the functions $1/x^2$ and $1/x$ in the interval a,b is the harmonic mean between a and b $(0 < a < b)$

Solution : $f(x)$ and $g(x)$ are both continuous in $[a,b]$

$f'(x) = \frac{-2}{x^3}$; $g'(x) = \frac{-1}{x^2}$ are defined $\forall x \in (a,b)$

$\therefore$ $f(x)$ and $g(x)$ are both differentiable in (a,b)

$g'(x) = \frac{-1}{x^2} \neq 0 \;\forall x \in (a,b)$ $\quad \exists c$ such that $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\Rightarrow \frac{-2/c^3}{-1/c^3} = \frac{1/b^2 - 1/a^2}{1/b - 1/a} \;\Rightarrow \frac{2}{c} = \frac{a^2 - b^2 / a^2b^2}{a - b / ab} \Rightarrow \frac{2}{c} = \frac{(a-b)(a+b)ab}{(a-b)(a^2b^2)}$$

$c = \frac{2ab}{a+b}$ is the harmonic mean between a and b, $c \in (a,b)$.Thus verified

Taylor's and MacLaurin's series expansions for function of one variable.

Taylor's Theorem

If $f(x)$ and its first $n-1$derivatives are continuous in a closed interval $\left[a, b\right]$, $f^{n-1}(x)$ is differentiable in the open interval (a, b) then there exists at least one point $'c'$ in the open interval $\left(a, b\right)$ such that $f(b)= f(a)+(b-a)f^{'}(a)+\dfrac{(b-a)^2}{2!}f^{''}(a)+---+\dfrac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a)+\dfrac{(b-a)^n}{n1}f^n(c)$.-------(1)

Another form of Taylor's Theorem

Taking $b-a=h$ and $c=a+\theta h$ where $\theta=\dfrac{c-a}{b-a}$ the result (1) may be put it in the following alternative form. $f(a+h)= f(a)+hf^{'}(a)+\dfrac{h^2}{2!}f^{''}(a)+---+\dfrac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\dfrac{h^n}{n1}f^n(a+\theta h)$, $0<\theta<1$ ---(2)

Taylor's Series

Taking $a+h=x$ in expression (2) we obtain

$f(x)= f(a)+(x-a)f^{'}(a)+\dfrac{(x-a)^2}{2!}f^{''}(a)+........+\dfrac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a)+\dfrac{(x-a)^n}{n!}f^n\left[a+\theta\left(x-a\right)\right]$------(3)

This expression gives the expansion of $f(x)$ in powers of $(x-a)$ and the expansion contains $n+1$terms. Higher order terms can be neglected for the approximation.

Problems on Taylor's Series

15. Verify Taylor's Theorem with $n=2$ for $f\ (x)=(1-x)^{5/2}$ in $\left[0,\dfrac{1}{3}\right]$.

Solution: We get $f\ '(x)=-\dfrac{5}{2}(1-x)^{3/2}$ and $f\ ''(x)=\dfrac{5}{2}\dfrac{3}{2}(1-x)^{1/2}$

$f(x)$ and $f\ '(x)$ are defined $\forall\ x\in\left[0,\ \dfrac{1}{3}\right]$ $f\ '$and f are continuous in $\left[0,\ \dfrac{1}{3}\right]$

$f\ ''(x)$ is defined $\forall\ x\ in\left(0,\dfrac{1}{3}\right)$. Therefore $f\ '(x)$ is differentiable in $\left(0,\dfrac{1}{3}\right)$

$$f\left(\frac{1}{3}\right)= f\ (0)+\left(\frac{1}{3}-0\right)f\ '(0)+\frac{\left(\frac{1}{3}-0\right)^2}{2!}f\ ''(c)\ ;\left(\frac{2}{3}\right)^{5/2}=1-\frac{5}{6}+\frac{5}{24}(1-c)^{1/2}$$

$c=0.1129\in\left(0,\dfrac{1}{3}\right)\ \therefore$ Taylor's Theorem is verified.

16. Obtain Taylor Series expansion of $\sin x$ in powers of $x-\dfrac{\pi}{2}$ up to the term containing $\left(x-\dfrac{\pi}{2}\right)^4$.

Solution: Let us expand $\sin x$ in powers of $x-\dfrac{\pi}{2}$ for that we have

$$f(x)=\sin x,\ f\left(\frac{\pi}{2}\right)=1,\ f\ '(x)=\cos x,\ f\ '\left(\frac{\pi}{2}\right)=0,\ f\ ''(x)=-\sin x,\ f\ ''\left(\frac{\pi}{2}\right)=-1,$$

$$f\ '''(x)=-\cos x,\ f\ '''\left(\frac{\pi}{2}\right)=0,\ f^{\ iv}(x)=\sin x,\ f^{\ iv}\left(\frac{\pi}{2}\right)=1,$$

Now using Taylor's series

$$f(x)=f\left(\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)}{1!}f'\left(\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)^2}{2!}f''\left(\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)^3}{3!}f'''\left(\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)^4}{4!}f^{IV}\left(\frac{\pi}{2}\right)+\ldots$$

$$f(x)=1+\frac{\left(x-\frac{\pi}{2}\right)}{1!}(0)+\frac{\left(x-\frac{\pi}{2}\right)^2}{2!}(-1)+\frac{\left(x-\frac{\pi}{2}\right)^3}{3!}(0)+\frac{\left(x-\frac{\pi}{2}\right)^4}{4!}(1)+\ldots$$

$$\sin x=1-\frac{\left(x-\frac{\pi}{2}\right)^2}{2!}+\frac{\left(x-\frac{\pi}{2}\right)^4}{4!}+\ldots$$

17. Obtain the Taylor's series expansion of $\log x$ **about** $x=1$ **up to the term containing fourth degree and hence obtain** $\log(1.1)$.

Solution: Taylor's series expansion about $x=a$ is given by,

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+.\frac{(x-a)^3}{3!}f''(a)\ldots\ldots\ldots \qquad (1)$$

Let $f(x)=\log x$ and $a=1$

$$\therefore f(x)=f(1)+(x-1)f'(1)+\frac{(x-1)^2}{2!}f''(1)+\frac{(x-1)^3}{3!}f'''(1)+\frac{(x-1)^4}{4!}f^{IV}(1)\ldots\ldots\ldots$$

Let $f(x)=\log x \Rightarrow f(1)=\log 1=0 \Rightarrow f'(x)=\frac{1}{x}\Rightarrow f'(1)=1$

$$f''(x)=-\frac{1}{x^2}\Rightarrow f''(1)=-1 \Rightarrow f'''(x)=\frac{2}{x^3}\Rightarrow f'''(1)=2 \Rightarrow f^{IV}(x)=-\frac{6}{x^4}\Rightarrow f^{IV}(1)=-6.$$

Substituting the derivative values in Taylor's series expansion we get,

$$\log x=0+(x-1).1+\frac{(x-1)^2}{2}(-1)+\frac{(x-1)^3}{6}(2)+\frac{(x-1)^4}{24}(-6)+\ldots\ldots.$$

Thus $\log x=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\ldots\ldots.$

For $x=1.1$, $\log(1.1)=0.1-\frac{(0.1)^2}{2}+\frac{(0.1)^3}{3}-\frac{(0.1)^4}{4}=0.0953$

18. Expand $\sin\left(\frac{x}{4}+h\right)$ **in ascending powers of '**h**' upto the term containing** h^4 **and hence deduce an approximate value of sin 50^0.**

Solution: Taylor's series expansion is given by

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}+\ldots.;\ \text{Taking } a=\frac{\pi}{4},\ f\left(\frac{\pi}{4}+h\right)=\sin\left(\frac{\pi}{4}+h\right)\Rightarrow f(x)=\sin x$$

Therefore, $f\left(\frac{\pi}{4}+h\right)=f\left(\frac{\pi}{4}\right)+hf'\left(\frac{\pi}{4}\right)+\frac{h^2}{2!}f''\left(\frac{\pi}{4}\right)+\frac{h^3}{3!}f'''\left(\frac{\pi}{4}\right)+\frac{h4}{4!}f^{IV}\left(\frac{\pi}{4}\right)+\ldots\ldots$

Consider $f(x)=\sin x \Rightarrow f\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}} \Rightarrow f'(x)=\cos x\ f'\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

$$f''(x)=-\sin x \; y_2\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}} \;\Rightarrow\; f'''(x)=-\cos x \; f'''\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$$

$$f^{IV}(x)=\sin x \Rightarrow f^{IV}\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}} \;\Rightarrow \sin\left(\frac{\pi}{4}+h\right)=\frac{1}{\sqrt{2}}\left\{1+h-\frac{h^2}{2!}-\frac{h^3}{3!}+\frac{h^4}{4!}\right\}$$

Taking $h=5^\circ$, that is $h=5.\left(\frac{\pi}{180}\right)=0.087 \Rightarrow \sin 50^\circ=0.7659$.

19. Obtain Taylor's series for the function $\log\cos x$ about $x=\frac{\pi}{3}$ up to the fourth degree term.

Solution: Taylor's series expansion at $x=\frac{\pi}{3}$ is given by,

$$y(x)=y\left(\frac{\pi}{3}\right)+\left(x-\frac{\pi}{3}\right)y_1\left(\frac{\pi}{3}\right)+\frac{(x-\pi/3)^2}{2!}y_2\left(\frac{\pi}{3}\right)+\frac{(x-\pi/3)^3}{3!}y_3\left(\frac{\pi}{3}\right)+\frac{(x-\pi/3)^4}{4!}y_4\left(\frac{\pi}{3}\right)+\ldots..$$

Let $y(x)=\cos x$ $\Rightarrow$Therefore $y\left(\frac{\pi}{3}\right)=\log\left[\cos\left(\frac{\pi}{3}\right)\right]=\log\left(\frac{1}{2}\right)=-\log 2$

$$y_1=\frac{1}{\cos x}(-\sin x) \;\Rightarrow\; y_1=-\tan x \Rightarrow y_1\left(\frac{\pi}{3}\right)=-\tan\left(\frac{\pi}{\sqrt{3}}\right)=-\sqrt{3}$$

$$y_2=-\sec^2 x=-\left(1+\tan^2 x\right)\Rightarrow y_2=-\left(1+y_1^2\right)\Rightarrow y_2\left(\frac{\pi}{3}\right)=-\left[1+\left(\sqrt{3}\right)^2\right]=-4$$

$$y_3=-2y_1y_2 \Rightarrow y_3\left(\frac{\pi}{3}\right)=-2-\sqrt{3}(-4)=-8\sqrt{3} \;\Rightarrow\; y_4=-2\left[y_1y_3+y_2^2\right]$$

$$\Rightarrow y_4\left(\frac{\pi}{4}\right)=-2\left[-\sqrt{3}\left(-8\sqrt{3}+16\right)\right]=-80$$

$$\log\cos x=-\log 2-\left(x-\frac{\pi}{3}\right)\sqrt{3}-\frac{\left(x-\frac{\pi}{3}\right)^2}{2}.4-\frac{\left(x-\frac{\pi}{3}\right)^3}{6}.8\sqrt{3}-\frac{\left(x-\frac{\pi}{3}\right)^4}{24}.80+\ldots\ldots$$

Thus $\log\cos x=-\log 2-\sqrt{3}\left(x-\frac{\pi}{3}\right)-2\left(x-\frac{\pi}{3}\right)^2-\frac{4}{\sqrt{3}}\left(x-\frac{\pi}{3}\right)^3-\frac{10}{3}\left(x-\frac{\pi}{3}\right)^4$

MacLaurin's series:

The MacLaurin's expansion is particular case of Taylor's series so by putting $a=0$ in the result (3) we get,

$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+--+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\frac{x^n}{n!}f^n(\theta x)$ is the MacLaurin's Series.

Problems on MacLaurin's series:

20. Obtain the MacLaurin's series of expansion for the function $e^{\cos x}$

Solution: Let $y=e^{\cos x} \Rightarrow y(0)=e^{\cos 0}=e \;\Rightarrow\; y_1=e^{\cos x}(-\sin x) \Rightarrow y_1(0)=0$

$$y_2=(\sin^2 x)e^{\cos x}-e^{\cos x}\cos x, \Rightarrow y_2(0)=-e \;\Rightarrow\; y_3=e^{\cos x}\left(\frac{3}{2}\sin 2x+\sin x-\sin^3 x\right)\Rightarrow y_3(0)=0$$

$$y_4=e^{\cos x}(-\sin x)\left(+\frac{3}{2}\sin 2x+\sin x-\sin^3 x\right)+e^{\cos x}\left(\frac{3}{2}\,2\cos 2x+\cos x-3\sin^2 x\cos x\right)$$

$$y_4(0)=e(4)=4e \;\Rightarrow e^{\cos x}=y(0)+xy_1(0)+\frac{x^2}{2!}y_2(0)+\frac{x^3}{3!}y_3(0)+\frac{x^4}{4!}y_4(0)+\ldots\ldots\ldots\ldots$$

$$e^{\cos x}=e-\frac{x^2}{2!}(e)+\frac{x^4}{4!}(4e)-+---$$

21. Obtain the MacLaurin's series of expansion for the function $\log(1+\sin x)$.

Solution: Let $y=\ln(1+\sin x) \Rightarrow y_1=\dfrac{\cos x}{1+\sin x}, y_2=\dfrac{-1}{1+\sin x}, y_3=\dfrac{\cos x}{(1+\sin x)^2}$;

We get $y(0)=0$, $y_1(0)=1$, $y_2(0)=-1$ & $y_3=-2$

$$y=y(0)+xy_1(0)+\frac{x^2}{2!}y_2(0)+\frac{x^3}{3!}y_3(0)+..........$$

$$y=0+x(1)+\frac{x^2}{2}(-1)+\frac{x^3}{3\,.\,2}(-2)+...... \Rightarrow y=x-\frac{x^2}{2}-\frac{x^3}{3}+..........$$

22. Obtain the MacLaurin's series of expansion for the function $\sin^{-1}x$.

Solution: $y=\sin^{-1}x \Rightarrow y(0)=0 \Rightarrow y_1=\dfrac{1}{\sqrt{1-x^2}} \Rightarrow y_1(0)=1$

$\sqrt{1-x^2}\;y_1=1$ or $(1-x^2)\;y_1^2=1$

Differentiate with respect to x again we get

$(1-x^2)\,2\,y_1y_2-2\,y_1^{\,2}x=0$ or $(1-x^2)\,y_2-y_1x=0 \Rightarrow y_2(0)=0$

Applying Leibniz's' Rule we get

$$(1-x^2)\,y_{n+2}+n(1-2x)\,y_{n+2}+\frac{n(n-1)}{2}(-2)\,y_n-\left[x\,y_{n+1}+n(1)\,y_n\right]=0$$

Put $x=0 \Rightarrow y_{n+2}(0)-n(n-1)\,y_n(0)-n\,y_n(0)=0 \Rightarrow y_{n+2}(0)=n^2y_n(0)$

For $n=0,1,2,3,......$ we get

$y_2(0)=0^2\,y(0)=0 \quad y_4(0)=2^2\,y_2(0)=0, \quad y_5(0)=3^2\,y_3(0)=9, \; y_6(0)=4^2\,y_4(0)=0$,

$y_7(0)=5^2\,y_5(0)=225$, $y_8(0)=0 \Rightarrow \sin^{-1}x=x(1)+\dfrac{x^3}{3!}1^2+\dfrac{x^5}{5!}3^21^2+\dfrac{x^7}{7!}5^23^21^2+.......$

L'Hospital's rule (without proof)

If an expression $f(x)$ at $x=a$ assumes forms like $0/0,\ \infty/\infty,\ 0\times\infty,\ \infty-\infty,\ 0^0,\ \infty^0, 1^\infty$ which do not represent any value are called Indeterminate forms. The concept of limit give a meaningful value for the function $f(x)$ at $x=a$ overcoming these indeterminate forms. Students alredy familiar with the evaluation of limit mostly in the case of $0/0,\ \infty/\infty$ without the involvement of differentiation. Few more indeterminate forms: $0\times\infty,\ \infty-\infty,\ 0^0,\ \infty^0, 1^\infty$ can be reduced to $0/0,\ \infty/\infty$, then the limit is found passing through a process of differentiation warranted by a very simple rule called L' Hospital's (French Mathematician) rule which is established by using Cauchy's Mean value theorem.

L' Hospital Rule (Theorem)

Let $f(x)$ and $g(x)$ be functions such that $\underset{x\to a}{Lim}\,f(x)=0$ and $\underset{x\to a}{Lim}\,g(x)=0$ i.e. $f(a)=0$ and $g(a)=0$

$f'(x)$ and $g'(x)$ exists and $g'(a)\neq 0$, then $\underset{x\to a}{Lim}\dfrac{f(x)}{g(x)}=\underset{x\to a}{Lim}\dfrac{f'(x)}{g'(x)}$.

Note: Extension of theorem

If $f'(a)=0$ and $g'(a)=0$ then we have $\underset{x\to a}{Lim}\dfrac{f(x)}{g(x)}=\underset{x\to a}{Lim}\dfrac{f'(x)}{g'(x)}=\underset{x\to a}{Lim}\dfrac{f''(x)}{g''(x)}$ and soon.

Working Rule:

a. The rule is applicable for the form $0/0$. It can be applied for the form ∞/∞ as we write $\underset{x\to a}{Lim}\frac{f(x)}{g(x)}=\underset{x\to a}{Lim}\frac{1/g(x)}{1/f(x)}$, where $f(x)\to\pm\infty$ and $g(x)\to\pm\infty$ as $x\to a$

b. Differentiate Numerator $f(x)$ and Denominator $g(x)$ separately and put $x=a$, if this reduces to indeterminate form $0/0$ continue the above procedure until a finite value is obtained.

c. In case where the expansions of functions, involved in indeterminate form are known or some of the Standard limits are known, may be used to simplify the work.

d. The following four standard limits and well known simple properties connected with limits can be readily used: (i) $\underset{x\to 0}{Lim}\frac{\sin x}{x}=1$ (ii) $\underset{x\to 0}{Lim}\frac{x}{\sin x}=1$ (iii) $\underset{x\to 0}{Lim}\frac{\tan x}{x}=1$ (iv) $\underset{x\to 0}{Lim}\frac{x}{\tan x}=1$

Indeterminate form $(0/0)$

The rule can be applied directly in case of $0/0$ and ∞/∞. In case of $\infty-\infty$ and $\infty\times 0$, we have to employ methods (taking L.C.M, using equivalent trignometric expressions etc) to simplify the given expression in bringing it to the form $0/0$ or ∞/∞ so that L' Hospital rule can be employed.

23. **Evaluate** $\lim\limits_{x\to 0}\frac{x-\log(1+x)}{1-\cos x}$

Solution: Let $y=\lim\limits_{x\to 0}\frac{x-\log(1+x)}{1-\cos x}\ \left(\frac{0}{0}\ form\right)$

On Applying L'Hospital's Rule, We get

$y=\lim\limits_{x\to 0}\frac{\left(1-\frac{1}{(1+x)}\right)}{\sin x}\quad\left(\frac{0}{0}\ form\right)$ On Applying L'Hospital's Rule, We get

$y=\lim\limits_{x\to 0}\frac{\left(0+\frac{1}{(1+x)^2}\right)}{\cos x}\Rightarrow y=1\Rightarrow$ Thus, $\lim\limits_{x\to 0}\frac{x-\log(1+x)}{1-\cos x}=1$

24. **Evaluate** $\lim\limits_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}$.

Solution: Let $y=\lim\limits_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}$

$y=\lim\limits_{x\to 0}\frac{\sin x\{\sec x-1\}}{\sin^3 x}\Rightarrow y=\lim\limits_{x\to 0}\frac{\sec x-1}{\sin^2 x}\left(\frac{0}{0}\ form\right)$

on Applying L'Hospital's Rule, We get

$y=\lim\limits_{x\to 0}\frac{\sec x-1}{x^2}.\frac{x^2}{\sin^2 x}$ Can be written as $y=\lim\limits_{x\to 0}\frac{\sec x\tan x}{2x}\ \left(\frac{0}{0}\ form\right)$

as $\because\lim\limits_{x\to 0}\frac{x^2}{\sin^2 x}=1$; on Applying L'Hospital's Rule, We get

$y=\lim\limits_{x\to 0}\frac{\sec x}{2}\ \left(\because\lim\limits_{x\to 0}\frac{\tan x}{x}=1\right)\Rightarrow y=\frac{1}{2}\Rightarrow$Thus, $\lim\limits_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}=\frac{1}{2}$

25. **Evaluate** $\lim\limits_{x\to 0}\frac{\sin x-x}{\tan^3 x}$

Solution: $\lim\limits_{x\to 0}\frac{\sin x-x}{\tan^3 x}\left(\frac{0}{0}\right)=\lim\limits_{x\to 0}\frac{\cos x-1}{3\tan^2 x\sec^2 x}\left(\frac{0}{0}\right)=\lim\limits_{x\to 0}\frac{-\sin x}{6\tan x\sec^4 x+6\tan^3 x\sec^2 x}\left(\frac{0}{0}\right)$

$$=\lim_{x\to 0}\frac{-\cos x}{6\sec^6 x+24\tan^2 x\sec^4 x+18\tan^2 x\sec^4 x+12\tan^4 x\sec^2 x}=-\frac{1}{6}$$

26. **Evaluate** $\lim_{x\to 0}\frac{a^x-b^x}{x}$

Solution: $\lim_{x\to 0}\frac{a^x-b^x}{x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{a^x\log a-b^x\log b}{1}=\log a-\log b=\log\frac{a}{b}$

27. **Evaluate** $\lim_{x\to 0}\frac{\cos x-\log(1+x)-1+x}{\sin^2 x}$

Solution: $\lim_{x\to 0}\frac{\cos x-\log(1+x)-1+x}{\sin^2 x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-\sin x-\frac{1}{1+x}+1}{\sin 2x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-\cos x+\frac{1}{(1+x)^2}}{2\cos 2x}=\frac{-1+1}{2}=0$

28. **Evaluate** $\lim_{x\to 1}\frac{x^x-x}{x-1-\log x}$

Solution: $\lim_{x\to 1}\frac{x^x-x}{x-1-\log x}\left(\frac{0}{0}\right)=\lim_{x\to 1}\frac{x^x(1+\log x)-1}{1-\frac{1}{x}}\left(\frac{0}{0}\right)=\lim_{x\to 1}\frac{x^x(1+\log x)^2+x^{x-1}}{\frac{1}{x^2}}=\frac{1+1}{1}=2$

29. **Evaluate** $\lim_{x\to 0}\frac{e^{ax}-e^{-ax}}{\log(1+bx)}$

Solution: $\lim_{x\to 0}\frac{e^{ax}-e^{-ax}}{\log(1+bx)}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{ae^{ax}+ae^{-ax}}{b/(1+bx)}=\frac{a+a}{b}=\frac{2a}{b}$

30. **Evaluate** $\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}$

Solution: $\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\sec^2 x-\cos x}{3\sin^2 x\cos x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{2\sec^2 x\tan x+\sin x}{6\sin x\cos^2 x-3\sin^3 x}\left(\frac{0}{0}\right)$

$$=\lim_{x\to 0}\frac{4\sec^2 x\tan^2 x+2\sec^4 x+\cos x}{6\cos^3 x-12\sin^2 x\cos x-9\sin^2 x\cos x}=\frac{0+2+1}{6-0-0}=\frac{3}{6}=\frac{1}{2}$$

Alternate Method:

$$\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}=\lim_{x\to 0}\frac{\frac{\tan x-\sin x}{x^3}}{\left(\frac{\sin x}{x}\right)^3}=\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}\left(\frac{0}{0}\right)\quad\because\lim_{x\to 0}\frac{\sin x}{x}=1$$

$$=\lim_{x\to 0}\frac{\sec^2 x-\cos x}{3x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{2\sec^2 x\tan x+\sin x}{6x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{4\sec^2 x\tan^2 x+2\sec^4 x+\cos x}{6}=\frac{0+2+1}{6}=\frac{3}{6}=\frac{1}{2}$$

Indeterminate form (∞/∞)

The given form or its simplified form will be in $0/0$ form when $x=0$ as $x\to 0$ but will inolve terms of the form $x^2\sin x, x\sin^3 x, x\tan^2 x$ etc. In the event of applying the rule, the differentiation becomes tedious and we should not venture to do so. We can conveniently modify such terms so as to involve $(\sin x/x)^k$ or $(\tan x/x)^k$ which can be seperated out from the given expression. These terms become 1 as $x\to 0$ with the result we will be left with a simple expression (product gets eliminated) in the $0/0$ form for the application of L' Hospital's rule. Simplification at each step has to be explored.

31. **Evaluate** $\lim_{x\to 0}\frac{\tan x - x}{x^2 \tan x}$

Solution: $\lim_{x\to 0}\frac{\tan x - x}{x^2\tan x} = \lim_{x\to 0}\frac{\frac{\tan x - x}{x^3}}{\left(\frac{\tan x}{x}\right)} = \lim_{x\to 0}\frac{\tan x - x}{x^3}\left(\frac{0}{0}\right) \qquad \because \lim_{x\to 0}\frac{\tan x}{x} = 1$

$= \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{2\sec^2 x \tan x}{6x}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{4\sec^2 x\tan^2 x + 2\sec^4 x}{6} = \frac{0+2}{6} = \frac{1}{3}$

32. **Evaluate** $\lim_{x\to 0}\frac{\log x}{\cot x}$

Solution: Let $y = \lim_{x\to 0}\frac{\log x}{\cot x} \quad \left(\frac{\infty}{\infty}\ form\right)$

on Applying L'Hospital's Rule, We get

$y = \lim_{x\to 0}\frac{\frac{1}{x}}{-\cos ec^2 x} \Rightarrow y = \lim_{x\to 0}\frac{-\sin^2 x}{x} \Rightarrow y = \lim_{x\to 0}\frac{-\sin x}{x}.\sin x \Rightarrow y = 0$

Thus, $\lim_{x\to 0}\frac{\log x}{\cot x} = 0$

33. **Evaluate** $\lim_{x\to 0}\frac{\log x}{\cos ecx}$

Solution: Let $y = \lim_{x\to 0}\frac{\log x}{\cos ecx}\left(\frac{\infty}{\infty}\ form\right)$

on Applying L'Hospital's Rule, We get

$y = \lim_{x\to 0}\frac{\frac{1}{x}}{-\cos ecx\ \cot x} \Rightarrow y = \lim_{x\to 0}\frac{-\sin x}{x}.\tan x \Rightarrow y = -1\times 0 \Rightarrow y = 0$

Thus, $\lim_{x\to 0}\frac{\log x}{\cos ecx} = 0$

34. **Evaluate** $\lim_{x\to a}\frac{\log(x-a)}{\log(e^x - e^a)}$

Solution: Let $y = \lim_{x\to a}\frac{\log(x-a)}{\log(e^x - e^a)} \qquad \left(\frac{\infty}{\infty}\ form\right)$

on Applying L'Hospital's Rule, We get

$y = \lim_{x\to a}\frac{\frac{1}{x-a}}{\frac{e^x}{(e^x - e^a)}} \Rightarrow y = \lim_{x\to a}\frac{e^x - e^a}{e^x(x-a)}\left(\frac{0}{0}\ form\right)$

on Applying L'Hospital's Rule, We get

$y = \lim_{x\to a}\frac{e^x}{e^x(x-a)+e^x} \Rightarrow y = 1 \Rightarrow$ Thus, $\lim_{x\to a}\frac{\log(x-a)}{\log(e^x - e^a)} = 1$

35. **Evaluate** $\lim_{x\to 0}\frac{\log(\sin 2x)}{\log(\sin x)}$

Solution: $\lim_{x\to 0}\frac{\log(\sin 2x)}{\log(\sin x)}\left(\frac{\infty}{\infty}\right) = \lim_{x\to 0}\frac{(2\cos 2x/\sin 2x)}{(\cos x/\sin x)} = \lim_{x\to 0}\frac{2\cot 2x}{\cot x} = \lim_{x\to 0}\frac{2\tan x}{\tan 2x}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{2\sec^2 x}{2\sec^2 2x} = \frac{2}{2} = 1$

36. Evaluate $\lim\limits_{x\to 0}\dfrac{\log x}{\cos ecx}$

Solution: $\lim\limits_{x\to 0}\dfrac{\log x}{\cos ecx}\left(\dfrac{\infty}{\infty}\right)=\lim\limits_{x\to 0}\dfrac{1\backslash x}{-\cos ec\,x.\cot x}=\lim\limits_{x\to 0}\dfrac{-\sin^2 x}{x\cos x}\left(\dfrac{0}{0}\right)=\lim\limits_{x\to 0}\dfrac{-2\sin 2x}{\cos x-x\sin x}=\dfrac{0}{1-0}=0$

37. Evaluate $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\log\cos x}{\tan x}$

Solution: $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\log\cos x}{\tan x}\left(\dfrac{\infty}{\infty}\right)=\lim\limits_{x\to\frac{\pi}{2}}\dfrac{-\tan x}{\sec^2 x}=-\lim\limits_{x\to\frac{\pi}{2}}\dfrac{-\tan x}{\sec^2 x}=\lim\limits_{x\to\frac{\pi}{2}}\dfrac{-\sin x\cos x}{1}=\dfrac{-0}{1}=0$

38. Evaluate $\lim\limits_{x\to 0}\log_{\tan 2x}\tan 3x$

Solution: $\lim\limits_{x\to 0}\log_{\tan 2x}\tan 3x=\lim\limits_{x\to 0}\left(\dfrac{\log\tan 3x}{\log\tan 2x}\right)\left(\dfrac{\infty}{\infty}\right)$ $\qquad\because \log_b a=\dfrac{\log_e a}{\log_e b}$

$$=\lim_{x\to 0}\left(\frac{3\sec^2 3x/\tan 3x}{2\sec^2 2x/\tan 2x}\right)=\lim_{x\to 0}\left(\frac{3/\sin 3x.\cos 3x}{2/\sin 2x.\cos 2x}\right)=\lim_{x\to 0}\left(\frac{3/\sin 3x.\cos 3x}{2/\sin 2x.\cos 2x}\right)$$

$$=\lim_{x\to 0}\left(\frac{6/\sin 6x}{4/\sin 4x}\right)=\lim_{x\to 0}\left(\frac{6\sin 4x}{4\sin 6x}\right)\left(\frac{0}{0}\right)=\lim_{x\to 0}\left(\frac{24\cos 4x}{24\cos 6x}\right)=\frac{24}{24}=1$$

39. Evaluate $\lim\limits_{x\to a}\dfrac{\log(x-a)}{\log(e^x-e^a)}$

Solution: $\lim\limits_{x\to a}\dfrac{\log(x-a)}{\log(e^x-e^a)}\left(\dfrac{\infty}{\infty}\right)=\lim\limits_{x\to a}\dfrac{1/(x-a)}{e^x/(e^x-e^a)}=\lim\limits_{x\to a}\dfrac{(e^x-e^a)}{e^x(x-a)}\left(\dfrac{0}{0}\right)=\lim\limits_{x\to a}\dfrac{e^x}{e^x(x-a)+e^x}=\dfrac{e^a}{e^a}=1$

Indeterminate form $(0\times\infty)$

To evaluate the limits of the form $(0\times\infty)$, we rewrite the given expression to obtain either $\left(\dfrac{0}{0}\right)$ or $\left(\dfrac{\infty}{\infty}\right)$ form and then apply the L'Hospital's Rule.

40. Evaluate $\lim\limits_{x\to\frac{\pi}{2}}(1-\sin x)\tan x$

Solution: Let $y=\lim\limits_{x\to\frac{\pi}{2}}(1-\sin x)\tan x\quad(0\times\infty\ \textit{form})$

$$y=\lim_{x\to\frac{\pi}{2}}\frac{(1-\sin x)}{\cot x}\qquad\left(\frac{0}{0}\ \textit{form}\right)$$

on Applying L'Hospital's Rule, We get

$$y=\lim_{x\to\frac{\pi}{2}}\frac{-\cos x}{-\cos ec^2 x}\Rightarrow y=\frac{0}{1}\Rightarrow y=0\ \Rightarrow\text{Thus, }\lim_{x\to\frac{\pi}{2}}(1-\sin x)\tan x=0$$

41. Evaluate $\lim\limits_{x\to 1}\left(\sec\dfrac{\pi}{2x}\right)\log x$

Solution: Let $y=\lim\limits_{x\to 1}\left(\sec\dfrac{\pi}{2x}\right)\log x\ (\infty\times 0\ \textit{form})$

$$y=\lim_{x\to 1}\frac{\log x}{\cos\dfrac{\pi}{2x}}\ \left(\frac{0}{0}\ \textit{form}\right)$$

on Applying L'Hospital's Rule, We get

$$y=\lim_{x\to1}\frac{1/x}{\frac{\pi}{2x^2}\sin\frac{\pi}{2x}}\Rightarrow y=\frac{2}{\pi}\Rightarrow \text{Thus, }\lim_{x\to1}\left(\sec\frac{\pi}{2x}\right)\log x=\frac{2}{\pi}$$

42. **Evaluate** $\lim\limits_{x\to\infty}(a^{\frac{1}{x}}-1)x$

Solution: $$\lim_{x\to\infty}(a^{\frac{1}{x}}-1)x\ \left(0\times\infty\ form\right)=\lim_{x\to\infty}\frac{(a^{\frac{1}{x}}-1)}{\left(\frac{1}{x}\right)}\left(\frac{0}{0}\right)=\lim_{x\to\infty}\frac{a^{\frac{1}{x}}(\log a)\left(\frac{-1}{x^2}\right)}{\left(\frac{-1}{x^2}\right)}$$

$$=\lim_{x\to\infty}a^{\frac{1}{x}}(\log a)=a^0\log a=\log a$$

43. **Evaluate** $\lim\limits_{x\to\frac{\pi}{2}}(1-\sin x)\tan x$

Solution: $$\lim_{x\to\frac{\pi}{2}}(1-\sin x)\tan x\ \left(0\times\infty\ form\right)=\lim_{x\to\frac{\pi}{2}}\frac{(1-\sin x)}{\cot x}\left(\frac{0}{0}\right)=\lim_{x\to\frac{\pi}{2}}\frac{-\cos x}{-\cos ec^2x}=\frac{0}{1}=0$$

44. **Evaluate** $\lim\limits_{x\to1}\sec\frac{\pi}{2x}.\log x$

Solution: $$\lim_{x\to1}\sec\frac{\pi}{2x}.\log x\ \left(\infty\times0\ form\right)=\lim_{x\to1}\frac{\log x}{\cos\frac{\pi}{2x}}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{1/x}{-\frac{\pi}{2}\left(\sin\frac{\pi}{2x}\right)\left(\frac{-1}{x^2}\right)}=\lim_{x\to1}\frac{2x}{\pi\sin\frac{\pi}{2x}}=\frac{2}{\pi}$$

45. **Evaluate** $\lim\limits_{x\to0}x\log\tan x$

Solution: $$\lim_{x\to0}x\log\tan x\ \left(0\times\infty\ form\right)=\lim_{x\to0}\frac{\log\tan x}{(1/x)}\left(\frac{\infty}{\infty}\right)$$

$$=\lim_{x\to0}\frac{\sec^2x/\tan x}{\left(\frac{-1}{x^2}\right)}=\lim_{x\to0}\frac{-x^2}{\sin x.\cos x}\Rightarrow=\lim_{x\to0}\frac{-2x^2}{\sin2x}\left(\frac{0}{0}\right)=\lim_{x\to0}\frac{-4x}{2\cos2x}=\frac{0}{2}=0$$

46. **Evaluate** $\lim\limits_{x\to1}(1-x^2)\tan\frac{\pi x}{2}$

Solution: $$\lim_{x\to1}(1-x^2)\tan\frac{\pi x}{2}\ \left(0\times\infty\ form\right)=\lim_{x\to1}\frac{1-x^2}{\cot\frac{\pi x}{2}}\left(\frac{0}{0}\right)=\lim_{x\to1}\frac{-2x}{-\frac{\pi}{2}\cos ec^2\frac{\pi x}{2}}=\frac{2}{\left(\frac{\pi}{2}\right)}=\frac{4}{\pi}$$

47. **Evaluate** $\lim\limits_{x\to0}\tan x.\log x$

Solution: $$\lim_{x\to0}\tan x.\log x\left(0\times\infty\ form\right)=\lim_{x\to0}\frac{\log x}{\cot x}\left(\frac{\infty}{\infty}\right)$$

$$=\lim_{x\to0}\frac{1/x}{-\cos ec^2x}=\lim_{x\to0}\frac{-\sin^2x}{x}\left(\frac{0}{0}\right)=\lim_{x\to0}\frac{-\sin2x}{1}=\frac{0}{1}=0$$

Indeterminate form $(\infty-\infty)$

To evaluate the limits of the form $(\infty-\infty)$, we take L.C.M. and rewrite the given expression to obtain either $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ form and then apply the L'Hospital's Rule.

48. Evaluate $\lim_{x\to 0}\left[\frac{1}{x}-\cot x\right]$

Solution: $\lim_{x\to 0}\left[\frac{1}{x}-\cot x\right]=\lim_{x\to 0}\left[\frac{\sin x - x\cos x}{x\sin x}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{\cos x-\cos x+x\sin x}{\sin x+x\cos x}\right]$

$$=\lim_{x\to 0}\left[\frac{\sin x - x\cos x}{x\sin x}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{\cos x-\cos x+x\sin x}{\sin x+x\cos x}\right]$$

$$=\lim_{x\to 0}\left[\frac{x\sin x}{\sin x+x\cos x}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{\sin x+x\cos x}{\cos x+\cos x-x\sin x}\right]=\frac{0+0}{1+1-0}=0$$

49. Evaluate $\lim_{x\to\frac{\pi}{2}}\left[\sec x-\tan x\right]$

Solution: $\lim_{x\to\frac{\pi}{2}}\left[\sec x-\tan x\right]=\lim_{x\to\frac{\pi}{2}}\left[\frac{1}{\cos x}-\frac{\sin x}{\cos x}\right]\left(\infty-\infty\ form\right)$

$$=\lim_{x\to\frac{\pi}{2}}\left[\frac{1-\sin x}{\cos x}\right]\left(\frac{0}{0}\right)=\lim_{x\to\frac{\pi}{2}}\left[\frac{-\cos x}{-\sin x}\right]=\frac{0}{1}=0$$

50. Evaluate $\lim_{x\to 1}\left[\frac{1}{\log x}-\frac{x}{x-1}\right]$

Solution: $\lim_{x\to 1}\left[\frac{1}{\log x}-\frac{x}{x-1}\right]\left(\infty-\infty\ form\right)=\lim_{x\to 1}\left[\frac{(x-1)-x\log x}{(x-1)\log x}\right]\left(\frac{0}{0}\right)$

$$=\lim_{x\to 1}\left[\frac{1-1-\log x}{\frac{x-1}{x}+\log x}\right]=\lim_{x\to 1}\left[\frac{-\log x}{1-\frac{1}{x}+\log x}\right]\left(\frac{0}{0}\right)=\lim_{x\to 1}\left[\frac{-1/x}{\frac{1}{x^2}+\frac{1}{x}}\right]=\frac{-1}{1+1}=\frac{-1}{2}$$

51. Evaluate $\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{e^x-1}\right]$

Solution: $\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{e^x-1}\right]\left(\infty-\infty\ form\right)=\lim_{x\to 0}\left[\frac{(e^x-1)-x}{x(e^x-1)}\right]\left(\frac{0}{0}\right)$

$$=\lim_{x\to 0}\left[\frac{e^x-1}{(e^x-1)+xe^x}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{e^x}{e^x+e^x+xe^x}\right]=\frac{1}{1+1+0}=\frac{1}{2}$$

52. Evaluate $\lim_{x\to 0}\left[\frac{1}{\sin x}-\frac{1}{x}\right]$

Solution: $\lim_{x\to 0}\left[\frac{1}{\sin x}-\frac{1}{x}\right]\left(\infty-\infty\ form\right)=\lim_{x\to 0}\left[\frac{x-\sin x}{x\sin x}\right]\left(\frac{0}{0}\right)$

$$=\lim_{x\to 0}\left[\frac{1-\cos x}{\sin x+x\cos x}\right]\left(\frac{0}{0}\right)==\lim_{x\to 0}\left[\frac{\sin x}{\cos x+\cos x-x\sin x}\right]=\frac{0}{1+1}=0$$

53. Evaluate $\lim_{x\to 0}\left[\frac{1}{x}-\frac{\log(1+x)}{x^2}\right]$

Solution: $\lim_{x\to 0}\left[\frac{1}{x}-\frac{\log(1+x)}{x^2}\right]=\lim_{x\to 0}\left[\frac{x-\log(1+x)}{x^2}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{1-\frac{1}{1+x}}{2x}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{\frac{1}{(1+x)^2}}{2}\right]=\frac{1}{2}$

54. **Evaluate** $\lim\limits_{x\to 0}\left[\frac{a}{x}-\cot\frac{x}{a}\right]$

Solution: $\lim\limits_{x\to 0}\left[\frac{a}{x}-\cot\frac{x}{a}\right]=\lim\limits_{x\to 0}\left[\frac{a}{x}-\frac{\cos\frac{x}{a}}{\sin\frac{x}{a}}\right](\infty-\infty\ form)=\lim\limits_{x\to 0}\left[\frac{a}{x}-\frac{\cos\frac{x}{a}}{\sin\frac{x}{a}}\right]\left(\frac{0}{0}\right)$

$$=\lim_{x\to 0}\left[\frac{a\sin\frac{x}{a}-x\cos\frac{x}{a}}{x\sin\frac{x}{a}}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{a.\frac{1}{a}\cos\frac{x}{a}-\cos\frac{x}{a}+\frac{x}{a}\sin\frac{x}{a}}{\sin\frac{x}{a}+\frac{x}{a}\cos\frac{x}{a}}\right]$$

$$=\lim_{x\to 0}\left[\frac{\frac{x}{a}\sin\frac{x}{a}}{\sin\frac{x}{a}+\frac{x}{a}\cos\frac{x}{a}}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{\frac{1}{a}\sin\frac{x}{a}+\frac{x}{a}.\frac{1}{a}.\cos\frac{x}{a}}{\frac{1}{a}\cos\frac{x}{a}+\frac{1}{a}\cos\frac{x}{a}-\frac{x}{a^2}\sin\frac{x}{a}}\right]=\frac{0+0}{\frac{1}{a}+\frac{1}{a}-0}=0$$

55. **Evaluate** $\underset{x\to 0}{Lim}\left[\frac{1}{x^2}-\frac{1}{\sin^2 x}\right](\infty-\infty)$

Solution: Let $k=\underset{x\to 0}{Lim}\left[\frac{1}{x^2}-\frac{1}{\sin^2 x}\right](\infty-\infty)$

$$=\underset{x\to 0}{Lim}\left[\frac{\sin^2 x-x^2}{x^2\sin^2 x}\right]\left(\frac{0}{0}\right)\Rightarrow \underset{x\to 0}{Lim}\left[\frac{\sin^2 x-x^2}{x^4}\right].\underset{x\to 0}{Lim}\left(\frac{x}{\sin x}\right)^2$$

$\Rightarrow \underset{x\to 0}{Lim}\left[\frac{\sin^2 x-x^2}{x^4}\right]$.1**,** Applying L Hospital's rule

Applying L Hospital's rule $\Rightarrow \underset{x\to 0}{Lim}\left[\frac{2\sin x\cos x-2x}{4x^3}\right]\Rightarrow \underset{x\to 0}{Lim}\left[\frac{\sin 2x-2x}{4x^3}\right]\left(\frac{0}{0}\right)$

$$\Rightarrow \underset{x\to 0}{Lim}\left[\frac{2\cos 2x-2}{12x^2}\right]\left(\frac{0}{0}\right)\Rightarrow \underset{x\to 0}{Lim}\left[\frac{\cos 2x-1}{6x^2}\right]\Rightarrow \underset{x\to 0}{Lim}\left[\frac{-2\sin^2 x}{6x^2}\right]$$

$$\Rightarrow \frac{-1}{3}\underset{x\to 0}{Lim}\left[\frac{\sin x}{x}\right]^2\Rightarrow k=\frac{-1}{3}$$

56. **Evaluate** $\lim\limits_{x\to 1}\left\{\frac{2}{x^2-1}-\frac{1}{x-1}\right\}\ (\infty-\infty)$

Solution: Let $y=\lim\limits_{x\to 1}\left\{\frac{2}{x^2-1}-\frac{1}{x-1}\right\}\ (\infty-\infty)$

$y=\lim\limits_{x\to 1}\frac{2-(x+1)}{x^2-1}$

On Applying L'Hospital's Rule, We get $y=\lim\limits_{x\to 1}\frac{-1}{2x}\Rightarrow y=-\frac{1}{2}$

Thus, $\lim\limits_{x\to 1}\left\{\frac{2}{x^2-1}-\frac{1}{x-1}\right\}=-\frac{1}{2}$

57. **Evaluate** $\lim\limits_{x\to 0}\left\{\frac{1}{x^2}-\frac{1}{x\tan x}\right\}$

Solution: Let $y=\lim\limits_{x\to 0}\left\{\frac{1}{x^2}-\frac{1}{x\tan x}\right\}\ (\infty-\infty\ form)$

$$y=\lim_{x\to 0}\frac{\tan x - x}{x^2\tan x} \qquad \left(\frac{0}{0}\ form\right)$$

on Applying L'Hospital's Rule, We get

$$y=\lim_{x\to 0}\frac{\sec^2 x - 1}{2x\tan x + x^2\sec^2 x} \qquad \left(\frac{0}{0}\ form\right)$$

$$y=\lim_{x\to 0}\frac{\tan^2 x}{2x\ \tan x + x^2 + x^2\tan^2 x} \Rightarrow y=\lim_{x\to 0}\frac{\tan^2 x}{x^2\left(\frac{2\tan x}{x}+\sec^2 x\right)}$$

$$y=\lim_{x\to 0}\frac{1}{\frac{2\tan x}{x}+\sec^2 x} \Rightarrow y=\frac{1}{3} \Rightarrow \text{Thus, } \lim_{x\to 0}\left\{\frac{1}{x^2}-\frac{1}{x\tan x}\right\}=\frac{1}{3}$$

58. Find the value of 'a' such that $\lim_{x\to 0}\frac{\sin 2x + a\sin x}{x^3}$ is finite. Also find the value of the limit.

Solution: Let $A=\lim_{x\to 0}\frac{\sin 2x + a\sin x}{x^3}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{2\cos 2x + a\cos x}{3x^2}=\frac{2+a}{0}\neq finite$

We can continue to apply the L'Hospital's Rule, if 2+a=0 i.e., a= -2 .

$For\ a=-2$, $A=\lim_{x\to 0}\frac{2\cos 2x - 2\cos x}{3x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-4\sin 2x + 2\sin x}{6x}\left(\frac{0}{0}\right)$

$=\lim_{x\to 0}\frac{-8\cos 2x + 2\cos x}{6}=\frac{-8+2}{6}=-1$

$\therefore$ *The given* lim *it will have a finite value when* $a=-2$ *and it is* -1.

59. Find the values of 'a' and 'b' such that $\lim_{x\to 0}\frac{x(1-a\cos x)+b\sin x}{x^3}=\frac{1}{3}$

Solution: *Let* $A=\lim_{x\to 0}\frac{x(1-a\cos x)+b\sin x}{x^3}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{(1-a\cos x)+ax\sin x+b\cos x}{3x^2}=\frac{1-a+b}{0}\neq finite$

We can continue to apply the L'Hospital's Rule, if 1-a+b=0 i.e., a-b = 1 .

$For\ a-b=1$,

$A=\lim_{x\to 0}\frac{(1-a\cos x)+ax\sin x+b\cos x}{3x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{2a\sin x+ax\cos x-b\sin x}{6x}\left(\frac{0}{0}\right)$

$=\lim_{x\to 0}\frac{3a\cos x-ax\sin x-b\cos x}{6}=\frac{3a-b}{6}=\ finite$

This finite value is given as $\frac{1}{3}$. *i.e.,* $\frac{3a-b}{6}=\frac{1}{3}\Rightarrow 3a-b=2$

Solving the equations $a-b=1$ *and* $3a-b=2$ *we obtain* $a=\frac{1}{2}$ *and* $b=-\frac{1}{2}$.

60. Find the values of 'a' and 'b' such that $\lim_{x\to 0}\frac{a\cosh x-b\cos x}{x^2}=1$

Solution: *Let* $A=\lim_{x\to 0}\frac{a\cosh x-b\cos x}{x^2}=\frac{a-b}{0}\neq finite$

We can continue to apply the L'Hospital'sRule, if a-b=0, since the denominator=0 .

$For\ a-b=0$,

$A=\lim_{x\to 0}\frac{a\cosh x-b\cos x}{x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{a\sinh x+b\sin x}{2x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{a\cosh x+b\cos x}{2}=\frac{a+b}{2}$

But this is given as 1 $\therefore a+b=2$

solving equations $a-b=0$ and $a+b=2$ we get $a=1$ and $b=1$

Indeterminate form $\left(0^0, \infty^0, 1^\infty\right)$

It is evident that the function involved will be of the form $\left[f(x)\right]^{g(x)}$ and we have to find the limit as $x \to a$.

Let, $k = \underset{x\to a}{Lim}\left[f(x)\right]^{g(x)}$; Take logarithms on both side we have, $\log_e k = \underset{x\to a}{Lim}\, g(x).\log\left[f(x)\right]$, we can evaluate the limit on R.H.S as already discussed and let us suppose that limit is equal to "L" i.e. $\log_e k = L \Rightarrow k = e^L$ which is required limit.

Remark: One of the common question is that why 1^∞ is indeterminate?

Reason: Let $k = \underset{x\to a}{Lim}\, g(x).\log\left[f(x)\right] - - - - - (1^\infty)$

$$\Rightarrow \log_e k = \lim_{x\to a} g(x)\log f(x) - - - - \left[\infty \times \log 1\right]\ or \left(\infty \times 0\right) \text{ which is indeterminate.}$$

on other hand if $k = \underset{x\to a}{Lim}\, g(x).\log\left[f(x)\right]$ is of the form c^∞ where $c \neq 1$ we have

$$\log_e k = \lim_{x\to a} g(x)\log f(x) = \infty \times \log c = \infty.$$

61. Evaluate $\lim\limits_{x\to 0}(\sin x)^{\sin x}$

Solution: Let $y = \lim\limits_{x\to 0}(\sin x)^{\sin x}$ $\left(0^o\ form\right)$

On Applying log on both sides

$$\log y = \lim_{x\to 0} \log(\sin x)^{\sin x}$$

$$\log y = \lim_{x\to 0} \sin x \log(\sin x) \qquad (0.\infty\ form)$$

$$\log y = \lim_{x\to 0} \frac{\log \sin x}{\operatorname{cosec} x} \qquad \left(\frac{\infty}{\infty}\ form\right)$$

On Applying L'Hospital's Rule, We get

$$\log y = \lim_{x\to 0} \frac{\cot x}{-\operatorname{cosec} x \cot x} \qquad \left(\frac{\infty}{\infty}\ form\right)$$

On Applying L'Hospital's Rule, We get

$$\log y = \lim_{x\to 0} \frac{-\sin x}{1} \Rightarrow \log y = 0 \Rightarrow y = e^0 \Rightarrow y = 1, \text{ Thus, } \lim_{x\to 0}(\sin x)^{\sin x} = 1$$

62. Evaluate $\lim\limits_{x\to 0}(\cot x)^{1/\log x}$

Solution: Let $y = \lim\limits_{x\to 0}(\cot x)^{1/\log x}$ $\left(\infty^o\ form\right)$

$$\log y = \lim_{x\to 0} \frac{1}{\log x}\log(\cot x) \qquad \left(\frac{\infty}{\infty}\ form\right)$$

on Applying L'Hospital's Rule, We get

$$\log y = \lim_{x\to 0} \frac{\frac{1}{\cot x}(-\operatorname{cosec}^2 x)}{1/x} \Rightarrow \log y = \lim_{x\to 0} \frac{-x\operatorname{cosec}^2 x}{\cot x}$$

$$\log y = \lim_{x\to 0} -x\frac{1}{\sin^2 x}.\frac{\sin x}{\cos x} \Rightarrow \log y = \lim_{x\to 0} \frac{-x^2}{\sin^2 x}.\frac{\sin x}{x}.\frac{1}{\cos x}$$

$$\therefore \log y = -1 \Rightarrow y = \frac{1}{e} \Rightarrow \text{Thus, } \lim_{x\to 0}(\cot x)^{1/\log x} = \frac{1}{e}$$

63. Evaluate $\lim_{x\to 0}\left(\frac{1}{x}\right)^{2\sin x}$

Solution: Let $y=\lim_{x\to 0}\left(\frac{1}{x}\right)^{2\sin x}$ $\quad(\infty^{o}\ \ form)$

$$\log\ y=\lim_{x\to 0}\log\left(\frac{1}{x}\right)^{2\sin x}\Rightarrow \log y=\lim_{x\to 0}2\sin x\log\left(\frac{1}{x}\right)$$

$$\Rightarrow \log y=\lim_{x\to 0}\frac{2\log\left(\frac{1}{x}\right)}{\cos ecx}\quad\left(\frac{\infty}{\infty}\ form\right)$$

On Applying L'Hospital's Rule, We get

$$\log y=\lim_{x\to 0}\frac{2x\left(\frac{-1}{x^{2}}\right)}{-\cos ecx\cot x}\quad\left(\frac{\infty}{\infty}\ form\right)$$

$$\log y=\lim_{x\to 0}\frac{2\tan x}{x}\times\sin x\Rightarrow \log y=\lim_{x\to 0}2\sin x\left(\sin ce\ \ \underset{x\to 0}{lt}\frac{\tan x}{x}=1\right)$$

$$\log y=0\Rightarrow y=e^{0}\Rightarrow y=1\Rightarrow \text{Thus, }\lim_{x\to 0}\left(\frac{1}{x}\right)^{2\sin x}=1$$

64. Evaluate $\lim_{x\to 0}(a^{x}+x)^{1/x}$

Solution: Let $y=\lim_{x\to 0}(a^{x}+x)^{1/x}$ $(1^{\infty}\ form)\Rightarrow \log y=\lim_{x\to 0}\frac{1}{x}\ \log(a^{x}+x)\ \left(\frac{\infty}{\infty}\ form\right)$

on Applying L'Hospital's Rule, We get

$$\log y=\lim_{x\to 0}\frac{\frac{1}{a^{x}+x}(a^{x}\ \log a+1)}{1}\Rightarrow y=\lim_{x\to 0}\frac{(\log a.a^{x})+1}{a^{x}+x}\Rightarrow y=\frac{\log a+1}{1}$$

$$\log y=\log a+\log e\ \Rightarrow \log y=\ln ae\Rightarrow y=ae\Rightarrow \text{Thus, }\lim_{x\to 0}(a^{x}+x)^{1/x}=ae$$

65. Evaluate $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x}$

Solution: Let $y=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x}$ $\quad(1^{\infty}\ form)$

$$\log y=\lim_{x\to 0}\log\left\{\frac{\sin x}{x}\right\}^{\frac{1}{x}}\Rightarrow \log y=\lim_{x\to 0}\frac{1}{x}\log\left\{\frac{\sin x}{x}\right\}$$

$$\log y=\lim_{x\to 0}\frac{\frac{x}{\sin x}\left\{\frac{x\cos x-\sin x}{x^{2}}\right\}}{1}\Rightarrow \log y=\lim_{x\to 0}\frac{x\cos x-\sin x}{x^{2}}\ \left(\frac{0}{0}\ form\right)$$

on Applying L'Hospital's Rule, We get

$$\log y=\lim_{x\to 0}\frac{-x\ \sin x+\cos x-\cos x}{2x}\ \left(\frac{0}{0}\ form\right)$$

on Applying L'Hospital's Rule, We get

$$\log y=\lim_{x\to 0}\frac{-x\cos x-\sin x}{2}\Rightarrow \log y=0\Rightarrow y=e^{0}\Rightarrow y=1,\text{Thus, }\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x}=1$$

Question Bank

Mean Value Theorems

Sl.No	Problems	Answers
1	Verify Rolle's Theorem for $f(x)=\sin x/e^x$ in $[0,\pi]$	$c=\pi/4$
2	Verify Rolle's Theorem for $f(x)=x(x+3)e^{-x/2}$ in $[-3,0]$	$c=-2$
3	Verify Rolle's Theorem for $f(x)=(x-a)[(x-a)(x-b)]^2$ in $[a,b]$	$c=2a+3b/5$
4	Verify Lagrange's Mean Value Theorem for $f(x)=\log(x)$ in $\left[e,e^2\right]$	$c=e^2-e$
5	Verify Lagrange's Mean Value Theorem for $f(x)=\tan^{-1}x$ in $[0,1]$	$c\approx 0.523$
6	Verify Lagrange's Mean Value Theorem for $f(x)=\cos^2 x$ in $\left[0,\frac{\pi}{2}\right]$	$c=0.345$
7	Verify Lagrange's Mean Value Theorem for $f(x)=(x-1)(x-2)(x-3)$ in $[0,4]$	$c=2\pm 1.1547$
8	find θ of Lagrange's mean value theorem for the function $f(x)=e^x$ in (0,1)	$\theta=0.53$
9	Verify Cauchy's Mean Value Theorem for $f(x)=\sin x$ and $g(x)=\cos x$ in $\left[\frac{\pi}{4},\frac{3\pi}{4}\right]$	$c=\pi$
10	Verify Cauchy's Mean Value Theorem for $f(x)=e^x$ and $g(x)=1/e^x$ in $[a,b]$	$c=a+b/2$
11	Verify Cauchy's Mean Value Theorem for $f(x)$ and $2f'(x)$ in $[a,b]$, where $f(x)=\sqrt{x}$	$c=\sqrt{ab}$
12	show that if $0<a<b$, $\frac{b-a}{\sqrt{1+b^2}}<\tan^{-1}b-\tan^{-1}a<\frac{b-a}{\sqrt{1+a^2}}$ & hence deduce that $\frac{5\pi+4}{20}<\tan^{-1}2<\frac{\pi+2}{4}$	

Taylor's and MacLaurin's Series expansion of function of one variables

Sl.No	Problems and Answers
1	Expand $\tan x$ about the point $x=\pi/4$ upto third degree terms and hence find $\tan 46^0$ **Ans:** $1+2(x-\pi/4)+2(x-\pi/4)^2+8/3(x-\pi/4)^3$; $\tan 46^0=1.035$
2	Show that $\sqrt{x}=\sqrt{2}\left[1+(x-2)/4-(x-2)^2/32+(x-2)^3/128+\ldots\right]$
3	Expand $\tan^{-1}x$ in power of $(x-1)$ upto the term containing fourth degree. **Ans:** $\tan^{-1}x=(\pi/4)+1/2\left\{(x-1)-(x-1)^2/2+(x-1)^3/6\right\}$
4	Using MacLaurin's Series prove that $\sqrt{1+\sin 2x}=1+x-\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}\ldots$
5	Obtain MacLaurin's Series expansion of function $\log(1+x)$ and hence deduce that $\log\sqrt{\frac{1+x}{1-x}}=x+\frac{x^3}{3}+\frac{x^5}{5}+..$ **Ans:** $\log(1+x)=x-x^2/2+x^3/3-x^4/4+x^5/5-\ldots$
6	Obtain MacLaurin's expansion of a^x **Ans:** $a^x=1+x\log a+x^2/2(\log a)^2+x^3/6(\log a)^3+\ldots$
7	Expand $e^{x\sin x}$ in ascending power of x upto fourth degree term.

	Ans: $e^{x\sin x}=1+x^2+x^4/3+...$
8	Expand $e^{\tan^{-1}x}$ upto the term containing x^5 **Ans:** $e^{\tan^{-1}x}=1+x+x^2/2-x^3/6-7x^4/24+x^5/24$
9	Obtain MacLaurin's Series expansion of $\log(\sec x+\tan x)$ upto the first three non vanishing terms. **Ans:** $\log(\sec x+\tan x)=x+x^3/6+x^5/24$
10	Expand $\tan(\pi/4+x)$ upto fourth degree term. **Ans:** $\tan(\pi/4+x)=1+2x+2x^2+8x^3/3+10x^4/3$

Indeterminate Forms

Sl.No	Problems	Answers
1	Evaluate $\underset{x\to 0}{Lim}\dfrac{x\cos x-\log(1+x)}{x^2}$	1/2
2	Evaluate $\underset{x\to 0}{Lim}\dfrac{x-\log(1+x)}{1-\cos x}$	1
3	Evaluate $\underset{x\to \pi/4}{Lim}\dfrac{\sec^2 x-2\tan x}{1+\cos 4x}$	1/2
4	Evaluate $\underset{x\to 0}{Lim}\dfrac{\sin x.\sin^{-1}x}{x^2}$	1
5	Evaluate $\underset{x\to 1}{Lim}\dfrac{\log(1-x)}{\cot \pi x}$	0
6	Evaluate $\underset{x\to \infty}{Lim}\dfrac{x\cos(1/x)}{1+x}$	1
7	Evaluate $\underset{x\to \pi/2}{Lim}\dfrac{\log(\cos x)}{\tan x}$	0
8	Evaluate $\underset{x\to 1}{Lim}\left[\dfrac{1}{\log x}-\dfrac{x}{\log x}\right]$	-1
9	Evaluate $\underset{x\to \pi/4}{Lim}\left(\tan x\right)^{\tan 2x}$	1/e
10	Evaluate $\underset{x\to 0}{Lim}\dfrac{a^x+b^x+c^x}{3}$	$(abc)^{1/3}$
11	Evaluate $\underset{x\to 0}{Lim}\left[\dfrac{\tan x}{x}\right]^{1/x^2}$	$e^{1/3}$
12	Find the constant a and b such that $\underset{x\to 0}{Lim}\dfrac{x(1+a\cos x)-b\sin x}{x^3}$ may be equal to unity.	a=-5/2, b=-3/2

Module-3
MULTIVARIABLE CALCULUS (DIFFERENTIATION)

- Partial derivatives
- Total derivatives
- Jacobians
- Maxima and Minima
- Method of Lagrange multipliers
- Directional derivatives
- Gradient, Curl, Divergence

Partial derivatives

The functions which depend on more than one independent variable are called functions of several variables. Partial derivatives of a function of several variables are the derivative with respect to one of the variables when all the remaining variables are kept constant.

Let $z = f(x, y)$ be a function of two variables, keeping y constant and varying x only, the partial derivative of z with respect to x is defined as

$\frac{\partial z}{\partial x} = \underset{\Delta x \to 0}{lt} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$, which is denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or $z_x(x, y)$ or $f_x(x, y)$ or z_x or f_x

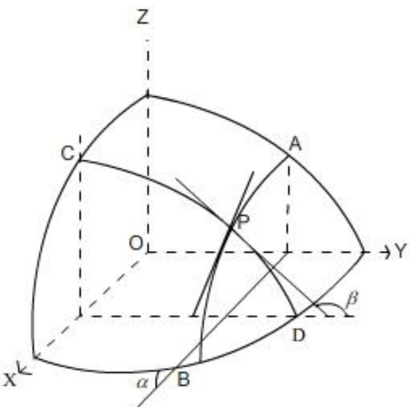
Similarly Partial Derivative of z w.r.t y can be defined as

$\frac{\partial z}{\partial y} = \underset{\Delta y \to 0}{lt} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$, which is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or $z_y(x, y)$ or $f_y(x, y)$ or z_y or f_y

Partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ or $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are known as first-order partial derivatives, while $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ are second order partial derivatives.

Geometrical Interpretation

The function $z = f(x, y)$ represents the equation of a surface in xyz- co-ordinate system.



Let APB the curve, which a plane through any point P on the surface parallel to xz-plane, cuts. As point P moves along this curve APB, its co-ordinates z and x vary while y remains constant. The slope of the tangent line at P along APB represents the rate at which z changes with respect to x. Similarly, the slope of the tangent line at P along CPD represents the rate at which z changes with respect to y .

$\frac{\partial z}{\partial x} = \tan \alpha$ = slope of the curve APB at point P.(α is the angle between tangent drawn to the curve P along APB at and the x- axis) $\frac{\partial z}{\partial y} = \tan \beta$ = slope of the curve CPD at point P. (β is the angle between tangent drawn to the curve at P along CPD and the y - axis)

Note: 1 Similarly higher order partial derivatives can be obtained for several independent variables

Note: 2 $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y\, \partial x}$

Problems

1. Find the first order Partial derivatives $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ for the functions

i. $u = e^x \cos y$ ii. $u = \tan^{-1} \dfrac{y}{x}$

Solution:

i. $u = e^x \cos y$ --- (1)

Differentiate partially with respect to x equation (1) $\dfrac{\partial u}{\partial x} = e^x \cos y$

Differentiate partially with respect to y equation (1) $\dfrac{\partial u}{\partial y} = -e^x \sin y$

ii. $u = \tan^{-1} \dfrac{y}{x}$ --- (1)

Differentiate partially with respect to x equation (1)

$$\frac{\partial u}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{-y}{x^2+y^2}$$

Differentiate partially with respect to y equation (1)

$$\frac{\partial u}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x} = \frac{x}{x^2+y^2}$$

2. If $u = (x-y)^4 + (y-z)^4 + (z-x)^4$, show that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.

Solution:

Given $u = (x-y)^4 + (y-z)^4 + (z-x)^4$ **--- (1)**

Differentiate the equation (1) partially u w.r.t. x and y

$$\frac{\partial u}{\partial x} = 4(x-y)^3 - 4(z-x)^3 \;;\; \frac{\partial u}{\partial y} = -4(x-y)^3 + 4(y-z)^3 \;;\; \frac{\partial u}{\partial z} = -4(y-z)^3 + 4(z-x)^3$$

Now consider,

$$LHS = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$= 4(x-y)^3 - 4(z-x)^3 - 4(x-y)^3 + 4(y-z)^3 - 4(y-z)^3 + 4(z-x)^3$$

$$= 0$$

3. If $u = \sin^{-1}\left(\dfrac{y}{x}\right)$ then, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

Solution:

Given $u = \sin^{-1}\left(\dfrac{y}{x}\right)$ **--- (1)**

Differentiate the equation (1) partially u w.r.t. x, and y **.**

$$\frac{\partial u}{\partial x}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{1}{y}=\frac{y}{\sqrt{y^2-x^2}}\,;\quad \frac{\partial u}{\partial y}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{-x}{y^2}=\frac{-x}{\sqrt{y^2-x^2}}$$

Now consider,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{xy}{\sqrt{y^2-x^2}}-\frac{xy}{\sqrt{y^2-x^2}}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.$$

4. If $z=\log\sqrt{x^2+y^2}$ then, show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=1$.

Solution:

Given $z=\log\sqrt{x^2+y^2}$ **--- (1)**

Differentiate the equation (1) partially z w.r.t. x, and y .

$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{x^2+y^2}}\frac{1}{2\sqrt{x^2+y^2}}2x\,;\quad \frac{\partial z}{\partial y}=\frac{1}{\sqrt{x^2+y^2}}\frac{1}{2\sqrt{x^2+y^2}}2y$$

Now consider,

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{x^2+y^2}{x^2+y^2}$$

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=1.$$

5. Given that $x=r\cos\theta$ and $y=r\sin\theta$ then find r_x, r_y, θ_x and θ_y. $\left(r^2=x^2+y^2\ \&\ \theta=\tan^{-1}\frac{y}{x}\right)$

Solution:

Given $x=r\cos\theta$ and $y=r\sin\theta$ can be written $r=\sqrt{x^2+y^2}$ and $\theta=\tan^{-1}\frac{y}{x}$

($x=r\cos\theta$ and $y=r\sin\theta$ are the parametric equations of $r=\sqrt{x^2+y^2}$ and $\theta=\tan^{-1}\frac{y}{x}$)

$r=\sqrt{x^2+y^2}$ --- (1)

Differentiate (1) partially r w.r.t. x, and y .

$$\frac{\partial r}{\partial x}=\frac{1}{2\sqrt{x^2+y^2}}2x=\frac{r\cos\theta}{r}=\cos\theta\,;$$

$$\frac{\partial r}{\partial y}=\frac{1}{2\sqrt{x^2+y^2}}2y=\frac{r\sin\theta}{r}=\sin\theta\,;$$

$\theta=\tan^{-1}\frac{y}{x}$ --- (2)

Differentiate (2) partially θ w.r.t. x, and y .

$$\frac{\partial \theta}{\partial x}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{-y}{x^2}=\frac{-y}{x^2+y^2}\,;$$

$$\frac{\partial \theta}{\partial y}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x}=\frac{x}{x^2+y^2}\,;$$

6. If $z=f(ax+by)$ then, show that $b\frac{\partial z}{\partial x}-a\frac{\partial z}{\partial y}=0$.

Solution:

Given $z=f(ax+by)$ **--- (1)**

Differentiate the equation (1) partially z w.r.t. x, and y .

$$\frac{\partial z}{\partial x}=f^1(ax+by)a\,;\quad \frac{\partial z}{\partial y}=f^1(ax+by)b$$

Now consider,

$$b\frac{\partial z}{\partial x}-a\frac{\partial z}{\partial y}=abf^1(ax+by)-baf^1(ax+by)$$

$$b\frac{\partial z}{\partial x}-a\frac{\partial z}{\partial y}=0$$

7. If $z=e^{ax+by}f(ax-by)$**, thenprove that** $b\frac{\partial z}{\partial x}+a\frac{\partial z}{\partial y}=2abz$.

Solution:

Given $z=e^{ax+by}f(ax-by)$ ------ (*)

$$\frac{\partial z}{\partial x}=ae^{ax+by}f(ax-by)+ae^{ax+by}f^1(ax-by) \quad \text{---- (1)}$$

$$\frac{\partial z}{\partial y}=be^{ax+by}f(ax-by)-be^{ax+by}f^1(ax-by) \quad \text{---- (2)}$$

Now consider

$$b\frac{\partial z}{\partial x}+a\frac{\partial z}{\partial y}=abe^{ax+by}f(ax-by)+abe^{ax+by}f^1(ax-by)+abe^{ax+by}f(ax-by)-abe^{ax+by}f^1(ax-by)$$

$$b\frac{\partial z}{\partial x}+a\frac{\partial z}{\partial y}=2abe^{ax+by}f(ax-by)$$

Using (*) we get,

$$b\frac{\partial z}{\partial x}+a\frac{\partial z}{\partial y}=2abz.$$

Homogeneous functions and Euler's theorem

A polynomial in x and y, i.e. $f(x,y)=a_0x^n+a_1x^{n-1}y+a_2x^{n-2}y^2+..........+a_ny^n$ is said to be homogeneous if all its terms are of same degree.

In general, A function $f(x,y)$ is said to be homogenous of degree n if it can be expressed as $x^n\varphi\left(\frac{y}{x}\right)$ or $y^n\varphi\left(\frac{x}{y}\right)$ where 'n' can be positive, negative or zero.

Euler's Theorem for homogeneous function.

Statement

If u be a homogeneous function of degree n in x and y then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$

Proof:

Since u is a homogenous function of degree n in x and y

$$u=x^nf\left(\frac{y}{x}\right) \quad \text{………….. (1)}$$

Differentiate partially (1) u with respect to x

$$\frac{\partial u}{\partial x}=nx^{n-1}f\left(\frac{y}{x}\right)+x^nf^1\left(\frac{y}{x}\right)*\left(\frac{-y}{x^2}\right) \quad \text{………….. (2)}$$

Differentiate partially (1) u with respect to y, we get

$$\frac{\partial u}{\partial y}=x^nf^1\left(\frac{y}{x}\right)\frac{1}{x} \quad \text{………….. (3)}$$

Multiply equation (2) by x and (3) by y,we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nx^n f\left(\frac{y}{x}\right)$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

Hence the Euler's theorem proved.

Problems

8. Verify the Euler's theorem for the function $u=\log(x^2+xy+y^2)$

Solution:

Given $u=\log(x^2+xy+y^2)$

u is a not homogeneous function.

So Euler's theorem can't be applied.

9. Verify the Euler's theorem for the function $u=\dfrac{1}{\sqrt{x^2+y^2+z^2}}$**.**

Solution: Given $u=\dfrac{1}{\sqrt{x^2+y^2+z^2}}$ --- (1)

$$u=\frac{1}{x\sqrt{1+\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2}}$$

$$u=x^{-1}\ f\left(\frac{y}{x},\frac{z}{x}\right)$$

u is a homogeneous function in x with degree -1.

By Euler's theorem $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=-u$. --- (2)

Differentiate partially (1) u with respect to x, y and z, we get

$$\frac{\partial u}{\partial x}=-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\ 2x=\frac{-x}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial u}{\partial y}=-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\ 2y=\frac{-y}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial u}{\partial z}=-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\ 2z=\frac{-z}{(x^2+y^2+z^2)^{3/2}}$$

Now consider, $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=\dfrac{-x^2}{(x^2+y^2+z^2)^{3/2}}+\dfrac{-y^2}{(x^2+y^2+z^2)^{3/2}}+\dfrac{-z^2}{(x^2+y^2+z^2)^{3/2}}$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-\frac{(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-\frac{1}{\sqrt{(x^2+y^2+z^2)}}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-u \quad \text{--- (3)}$$

From equations (2) and (3), Euler's theorem verified.

10. **Verify the Euler's theorem for the function** $u = ax^2 + 2hxy + b^2y^2$

Solution:

Given $u = ax^2 + 2hxy + b^2$

$$u = x^2\left(a + 2h\frac{y}{x} + b\frac{y^2}{x^2}\right) \qquad \text{--- (1)}$$

$$u = x^2 \ f\left(\frac{y}{x}\right)$$

u is a homogeneous function in x with degree 2.

By Euler's theorem $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$. --- (2)

Differentiate partially (1) u with respect to x and y, we get

$$\frac{\partial u}{\partial x} = 2ax + 2hy + 0$$

$$\frac{\partial u}{\partial y} = 0 + 2hx + 2by$$

Now consider, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + = 2ax^2 + 2hxy + 2ay^2 + 2hxy$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + = 2(ax^2 + 2hxy + 2ay^2)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u \qquad \text{--- (3)}$$

From equations (2) and (3), Euler's theorem verified.

11. **Show that** $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u$ **, where** $\log u = \frac{x^3 + y^3}{3x + 4y}$ **.**

Solution:

Let $z = \frac{x^3 + y^3}{3x + 4y} = x^2 . \frac{1 + \left(\frac{y}{x}\right)^3}{3 + 4\left(\frac{y}{x}\right)}$ where $z = \log u$ --- (1)

z is a homogenous function of degree 2 in x and y

By Euler's theorem, we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \qquad \text{--- (2)}$$

Differentiate partially (1) w.r.t. x**,** and y **.**

$$\frac{\partial z}{\partial x} = \frac{1}{u}\frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \frac{1}{u}\frac{\partial u}{\partial y}$$

Substituting these in equation (2), we get

$$x\frac{1}{u}\frac{\partial u}{\partial x} + y\frac{1}{u}\frac{\partial u}{\partial y} = 2\ln u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u \text{ .}$$

12. If $u = e^{x^2y^2/(x+y)}$ then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3u\log u$.

Solution:

Given $u = e^{x^2y^2(x+y)}$

Apply ln on both sides

$$\ln u = \ln e^{x^2y^2/(x+y)}$$

$$\ln u = \frac{x^2y^2}{(x+y)}$$

Let $z = \dfrac{x^2y^2}{(x+y)} = \dfrac{x^4\left(\dfrac{y^2}{x^2}\right)}{x\left(1+\dfrac{y}{x}\right)} = x^3\dfrac{\left(\dfrac{y^2}{x^2}\right)}{\left(1+\dfrac{y}{x}\right)}$, where $z = \log u$ --- (1)

z is a homogenous function in x with degree 3.

By Euler's theorem, we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z \quad \text{-- (2)}$$

Differentiate partially (1) w.r.t. x, and y.

$$\frac{\partial z}{\partial x} = \frac{1}{u}\frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \frac{1}{u}\frac{\partial u}{\partial y}$$

Substituting these in equation (2), we get

$$x\frac{1}{u}\frac{\partial u}{\partial x} + y\frac{1}{u}\frac{\partial u}{\partial y} = 3\ln u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u\log u.$$

13. If $u = \tan^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{4}\sin 2u$.

Solution:

Given $u = \tan^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$

$$\tan u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

Let $z = \dfrac{x+y}{\sqrt{x}+\sqrt{y}} = \dfrac{x\left(1+\dfrac{y}{x}\right)}{\sqrt{x}\left(1+\dfrac{\sqrt{y}}{\sqrt{x}}\right)} = \sqrt{x}\dfrac{\left(1+\dfrac{y}{x}\right)}{\left(1+\sqrt{\dfrac{y}{x}}\right)}$, where $z = \tan u$ --- (1)

z is a homogenous function in x with degree 1/2.

By Euler's theorem, we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}z \quad \text{--- (2)}$$

Differentiate partially (1) w.r.t. x, and y.

$$\frac{\partial z}{\partial x} = \sec^2 u\frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \sec^2 u\frac{\partial u}{\partial y}$$

Substituting these in equation (2), we get

$$x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=\frac{1}{2}\tan u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\frac{\sin u}{\cos u}\frac{1}{\sec^2 u}.$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\frac{2}{2}\sin u\cos u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{4}\sin 2u.$$

Extension of Euler's Theorem

Statement

If $z=f(x,y)$ is a homogeneous function of x, y of degree 'n', then $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z$

Proof:

By Euler's Theorem, we have

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz \qquad \text{-- (1)}$$

Differentiate equation (1) partially with respect to x

$$\frac{\partial z}{\partial x}+x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y}=n\frac{\partial z}{\partial x} \qquad \text{--- (2)}$$

Differentiate equation (1) partially with respect to y

$$\frac{\partial z}{\partial y}+y\frac{\partial^2 z}{\partial y^2}+x\frac{\partial^2 z}{\partial y\partial x}=n\frac{\partial z}{\partial y} \qquad \text{--- (3)}$$

Multiply equation 2 by x and 3 by y, we get

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z.$$

Problems

14. If $u=\tan^{-1}\sqrt{x^4+y^4}$ then prove that

i. $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$ **and**

ii. $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\sin 4u-\sin 2u$

Solution:

Given $u=\tan^{-1}\sqrt{x^4+y^4}$

$\tan u=\sqrt{x^4+y^4}$

Let $z=\sqrt{x^4+y^4}=\sqrt{x^4\left(1+\left(\frac{y}{x}\right)^4\right)}=x^2\sqrt{\left(1+\left(\frac{y}{x}\right)^4\right)}$, where $z=\tan u$ --- (1)

z is a homogenous function in x with degree 2.

By Euler's theorem, we get

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z \qquad \text{--- (2)}$$

Differentiate partially (1) w.r.t. x, and y.

$$\frac{\partial z}{\partial x}=\sec^2 u\frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y}=\sec^2 u\frac{\partial u}{\partial y}$$

Substituting these in equation (2), we get

$$x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=2\tan u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\frac{\sin u}{\cos u}\frac{1}{\sec^2 u}.$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\sin u\cos u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u. \qquad \text{--- (3)}$$

Differentiate partially equation (3) w.r.t x and y again, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u\frac{\partial u}{\partial x}$$

Multiply x on both sides in above equation, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u\ x\frac{\partial u}{\partial x} \qquad \text{--- (4)}$$

Similarly,

$$y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y\partial x}=2\cos 2u\frac{\partial u}{\partial y}$$

Multiply y on both sides in above equation, we get

$$y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial x}+yx\frac{\partial^2 u}{\partial y\partial x}=2\cos 2u\ y\frac{\partial u}{\partial y} \qquad \text{--- (5)}$$

Adding equations (4) and (5), we get

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial x\partial y}\ y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial x}+yx\frac{\partial^2 u}{\partial y\partial x}=2\cos 2u\ y\frac{\partial u}{\partial y}+2\cos 2u\ x\frac{\partial u}{\partial x}$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}\ y^2\frac{\partial^2 u}{\partial y^2}+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial x}\right)=2\cos 2u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial x}\right) \qquad \text{---- (6)}$$

Using equation (3) in (6)

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}\ y^2\frac{\partial^2 u}{\partial y^2}+\sin 2u=2\cos 2u\sin 2u$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}\ y^2\frac{\partial^2 u}{\partial y^2}=2\ \cos 2u\sin 2u-\sin 2u$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}\ y^2\frac{\partial^2 u}{\partial y^2}=\sin 4u-\sin 2u$$

15. **If** $u=\tan^{-1}\left(\frac{y}{x}\right)+y\sin^{-1}\left(\frac{x}{y}\right)$ **then prove that** $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}y^2\frac{\partial^2 u}{\partial y^2}=0$

Solution:

Given $u=\tan^{-1}\left(\frac{y}{x}\right)+y\sin^{-1}\left(\frac{x}{y}\right)$ --- (1)

Let $u=v+w$, where $v=\tan^{-1}\left(\frac{y}{x}\right)$, $w=y\sin^{-1}\left(\frac{x}{y}\right)$ --- (*)

Consider $v=\tan^{-1}\left(\frac{y}{x}\right)$

$$v=x^0\tan^{-1}\left(\frac{y}{x}\right)$$

Implies that v is homogeneous function in x with degree 0, then by Euler's extension theorem we can write $x^2\frac{\partial^2 v}{\partial x^2}+2xy\frac{\partial^2 v}{\partial x\partial y}y^2\frac{\partial^2 v}{\partial y^2}=0$ --- (2)

Again consider $w=y\sin^{-1}\left(\frac{x}{y}\right)$

$$w=x^1\ \frac{y}{x}\sin^{-1}\left(\frac{1}{\frac{y}{x}}\right)$$

Implies that w is homogeneous function in x with degree 1, then by Euler's extension theorem we can write $x^2\frac{\partial^2 w}{\partial x^2}+2xy\frac{\partial^2 w}{\partial x\partial y}y^2\frac{\partial^2 w}{\partial y^2}=1(1-1)w$

i.e. $x^2\frac{\partial^2 w}{\partial x^2}+2xy\frac{\partial^2 w}{\partial x\partial y}y^2\frac{\partial^2 w}{\partial y^2}=0$ --- (3)

Adding (2) and (3) we get

$$x^2\frac{\partial^2 v}{\partial x^2}+2xy\frac{\partial^2 v}{\partial x\partial y}y^2\frac{\partial^2 v}{\partial y^2}+x^2\frac{\partial^2 w}{\partial x^2}+2xy\frac{\partial^2 w}{\partial x\partial y}y^2\frac{\partial^2 w}{\partial y^2}=0$$

$$x^2\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}\right)+2xy\left(\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x\partial y}\right)+y^2\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\right)=0$$

$$x^2\frac{\partial^2}{\partial x^2}(v+w)+2xy\frac{\partial^2}{\partial x\partial y}(v+w)+y^2\frac{\partial^2}{\partial y^2}(v+w)=0$$

Using equation * we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}y^2\frac{\partial^2 u}{\partial y^2}=0$$

Total differentiation, differentiation of composite and implicit functions

Total Differentials

Consider a function $f=f(x,y)$ of two independent variable x and y then the total differential is defined as $df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$. Similarly, if a function $f=f(x,y,z)$ of three independent variable x, y and z then the total differential is defined as $df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz$.

Total Derivatives

Consider a function $f = f(x(t), y(t))$ where x and y are functions of t, then total derivative of f with respect to t is defined as $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

Similarly,if a function $f = f(x(t)y(t)z(t))$ where x, y and z are functions of t then total derivative of f with respect to t is defined as $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$

Problems

16. If $u = x^2 + y^2 + z^2$ **and** $x = e^{2t}, y = e^{2t}\cos 3t$ **,** $z = e^{2t}\sin 3t$ **then, Find** $\frac{du}{dt}$ **as a total derivative and verify the result by direct substitution**

Solution:

Given $u = x^2 + y^2 + z^2$, $x = e^{2t}, y = e^{2t}\cos 3t$ and $z = e^{2t}\sin 3t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= 4x\, e^{2t} + 2y(-e^{2t}3\sin 3t + 2\cos 3t\, e^{2t}) + 2z(e^{2t}\cos 3t + \sin 3t\, 2e^{2t})$$

$$= 4x\, e^{2t} - 6y\, e^{2t}\sin 3t + 4y\cos 3t\, e^{2t}) + 6z\, e^{2t}\cos 3t + 4z\, e^{2t}\sin 3t)$$

$$= 4xx - 6yz + 4yy + 6zy + 4zz$$

$$= 4x^2 - 6yz + 4y^2 + 6zy + 4z^2$$

$$= 4(x^2 + y^2 + z^2)$$

$$= 4u$$

$$\frac{du}{dt} = 4(e^{4t} + e^{4t}(\cos^2 t + \sin^2 t))$$

$$= 4(e^{4t} + e^{4t})$$

$$= 8e^{4t}$$

17. If $u = \sin^{-1}(x - y)$**,** $x = 3t$ **and** $y = 4t^3$ **then show that** $\frac{\mathrm{du}}{\mathrm{dt}} = 3(1 - \mathrm{t}^2)^{-1/2}$.

Solution:

Given $u = \sin^{-1}(x - y)$, $x = 3t$ and $y = 4t^3$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$\frac{du}{dt} = \frac{1}{\sqrt{1-(x-y)^2}}3 + \frac{-1}{\sqrt{1-(x-y)^2}}12t^2$$

$$\frac{du}{dt} = \frac{3-12t^2}{\sqrt{1-(x-y)^2}}$$

$$x - y = 3t - 4t^3$$

$$x - y = t(3 - 4t^2)$$

$$(x-y)^2 = \left(t(3-4t^2)\right)^2$$

$$(x-y)^2 = t^2(9 + 16t^4 - 24t^2)$$

$$\frac{du}{dt}=\frac{3(1-4t^2)}{\sqrt{1-9t^2-16t^6+24t^4}}$$

$$\frac{du}{dt}=\frac{3(1-4t^2)}{\sqrt{1-8t^2-t^2-16t^6+16t^4+8t^4}}$$

$$\frac{du}{dt}=\frac{3(1-4t^2)}{\sqrt{1-8t^2+16t^4-t^2-16t^6+8t^4}}$$

$$\frac{du}{dt}=\frac{3(1-4t^2)}{\sqrt{(1-8t^2+16t^4)-t^2(1-8t^2+16t^4)}}$$

$$\frac{du}{dt}=\frac{3(1-4t^2)}{\sqrt{(1-8t^2+16t^4)(1-t^2)}}$$

$$\frac{du}{dt}=\frac{3(1-4t^2)}{\sqrt{(1-4t^2)^2(1-t^2)}}$$

$$\frac{du}{dt}=\frac{3}{\sqrt{(1-t^2)}}$$

$$\frac{du}{dt}=3(1-t^2)^{-1/2}$$

Alternatively

Given $u=\sin^{-1}(x-y)$, $x=3t$ and $y=4t^3$

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$\frac{du}{dt}=\frac{1}{\sqrt{1-(x-y)^2}}3+\frac{-1}{\sqrt{1-(x-y)^2}}12t^2$$

$$\frac{du}{dt}=\frac{3-12t^2}{\sqrt{1-(x-y)^2}}$$

$$x-y=3t-4t^3$$

$$x-y=t(3-4t^2)$$

$$\left(x-y\right)^2=\left(t(3-4t^2)\right)^2$$

$$\left(x-y\right)^2=t^2(9+16t^4-24t^2)$$

$$\frac{du}{dt}=\frac{3}{\sqrt{\frac{1-9t^2-16t^6+24t^4}{(1-4t^2)^2}}}$$

$$\frac{du}{dt}=\frac{3}{\sqrt{\frac{1-9t^2-16t^6+24t^4}{1+16t^4-8t^2}}}$$

$1+16t^4-8t^2)-16t^6+24t^4-9t^2+1(-t^2+1$

$-16t^6+8t^4-t^2$

$+\quad -\quad +$

Sub $16t^4-8t^2+1$

$16t^4-8t^2+1$

$-\quad +\quad -$

Sub 0

$$\frac{-16t^6+24t^4-9t^2+1}{1+16t^4-8t^2}=-t^2+1$$

$$\frac{du}{dt}=\frac{3}{\sqrt{(1-t^2)}}$$

$$\frac{du}{dt}=3(1-t^2)^{-1/2}$$

18. If $u=\tan^{-1}(y/x)$ where $x=e^t-e^{-t}$ and $y=e^t+e^{-t}$ then find $\frac{du}{dt}$

Solution:

Given $u=\tan^{-1}(y/x)$ where $x=e^t-e^{-t}$ and $y=e^t+e^{-t}$

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$\frac{du}{dt}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{-y}{x^2}(e^t+e^{-t})+\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x}(e^t-e^{-t})$$

$$\frac{du}{dt}=\frac{-y}{x^2+y^2}y+\frac{x}{x^2+y^2}x$$

$$\frac{du}{dt}=\frac{-y^2}{x^2+y^2}+\frac{x^2}{x^2+y^2}$$

$$\frac{du}{dt}=\frac{x^2-y^2}{x^2+y^2}$$

$$x^2=e^{2t}+e^{-2t}-2$$

$$y^2=e^{2t}+e^{-2t}+2$$

$$x^2+y^2=2(e^{2t}+e^{-2t})$$

$$x^2-y^2=-4$$

$$\frac{du}{dt}=\frac{-4}{2(e^{2t}+e^{-2t})}$$

$$\frac{du}{dt}=\frac{-2}{(e^{2t}+e^{-2t})}$$

Differentiation of implicit functions

If $f(x,y)=c$ be an implicit function of x & y then derivative of the function f w r t x is given by $\frac{df}{dx}=0$ ... (1)

From the definition of total derivatives we have $\frac{df}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}$ ... (2)

From (1) and (2) we get $\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}$.

Problems

19. If $u=x\log xy$ where $x^3+y^3+3xy=1$ then, find $\frac{du}{dx}$.

Solution:

Let $f(x,y)=x^3+y^3+3xy-1$

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}=-\frac{(3x^2+3y)}{(3y^2+3x)}=-\frac{x^2+y}{y^2+x}$$

$$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$\frac{du}{dx}=1+\log xy+\frac{\partial u}{\partial y}\left(-\frac{x^2+y}{y^2+x}\right)$$

$$\frac{du}{dx}=1+\log xy-\frac{x}{y}\left(\frac{x^2+y}{y^2+x}\right)$$

20. If $u=e^{x^2+y^2}$ where $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then, find $\frac{du}{dx}$.

Solution:

Let $f(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}=-\frac{2x/a^2}{2y/b^2}=-\frac{b^2}{a^2}\frac{x}{y}$$

$$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$\frac{du}{dx}=2x\ e^{x^2+y^2}+2y\ e^{x^2+y^2}\left(-\frac{b^2}{a^2}\frac{x}{y}\right)$$

$$\frac{du}{dx}=\frac{2x}{a^2}(a^2-b^2)e^{x^2+y^2}.$$

21. If $u=\cos(x^2-y^2)$ where $a^2x^2+b^2y^2=c^2$ then, find $\frac{du}{dx}$.

Solution:

Let $f(x,y)=a^2x^2+b^2y^2-c^2$

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}=-\frac{2xa^2}{2yb^2}=-\frac{a^2}{b^2}\frac{x}{y}$$

$$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$\frac{du}{dx}=-2x\sin(x^2-y^2)-2y\ \sin(x^2-y^2)\left(-\frac{a^2}{b^2}\frac{x}{y}\right)$$

$$\frac{du}{dx}=-2x\left(1+\frac{a^2}{b^2}\right)\sin(x^2-y^2).$$

22. If $x^m y^n=(x+y)^{m+n}$ prove that $\frac{dy}{dx}=\frac{y}{x}$.

Solution:

Let $f(x,y)=x^m y^n-(x+y)^{m+n}$

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}=-\frac{m\,x^{m-1}y^n-(m+n)(x+y)^{m+n-1}}{n\,x^m y^{n-1}-(m+n)(x+y)^{m+n-1}}$$

$$\frac{dy}{dx}=-\frac{m\,x^{m-1}y^{n}-\dfrac{(m+n)(x+y)^{m+n}}{(x+y)}}{n\,x^{m}y^{n-1}-\dfrac{(m+n)(x+y)^{m+n}}{(x+y)}}$$

$$\frac{dy}{dx}=-\frac{\dfrac{m\,x^{m}y^{n}+mx^{m-1}y^{n+1}-(m+n)(x+y)^{m+n}}{(x+y)}}{\dfrac{n\,x^{m+1}y^{n-1}+nx^{m}y^{n}-(m+n)(x+y)^{m+n}}{(x+y)}}$$

$$\frac{dy}{dx}=-\frac{m\,x^{m}y^{n}+mx^{m-1}y^{n+1}-(m+n)x^{m}y^{n}}{n\,x^{m+1}y^{n-1}+nx^{m}y^{n}-(m+n)x^{m}y^{n}}$$

$$\frac{dy}{dx}=-\frac{x^{m}y^{n}}{x^{m}y^{n}}\frac{m+m\dfrac{y}{x}-m-n}{n\dfrac{x}{y}+n-m-n}$$

$$\frac{dy}{dx}=-\frac{m\dfrac{y}{x}-n}{n\dfrac{x}{y}-m}$$

$$\frac{dy}{dx}=-\frac{\dfrac{my-nx}{x}}{\dfrac{-(my-nx)}{y}}$$

$$\frac{dy}{dx}=\frac{y}{x}.$$

Differentiation of Composite functions

Consider a function $f=f\big(x(u,v),y(u,v)\big)$ where f is a function of x and y further x and y is a function u and v then the differentiation of composite function f is defined as $\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$ or $f_u=f_x x_u+f_y y_u$ and $\frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$ or $f_v=f_x x_v+f_y y_v$

Similarly, Consider a function $f=f\big(u(x,y),v(x,y)\big)$ where f is a function of u and v further u and v is a function of x and y then the differentiation of composite function f is defined as $\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$ or $f_x=f_u u_x+f_v v_x$ and $\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$ or $f_y=f_u u_y+f_v v_y$.

In the same manner composite function can be defined to the desired functions, here some following examples can go through

If $f=f\big(x(u,v),y(u,v),z(u,v)\big)$ **then** $f_u=f_x x_u+f_y y_u+f_z z_u$, $f_v=f_x x_v+f_y y_v+f_z z_v$.

If $f=f\big(x(u,v,w),y(u,v,w),z(u,v,w)\big)$ **then** $f_u=f_x x_u+f_y y_u+f_z z_u$, $f_v=f_x x_v+f_y y_v+f_z z_v$

$$f_w=f_x x_w+f_y y_w+f_z z_w.$$

If $f=f\big(x(u,v,w),y(u,v,w)\big)$ **then** $f_u=f_x x_u+f_y y_u$, $f_v=f_x x_v+f_y y_v$, $f_w=f_x x_w+f_y y_w$.

Problems

23. **If** $z=\dfrac{\cos y}{x}$ **and** $x=u^2-v$ **,** $y=e^v$**. Prove that** $\dfrac{\partial z}{\partial v}=\dfrac{\cos y-xy\sin y}{x^2}$**.**

Solution:

Given $z=\dfrac{\cos y}{x}$ where $x=u^2-v$, $y=e^v$

By the definition of composite function, we have

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial v}=-\frac{1}{x^2}(-\cos y)+\left(-\frac{\sin y}{x}\right)e^v$$

$$\frac{\partial z}{\partial v}=\frac{\cos y-xy\sin y}{x^2}.$$

24. **If** $u=x^2-y^2$ **and** $x=2r-3s+4$**,** $y=-r+8s-5$**. Prove that** $\dfrac{\partial u}{\partial r}=4x+2y$**.**

Solution:

Given $z=\dfrac{\cos y}{x}$ where $x=u^2-v$, $y=e^v$

By the definition of composite function, we have

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial r}=2x.2+(-2y)(-1)$$

$$\frac{\partial u}{\partial r}=4x+2y$$

25. **If** $z=f(u,v)$ **where,** $u=e^x\cos y$ **and,** $v=e^x\sin y$**, then show that** $\dfrac{\partial z}{\partial x}=u\dfrac{\partial z}{\partial u}+v\dfrac{\partial z}{\partial v}$**.**

Solution:

Given $z=f(u,v)$ where, $u=e^x\cos y$ and, $v=e^x\sin y$,

By the definition of composite function, we have

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}e^x\cos y+\frac{\partial z}{\partial v}e^x\sin y$$

$$\frac{\partial z}{\partial x}=u\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}v.$$

26. **If** $u=\frac{x}{y}$ **,** $v=\frac{y}{z}$ **,** $w=z$ **and** $f=f(u,v,w)$ **then show that** $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=w\frac{\partial f}{\partial w}$ **.**

Solution:

Given $f=f(u,v,w)$ where, $u=\frac{x}{y}$, $v=\frac{y}{z}$, $w=z$,

By the definition of composite function, we have

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$$

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\frac{1}{z}+\frac{\partial f}{\partial v}0+\frac{\partial f}{\partial w}0 \qquad \text{--- (1)}$$

$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial f}{\partial w}\frac{\partial w}{\partial y}$$

$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}0+\frac{\partial f}{\partial v}\frac{1}{z}+\frac{\partial f}{\partial w}0 \qquad \text{--- (2)}$$

$$\frac{\partial f}{\partial z}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial z}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial z}+\frac{\partial f}{\partial w}\frac{\partial w}{\partial z}$$

$$\frac{\partial f}{\partial z}=\frac{\partial f}{\partial u}\left(-\frac{x}{z^2}\right)+\frac{\partial f}{\partial v}\left(-\frac{y}{z^2}\right)+\frac{\partial f}{\partial w}1 \qquad \text{--- (3)}$$

From (1), (2) and (3), we get

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=x\left(\frac{\partial f}{\partial u}\frac{1}{z}+\frac{\partial f}{\partial v}0+\frac{\partial f}{\partial w}0\right)+y\left(\frac{\partial f}{\partial u}0+\frac{\partial f}{\partial v}\frac{1}{z}+\frac{\partial f}{\partial w}0\right)+z\left(\frac{\partial f}{\partial u}\left(-\frac{x}{z^2}\right)+\frac{\partial f}{\partial v}\left(-\frac{y}{z^2}\right)+\frac{\partial f}{\partial w}1\right)$$

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=\frac{x}{z}\frac{\partial f}{\partial u}+\frac{y}{z}\frac{\partial f}{\partial v}-\frac{x}{z}\frac{\partial f}{\partial u}-\frac{y}{z}\frac{\partial f}{\partial v}z\frac{\partial f}{\partial w}$$

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=w\frac{\partial f}{\partial w}$$ (because $z=w$)

27. **If** $x=u+v+w$ **,** $y=uv+vw+wu$ **,** $z=uvw$ **and** $f=f(x,y,z)$ **then show that** $u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}+w\frac{\partial f}{\partial w}=x\frac{\partial f}{\partial x}+2y\frac{\partial f}{\partial y}+3z\frac{\partial f}{\partial z}$ **.**

Solution:

Given $f=f(x,y,z)$ where, $x=u+v+w$, $y=uv+vw+wu$, $z=uvw$,

By the definition of composite function, we have

$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(v+w)+\frac{\partial f}{\partial z}vw \qquad \text{--- (1)}$$

$$\frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$$

$$\frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(u+w)+\frac{\partial f}{\partial z}uw \qquad \text{--- (2)}$$

$$\frac{\partial f}{\partial w}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial w}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial w}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial w}$$

$$\frac{\partial f}{\partial w}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(u+v)+\frac{\partial f}{\partial z}uv \qquad \text{--- (3)}$$

From equations (1),(2) and (3), we get

$$u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}+w\frac{\partial f}{\partial w}=u\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(v+w)+\frac{\partial f}{\partial z}vw\right)+v\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(u+w)+\frac{\partial f}{\partial z}uw\right)+w\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(u+v)+\frac{\partial f}{\partial z}uv\right)$$

$$u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}+w\frac{\partial f}{\partial w}=u\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(uv+uw)+\frac{\partial f}{\partial z}uvw+v\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(uv+vw)+\frac{\partial f}{\partial z}uvw+w\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(uw+vw)+\frac{\partial f}{\partial z}uvw$$

$$u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}+w\frac{\partial f}{\partial w}=(u+v+w)\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(uv+uw+uv+vw+uw+vw)+\frac{\partial f}{\partial z}(uvw+uvw+uvw)$$

$$u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}+w\frac{\partial f}{\partial w}=(u+v+w)\frac{\partial f}{\partial x}+2(uv+vw+wu)\frac{\partial f}{\partial y}+3uvw\frac{\partial f}{\partial z}$$

$$u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}+w\frac{\partial f}{\partial w}=x\frac{\partial f}{\partial x}+2y\frac{\partial f}{\partial y}+3z\frac{\partial f}{\partial z}.$$

28. If $u=x+y$, $v=xy$ **and,** $f=f(u,v)$ **, then show that** $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=u\frac{\partial f}{\partial u}+2v\frac{\partial f}{\partial v}$**.**

Solution:

Given $f=f(u,v)$ where $u=x+y$, $v=xy$

By the definition of composite function, we have

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \qquad \frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}=\left(\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}y\right)x+\left(\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}x\right)y$$

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}=x\frac{\partial f}{\partial u}+xy\frac{\partial f}{\partial v}y+y\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}xy$$

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}=(x+y)\frac{\partial f}{\partial u}+2xy\frac{\partial f}{\partial v}.$$

29. If $z=f(x,y)$ **where,** $x=e^{u}-e^{-v}$ **and** $y=-e^{u}+e^{-v}$**, show that** $\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$

Solution:

$$x=e^{u}-e^{-v} \qquad y=-e^{u}+e^{-v}$$

$$\frac{\partial x}{\partial u} = e^{u} \qquad \frac{\partial y}{\partial u} = -e^{u}$$

$$\frac{\partial x}{\partial v} = e^{-v} \qquad \frac{\partial y}{\partial v} = -e^{-v}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = e^{u}\frac{\partial z}{\partial x} - e^{u}\frac{\partial z}{\partial y} \qquad \text{--- (1)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = e^{-v}\frac{\partial z}{\partial x} - e^{-v}\frac{\partial z}{\partial y} \qquad \text{--- (2)}$$

Equation (1)-(2) gives

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^{u} - e^{-v})\frac{\partial z}{\partial x} + (-e^{u} + e^{-v})\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

30. **If** $u = F(x-y, y-z, z-x)$**, then prove that** $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ **.**

Solution:

Given $u = F(x-y, y-z, z-x)$

Let $p = x-y$, $q = y-z$, $r = z-x$.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}(1) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}(-1) \qquad \text{----- (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}(-1) + \frac{\partial u}{\partial q}(1) + \frac{\partial u}{\partial r}(0) \qquad \text{----- (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}(-1) + \frac{\partial u}{\partial r}(1) \qquad \text{----- (3)}$$

Now consider,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}(-1) + \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial p}(-1) + \frac{\partial u}{\partial q}(1) + \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial p}(1) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}(-1)$$

Using (1) (2) and (3)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

31. If $u = F(xz, y/z)$ then show that $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} - z\frac{\partial u}{\partial z} = 0$.

Solution:

Given $u = F(xz, y/z)$

Let $v = xz$, $w = y/z)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}(z) + \frac{\partial u}{\partial w}(0) \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}(0) + \frac{\partial u}{\partial w}\left(\frac{1}{z}\right) \quad \text{--- (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v}(x) + \frac{\partial u}{\partial w}\left(\frac{-y}{z^2}\right) \quad \text{--- (3)}$$

Now consider,

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} - z\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v}(xz) + \frac{\partial u}{\partial w}(0) - \frac{\partial u}{\partial v}(0) - \frac{\partial u}{\partial w}\left(\frac{y}{z}\right) - \frac{\partial u}{\partial v}(xz) + \frac{\partial u}{\partial w}\left(\frac{-y}{z}\right)$$

Using (1) (2) and (3)

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} - z\frac{\partial u}{\partial z} = 0.$$

32. **If** $f = f(x, y)$ and $x = u^2 + v^2$, $y = 2uv$, Show that $u\frac{\partial f}{\partial u} - v\frac{\partial f}{\partial v} = 2(x^2 - y^2)^{1/2}\frac{\partial f}{\partial x}$.

Solution:

Given $f = f(x, y)$ where $x = u^2 + v^2$, $y = 2uv$

Consider $x^2 - y^2 = \left(u^2 + v^2\right)^2 - (2uv)^2$

$$\sqrt{x^2 - y^2} = \sqrt{\left(u^2 + v^2\right)^2 - (2uv)^2}$$

$$\sqrt{x^2 - y^2} = \sqrt{u^4 + v^4 + 2u^2v^2 - 4u^2v^2}$$

$$\sqrt{x^2 - y^2} = \sqrt{u^4 + v^4 - 2u^2v^2}$$

$$\sqrt{x^2 - y^2} = \sqrt{\left(u^2 - v^2\right)^2}$$

$$\sqrt{x^2-y^2}=u^2-v^2 \quad \text{--- (1)}$$

By the definition of composite functions, we have

$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \qquad \frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}2u+\frac{\partial f}{\partial y}2v \qquad \frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}2v+\frac{\partial f}{\partial y}2u$$

$$u\frac{\partial f}{\partial u}-v\frac{\partial f}{\partial v}=u\left(\frac{\partial f}{\partial x}2u+\frac{\partial f}{\partial y}2v\right)-\left(v\frac{\partial f}{\partial x}2v+\frac{\partial f}{\partial y}2u\right)$$

$$u\frac{\partial f}{\partial u}-v\frac{\partial f}{\partial v}=2u^2\frac{\partial f}{\partial x}+2uv\frac{\partial f}{\partial y}-2v^2\frac{\partial f}{\partial x}-\frac{\partial f}{\partial y}2uv$$

$$u\frac{\partial f}{\partial u}-v\frac{\partial f}{\partial v}=2\left(u^2-v^2\right)\frac{\partial f}{\partial x}. \quad \text{--- (2)}$$

From (1) and (2), we get

$$u\frac{\partial f}{\partial u}-v\frac{\partial f}{\partial v}=2(x^2-y^2)^{1/2}\ \frac{\partial f}{\partial x}.$$

33. **If** $v=x+ct$**,** $w=x-ct$ **, and** $u=f(v,w)$ **, then prove that** $\frac{\partial^2 u}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=4\frac{\partial^2 u}{\partial v\partial w}$**.**

Solution:

Given $u=f(v,w)$ where $v=x+ct$, $w=x-ct$.

By the definition of composite function, we have

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial x} \qquad \frac{\partial u}{\partial t}=\frac{\partial u}{\partial v}\frac{\partial v}{\partial t}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial t}$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial v}+\frac{\partial u}{\partial w} \ \text{-------}\mathbf{(1)} \qquad \frac{\partial u}{\partial t}=\frac{\partial u}{\partial v}c+\frac{\partial u}{\partial w}(-c)\ \text{------}\mathbf{(2)}$$

Now differentiate equation (1) and (2) partially again w.r.t x and t respectively, we get

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial v}+\frac{\partial u}{\partial w}\right)\frac{\partial v}{\partial x}+\frac{\partial}{\partial w}\left(\frac{\partial u}{\partial v}+\frac{\partial u}{\partial w}\right)\frac{\partial w}{\partial x} \qquad \frac{\partial^2 u}{\partial t^2}=c\left(\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial v}-\frac{\partial u}{\partial w}\right)\frac{\partial v}{\partial t}+\frac{\partial}{\partial w}\left(\frac{\partial u}{\partial v}+-\frac{\partial u}{\partial w}\right)\frac{\partial w}{\partial t}\right)$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial v^2}+\frac{\partial^2 u}{\partial v\partial w}+\frac{\partial^2 u}{\partial v\partial w}+\frac{\partial^2 u}{\partial w^2} \quad \text{----}\ \mathbf{(3)} \qquad \frac{\partial^2 u}{\partial t^2}=c\left(\left(\frac{\partial^2 u}{\partial v^2}-\frac{\partial^2 u}{\partial v\partial w}\right)c+\left(\frac{\partial^2 u}{\partial v\partial w}-\frac{\partial^2 u}{\partial w^2}\right)(-c)\right)$$

$$\frac{\partial^2 u}{\partial t^2}=c^2\left(\frac{\partial^2 u}{\partial v^2}-\frac{\partial^2 u}{\partial v\partial w}-\frac{\partial^2 u}{\partial v\partial w}+\frac{\partial^2 u}{\partial w^2}\right)$$

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial v^2}-\frac{\partial^2 u}{\partial v\partial w}-\frac{\partial^2 u}{\partial v\partial w}+\frac{\partial^2 u}{\partial w^2} \quad \text{---}\ \mathbf{(4)}$$

From (3) and (4), we have

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial v^2}+\frac{\partial^2 u}{\partial v\partial w}+\frac{\partial^2 u}{\partial v\partial w}+\frac{\partial^2 u}{\partial w^2}-\left(\frac{\partial^2 u}{\partial v^2}-\frac{\partial^2 u}{\partial v\partial w}-\frac{\partial^2 u}{\partial v\partial w}+\frac{\partial^2 u}{\partial w^2}\right)$$

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial v^2}+\frac{\partial^2 u}{\partial v\partial w}+\frac{\partial^2 u}{\partial v\partial w}+\frac{\partial^2 u}{\partial w^2}-\frac{\partial^2 u}{\partial v^2}+\frac{\partial^2 u}{\partial v\partial w}+\frac{\partial^2 u}{\partial v\partial w}-\frac{\partial^2 u}{\partial w^2}$$

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=4\frac{\partial^2 u}{\partial v\partial w}.$$

34. **If** $z=f(u,v)$ **and** $u=ax+by$, $v=ay-bx$ **,then show that** $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=(a^2+b^2)\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}\right)$

Solution:

Given $z=f(u,v)$ where $u=ax+by$, $v=ay-bx$

By the definition of composite function, we have

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}a+\frac{\partial z}{\partial v}(-b) \text{ ------ } \mathbf{(1)} \qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}b+\frac{\partial z}{\partial v}a \text{ ------ } \mathbf{(2)}$$

Now differentiate equation (1) and (2) partially again w.r.t x and y respectively, we get

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(a\frac{\partial z}{\partial u}-b\frac{\partial z}{\partial v}\right) \qquad \frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(b\frac{\partial z}{\partial u}+a\frac{\partial z}{\partial v}\right)$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial u}\left(a\frac{\partial z}{\partial u}-b\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(a\frac{\partial z}{\partial u}-b\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x} \qquad \frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial u}\left(b\frac{\partial z}{\partial u}+a\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(b\frac{\partial z}{\partial u}+a\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial u}\left(a\frac{\partial z}{\partial u}-b\frac{\partial z}{\partial v}\right)a+\frac{\partial}{\partial v}\left(a\frac{\partial z}{\partial u}-b\frac{\partial z}{\partial v}\right)(-b) \qquad \frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial u}\left(b\frac{\partial z}{\partial u}+a\frac{\partial z}{\partial v}\right)b+\frac{\partial}{\partial v}\left(b\frac{\partial z}{\partial u}+a\frac{\partial z}{\partial v}\right)a$$

$$\frac{\partial^2 z}{\partial x^2}=a^2\frac{\partial^2 z}{\partial u^2}-ab\frac{\partial^2 z}{\partial u\partial v}-ab\frac{\partial^2 z}{\partial v\partial u}+b^2\frac{\partial^2 z}{\partial v^2} \text{ ----- } \mathbf{(3)} \qquad \frac{\partial^2 z}{\partial y^2}=b^2\frac{\partial^2 z}{\partial u^2}+ab\frac{\partial^2 z}{\partial u\partial v}+ab\frac{\partial^2 z}{\partial v\partial u}+a^2\frac{\partial^2 z}{\partial v^2} \text{ --- } \mathbf{(4)}$$

From (3) and (4), we have

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=a^2\frac{\partial^2 z}{\partial u^2}-ab\frac{\partial^2 z}{\partial u\partial v}-ab\frac{\partial^2 z}{\partial v\partial u}+b^2\frac{\partial^2 z}{\partial v^2}+b^2\frac{\partial^2 z}{\partial u^2}+ab\frac{\partial^2 z}{\partial u\partial v}+ab\frac{\partial^2 z}{\partial v\partial u}+a^2\frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\left(a^2+b^2\right)\frac{\partial^2 z}{\partial u^2}+\left(a^2+b^2\right)\frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\left(a^2+b^2\right)\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}\right).$$

Jacobians and their properties (without proof)

Jacobian is a functional determinant (whose elements are functions) which is useful in transformation of variables from Cartesian to polar, cylindrical and spherical co-ordinates in multiple integrals.

The Jacobian of u, v with respect to x, y denoted $J\left(\frac{u,v}{x,y}\right)$ by or $\frac{\partial(u,v)}{\partial(x,y)}$ is a second order functional determinant defined as $J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

Similarly the Jacobian of three functions u, v, w of three independent variables x, y, z is defined as

$$J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Note: 1 If $J\left(\frac{u,v}{x,y}\right) = 0$ then, u and v are not functionally independent. In fact they are mutually dependent

2 If $J\left(\frac{u,v}{x,y}\right) \neq 0$ then, we can express x and y in terms of u and v (explicitly) as i.e. x=x(u,v) and y=y(u,v).

3 Consequently, if $J\left(\frac{u,v}{x,y}\right) \neq 0$ then we can define the Jacobian of x and y w.r.t u and v as follows $J^*\left(\frac{x,y}{u,v}\right) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$.

Two Important Properties of Jacobians

1. **If** $J = \frac{\partial(u,v)}{\partial(x,y)}$ **and** $J^* = \frac{\partial(x,y)}{\partial(u,v)}$ **then,** $JJ^* = 1$ **i.e.** $J = \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{J^*} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$

Proof:

Let $u = f(x,y)$ and $v = g(x,y)$ be two functions

Suppose on solving for x and y, we get

$x = \varphi(u,v)$ and $y = \psi(u,v)$ --- (1)

Diff partially x and y w r t u and v, we get

$1 = \frac{\partial u}{\partial v} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} = u_x \; x_u + u_y y_u$ ---

(2) $0 = \frac{\partial u}{\partial v} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} = u_x \; x_v + u_y y_v$ --- (3)

$$0=\frac{\partial v}{\partial u}=\frac{\partial v}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial v}=v_x x_v+v_y y_v \qquad \text{---}$$

$$(4) 1=\frac{\partial v}{\partial v}=\frac{\partial v}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial v}=v_x x_v+v_y y_v \qquad \text{--- (5)}$$

Consider, $JJ^*=\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

Interchanging rows and columns in 2^{nd} determinant

$$=\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}\begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}$$

Multiplying the determinant row-wise

$$jj^*=\begin{vmatrix} u_x x_u+u_y y_u & u_x x_v+u_y y_v \\ v_x x_u+v_y y_u & v_x x_v+v_y y_v \end{vmatrix}$$

Using (2), (3), (4) and (5) we get,

$$JJ^*=\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}=1$$

2. If u **,** v **are functions of** r,s **and** r,s **are functions of** x,y **then** $\frac{\partial(u,v)}{\partial(x,y)}=\frac{\partial(u,v)}{\partial(r,s)}\frac{\partial(r,s)}{\partial(x,y)}$ **i.e.**

$J\left(\frac{u,v}{x,y}\right)=J\left(\frac{u,v}{r,s}\right).J\left(\frac{r,s}{x,y}\right)$ **(Chain Rule for Jacobians)**

Proof: Differentiating u , v partially wrt x,y then

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial x} \qquad \text{--- (1)}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial y} \qquad \text{--- (2)}$$

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial v}{\partial s}\frac{\partial s}{\partial x} \qquad \text{--- (3)}$$

$$\frac{\partial v}{\partial y}=\frac{\partial v}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial v}{\partial s}\frac{\partial s}{\partial y} \qquad \text{--- (4)}$$

From definition of Jacobians, we have

$$\frac{\partial(u,v)}{\partial(r,s)}\cdot\frac{\partial(r,s)}{\partial(x,y)}=\begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix}\begin{vmatrix} r_x & r_y \\ s_x & s_y \end{vmatrix}$$

By interchanging the rows and columns in 2^{nd} determinant

$$=\begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix}\begin{vmatrix} r_x & s_x \\ r_y & s_y \end{vmatrix}$$

Multiplying the determinant row-wise

$$=\begin{vmatrix} u_r r_x+u_s r_y & u_r s_x+u_s s_y \\ v_r r_x+v_s r_y & v_r s_x+v_s s_y \end{vmatrix}$$

Using (1), (2), (3) and (4) we get

$$=\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)}=\frac{\partial(u,v)}{\partial(r,s)}\frac{\partial(r,s)}{\partial(x,y)}$$

Problems

In each of the following cases , find the Jacobians $J=\dfrac{\partial(u,v)}{\partial(x,y)}$ **and** $J^{1}=\dfrac{\partial(x,y)}{\partial(u,v)}$ **also verify that** $JJ^{1}=1$**.**

34. $u=x+y\,,\; v=xy$

Solution:

Given that $u=x+y\,,\; v=xy$

$$\frac{\partial u}{\partial x}=1 \quad \frac{\partial u}{\partial y}=1$$

$$\frac{\partial v}{\partial x}=y \quad \frac{\partial v}{\partial y}=x$$

$$J=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}1 & 1\\ y & x\end{vmatrix}=x-y$$

Now, $u=x+y$

$$x=u-y$$

$$x=u-\frac{v}{x} \qquad \left(y=\frac{v}{x}\right)$$

$$x^2=ux-v$$

$$x^2-ux+v=0$$

$$x=\frac{u\pm\sqrt{u^2-4v}}{2} \qquad y=u-\frac{u\mp\sqrt{u^2-4v}}{2}\,. \qquad (y=u-x)$$

Consider, $x=\dfrac{u+\sqrt{u^2-4v}}{2}$ $\qquad y=u-\dfrac{u+\sqrt{u^2-4v}}{2}$

$$y=\frac{u-\sqrt{u^2-4v}}{2}$$

$$\frac{\partial x}{\partial u}=\frac{1}{2}\left(1+\frac{u}{\sqrt{u^2-4v}}\right) \quad \frac{\partial x}{\partial v}=\frac{-1}{\sqrt{u^2-4v}}$$

$$\frac{\partial y}{\partial u}=\frac{1}{2}\left(1-\frac{u}{\sqrt{u^2-4v}}\right) \quad \frac{\partial y}{\partial v}=\frac{1}{\sqrt{u^2-4v}}$$

$$J^{1}=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{1}{2}\left(1+\frac{u}{\sqrt{u^2-4v}}\right) & \frac{-1}{\sqrt{u^2-4v}}\\ \frac{1}{2}\left(1-\frac{u}{\sqrt{u^2-4v}}\right) & \frac{1}{\sqrt{u^2-4v}}\end{vmatrix}$$

$$=\frac{1}{2}\frac{1}{\sqrt{u^2-4v}}+\frac{1}{2}\frac{u}{\left(\sqrt{u^2-4v}\right)^2}+\frac{1}{2}\frac{1}{\sqrt{u^2-4v}}-\frac{1}{2}\frac{u}{\left(\sqrt{u^2-4v}\right)^2}$$

$$=\frac{u}{\sqrt{u^2-4v}}$$

$$JJ^{1}=x-y\frac{1}{\sqrt{u^2-4v}}=\frac{\sqrt{u^2-4v}}{\sqrt{u^2-4v}}=1$$

35. $u = x^2 - 2y$, $v = x + y$

Solution:

Given that $u = x + y$, $v = xy$

$$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial u}{\partial y} = -2$$

$$\frac{\partial v}{\partial x} = 1 \quad \frac{\partial v}{\partial y} = 1$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2 \\ 1 & 1 \end{vmatrix} = 2(x+1)$$

Now, $u = x^2 - 2y$

$$u = x^2 - 2(v - x) \qquad (y = v - x)$$

$$x^2 + 2x - (2v + u) = 0$$

$$x = -1 \pm \sqrt{1+u+2v} \qquad y = v + 1 \mp \sqrt{1+u+2v}. \qquad (y = v - x)$$

Consider, $x = -1 + \sqrt{1+u+2v}$ $\qquad y = v + 1 - \sqrt{1+u+2v}$

$$\frac{\partial x}{\partial u} = \frac{1}{2}\frac{1}{\sqrt{1+u+2v}} \qquad \frac{\partial x}{\partial v} = \frac{1}{\sqrt{1+u+2v}}$$

$$\frac{\partial y}{\partial u} = \frac{1}{2}\frac{1}{\sqrt{1+u+2v}} \qquad \frac{\partial y}{\partial v} = 1 + \frac{1}{\sqrt{1+u+2v}}$$

$$J^1 = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2}\frac{1}{\sqrt{1+u+2v}} & \frac{1}{\sqrt{1+u+2v}} \\ \frac{1}{2}\frac{1}{\sqrt{1+u+2v}} & 1 + \frac{1}{\sqrt{1+u+2v}} \end{vmatrix}$$

$$= \frac{1}{2}\frac{1}{\sqrt{1+u+2v}} - \frac{1}{2}\frac{1}{\left(\sqrt{1+u+2v}\right)^2} + \frac{1}{2}\frac{1}{\left(\sqrt{1+u+2v}\right)^2}$$

$$= \frac{1}{2}\frac{1}{\sqrt{1+u+2v}}$$

$$JJ^1 = 2(x+1)\frac{1}{2}\frac{1}{\sqrt{1+u+2v}} = \frac{\sqrt{1+u+2v}}{\sqrt{1+u+2v}} = 1.$$

36. $u = x + \frac{y^2}{x}$, $v = \frac{y^2}{x}$

Solution:

Given that $u = x + \frac{y^2}{x}$, $v = \frac{y^2}{x}$

$$\frac{\partial u}{\partial x} = 1 - \frac{y^2}{x^2} \quad \frac{\partial u}{\partial y} = \frac{2y}{x}$$

$$\frac{\partial v}{\partial x} = -\frac{y^2}{x^2} \quad \frac{\partial v}{\partial y} = \frac{2y}{x}$$

$$J=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}1-\frac{y^2}{x^2} & \frac{2y}{x}\\ -\frac{y^2}{x^2} & \frac{2y}{x}\end{vmatrix}=\frac{2y}{x}$$

Now, $u=x+v$ $\left(v=\frac{y^2}{x}\right)$

$x=u-v$ $\quad y^2=vx.$

$$y^2=v(u-v)$$
$$y^2=vu-v^2$$
$$y=\sqrt{vu-v^2}$$

Then,

$$\frac{\partial x}{\partial u}=1 \qquad \frac{\partial x}{\partial v}=-1$$

$$\frac{\partial y}{\partial u}=\frac{v}{2\sqrt{uv-v^2}} \qquad \frac{\partial y}{\partial v}=\frac{u-2v}{2\sqrt{uv-v^2}}$$

$$J^1=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}1 & -1\\ \frac{v}{2\sqrt{uv-v^2}} & \frac{u-2v}{2\sqrt{uv-v^2}}\end{vmatrix}$$

$$=\frac{u-2v}{2\sqrt{uv-v^2}}+\frac{v}{2\sqrt{uv-v^2}}$$

$$=\frac{u-v}{2\sqrt{uv-v^2}}$$

$$JJ^1=\frac{2y}{x}\frac{u-v}{2\sqrt{uv-v^2}}=\frac{2y}{x}\frac{x}{2y}=1. \qquad \begin{pmatrix}u-v=x\\ \sqrt{uv-v^2}=y\end{pmatrix}$$

37. $u=\sqrt{xy}$, $v=\sqrt{\frac{y}{x}}$

Solution:

Given that $u=\sqrt{xy}$, $v=\sqrt{\frac{y}{x}}$

$$\frac{\partial u}{\partial x}=\frac{y}{2\sqrt{xy}} \qquad \frac{\partial u}{\partial y}=\frac{x}{2\sqrt{xy}}$$

$$\frac{\partial v}{\partial x}=\frac{1}{2\sqrt{xy}} \qquad \frac{\partial v}{\partial y}=-\frac{\sqrt{x}}{2y\sqrt{y}}$$

$$J=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}\frac{y}{2\sqrt{xy}} & \frac{x}{2\sqrt{xy}}\\ \frac{1}{2\sqrt{xy}} & -\frac{\sqrt{x}}{2y\sqrt{y}}\end{vmatrix}=\left(\frac{-y}{1\sqrt{xy}}\right)\frac{\sqrt{x}}{2y\sqrt{y}}-\frac{x}{2\sqrt{xy}}\frac{1}{2\sqrt{xy}}=-\frac{1}{4y}-\frac{1}{4y}=-\frac{1}{2y}$$

Now, $u=\sqrt{xy}$ $\quad v=\sqrt{\frac{x}{y}}$

$uv=\sqrt{x}\sqrt{y}\frac{\sqrt{x}}{\sqrt{y}}$ $\quad y^2=vx.$

$uv = x$

$$\sqrt{y} = \frac{\sqrt{x}}{v}$$

$$\sqrt{y} = \frac{\sqrt{uv}}{v}$$

$$\sqrt{y} = \frac{\sqrt{u}\sqrt{v}}{v}$$

$$\sqrt{y} = \frac{\sqrt{u}}{\sqrt{v}}$$

$$y = \frac{u}{v}$$

Then,

$$\frac{\partial x}{\partial u} = v \qquad \frac{\partial x}{\partial v} = u$$

$$\frac{\partial y}{\partial u} = \frac{1}{v} \quad \frac{\partial y}{\partial v} = -\frac{u}{v^2}$$

$$J^1 = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix}$$

$$= -\frac{u}{v} - \frac{u}{v}$$

$$= -\frac{2u}{v}$$

$$JJ^1 = \left(-\frac{1}{2y}\right)\left(-\frac{2u}{v}\right) = \frac{1}{y} y = 1. \qquad \left(\frac{u}{v} = y\right)$$

38. $u = \sqrt{x^2 + y^2}$, $v = \sqrt{x^2 - y^2}$

Solution:

Given that $u = \sqrt{x^2 + y^2}$, $v = \sqrt{x^2 - y^2}$

$$\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} \qquad \frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial v}{\partial x} = \frac{x}{\sqrt{x^2 - y^2}} \qquad \frac{\partial v}{\partial y} = -\frac{y}{\sqrt{x^2 - y^2}}$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{x}{\sqrt{x^2-y^2}} & -\frac{y}{\sqrt{x^2-y^2}} \end{vmatrix} = -\frac{xy}{\sqrt{(x^2+y^2)(x^2-y^2)}} - \frac{xy}{\sqrt{(x^2+y^2)(x^2-y^2)}}$$

$$= -\frac{2xy}{\sqrt{x^4 - y^4}}$$

Now, $u = \sqrt{x^2 + y^2}$, $v = \sqrt{x^2 - y^2}$

$$u^2 + v^2 = x^2 + y^2 + x^2 - y^2 \qquad u^2 - v^2 = x^2 + y^2 - x^2 + y^2.$$

$$x^2 = \frac{u^2 + v^2}{2} \qquad y^2 = \frac{u^2 - v^2}{2}$$

Then,

$$2x\frac{\partial x}{\partial u}=\frac{1}{2}2u \qquad 2x\frac{\partial x}{\partial v}=\frac{1}{2}2v$$

$$\frac{\partial x}{\partial u}=\frac{u}{\sqrt{2}\sqrt{u^2+v^2}} \qquad \frac{\partial x}{\partial v}=\frac{v}{\sqrt{2}\sqrt{u^2+v^2}}$$

$$2y\frac{\partial y}{\partial u}=\frac{1}{2}2u \qquad 2y\frac{\partial y}{\partial v}=\frac{1}{2}(-2v)$$

$$\frac{\partial y}{\partial u}=\frac{u}{\sqrt{2}\sqrt{u^2-v^2}} \qquad \frac{\partial y}{\partial u}=-\frac{v}{\sqrt{2}\sqrt{u^2-v^2}}$$

$$J^1=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{u}{\sqrt{2}\sqrt{u^2+v^2}} & \frac{v}{\sqrt{2}\sqrt{u^2+v^2}}\\ \frac{u}{\sqrt{2}\sqrt{u^2-v^2}} & -\frac{v}{\sqrt{2}\sqrt{u^2-v^2}}\end{vmatrix}$$

$$=-\frac{uv}{2\sqrt{u^4-v^4}}-\frac{uv}{2\sqrt{u^4-v^4}}$$

$$=-\frac{uv}{\sqrt{u^4-v^4}}$$

$$JJ^1=\left(-\frac{2xy}{\sqrt{x^4-y^4}}\right)\left(-\frac{uv}{\sqrt{u^4-v^4}}\right)=\frac{2xy}{\sqrt{x^4-y^4}}\frac{\sqrt{x^4-y^4}}{2xy}=1. \qquad \left(\begin{matrix}uv=\sqrt{x^4-y^4}\\ 2xy=\sqrt{u^4-v^4}\end{matrix}\right)$$

39. $x=e^u\cos v$, $y=e^u\sin v$

Solution:

Given that $x=e^u\cos v$, $y=e^u\sin v$

$$\frac{\partial x}{\partial u}=e^u\cos v \qquad \frac{\partial x}{\partial v}=-e^u\sin v$$

$$\frac{\partial y}{\partial u}=e^u\sin v \qquad \frac{\partial y}{\partial v}=e^u cos v$$

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}e^u\cos v & -e^u\sin v\\ e^u\sin v & e^u cos v\end{vmatrix}$$

$$=e^{2u}\cos^2 v+e^{2v}\sin^2 v$$

$$=e^{2u}$$

Now, $x=e^u\cos v$, $y=e^u\sin v$

$$\frac{y}{x}=\tan v, \qquad x^2+y^2=e^{2u}.$$

$$v=\tan^{-1}\frac{y}{x} \qquad u=\frac{1}{2}\ln(x^2+y^2)$$

Then,

$$\frac{\partial u}{\partial x}=\frac{x}{x^2+y^2} \qquad \frac{\partial u}{\partial y}=\frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x}=-\frac{-y}{x^2+y^2} \qquad \frac{\partial v}{\partial y}=\frac{x}{x^2+y^2}$$

$$J^1 = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ -\frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{\left(x^2+y^2\right)^2} + \frac{y^2}{\left(x^2+y^2\right)^2}$$

$$= \frac{1}{x^2+y^2}$$

$$JJ^1 = e^{2u}\frac{1}{x^2+y^2} = \frac{x^2+y^2}{x^2+y^2} = 1. \qquad \left(x^2+y^2 = e^{2u}\right)$$

Prove the following

40. **If** $x = u(1+v)$ **,** $y = v(1+u)$ **then** $\frac{\partial(u,v)}{\partial(x,y)} = 1+u+v$**.**

Solution:

$$x = u+uv\ , \qquad y = v+uv$$

$$\frac{\partial x}{\partial u} = 1+v \qquad \frac{\partial x}{\partial v} = u$$

$$\frac{\partial y}{\partial u} = v \qquad \frac{\partial y}{\partial v} = 1+u$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$

$$= (1+v)(1+u) - uv$$

$$= 1+u+v+uv-uv$$

$$= 1+u+v$$

41. **If** $x = \frac{u^2}{v}$ **,** $y = \frac{v^2}{u}$ **then** $\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{3}$**.**

Solution:

$$x = \frac{u^2}{v}\ , \qquad y = \frac{v^2}{u}$$

$$v = \frac{u^2}{x} \qquad u = \frac{v^2}{y}$$

$$v = \frac{v^4}{y^2 x} \qquad u = \frac{u^4}{x^2 y}$$

$$v^3 = xy^2 \qquad u^3 = x^2 y \qquad \begin{pmatrix} u^3v^3 = x^3y^3 \\ \Rightarrow uv = xy \end{pmatrix}$$

$$3u^2\frac{\partial u}{\partial x}=2xy \quad 3u^2\frac{\partial u}{\partial y}=x^2$$

$$\frac{\partial u}{\partial x}=\frac{2xy}{3u^2} \quad \frac{\partial u}{\partial y}=\frac{x^2}{3u^2}$$

$$3v^2\frac{\partial v}{\partial x}=y^2 \quad 3v^2\frac{\partial v}{\partial y}=2xy$$

$$\frac{\partial v}{\partial x}=\frac{y^2}{3v^2} \quad \frac{\partial v}{\partial y}=\frac{2xy}{3v^2}$$

$$J^1=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{2xy}{3u^2} & \frac{x^2}{3u^2}\\ \frac{y^2}{3v^2} & \frac{2xy}{3v^2}\end{vmatrix}$$

$$=\frac{4x^2y^2}{9u^2v^2}-\frac{x^2y^2}{9u^2v^2}$$

$$=\frac{3x^2y^2}{9u^2v^2}$$

$$=\frac{1}{3} \qquad (\because uv=xy)$$

42. Using Jacobians, prove that $u=x+y$ and $v=\frac{1}{x+y}$ are functionally dependent.

Solution:

Given $u=x+y$ and $v=\frac{1}{x+y}$

$$\frac{\partial u}{\partial x}=1 \qquad \frac{\partial u}{\partial y}=1$$

$$\frac{\partial v}{\partial x}=-\frac{1}{(x+y)^2} \quad \frac{\partial v}{\partial y}=-\frac{1}{(x+y)^2}$$

$$J=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}1 & 1\\ -\frac{1}{(x+y)^2} & -\frac{1}{(x+y)^2}\end{vmatrix}$$

$$=-\frac{1}{(x+y)^2}+\frac{1}{(x+y)^2}$$

$$=0$$

Implies that u and v are functionally dependent.

$v=\frac{1}{u}$.

Prove the following

43. If $u=2axy$ and $v=a(x^2-y^2)$ where $x=r\cos\theta$ and $y=r\sin\theta$ then $\frac{\partial(u,v)}{\partial(r,\theta)}=-4a^2r^3$.

Solution:

Given $u=2axy$ and $v=a(x^2-y^2)$

$u = 2ar^2 \sin\theta\cos\theta$ $\qquad$ $u = a\left(r^2\cos^2\theta - r^2\sin^2\theta\right)$

$u = ar^2 \sin 2\theta$ $\qquad$ $u = ar^2\cos 2\theta$

$\frac{\partial u}{\partial r} = 2ar\sin 2\theta \quad \frac{\partial u}{\partial \theta} = 2ar^2\cos 2\theta$

$\frac{\partial v}{\partial x} = 2ar\cos 2\theta \quad \frac{\partial v}{\partial y} = -2ar^2\sin 2\theta$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2ar\sin 2\theta & 2ar^2\cos 2\theta \\ 2ar\cos 2\theta & -2ar^2\sin 2\theta \end{vmatrix}$$

$= -4a^2r^3\sin^2 2\theta - 4a^2r^3\cos^2 2\theta$

$= -4a^2r^3$

44. **If** $u = x^2 - 2y^2$ **and** $v = 2x^2 - y^2$ **where** $x = r\cos\theta$ **and** $y = r\sin\theta$ **then** $\frac{\partial(u,v)}{\partial(r,\theta)} = 6r^3\sin 2\theta$**.**

Solution:

Given $u = x^2 - 2y^2$ and $v = 2x^2 - y^2$

$u = r^2\cos^2\theta - 2r^2\sin^2\theta$ $\qquad$ $u = 2r^2\cos^2\theta - r^2\sin^2\theta$

$u = ar^2\sin 2\theta$ $\qquad$ $u = ar^2\cos 2\theta$

$\frac{\partial u}{\partial r} = 2r\cos^2\theta - 4r\sin^2\theta \quad \frac{\partial u}{\partial \theta} = -2r^2\cos\theta\sin\theta - 4r^2\sin\theta\cos\theta$

$\frac{\partial v}{\partial x} = 4r\cos^2\theta - 2r\sin^2\theta \quad \frac{\partial v}{\partial y} = -4r^2\cos\theta\sin\theta - 2r^2\sin\theta\cos\theta$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2r\cos^2\theta - 4r\sin^2\theta & -2r^2\cos\theta\sin\theta - 4r^2\sin\theta\cos\theta \\ 4r\cos^2\theta - 2r\sin^2\theta & -4r^2\cos\theta\sin\theta - 2r^2\sin\theta\cos\theta \end{vmatrix}$$

$= \left(2r\cos^2\theta - 4r\sin^2\theta\right)\left(-4r^2\cos\theta\sin\theta - 2r^2\sin\theta\cos\theta\right)$

$-\left(-2r^2\cos\theta\sin\theta - 4r^2\sin\theta\cos\theta\right)\left(4r\cos^2\theta - 2r\sin^2\theta\right)$

$= -8r^3\cos^3\theta\sin\theta + 16r^3\cos\theta\sin^3\theta - 4r^3\sin\theta\cos^3\theta + 8r^3\sin^3\theta\cos\theta$

$+8r^3\cos^3\theta\sin\theta - 4r^3\sin^3\theta\cos\theta + 16r^3\cos^3\theta\sin\theta - 8r^3\sin^3\theta\cos\theta$

$= 16r^3\cos\theta\sin\theta(\sin^2\theta + \cos^2\theta) - 4r^3\sin\theta\cos\theta(\sin^2\theta + \cos^2\theta)$

$= 4r^3\sin\theta\cos\theta(4-1)$

$= 6r^3\sin 2\theta$.

In each of the following cases, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

45. $u = xy^2$**,** $v = yz^2$, $w = zx^2$

Solution:

Given $u = xy^2$**,** $v = yz^2$, $w = zx^2$

$\frac{\partial u}{\partial x} = y^2 \quad \frac{\partial u}{\partial y} = 2xy \quad \frac{\partial u}{\partial z} = 0$

$\frac{\partial v}{\partial x} = 0 \quad \frac{\partial v}{\partial y} = z^2 \quad \frac{\partial v}{\partial z} = 2yz$

$\frac{\partial w}{\partial x} = 2xz \quad \frac{\partial w}{\partial y} = 0 \quad \frac{\partial w}{\partial z} = x^2$

$$J=\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}$$

$$=\begin{vmatrix}y^2 & 2xy & 0\\ 0 & z^2 & 2yz\\ 2xz & 0 & x^2\end{vmatrix}$$

$$=y^2(x^2z^2)+2xz(4xy^2z)$$

$$=9x^2y^2z^2 .$$

$$=(3xyz)^2$$

46. $u=x(1-r^2)^{-\frac{1}{2}}$**,** $v=y(1-r^2)^{-\frac{1}{2}}$**,** $w=z(1-r^2)^{-\frac{1}{2}}$**, where** $r^2=x^2+y^2+z^2$ **.**

Solution:

Given $u=x(1-r^2)^{-\frac{1}{2}}$**,** $v=y(1-r^2)^{-\frac{1}{2}}$, $w=z(1-r^2)^{-\frac{1}{2}}$

$$u=x(1-x^2-y^2-z^2)^{-\frac{1}{2}},$$

$$\frac{\partial u}{\partial x}=(1-x^2-y^2-z^2)^{-\frac{1}{2}}+x^2(1-x^2-y^2-z^2)^{-\frac{3}{2}}$$

$$\frac{\partial v}{\partial x}=xy(1-x^2-y^2-z^2)^{-\frac{3}{2}}$$

$$\frac{\partial w}{\partial x}=xz(1-x^2-y^2-z^2)^{-\frac{3}{2}}$$

$$v=y(1-x^2-y^2-z^2)^{-\frac{1}{2}}$$

$$\frac{\partial u}{\partial y}=xy(1-x^2-y^2-z^2)^{-\frac{3}{2}}$$

$$\frac{\partial v}{\partial y}=(1-x^2-y^2-z^2)^{-\frac{1}{2}}+y^2(1-x^2-y^2-z^2)^{-\frac{3}{2}}$$

$$\frac{\partial w}{\partial y}=yz(1-x^2-y^2-z^2)^{-\frac{3}{2}}$$

$$w=z(1-x^2-y^2-z^2)^{-\frac{1}{2}}$$

$$\frac{\partial u}{\partial z}=xz(1-x^2-y^2-z^2)^{-\frac{3}{2}}$$

$$\frac{\partial v}{\partial z}=yz(1-x^2-y^2-z^2)^{-\frac{3}{2}}$$

$$\frac{\partial w}{\partial z}=(1-x^2-y^2-z^2)^{-\frac{1}{2}}+z^2(1-x^2-y^2-z^2)^{-\frac{3}{2}}$$

$$J=\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}$$

$$=\begin{vmatrix}(1-r^2)^{-\frac{1}{2}}+x^2(1-r^2)^{-\frac{3}{2}} & xy(1-r^2)^{-\frac{3}{2}} & xz(1-r^2)^{-\frac{3}{2}}\\ xy(1-r^2)^{-\frac{3}{2}} & (1-r^2)^{-\frac{1}{2}}+y^2(1-r^2)^{-\frac{3}{2}} & yz(1-r^2)^{-\frac{3}{2}}\\ xz(1-r^2)^{-\frac{3}{2}} & yz(1-r^2)^{-\frac{3}{2}} & (1-r^2)^{-\frac{1}{2}}+z^2(1-r^2)^{-\frac{3}{2}}\end{vmatrix}$$

$$=\begin{vmatrix}(1-r^2)^{-\frac{3}{2}}(1-y^2-z^2) & xy(1-r^2)^{-\frac{3}{2}} & xz(1-r^2)^{-\frac{3}{2}}\\ xy(1-r^2)^{-\frac{3}{2}} & (1-r^2)^{-\frac{3}{2}}(1-x^2-z^2) & yz(1-r^2)^{-\frac{3}{2}}\\ xz(1-r^2)^{-\frac{3}{2}} & yz(1-r^2)^{-\frac{3}{2}} & (1-r^2)^{-\frac{3}{2}}(1-y^2-x^2)\end{vmatrix}$$

$$=\frac{(1-r^2)^{-\frac{9}{2}}}{x}\begin{vmatrix}x(1-y^2-z^2) & xxy & xxz\\ xy & (1-x^2-z^2) & yz\\ xz & yz & (1-y^2-x^2)\end{vmatrix}$$

$$=\frac{(1-r^2)^{-\frac{9}{2}}}{x}\begin{vmatrix}x & y & z\\ xy & \frac{y(1-x^2-z^2)}{y} & yz\\ xz & yz & \frac{z(1-y^2-x^2)}{z}\end{vmatrix} R_1^1=R_1+yR_2+zR_3$$

$$=\frac{(1-r^2)^{-\frac{9}{2}}}{x}xyz\begin{vmatrix}1 & 1 & 1\\ y & \frac{(1-x^2-z^2)}{y} & z\\ z & z & \frac{(1-y^2-x^2)}{z}\end{vmatrix} R_1^1=R_1+yR_2+zR_3$$

$$=(1-r^2)^{-\frac{9}{2}}yz\begin{vmatrix}1 & 0 & 0\\ y & \frac{(1-x^2-z^2)}{y}-y & 0\\ z & 0 & \frac{(1-y^2-x^2)}{z}-z\end{vmatrix}$$

$$=(1-r^2)^{-\frac{9}{2}}yz\left(\left(\frac{1-z^2-x^2-y^2}{y}\right)\left(\frac{1-z^2-x^2-y^2}{y}\right)\right)$$

$$=(1-r^2)^{-\frac{9}{2}}(1-r^2)(1-r^2)$$

$$=(1-r^2)^{-\frac{5}{2}}$$

In each of the following cases find $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

47. $x=u$, $y=u\tan v$, $z=w$

Solution:

Given $x=u$ $\quad y=u\tan v$ $\quad z=w$

$$\frac{\partial x}{\partial u}=1 \qquad \frac{\partial x}{\partial v}=0 \qquad \frac{\partial x}{\partial w}=0$$

$$\frac{\partial y}{\partial u}=\tan v \qquad \frac{\partial y}{\partial v}=u\sec^2 v \qquad \frac{\partial y}{\partial w}=0$$

$$\frac{\partial z}{\partial u}=0 \qquad \frac{\partial z}{\partial v}=0 \qquad \frac{\partial z}{\partial w}=1$$

$$J=\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 0 & 0 \\ \tan v & u\sec^2 v & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$=u\sec^2 v$$

48. $x=u(1-v)$, $y=uv(1-w)$, $z=uvw$

Solution:

Given $x=u(1-v)$, $\quad y=uv(1-w)$, $\quad z=uvw$

$$\frac{\partial x}{\partial u}=1-v \qquad \frac{\partial x}{\partial v}=-u \qquad \frac{\partial x}{\partial w}=0$$

$$\frac{\partial y}{\partial u}=v(1-w) \qquad \frac{\partial y}{\partial v}=u(1-w) \qquad \frac{\partial y}{\partial w}=-uv$$

$$\frac{\partial z}{\partial u}=vw \qquad \frac{\partial z}{\partial v}=uw \qquad \frac{\partial z}{\partial w}=uv$$

$$J=\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$=\begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$$=(1-v)\left[u^2v(1-w)+u^2vw\right]+u\left[uv^2(1-w)+uv^2w\right]$$

$$=(1-v)\left[u^2v-u^2vw+u^2vw\right]+u^2v^2(1-w)+u^2v^2w$$

$$=(1-v)\left[u^2v-u^2vw+u^2vw\right]+u^2v^2(1-w)+u^2v^2w$$

$$=u^2v-u^2v^2+u^2v^2$$

$$=u^2v$$

49. **If** $u = x+y+z$, $uv = y+z$, $z = uvw$ **then show that** $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$.

Solution:

Given $u = x+y+z$, $uv = y+z$, $z = uvw$

$x = u - uv$, $y = uv - uvw$, $z = uvw$

$$\frac{\partial x}{\partial u} = 1-v \qquad \frac{\partial x}{\partial v} = -u \qquad \frac{\partial x}{\partial w} = 0$$

$$\frac{\partial y}{\partial u} = v(1-w) \qquad \frac{\partial y}{\partial v} = u(1-w) \qquad \frac{\partial y}{\partial w} = -uv$$

$$\frac{\partial z}{\partial u} = vw \qquad \frac{\partial z}{\partial v} = uw \qquad \frac{\partial z}{\partial w} = uv$$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= (1-v)\left[u^2 v(1-w) + u^2 vw\right] + u\left[uv^2(1-w) + uv^2 w\right]$$

$$= (1-v)\left[u^2 v - u^2 vw + u^2 vw\right] + u^2 v^2 (1-w) + u^2 v^2 w$$

$$= (1-v)\left[u^2 v - u^2 vw + u^2 vw\right] + u^2 v^2 (1-w) + u^2 v^2 w$$

$$= u^2 v - u^2 v^2 + u^2 v^2$$

$$= u^2 v$$

50. **Prove that the functions** $x = u^2 - v^2$, $y = v^2 - w^2$, $z = w^2 - u^2$ **are functionally dependent.**

Solution:

Given $x = u^2 - v^2$, $y = v^2 - w^2$, $z = w^2 - u^2$

$$\frac{\partial x}{\partial u} = 2u \qquad \frac{\partial x}{\partial v} = -2v \qquad \frac{\partial x}{\partial w} = 0$$

$$\frac{\partial y}{\partial u} = 0 \qquad \frac{\partial y}{\partial v} = 2v \qquad \frac{\partial y}{\partial w} = -2w$$

$$\frac{\partial z}{\partial u} = -2u \qquad \frac{\partial z}{\partial v} = 0 \qquad \frac{\partial z}{\partial w} = 2w$$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 2u & -2v & 0 \\ 0 & 2v & -2w \\ -2u & 0 & 2w \end{vmatrix}$$

$$= 2u(4vw) + 2v(-4uw)$$
$$= 8uvw - 8uvw$$
$$= 0$$

This implies that the functions are functionally dependent and is connected by the relation

$$x + y + z = u^2 - v^2 + v^2 - w^2 + w^2 - u^2$$
$$x + y + z = 0.$$

Standard Jacobians

In polar co-ordinates if $x = r\cos\theta$**,** $y = r\sin\theta$ **then show that** $\frac{\partial(x,y)}{\partial(r,\theta)} = r$

Proof : $x = r\cos\theta$, $\quad y = r\sin\theta$

$$x_r = \frac{\partial x}{\partial y} = \cos\theta; \qquad y_r = \frac{\partial y}{\partial r} = \sin\theta$$

$$x_\theta = \frac{\partial x}{\partial \theta} = -r\sin\theta; \qquad y_\theta = \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r$$

In cylindrical co-ordinate $x = r\cos\phi$**,** $y = r\sin\phi$ **and** $z = z$ **then Show that** $\frac{\partial(x,y,z)}{\partial(r,\phi,z)} = r$

Proof : $x = r\cos\phi$, $\quad y = r\sin\phi \quad z = z$

$$\frac{\partial x}{\partial r} = \cos\phi \qquad \frac{\partial y}{\partial r} = \sin\phi \qquad \frac{\partial z}{\partial r} = 0$$

$$\frac{\partial x}{\partial \phi} = -r\sin\phi \qquad \frac{\partial y}{\partial \phi} = r\cos\phi \qquad \frac{\partial z}{\partial \phi} = 0$$

$$\frac{\partial x}{\partial z} = 0 \qquad \frac{\partial y}{\partial z} = 0 \qquad \frac{\partial z}{\partial z} = 1$$

$$\frac{\partial(x,y,z)}{\partial(r,\phi,z)} = \begin{vmatrix} x_r & x_\phi & x_z \\ y_y & y_\phi & y_z \\ z_r & z_\phi & z_z \end{vmatrix} = \begin{vmatrix} \cos\phi & -r\sin\phi & 0 \\ \sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\frac{\partial(x,y,z)}{\partial(r,\phi,z)} = r$$

Similarly in spherical co-ordinates $x = r\cos\phi\sin\theta$**,** $y = r\sin\phi\sin\theta$ **and** $z = z\cos\theta$ **We can show that** $\frac{\partial(\mathrm{x,y,z})}{\partial(\mathrm{r},\theta,\phi)} = \mathrm{r}^2\sin\theta$**.**

Jacobian of implicit functions If u_1, u_2, u_3 and x_1, x_2, x_3 are implicitly connected by 3 equations as $f_1(u_1,u_2,u_3,x_1,x_2,x_3) = 0$, $f_2(u_1,u_2,u_3,x_1,x_2,x_3) = 0$, $f_3(u_1,u_2,u_3,x_1,x_2,x_3) = 0$ then

$$\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x_1,x_2,x_3)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u_1,u_2,u_3)}}$$

Problems

51. If $u = xyz$**,** $v = x^2 + y^2 + z^2$ **and** $w = x + y + z$**, then find** $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

Solution:

Let $f_1 = u - xyz$; $f_2 = v - x^2 - y^2 - z^2$; $f_3 = w - (x+y+z)$

We have $\dfrac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = (-1)^3 \dfrac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x_1,x_2,x_3)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u_1,u_2,u_3)}}$ -------- (1)

$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \qquad \text{-------- (2)}$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} -yz & -xz & -xy \\ -zx & -2y & -2z \\ -1 & -1 & -1 \end{vmatrix} \qquad \text{-------- (3)}$$

$$= 2(x-y)(y-z)(z-x)$$

Using (2) and (3) in (1)

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{(-1)^3}{-2(x-y)(y-z)(z-x)} = \frac{1}{2(x-y)(y-z)(z-x)}$$

Maxima and Minima for functions of two variables

Let $z = f(x,y)$ be a function of two independent variables x and y.

Relative Maximum

The function $f(x,y)$ is said to have a relative maximum at a point (a,b) if $f(a,b) > f(a+h,b+k)$ for small positive or negative values of h and k.

Relative Minimum

The function $f(x,y)$ is said to have a relative minimum at a point (a,b) if $f(a,b) < f(a+h,b+k)$ for small positive or negative values of h and k.

Let $\Delta = f(a+h,b+k) - f(a,b)$

$f(a,b)$ is maximum if $\Delta < 0$ for all small values of h and k

$f(a,b)$ is minimum if $\Delta > 0$ for all small values of h and k

Extremum or Extreme Value

A maximum or minimum value of a function is called its extreme value.

Saddle Point

Saddle point is a point where function is neither maximum nor minimum. At that point f is maximum in or direction while minimum in another direction.

Conditions for $f(x,y)$ to be maximum or minimum

By Taylor's theorem, we have

$$f(a+h,b+k) = f(a,b)+[hf_x(a,b)+kf_y(a,b)]+\frac{1}{2!}[h^2 f_{xx}(a,b)+2hkf_{xy}+k^2 f_{yy}(a,b)]+\ldots \quad \text{--- (1)}$$

Neglecting higher order terms of $h^2, hk, k^2\ldots$ etc.

$$(1) \Rightarrow \Delta = f(a+h,b+k)-f(a,b)$$
$$= hf_x(a,b)+kf_y(a,b) \quad \text{--- (2)}$$

Hence necessary conditions for $f(x,y)$ to have a maximum or a minimum at (a,b) are $f_x(a,b)=0$ and $f_y(a,b)=0$

If the conditions are satisfied then by neglecting the higher orders terms $h^3, k^3\ldots$

$$(1) \Rightarrow \Delta = \frac{1}{2!}[h^2 f_{xx}(a,b)+2hkf_{xy}(a,b)+k^2 f_{yy}(a,b)] \quad \text{--- (3)}$$

Denote $f_{xx}(a,b)=r$, $f_{xy}(a,b)=s$ and $f_{yy}(a,b)=t$

$$\text{Sign of } \Delta = \text{Sign of } \left[\frac{1}{2}[h^2 r+2hks+k^2 t]\right]$$
$$= \text{Sign of } \left[\frac{h^2r^2+2hksr+k^2tr}{2!\,r}\right]$$
$$= \text{Sign of } \frac{1}{2r}\left[(hr+ks)^2+k^2(rt-s^2)\right]$$

Conditions for Extremes are

- f has a maximum value at (a,b) if $rt-s^2>0$ and $r<0$
- f has a minimum value at (a,b) if $rt-s^2>0$ and $r>0$
- Saddle point: If $rt-s^2<0$ then $\Delta>0$ *or* $\Delta<0$ depending on h and k

  $\therefore$ f has a saddle point at (a,b) if $rt-s^2<0$
- If $rt-s^2=0$, further investigation is needed to determine the nature of function of f .

Working rule to find the Maximum and Minimum value of $f(x,y)$

i. Find $\frac{\partial f}{\partial x}$ *or* f_x and $\frac{\partial f}{\partial y}$ *or* f_y and equate each to zero. Solve these as simultaneous equations in x and y . Let (a,b), (c,d),… be the pairs of values.

ii. Calculate the value of $f_{xx}=r$, $f_{xy}=s$ and $f_{yy}=t$ for each pair of values.

iii.

a. If $rt-s^2>0$ and $r<0$ r < 0 at (a,b) then $f(a,b)$ has maximum value

b. If $rt-s^2>0$ and $r>0$ at (a,b) then $f(a,b)$ has minimum value

c. If $rt-s^2<0$ at (a,b) then $f(a,b)$ is not an extreme value, i.e., (a,b) is a saddle point.

iv. If $rt-s^2=0$ at (a,b), further investigation needed.

Problems

52. **Show that minimum value of** $u=xy+\frac{a^3}{x}+\frac{a^3}{y}$ **is** $3a^2$

Solution:

Let $u=xy+\frac{a^3}{x}+\frac{a^3}{y}$

$\therefore\ p=\frac{\partial u}{\partial x}=y-\frac{a^3}{x^2};\ q=\frac{\partial u}{\partial y}=x-\frac{a^3}{y^2}$

$r=\frac{\partial^2 u}{\partial x^2}=\frac{2a^3}{x^3};\ s=\frac{\partial^2 u}{\partial x\partial y}=1; t=\frac{\partial^2 u}{\partial y^2}=\frac{2a^3}{y^3}$

For maximum or minimum, we must have $\frac{\partial u}{\partial x}=0=\frac{\partial u}{\partial y}$

$\frac{\partial u}{\partial x}=0\Rightarrow y-\frac{a^3}{y^2}=0\ or\ x^2y=a^3$ --- (1)

$\frac{\partial u}{\partial y}=0\Rightarrow x-\frac{a^3}{y^2}=0\ or\ xy^2=a^3$ --- (2)

Solving (1) and (2), we get

$xy(x-y)=0$ or $x=0, y=0$ and $x=y$

From (1) and (2) $\Rightarrow x=0$ and $y=0$ do not hold.

$\therefore\ x=y$ from (1) we get $x=a$

$\therefore\ x=y=a$

at $x=y=a$, we get $r=\frac{2a^3}{a^3}=2,\ s=1,\ t=2$,

$rt-s^2=2(2)-1^2=3>0$

Also $r=2>0$

Hence there is a minima at $x=y=a$

Hence minimum value of $u=a+a\frac{a^3}{a}+\frac{a^3}{a}=3a^2$

53. **Show that the function** $f(x,y)=x^3+y^3-3xy+1$ **is minimum at** $(1,1)$**.**

Solution:

Let $f(x,y)=x^3+y^3-3xy+1$

$$\frac{\partial f}{\partial x} = 3x^2 - 3y \; ; p = \left(\frac{\partial f}{\partial x}\right)_{(1,1)} = 0 \qquad \frac{\partial f}{\partial y} = 3y^2 - 3x \; ; q = \left(\frac{\partial f}{\partial y}\right)_{(1,1)} = 0 \qquad \frac{\partial^2 f}{\partial x^2} = 6x \quad ;$$

$$r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{(1,1)} = 6 \qquad \frac{\partial^2 f}{\partial x \partial y} = -3 \quad ; s = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(1,1)} = -3 \qquad \frac{\partial^2 f}{\partial y^2} = 6y$$

$$t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{(1,1)} = 6$$

We have, $r = 6 > 0$ and $rt - s^2 = 36 - 9 = 27 > 0$

$\therefore$ $f(x, y)$ is minimum at $(1,1)$.

54. Find the extreme values of the function $f(x, y) = x^2 + y^2 + 6x - 12$.

Solution:

Given $f(x, y) = x^2 + y^2 + 6x - 12$

$$\frac{\partial f}{\partial x} = 2x + 6, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial x \partial y} = 2, \quad \frac{\partial^2 f}{\partial x \partial y} = 0$$

Solving $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$

We get $x = -3$ *and* $y = 0$

Critical point is $(-3, 0)$

We have $r = 2 > 0$ and $rt - s^2 = 2(2) - 0 = 4 > 0$

$\therefore$ $f(x, y)$ is minimum at $(-3, 0)$

$Min\ f(x, y) = (-3)^2 + 6(-3) - 12 = -21$

55. Show that the function $f(x, y) = xy(a - x - y),\ a > 0$ is maximum at the point $\left(\frac{a}{3}, \frac{a}{3}\right)$.

Solution:

Let $f(x, y) = axy - x^2 y - xy^2$

$$\frac{\partial f}{\partial x} = ay - 2yx - y^2, \quad \frac{\partial f}{\partial y} = ax - x^2 - 2xy,$$

$p = 0$ and $q = 0$ at $\left(\frac{a}{3}, \frac{a}{3}\right)$.

$$\frac{\partial^2 f}{\partial x^2} = -2y \qquad \frac{\partial^2 f}{\partial x\, \partial y} = a - 2x - 2y \qquad \frac{\partial^2 f}{\partial y^2} = -2x$$

$$r = \frac{-2a}{3}, \qquad s = \frac{-a}{3} \qquad t = \frac{-2a}{3}$$

Now

$$r = \frac{-2a}{3} < 0 \because a > 0 \quad \text{and} \quad rt - s^2 = \left(\frac{-2a}{3} \cdot \frac{-2a}{3}\right) - \frac{a^2}{9} = \frac{a^2}{3} > 0$$

$\therefore$ $f(x, y)$ is maximum at $\left(\frac{a}{3}, \frac{a}{3}\right)$

56. **Find the extreme values of** $f(x,y)=\sin x\sin y\sin(x+y),\ where\ 0<x<\frac{\pi}{2},\ 0<y<\frac{\pi}{2}.$

Solution:

Let $f(x,y)=\sin x\sin y\sin(x+y),$

$\therefore \frac{\partial f}{\partial x}=\sin y\ \sin x\ \cos(x+y)+\sin y\sin(x+y)\ \cos x$

$=\sin y\sin(2x+y)$

$\frac{\partial f}{\partial y}=\sin x\cos y\sin(x+y)+\sin x\sin y\cos(x+y)$

$=\sin x\sin(x+2y)$

$\frac{\partial^2 f}{\partial x^2}=2\sin y\cos(2x+y),\ \frac{\partial^2 f}{\partial x\partial y}=\sin y\cos(2x+y)+\sin(2x+y)\sin y$

$=\sin(2x+2y)$

$\frac{\partial^2 f}{\partial y^2}=2\sin x\cos(x+2y)$

Solve $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$

$\Rightarrow \sin y\ \sin(2x+y)=0$ and $\sin x\ \sin(x+2y)=0$

$\therefore 2x+y=\pi$

We get $x=\frac{\pi}{3}$ and $y=\frac{\pi}{3}$ so the critical point is $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$

Now $r=2\sin\frac{\pi}{3}\cos\pi=-\sqrt{3}<0$ and $rt-s^2=(-\sqrt{3})(-\sqrt{3})-\left(-\frac{\sqrt{3}}{2}\right)^2=\frac{9}{4}>0$

$\therefore\ f(x,\ y)$ is maximum at $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$

$\therefore\quad Max\ f(x,\ y)=\sin\frac{\pi}{3}\sin\frac{\pi}{3}\sin\frac{2\pi}{3}=\frac{3\sqrt{3}}{8}.$

57. **Examine the function** $f(x,y)=1+\sin(x^2+y^2)$ **for extreme values.**

Solution:

$f(x,y)=1+\sin(x^2+y^2)$

$\frac{\partial f}{\partial x}=\cos(x^2+y^2)\ (2x),\ \frac{\partial f}{\partial y}=\cos(x^2+y^2)\ (2y)$

$\frac{\partial^2 f}{\partial x^2}=2\cos(x^2+y^2)-\sin(x^2+y^2)\ .\ 4x^2,\ \frac{\partial^2 f}{\partial x\partial y}=-2x\sin(x^2+y^2)\ (2y)$

$\frac{\partial^2 f}{\partial y^2}=2\cos(x^2+y^2)-\sin(x^2+y^2)\ 4y^2$

Solve $\frac{\partial f}{\partial x}=0\ and\ \frac{\partial f}{\partial y}=0$

$\Rightarrow \cos(x^2+y^2)2x=0$

$\therefore\ x=0$ or $\cos(x^2+y^2)=0$ and $2y\cos(x^2+y^2)=0$

$\therefore\ x=0$ and $y=0$ or $\cos(x^2+y^2)=0$

$\therefore (0,0)$ and (a,b) are critical points such that $a^2+b^2=\frac{\pi}{2}$

Case (i) at $(0,0)$

$p=0, q=0, r=2, s=0, \ t=2$

Since $r>0$ and $rt-s^2=4-0=4>0$

$\therefore \ f(x,y)$ is minimum at $(0,0)$

Case (ii) at (a,b)

$p=\cos(a^2+b^2)\ 2a=\cos\frac{\pi}{2}\ .\ 2a \ \ p=0$

$q=\cos(a^2+b^2)\ 2b=0, \ r=2\cos(a^2+b^2)-\sin(a^2+b^2)4a^2=-4a^2$

$s=-4ab$ and $t=-4b^2$

Since $rt-s^2=0$, no conclusion can be made.

Further investigation is required.

But $f(a,b)=1+\sin(a^2+b^2)=2$

$\because a^2+b^2=\frac{\pi}{2}$

$\therefore$ Maximum value of $f(x,y)$ is 2.

58. **Find the extreme value of the function** $f(x,y)=x^3+3xy^2-3x^2-3y^2+4.$

Solution:

Let $f(x,y)=x^3+3xy^2-3x^2-3y^2+4.$

We get $f_x=3x^2+3y^2-6x$ $\quad f_y=6xy-6y$

$f_{xx}=6x-6,\quad f_{xy}=6y,\quad f_{yy}=6x-6$

Solving $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$

i.e., $3x^2+3y^2-6x=0$ ---(1)

and $6xy-6y=0$ ---(2)

From (2) we get $\Rightarrow x=1$ or $y=0$

Put $y=0$ in (1) $\Rightarrow 3x^2-6x=0$

$\Rightarrow 3x(x-2)=0$

$\Rightarrow x=0$ and $x=2$

The critical points: $(0,0)$ and $(2,0)$

Put $x=1$ in (1) $\Rightarrow 3y^2-3=0$

$\Rightarrow y=\pm 1$

The other critical points are $(1,1)$, $(1,-1)$

To examine the nature of the critical points observe the table.

Critical point	r	$rt-s^2$	**Nature of the critical point**
$(0,0)$	$-6<0$	$36>0$	Maxima
$(2,0)$	$6>0$	$36>0$	Minima
$(1,1)$	0	-36	Saddle point
$(1,-1)$	0	-36	Saddle point

$\therefore$ Max $f(x,y)=f(0,0)=4$ and Min $f(x,y)=f(2,0)=0$

59. **Examine the function** $f(x,y)=x^4+y^4-2(x-y)^2$ **for extreme values.**

Solution:

Given $f(x,y)=x^4+y^4-2(x-y)^2$

We have, $p=f_x=4x^3-4(x-y)$, $q=f_y=4y^3-4(x-y)$,

$r=f_{xx}=12x^2-4$, $s=f_{xy}=4$, $t=f_{yy}=12y^2+4$

Solving $p=0$ and $q=0$

$4x^3-4(x-y)=0$ and --- (1)

$4y^3+4(x-y)=0$ --- (2)

From (1) and (2), we get

$x^3+y^3=0$

i.e. $(x+y)(x^2-xy+y^2)=0$

$\therefore\ y=-x$ or $x^2-xy+y^2=0$

Put $y=-x$ in (1) $\Rightarrow x^3=x-y$

$\Rightarrow x^3=2x$

$\Rightarrow x(x^2-2)=0$

$\Rightarrow x=0$ or $x=\pm\sqrt{2}$

When $x=0$, $y=0$

$x=\sqrt{2}$, $y=-\sqrt{2}$ and $x=-\sqrt{2}$, $y=\sqrt{2}$

To examine the nature of the critical points the following table is considered.

Critical point	r	s	t
$(0,0)$	-4	4	-4
$(\sqrt{2},-\sqrt{2})$	20	4	20
$(-\sqrt{2},\sqrt{2})$	20	4	20

$\therefore$ Min $f(x,y)=f(-\sqrt{2},\sqrt{2})=4+4-2(8)=-8$.

60. **Examine the function** $f(x,y)=x^2y^2-5x^2-8xy-5y^2$ **for extreme values**.

Solution:

Let $f(x,y)=x^2y^2-5x^2-8xy-5y^2$

$p=f_x=2y^2x-10x-8y$, $q=f_y=2x^2y-8x-10y$

and $r=f_{xx}=2y^2-10$, $s=f_{xy}=4xy-8$, $t=f_{yy}=2x^2-10$

Solving $p=0$ and $q=0$

$2y^2x-10x-8y=0$ … (1) $2x^2y-8x-10y=0$ … (2)

From (1) and (2)

$x^2=y^2\Rightarrow x=\pm y$

Case 1:

Put $x=y$ in --- (1)

$\Rightarrow 2y^3-18y=0$

$\Rightarrow y^3-9y=0$

Case 2:

Put $x=-y$ in --- (1)

$\Rightarrow -2y^3+10y-8y=0$

$\Rightarrow -2y^3+2y=0$

$\Rightarrow y(y^2-9)=9$ $\qquad\qquad$ $\Rightarrow y^3-y=0$

$\Rightarrow y=0,\ \pm 3$ $\qquad\qquad$ $\Rightarrow y=0,\ \pm 1$

∴ The Critical points are $(0,0)$, $(3,3)$, $(-3,-3)$, $(1,-1)$, $(-1,1)$.

Consider the table for examining the nature of the critical points.

Critical point	r	s	t	$rt-s^2$
$(0,0)$	-10 < 0	-8	-10	36 > 0
$(3,3)$	8 > 0	28	8	-720 < 0
$(-3,-3)$	8 > 0	28	8	-720 < 0
$(1,-1)$	-8 < 0	-12	-8	-80 < 0

∴Max $f(x,y)=f(0,0)=0$.

61. Obtain the critical points for the function $f(x,y)=x^4+y^4-x^2-y^2+1$

Solution:

We get $p=f_x=4x^3-2x$, $q=f_y=4y^3-2y$,

$r=f_{xx}=12x^2-2$, $\qquad s=f_{xy}=0$, $\qquad t=f_{yy}=12y^2-2$

Now Solving $p=0$ and $q=0$

i.e. $4x^3-2x=0$ and $4y^3-2y=0$

$x=0,\ \pm\frac{1}{\sqrt{2}}$ or $y=0,\ \pm\frac{1}{2}$

Therefore the critical points are: $(0,0)$, $\left(\pm\frac{1}{\sqrt{2}},\ \pm\frac{1}{\sqrt{2}}\right)$, $\left(0,\ \pm\frac{1}{\sqrt{2}}\right)$, $\left(\pm\frac{1}{\sqrt{2}},\ 0\right)$

Lagrange's Method of Undetermined Multipliers

Let u = f(x,y,z)…(1) be a function of 3 variables x,y,z which are connected by the relation ϕ(x,y,z) =0…(2)

For u to have stationary values, if is necessary that

$$\frac{\partial u}{\partial x}=0,\ \frac{\partial u}{\partial y}=0,\ \frac{\partial u}{\partial z}=0 \qquad \ldots(3)$$

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=d\varphi=0 \qquad \ldots(4)$$

Multiply (4) by a parameter λ and add to (3)

$$\left(\frac{\partial u}{\partial x}+\lambda\frac{\partial\varphi}{\partial x}\right)dx+\left(\frac{\partial u}{\partial y}+\lambda\frac{\partial\varphi}{\partial y}\right)dy+\left(\frac{\partial u}{\partial z}+\lambda\frac{\partial\varphi}{\partial z}\right)dz=0$$

This equation will be satisfied if $\frac{\partial u}{\partial x}+\lambda\frac{\partial\varphi}{\partial x}=0;\ \frac{\partial u}{\partial y}+\frac{\lambda\partial\varphi}{\partial y}=0;\ \frac{\partial u}{\partial z}+\lambda\frac{\partial\varphi}{\partial z}=0$

These three equations together with (2) will determine the values of x, y, z and λ for which u is stationary.

Working Rule

- Write F = f(x, y, z) + λϕ(x, y, z)
- Obtain the equations $\frac{\partial f}{\partial x}=0,\ \frac{\partial f}{\partial y}=0,\ \frac{\partial f}{\partial z}=0$
- Solve the above equations together with ϕ(x, y, z) = 0
- The values of x, y, z so obtained will give the stationary value of f(x,y, z)

62. Find the stationary value of $x^2y^3z^4$ subject to the condition x + y + z = 5.

Solution:

$Let\ f = x^2y^3z^4$ and $\varphi(x,y,z) = x+y+z$

$x+y+z=5 \qquad ....(1)$

$Let\ F = f+\lambda\varphi \Rightarrow F = x^2y^3z^4 + \lambda(x+y+z)$

$F_x = 2xy^3z^4+\lambda=0 \qquad ....(2)\,, F_y = 3x^2y^2z^4+\lambda=0 \qquad ....(3)\ \ F_z = 4x^2y^3z^3+\lambda=0 \qquad ....(4)$

$(2)\Rightarrow \lambda = -2xy^3z^4,\ \ (3)\Rightarrow \lambda=-3x^2y^2z^4,\ (4)\Rightarrow \lambda = -4x^2y^3z^3$

From (2) and (3) $\ -2xy^3z^4 = -3x^2y^2z^4 \Rightarrow y = \dfrac{3}{2}x$

$From\ (2)\ and\ (4)\ -2xy^3z^4 = -4x^2y^3z^4 \Rightarrow z = 2x$

Using this in (1) $\ x+\dfrac{3x}{2}+2x = 5 \Rightarrow 9x = 10$

$\therefore\ x = \dfrac{10}{9},\ y=\dfrac{5}{3},\ z=\dfrac{20}{9}$.The stationary value of f(x, y, z) is $\left(\dfrac{10}{9}\right)^2.\left(\dfrac{5}{3}\right)^2.\left(\dfrac{20}{9}\right)^4$

63. The temperature T at any point (x, y, z) in space is given by T = 400 xyz^2. Find the highest temperature at the surface of a unit sphere $x^2 + y^2 + z^2 = 1$.

Solution: $Let\ \varphi(x,y,z) = x^2+y^2+z^2$

$x^2+y^2+z^2=1 \qquad ....(1)$

$T = 400\,xyz^2 \qquad Let\ F = T+\lambda\varphi \ \ \therefore F = 400\,xyz^2+\lambda(x^2+y^2+z^2)$

$F_x = 400\,yz^2+2x\lambda=0 \qquad ....(2)$

$F_y = 400\,xz^2+2y\lambda=0 \qquad ....(3)\ F_z = 800\,xyz+2z\lambda=0 \qquad ....(4)$

$(2)\Rightarrow \lambda = \dfrac{-200\,yz^2}{x}, (3)\Rightarrow \lambda = \dfrac{-200\,xz^2}{y}, (4)\Rightarrow \lambda = -400\,xy$

$From\ (2)\ and\ (3)\ \dfrac{-200\,yz^2}{x} = \dfrac{-200\,xz^2}{y} \Rightarrow x^2 = y^2 \Rightarrow x = y$

$From\ (2)\ and\ (4)\ \dfrac{-200\,yz^2}{x} = -400\,xy \Rightarrow z^2 = 2x^2 \Rightarrow z = \sqrt{2}x$

$(1)\Rightarrow\ x^2+x^2+2x^2 = 1 \Rightarrow 4x^2 = 1 \qquad or\ x = \dfrac{1}{2}, \qquad \therefore y = \dfrac{1}{2},\ z = \dfrac{1}{\sqrt{2}}$

Critical point is : $\left(\dfrac{1}{2},\dfrac{1}{2},\dfrac{1}{\sqrt{2}}\right)$ The maximum temperature is $= 400\times\dfrac{1}{2}\times\dfrac{1}{2}\times\dfrac{1}{2} = 50\ units$

64. Find the maximum & minimum distances of the point (1,2,3) from the sphere $(x^2 + y^2+z^2) = 56$.

Solution:

Let A = (1, 2, 3)

Let P (x, y, z) be a point on the sphere $(x^2 + y^2 + z^2) = 56 \qquad ….(1)$

Let $\varphi = x^2+y^2+z^2$

$Let\ f = (AP)^2 = (x-1)^2+(y-2)^2+(z-3)^2$

Now $F = f+\lambda\varphi \Rightarrow F = (x-1)^2+(y-2)^2+(z-3)^2+\lambda(x^2+y^2+z^2)$

$F_x = 2(x-1)+2\lambda x = 0 \qquad ....(2)$

$F_y = 2(y-2)+2\lambda y = 0 \qquad ....(3)$

$F_z = 2(z-3) + 2\lambda z = 0$ ....(4)

$(2) \Rightarrow \lambda = \frac{1-x}{x}, (3) \Rightarrow \lambda = \frac{2-y}{y}, (4) \Rightarrow \lambda = \frac{3-z}{z}$

From (2) *and* (3)

$\frac{1-x}{x} = \frac{2-y}{y}$

$\Rightarrow y = 2x$

From (2) *and* (4)

$\frac{1-x}{x} = \frac{3-z}{z}$

$\Rightarrow \quad z = 3x$

Using in (1) $x^2 + 4x^2 + 9x^2 = 56$ or x = ± 2

$\therefore$ y = 4, z = 6 for x = 2, y = -4, z = -6 for x = -2

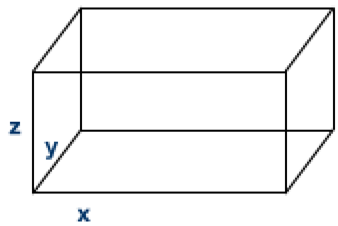
The critical points are (2, 4, 6) and (-2, -4, -6).

At (2, 4, 6) $f_{\min} = 14$ and the minimum distance is $\sqrt{14}$

At (-2, -4, -6) $\therefore$ f_{max} = 126 and the maximum distance is $3\sqrt{14}$.

65. A rectangular box open at the top is to have a volume of 32 cubic feet. Find its dimension if the total surface area is minimum.

Solution:



Let x, y and z be the dimensions of the box.
Given: xyz = 32 ….(1)
$\therefore$ ϕ(x,y, z)= xyz
Let f = xy +2yz + 2xz (be the surface area)
Let F = f + $\lambda\phi$= xy + 2yz + 2xz + λ xyz
Solve $F_x = y + 2z + \lambda yz = 0$ ....(2)

$F_y = x + 2z + \lambda xz = 0$ ....(3)

$F_z = 2y + 2x + \lambda xy = 0$ ....(4)

$(2) \Rightarrow \lambda = -\frac{(y+2z)}{yz}, (3) \Rightarrow \lambda = -\frac{(x+2z)}{xz}, (4) \Rightarrow \lambda = -\frac{2(x+y)}{xy}$

From (2) *and* (3)

$\frac{y+2z}{yz} = \frac{x+2z}{xz}$

$xyz + 2xz^2 = xyz + 2z^2 y$

Or $x = y$

From (2) *and* (4)

$\frac{y+2z}{yz} = \frac{2x+2y}{xy}$

$xy^2 + 2xyz = 2xyz + 2y^2 z$

or $x = 2z$

Using in (1) x = 4, y = 4,z = 2

66. Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution:

Let f = xyz (be the volume)

Given: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ....(1) $\therefore \varphi(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$

Let F = f + λϕ$\Rightarrow F = xyz+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$

From the equations $F_x = 0$, $F_y = 0$, $F_z = 0$.

$F_x = yz+\lambda\frac{2x}{a^2}=0$ ....(2) , $F_y = xz+\lambda\frac{2y}{b^2}=0$ ....(3) , $F_z = xy+\lambda\frac{2z}{c^2}=0$ ....(4)

From (2) $\Rightarrow \lambda = -\frac{a^2yz}{2x}$,From (3) $\Rightarrow \lambda = -\frac{b^2xz}{2y}$,From (4) $\Rightarrow \lambda = -\frac{c^2xy}{2z}$

From (2) *and* (3) *From* (2) *and* (4)

$\frac{a^2yz}{2x}=\frac{b^2xz}{2y}$ $\frac{a^2yz}{x}=\frac{c^2xy}{z}$

$\Rightarrow a^2y^2z=b^2x^2z$ $\Rightarrow a^2yz^2=c^2x^2y$

$\Rightarrow y=\frac{b}{a}x$ $\Rightarrow z=\frac{c}{a}x$

Using in (1) $x=\frac{a}{\sqrt{3}}$, $y=\frac{b}{\sqrt{3}}$, $z=\frac{c}{\sqrt{3}}$ $Volume = f = \frac{a}{\sqrt{3}}\frac{b}{\sqrt{3}}\frac{c}{\sqrt{3}}=\frac{abc}{3\sqrt{3}}$

Vectors

Definitions

Vector is a quantity which has both magnitude and direction. Vector quantities like force, velocity, acceleration etc. have lot of significance in physical and engineering fields.
A quantity which has only magnitude but no direction is called a scalar e.g. Length, Volume etc.
If $\vec{a}=\vec{AB}$ is a vector, then the magnitude of the vector is $\left|\vec{a}\right| = AB$. The magnitude of the vector is also known as length or module of the vector.
If the magnitude of the vector is one unit, then it is called a unit vector. A unit vector in the direction of the vector $\vec{a}$ is usually denoted by $\hat{a}$ and is given by $\hat{a}=\frac{\vec{a}}{\left|\vec{a}\right|}$.

Vectors lying in the same plane are called the co-planer vectors.

The position vector of a point P w.r.t. the fixed point O is the vector $\vec{r}=\vec{OP}$.

Position vector of a point is also written as $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along x, y and z axes respectively.

Scalar Product

- $\vec{a}\vec{b}=a_1b_1+a_2b_2+a_3b_3=|\vec{a}||\vec{b}|\cos\theta$, θ is the angle between $\vec{a}$ and $\vec{b}$
- if $\theta=90^0$ then $\vec{a}.\vec{b}=0$
- $\hat{i}.\hat{i}=1$, $\hat{j}.\hat{j}=1$, $\hat{k}.\hat{k}=1$, $\hat{i}.\hat{j}=0$, $\hat{j}.\hat{k}=0$, $\hat{k}.\hat{i}=0$.

Projection

- Projection of $\vec{a}$ on $\vec{b}$ is $\vec{a}\,\hat{b}$

Vector Product

- $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where $\hat{n}$ is the unit vector $\perp$ to both $\vec{a}$ and $\vec{b}$

- $\hat{i}\times\hat{i}=\vec{0}$, $\hat{j}\times\hat{j}=\vec{0}$, $\hat{k}\times\hat{k}=\vec{0}$, $\hat{i}\times\hat{j}=\hat{k}$, $\hat{j}\times\hat{k}=\hat{i}$, $\hat{k}\times\hat{i}=\hat{j}$, $\hat{i}\times\hat{k}=-\hat{j}$, $\hat{j}\times\hat{i}=-\hat{k}$ and $\hat{i}\times\hat{k}=-\hat{j}$

Scalar Triple Product

- $[\vec{a},\vec{b},\vec{c}] = \vec{a}.(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & c_2 & b_3 \\ c_1 & c_3 & c_3 \end{vmatrix}$

- Equation of line passing through (x_1, y_1, z_1) and direction ratios a,b,c are $\dfrac{x-x_1}{a}=\dfrac{y-y_1}{b}=\dfrac{z-z_1}{c}$

- Equation of a plane: $ax+by+cz+d=0$ where a,b,c are the d r's of the normal.

Vector and Scalar Functions and Fields

Let the position vector of a point P(x, y, z) in space be $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$. If x, y, z are all functions of a single parameter 't', then $\vec{r}$ is said to be a vector function of 't' (also called vector point function) denoted as $\vec{r}=\vec{r}(t)$.

$\therefore$ Vector functions whose values are vectors.

$v=v(P)=\left[v_1(p),\ v_2(p),\ v_3(p)\right]$ Is depending on the points P in space.

Scalar valued function is a function that takes one or more values but returns a single value.

For example: $f(x,\ y,\ z)\ =\ x^2+\ 2yz^5$ is a scalar valued function. A variable scalar valued function acts as a map from the space R^n to the real number line i.e. $f:R^n\longrightarrow R$

Therefore, Scalar functions, whose values are scalars $f=f(P)$, is depending on P. In applications, the domain of definition for such a function is a region of space or a surface in space or a curve in space. A vector function defines a vector field in that region (or on that surface or curve). A scalar function defines a scalar field in a region or on a surface or a curve.

Scalar field

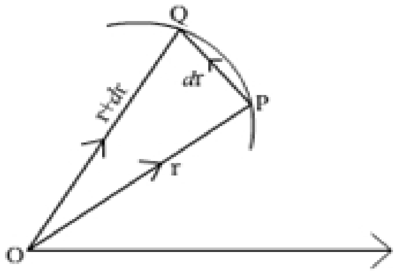
The temperature at any instant, Density of a body, Potential due to gravitational matter

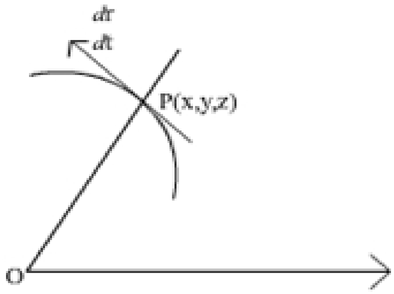
Vector Field

Velocity of a moving fluid at any instant the gravitational intensity of force.

Derivative of a Vector Function

A vector function $\vec{r}(t)$ $[or\ \vec{v}(t)]$ is said to be differentiable at a point t if the following limit exists:





$$\vec{r}\,'(t) = \frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{\vec{r}(t+\delta t) - \vec{r}(t)}{\delta t}$$

The vector $\vec{r}\,'(t)$ is called the derivative of $\vec{r}(t)$.In terms of components w.r.t a given Cartesian coordinate system, $\vec{r}(t)$ is differentiable at a point tiff its three components are differentiable at t, and then the derivative $\vec{r}\,'(t)$ is obtained by differentiating each component separately, $r'(t) = \left[x'(t), y'(t), z'(t)\right]$.

If follows that the familiar rules of differentiation yield corresponding rules for differentiating vector functions. $(C\vec{r})' = C\vec{r}\,'$ and $(\vec{r}_1 + \vec{r}_2)' = \vec{r}_1\,' + \vec{r}_2\,'$

In particular

$$(\vec{r}_1 . \vec{r}_2)' = \vec{r}_1\,' . \vec{r}_2 + \vec{r}_1 . \vec{r}_2\,'$$

$$(\vec{r}_1 \times \vec{r}_2)' = \vec{r}_1\,' \times \vec{r}_2 + \vec{r}_1 \times \vec{r}_2\,'$$

$$\left(\vec{r}_1\, \vec{r}_2\, \vec{r}_3\right)' = \left(\vec{r}_1\,'\ \vec{r}_2\, \vec{r}_3\right) + \left(\vec{r}_1\, \vec{r}_2\,'\ \vec{r}_3\right) + \left(\vec{r}_1\, \vec{r}_2\, \vec{r}_3\,'\right)$$

[The order of the vectors must be carefully observed because cross product is not commutative].

Derivative of a vector function $\vec{r}(t)$ constant length is either the zero vector or is $\perp^r$ to $\vec{r}(t)$.

$\because$ Let $|\vec{r}| = C$, then $|\vec{r}|^2 = \vec{r}.\vec{r} = C^2$ and $(\vec{r}.\vec{r})' = 2(\vec{r}.\vec{r}) = 0$

Definition of Gradient

The gradient grad f of a given scalar function $f(x, y, z)$ is the vector function defined by $grad\ f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

Here, we assume that f is differentiable. It has become popular with physicists and engineers to introduce the differential operator.

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial t}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Read ∇ as or del and write $grad f$ or ∇f $grad\ f = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$ … (2)

67. If $f(x, y, z) = 2x + yz - 3y$ then find $grad\ f$.

Solution:

Given $f(x, y, z) = 2x + yz - 3y$

Then $grad\ f = \nabla f$

$$= \hat{i}\frac{\partial}{\partial x}(2x+yz-3y^2)+\hat{j}\frac{\partial}{\partial y}(2x+yz-3y^2)+\hat{k}\frac{\partial}{\partial z}(2x+yz-3y^2)$$

$$= 2\hat{i}+(z-6y)\hat{i}+y\hat{k}$$

Divergence of a Vector Field

Let $\vec{v}(x,\ y,\ z)$ be a differentiable vector function, where x, y, z are Cartesian Coordinates and let $v_1,\ v_2,\ v_3$ be the components of $\vec{v}$.

Then the function $div.\vec{v} = \frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}$ … (1) is called the divergence of $\vec{v}$. Another common notation for the divergence of $\vec{v}$ is $\nabla.\vec{v}$

$div.\vec{v} = \nabla.\vec{v} = \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right).(v_1\hat{i}+v_2\hat{j}+v_3\hat{k}) = \frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}$ with the understanding that the "product" $\left(\frac{\partial}{\partial x}\right)v_1$ in the dot product means the partial derivative $\frac{\partial v_1}{\partial x}$ etc.

Remark: $\nabla.\vec{v}$ means the scalar $div.\vec{v}$, where as ∇f means the vector $grad.f$.

68. If $\vec{v} = 3xz\hat{i}+2xy\hat{j}-yz^2\hat{k}$ **then find** $div\ \vec{v}$ **.**

Solution:

$$\vec{v} = 3xz\hat{i}+2xy\hat{j}-yz^2\hat{k}$$

$$div\,\vec{v} = \frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}$$

$$= 3z+2x-2yz$$

Physical Meaning of Divergence

The values of $div.\vec{v}$ depend only on the points in space (and, of course on $\vec{v}$) but not on the particular choice of the coordinates in (1), so that w.r.t other Cartesian coordinates $x^*y^*z^*$ and corresponding components v_1^*,v_2^*,v_3^*

$$div\,\vec{v} = \frac{\partial v_1^*}{\partial x^*}+\frac{\partial v_2^*}{\partial y^*}+\frac{\partial v_3^*}{\partial z^*}$$

Thus, if $\vec{v}(x,y,z)$ represents any physical quantity, the divergence of $\vec{v}$ gives the rate at which the physical quantity is originating at that point per unit volume.

Let us suppose that a fluid is moving such that its velocity at any point P(x, y, z) is given by the vector point function $\vec{v}(x,y,z)$ consider a small parallelepiped of volume $\delta x\delta y\delta z$ through which the fluid is passing.

If $\vec{v}(x,y,z)=v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$

Then $div\,\vec{v}=\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}$ gives the total gain in the volume of the fluid per unit volume per unit time.

$div\,\vec{v}=0$ is called as the continuity equation of an incompressible fluid.

A vector $\vec{v}$ whose divergence is zero is called a solenoidal vector.

Remarks

If $f(x,y,z)$ is a twice differentiable scalar function, then $grad\ f=\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}$ and by

(1) $div(grad f)=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\Rightarrow div\left(grad\ f\right)=\nabla^2 f$

$\nabla^2 f$ is called the Laplacian of f.

Curl of a Vector Field

Let x,y,z be right-handed Cartesian coordinates and let $\vec{v}(x,y,z)=v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$ be a differentiable vector function. Then curl of the vector field denoted by $Curl\,\vec{v}=\nabla\times\vec{v}$ is defined as

$$Curl\,\vec{v}=\nabla\times\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_1 & v_2 & v_3\end{vmatrix}=\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)\hat{i}+\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)\hat{j}+\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)\hat{k}$$

69. If $\vec{v}=yz\,\hat{i}+3z\,\hat{j}+z\,\hat{k}$ find $Curl\,\vec{v}$ with respect to right-handed Cartesian coordinates.

Solution:

Let $\vec{v}=yz\,\hat{i}+3z\,\hat{j}+z\,\hat{k}$ then $curl\,\vec{v}=\begin{vmatrix}i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ yz & 3zx & z\end{vmatrix}$

$$=-3x\,\hat{i}+y\,\hat{j}+2z\,\hat{k}$$

Remark

For any twice differentiable scalar field f, $Curl\left(grad\ f\right)=0$ … (2)

i.e. If a vector function is the gradient of a scalar function, its curl is the zero vector.

Since the curl characterizes the rotation in a field, we also say more briefly that gradient fields describing a motion are irrotational. Also if $\vec{v}$ is the velocity of a particle in a rigid body rotating about a fixed axes with uniform angular velocity then $\frac{1}{2}curl\,\vec{v}$ is equal

Some basic formulae for grad, div and curl:

i. $\nabla(fg)=f\nabla g+g\nabla f$, $\nabla\left(\frac{f}{g}\right)=\left(\frac{1}{g^2}\right)\left(g\nabla f-f\nabla g\right)$

ii. $div(f\,\vec{v})=f\,div\,\vec{v}+v.\nabla f$, $div(f\nabla g)\ =\ f\nabla^2 g\ +\nabla g.\nabla g$

iii. $\nabla^2 f = div(\nabla f), \nabla^2(fg) = g\nabla^2 f + 2\nabla f.\nabla g + f\nabla^2 g$

iv. $curl(f\vec{v}) = \nabla f \times \vec{v} + f\, curl\,\vec{v}, div(\vec{u}\times\vec{v}) = \vec{v}.curl\,\vec{u} - \vec{u8}.curl\,\vec{v}$

v. $Curl(\nabla f) = 0$, $div(curl\,\vec{v}) = 0$.

Here f and g are scalar fields.

Remark: A vector field $\vec{v}$ is solenoidal if $div\,\vec{v} = 0$ and irrotational if $curl\,\vec{v} = \vec{0}$

70. Find the unit vector normal to the surface $xy^3z^2 = 4\ at\ (-1, -1, 2)$

Solution:

Let $f = xy^3z^2$ $Grad\ f$ (or ∇f) is a vector normal to the surface.

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = y^3z^2\hat{i} + 3xy^2z^2\hat{j} + 2xy^3z\hat{k}$$

$$\therefore (\nabla f)_{(-1,-1,2)} = -4\hat{i} - 12\hat{j} + 4\hat{k} = -4(\hat{i} + 3\hat{j} - \hat{k})$$

Hence the required unit vector normal $\hat{n} = \frac{\nabla f}{|\nabla f|}$ i.e., $\hat{n} = \frac{-4(\hat{i}+3\hat{j}-\hat{k})}{\sqrt{4^2(1^2+3^2+1^1)}} = -\frac{(\hat{i}+3\hat{j}-\hat{k})}{\sqrt{11}}$

71. Find the directional derivative of $f = x^2yz + 4xz^2 at\ (1,-2,-1)$ **along** $2\hat{i} - \hat{j} - 2\hat{k}$

Solution:

Let $f = x^2yz + 4xz^2$

$$\therefore \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = (2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y + 8xz)\hat{k}$$

$$(\nabla f)_{(1,-2,-1)} = 8\hat{i} - \hat{j} - 10\hat{k}$$

The unit vector in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$ is $\hat{n} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4+1+4}} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{3}$

$\therefore$ The required directional derivative is $\nabla f.\hat{n} = (8\hat{i} - \hat{j} - 10\hat{k}).\frac{(2\hat{i} - \hat{j} - 2\hat{k})}{3}$

$$= \frac{37}{3}$$

72. In which direction the directional derivative of x^2yz^3 **is maximum at** $(2,1,-1)$ **and find the magnitude of this maximum.**

Solution:

We know that the directional derivative is maximum along the normal vector ∇f .

Let $f = x^2yz^3$ $\therefore \nabla f = 3xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$

$[\nabla f]_{(2,1,-1)} = -4i - 4j + 12k$ which is the direction in which the directional derivative is maximum and its magnitude is $\sqrt{4^2(1+1+9)} = 4\sqrt{11}$.

73. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ **at the point** $(2, -1, 2)$.

Solution:

The angle between the surfaces is the angle between their normals.

We know ∇f is a vector normal to the surface.
Equation of two surfaces be $f_1 = x^2 + y^2 + z^2$ and $f_2 = x^2 + y^2 - z$

$$\therefore \nabla f_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k} \text{ and } \nabla f_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$(\nabla f_1)_{(2,-1,2)} = 4\hat{i} - 2\hat{j} + 4\hat{k} \text{ and } (\nabla f_2)_{(2,-1,2)} = 4\hat{i} - 2\hat{j} - \hat{k} = 2(2\hat{i} - \hat{j} + 2\hat{k})$$

Let θ be the angle between the normals,

$$\therefore \cos\theta = \frac{\nabla f_1 . \nabla f_2}{|\nabla f_1||\nabla f_2|} = \frac{2(8+2-2)}{\sqrt{2^2(4+1+4)}\sqrt{(16+4+1)}} = \frac{8}{3\sqrt{21}}$$

or $$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

74. If f and g are scalar point functions of x, y, z prove that

a) $\nabla(fg) = f\nabla g + g\nabla f$

b) $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}, g \neq 0$

Solution:

a) $$\nabla(fg) = \sum \frac{\partial}{\partial x}(fg)\hat{i} = \sum\left(\mathrm{f}\frac{\partial \mathrm{g}}{\partial \mathrm{x}} + \mathrm{g}\frac{\partial \mathrm{f}}{\partial \mathrm{x}}\right)\hat{\mathrm{i}} = f\sum\frac{\partial g}{\partial x}\hat{i} + g\sum\frac{\partial f}{\partial x}\hat{i} = f(\nabla g) + g(\nabla f)$$

b) $$\nabla(\mathrm{f}/\mathrm{g}) = \sum\frac{\partial}{\partial \mathrm{x}}\left(\frac{\mathrm{f}}{\mathrm{g}}\right)\hat{\mathrm{i}} = \sum\frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}\hat{i} = \frac{1}{g^2}\left\{g\sum\frac{\partial f}{\partial x}\hat{i} - f\sum\frac{\partial g}{\partial x}\hat{i}\right\} = \frac{1}{g^2}\{g\nabla f - f\nabla g\}$$

75. Find $div.\vec{v}$ and $Curl\,\vec{v}$ where $\vec{v} = \nabla(x^3 + y^3 + z^3 - 3xyz)$.

Solution:

Let $f = x^3 + y^3 + z^3 - 3xyz$

$$\therefore \vec{v} = \nabla f = (3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}$$

$$div\,\vec{v} = \nabla.\vec{v}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left\{(3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}\right\}$$

$$= \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy).$$

$$\therefore div\,\vec{v} = 6x + 6y + 6z = 6(x + y + z)$$

Also $$curl\,\vec{v} = \nabla \times \vec{v}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x^2 - 3yz) & (3y^2 - 3xz) & (3z^2 - 3xy) \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3y^2 - 3xz)\right\} - \hat{j}\left(\frac{\partial}{\partial x}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3x^2 - 3yz)\right)$$

$$+\hat{k}\left(\frac{\partial}{\partial x}3y^2-3xz)-\frac{\partial}{\partial y}(3x^2-3yz)\right)$$

$$=\hat{i}\{-3x-(-3x)\}-\hat{j}\{-3y-(-3y)\}+\hat{k}\{-3z-(-3z)\}$$

$$=\vec{0}$$

76. If $\vec{A}$ is a constant vector, prove that div $(\vec{A}\times\vec{r})=0$.

Solution:

Let $\vec{A}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ be a constant vector and $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

$$\therefore \vec{A}\times\vec{r}=\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}=\sum \hat{i}(a_2z-a_3y)$$

$$\therefore div(\vec{A}\times\vec{r})=\nabla.(\vec{A}\times\vec{r})=\left(\sum\frac{\partial}{\partial x}i\right).\sum i(a_2z-a_3y)=\sum\frac{\partial}{\partial x}(a_2z-a_3y)\ =0$$

$$\therefore div(\vec{A}\times\vec{r})=0$$

77. Show that $\nabla^2[f(r)]=f''(r)+\frac{2}{r}f'(r)$ with usual meanings and hence deduce the expressions for $\nabla^2\left(\mathbf{e^r}\right)$ and $\nabla^4\left(\mathbf{e^r}\right)$.

Solution:

Since $r^2=x^2+y^2+z^2 \Rightarrow \frac{\partial r}{\partial x}=\frac{x}{r}$

$$=\nabla^2 f(r)\sum\frac{\partial^2}{\partial x^2}[f(r)]=\sum\frac{\partial}{\partial x}\frac{\partial}{\partial x}[f(r)]=\sum\frac{\partial}{\partial x}\left\{f^1(r)\frac{\partial r}{\partial x}\right\}=\sum\frac{\partial}{\partial x}\left\{f'(r)\frac{x}{r}\right\}$$

$$=\sum\left\{f'(r)\left[\frac{r-x\frac{\partial r}{\partial x}}{r^2}\right]+f''(r)\frac{\partial r}{\partial x}\frac{x}{r}\right\}=\sum f'(r)\left[\frac{r-x\left(\frac{x}{r}\right)}{r^2}\right]+\sum f''(r)\frac{x}{r}.\frac{x}{r}$$

$$=\sum\frac{f'(r)}{r^3}[r^2-x^2]+\sum f''(r)\frac{x^2}{r^2}=\frac{f'(r)}{r^3}=(3r^2-r^2)+\frac{f''(r)}{r^2}.r^2$$

$$=\frac{f'(r)}{r^3}2r^2+f''(r)=\frac{2}{r}f'(r)+f''(r)$$

$$\nabla^2[f(r)]=\frac{2}{r}f'(r)+f''(r)$$

Let $f(r)=e^r$

$$\therefore f'(r)\ =\ e^r=f''(r)$$

$$\therefore \nabla^2[f(r)]=\frac{2}{r}e^r+e^r=e^r\left(\frac{2}{r}+1\right)$$

Also $\nabla^2\nabla^4\left(e^r\right)\ =\nabla^2\nabla^2\left(e^r\right)\ =\left[e^r\left(\frac{2}{r}+1\right)\right]$

Taking $f(r)=e^r\left(\frac{2}{r}+1\right)$, $f'(r)=e^r\left(\frac{-2}{r^2}\right)+e^r\left(\frac{2}{r}+1\right)$, $f''(r)==\frac{4}{r^3}e^r-\frac{4}{r^2}e^r+\frac{2}{r}e^r+e^r$

$$\therefore \nabla^2\left\{e^r\left(\frac{2}{r}+1\right)\right\}=\frac{2}{r}\left\{\frac{-2e^r}{r^2}+\frac{2e^r}{r}+e^r\right\}$$

$$\Rightarrow \nabla^2\left\{\nabla^2(e^r)\right\} = e^r\left(\frac{4}{r}+1\right).$$

78. **Show that** $\vec{F} = (2xy^2 + yz)\hat{i} + (2x^2y + xy + 2yz^2)\hat{j} + (2y^2z + xy)\hat{k}$ **is a conservative force field. Find its Scalar potential.**

Solution:

$$curl\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2xy^2 + yz) & (2x^2y + xy + 2yz^2) & (2y^2z + xy) \end{vmatrix}$$

Hence $\vec{F}$ is a conservative force field.

Now consider $\vec{F} = \nabla\varphi$, where φ a scalar potential

To determine, we have

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k} = (2xy^2 + yz)\hat{i} + (2x^2y + xy + 2yz^2)\hat{j} + (2y^2z + xy)\hat{k}$$

Therefore,

$$\frac{\partial\varphi}{\partial x} = 2xy^2 + yz \quad \frac{\partial\varphi}{\partial y} = 2x^2y + xy + 2yz^2 \quad \frac{\partial\varphi}{\partial z} = 2y^2z + xy$$

Which gives

$$\varphi = x^2y^2 + xyz + f_1(y,z) \quad , \varphi = x^2y^2 + xyz + y^2z^2 + f_2(x,z)$$

$$\varphi = y^2z^2 + xyz + f_3(x,y)$$

Choose $f_1(y,z) = y^2z^2$, $f_2(x,z) = 0$, $f_3(x,y) = x^2y^2$ .Thus

$\varphi = x^2y^2 + xyz + y^2z^2$ is the required scalar potential.

Exercise

Sl.No	Questions	Answers
1	Find the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ when $u=3x+5y, v=4x-3y$	-29
2	Find the Jacobian $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ if $u=x^2, v=\sin y, w=e^{-3z}$	$-6e^{-3z}x\cos y$
3	Calculate Jacobian of u,v,w w.r.t. x,y,z when $u=\frac{yz}{x}, v=\frac{zx}{y}, w=\frac{xy}{z}$.	4
4	If $X=u^2v, Y=uv^2$ and $u=x^2-y^2, v=xy$ find $\frac{\partial(X,Y)}{\partial(x,y)}$ $\left(H\text{ int}: use\ \frac{\partial(X,Y)}{\partial(x,y)}=\frac{\partial(X,Y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(x,y)}\right)$	$6x^2y^2(x^2+y^2)(x^2-y^2)^2$
5	Find the directional derivative of $f(x,y,z)=4e^{2x-y+z}$ at the point (1,1,-1) in the direction toward the point (-3,5,6)	$\frac{-20}{9}$
6	Find the directional derivative of $f=xy+yz+zx$ in the direction of vector $i+2j+2k$ at the point (1,2,0)	$\frac{10}{3}$
7	Find the curl of $\vec{V}=e^{xyz}(i+j+k)$ at the point (1,2,3)	$e^6(i-4j+3k)$
8	Calculate $\nabla^2 f$ when $f=3x^2z-y^2z^3+4x^3y+2x-3y-5$ at the point (1,1,0)	24
9	Evaluate divergence of $(2x^2zi-xy^2zj+3yz^2k)$ at the point (1,1,1)	8
10	Show that $\vec{A}=3y^4z^2i+4x^3z^2j-3x^2y^2k$ is solenoidal	0
11	Prove that $(y^2-z^2+3yz-2x)i+(3xz+2xy)j+(3xy-2xz+2z)$ is both solenoidal and irrotational.	0
12	Find the maximum and minimum values of $f(x,y)=x^3+3xy^2-15x^2-15y^2+72x.$	Maximum value = $2\frac{7}{16}$
13	Find the maximum and minimum values of $3x^4-2x^3-6x^2+6x+1$ in the internal (0,2).	Minimum value = 2
14	If $u=\frac{y}{z}+\frac{z}{x}$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0.$	0
15	If $v=\log(x^2+y^2+z^2)$, prove that $(x^2+y^2+z^2)\left(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}\right)=2$	2
16	If $u=e^{xyz}$, find the value of $\frac{\partial^3u}{\partial x\partial y\partial z}$	$e^{xyz}(x^2y^2z^2+3xyz+1)$
17	Show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u\log u$, where $u=e^{x^2+y^2}$	
18	If $u=\tan^{-1}\frac{x^3+y^3}{x+y}$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\sin 2u.$ and $x^2\frac{\partial^2u}{\partial x^2}+2xy\frac{\partial^2u}{\partial x\partial y}+y^2\frac{\partial^2u}{\partial y^2}=2\cos 3u\sin 2u$	

Module-4
ORDINARY DIFFERENTIAL EQUATIONS EQUATION

- Exact
- Linear and Bernoulli's differential equations
- Higher order linear differential equations with constant coefficients
- Method of variation of parameters
- Cauchy and Legendre's differential equation
- Power series solution of differential equation

Introduction

Engineering design focuses on the use of models in developing predictions of natural phenomena. These models are developed by determining relationships between key parameters of the problem. Usually, it is difficult to find immediately the functional dependence between needed quantities in the model; at the same time, often, it is easy to establish relationships for the rates of change of these quantities using empirical laws.

For example, if we are asked to find the path x(t) of a particle of mass m moving under a given time-dependent force), $F(t)$ it is not easy to find it directly, however, Newton's second law (acceleration is proportional to the force) gives a differential equation describing this motion: $m\frac{d^2x}{dt^2} = F(t)$.

The solution of the above equation helps us to establish the dependence of path x(t) on the acting force. Likewise, in many cases, the governing equation for a physical model can be expressed in the form of a differential equation.

"There is nothing permanent except change" explains that differential equations are of fundamental importance. Such relationships form the basis for studying phenomena in the field of science, engineering and businesses.

In the study of differential equations we go through three phases:

The formulation differential equation from the given physical phenomena, called mathematical modelling.

The next phase is to find the solution of differential equation and then to evaluate the arbitrary constants from the given conditions.

Physical interpretation of the obtained solution.
In the unit 1 ,we have seen that how different phenomena such as cooling or warming of body, population growth, radioactive decay, mass spring systems, series electric circuits can be formulated into differential equations.

In this unit we discuss the various methods of obtaining solutions of differential equations of first order.

Basic Definitions

A differential equation is an equation involving an unknown function of one or more independent variables and its derivatives.
Differential equations are classified into two types according as whether dependent variable is a function of a single independent variable or it is a function of more than one independent variable.

Ordinary Differential Equation

An ordinary differential equation (O.D.E) is an equation in which the dependent variable depends only on single independent variable. The general form of an O.D.E is $F\left(x, y(x), \frac{dy}{dx}, \frac{d^2y}{dx^2}, ... \frac{d^n y}{dx^n}\right) = 0$ .Here x is independent variable and y is the dependent variable.

Note

The first derivative $\frac{dy}{dx}$ can be denoted as y' , the second order derivative $\frac{d^2y}{dx^2}$ can be denoted as y'' & so on.

Order and Degree of an Ordinary Differential Equation

Order

The order of an ordinary differential equation is the order of the highest derivative appearing in the equation.

Degree

The degree of a differential equation is the positive integral power to which the highest order derivative present in the equation is raised

Here are a few more examples of differential equations:

1. $\left(\frac{dy}{dx}\right)^2 + x^2 \frac{dy}{dx} + \cos y = 0$ Here order is 1 and degree is 2.

   Dependent variable involving with exponential, logarithmic, and Trigonometric functions etc. doesn't affect the degree of the differential equation.

2. $\left(\frac{d^2y}{dt^2}\right)^3 + e^{2t} = \frac{dy}{dt}$ Here order is 2 and degree is 3.

Note

- The degree of a differential equation, involving fractional power of the derivatives, cannot be decided directly. In that case, we should rewrite the differential equation by removing the fractional power and reduce it to the least positive integral power.

  Example: $\frac{d^2y}{dx^2} = K\sqrt{1 + \left(\frac{dy}{dx}\right)^3}$

  Squaring on both sides, $\left(\frac{d^2y}{dx^2}\right)^2 = K^2\left[1 + \left(\frac{dy}{dx}\right)^3\right]$ Here order is 2 and degree is 2.

- 2) Degree of the differential equation does not exist when the differential co-efficient involving with exponential functions, logarithmic functions, and Trigonometric functions.

  Examples:

  a. There is no degree for the differential equation $e^{2\frac{dy}{dx}} + 5 = 0$

  b. There is no degree for the differential equation $\log_{10}\left(\frac{d^3y}{dx^3}\right) + 2 = 0$

c. There is no degree for the differential equation $\cos\left(\frac{d^3y}{dx^3}\right)+y=0$

- The degree of the differential equation is always a positive integer, but it never be a negative (or) zero (or) fraction.

Linear and Non Linear Differential Equations:

An n^{th} order ordinary differential equation in the dependent variable y is said to be linear in y if

1. No products of the function y and its derivatives and neither the function nor its derivatives occur to any power other than the first power.
2. No transcendental functions of y and/or its derivatives occur.

The general form of nth order linear differential equation in y is

$$a_n\frac{d^ny}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+........+a_1\frac{dy}{dx}+a_oy=g(x)\ .$$

The coefficients $a_n, a_{n-1},......a_1, a_0$ and $g(x)$ can be zero or non-zero functions, constant or non-constant functions, linear or non-linear functions of x. If a differential equation is not linear then it is called a **non-linear** differential equation.

Examples: Linear differential equations:

1. $\frac{dy}{dx}=x^4+y$ 2. $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=e^x$

Non linear differential equations:

1. $\frac{dy}{dx}=x+y^2$ 2. $\frac{dy}{dx}=x+\sin y$

Solution (Integral/Primitive) of an Ordinary Differential Equation

Any n times differentiable function y(x) which satisfies a differential equation F(x, y, y′,..., y (n))= 0 is called a solution of the differential equation. Here "satisfies" means that substitution of the solution into the equation turns it into an identity. More precisely we can say a **solution** of an ordinary differential equation is a function $y=f(x)$ that when substituted into the equation satisfies it over the interval on which the differential equation is defined.

A differential equation may have unique solution or several solution, or no solution.

The solution can be general or particular solution. The '**general solution**' of a differential equation is that in which the number of arbitrary constants are equal to the order of the equation.

If the arbitrary constants in the general solution are assigned specific values, the resultant solution is called a **particular solution** of the differential equation.

Initial Value Problem

An **Initial Value Problem** (or **IVP**) is a differential equation in which solution to the equation is obtained subject to conditions on the unknown function and its derivative specified at one or more values of independent variable. Such conditions are called initial conditions.

1. Find the order and degree of the following differential equations

Solution:

i) $\frac{d^3y}{dx^3}+5\frac{d^2y}{dx^2}-3\frac{dy}{dx}+\log y=0$ Here order is 3 and degree is 1.

ii) $x^3\left(\frac{d^2y}{dx^2}\right)^3+y\left(\frac{dy}{dx}\right)^4+x^4=0$ Here order is 2 and degree is 4.

iii) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{7}{2}}=\frac{d^2y}{dx^2}$

Squaring on both sides, $\left[1+\left(\frac{dy}{dx}\right)^2\right]^7=\left(\frac{d^2y}{dx^2}\right)^2$, here order is 2 and degree is 2.

Differential Equations of First Order and First Degree

Let x be independent variable and y be dependent variable. The general form of first order and first degree differential equation is $f\left(x, y(x), \frac{dy}{dx}\right)=0$. Here $\frac{dy}{dx}$ appears in first degree.

The above equation can be rewritten explicitly $\frac{dy}{dx}=f(x,y)$.

The general solution of this differential equation is of the form $F(x,y,c)=0$, where c is arbitrary constant. In this unit, we consider some standard forms of first order first degree differential equations which can be solved by standard techniques.

We discuss the following forms:

1 Exact differential equations

2 Linear and Bernoulli's first order differential equations

Exact Differential Equations

The first order differential equation expressed in the form $M(x,y)dx+N(x,y)dy=0$ is said to be exact differential equation, if there exists a function $u(x,y)$ such that the total $du=M(x,y)dx+N(x,y)dy.$ where $M(x,y), N(x,y)$ and $u(x,y)$ are continuous functions.

The general solution of this equation is $u(x,y)=C.$

The question then becomes, if we have a general differential equation of the form $du=M(x,y)dx+N(x,y)dy.$ how do we know if it is exact? This is answered in the following theorem.

Theorem

The necessary and sufficient condition for the differential equation $Mdx+Ndy=0$ to be exact is $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Proof:

Condition is necessary

The differential equation $Mdx+Ndy=0$ will be exact if $Mdx+Ndy=du$ **(1)**

where u is function of x and y.

But $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$ **(2)**

Equating the coefficients of dx and dy in the above two equations, we get $M=\frac{\partial u}{\partial x}$.

and $N=\frac{\partial u}{\partial y}$ $\therefore \frac{\partial M}{\partial y}=\frac{\partial^2 u}{\partial y \partial x}$ and $\frac{\partial N}{\partial x}=\frac{\partial^2 u}{\partial x \partial y}$

But $\frac{\partial^2 u}{\partial x \partial y}=\frac{\partial^2 u}{\partial y \partial x}$. Therefore $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ is the necessary condition for exactness of differential equation.

Condition is sufficient:If $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ then $Mdx+Ndy=0$ is exact.

Let $\int Mdx=u$ and assume y as constant while doing integration.

We get, $\frac{\partial}{\partial x}\int Mdx=\frac{\partial u}{\partial x}\Rightarrow M=\frac{\partial u}{\partial x}$,

$\therefore \frac{\partial M}{\partial y}=\frac{\partial^2 u}{\partial y \partial x}$ or $\frac{\partial N}{\partial x}=\frac{\partial^2 u}{\partial x \partial y}$ **(3)**

Because $\frac{\partial^2 u}{\partial x \partial y}=\frac{\partial^2 u}{\partial y \partial x}$ and $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

Integrate equation (3) with respect to x assuming y as constant.

$N=\frac{\partial u}{\partial y}+F(y)$ where $F(y)$ is function of y alone.

$$Mdx+Ndy=\frac{\partial u}{\partial x}dx+\left(\frac{\partial u}{\partial y}+F(y)\right)dy=du+F(y)dy=d[u+F(y)] \qquad (4)$$

This shows that $Mdx+Ndy=0$ is exact.

By equation (4) we have $Mdx+Ndy=d[u+F(y)]=0$

Integrate we get, $u+\int f(y)dy=0$

But $u=\int_{y\to constant} Mdx$ and $f(y)$ = terms of N containing x.

Solution of exact differential equation $Mdx+Ndy=0$ is

$$\int_{y=constant} Mdx+\int terms\ of\ N\ not\ containing\ x\, dy=C$$

Working rule:

1. Check for the exactness using the necessary condition $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

2. If $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ then differential equation is exact.

   And the solution of exact differential equation $Mdx+Ndy=0$ is

$$\int_{y=constant} Mdx+\int terms\ of\ N\ not\ containing\ x\, dy=C$$

2. Solve $(x+3y-4)dx+(3x+9y-2)dy=0.$

Solution: The given equation is in the form of $Mdx+Ndy=0$.

Here $M=(x+3y-4)$ and $N=(3x+9y-2)$

$\frac{\partial M}{\partial y}=3$, $\frac{\partial N}{\partial x}=3.$ Clearly $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. The given differential equation is exact.

And the solution is $\int_{y=constant} Mdx+\int N(y)dy=C$

$$\int_{y\to cons\tan t}(x+3y-4)dx+\int 9ydy=C \Rightarrow \frac{x^2}{2}+3\frac{y^2}{2}-4x+9\frac{y^2}{2}=C$$

$$\Rightarrow x^2+3y^2-8x+9y^2=k, \qquad k=2C.$$

3. **Solve** $y\sin 2xdx-(y^2+\cos^2 x)dy=0.$

Solution: The given equation is in the form of $Mdx+Ndy=0$ **.**

Here $M=y\sin 2x$ and $N=-(y^2+\cos^2 x)$

$\frac{\partial M}{\partial y}=\sin 2x$, $\frac{\partial N}{\partial x}=2\cos x\sin x=\sin 2x.$ Clearly $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ **.**

The given differential equation is exact.

And the solution is $\int_{y=cons\tan t} Mdx+\int N(y)dy=C$

$$\int_{y\to cons\tan t}(y\sin 2x)dx+\int -y^2dy=C \Rightarrow y\frac{\cos 2x}{2}-\frac{y^3}{3}=C\Rightarrow 3y\cos 2x-2y^3=6C.$$

4. **Solve** $\{3x^2+2y+2\cosh(2x+3y)\}dx+\{2x+2y+3\cosh(2x+3y)\}dy=0, y(0)=0$

Solution: The given equation is in the form of $Mdx+Ndy=0$ **.**

Here $M=$ and $N=2x+2y+3\cosh(2x+3y)$

$\frac{\partial M}{\partial y}=3x^2+2y+2\cosh(2x+3y)$, $\frac{\partial N}{\partial x}=2\cos x\sin x=\sin 2x.$ Clearly $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ **.**

The given differential equation is exact.

And the solution is $\int_{y=cons\tan t} Mdx+\int N(y)dy=C$

$$\int_{y\to cons\tan t}3x^2+2y+2\cosh(2x+3y)dx+\int 2ydy=C \Rightarrow$$

$$x^3+2xy+\sinh(2x+3y)+y^2=C$$

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The above is the general solution of given differential equation.

$y(0)=0\Rightarrow c=0.$ The particular solution is $x^3+2xy+\sinh(2x+3y)+y^2=0$

5. **Solve** $[4x^3y^2+y\cos(xy)]dx+[2x^4y+x\cos(xy)]dy=0$

Solution: The given equation is in the form of $Mdx+Ndy=0$ **.**

Here $M=4x^3y^2+y\cos(xy)$ and $N=2x^4y+x\cos(xy)$

$\frac{\partial M}{\partial y}=8x^3y+\cos(xy)-xy\sin(xy)$, $\frac{\partial N}{\partial x}=8x^3y+\cos(xy)-xy\sin(xy)$, Clearly $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ **.**

The given differential equation is exact.

And the solution is $\int_{y=cons\tan t} Mdx+\int N(y)dy=C$

$$\int_{y\to cons\tan t}4x^3y^2+y\cos(xy)dx+\int 0dy=C \Rightarrow y^2x^4+\sin(xy)=C \backslash$$

6. **Solve** $3x(xy-2x)dx+(x^3+2y)dy=0.$

Solution: The given equation is in the form of $Mdx+Ndy=0$ **.**

Here $M=3x(xy-2x)$ and $N=(x^3+2y)$

$\frac{\partial M}{\partial y} = 3x^2$, $\frac{\partial N}{\partial x} = 3x^2$, Clearly $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ **.**The given differential equation is exact.

And the solution is $\int_{y=cons\tan t} Mdx + \int N(y)dy = C$

$$\int_{y\to cons\tan t} (3x^2 y - 6x^2)dx + \int 2ydy = C$$

$$x^3 y - 3x^2 + y^2 = C.$$

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Equations reducible to Exact form:

Most of the differential equations of the form $Mdx + Ndy = 0$ are not exact on the domain of definition but can be reduced to exact differential equations by multiplying with a suitable factor $f(x, y)$ called an integrating factor.

Example: The differential equation $-ydx + xdy = 0.$ is not exact. But if we multiply it by $\frac{1}{x^2}$,we get an exact equation.

Rules for finding integrating factors:

Rule 1: If $Mdx + Ndy = 0$ is a homogeneous equation in x and y, then $\frac{1}{Mx + Ny}$ is an integrating factor provided $Mx + Ny \neq 0.$

{Homogeneous equation: In the differential equation $Mdx + Ndy = 0$,if both M and N are both homogeneous functions}

Rule 2: If the given differential equation $Mdx + Ndy = 0$ is such that $M = yf_1(xy)$ and $N = xf_2(xy)$i.e. if the equation is of the form $f_1(xy)ydx + f_2(xy)xdy = 0$ form then $\frac{1}{Mx - Ny}$ is an integrating factor provided $Mx - Ny \neq 0$.

Rule 3: In the differential equation $Mdx + Ndy = 0$,

a. If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M}$ is constant or a function of y alone, say $f(y)$ then the integrating factor is $e^{\int -f(y)dy}$

b. If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is constant or a function of x alone, say $g(x)$ then the integrating factor is $e^{\int g(x)dx}$.

7. **Solve** $(x^2 - 3xy + 2y^2)dx + (3x^2 - 2xy)dy = 0.$

Solution: The equation is in the form of $Mdx + Ndy = 0$.

Here $M = (x^2 - 3xy + 2y^2)$ and $N = (3x^2 - 2xy)$

$\frac{\partial M}{\partial y} = -3x + 4y, \frac{\partial N}{\partial x} = 6x - 2y.$ Therefore the given differential equation is not exact as $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ .And the given equation is homogeneous, and $Mx + Ny = x^3 \neq 0.$

$\therefore$ Integrating factor=$\frac{1}{x^3}$,Multiply the given differential equation by this factor and the equation reduces to $\left(\frac{1}{x} - 3\frac{y}{x^2} + 2\frac{y^2}{x^3}\right)dx + \left(\frac{3}{x} - \frac{2y}{x^2}\right)dy = 0.$

And now $\frac{\partial M}{\partial y} = -\frac{3}{x^2} + \frac{6y}{x^3} = \frac{\partial N}{\partial x}$.

The general solution of this exact differential equation is $\int_{y=cons\tan t} Mdx + \int N(y)dy = C$

$$\int_{y=cons\tan t}\left(\frac{1}{x}-3\frac{y}{x^2}+2\frac{y^2}{x^3}\right)dx+\int 0dy=C\Rightarrow \ln x+\frac{3y}{x}-\frac{y^2}{x^2}=C.$$

8. **Solve** $y(x^2-2xy)dx-x^2(x-3y)dy=0$

Solution: The equation is in the form of $Mdx+Ndy=0$.

Here $M=yx^2-2xy^2$ and $N=-x^3+3x^2y$

$\frac{\partial M}{\partial y}=x^2-4xy, \frac{\partial N}{\partial x}=-3x^2+6xy$.Therefore the given differential equation is not exact as $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$.

And the given equation is homogeneous, and $Mx+Ny=yx^3-2x^2y^2-x^3y+3x^2y^2=x^2y^2\neq 0$.

$\therefore$ Integrating factor $=\frac{1}{x^2y^2}$,Multiply the given differential equation by this factor and the equation reduces to

$\left(\frac{1}{y}-\frac{2}{x}\right).dx-\left(\frac{x}{y^2}-\frac{3}{y}\right)dy=0$ And now $\frac{\partial M}{\partial y}=-\frac{1}{y^2}=\frac{\partial N}{\partial x}$. The differential equation is exact.

The general solution of this exact differential equation is $\int_{y=cons\tan t}Mdx+\int N(y)dy=C$

$$\int_{y=cons\tan t}\left(\frac{1}{y}-\frac{2}{x}\right)dx+\int \frac{3}{y}dy=C\Rightarrow \frac{x}{y}-2\ln x+3\ln y=C$$

9. **Solve** $xydx+(2x^2+3y^2+40)dy=0$

Solution: The equation is in the form of $Mdx+Ndy=0$.

Here $M=xy$ and $N=(2x^2+3y^2+40)$

$\frac{\partial M}{\partial y}=x, \frac{\partial N}{\partial x}=4x$. Therefore the given differential equation is not exact as $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$

Consider $\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=x-4x=-3x$ close to M

Now let us consider $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=-\frac{3x}{xy}=-\frac{3}{y}$ is a function of y alone.

$\therefore$ Integrating factor $=e^{\int -f(y)dy}=e^{\int \frac{3}{y}dy}=y^3$,Multiply the given differential equation by this factor and the equation reduces to

$xy^4dx+(2x^2y^3+3y^5-40y^3)dy=0$

And now $\frac{\partial M}{\partial y}=4xy^3=\frac{\partial N}{\partial x}$. The differential equation is exact.

The general solution of this exact differential equation is $\int_{y=cons\tan t}Mdx+\int N(y)dy=C$

$$\int_{y=cons\tan t}xy^4dx+\int (3y^5-40y^3)dy=C$$

$$\frac{x^2y^4}{2}.+\frac{1}{2}y^6-\frac{40y^4}{4}=C$$

10. **Solve** $(3x^2y^4+2xy)dx-(2x^3y^3-x^2)=0$

Solution: The equation is in the form of $Mdx+Ndy=0$.

Here $M=\left(3x^2y^4+2xy\right)=xy\left(3xy^3+2\right)$ and $N=-2x^3y^3+x^2$

$\frac{\partial M}{\partial y}=12x^2y^3+2x, \frac{\partial N}{\partial x}=6x^2y^3-2x.$ Therefore the given differential equation is not exact as $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ .Consider $\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=12x^2y^3+2x-6x^2y^3+2x=2x\left(3xy^3+2\right)$ close to M

Now let us consider $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=\frac{2x\left(3xy^3+2\right)}{xy\left(3xy^3+2\right)}=\frac{2}{y}$ is a function of y alone.

$\therefore$ Integrating factor $=e^{\int -f(y)dy}=e^{\int -\frac{2}{y}dy}=\frac{1}{y^2}$

Multiply the given differential equation by this factor and the equation reduces to

$$\left(3x^2y^2+\frac{2x}{y}\right)dx+\left(2x^3y-\frac{x^2}{y^2}\right)dy=0$$

And now $\frac{\partial M}{\partial y}=6x^2y-\frac{2x}{y^2}=\frac{\partial N}{\partial x}$. The differential equation is exact.

The general solution of this exact differential equation is $\int_{y=cons\tan t} Mdx+\int N(y)dy=C$

$$\int_{y=cons\tan t}\left(3x^2y^2+\frac{2x}{y}\right)dx+\int 0dy=C$$

$$x^3y^2+\frac{x^2}{y}=C.$$

11. **Solve** $\left(x^3+y^3+x\right)dx+xy^2dy=0$

Solution: The equation is in the form of $Mdx+Ndy=0$.

Here $M=\left(x^3+y^3+x\right)$ and $N=xy^2$

$\frac{\partial M}{\partial y}=3y^2, \frac{\partial N}{\partial x}=y^2$ Therefore the given differential equation is not exact as $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$

Consider $\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=3y^2-y^2=2y^2$ close to N

Now let us consider $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{2y^2}{xy^2}=\frac{2}{x}$ is a function of x alone.

$\therefore$ Integrating factor $=e^{\int g(x)dx}=e^{\int \frac{2}{x}dx}=e^{2\ln x}=x^2$

Multiply the given differential equation by this factor and the equation reduces to

$$\left(x^5+x^2y^3+x^3\right)dx+x^3y^2=0$$

And now $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. The differential equation is exact.

The general solution of this exact differential equation is $\int_{y=cons\tan t} Mdx+\int N(y)dy=C$

$$\int_{y=cons\tan t}\left(x^5+x^2y^3+x^3\right)dx+\int 0dy=C\Rightarrow\frac{x^6}{6}+\frac{x^3y^3}{3}+\frac{x^4}{4}=C$$

12. **Solve** $\left(6x^2+4y^3+12y\right)dx+3x\left(1+y^2\right)dy=0$

Solution: The equation is in the form of $Mdx+Ndy=0$.

Here $M=\left(6x^2+4y^3+12y\right)$ and $N=3x\left(1+y^2\right)$

$\frac{\partial M}{\partial y}=12y^2+12, \frac{\partial N}{\partial x}=3+3y^2$ Therefore the given differential equation is not exact as $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$

Consider $\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=12y^2+12-3-3y^2=9+9y^2=9(1+y^2)$ close to N

Now let us consider $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{9(1+y^2)}{3x(1+y^2)}=\frac{3}{x}$ is a function of x alone.

$\therefore$ Integrating factor $=e^{\int g(x)dx}=e^{\int \frac{3}{x}dx}=e^{3\ln x}=x^3$,Multiply the given differential equation by this factor and the equation reduces to

$(6x^5+4y^3x^3+12x^3y)dx+3x^4(1+y^2)dy=0$, And now $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. The differential equation is exact.

The general solution of this exact differential equation is $\int_{y=cons\tan t} Mdx+\int N(y)dy=C$

$\int_{y=cons\tan t}(6x^5+4y^3x^3+12x^3y)dx+\int 0dy=C\Rightarrow x^6+x^4y^3+3x^4y=C$

13. **Solve** $y(1-xy)dx-x(1+xy)dy=0$

Solution: The equation is in the form of $Mdx+Ndy=0$.

Here $M=y-xy^2$ and $N=-x^2-x^2y$

$\frac{\partial M}{\partial y}=1-2xy, \frac{\partial N}{\partial x}=-2x-2xy$.Therefore the given differential equation is not exact as $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ .But the differential equation is in the form of $f_1(xy)ydx+f_2(xy)xdy=0$.

Consider $Mx-Ny=xy-x^2y^2+xy+x^2y^2=2xy\neq 0.$

$\therefore$ Integrating factor $=\frac{1}{Mx-Ny}=\frac{1}{2xy}$,Multiply the given differential equation by this factor and the equation reduces to

$\frac{1}{2x}(1-xy)dx-\frac{1}{2y}(1+xy)dy=0$, And now $\frac{\partial M}{\partial y}=-\frac{1}{2}=\frac{\partial N}{\partial x}$. The differential equation is exact.

The general solution of this exact differential equation is $\int_{y=cons\tan t} Mdx+\int N(y)dy=C$

$\int_{y=cons\tan t}\left(\frac{1}{2x}-\frac{y}{2}\right)dx+\int\left(-\frac{1}{2y}\right)dy=C\Rightarrow\frac{1}{2}\ln x-\frac{xy}{2}-\frac{1}{2}\ln y=C$

14. Solve $(x^2y^2+5xy+2)ydx+(x^2y^2+4xy+2)xdy=0$

Solution: The equation is in the form of $Mdx+Ndy=0$.

Here $M=(x^2y^2+5xy+2)y$ and $N=(x^2y^2+4xy+2)x$

$\frac{\partial M}{\partial y}=3x^2y^2+10xy+2, \frac{\partial N}{\partial x}=3x^2y^2+8xy+2$.Therefore the given differential equation is not exact as $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ .But the differential equation is in the form of $f_1(xy)ydx+f_2(xy)xdy=0$.

Consider $Mx - Ny = x^2 y^2 \neq 0.$

$\therefore$ Integrating factor $= \dfrac{1}{Mx - Ny} = \dfrac{1}{x^2 y^2}$,Multiply the given differential equation by this factor and the equation reduces to $\left[y + \dfrac{5}{x} + \dfrac{2}{x^2 y} \right] dx + \left[x + \dfrac{4}{y} + \dfrac{2}{xy^2} \right] dy = 0$

And now $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. The differential equation is exact.

The general solution of this exact differential equation is $\int_{y=cons\tan t} Mdx + \int N(y)dy = C$

$$\int_{y=cons\tan t} \left(y + \frac{5}{x} + \frac{2}{x^2 y} \right) dx + \int \frac{4}{y} dy = C \Rightarrow xy + 5\ln x - \frac{2}{xy} + 4\ln y = C$$

Linear and Bernoulli's Differential Equations:

a. Linear Differential Equations:

We know that a differential equation is said to be linear if the dependent variable and its differential coefficients occur only in the first degree and not multiplied together.

General form of linear differential equation of first order (Leibnitz's form):

A differential equation of the form $\dfrac{dy}{dx} + Py = Q$, where P and Q are functions of x is called linear differential equation of first order in y. Example: $\dfrac{dy}{dx} + 2xy = \sin x.$

Similarly, a differential equation of the form $\dfrac{dx}{dy} + Px = Q$, where P and Q are functions of y is called Linear differential equation of first order in y. Example: $\dfrac{dx}{dy} - \dfrac{2}{y} x = y^2 e^{-y}.$

Method of solution:

Consider first order linear equation in y i.e. $\dfrac{dy}{dx} + Py = Q$ (1)

Let R a function of x be integrating factor of the above differential equation.
Multiply equation (1) by integrating factor, we get

$R\dfrac{dy}{dx} + RPy = RQ$ which must be exact.

Let us choose R such that LHS of above equation is differential coefficient of yR.

i.e. $\dfrac{d}{dx}(yR) = R\dfrac{dy}{dx} + y\dfrac{dR}{dx}.$

$$R\frac{dy}{dx} + PRy = LHS = R\frac{dy}{dx} + y\frac{dR}{dx} \Rightarrow \therefore PRy = y\frac{dR}{dx} \Rightarrow \frac{dR}{R} = Pdx \Rightarrow R = e^{\int Pdx}$$

Now multiply equation (1) with integrating factor $e^{\int Pdx}$,we get

$$e^{\int Pdx}\frac{dy}{dx} + Pe^{\int Pdx} y = Qe^{\int Pdx} \Rightarrow \frac{d}{dx}\left(e^{\int Pdx} y \right) = e^{\int Pdx} Q$$

Integrating with respect to x, $ye^{\int Pdx} = \int Qe^{\int Pdx} dx + C$

Similarly, for the linear differential equation of the form $\frac{dx}{dy}+Px=y$ (Linear in x)the integrating factor is $e^{\int Pdy}$.

15. **Solve the initial value problem** $\cos x\frac{dy}{dx}+y=\sin x,\ y(0)=2.$

Solution: The equation can be written as $\frac{dy}{dx}+\frac{y}{\cos x}=\tan x.$

The equation is in the form of $\frac{dy}{dx}+Py=Q$ (linear in y)

$P=\sec x, Q=\tan x. \therefore I.F.=e^{\int Pdx}=e^{\int \sec xdx}e^{\ln(\sec x+\tan x)}=(\sec x+\tan x.)$

The general solution of the given differential equation is $ye^{\int Pdx}=\int Qe^{\int Pdx}dx+C$.

$y(\sec x+\tan x)=\int \tan x(\sec x+\tan x)dx+C \Rightarrow y(\sec x+\tan x)=\sec x+\tan x-x+C.$

Given $y(0)=2$, we get C=1.Hence the particular solution is $y(\sec x+\tan x)=\sec x+\tan x-x+1$.

16. **Solve** $(1+x^2)\frac{dy}{dx}+xy=\sinh^{-1}x$

Solution: The equation can be written as $\frac{dy}{dx}+\frac{x}{1+x^2}y=\sinh^{-1}x$

The equation is in the form of $\frac{dy}{dx}+Py=Q$ (linear in y)

$P=\frac{x}{1+x^2}, Q=\sinh^{-1}x. \Rightarrow \therefore I.F.=e^{\frac{1}{2}\int \frac{2x}{1+x^2}dx}=e^{\ln\sqrt{1+x^2}}=\sqrt{1+x^2}.$

The general solution of the given differential equation

is $ye^{\int \frac{x}{1+x^2}dx}=\int \frac{\sinh^{-1}x}{\sqrt{1+x^2}}dx+C$.

Put $t=\sinh^{-1}x \Rightarrow dt=\frac{1}{\sqrt{1+x^2}}dx.$

The equation reduces to $y\sqrt{1+x^2}=\frac{t^2}{2}+C \Rightarrow y\sqrt{1+x^2}=\frac{(\sinh^{-1}x)^2}{2}+C$

17. **Solve** $(1+y^2)dx=(\tan^{-1}y-x)dy$

Solution: The equation can be written as $\frac{dx}{dy}+\frac{1}{1+y^2}x=\frac{\tan^{-1}y}{1+y^2}$.

It is in the form of $\frac{dx}{dy}+Px=Q$ (linear in x).

Here $P=\frac{1}{1+y^2}, Q=\frac{\tan^{-1}y}{1+y^2}. \therefore I.F.=e^{\int \frac{1}{1+y^2}}=e^{\tan^{-1}y}$

The general solution of the given differential equation is $xe^{\tan^{-1}y}=\int \frac{\tan^{-1}y}{1+y^2}e^{\tan^{-1}y}dy+C$.

Put $t=\tan^{-1}y$, $dt=\frac{1}{1+y^2}dy$,equation becomes, $x\tan^{-1}y=\int te^t dt+C.$

$x\tan^{-1}y=(t-1)e^t+C \Rightarrow x\tan^{-1}y=(\tan^{-1}y-1)e^{\tan^{-1}y}+C.$

18. **Solve** $(x+2y^3)\frac{dy}{dx}=y$

Solution: The equation can be written as $\frac{dy}{dx}=\frac{y}{x+2y^3}$

$\Rightarrow \frac{dx}{dy}-\frac{1}{y}x=2y^2$ The equation is in the form of $\frac{dx}{dy}+Px=Q,$ Here

$$P=-\frac{1}{y},Q=2y^2.\therefore I.F.=e^{\int -\frac{1}{y}dy}=e^{-\ln y}=\frac{1}{y}$$

The general solution of the given differential equation is $\frac{x}{y}=\int 2y^2\frac{1}{y}dy+C$.

19. Solve $\left(e^{-2\sqrt{x}}-y\right)dx=\sqrt{x}dy$

Solution: The equation can be written as $\frac{dx}{dy}=\frac{\sqrt{x}}{\left(e^{-\sqrt{x}}-y\right)}\Rightarrow \frac{dy}{dx}+\frac{1}{\sqrt{x}}y=\frac{e^{-\sqrt{x}}}{\sqrt{x}}$

The equation is in the form of $\frac{dx}{dy}+Px=Q,$

$$P=\frac{1}{\sqrt{x}},Q=\frac{e^{-\sqrt{x}}}{\sqrt{x}}.\therefore I.F.=e^{\int \frac{1}{\sqrt{x}}dx}=e^{2\sqrt{x}}$$

The general solution of the given differential equation

is $ye^{2\sqrt{x}}=\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}e^{2\sqrt{x}}dx+C$.

$ye^{2\sqrt{x}}=\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx+C$,Put $t=\sqrt{x}\Rightarrow dt=\frac{1}{2\sqrt{x}}dx$

The equation reduces to $ye^{2\sqrt{x}}=\int 2e^t dt+C\Rightarrow ye^{2\sqrt{x}}=2e^t+C.$

20. Solve $\left(x+\tan y\right)dy=\sin 2y dx$

Solution: The equation can be written as $\frac{dy}{dx}=\frac{\sin 2y}{\left(x+\tan y\right)}\Rightarrow \frac{dx}{dy}-\frac{1}{\sin 2y}x=\frac{\tan y}{\sin 2y}$

The equation is in the form of $\frac{dx}{dy}+Px=Q,$

$$P=-\frac{1}{\sin 2y},Q=\frac{1}{2}.\sec^2 y\therefore I.F.=e^{\int -\operatorname{cosec}2y dy}=e^{\frac{-1}{2}\ln(\tan y)}=\frac{1}{\sqrt{\tan y}}$$

The general solution of the given differential equation

is $x\frac{1}{\sqrt{\tan y}}=\frac{1}{2}\int \frac{1}{\sqrt{\tan y}}\sec^2 y dy+C$.

$$\frac{x}{\sqrt{\tan y}}=\sqrt{\tan y}+C.$$

21. A tank contains 300 liters of fluid in which 20 grams of salt are dissolved. Brine containing 1 gm of salt per liter is then pumped into the tank at a rate of 4 L/min; the well-mixed solution is pumped out at the same rate. Find the number N(t) of grams of salt in the tank at time t.

Solution: Let A(t) denotes the amount of salt (in pounds) in the tank at time t ,then

the rate at which A(t) changes is a net rate:

$$\frac{dA}{dt} = Salt\ Inflow\ rate - Salt Outflow\ rate$$

$$\frac{dA}{dt} = 4 - \frac{A}{75} \Rightarrow \frac{dA}{dt} + \frac{A}{75} = 4\ ,$$

$$I.F. = e^{\int \frac{1}{75}dt} = e^{\frac{1}{75}t} \Rightarrow Ae^{\frac{1}{75}t} = \int 4e^{\frac{1}{75}t}dt + C$$

$$A(t) = 300 + ce^{-\frac{1}{75}t}$$

Given $A(0) = 20 \Rightarrow c = -280, \therefore A(t) = 300 - 280e^{-\frac{1}{75}t}$

22. A 100-volt electromotive force is applied to an RC series circuit in which the resistance is 250 ohms and the capacitance is 10^{-3} farads. Find the charge q(t) on the capacitor if q (0)=0. Find the current at any time t.

Solution: Series RC circuit(circuit diagram)

Apply Kirchoff's law to the circuit ,we get, $R\frac{dq}{dt} + \frac{q}{c} = E(t)$

$$R = 250\Omega, E = 100V, C = 0.001F$$

$$\therefore 250\frac{dq}{dt} + \frac{1}{10^{-3}}q = 100 \Rightarrow \frac{dq}{dt} + 4q = 0.4.$$

$$I.F. = e^{\int 4dt} = e^{4t} \Rightarrow qe^{4t} = \int 0.4e^{4t}dt + C$$

$$q(t) = 0.1 + ce^{-4t}$$

Given $q(0) = 0 \Rightarrow c = -0.1 \Rightarrow q(t) = 0.1[1 - e^{-4t}]$

Equations reducible to Leibnitz's form:

An equation of the form $f'(y)\frac{dy}{dx} + f(y)P = Q$ can be solved by substitution $t = f(y)$, $\frac{dt}{dx} = f'(y)\frac{dy}{dx}$.The above equation reduces to linear equation in t, $\frac{dt}{dx} + ty = Q$ which can be solved by the method discussed above. Similarly an equation of the form $f'(x)\frac{dx}{dy} + f(x)P = Q$ can be solved by substitution $t = f(x)$

b. **Bernoulli's differential equation:**

The Bernoulli equation is a non linear first order differential equation. Its standard form is $\frac{dy}{dx} + Py = Qy^n$ where P and Q are functions of x.

The above equation is separable if n=1 and linear if n=0 and Bernoulli equation with $n \neq 1$
And this differential equation transforms to Leibnitz's linear equation under the change of variable $z = y^{n-1}$.

Consider $\frac{dy}{dx} + Py = Qy^{n-1}$

Divide both sides of the equation by y^n, we get $y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$

Put $z = y^{1-n}$, so that the equation reduces to $(1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$

$$\Rightarrow \frac{1}{1-n}\frac{dz}{dx} + Pz = Q \Rightarrow \frac{dz}{dx} + P(1-n)z = Q(1-n).$$

The above equation is in Leibnit'z form which can be solved by the method as discussed earlier. Similarly Bernoulli's equation in x is of the form $\frac{dx}{dy}+Px=Qx^n$ which can be reduced to linear equation by taking $z=x^{n-1}$.

23. **Solve** $\cos x\frac{dy}{dx}+(\sin x)y=\sqrt{y\sec x}$ **.**

Solution: The given differential equation can be written as $\frac{dy}{dx}+(\tan x)y=\left(\sec^{\frac{3}{2}}x\right)y^{\frac{1}{2}}$

And this equation is Bernoulli's equation in y.

Divide both sides of the equation by $y^{\frac{1}{2.}}$,we get, $y^{-\frac{1}{2}}\frac{dy}{dx}+(\tan x)y^{\frac{1}{2}}=\sec^{\frac{3}{2}}x$.

Put $z=y^{1-n}=y^{\frac{1}{2}}\Rightarrow\frac{dz}{dx}=\frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx}$. The equation reduces to $\frac{dz}{dx}+\frac{1}{2}\tan x.z=\frac{1}{2}\sec^{\frac{3}{2}}x$ which is linear in z. $P=\frac{1}{2}\tan x, Q=\frac{1}{2}\sec^{\frac{3}{2}}x\therefore I.F.=e^{\int Pdx}=e^{\int\frac{1}{2}\tan xdx}=e^{\ln\sqrt{\sec x}}=\sqrt{\sec x}.$

The general solution is $z\sqrt{\sec x}=\int\sqrt{\sec x}\frac{1}{2}\sec^{\frac{3}{2}}xdx$

$z\sqrt{\sec x}=\frac{1}{2}\int\sec^2 xdx+C=\frac{1}{2}\tan x+C\Rightarrow\sqrt{y\sec x}=\frac{1}{2}\tan x+C$

24. **Solve** $\tan y\frac{dy}{dx}+\tan x=\cos y\cos^3 x.$

Solution: Divide the given differential equation by $\cos y$

$\sec y\tan y\frac{dy}{dx}+(\sec y)\tan x=\cos^3 x.$

This equation takes the form $f'(y)\frac{dy}{dx}+f(y)P=Q$

Put $t=\sec y\Rightarrow\frac{dt}{dx}=(\sec y\tan y)\frac{dy}{dx}$.

The equation reduces to $\frac{dt}{dx}+(\tan x)t=\cos^3 x$ which is linear in t.

$P=\tan x, Q=\cos^3 x\therefore I.F.=e^{\int Pdxx}=e^{\int\tan dx}=e^{\ln\sec x}=\sec x$

The general solution is $t\sec x=\int\sec x\cos^3 xdx+C$

$t\sec x=\int\cos^2 xdx+C\Rightarrow t\sec x=\frac{1}{2}x+\frac{1}{4}\sin 2x+C.$

25. **Solve** $e^y.\frac{dy}{dx}+e^y=e^{2x}$

Solution: The given differential equation is in the form of $f'(y)\frac{dy}{dx}+f(y)P=Q$

Put $t=e^y\Rightarrow\frac{dt}{dx}=e^y\frac{dy}{dx}$.The equation reduces to $\frac{dt}{dx}+t=e^{2x}$ which is linear in t.

$P=1, Q=e^{2t}\therefore I.F.=e^{\int Pdxx}=e^{\int 1dx}=e^x$

The general solution is

$te^x=\int e^{3x}dx+C$

26. **Solve** $r\sin\theta-\frac{dr}{d\theta}\cos\theta=r^2$

Solution: The given equation can be written as $-\frac{dr}{d\theta}\cos\theta + r\sin\theta = r^2$

Divide the equation by $r^2\cos\theta$,

$-\frac{1}{r^2}\frac{dr}{d\theta}+\frac{1}{r}\tan\theta=\sec\theta \Rightarrow$ Put $t=\frac{1}{r}\Rightarrow \frac{dt}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$.

The equation reduces to $\frac{dt}{d\theta}+t(\tan\theta)=\sec\theta$ which is linear in t.

$$P=\tan\theta, Q=\sec\theta \therefore I.F.=e^{\int \tan\theta d\theta}=e^{\ln(\sec\theta)}=\sec\theta$$

Solution is $t\sec\theta=\int (\sec^2\theta)d\theta + C \Rightarrow t\sec\theta=\tan\theta + C \Rightarrow \frac{\sec\theta}{r}=\tan\theta + C$

27. **Solve** $\frac{dy}{dx}+\left(2x\tan^{-1}y-x^3\right)\left(1+y^2\right)=0$

Solution: The given equation can be written as $\frac{1}{1+y^2}\frac{dy}{dx}+2x\left(\tan^{-1}y\right)=x^3$

Put $t=\tan^{-1}y\Rightarrow\frac{dt}{dx}=\frac{1}{1+y^2}\frac{dy}{dx}$.The equation reduces to $\frac{dt}{dx}+t(2x)=x^3$ which is linear in t.

$P=2x, Q=x^3 \therefore I.F.=e^{\int 2xdx}=e^{x^2}$ The general solution is $te^{x^2}=\int x^3e^{x^2}dx+C$

Put $u=x^2\Rightarrow te^{x^2}=\frac{1}{2}\int ue^u du + C\Rightarrow \tan^{-1}y=\frac{1}{2}\left(x^2-1\right)+C$

28. **Solve** $x^3\frac{dy}{dx}-x^2y=-y^4\cos x$

Solution: Divide the given equation by $-x^3y^4$,we get,

,

$$-\frac{1}{y^4}\frac{dy}{dx}+\frac{1}{xy^3}=\cos x$$

Put $t=\frac{1}{y^3}\Rightarrow\frac{dt}{dx}=-\left(\frac{3}{y^4}\right)\frac{dy}{dx}$.

The equation reduces to $\frac{dt}{dx}+\frac{3t}{x}=\frac{3}{x^3}\cos x$ which is linear in t.

$$P=\frac{3}{x}, Q=\frac{3}{x^3}\cos x \therefore I.F.=e^{\int\frac{3}{x}dx}=x^3$$

General solution is $tx^3=\int\left(\frac{3}{x^3}\cos x\right)x^3dx+C\Rightarrow tx^3=\sin x+C\Rightarrow\frac{x^3}{y^3}=3\sin x+C$

29. **Solve** $\frac{dy}{dx}=\frac{y}{x}+\frac{y^2}{3x^2}$

Solution: Divide the given equation by y^2,we get,

$$\frac{1}{y^2}\frac{dy}{dx}-\frac{1}{xy}=\frac{1}{3x^2},$$

This is Bernoulli's equation in y.

Put $t=\frac{1}{y^2}\Rightarrow\frac{dt}{dx}=-\left(\frac{3}{y^4}\right)\frac{dy}{dx}$.The equation reduces to $\frac{dt}{dx}+\frac{t}{x}=\frac{1}{3x^2}$ which is linear in t.

$$P = \frac{1}{x}, Q = \frac{1}{3x^2} \therefore I.F. = e^{\int \frac{1}{x}dx} = x$$

The general solution is $tx = \int \frac{1}{3x^2} x dx + C \Rightarrow tx = \frac{1}{3}\ln x + C \Rightarrow \frac{1}{y^2}x - \frac{1}{3}\ln x = C$

30. Solve $\frac{dx}{dy} - xy = y^3 x^2$

Solution: Divide the given equation by x^2, we get, $\frac{1}{x^2}\frac{dx}{dy} - \frac{y}{x} = y^3$

Put $t = -\frac{1}{x} \Rightarrow \frac{dt}{dy} = \frac{1}{x^2}\frac{dx}{dy}$. The equation reduces to $\frac{dt}{dy} + ty = y^3$ which is linear in t.

$$P = y, Q = y^3 \therefore I.F. = e^{\int ydy} = e^{\frac{y^2}{2}}.$$

The general solution is $te^{\frac{y^2}{2}} = \int y^3 e^{\frac{y^2}{2}} dy + C$

Put

$$v = \frac{1}{2}y^2 \Rightarrow \frac{1}{x}e^{\frac{y^2}{2}} = 2\int ve^v dv + C \Rightarrow -\frac{1}{x}e^{\frac{y^2}{2}} = 2(v-1)e^v + C \Rightarrow -\frac{1}{x}e^{\frac{y^2}{2}} = (y^2 - 2)e^{\frac{y^2}{2}} + C$$

Higher Order Linear differential equations with constant coefficients

Introduction

We have studied methods of solving ordinary differential equations of first order and first degree (in Chapter 2). In this unit, we're going to learn how to solve second and higher order linear ordinary differential equations with constant coefficients. Differential equations of higher order arise very often in physical problems.

A differential equation in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together is called a **linear differential equation**.

Thus, the general linear differential equations with constant co-efficient of the n^{th} order is of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots\ldots\ldots + a_n y = r(x) \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

Where, $a_{1,} a_2, a_3, \ldots\ldots . a_n$ are constants and $r(x)$ are functions of x only.

If $r(x) = 0$, then the equation (1) is called a **homogeneous equation**. Otherwise, it is a **non-homogeneous equation**.

Operator D:

Let us denote $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2, \ldots\ldots\ldots\ldots, \frac{d^n}{dx^n} = D^n$

$$\therefore \frac{dy}{dx} = Dy, \frac{dy^2}{dx^2} = D^2 y, \frac{dy^3}{dx^3} = D^3 y, \ldots\ldots\ldots\ldots$$

The above equation (1) can be written as

$$(D^n + a_1 D^{n-1} + ... + a_n)y = r(x)$$

i.e. $f(D)y = r(x)$ where, $f(D) = D^n + a_1 D^{n-1} + ... + a_n$, is a polynomial in D.

Here the symbol D is called the **differential operator** and it stands for the operation of differentiation.

General solution of a homogeneous differential equation with constant co-efficient

Let us consider a second order differential equation to explain the method of solving a homogeneous linear differential equation

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0 \quad \text{, where } a_1, a_2 \text{ are constants} \qquad ……….. (2)$$

i.e. $(D^2 + a_1 D + a_2)y = 0$

Or $f(D)y = 0$ where, $f(D) = D^2 + a_1 D + a_2$ …...……(3)

Note

A set of functions $f_1(x), f_2(x), ..., f_n(x)$ is **linearly independent** on a ≤ x ≤ b, if constants $c_1, c_2, ..., c_n$ must be equal to zero in order to satisfy $c_1 f_1(x) + c_2 f_2(x) + ... + c_n f_n(x) = 0$ in $a \leq x \leq b$. In other words, one function cannot be expressed in terms of other functions.
Now we shall prove the following theorem.

Theorem

If y_1 and y_2 are linearly independent solutions (one cannot be expressed in terms of the other) of (3) then $c_1 y_1 + c_2 y_2$ is also a solution of (3), where c_1 and c_2 are arbitrary constants.

Proof : Since y_1 and y_2 are solutions of (3) we have

$$f(D)y_1 = 0 \text{ and } f(D)y_2 = 0 \qquad ……………. (4)$$

$$\therefore \quad \begin{aligned} f(D)(c_1 y_1 + c_2 y_2) &= c_1 f(D)\, y_1 + c_2 f(D)\, y_2 \\ &= c_1.0 + c_2.0 = 0 \end{aligned} \qquad \text{(Using 4)}$$

Thus $f(D)(c_1 y_1 + c_2 y_2) = 0$

This shows that $c_1 y_1 + c_2 y_2$ is a solution of (3)

Therefore, $y = c_1 y_1 + c_2 y_2 = y_c$ which contains two arbitrary constants c_1 and c_2, is called the **General Solution** of second order differential equation (3).

y_c is called as the **Complimentary Function** (C.F)

For nth order homogeneous differential equation,

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + a_n y = 0$$

If $y_1, y_2, y_3, ..., y_n$ are the solutions of this differential equation, their linear combinations form the **general solution** of this equation. i.e. $y(x) = c_1 y_1(x) + c_2 y_2(x) + c_3 y_2(x) + c_3 y_3(x) + ... + c_n y_n(x)$ is its complete solution. Where, $c_1, c_2, c_3, ..., c_n$ are arbitrary constants.

Methods of finding Complementary function

Consider a second order homogeneous differential equation

$$\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_2y=0 \qquad \text{..............} (5)$$

Using operator D, the above equation can be written as

$$(D^2+a_1D+a_2)y=0 \;\Rightarrow f(D)y=0 \qquad \text{..............} (6)$$

Let $y=e^{mx}$

$\therefore Dy=me^{mx}$ and $D^2y=m^2e^{mx}$

Using this in (5), we get $(m^2+a_1m+a_2)e^{mx}=0$

i.e. $m^2+a_1m+a_2=0$ .... (7)

This is called **Auxiliary Equation (A.E) or characteristic equation.**

Equation (7) is a quadratic which has two roots that may be (i) real and distinct (ii) real and repeated (iii) complex

Case (i) Suppose the roots, m_1 and m_2 are real and distinct then the two independent solution of (5) are $y=e^{m_1x}$ and $y=e^{m_2x}$

By theorem, $y=c_1e^{m_1x}+c_2e^{m_2x}$ is the general solution of (5).

In general, the solution of the n^{th} order differential equation is given by

$$y=c_1e^{m_1x}+c_2e^{m_2x}+...............+c_ne^{m_nx}$$

Where, $c_1,c_2,c_3,..............,c_n$ are arbitrary constants

Case (ii) Suppose the roots are real and repeated i.e. $m_1=m_2$ then the differential equation is $(D-m_1)^2y=0$

Let $(D-m_1)y=p$

$\therefore$ The D.E becomes $\frac{dp}{dx}-m_1p=0$

This is a linear D.E in p

The Integrating factor is $e^{\int -m_1\,dx}=e^{-m_1x}$

The solution of linear D.E is $pe^{-m_1x}=\int e^{-m_1x}(0)dx+c_2 \Rightarrow p=c_2e^{m_1x}$

But $p=(D-m_1)y \Rightarrow \frac{dy}{dx}-m_1y=c_2e^{m_1x}$

Integrating factor for the above equation is $e^{\int -m_1\,dx}=e^{-m_1x}$ and the solution is

$y\,e^{-m_1x}=\int c_2\,e^{m_1x}\,e^{-m_1x}\,dx+c_1 \Rightarrow ye^{-m_1x}=c_2\,x+c_1$

Therefore, $y=(c_1+c_2x)e^{m_1x}$ is the general solution when the roots are repeated.

If three roots are repeated then the general solution is $y=(c_1+c_2x+c_3x^2)e^{m_1x}$

If four roots are repeated then the complete solution is $y=(c_1+c_2x+c_3x^2+c_4x^3)e^{m_1x}$

Case (iii) Suppose the roots are complex i.e. $m=\alpha\pm i\beta$ then the general solution of (5) is $y=c_1\,e^{m_1x}+c_2\,e^{m_2x}=c_1\,e^{(\alpha+i\beta)x}+c_2\,e^{(\alpha-i\beta)x}$

i.e. $y = e^{\alpha x}\left(c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right)$

This can be written as $y = e^{\alpha x}\left[c_1\left(\cos\beta x + i\sin\beta x\right) + c_2\left(\cos\beta x - i\sin\beta x\right)\right]$
$= e^{\alpha x}\left[\left(c_1 + c_2\right)\cos\beta x + i\left(c_1 - c_2\right)\sin\beta x\right]$

$\therefore\ y = e^{\alpha x}\left(A\cos\beta x + B\sin\beta x\right)$, where $A = c_1 + c_2$ and $B = i\left(c_1 - c_2\right)$ are arbitrary constants.

If the complex root $\alpha \pm i\beta$ is repeated n times then the general solution is $y = e^{\alpha x}\left[(a_1 + a_2 x + a_3 x^2 + \ldots\ldots\ldots\ldots + a_{n-1}x^{n-1})\cos\beta x + (b_1 + b_2 x + b_3 x^2 + \ldots\ldots + b_{n-1}x^{n-1})\sin\beta x\right]$

31. Find the General solution for the differential equation $4y'' + 4y' - 3y = 0$**.**

Solution: Auxiliary equation is $4m^2 + 4m - 3 = 0$

Roots are $m_1 = \frac{1}{2}, m_2 = -\frac{3}{2}$

$\therefore$ The complementary function is $y = c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{3}{2}x}$

32. Solve: $2y'' - gy' = 0$**.**

Solution: Auxiliary equation is $2m^2 - gm = 0$

$m_1 =$ m (2m-g) =0 $\Rightarrow\ m_1 = 0, m_2 = \frac{g}{2}$

The general solution is $y = c_1 e^{0x} + c_2 e^{\frac{g}{2}x}$.

33. Solve: $y'' + 2ky' + k^2 y = 0$

Solution: $(m + k)^2 = 0$

i.e. $m = -k, -k$

Roots are repeated. Hence the general solution is y= $(c_1 + c_2 x)e^{-kx}$

34. Solve the initial value problem $\left(D^3 - D^2 - 4D + 4\right)y = 0$**,** $y(0) = 1, y'(0) = 2\ and\ y''(0) = 1$

Solution: Auxiliary equation is $m^3 - m^2 - 4m + 4 = 0$

$\Rightarrow m^2(m-1) - 4(m-1) = 0$

$\Rightarrow (m-1)\left(m^2 - 4\right) = 0$

$\Rightarrow m = 1, m = \pm 2$

General solution is $y = c_1 e^x + c_2 e^{2x} + c_3^{-2x}$ ………………… (1)

Differentiating (1) w.r.t x

$y' = c_1 e^x + 2c_2 e^{2x} - 2c_3 e^{-2x}$ ………………… (2)

Differentiating (2) w.r.t x

$y'' = c_1 e^x + 4c_2 e^{2x} + 4c_3 e^{-2x}$ ..……………….. (3)

Using the conditions in (1), (2) and (3), we obtain

$$\left.\begin{aligned} c_1 + c_2 + c_3 &= 1 \\ c_1 + 4c_2 + 4c_3 &= 1 \\ c_1 + 2c_2 - 2c_3 &= 2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots.... (4)$$

Solving above system of equations, we obtain $c_1 = 1, c_2 = \frac{1}{4}$ and $c_3 = -\frac{1}{4}$

Hence the general solution is $y = e^x + \frac{e^{2x}}{4} - \frac{e^{-2x}}{4}$

35. **Solve:** $(D^3 - 2D + 4)^2$ **y=0**

Solution: Auxiliary equation is $(m^3 - 2m + 4)^2 = 0$

$(m^3 - 2m + 4)\left(m^3 - 2m + 4\right) = 0$

i.e. $(m^3 - 2m + 4) = 0$; $\left(m^3 - 2m + 4\right) = 0$

$\Rightarrow m = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm i\,2\sqrt{3}}{2} = 1 \pm i\sqrt{3}$

$\therefore\ m = 1 \pm i\sqrt{3}$, $m = 1 \pm i\sqrt{3}$

Hence the complete solution is $y = e^x\left[(c_1 + c_2 x)\cos\sqrt{3}x + (c_3 + c_4 x)\sin\sqrt{3}x)\right]$

36. **Solve** $y''' - 5y'' + 7y' - 3y = 0$

Solution: The auxiliary equation is $m^3 - 5m^2 + 7m - 3 = 0$

$m = 1$ is a root by inspection

By Synthetic division

$$\begin{array}{c|cccc} 1 & 1 & -5 & 7 & -3 \\ & 0 & 1 & -4 & 3 \\ \hline & 1 & -4 & 3 & \boxed{0} \end{array}$$

Now we have $(m-1)(m^2 - 4m + 3) = 0$

i.e. $\left(m-1\right) = 0, \left(m^2 - 4m + 3\right) = 0 \Rightarrow m = 1,\ m = 1, 3$

The roots are $m = 1, 1, 3$

Therefore the general solution is $y = \left(c_1 + c_2 x\right)e^x + c_3\, e^{3x}$

37. **Solve** $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$

Solution: The auxiliary equation is $m^3 - 3m^2 + 3m - 1 = 0$

$m = 1$ is a root by inspection

By Synthetic division

$$\begin{array}{c|cccc} 1 & 1 & -3 & 3 & -1 \\ & 0 & 1 & -2 & -1 \\ \hline & 1 & -2 & 1 & \boxed{0} \end{array}$$

Now we have $(m-1)(m^2 - 2m + 1) = 0$

i.e. $\left(m-1\right) = 0, \left(m^2 - 2m + 1\right) = 0 \Rightarrow m = 1,\ \left(m-1\right)^2 = 0$

The roots are $m = 1, 1, 1$

Roots are real and repeated. Therefore the general solution is

$$y = \left(c_1 + c_2 x + c_3 x^2\right)e^x$$

38. **Solve** $\left(D^4 - 2D^3 + 2D^2 - 2D + 1\right)y = 0$

Solution: The auxiliary equation is $m^4 - 2m^3 + 2m^2 - 2m + 1 = 0$

$m = 1$ is a root by inspection

By Synthetic division

$$\begin{array}{r|rrrrr} 1 & 1 & -2 & 2 & -2 & 1 \\ & 0 & 1 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & -1 & \multicolumn{1}{|r}{0} \end{array}$$

Thus we obtain $m^3 - m^2 + m - 1 = 0$

$m = 1$ is a root by inspection

Again by Synthetic division,

$$\begin{array}{r|rrrr} 1 & 1 & -1 & 1 & -1 \\ & 0 & 1 & 0 & 1 \\ \hline & 1 & 0 & 1 & \multicolumn{1}{|r}{0} \end{array}$$

Now we have, $m^2 + 1 = 0 \Rightarrow m = \pm i$

The roots are $m = 1, 1, \pm i$

Thus the General solution is $y = \left(c_1 + c_2 x\right)e^x + c_3 \cos x + c_4 \sin x$

Non-Homogeneous Equations

A differential equation of the form $\dfrac{d^n y}{dx^n} + a_1 \dfrac{d^{n-1}y}{dx^{n-1}} + a_2 \dfrac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots\ldots\ldots + a_n y = r(x)$ is called as the Non-Homogeneous Linear Differential Equation of nth order with constant coefficients, where $a_1, a_2, a_3, \ldots\ldots a_n$ are real constants. Let us denote $\dfrac{d}{dx} = D, \dfrac{d^2}{dx^2} = D^2, \dfrac{d^3}{dx^3} = D^3$etc, then above equation becomes $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots\ldots\ldots + a_n)y = r(x)$

This is in the form of $f(D)y = r(x)$, where $f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots\ldots\ldots + a_n$

The **General Solution** of the above equation is $y = C.F + P.I$, where, C.F is Complementary function and P.I is particular integral.

i.e $\quad y = y_c + y_p$

Particular Integral

The evaluation of $\dfrac{1}{f(D)} r(x)$ is called as Particular Integral and it is denoted by y_p or φ_p

i.e. $y_p = \dfrac{1}{f(D)} r(x)$

$\dfrac{1}{f(D)} r(x)$ satisfies the equation $f(D) y = r(x)$

i.e. $f(D) \left\{ \dfrac{1}{f(D)} r(x) \right\} = r(x)$

Therefore, it is the particular integral.

Hence, $f(D)$ and $\dfrac{1}{f(D)}$ are **inverse operators.**

Let us prove the following two results:

(a) $\dfrac{1}{D} r(x) = \int r(x) dx$ **(b)** $\dfrac{1}{D-a} r(x) = e^{ax} \int r(x) e^{-ax} dx$

(a)Proof: Let $\dfrac{1}{D} r(x) = y$ ………….. (1)

Operating (1) by D, $D \dfrac{1}{D} r(x) = Dy \Rightarrow r(x) = \dfrac{dy}{dx}$

i.e. $\int r(x) dx = \int \dfrac{dy}{dx} dx \Rightarrow y = \int r(x)\, dx$

[Here no constant is added as the equation (1) does not contain any constant]

Using (1), $\dfrac{1}{D} r(x) = \int r(x)\, dx$

(b)Proof: Let $\dfrac{1}{D-a} r(x) = y$. …………… (2)

Operating by $D-a$, $(D-a) \dfrac{1}{D-a} r(x) = (D-a) y$.

$r(x) = \dfrac{dy}{dx} - ay \Rightarrow \dfrac{dy}{dx} - ay = r(x)$ is a linear equation.

$\therefore$ I.F$= e^{\int -a\, dx} = e^{-ax}$ and the solution of linear equation is

$y(I.F) = \int r(x)\,(I.F) dx \Rightarrow y e^{-ax} = \int r(x) e^{-ax}\, dx$

$\therefore \quad y = e^{ax} \int r(x) e^{-ax} dx$

Using (2), $\dfrac{1}{D-a} r(x) = e^{ax} \int r(x) e^{-ax} dx$

Methods to find Particular Integral

Method 1: Method to find P.I of f (D) y=r(x) where r(x) =e^ax, where a is a constant.

We know that $y=\frac{1}{f(D)}r(x)=\frac{1}{f(D)}e^{ax}$

$\therefore y=\frac{1}{f(a)}e^{ax}$ if $f(a)\neq 0$ [Directly substitute a in place of D]

If $f(a)=0$, then the above rule fails and we proceed further as

i.e. $\frac{1}{f(D)}e^{ax}=x\frac{1}{f'(a)}e^{ax}$

If $f'(a)$=0, we get

$\frac{1}{f(D)}e^{ax}=x^2\frac{1}{f''(a)}e^{ax}$, provided $f'(a)\neq 0$ and so on.

39. **Find the ϕ_p of** $(D^2+5D+6)y=e^x$

Solution: $\phi_p=\frac{1}{D^2+5D+6}e^{3x}$

Replace D by a i.e. 3,

$$\phi_p=\frac{1}{3^2+5(3)+6}=\frac{e^x}{30}\Rightarrow \phi_p=\frac{e^x}{30}$$

40. **Find the y_p of** $(D+2)(D-1)^2y=e^{-2x}+2\sinh x$

Solution: $y_p=\frac{1}{(D+2)(D-1)^2}\{e^{-2x}+2\sinh x\}=\frac{1}{(D+2)(D-1)^2}\left[e^{-2x}+2\left(\frac{e^x-e^{-x}}{2}\right)\right]$

$$y_p=\frac{1}{(D+2)(D-1)^2}e^{-2x}+\frac{1}{(D+2)(D-1)^2}e^{x}-\frac{1}{(D+2)(D-1)^2}e^{-x}$$

Let us evaluate three terms separately

Consider $\frac{1}{(D+2)(D-1)^2}e^{-2x}=\frac{1}{(D+2)}\left[\frac{1}{(D-1)^2}e^{-2x}\right]$

Replace D by -2, we have

$$=\frac{1}{(D+2)}\frac{1}{(-2-1)^2}e^{-2x}=\frac{1}{9}\frac{1}{(D+2)}e^{-2x}$$

By using the result (2) i.e. $\frac{1}{D-a}r(x)=e^{ax}\int r(x)e^{-ax}dx$,we get

$$=\frac{1}{9}e^{-2x}\int e^{-2x}e^{2x}\,dx=\frac{1}{9}e^{-2x}x$$

Now consider the second term ,

$\frac{1}{(D+2)(D-1)^2}e^{x}=\frac{1}{(D-1)^2}\left[\frac{1}{D+2}e^{x}\right]=\frac{1}{3}\frac{1}{(D-1)^2}e^{x}$ [Replacing D by 1]

$$=\frac{1}{3}x\frac{1}{2(D-1)}e^{x}=\frac{x^2}{6}e^{x}$$

And consider the last term $\frac{1}{(D+2)(D-1)^2}e^{-x}=\frac{1}{(-1+2)}\frac{1}{(-1-1)^2}e^{-x}=\frac{1}{4}e^{-x}$

Hence $y_p = x\frac{e^{-2x}}{9} + \frac{x^2}{6}e^x + \frac{1}{4}e^{-x}$

Method 2: Method to find P.I of f (D) y=r(x) where r(x) =sin (ax+b) (or) cos (ax+b), a is constant.

Here $y_p = \frac{1}{f(D^2)}\sin(ax+b)$

In this case replace D^2 by $-a^2$

$$y_p = \frac{1}{f(-a^2)}\sin(ax+b) \text{ provided } f(-a^2) \neq 0$$

If $f(-a^2) = 0$, the above rules fails and we proceed further as

$$y_p = \frac{1}{f(D^2)}\sin(ax+b) = x\frac{1}{f'(-a^2)}\sin(ax+b) \quad \text{provided } f(-a^2) \neq 0 \text{ and so on.}$$

If $f'(-a^2) = 0, \frac{1}{f(D^2)}\sin(ax+b) = x^2\frac{1}{f''(-a^2)}\sin(ax+b)$ provided $f''(-a^2) \neq 0$

Note: Same rule applies even when r(x) =cos (ax+b) or cos(ax)

41. Find y_p of $(D^2 - 2D + 2)y = \cos(x-1)$

Solution: The particular integral is $y_p = \frac{1}{(D^2 - 2D + 2)}\cos(x-1)$

Replace D^2 by -1^2, we get

$$y_p = \frac{1}{(-1-2D+2)}\cos(x-1) = \frac{1}{(1-2D)}\cos(x-1)$$

$$= \frac{(1+2D)}{(1-2D)(1+2D)}\cos(x-1) = \frac{(1+2D)}{(1-4D^2)}\cos(x-1)$$

Now replace D^2 by -1^2, we get

$$= \frac{(1+2D)}{(1-4(-1))}\cos(x-1) = \frac{1}{5}(1+2D)\cos(x-1)$$

$$\therefore y_p = \frac{1}{5}[\cos(x-1) - 2\sin(x-1)]$$

Method 3: Method to find P.I of f (D) y=r(x) where r(x) =x^m , $m \in Z^+$.

Here $y_p = \frac{1}{f(D)}x^m = [f(D)]^{-1}x^{m.}$

Expand $[f(D)]^{-1}$ in ascending powers of D as far as the term in D^m and operate on x^m term by term. Since $(m+1)^{th}$ and the higher derivatives of x^m are zero, we need not consider terms by D^m.

Important Formulae:

1. $(1-D)^{-1} = 1 + D + D^2 + \ldots\ldots\ldots..$
2. $(1+D)^{-1} = 1 - D + D^2 - \ldots\ldots\ldots..$
3. $(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3\ldots\ldots\ldots..$
4. $(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3\ldots\ldots\ldots..$
5. $(1-D)^{-3} = 1 + 3D + 6D^2 + \ldots\ldots\ldots..$
6. $(1+D)^{-3} = 1 - 3D + 6D^2 - \ldots\ldots\ldots..$

42. Find y_p of $(D^2 + D)y = x^2 + 2x + 25.$

Solution: $y_p = \dfrac{1}{D(D+1)}(x^2+2x+25) = \dfrac{1}{D}(1+D)^{-1}(x^2+2x+25)$

$$= \frac{1}{D}(1-D+D^2-.........)(x^2+2x+25)$$

$$= \frac{1}{D}\{x^2+2x+25-(2x+2)-2\}$$

$$= \int (x^2+21)dx = \frac{x^3}{3}+21x$$

$$\therefore y_p = \frac{x^3}{3}+21x$$

Method 4: Method to find P.I of f (D) y=r(x), where r(x) =e^{ax} V, V being the function of x.

Here $y_p = \dfrac{1}{f(D)}e^{ax}V$

In such cases, first take e^{ax} term outside the operator, by substituting $(D+a)$ in place of D.

$$\Rightarrow y_p = e^{ax}\frac{1}{f(D+a)}V$$

Depending upon the nature of V we will solve further.

43. **Find** y_p **of** $(D^2-2D+4)y = e^x\cos x.$

Solution: $y_p = \dfrac{1}{(D^2-2D+4)}e^x\cos x.$

Here add 1 to D ,

$$y_p = e^x\frac{1}{(D+1)^2-2(D+1)+4}\cos x$$

$$= e^x\frac{1}{D^2+3}\cos x$$

Now replace D^2 by -1^2

$$= e^x\frac{1}{-1^2+3}\cos x = e^x\frac{1}{2}\cos x$$

Method 5: Method to find P.I of f (D) y=r(x), where r(x) =x^{m}V ,V is a function of x.

Here $y_p = \dfrac{1}{f(D)}x^mV$

Let $m\neq 0$ and $V=\sin ax$

Then $y_p = \dfrac{1}{f(D)}x^m\sin ax$

W.K.T. $e^{i\theta} = \cos\theta + i\sin\theta$

i.e. $\sin\theta = I.P.\left(e^{i\theta}\right)$

$$\therefore y_p = \frac{1}{f(D)}x^m I.P(e^{iax})$$

$$= I.P.\frac{1}{f(D)}e^{iax}x^m$$

$$y_p = I.P.e^{iax}\frac{1}{f(D+ia)}x^m$$

Let $m\neq 0$ and $V=\cos ax$

Then $y_p = \dfrac{1}{f(D)}x^m\cos ax$

W.K.T. $e^{i\theta} = \cos\theta + i\sin\theta$

i.e. $\cos\theta = R.P.\left(e^{i\theta}\right)$

$$\therefore y_p = \frac{1}{f(D)}x^m R.P(e^{iax})$$

$$= R.P.\frac{1}{f(D)}e^{iax}x^m$$

$$y_p = R.P.e^{iax}\frac{1}{f(D+ia)}x^m$$

We will solve further by using previous methods and finally substitute $e^{iax} = \cos ax + i\sin ax$.

44. **Solve** $(D^2+4)y = e^{4x}$

Solution: The auxiliary equation is $m^2+4=0$

The roots are $m=\pm 2i$

The complementary function is $y_c=c_1\cos 2x+c_2\sin 2x$

The particular integral is $y_p=\dfrac{1}{(D^2+4)}e^{4x}$

Replacing D by 4, we get

$$y_p=\frac{1}{(4^2+4)}e^{4x}=\frac{1}{20}e^{4x}$$

$\therefore$ The general solution is $y=y_c+y_p$

i.e. $y=c_1\cos 2x+c_2\sin 2x+\dfrac{1}{20}e^{2x}$

45. **Solve** $\dfrac{d^4x}{dt^4}+4x=\sinh t$

Solution: We have $(D^4+4)x=\sinh t$

Auxiliary equation is $m^4+4=0$

i.e. $(m^2+2)^2-4m^2=0$

Or $[(m^2+2)-2m]\,[(m^2+2)+2m]=0$

i.e. $m^2-2m+2=0$; $m^2+2m+2=0$

$$\therefore m=\frac{-2\pm 2i}{2}=-1\pm i \quad ; \quad m=\frac{2\pm 2i}{2}=1\pm i$$

The complementary function is $x_c=e^t(c_1\cos t+c_2\sin t)+e^{-t}(c_3\cos t+c_4\sin t)$

The particular integral is

$$x_p=\frac{1}{D^4+4}\sinh t=\frac{1}{2}\left[\frac{e^t-e^{-t}}{D^4+4}\right]=\frac{1}{2}\left[\frac{e^t}{D^4+4}-\frac{e^{-t}}{D^4+4}\right]$$

$$=\frac{1}{2}\left[\frac{e^t}{1+4}-\frac{e^{-t}}{1+4}\right]=\frac{\sinh t}{5}$$

Thus the general solution is $y=y_c+y_p$

$$\therefore \quad y=e^t(c_1\cos t+c_2\sin t)+e^{-t}(c_3\cos t+c_4\sin t)+\sinh t/5$$

46. **Solve** $(D^4-18D^2+18)y=36e^{3x}+8^x$

Solution: Auxiliary equation is $m^4-18m^2+18=0$

i.e. $(m^2-9)^2=0$

$(m-3)^2(m+3)^2=0$

The roots are $m=3,3,-3,-3$

$\therefore y_c=(c_1+c_2x)e^{3x}+(c_3+c_4x)e^{-3x}$

$$y_p=\frac{36}{D^4-18D^2+81}e^{3x}+\frac{1}{D^4-18D^2+81}8^x$$

$$=36\frac{1}{D^4-18D^2+81}e^{3x}+\frac{1}{(D^4-18D^2+81)}e^{(\log 8)x}$$

Replacing D by a , we get

$$=36\frac{1}{3^4-18(3^2)+81}e^{3x}+\frac{1}{(\log 8)^4-18(\log 8)^2+81)}e^{(\log 8)x}$$

Denominator is zero in the first term , it follows that

$$\therefore y_p=36x\frac{1}{4D^3-36D}e^{3x}+\frac{8^x}{(\log 8)^4-18(\log 8)^2+81}$$

Replacing D by 3 in the first term , we get

$$= 36x\frac{1}{4(3)^3 - 36(3)}e^{3x} + \frac{8^x}{(\log 8)^4 - 18(\log 8)^2 + 81)}$$

Again the denominator is zero in the first term , hence it follows that

$$y_p = 36x^2\frac{1}{12D^2 - 36}e^{3x} + \frac{8^x}{(\log 8)^4 - 18(\log 8)^2 + 81)}$$

Thus $y_p = 36x^2\frac{e^{3x}}{72} + \frac{8^x}{(\log 8)^4 - 18(\log 8)^2 + 81)}$

The general solution is

$$y = (c_1 + c_2x)e^{3x} + (c_3 + c_4x)e^{-3x} + x^2\frac{e^{3x}}{2} + \frac{8^x}{(\log 8)^4 - 18(\log 8)^2 + 81}$$

47. Solve $(D^4 + 4)y = 4\sin 2x + \cos 5x$

Solution: Auxiliary equation is $m^2 + 4 = 0$

The roots are $m = \pm 2i$

The complementary function is $y_c = c_1\cos 2x + c_2\sin 2x$

$$y_p = \frac{1}{D^2 + 4}(4\sin 2x + \cos 5x)$$

$$= 4\frac{1}{D^2 + 4}\sin 2x + \frac{1}{D^2 + 4}\cos 5x = P_1 + P_2$$

Consider $P_1 = 4\frac{1}{D^2 + 4}\sin 2x$

Replacing D^2 by -2^2, we get

$P_1 = 4\ \frac{1}{-(2)^2 + 4}\sin 2x$ [Denominator is zero]

It follows that $P_1 = 4x\frac{1}{2D}\sin 2x = 2x\int \sin 2x\ dx = -x\ \cos 2x$

Now consider $P_2 = \frac{1}{D^2 + 4}\cos 5x$

Replacing D^2 by -5^2, we get

$$P_2 = \frac{1}{-(5)^2 + 4}\cos 5x = \frac{-1}{21}\cos 5x$$

Thus $y_p = -x\ \cos 2x - \frac{1}{21}\cos 5x$

Hence, the general solution is $y = c_1\cos 2x + c_2\sin 2x - x\cos 2x - \frac{1}{21}\cos 5x$

48. Solve $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 4\sin^2 x$

Solution: Auxiliary equation is $m^4 + 8m^2 + 16 = 0$

Or $(m^2 + 4)^2 = 0 \Rightarrow m = \pm 2i, \pm 2i$

$\therefore\ y_c = (c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x$

$$y_p = \frac{1}{D^4 + 8D^2 + 16}4\sin^2 x = \frac{1}{D^4 + 8D^2 + 16}2(1 - \cos 2x)$$

$$y_p = 2\frac{1}{D^4 + 8D^2 + 16} - \frac{1}{D^4 + 8D^2 + 16}2\cos 2x = P_1 - P_2$$

Consider $P_1 = 2\frac{1}{D^4 + 8D^2 + 16}e^{0x} = \frac{2}{0 + 0 + 16}e^{0x} = \frac{1}{8}$

Now consider $P_2 = 2\frac{1}{D^4 + 8D^2 + 16}\cos 2x$

Now replacing D^2 by $-(2)^2$ i.e. -4

$$P_2 = 2\frac{1}{(-4)^2 + 8(-4) + 16}\cos 2x = \frac{2}{32-32}\cos 2x \quad \text{(Denominator is zero)}$$

$$\therefore P_2 = 2x\frac{1}{4D^3 + 16D}\cos 2x = \frac{2x}{4}\frac{1}{(D^2 D + 4D)}\cos 2x$$

Replacing D^2 by -4, we get

$$= \frac{x}{2}\frac{1}{(-4D+4D)}\cos 2x \qquad \text{(Again the denominator is zero)}$$

$$\therefore P_2 = \frac{x^2}{2}\frac{1}{(3D^2+4)}\cos 2x$$

Replacing D^2 by -4, we get $P_2 = \frac{x^2}{2}\frac{1}{(-12+4)}\cos 2x = \frac{-x^2\cos 2x}{16}$

Substitute P_1 and P_2 in y_p,

Thus $y_p = \frac{1}{8} - \frac{(-x^2\cos 2x)}{16} = \frac{1}{8} + \frac{x^2\cos 2x}{16}$

$\therefore$ The general solution is $y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x + \frac{1}{8} + \frac{x^2\cos 2x}{16}$

Solve $(D^2 + 5D - 6)y = \cos 4x \cos x$

Solution: A. E is $m^2 + 5m - 6 = 0$

Roots are $m = 1, -6$

The Complementary function is $y_c = c_1 e^x + c_2 e^{-6x}$

The Particular Integral is $y_p = \frac{1}{D^2 + 5D - 6}\cos 4x \cos x$

$$= \frac{1}{(D^2+5D-6)}\frac{1}{2}(\cos 5x + \cos 3x)$$

$$= \frac{1}{2}\left\{\frac{1}{(D^2+5D-6)}\cos 5x + \frac{1}{(D^2+5D-6)}\cos 3x\right\}$$

$$y_p = \frac{1}{2}(P_1 + P_2)$$

Consider $P_1 = \frac{1}{D^2+5D-6}\cos 5x$

Replacing D^2 by -5^2, we get

$$P_1 = \frac{1}{-25+5D-6}\cos 5x = \frac{1}{5D-31}\cos 5x = \frac{1}{5D-31}\cos 5x$$

$$= \frac{(5D+31)}{(5D-31)(5D+31)}\cos 5x = \frac{(5D+31)}{(25D^2-961)}\cos 5x$$

Now replacing D^2 by -5^2, we get

$$P_1 = \frac{(5D+31)}{(-625-961)}\cos 5x = -\frac{1}{1586}(5D+31)\cos 5x$$

$$= -\frac{1}{1586}(-25\sin 5x + 31\cos 5x) = \frac{1}{1586}(25\sin 5x - 31\cos 5x)$$

Now consider $P_2 = \frac{1}{D^2+5D-6}\cos 3x$

Replacing D^2 by -3^2, we get

$$P_2 = \frac{1}{(-9+5D-6)}\cos 3x = \frac{1}{5D-15}\cos 3x = \frac{1}{5(D-3)}\cos 3x$$

$$= \frac{(D+3)}{5(D-3)(D+3)}\cos 3x = \frac{1}{5}\frac{(D+3)}{(D^2-9)}\cos 3x$$

Replacing D^2 by -3^2, we get

$$P_2 = \frac{1}{5}\frac{(D+3)}{(-9-9)}\cos 3x = \frac{-1}{90}(D+3)\cos 3x = \frac{3}{90}\sin 3x - \frac{3}{90}\cos 3x$$

$$\therefore P_2 = \frac{1}{30}\sin 3x - \frac{1}{30}\cos 3x$$

Substitute P_1 and P_2 in y_p,

Thus $y_p = \frac{1}{2}\left[\frac{1}{1586}(25\sin 5x - 31\cos 5x) + \frac{1}{30}(\sin 3x - \cos 3x)\right]$

Hence the general solution is $y = y_c + y_p$

i.e. $y = c_1e^x + c_2e^{-6x} + \frac{1}{2}\left[\frac{1}{1586}(25\sin 5x - 31\cos 5x) + \frac{1}{30}(\sin 3x - \cos 3x)\right]$

49. Solve $(D^2 - D)y = x^2 - 2x - 32$

Solution: A. E is $m^2 - m = 0$

i.e. $m(m-1) = 0$

The roots are $m = 0, 1$

The complementary function is $y_c = c_1e^{0x} + c_2e^x$

Particular Integral is $y_p = \frac{1}{D(D-1)}(x^2 - 2x - 32)$

$$= \frac{1}{D}(D-1)^{-1}(x^2-2x-32) = \frac{1}{D}(1+D+D^2)(x^2-2x-32)$$

$$= \frac{1}{D}\left[x^2 - 2x - 32 + (2x-2) + (2)\right] = \int (x^2 - 32)dx = \frac{x^3}{3} - 32x$$

$\therefore$ The general solution is $y = c_1 + c_2e^{-x} + \frac{x^3}{3} - 32x$

50. Solve $y'' + 3y' + 2y = \sin x + e^x + 2x^2$

Solution: A. E is $m^2 + 3m + 2 = 0$

The roots are $m = -1, -2$

The complementary function is $y_c = c_1e^{-x} + c_2e^{-2x}$

Particular Integral is $y_p = \frac{1}{(D^2+3D+2)}(\sin x + e^x + 2x^2)$

$$= \frac{1}{(D^2+3D+2)}\sin x + \frac{1}{(D^2+3D+2)}e^x + \frac{1}{(D^2+3D+2)}2x^2$$

$y_p = P_1 + P_2 + P_3$

Consider $P_1 = \frac{1}{(D^2+3D+2)}\sin x$

Here replacing D^2 by -1^2, we get

$$P_1 = \frac{1}{(-1+3D+2)}\sin x = \frac{1}{3D+1}\sin x = \frac{(3D-1)}{(9D^2-1)}\sin x$$

Now replacing D^2 by -1^2, we get

$$P_1 = -\frac{1}{10}(3D-1)\sin x = -\frac{3\cos x}{10} + \frac{\sin x}{10}$$

Now consider $P_2 = \frac{1}{(D^2+3D+2)} e^x$

Replacing D by1,we get $P_2 = \frac{1}{(1+3+2)} e^x = \frac{e^x}{6}$

Consider $P_3 = \frac{1}{(D^2+3D+2)} 2x^2 = 2\frac{1}{2\left[1+\left(\frac{D^2+3D}{2}\right)\right]} x^2$

$$= \left[1+\left(\frac{D^2+3D}{2}\right)\right]^{-1} x^2 = \left[1-\left(\frac{D^2+3D}{2}\right)+\left(\frac{D^2+3D}{2}\right)^2\right]x^2$$

$$= x^2 - \left(\frac{2}{2}+\frac{6x}{2}\right)+\left(\frac{D^4+9D^2+6D^3}{4}\right)x^2$$

$$\therefore P_3 = x^2 - (1+3x) + \frac{18}{4} = x^2 - 3x + \frac{7}{2}$$

Thus $y_p = P_1 + P_2 + P_3 = \frac{\sin x}{10} - \frac{3\cos x}{10} + \frac{e^x}{6} + x^2 - 3x + \frac{7}{2}$

Hence the general solution is $y = c_1 e^{-x} + c_2 e^{-2x} + \frac{\sin x}{10} - \frac{3\cos x}{10} + \frac{e^x}{6} + x^2 - 3x + \frac{7}{2}$

51. Solve $(D^2+1)^2 y = x^4 + 2\sin x \cos 3x$

Solution: A. E is $(m^2+1)^2 = 0 \Rightarrow (m^2+1)(m^2+1) = 0$

The roots are $m = \pm i, \pm i$

The complementary function is $y_c = e^{0x}[(c_1+c_2x)\cos x + (c_3+c_4x)\sin x]$

Particular Integral is $y_p = \frac{1}{(D^2+1)^2}(x^4 + 2\sin x \cos 3x)$

$$y_p = \frac{1}{(D^2+1)^2} x^4 + \frac{2}{(D^2+1)^2}\frac{1}{2}(\sin 4x - \sin 2x)$$

$$= (1+D^2)^{-2} x^4 + \frac{1}{(D^2+1)^2}\sin 4x - \frac{1}{(D^2+1)^2}\sin 2x$$

$$= [1 - 2D^2 + 3D^4 - 4D^6 + \dots\dots\dots]x^4 + \frac{1}{(-4^2+1)^2}\sin 4x - \frac{1}{(-2^2+1)^2}\sin 2x$$

$$y_p = [x^4 - 2(12x^2) + 3(24)] + \frac{1}{225}\sin 4x - \frac{1}{9}\sin 2x$$

$$y = [(c_1+c_2x)\cos x + (c_3+c_4x)\sin x] + x^4 - 24x^2 + 72 + \frac{\sin 4x}{225} - \frac{\sin 2x}{9}$$

52. Solve $(D^2+4D+5)y = e^{-2x}\cos x$

Solution: A. E is $m^2 + 4m + 5 = 0$

i.e. $m = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$

The roots are $m = -2 \pm i$

The complementary function is $y_c = e^{-2x}[c_1 \cos x + c_2 \sin x]$

Particular Integral is $y_p = \frac{1}{(D^2+4D+5)} e^{-2x}\cos x$

Add -2 to D , we get

$$y_p = e^{-2x} \frac{1}{\left((D-2)^2+4(D-2)+5\right)}\cos x = e^{-2x}\frac{1}{(D^2+1)}\cos x$$

Replacing D^2 by -1^2, denominator is zero

$$\therefore\ y_p = e^{-2x} x \frac{1}{2D}\cos x = \frac{x}{2}e^{-2x}\int \cos x\, dx$$

$$y_p = \frac{x}{2}e^{-2x}\ \sin x$$

Hence, the general solution is

$$y = e^{-2x}[c_1 \cos x + c_2 \sin x] + \frac{x}{2}e^{-2x}\ \sin x$$

53. **Solve** $(D^3 - 7D - 6)y = (1+x)e^{2x}$

Solution: A. E is $m^3 - 7m - 6 = 0$

$m = -1$ is a root by inspection.

By Synthetic Division,

$$\begin{array}{r|rrrr}
-1 & 1 & 0 & -7 & -6 \\
 & 0 & -1 & 1 & 6 \\
\hline
 & 1 & -1 & -6 & \boxed{0}
\end{array}$$

Thus we obtain $(m+1)(m^2 - m - 6) = 0 \Rightarrow (m+1)(m-3)(m+2) = 0$

The roots are $m = -1, 3, -2$

The Complementary function is $y_c = c_1 e^{-x} + c_2 e^{3x} + c_3 e^{-2x}$

The Particular Integral is $y_p = \frac{1}{(D^3 - 7D - 6)}\left(e^{2x} + xe^{2x}\right)$

$$y_p = \frac{1}{(D^3-7D-6)}e^{2x} + \frac{1}{(D^3-7D-6)}xe^{2x} = P_1 + P_2$$

Consider $P_1 = \frac{1}{(D^3-7D-6)}e^{2x}$

Replace D by a i.e. 2,

$$\therefore P_1 = -\frac{e^{2x}}{12}$$

Now let us consider $P_2 = \frac{1}{(D^3-7D-6)}xe^{2x}$

Add 2 to D, we get $P_2 = e^{2x}\frac{1}{(D+2)^3 - 7(D+2) - 6}\ x$

$$= e^{2x}\frac{1}{(D^3+6D^2+5D-12)}x = e^{2x}\frac{1}{(-12)}\frac{1}{\left[1-\left(\frac{D^3+6D^2+5D}{12}\right)\right]}\ x$$

$$= -\frac{e^{2x}}{12}\left[1-\left(\frac{D^3+6D^2+5D}{12}\right)\right]^{-1} x$$

$$= -\frac{e^{2x}}{12}\left[1+\left(\frac{D^3+6D^2+5D}{12}\right)\right]x = -\frac{e^{2x}}{12}\left[x+\frac{5}{12}\right]$$

$$\therefore y_p = -\frac{e^{2x}}{12} - \frac{e^{2x}}{12}\left(x + \frac{5}{12}\right) = -\frac{e^{2x}}{12}\left(x + \frac{7}{12}\right)$$

Hence the general solution is $y = c_1 e^{-x} + c_2 e^{3x} + c_3 e^{-2x} - \frac{e^{2x}}{12}\left(x + \frac{7}{12}\right)$

54. **Solve** $y'' + 9y = x \cos 2x$

Solution: A.E is $m^2 + 9 = 0$

$\Rightarrow m = \pm 3i$

The Complementary function is $y_c = c_1 \cos 3x + c_2 \sin 3x$

The Particular Integral is $y_p = \frac{1}{(D^2+9)} x \cos 2x$

$$= \frac{1}{(D^2+9)} x \; R.P\left(e^{i2x}\right)$$

$$= R.P\left[e^{i2x} \frac{1}{(D+i2)^2 + 9} x\right] = R.P\left[e^{i2x} \frac{1}{(D^2 + 4i^2 + 4Di + 9)} x\right]$$

$$= R.P\left[e^{i2x} \frac{1}{(D^2 + 4iD + 5)} x\right] = R.P.\left[e^{i2x} \frac{1}{5} \frac{1}{\left(1 + \left(\frac{D^2 + 4iD}{5}\right)\right)} x\right]$$

$$= \frac{1}{5} R.P.\left[e^{i2x}\left(1 + \frac{D^2 + 4iD}{5}\right)^{-1} x\right] = \frac{1}{5} R.P.\left[e^{i2x}\left(x - \frac{4i}{5}\right)\right]$$

$$= \frac{1}{25} R.P.[(\cos 2x + i \sin 2x)(5x - 4i)] = \frac{1}{25} R.P.[(5x \; \cos 2x + 4 \sin 2x) + i(5x \; \sin 2x - 4 \cos 2x)]$$

$$y_p = \frac{1}{25}(5x \cos 2x + 4 \sin 2x)$$

The general solution is $y = c_1 \cos 4x + c_2 \sin 4x + \frac{1}{5} x \cos 2x + \frac{4}{25} \sin 2x$

55. **Solve** $(D^2 - 1)y = x^2 \sin 3x$

Solution: A.E is $m^2 - 1 = 0 \Rightarrow m^2 = 1$

The roots are $m = 1, -1$

The complementary function is $y_c = c_1 e^x + c_2 e^{-x}$

And the particular integral is $y_p = \frac{1}{(D^2 - 1)} x^2 \sin 3x$

$$= \frac{1}{(D^2 - 1)} x^2 \; I.P.\left(e^{i3x}\right)$$

Changing D to (D+i3), we get

$$= I.P\left[e^{i3x} \frac{1}{((D+i3)^2 - 1)} x^2\right] = I.P.\left[e^{i3x} \frac{1}{(D^2 - 9 + 6Di - 1)} x^2\right]$$

$$= I.P\left[e^{i3x}\frac{1}{\left(D^2+6Di-10\right)}x^2\right] = I.P\left[e^{i3x}\frac{1}{(-10)}\frac{1}{\left(1-\left(\frac{D^2+6Di}{10}\right)\right)}x^2\right]$$

$$= -\frac{1}{10}\ I.P\left[e^{i3x}\left(1-\left(\frac{D^2+6Di}{10}\right)\right)^{-1}x^2\right]$$

$$= -\frac{1}{10}\ I.P\left[e^{i3x}\ \left(1+\left(\frac{D^2+6Di}{10}\right)+\left(\frac{D^2+6Di}{10}\right)^2\right)x^2\right]$$

$$= -\frac{1}{10}\ I.P\left[e^{i3x}\ \left(x^2+\frac{2}{10}+\frac{12xi}{10}-\frac{72}{100}\right)\right] = -\frac{1}{10}I.P.\left[\left(\cos 3x + i\sin 3x\right)\left(x^2-\frac{26}{50}+\frac{6xi}{5}\right)\right]$$

$$\therefore y_p = -\frac{1}{10}\left[\frac{6x}{5}\cos 3x + x^2\sin 3x - \frac{13}{25}\sin 3x\right] = \frac{13}{250}\sin 3x - \frac{x^2}{10}\sin 3x - \frac{3x}{25}\cos 3x$$

Hence, the general solution is $y = y_c + y_p$

i.e. $y = c_1e^{x} + c_2e^{-x} + \frac{13}{250}\sin 3x - \frac{x^2}{10}\sin 3x - \frac{3x}{25}\cos 3x$

Method of variation of parameters:

Introduction

We are familiar with the method of solving linear differential equations with constant coefficients. In this chapter we discuss how to reduce linear differential equation with variable coefficients to equations with constant coefficients and solve by the method discussed in last chapter.

And also we discuss one particular method known as method of variation of parameters to solve linear differential equations with constant coefficients of second degree.

In this section we discuss one method that can be used to obtain particular integral of linear differential equation of the form $y'' + a_1y' + a_2y = X$

This method can be used to solve the differential equations of the form $y'' + a_1y' + a_2y = X$ where a_1, a_2 may be functions of x or constants and X is function of x. The method is important because it solves the largest class of equations, specifically included are functions X like $\ln x,\ e^{x^m}, \tan ax, \sec ax, \cos\left(e^{-x}\right),\ \frac{e^{mx}}{x^n}$ etc.

Theorem

The particular integral of differential equation $y'' + a_1y' + a_2y = X$ is $-y_1\int\frac{y_2X}{W}dx + y_2\int\frac{y_1X}{W}dx$

Proof:

Let the complimentary function of differential equation be $y = c_1y_1 + c_2y_2$.

Replace c_1 and c_2 regarded as parameters by unknown functions $A(x)$ and $B(x)$

Let the P.I .be $y = Ay_1 + By_2$. ……………… (1)

Differentiate (1) w.r.t. x, we get

$y' = Ay_1' + By_2' + A'y_1 + B'y_2 = Ay_1' + By_2'$ ……………… (2)

Assume that $A'y_1 + B'y_2 = 0$ ……………… (3)

Differentiate (3) and substitute in (1).Then we obtain $A'y_1' + B'y_2' = X$ ..……………. (4)

Solve (3) and (4), we get

$A' = -\frac{y_2 X}{y_1 y_2' - y_2 y_1'}, B' = \frac{y_1 X}{y_1 y_2' - y_2 y_1'}$.

Let W= $y_1 y_2' - y_2 y_1'$. is called Wronskian of the functions y_1 and y_2

Integrate $A = -\int \frac{y_2 X}{W} dx, B = \int \frac{y_1 X}{W} dx$.

Substitute the values of u and v in (1) we get the P.I.

Finally the general solution is given by $y = CF + PI$.

Working Rule:

Consider the differential equation of the form $y'' + a_1 y' + a_2 y = X$

1. Find complimentary function of the above equation $y_c = c_1 y_1 + c_2 y_2$.
2. Assume $y = Ay_1 + By_2$ be the complete solution of differential equation where A, B are functions of x.
3. We compute y_1', y_2'' and W= $y_1 y_2' - y_2 y_1'$. $\left[W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \right]$
4. A= $-\int \frac{y_2 X}{W} dx$,B = $\int \frac{y_1 X}{W} dx$ and substitute in $y = Ay_1 + By_2$ and the complete solution is $y = CF + PI$.

56. Solve by the method of variation of parameters, $\frac{d^2 y}{dx^2} - 8\frac{dy}{dx} + 16y = \frac{e^{4x}}{x^2}$

Solution: Given $(D^2 - 8D + 16)y = \frac{e^{4x}}{x^2}$.

AE: $m^2 - 8m + 16 = 0 \quad or \quad (m-4)^2 = 0 \qquad \Rightarrow \quad m = 4,4.$

$y_c = (c_1 + c_2 x)e^{4x}$

Let the assumed PI be $y_p = Ae^{4x} + Bxe^{4x}$,where A and B are functions of x to be found .

We have A= $-\int \frac{y_2 X}{W} dx$, B= $\int \frac{y_1 X}{W} dx$

Here $y_1 = e^{4x}$, $y_2 = xe^{4x} \Rightarrow y'_1 = 4e^{4x}$, $y_2' = 4xe^{4x} + e^{4x}$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{4x} & xe^{4x} \\ 4e^{4x} & 4xe^{4x} + e^{4x} \end{vmatrix} = e^{8x}$

Thus P.I. $= -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx = \underline{-e^{4x} \int \frac{dx}{x} + xe^{4x} \int x^{-2} dx}$

$\Rightarrow -e^{3x}(\log x + 1)$. Hence the solution is y= $(c_1 + c_2 x)e^{4x} - e^{4x}(\log x + 1)$

57. Solve, by the method of variation of parameters $y'' - 2y' + y = \frac{e^t}{1+t^2}$

Solution: Given $(D^2 - 2D + 1)y = \frac{e^t}{1+t^2}$.

AE: $m^2 - 2m + 1 = 0 \quad or \quad (m-1)^2 = 0 \qquad \Rightarrow \quad m = 1,1.$

$y_c = (c_1 + c_2 t)e^t$

Let the assumed PI be $y_p = Ae^t + Bte^t$,where A and B are functions of x to be found .

We have A= $-\int \frac{y_2 X}{W} dt$, B= $\int \frac{y_1 X}{W} dt$

Here $y_1 = e^t$, $y_2 = te^t \Rightarrow y'_1 = e^t$, $y_2' = te^t + e^t$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = e^{2t}$

Thus P.I. $=-y_1\int \frac{y_2X}{W}dx+y_2\int \frac{y_1X}{W}dx$ $=\underline{-e^t\int \frac{t}{1+t^2}dt+te^t\int \frac{1}{1+t^2}dt}$

$\Rightarrow -\frac{1}{2}e^t\ln\left(1+t^2\right)+te^t\tan^{-1}t$.

Hence the solution is y=$(c_1+c_2t)e^t-\frac{1}{2}e^t\ln\left(1+t^2\right)+te^t\tan^{-1}t$

58. Solve, by the method of variation of parameters $y''-5y'+4y=\cos\left(e^{-x}\right)$.

Solution: Given $(D^2-5D+4)y=\cos\left(e^{-x}\right)$

AE: $m^2-5m+4=0 \;\; or \;\; (m-4)(m-1)=0 \Rightarrow m=1,4$

$y_c=c_1e^x+c_2e^{4x}$

Let the assumed PI be $y_p=Ae^x+Be^{4x}$,where A and B are functions of x to be found.

We have A=$-\int\frac{y_2X}{W}dx$, B=$\int\frac{y_1X}{W}dx$

Here $y_1=e^x$, $y_2=e^{4x}\Rightarrow y'_1=e^x$, $y_2'=4e^{4x}$ and X= $\cos\left(e^{-x}\right)$

$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=3e^{5x}$

P.I.=$-y_1\int \frac{y_2X}{W}dx+y_2\int \frac{y_1X}{W}dx$ $=-\frac{1}{3}e^x\int e^{-x}\cos\left(e^{-x}\right)dx+\frac{1}{3}e^{4x}\int e^{-4x}\cos\left(e^{-x}\right)dx$

$\Rightarrow -\frac{1}{3}e^x\sin\left(e^{-x}\right)-\frac{e^{-4x}}{3}\left[\sin\left(e^{-x}\right)\left(e^{-3x}-6e^{-x}\right)+\cos\left(e^{-x}\right)\left(3e^{-2x}-6\right)\right]$.

Hence the solution is $y=CF+PI$

59. Solve, by the method of variation of parameters $y''+3y'+2y=e^{e^x}$

Solution: Given $(D^2+3D+2)y=e^{e^x}$

AE: $m^2+3m+2=0 \;\; or \;\; (m+1)(m+2)=0 \Rightarrow m=-1,-2$

$y_c=c_1e^{-x}+c_2e^{-2x}$

Let the assumed PI be $y_p=Ae^{-x}+Be^{-2x}$,where A and B are functions of x to be found

We have A $=-\int\frac{y_2X}{W}dx$, B=$\int\frac{y_1X}{W}dx$

Here $y_1=e^{-x}$, $y_2=e^{-2x}\Rightarrow y'_1=e^{-x}$, $y_2'=-2e^{-2x}$

$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=-e^{-3x}$

Thus P.I. $=-y_1\int \frac{y_2X}{W}dx+y_2\int \frac{y_1X}{W}dx$ $=-e^{-x}\int e^xe^{e^x}dx+e^{-2x}\int e^{2x}e^{e^x}dx$

Hence the solution is y= $c_1e^x+c_2e^{2x}-e^{2x}\cos\left(e^{-x}\right)$.

60. Solve, by the method of variation of parameters $\left(D^2-1\right)y=2\left(1-e^{-2x}\right)^{-\frac{1}{2}}$

Solution: Given $\left(D^2-1\right)y=2\left(1-e^{-2x}\right)^{-\frac{1}{2}}$

AE: $m^2-1=0\Rightarrow m=1,-1.$

$y_c=c_1e^x+c_2e^{-x}$

Let the assumed PI be $y_p=Ae^x+Be^{-x}$,where A and B are functions of x to be found.

We have A=$-\int\frac{y_2X}{W}dx$, B=$\int\frac{y_1X}{W}dx$

Here $y_1 = e^x$, $y_2 = e^{-x} \Rightarrow y'_1 = e^x, y_2' = -e^{-x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -2$$

Thus P.I. $= -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$

$$A = -\int \frac{y_2 X}{W} dx = -\int \frac{e^{-x} dx}{\left(1 - e^{-2x}\right)^{\frac{1}{2}}}$$

$$e^{-x} = t, \; -e^{-x} dx = dt$$

$$A = \int \frac{dt}{\sqrt{1-t^2}} = -\sin^{-1} t = -\sin^{-1}\left(e^{-x}\right).$$

$$B = \int \frac{y_1 X}{W} dx = -\int \frac{e^x dx}{\sqrt{1-e^{-2x}}} = -\int \frac{e^{2x} dx}{\sqrt{e^{2x}-1}}$$

$$e^{2x} = t, 2e^{2x} dx = dt. \therefore B = -\frac{1}{2} \int \frac{dt}{\sqrt{t-1}} = -(t-1)^{\frac{1}{2}} = -\left(e^{2x}-1\right)^{\frac{1}{2}}$$

$$\therefore PI = -\sin^{-1}\left(e^{-x}\right)e^x + \left(e^{2x}-1\right)^{\frac{1}{2}} e^x$$

Hence the solution is $y = c_1 e^x + c_2 e^{-x} - \sin^{-1}\left(e^{-x}\right)e^x + \left(e^{2x}-1\right)^{\frac{1}{2}} e^x$

Here we have learnt one more method called "Method of variation of parameters" to solve linear differential equations of second and higher order. In this method, based on the complimentary function of the given differential equation we obtain the complete solution of the differential equation. The complete solution is obtained by replacing the arbitrary constants present in the C.F. by functions of the independent variable. This method can be considered as powerful one as it can be applied to all linear differential equations regardless of the nature of the coefficients and the RHS function, provided CF is known.

Linear differential equations with variable coefficients:

Here we consider linear differential equations with variable coefficients in some specific forms that can be reduced to differential equations with constant coefficients by suitable substitution and further the solution can be obtained by the methods discussed earlier.

Legendre's linear equation:

An equation of the form $(ax+b)^n \frac{d^n y}{dx^n} + k_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots\ldots\ldots + k_n y = X$, where k's are constants and X is a function of x is called **Legendre's linear equation.**

The equation can be reduced to LDE with constant coefficients by taking $ax + b = e^t \Rightarrow t = \log(ax+b)$

Then $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{a}{ax+b}\frac{dy}{dt}$ $\therefore (ax+b)\frac{dy}{dx} = aDy$ where $D = \frac{d}{dt}$

And $\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{a}{ax+b}\frac{dy}{dt}\right) = \frac{-a^2}{(ax+b)}\frac{dy}{dt} + \frac{a}{ax+b}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} = \frac{a^2}{(ax+b)^2}\left(\frac{d^2 y}{dt^2} - \frac{dy}{dt}\right)$

$\therefore (ax+b)^2 \frac{d^2 y}{dx^2} = a^2 D(D-1)y$. Similarly $(ax+b)^3 \frac{d^3 y}{dx^3} = a^3 D(D-1)(D-2)y$ and so on.

Substitute these in the differential equation, then it reduces to linear equation with constant coefficients which can be solved easily.

Cauchy's homogeneous linear equation

An equation of the form $x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots\ldots\ldots + k_n y = X$, where k's are constants and X is a function of x is called Cauchy's linear equation.

The above equation can be reduced to LDE with constant coefficients by taking $x = e^t \Rightarrow t = \log x$, so that $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt} \quad \therefore x\frac{dy}{dx} = Dy$ where $D = \frac{d}{dt}$

Similarly $x^2 \frac{d^2 y}{dx^2} = D(D-1)y$ and so on.

Substitute these in the DE, then it reduces to linear equation with constant coefficients which can be solved easily.

61. **Solve** $x^2 y'' + 3xy' + y = \frac{1}{(1-x)^2}$

Solution: The given differential is Cauchy's linear equation.

Put $x = e^z \Rightarrow z = \log x$ *so that* $x\frac{dy}{dx} = Dy,\ x^2\frac{d^2y}{dx^2} = D(D-1)y$

Then the given equation becomes

$$[D(D-1) + 3D + 1]y = \frac{1}{(1-e^z)^2} \Rightarrow (D^2 + 2D + 1)y = \frac{1}{(1-e^z)^2}$$

$$A.E.\ is\ D^2 + 2D + 1 = 0 \Rightarrow D = -1, -1 \Rightarrow \therefore C.F.\ is\ (c_1 + c_2 x)e^{-z} = (c_1 + c_2 \log x)\frac{1}{x}$$

$$PI = \frac{1}{(D+1)^2}\left[\frac{1}{(1-e^z)^2}\right] = \frac{1}{D+1}u,$$

Here $u = \frac{1}{D+1}.\frac{1}{(1-e^z)^2} \Rightarrow \frac{du}{dt} + u = (1-e^z)^{-2}$

The differential equation is Leibnitz linear equation having $I.F. = e^z$

$$ue^z = \int \frac{e^z}{(1-e^z)^2}dz = \frac{1}{1-e^z} \Rightarrow u = \frac{e^{-z}}{1-e^z}$$

$$\therefore P.I. = \frac{1}{D+1}\left(\frac{e^{-z}}{1-e^z}\right) = e^{-z}\int\frac{1}{1-e^z}dz = \frac{1}{x}\int\frac{dx}{x(1-x)} = \frac{1}{x}\int\frac{1}{x} + \frac{1}{(1-x)}dx$$

$$= \frac{1}{x}[\log x - \log(1-x)] = \frac{1}{x}\log\frac{x}{x-1}$$

Hence the solution is $y = \left\{c_1 + c_2\log x + \log\frac{x}{x-1}\right\}\frac{1}{x}$

62. **Solve:** $(2x-1)^2\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} - 2y = 8x^2 - 2x + 3$

Solution: This is a Legendre's linear equation.

Put $(2x-1) = e^z \Rightarrow z = \log(2x-1)$, that $(2x-1)\frac{dy}{dx} = 2Dy, (2x-1)^2\frac{d^2y}{dx^2} = 4D(D-1)y$

Then equation becomes,

$$4D(D-1)y+2Dy-2y=8\left(\frac{1+e^{z}}{2}\right)^{2}-2\left(\frac{1+e^{z}}{2}\right)+3,$$

$$\Rightarrow 2D^{2}y-Dy-y=e^{2z}+\frac{3}{2}e^{z}+2$$

$$A.E: 2D^{2}-D-1=0 \Rightarrow D=1,-1/2$$

$$\therefore C.F.=c_{1}e^{z}+c_{2}e^{-z/2}$$

$$P.I.=\frac{1}{2D^{2}-D-1}\left(e^{2z}+\frac{3}{2}e^{z}+2\right)=\frac{1}{5}e^{2z}+\frac{3}{2}\frac{z}{4D-1}e^{z}+2\frac{1}{2.0^{2}-0-1}e^{0z}$$

$$=\frac{1}{5}e^{2z}+\frac{3z}{2}\frac{1}{4-1}e^{z}-2=\frac{1}{5}e^{2z}+\frac{z}{2}e^{z}-2.$$

Hence the solution is $y=c_{1}e^{z}+c_{2}e^{-z/2}+\frac{1}{5}e^{2z}+\frac{z}{2}e^{z}-2$

Replacing z by $\log(2x-1)$

$$y=c_{1}(2x-1)+c_{2}(2x-1)^{-1/2}+\frac{1}{5}(2x-1)^{2}+\frac{1}{2}(2x-1)\log(2x-1)-2.$$

63. **Solve** $(3x+2)^{2}y''+3(3x+2)y'-36y=4x(2x+1)$

Solution: This is Legendre's equation. Put $t=\log(3x+2)\Rightarrow e^{t}=3x+2$

Then we have $(3x+2)y'=3.D_{y},(3x+2)^{2}y''=9.D(D-1)y$ where $D=\frac{d}{dt}$

And $x=\frac{1}{3}(e^{t}-2)$, Hence the given equation becomes

$$[9D(D-1)+9D-36]y=\frac{8}{9}(e^{t}-2)^{2}+\frac{4}{3}(e^{t}-2)\Rightarrow(D^{2}-4)y=\frac{8}{9}(e^{2t}+1)-\frac{10}{9}e^{t}$$

A.E. is $m^{2}-4=0\Rightarrow m=\pm2 \;\therefore CF=c_{1}e^{2t}+c_{2}e^{-2t}$

PI= $\left[\frac{8}{9}\frac{1}{D^{2}-4}e^{2t}+\frac{8}{9}\frac{1}{D^{2}-4}e^{0t}-\frac{10}{9}\frac{1}{D^{2}-4}e^{t}\right]\Rightarrow\frac{4}{9}te^{2t}+\frac{4}{27}$

Hence the solution is y= $c_{1}e^{2t}+c_{2}e^{-2t}+\frac{4}{9}te^{2t}+\frac{4}{27}$

64. **Solve** $x\frac{d^{3}y}{dx^{3}}+\frac{d^{2}y}{dx^{2}}=x^{-1}$

Solution: The above equation can be written as $x^{3}y'''+x^{2}y''=x$

Put $e^{t}=x\Rightarrow t=\log x$

Then we have $x\frac{dy}{dx}=Dy,\ x^{2}\frac{d^{2}y}{dx^{2}}=D(D-1)y, x^{3}\frac{d^{3}y}{dx^{3}}=D(D-1)(D-2)y$, where $D=\frac{d}{dt}$

Hence the given equation becomes $D(D-1)^{2}y=e^{t}$

A.E.= $m(m-1)^{2}=0\Rightarrow m=0,1,1\Rightarrow C.F.=c_{1}+(c_{2}+c_{3}t)e^{t}$

PI= $\frac{1}{D^{3}-2D^{2}+D}e^{t}\Rightarrow\frac{t^{2}e^{t}}{2}$,

Here we discussed how to solve LDE with variable coefficients of two specific forms by reducing to LDE with constant coefficients with constant coefficients.

65. **Solve** $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=\sin(\log x^2)$

Solution: The given differential equation is Cauchy's Linear equation.

Put $e^t = x \Rightarrow t=\log x$,

Then we have $x\frac{dy}{dx}=Dy,\ x^2\frac{d^2y}{dx^2}=D(D-1)y,$ where $D=\frac{d}{dt}$

Hence the given equation becomes $D(D-1)y+Dy+y=\sin(2z)\Rightarrow (D^2+1)y=\sin 2z$

$A.E=(m^2+1)=0\Rightarrow m=+i,-i$

$C.F.=c_1\cos z+c_2\sin z$

PI= $\frac{1}{D^2+1}\sin 2z=\frac{1}{-4+1}\sin 2z=-\frac{1}{3}\sin 2z$.

Hence the solution is $y=c_1\cos(\log x)+c_2\sin(\log x)-\frac{1}{3}\sin(\log x^2)$

Here we discussed how to solve linear differential equations with variable coefficients of two specific forms by reducing to differential equations with constant coefficients with constant coefficients.

Power series

Introduction:

Many differential equations arising from physical problems are linear but have variable coefficients and do not permit a general solution in terms of known functions. Such equations can be solved by numerical methods but in many cases it is easier to find a solution in the form of an infinite convergent series.

The series solution of certain differential equations gives rise to special functions such as Bessels function, Legendre's Polynomial, Hermite Polynomial, Cheybhshev Polynomials. These special functions have many applications in Engineering

An expression of the form $C_0+C_1(x-x_0)+C_2(x-x_0)^2+......+C_n(x-x_0)^n+...$ or in the summation form

$\sum_{n=0}^{\infty}C_n(x-x_0)^n$ is known as a Power series of the variable x in powers of x_0 is known as the center of expansion of the power series. Since n takes only positive integral values, the power series does not contain negative or fractional powers. So power series contains only positive powers

This helps in finding solution of a second order Homogeneous differential equation in the form of convergent infinite power series

Power series solution of a second order Ordinary Differential Equation

Consider the DE in the form

$$P_0(x)\frac{d^2y}{dx^2}+P_1(x)\frac{dy}{dx}+P_2(x)y=0 \qquad \text{.............(1)}$$

Where $P_0(x)$, $P_1(x)$ and $P_2(x)$ are polynomials in x with $P_0(x)\neq 0$ at $x=0$.

The method is explained step wise.

- We assume the solution of (1) in the form of a power series,

$$y=\sum_{r=0}^{\infty}a_r x^r \qquad \text{.....................(2)}$$

- Then, $\frac{dy}{dx} = y' = \sum_{r=0}^{\infty} a_r r x^{r-1}$ and $\frac{d^2y}{dx^2} = y'' = \sum_{r=0}^{\infty} a_r r(r-1)x^{r-2}$
- We substitute these along with $y = \sum_{r=0}^{\infty} a_r x^r$ in (1) which results in an infinite series with various powers of x equal to zero
- We equate the coefficients of various powers of x (starting from the lowest power of x) to zero. In general when the coefficient of x^r is equated to zero, we obtain a reccurrence relation which will helps us to determine the constants $a_{2,}\ a_{3,}\ a_{4,}\ a_{5,}\ \ldots\ldots$ interms of $a_{0,}$ and $a_{1,}$
- We substitute the value of $a_{2,}\ a_{3,}\ a_{4,},\ \ldots.$ In the expanded form of (2) $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots\ldots$
- Thus we get the power series solution of the ODE in the form,

  $y = a_0 F(x) + a_1 G(x)$ Where $F(x)$ and $G(x)$ are convergent infinite series in x

Problems

66. Solve $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \sin(\log x^2)$

Solution: The given differential equation is Cauchy's Linear equation.

Example 67: Obtain the series solution of the equation $y' - 2xy = 0$.

Solution:

We have $y' - 2xy = 0$ … (1)

[Note that the coefficient of $y' = 1 = P_0(x) \neq 0 \quad at \quad x = 0$]

Let $y = \sum_{r=0}^{\infty} a_r x^r$ … (2) be the series solution of the given D.E (1)

$$y' = \sum_{r=0}^{\infty} a_r r x^{r-1}$$

Now Equation (1) becomes,

$$\sum_0^{\infty} a_r r(r-1)x^{r-2} - 2x\sum_0^{\infty} a_r x^r = 0$$

$$\sum_0^{\infty} a_r r(r-1)x^{r-2} - 2\sum_0^{\infty} a_r x^{r+1} = 0$$

We equate the coefficients of various powers of x to zero

Coeff. Of x^{-1} : $a_0(0) = 0 \quad \Rightarrow a_0 \neq 0$

Coeff. Of x^0 : $a_1(1) = 0 \quad \Rightarrow a_1 \neq 0$

Now we shall equate the coefficient of $x^r\ (r \geq 0)$ to zero

i.e $\quad a_{r+1}(r+1) - 2a_{r-1} = 0$

or $\quad a_{r+1} = \frac{-2a_{r-1}}{((r+1)} \quad (r \geq 1)$ …………………(3)

By putting $r = 0,1,2,3,4.....$ in (3) we obtain

$$a_2 = \frac{-2a_0}{2} = a_0 \quad ; \quad a_3 = \frac{2a_1}{3} = 0, \text{ since } a_1 = 0$$

$$a_4 = \frac{2a_2}{4} = \frac{1}{2}a_0 \quad ; \quad a_5 = \frac{2a_3}{5} = 0 \quad \text{since } a_3 = 0$$

$$a_6 = \frac{2a_4}{6} = \frac{1}{3}.\frac{1}{2}a_0 = \frac{a_0}{6} \quad ; \quad a_7 = \frac{2a_5}{7} = 0 \text{ since } a_5 = 0 \text{ and so on ……}$$

We substitute these values in the expanded form of (2)

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \text{..........}$$

$$y = a_0 + a_2 x^2 + a_4 x^4 + \text{..........}$$

$$y = a_0\left[1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \text{........}\right]$$

This is the required series solution of the given d.e

67. Obtain the series solution of the equation $\frac{d^2y}{dx^2} + y = 0$.

Solution: We have $y'' + y = 0$ ...(1)

Let $y = \sum_{r=0}^{\infty} a_r x^r$ ...(2)

be the series solution of the given D.E (1)

$$y' = \sum_{r=0}^{\infty} a_r r x^{r-1}, \quad y'' = \sum_{0}^{\infty} a_r r(r-1)x^{r-2}$$

Now Eq (1) becomes, $\sum_{0}^{\infty} a_r r(r-1)x^{r-2} + \sum_{0}^{\infty} a_r x^r = 0$

We equate the coefficients of various powers of x to zero

Coeff. Of x^{-2} : $a_0(0)(1) = 0 \quad \Rightarrow a_0 \neq 0$

Coeff. Of x^{-1} : $a_1(1)(0) = 0 \quad \Rightarrow a_1 \neq 0$

Now we shall equate the coefficient of x^r $(r \geq 0)$ to zero

i.e $a_{r+2}(r+2)(r+1) + a_r = 0$

or $a_{r+2} = \frac{-a_r}{(r+2)(r+1)} \quad (r \geq 0)$ … (3)

By putting $r = 0,1,2,3,4.....$ in (3) we obtain

$$a_2 = \frac{-a_0}{2} \quad ; \quad a_3 = \frac{-a_1}{6}$$

$$a_4 = \frac{-a_2}{12} = \frac{-a_0}{24} \quad ; \quad a_5 = \frac{-a_3}{20} = \frac{-a_1}{120}$$

$$a_6 = \frac{-a_4}{30} = \frac{-a_0}{720} \quad ; \quad a_7 = \frac{-a_5}{42} = \frac{-a_1}{5040} \quad \text{and so on ……}$$

We substitute these values in the expanded form of (2)

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + ...$$

$$y = a_0\left[1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + ...\right] + a_1\left[x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + ...\right]$$

$$y = a_0\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...\right] + a_1\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...\right]$$ is the required series solution of the given D.E.

68. Develop the series solution of the equation $y'' + xy' + (x^2 + 2)y = 0$.

Solution:

$$y'' + xy' + (x^2 + 2)y = 0 \qquad \text{………………………(1)}$$

The coefficient of $y'' = 1 = P_0(x) \neq 0$ at $x = 0$

Let $y = \sum_{r=0}^{\infty} a_r x^r$ ……………………(2)

be the series solution of the given D.E (1)

$$y' = \sum_{r=0}^{\infty} a_r r x^{r-1} \quad , \quad y'' = \sum_{0}^{\infty} a_r r(r-1)x^{r-2}$$

Now (1) becomes

$$\sum_{0}^{\infty} a_r r(r-1)x^{r-2} + \sum_{0}^{\infty} a_r r x^r + \sum_{0}^{\infty} a_r x^{r+2} + 2\sum_{0}^{\infty} a_r x^r = 0$$

We equate the coefficients of various powers of x to zero. We first equate the coefficients of x^{-2}, x^{-1}, x^0, x^1 (except the third one) to zero

Coeff. Of x^{-2} : $a_0(0)(1) = 0 \Rightarrow a_0 \neq 0$

Coeff. Of x^{-1} : $a_1(1)(0) = 0 \Rightarrow a_1 \neq 0$

Coeff. Of x^0 : $a_2(2)(1) + a_0(0) + 2a_0 = 0$

$$2a_2 + 2a_0 = 0 \quad \Rightarrow a_2 = -a_0$$

Coeff. Of x^{-1} : $a_3(3)(2) + a_1(1) + 2a_1 = 0$

$$6a_3 + 3a_1 = 0 \quad \Rightarrow a_3 = \frac{-a_1}{2}$$

Now we shall equate the coefficient of x^r $(r \geq 2)$ to zero

$$a_{r+2}(r+2)(r+1) + a_r r + a_{r-2} + 2a_r = 0$$

$$a_{r+2} = \frac{-[a_{r-2} + (r+2)a_r]}{(r+2)(r+1)}$$

$$a_{r+2} = \frac{-a_{r-2}}{(r+2)(r+1)} - \frac{a_r}{(r+1)} \qquad (r \geq 2)$$

By putting in $r = ,2,3,4,5...$ (3) we obtain

$$a_4 = \frac{-a_0}{2} - \frac{a_2}{3} = \frac{-a_0}{12} + \frac{a_0}{3} = \frac{a_0}{4}$$

$$a_5 = \frac{-a_1}{20} - \frac{a_3}{4} = \frac{-a_1}{20} + \frac{a_1}{8} = \frac{3a_1}{40}$$

$$a_6 = \frac{-a_2}{30} - \frac{a_4}{5} = \frac{a_0}{30} - \frac{a_0}{20} = \frac{-a_0}{60} \text{ and so on....}$$

We substitute these values in the expanded form of (2)

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + ...$$

Thus, $y = a_0\left[1 - x^2 + \frac{x^4}{4} - \frac{x^6}{60} + ...\right] + a_1\left[x - \frac{x^3}{2} + \frac{3x^5}{40} + ...\right]$ is the required series solution

69. Solve in series the equation $(x-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ **subject to the conditions** $y(0) = 2$ **and** $y'(0) = -1$

Solution:

we have $(x-1)y'' + xy' + y = 0$ ... (1)

The coefficient of $y'' = (x-1) = P_0(x)$ and at $x = 0$, $P_0(x) = -1 \neq 0$

Let $y = \sum_{r=0}^{\infty} a_r x^r$ ...(2) be the series solution of the given D.E (1)

$$y' = \sum_{r=0}^{\infty} a_r r x^{r-1} \quad , \quad y'' = \sum_{0}^{\infty} a_r r(r-1)x^{r-2}$$

Now (1) becomes

$$\sum_{0}^{\infty} a_r r(r-1)x^{r-1} - \sum_{0}^{\infty} a_r r(r-1)x^{r-2} + \sum_{0}^{\infty} a_r r x^r + \sum_{0}^{\infty} a_r x^r = 0$$

We equate the coefficients of various powers of x to zero. We first equate the coefficients of x^{-2}, x^{-1} to zero

Coeff. Of x^{-2} : $-a_0(0)(1) = 0 \quad \Rightarrow a_0 \neq 0$

Coeff. Of x^{-1} : $a_0(0)(-1) - a_1(1)(0) = 0 = 0 \quad \Rightarrow a_1 \neq 0$

Now we shall equate the coefficient of x^r $(r \geq 0)$ to zero

$$a_{r+1}(r+1) - a_{r+2}(r+2)(r+1) + a_r r + a_r = 0$$

$$a_{r+1}(r+1) - a_{r+2}(r+2)(r+1) + a_r(r+1) = 0$$

$$a_{r+1} r - a_{r+2}(r+2) + a_r = 0$$

$$a_{r+2} = \frac{ra_{r+1} + a_r}{(r+2)} \quad (r \geq 0) \qquad \ldots(3)$$

By putting in $r = 0,1,2,3,..$ (3) we obtain

$$a_2 = \frac{a_0}{2} \ ; \quad a_3 = \frac{a_2 + a_1}{3} = \frac{a_0/2 + a_1}{3} = \frac{a_0}{6} + \frac{a_1}{3}$$

$$a_4 = \frac{2a_3 + a_2}{4} = \frac{a_0/3 + 2a_1/3 + a_0/2}{4} = \frac{5a_0}{24} + \frac{a_1}{6}$$

$$a_5 = \frac{3a_4 + a_3}{5} = \frac{5a_0/8 + a_1/2 + a_0/6 + a_1/3}{5} = \frac{19a_0}{120} + \frac{a_1}{6} \quad \text{and so on}\ldots.$$

We substitute these values in the expanded form of (2)

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

Thus, $y = a_0\left[1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{5x^4}{24} + \frac{19x^5}{120}\ldots\right] + a_1\left[x + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{6}\ldots\right]$ $\ldots(4)$

is the required general solutionof (1) in series.

To apply giveninitial condition. We differentiate (4) w.r.t x

$$y' = a_0\left[x + \frac{x^2}{2} + \frac{5x^3}{6}\ldots\right] + a_1\left[1 + x^2 + \frac{2x^3}{3} + \ldots\right] \qquad \ldots(5)$$

Using the condition, $y = 2$ and $y' = -1$ and $x = 0$, (4) and (5) respectively becomes $2 = a_0$ and $-1 = a_1$

Hence (4) becomes

$$y = 2\left[1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{5x^4}{24} + \frac{19x^5}{120}........\right] - 1\left[x + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{6}...............\right]$$

Thus $y = 2 - x + x^2 + \frac{x^4}{4} + \frac{3x^5}{20} + ...$ is the required particular solution in series

Exercise:

I. Solve the following differential equations:

1. $\left(5x^4+3x^2y^2-2xy^3\right)dx+\left(2x^3y-3x^2y^2-5y^4\right)dy=0.$

2. $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+\left[x+\log x-x\sin y\right]dy=0$

3. $\left[\cos x\tan y+\cos(x+y)\right]dx+\left[\sin x\sec^2 y+\cos(x+y)\right]dy=0$

4. $\left(x-\frac{y}{x^2+y^2}\right)dx+\left(y+\frac{x}{x^2+y^2}\right)dy=0 \quad y=1 \text{ when } x=1$

5. $xdx=y\left(x^2+y^2-1\right)dy$

6. $ye^{\frac{x}{y}}dx=\left(xe^{\frac{x}{y}}+y^2\right)dy$

7. $(x+1)\frac{dy}{dx}-y=e^x(x+1)^2$

8. $\left(x^3-x\right)\frac{dy}{dx}-\left(3x^2-y\right)=x^5-2x^3+x$

9. $\sin x\frac{dy}{dx}+2y=\tan^3\left(\frac{x}{2}\right)$

10. $\cot 3x\frac{dy}{dx}-3y=\cos 3x+\sin 3x,\ 0<x<\frac{\pi}{2}$

11. $(2x\log x-xy)dx+2ydy=0.$

12. $\left(y^4+2y\right)dx+\left(xy^3+2y^4-4x\right)dy=0$

13. $y\left(x^2y+e^x\right)dx-e^xdy=0.$

14. $y\left(xy+2x^2y^2\right)dx+x\left(xy-x^2y^2\right)dy=0.$

15. $y(1+xy)dx+x\left(1+xy+x^2y^2\right)dy=0.$

16. $\left(y^3-2x^2y\right)dx+\left(2xy^2-x^3\right)dy=0.$

17. $\left(3y-2xy^3\right)dx+\left(4x-3x^2y^2\right)dy=0.$

18. $x\frac{dy}{dx}+y\log y=xye^x.$

19. $y'+\frac{y}{x}=x^2$

20. $(1+x)\frac{dy}{dx}-xy=1-x$

21. $\frac{dy}{dx}+\left[\frac{1-2x}{x^2}\right]y=1$

22. $\sec^2 y\frac{dy}{dx}+x\tan y=x^3$

23. $\frac{dy}{dx}+2y\tan x=y^2$

II. Solve the following differential equations:

1. $2y''+y'-6y=0$

2. $4y''-4y'+y=0$

3. $y''-6y'+9y=0$, $y(0)=3$, $y'(0)=10$

4. $y''+4y'+9y=0$

5. $\left(D^3+7D^2+11D+5\right)y=0$

6. $D^3y-3Dy+2y=0$

7. $\frac{d^3x}{dt^3}-3\frac{d^2x}{dt^2}-\frac{dx}{dt}+3x=0$

8. $\frac{d^4y}{dx^4}+64y=0$

9. $4\frac{d^4y}{dx^4}-8\frac{d^3y}{dx^3}-7\frac{d^2y}{dx^2}+11\frac{dy}{dx}+6y=0$

10. $y'''-y''+y'-y=0$

11. $D^2(D-1)^3(D+1)y=0$

12. $(D^2+2D-8)y=e^{-2x}+e^{-4x}$

13. $\frac{d^2y}{dx^2}-4y=\cosh 2x$

14. $(D^2-4D+1)y=\sin^2 x$

15. $y''+16y=e^{-3x}+\cos 4x$

16. $(D^2-3D+2)y=2\sin 2x$

17. $y''+2y'+y=2x+x^2$

18. $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=x^3$

19. $y''-2y'+4y=2e^x\cos x$

20. $\frac{d^3y}{dx^3}+y=16x^2e^x$

21. $(D^2+4D+5)y=e^{-2x}\,2\cos^2\left(x/2\right)$

22. $(D^2+2D+1)y=x\sin x$

23. $y''-y=x^2\cos x$

24. $\frac{d^4y}{dx^4}-y=\cos x\cosh x$

III. A. Solve the following differential equations by the method of variation of parameters:

1. $\frac{d^2y}{dx^2}+y=\sec x\tan x.$
2. $\frac{d^2y}{dx^2}-y=\frac{2}{1+e^x}.$
3. $\frac{d^2y}{dx^2}+y=\tan x.$
4. $\frac{d^2y}{dx^2}+y=\frac{1}{1+\sin x}$
5. $\frac{d^2y}{dx^2}+4y=4\sec^2 2x$
6. $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2y=e^{-x}\sec^3 x$
7. $(D^2+3D+2)y=e^{e^x}$
8. $\frac{d^2y}{dx^2}-y=\frac{1}{(1+e^{-x})^2}$

B. Solve the following differential equations:

1. $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+y=\log x\frac{\sin(\log x)+1}{x}$
2. $x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+6y=\frac{42}{x^4}$
3. $x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0$
4. $x\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}=\frac{1}{x}$
5. $(3x-2)^2\frac{d^2y}{dx^2}-3(3x-2)\frac{dy}{dx}=9(3x-2)\sin\log(3x-2)$
6. $(2x+3)^2\frac{d^2y}{dx^2}-(2x+3)\frac{dy}{dx}-12y=6x$
7. $(3x+2)^2\frac{d^2y}{dx^2}+3(3x+2)\frac{dy}{dx}-36y=3x^2+4x+1$

8 . $(2+t)^2\dfrac{d^2y}{dt^2}+7(2+t)\dfrac{dy}{dt}-12y=3t^2+2t+7.$

IV. 1. Obtain the series solution of the equation $\dfrac{dy}{dx}=ky$.

2. Solve in series the equation $(1-x^2)y'-y=0$.

3. Obtain the power series solution for the equation $(1-x^2)y''-2xy'+2y=0$

I. **ANSWERS:**

$I.$ 1. $x^5+x^3y^2-x^2y^3-y^5=C$

2. $y(x+\log x)+x\cos y=C.$ 3. $\sin x\tan y+\sin(x+y)=C$

4. $\dfrac{x^2}{2}+\dfrac{y^2}{2}-\tan^{-1}\left(\dfrac{x}{y}\right)=1-\dfrac{\pi}{4}$ 5. $\log(x^2+y^2)-y^2=k$

6. $e^{\frac{x}{y}}-y=C$ 7. $\dfrac{y}{x+1}=e^x+C$

8. $y=(x^3-x)\log x+(x^3-x)C$ 9. $y\tan^2\dfrac{x}{2}=\dfrac{\tan^5\frac{x}{2}}{5}+C$

10. $y\cos 3x=\dfrac{1}{12}[6x-\sin 6x-\cos 6x]$

11. $2y\log x-\dfrac{1}{2}y^2=C$ 12. $x\left[y+\dfrac{2}{y^2}\right]+y^2=C.$

13. $\dfrac{x^3}{3}+\dfrac{e^x}{y}=C.$ 14. $-\dfrac{1}{xy}+2\log x-\log y=b.$

15. $\dfrac{1}{2x^2y^2}+\dfrac{1}{xy}-\log y=C.$ 16. $x^2y^4-x^4y^2=C$

17.. $x^3y^4-\dfrac{x^4y^6}{2}=C.$ 18. $x\log y=xe^x-e^x+C.$

19. $y=\dfrac{x^3}{4}+\dfrac{c}{x}$

20. $y=\dfrac{x}{1+x}+\dfrac{c}{1+x}e^{(1+x)}$

21. $y=x^2+cx^2e^{\frac{1}{x}}$

22. $\tan y=x^2-2+ce^{\frac{-x^2}{2}}$

23. $y=\dfrac{4\cos^2 x}{2x+\sin 2x+4c}$

II. Answers:

1. $y = c_1 e^{3x/2} + c_2 e^{-2x}$ **2.** $y = (c_1 + c_2 x)e^{x/2}$ **3.** $y = (3 + x)e^{3x}$

4. $y = e^{-2x}\left(c_1 \cos\sqrt{5}x + c_2 \sin\sqrt{5}x\right)$ **5.** $y = (c_1 + c_2 x)e^{-x} + c_3 e^{-5x}$

6. $y = (c_1 + c_2 x)e^{x} + c_3 e^{-2x}$ **7.** $y = c_1 e^{-t} + c_2 e^{t} + c_3 e^{3t}$

8. $y = e^{2x}(c_1 \cos 2x + c_2 \sin 2x) + e^{-2x}(c_3 \cos 2x + c_4 \sin 2x)$

9. $y = c_1 e^{-x/2} + c_2 e^{-x} + c_3 e^{3x/2} + c_4 e^{2x}$ **10.** $y = c_1 e^{x} + c_2 \sin x + c_3 \cos x$

11. $y = (c_1 + c_2 x) + \left(c_3 + c_4 x + c_5 x^2\right)e^{x} + c_6 e^{-x}$ **12.** $y = c_1 e^{2x} + c_2 e^{-4x} - \dfrac{e^{-2x}}{8} - \dfrac{x}{6} e^{-4x}$

13. $y = c_1 e^{2x} + c_2 e^{-2x} + \dfrac{x}{4}\sinh 2x$ **15.** $y = c_1 \cos 4x + c_2 \sin 4x + \dfrac{e^{-3x}}{25} + \dfrac{x}{8}\sin 4x$

14. $y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x} + \dfrac{1}{2}\left[1 - \dfrac{1}{73}(8\sin 2x + 3\cos 2x)\right]$

16. $y = c_1 e^{x} + c_2 e^{2x} + \dfrac{(3\cos 2x - \sin 2x)}{10}$ **17.** $y = (c_1 + c_2 x)e^{-x} + x^2 - 2x + 2$

18. $y = e^{-x/2}\left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right) + x^3 - 3x^2 + 6$

19. $y = e^{x}\left(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right) + e^{x}\cos x$

20. $y = c_1 e^{-x} + e^{x/2}\left(c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x\right) + 4e^{x}\left(2x^2 - 6x + 3\right)$

21. $y = e^{-2x}(c_1 \cos x + c_2 \sin x) + e^{-2x}\left(1 + \frac{x}{2}\sin x\right)$

22. $y = (c_1 + c_2 x)e^{-x} + \frac{1}{2}(\sin x - x\cos x + \cos x)$

23. $y = c_1 e^{x} + c_2 e^{-x} + x\sin x + \dfrac{\left(1 - x^2\right)}{2}\cos x$

24. $y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \frac{1}{5}\cos x \cosh x$

III. ANSWERS

A.

1. $y = A\cos x + B\sin x + x\cos x + \sin\log\sec x - \sin x.$

2. $y = Ae^{x} + Be^{-x} - 1 + e^{x}\log\left(e^{-x} + 1\right) - e^{-x}\log\left(e^{x} + 1\right)$

3. $y = A\cos x + B\sin x - \cos x \log(\sec x + \tan x)$

4. $y = A\cos x + B\sin x - (x\cos x + 1) + \sin x \log(1 + \sin x)$

5. $y = a\cos 2x + b\sin 2x - 1 + \sin 2x.\log(\sec 2x + \tan 2x)$

6. $y = e^{-x}(A\cos x + B\sin x) + \dfrac{e^{-x}\tan x \sin x}{2}$

7. $y = Ae^{-x} + Be^{-2x} + e^{-2x}e^{e^{x}}$

8. $y = Ae^{x} + Be^{-x} + e^{-x}\log\left(1 + e^{x}\right) - 1$

B.

$$1.\ y = Ax^{2+\sqrt{3}} + Bx^{2-\sqrt{3}} + \frac{1}{x}\left[\frac{382}{61}\cos\log x + \frac{54}{61}\sin(\log x) + 6\log x\cos(\log x) + 5\log x\sin(\log x) + \frac{1}{6x}\right]$$

$$2.\ y = Ax^2 + Bx^3 + \frac{1}{x^4}$$

$$3.\ y = (A + B\log x)x + \frac{c}{x^3}$$

$$4.\ y = A + (B + C\log x)x + \frac{x(\log x)^2}{2}$$

$$5.\ y = A + B(3x-2)^2 - \frac{1}{2}(3x-2)\sin[\log(3x-2)]$$

$$6.\ y = c_1(2x+3)^a + c_2(2x+3)^b - \frac{3}{14}(2x+3) + \frac{3}{4} \qquad where\ a,b = \frac{3 \pm \sqrt{57}}{4}$$

$$7.\ y = c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{108}\left[(3x+2)^2\log(3x+2) + 1\right]$$

$$8.\ y = A(2+t)^{-3+\sqrt{2}i} + B(2+t)^{-3-\sqrt{2}i} + \frac{3}{4}x^2 + 5x + \frac{23}{4}$$

IV. ANSWER

1. $y = a_0\left[1 + \frac{kx}{1!} + \frac{k^2x^2}{2!} + \frac{k^3x^3}{3!} + ...\right]$; $y = a_0e^{kx}$

2. $y = a_0\left[1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^4}{8}....\right]$

3. $y = a_0\left[1 - x^2 - \frac{x^4}{3} - \frac{x^{\backslash 6}}{5} - \frac{x^8}{7}....\right] + a_1x$

Module-5
MATRICES

- Echelon form
- Rank of a matrix
- System of linear equations
- Consistency
- Solution by Gauss Elimination
- Solution by Gauss-Siedel
- Eigenvalues and Eigenvectors
- Diagonalization of matrices.
- Conversion of an n^{th} order differential equation to a system of first order linear differential equations
- Solution of system of linear differential equations by diagonalization method
- Discuss the stability of the system.

Introduction:
Linear algebra is the study of vectors and linear functions. It comprises of the theory and application of linear system of equations, linear transformations, Eigen values and Eigen vectors, problems.

Elementary transformations of a matrix:
The following are the elementary row transformations of a matrix. The transformations can be applied for columns.

1. The interchange of any two rows (columns).
2. The multiplication of any row (column) by a non-zero constant.
3. The addition of a constant multiple of the elements of any row (column) to the corresponding elements of any other row (column).

Equivalent matrices:
Two matrices A and B of the same order are said to be equivalent if one matrix can be obtained from the other by a sequence of elementary row or column transformations. We use the notation ‘A~B’

Row reduced echelon form of a matrix:
A non – zero matrix A is said to be in row reduced echelon form if the following conditions prevail:
i) All the zero rows are below non zero rows.
ii) The first non zero entry in any non zero row is 1, and the entries below 1 in the same column zero.

Ex : $A=\begin{bmatrix}1 & 2 & 3 & 4\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 2 & 3 & 4\\ 0 & 1 & 1 & 2\\ 0 & 0 & 0 & 0\end{bmatrix}$,

Rank of a matrix:
If we select any ‘r’ rows and ‘r’ columns from any matrix A, deleting all the other rows and columns, then the determinant formed by these ‘r×r’ elements is called the minor of order ‘r’.
A matrix A is said to be of rank ‘r’ when

(i) It has at least one non-zero minor of order ‘r’ and,

(ii) Every minor of order higher than r vanishes.

Briefly, the rank of a matrix A is the highest order of any non-vanishing minor of the matrix. The rank or a matrix A is denoted by $\rho(A)$.

If a matrix has a non zero minor of order r, its rank $\geq r$.

If all minors of a matrix of order $r+1$ are zero, its rank its rank $\leq r$.

Notations:

- R_{ij} for the interchange of the ith and jth rows.
- KR_{ij} for multiplication of the ith and jth row by K.
- $R_i + pR_j$ for addition to the ith row, p times the jth row.

The corresponding column transformation will be denoted by writing C in place of R.

NOTE:

1. The **rank** of a matrix A in its **echelon form** is equal to the number of **non zero rows**.
2. Elementary transformations do not change either the order or rank of a matrix.
3. While reducing the given matrix to a row reduced echelon form, we prefer to have the leading element non zero. If it is zero, we can interchange with any row having the leading element is nonzero. Then we use that leading non zero element in every row to make all the elements in that column zero. The transformation in this process has to be performed for the entire row. Avoid fraction as far as possible during the process of elementary transformation.

Problems:

1. Find the rank of the following matrix by reducing it to the row echelon form.

$$A=\begin{bmatrix}0 & 2 & 3 & 4\\ 2 & 3 & 5 & 4\\ 4 & 8 & 13 & 12\end{bmatrix}$$

Solution: $A=\begin{bmatrix}2 & 3 & 5 & 4\\ 0 & 2 & 3 & 4\\ 4 & 8 & 13 & 12\end{bmatrix}$ $\quad R_1 \leftrightarrow R_2$

$$=\begin{bmatrix}2 & 3 & 5 & 4\\ 0 & 2 & 3 & 4\\ 0 & 2 & 3 & 4\end{bmatrix} \quad R_3 \to R_3 - 2R_1$$

$$=\begin{bmatrix}2 & 3 & 5 & 4\\ 0 & 2 & 3 & 4\\ 0 & 0 & 0 & 0\end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$=\begin{bmatrix}1 & 3/2 & 5/2 & 2\\ 0 & 1 & 3/2 & 2\\ 0 & 0 & 0 & 0\end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 \div 2\\ R_2 \to R_2 \div 2\end{array}$$

A is in the row echelon forms having two non – zero rows. $\therefore \rho(A)=2$

2. Find the rank of the following matrix by reducing it to the row echelon form.

Solution: $A=\begin{bmatrix}2 & 3 & -1 & -1\\ 1 & -1 & -2 & -4\\ 3 & 1 & 3 & -2\\ 0 & 0 & 0 & 0\end{bmatrix} R_1 \leftrightarrow R_2$

$$A=\begin{bmatrix}1 & -1 & -2 & -4\\ 2 & 3 & -1 & -1\\ 3 & 1 & 3 & -2\\ 0 & 0 & 0 & 0\end{bmatrix} R_4 \to R_4-(R_1+R_2+R_3)$$

$$A=\begin{bmatrix}1 & -1 & -2 & -4\\ 0 & 5 & 3 & 7\\ 0 & 4 & 9 & 10\\ 0 & 0 & 0 & 0\end{bmatrix} \quad R_3 \to -4R_2+5R_3$$

$$A=\begin{bmatrix}1 & -1 & -2 & -4\\ 0 & 5 & 3 & 7\\ 0 & 0 & 33 & 22\\ 0 & 0 & 0 & 0\end{bmatrix}. \quad \rho(A)=3$$

3. Find the rank of the following matrices by elementary row transformations

A = $\begin{bmatrix}2 & -1 & -3 & -1\\ 1 & 2 & 3 & -1\\ 1 & 0 & 1 & 1\\ 0 & 1 & 1 & -1\end{bmatrix}$

Solution:

$$A=\begin{bmatrix}1 & 2 & 3 & -1\\ 2 & -1 & -3 & -1\\ 1 & 0 & 1 & 1\\ 0 & 1 & 1 & -1\end{bmatrix} \quad \begin{matrix}R_2 \to -2R_1+R_2\\ R_3 \to -R_1+R_3\end{matrix}$$

$$A=\begin{bmatrix}1 & 2 & 3 & -1\\ 0 & -5 & -9 & 1\\ 0 & -2 & -2 & 2\\ 0 & 1 & 1 & -1\end{bmatrix} \quad R_2 \leftrightarrow R_4$$

$$A=\begin{bmatrix}1 & 2 & 3 & -1\\ 0 & 1 & 1 & -1\\ 0 & -2 & -2 & 2\\ 0 & -5 & -9 & 1\end{bmatrix} \quad \begin{matrix}R_3 \to 2R_2+R_3\\ R_4 \to 5R_2+R_4\end{matrix}$$

$$A=\begin{bmatrix}1 & 2 & 3 & -1\\ 0 & 1 & 1 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & -4 & -4\end{bmatrix} \quad \begin{matrix}R_3 \leftrightarrow R_4\\ and\ -1/4.R_3\end{matrix}$$

$$A=\begin{bmatrix}1 & 2 & 3 & -1\\ 0 & 1 & 1 & -1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

$\rho(A)=3$.

4. Find the rank of the following matrices by elementary row transformations

$$A=\begin{bmatrix}1 & 2 & 3 & 2\\ 2 & 3 & 5 & 1\\ 1 & 3 & 4 & 5\end{bmatrix}$$

Solution: $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix} \begin{matrix} R_2 \to -2R_1 + R_2 \\ R_3 \to -R_1 + R_3 \end{matrix}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_2 + R_3$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (-1)R_2$$

$$\rho(A) = 2.$$

5. Find the rank of the following matrices by elementary row transformations

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Solution: $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \quad \begin{matrix} R_3 \to -3R_1 + R_3 \\ R_4 \to -R_1 + R_4 \end{matrix}$$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_3 \to -R_2 + R_3 \\ R_4 \to -R_2 + R_4 \end{matrix}$$

$$\rho(A) = 2$$

6. Find the rank of the following matrix by reducing it to the normal form.

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$A=\begin{bmatrix}1 & 0 & 1 & 1\\ 0 & 1 & -3 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \qquad \begin{array}{l} R3\to R_3-3R_2\\ R_4\to R_4-R_2\end{array}$$

$$A=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & -3 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \qquad \begin{array}{l} C3\to C_3-C_1\\ C_4\to C_4-c_2\end{array}$$

$$A=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \qquad \begin{array}{l} C3\to C_3+3C_2\\ C_4\to C_4+C_2\end{array}$$

$$A=\begin{bmatrix}I_2 & 0\\ 0 & 0\end{bmatrix}.$$

$\therefore \quad \rho(A)=2$

Consistency of a system of linear equations

Consider a system of ‘m’ linear equations in $'n'$ unknowns.

$$\begin{array}{l} a_{11}\ x_1\ +a_{12}\ \ x_2\ \ +\text{.........}+a_{1n}\ \ x_n\ =b_1\\ a_{21}\ x_1\ +a_{22}\ \ x_2\ \ +\text{.........}+a_{2n}\ \ x_n\ =b_2\\ \qquad\ldots\ldots\ldots \qquad\qquad \ldots\ldots..\\ \qquad\ldots\ldots\ldots \qquad\qquad \ldots\ldots..\\ a_{m1}x_1+a_{m2}\ \ x_2\ \ +\text{..........}+a_{mn}\ \ x_n\ =b_m\end{array}$$

Where $a_{ij}'s$ and $b_i's$ are constants.

If $b_1, b_2,.........,b_m$ are all zero, the system is said to be homogeneous. Otherwise, it is said to be Non-homogeneous.

A system is said to be consistent if it possesses a solution. Otherwise it is said to be inconsistent.

The above system of equations can be written as the matrix equation A X = B, where

$$A=\begin{bmatrix}a_{11} & a_{12} & \ldots & a_{1n}\\ a_{21} & a_{22} & \ldots & a_{2n}\\ .. & .. & \ldots & ..\\ a_{m1} & a_{m2} & \ldots & a_{mn}\end{bmatrix},\quad X=\begin{bmatrix}x_1\\ x_2\\ ..\\ x_n\end{bmatrix}\quad B=\begin{bmatrix}b_1\\ b_2\\ ..\\ b_m\end{bmatrix}$$

$x_1=x_2=x_3=\text{..........}=x_n=0$ is a solution of the homogeneous system of equations and is called a **trivial solution.**

Problems

7. Solve the system of equations $x+2y+3z=0$, $3x+4y+4z=0$ and $7x+10y+12z=0$.

Solution: $[A:B]=\begin{bmatrix}1 & 2 & 3 & : & 0\\ 3 & 4 & 4 & : & 0\\ 7 & 10 & 12 & : & 0\end{bmatrix}$

$$[A:B]=\begin{bmatrix}1 & 2 & 3 & : & 0\\ 0 & -2 & -5 & : & 0\\ 7 & 10 & 12 & : & 0\end{bmatrix} \quad R_2 \to R_2 - 3R_1$$

$$[A:B]=\begin{bmatrix}1 & 2 & 3 & : & 0\\ 0 & -2 & -5 & : & 0\\ 0 & -4 & -9 & : & 0\end{bmatrix} \quad R_3 \to R_3 - 7R_1$$

$$[A:B]=\begin{bmatrix}1 & 2 & 3 & : & 0\\ 0 & -2 & -5 & : & 0\\ 0 & 0 & 1 & : & 0\end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

Here rank (r) = 3 and n = number of variables = 3 (i.e., r = n).

Therefore the equation has only a trivial solution: $x = y = z = 0$.

Note: If at least one $x_i\,(i = 1,2,....n)$ is not equal to zero, then it is called a non-trivial solution. The concept of the rank of a matrix helps to find

1. Whether the system is consistent or not.
2. Whether the system possess unique solution or many solutions.

Condition for consistency and types of solution

Consider a system of $'m'$ equations in $'n'$ unknowns in the matrix form $AX = B$, where

$$A=\begin{bmatrix}a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{2n}\\ .. & .. & \dots & ..\\ a_{m1} & a_{m2} & \dots & a_{mn}\end{bmatrix}$$ is called the coefficient matrix.

The matrix formed by appending to A an extra column consisting of the elements of B is called the augmented matrix denoted by $[A:B]$.

$$\therefore [A:B]=\begin{bmatrix}a_{11} & a_{12} & \dots & ; & b_1\\ a_{21} & a_{22} & \dots & ; & b_2\\ .. & .. & \dots & & ..\\ a_{m1} & a_{m2} & \dots & ; & b_m\end{bmatrix}$$

The system is consistent if $\rho(A)=\rho(A:B)$

Let $\rho(A)=\rho(A:B)=r$.

Then the conditions for various types of solutions are

1. Unique Solution: $\rho(A)=\rho(A:B)=r=n,$ n is the no. of unknowns.
2. Infinite solutions: In this case, $(n-r)$unknowns can take arbitrary values.
3. If $\rho(A)\neq\rho(A:B),$ then the system is said to be inconsistent.

Problems

8. Test for consistency and solve $5x_1 + x_2 + 3x_3 = 20$**,** $2x_1 + 5x_2 + 2x_3 = 18$ **and** $3x_1 + 2x_2 + x_3 = 14$.

Solution: The augmented matrix is $[A:B]=\begin{bmatrix}5 & 1 & 3 & : & 20\\ 2 & 5 & 2 & : & 18\\ 3 & 2 & 1 & : & 14\end{bmatrix}$

$$[A:B]=\begin{bmatrix}5 & 1 & 3 & :20\\ 0 & 23 & 4 & :50\\ 0 & 7 & -4 & :10\end{bmatrix}\quad \begin{matrix}R_2\to -2R_1+5R_2\\ R_3\to -3R_1+5R_3\end{matrix}$$

$$[A:B]=\begin{bmatrix}5 & 1 & 3 & : & 20\\ 0 & 23 & 4 & : & 50\\ 0 & 0 & -120 & : & -120\end{bmatrix}\quad R_3\to -> E_2+23R_3$$

We have $\rho(A)=3, \rho(A:B)=3=n$

$\therefore$ The system is consistent and will have a unique solution.

$\therefore$ By back substitution method, $5x_1+x_2+3x_3=20$, $23x_2+4x_3=50$, $-120x_3=-120$.

Solving we get $x_3=1,\ x_2=2,\ x_1=3$

$\therefore$ The unique solution is $x_1=3,\ x_2=2,\ x_3=1$

9. Test for consistency and solve: $x+2y+3z=14$, $4x+5y+7z=35$, $3x+3y+4z=21$

Solution: The augmented matrix is $[A:B]=\begin{bmatrix}1 & 2 & 3 & : & 14\\ 4 & 5 & 7 & : & 35\\ 3 & 3 & 4 & : & 21\end{bmatrix}$

$$[A:B]=\begin{bmatrix}1 & 2 & 3 & : & 14\\ 0 & -3 & -5 & : & -21\\ 0 & -3 & -5 & : & 21\end{bmatrix}\begin{matrix}R_2\to -4R_1+R_2\\ R_3\to -3R_1+R_3\end{matrix}$$

We have, $\rho(A)=2,\qquad \rho(A:B)=2=r,$

$\therefore \rho(A)=\rho(A:B)=2<3$, ie, $r<n$

The system is consistent and will have infinite solutions.

Here $n-r=1$. Hence one of the variables can take arbitrary values.

$\therefore x+2y+3z=14\ldots(1)$

$-3y-5z=-21\ldots\ldots(2)$

Let $z=k$ be arbitrary. $(2)\Rightarrow -3y-5k=-21\quad \Rightarrow y=7-\frac{5k}{3}\Rightarrow x=\frac{k}{3}$.

$\therefore$ The infinite solutions are $x=\frac{k}{3}, y=7-\frac{5k}{3}, z=k$, where k is arbitrary.

10. Show that the following system of equations is inconsistent.

$$5x+3y+7z=5$$
$$3x+26y+2z=9$$
$$7x+2y+10z=5$$

Solution: The augmented matrix is $[A:B]=\begin{bmatrix}5 & 3 & 7 & : & 5\\ 3 & 2b & 2 & : & 9\\ 7 & 2 & 10 & : & 5\end{bmatrix}$

$$[A:B]=\begin{bmatrix}5 & 3 & 7 & : & 5\\ 0 & 121 & -11 & : & 30\\ 0 & -11 & 1 & : & -10\end{bmatrix}\begin{matrix}R_2\to -3R_1+5R_2\\ R_3\to -7R_1+5R_3\end{matrix}$$

$$[A:B]=\begin{bmatrix}5 & 3 & 7 & : & 5\\ 0 & 121 & -11 & : & 30\\ 0 & 0 & 0 & : & -80\end{bmatrix}\quad R_3\to R_2+11R_3$$

We have, $\rho(A)=2, \rho(A:B)=3.$ $\therefore \rho(A) \neq \rho(A:B)$ $\therefore$ The system is inconsistent.

11. Find all the solutions of the following system of homogeneous equations:

$$\begin{aligned}x_1-2x_2+x_3-x_4&=0\\ x_1+x_2-2x_3+3x_4&=0\\ 4x_1+x_2-5x_3+8x_4&=0\\ 5x_1-7x_2+2x_3-x_4&=0\end{aligned}$$

Solution: The Augmented matrix is

$$[A:B]=\begin{bmatrix}1 & -2 & 1 & -1 & : & 0\\ 1 & 1 & -2 & 3 & : & 0\\ 4 & 1 & -5 & 8 & : & 0\\ 5 & -7 & 2 & -1 & : & 0\end{bmatrix}$$

$$[A:B]\sim\begin{bmatrix}1 & -2 & 1 & -1 & : & 0\\ 0 & 3 & -3 & 4 & : & 0\\ 0 & 9 & -9 & 12 & : & 0\\ 0 & 3 & -3 & 4 & : & 0\end{bmatrix}\begin{array}{l}R_2\to -R_1+R_2\\ R_3\to -4R_1+R_3\\ R_4\to -5R_1+R_4\end{array}$$

$$[A:B]\sim\begin{bmatrix}1 & -2 & 1 & -1 & : & 0\\ 0 & 3 & -3 & 4 & : & 0\\ 0 & 0 & 0 & 0 & : & 0\\ 0 & 0 & 0 & 0 & : & 0\end{bmatrix}\begin{array}{l}R_3\to -3R_2+R_3\\ R_4\to -R_2+R_4\end{array}$$

We have, $\rho(A:B)=2$ and $\rho(A)=2,$ $\therefore r=2$ also $n=4$

$$\therefore \rho(A:B)=\rho(A)=2<4.$$

The system is consistent and will have infinite solutions. Since $n-r=2$, for 2 variables can take arbitrary values.

Let $x_4=k_1,\ x_3=k_2$ arbitrarily.

$\therefore$ We have $x_1-2x_2+x_3-x_4=0$ ……(1), $3x_2-3x_3+4x_4=0$ ……….(2)

Let $x_4=k_1, x_3=k_2$

$(2)\Rightarrow 3x_2-3k_2+4k_1=0\Rightarrow x_2=k_2-\frac{4}{3}k_1$

$(1)\Rightarrow x_1-2k_2+\frac{8}{3}k_1+k_2-k_1=0\Rightarrow x_1=k_2-\frac{5}{3}k_1$

$\therefore$ The infinite solutions of the given system are given by $x_1=k_2-\frac{5}{3}k_1,\ x_2=k_2-\frac{4}{3}k_1,\ x_3=k_2,\ x_4=k_1.$

12. Investigate the values of λ and μ such that the system of equations

$x+y+z=6;\quad x+2y+3z=10;\quad x+2y+\lambda z=\mu,$ may have

a] Unique solution b] Infinite Solution c] No solution.

Solution: The augmented matrix is $[A:B]=\begin{bmatrix}1 & 1 & 1 & : & 6\\ 1 & 2 & 3 & : & 10\\ 1 & 2 & \lambda & : & \mu\end{bmatrix}$

$$[A:B]=\begin{bmatrix}1 & 1 & 1 & : & 6\\ 0 & 1 & 2 & : & 4\\ 0 & 1 & \lambda-1 & : & \mu-6\end{bmatrix}\begin{matrix}R_2\to -R_1+R_2\\ R_3\to -R_1+R_3\end{matrix}$$

$$[A:B]=\begin{bmatrix}1 & 1 & 1 & : & 6\\ 0 & 1 & 2 & : & 4\\ 0 & 0 & \lambda-3 & : & \mu-10\end{bmatrix}R_3\to -R_2+R_3$$

a] Unique solution: We must have $\rho(A)=\rho(A:B)=3$.and $\rho(A)$ will be 3 if $\lambda-3\neq 0$

Also $\rho(A:B)$ will be 3 if $\lambda-3\neq 0$ and for any value of μ.

$\therefore$ If $\lambda\neq 3$, then the system will have unique solution.

b] Infinite solutions: We must have $\rho(A)=\rho(A:B)=r$ and $r<n$.

Since $n=3$, we must have $r=2$.

If $\lambda-3=0$ and $\mu-10=0$, then we have $\rho(A)=\rho(A:B)<n$

$\therefore$ The system have infinite solutions if $\lambda=3$ and $\mu=10$.

c] No solution: If the system is inconsistent, then we have no solution.

$\therefore$ We must have $\rho(A)\neq\rho(A:B)$.

If $\lambda=3$, we have $\rho(A)=2$.

If $\mu\neq 10$, we have $\rho(A:B)=3$

$\therefore$ The system has no solution if $\lambda=3$ and $\mu\neq 10$.

Solution of a system of non homogeneous equations:

Linear simultaneous Equations occur in various engineering problems. We are already familiar with the methods, Cramer's rule and matrix method for solving such equations. But these methods are tedious for large systems.

Here we study now another two methods for solving such large systems.

They are **a)** Gauss elimination method

b) Gauss –Seidel method.

Gauss – Elimination Method

In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular system from which the unknowns are found by back substitution. This method is quite general and is well –adapted for computer operations.

13. Solve the following system of equations by Gauss elimination method.

$$x+y+z=9,\quad x-2y+3z=8,\qquad 2x+y-z=3$$

Solution: The augmented matrix of the systems is $[A:B]=\begin{bmatrix}1 & 1 & 1 & :9\\ 1 & -2 & 3 & :8\\ 2 & 1 & -1 & :3\end{bmatrix}$

$$[A:B]=\begin{bmatrix}1 & 1 & 1 & :9\\ 0 & -3 & 2 & :-1\\ 0 & -1 & -3 & :-15\end{bmatrix}\qquad\begin{matrix}R_2\to R_2-R_1\\ R_3\to R_3-2R_1\end{matrix}$$

$$[A:B]=\begin{bmatrix}1 & 1 & 1 & :9\\ 0 & -3 & 2 & :-1\\ 0 & 0 & 11 & :44\end{bmatrix} \qquad R_3 \to R_2 - 3R_3$$

Hence we have

$x+y+z=9, \quad -3y+2z=-1, \quad 11z=44 \therefore \quad z=4$

By back substitution, $y=3, x=2$

$\therefore$ The solution is $x=2, \quad y=3, \quad z=4$

14. Solve the following system of equations by Gauss elimination method.

$$2x+y+4z=12, \qquad 4x+11y-z=33, \qquad 8x-3y+2z=20$$

Solution:

The augmented matrix of the systems is $[A:B] = \begin{bmatrix}2 & 1 & 4 & :12\\ 4 & 11 & -1 & :33\\ 8 & -3 & 2 & :20\end{bmatrix} \begin{array}{l} R_2 \to R_2 - 2R_1, \\ R_3 \to R_3 - 4R_1 \end{array}$

$$[A:B] = \begin{bmatrix}2 & 1 & 4 & :12\\ 0 & 9 & -9 & :9\\ 0 & -7 & -14 & :-28\end{bmatrix} \begin{array}{l} R_2 = \frac{1}{9}R_2, \\ R_3 - \frac{1}{7}R_3, \end{array}$$

$$[A:B] = \begin{bmatrix}2 & 1 & 4 & :12\\ 0 & 1 & -1 & :1\\ 0 & 1 & 2 & :4\end{bmatrix} R_3 \to -R_2 + R_3$$

$$[A:B] = \begin{bmatrix}2 & 1 & 4 & :12\\ 0 & 1 & -1 & :1\\ 0 & 0 & 3 & :3\end{bmatrix}$$

Hence we have

$2x+y+4z=12; y-z=1; 3z=1$

By back substitution, $z=1, y=2$ and $x=3$

15. Solve the following system of equations by Gauss elimination method.

$$5x+y+z+w=4, \qquad x+7y+z+w=12, \qquad x+y+6z+w=-5, x+y+z+4w=-6$$

Solution: It is convenient to perform row transformations if leading entry is one. We shall write the augmented matrix by interchanging the first equation with the fourth equation.

The augmented matrix of the systems is $[A:B] = \begin{bmatrix}1 & 1 & 1 & 4 & : & -6\\ 1 & 7 & 1 & 1 & : & 12\\ 1 & 1 & 6 & 1 & : & -5\\ 5 & 1 & 1 & 1 & : & 4\end{bmatrix} \begin{array}{l} R_2 \to R_2 - R_1, \\ R_3 \to R_3 - R_1, \\ R_4 \to R_4 - 5R_1 \end{array}$

$$[A:B] = \begin{bmatrix}1 & 1 & 1 & 4 & : & -6\\ 0 & 6 & 0 & -3 & : & 18\\ 0 & 0 & 5 & -3 & : & 1\\ 0 & -4 & -4 & -19 & : & 34\end{bmatrix} R_2 = \frac{1}{3}R_2$$

$$[A:B] = \begin{bmatrix}1 & 1 & 1 & 4 & : & -6\\ 0 & 2 & 0 & -1 & : & 6\\ 0 & 0 & 5 & -3 & : & 1\\ 0 & -4 & -4 & -19 & : & 34\end{bmatrix} R_4 \to 2R_2 + R_4$$

$$[A:B]=\begin{bmatrix}1 & 1 & 1 & 4 & : & -6\\ 0 & 2 & 0 & -1 & : & 6\\ 0 & 0 & 5 & -3 & : & 1\\ 0 & 0 & -4 & -21 & : & 46\end{bmatrix} \quad R_4 \to 4R_3 + 5R_4$$

$$[A:B]=\begin{bmatrix}1 & 1 & 1 & 4 & : & -6\\ 0 & 2 & 0 & -1 & : & 6\\ 0 & 0 & 5 & -3 & : & 1\\ 0 & 0 & 0 & -117 & : & 234\end{bmatrix}$$

Hence we have $x+y+z+4w=-6;\ 2y-w=6; 5z-3w=1, -117w=234$

By back substitution, $w=-2, z=-1, y=2$ and $x=1$.

Gauss Seidel Method

This is a numerical iterative method giving approximate solution. This method cannot be applied to all the system of equations.

The method is applicable when the numerically large coefficients are along the leading diagonal of the coefficient matrix .Such a system is called a diagonally dominant system.

16. Solve the system of equations by Gauss – Seidel method. $10x+y+z=12;\ x+10y+z=12;$ and $x+y+10z=12$

Solution: The given system of equations is diagonally dominant and the equations are put in the form $x=\frac{1}{10}(12-y-z)$, $y=\frac{1}{10}(12-x-z)$, $z=\frac{1}{10}(12-x-y)$,

Let us start with trial solution $x=0, y=0, z=0.$

First iteration: $x^{(1)}=\frac{1}{10}(12-0-0)=1.2$

$y^{(1)}=\frac{1}{10}(12-1.2-0)=1.08$

$z^{(1)}=\frac{1}{10}(12-1.2-1.08)=0.972$

Second iteration: $x^{(2)}=\frac{1}{10}(12-1.08-0.972)=0.9948$

$y^{(2)}=\frac{1}{10}(12-0.9948-0.972)=1.00332$

$z^{(2)}=\frac{1}{10}(12-0.9948-1.00332)=1.000188$

Third iteration: $x^{(3)}=\frac{1}{10}(12-1.00332-1.000188)=0.99965\approx 1$

$y^{(3)}=\frac{1}{10}(12-0.99965-1.000188)=1.00002\approx 1$

$z^{(3)}=\frac{1}{10}(12-0.99965-1.00002)=1.00003\approx 1.$

Thus, $x=1, y=1, z=1$.

17. Solve the system of equations by Gauss – Seidel method

$$20x + y - 2z = 17; \quad 3x + 20y - z = -18; \quad 2x - 3y + 20z = 25,$$

Solution: The given systems of equations are diagonally dominant and the equations are put in the form

$$x = \frac{1}{20}(17 - y + 2z), \; y = \frac{1}{20}(-18 - 3x + z) \;, \; z = \frac{1}{20}(25 - 2x + 3y)$$

Let us start with trial solution $x = 0, y = 0, z = 0.$

First iteration: $x^{(1)} = \frac{1}{20}(17 - 0 - 0) = 0.85$

$y^{(1)} = \frac{1}{20}(-18 - 3(0.85) - 0) = -1.0275$

$z^{(1)} = \frac{1}{20}(25 - 2(0.85) + 3(-1.0275)) = 1.0109$

Second iteration: $x^{(2)} = \frac{1}{20}(17 - (-1.0275) + 2(1.0109)) = 1.0025$

$y^{(2)} = \frac{1}{20}(-18 - 3(1.0025) + 1.0109) = -0.9998$

$z^{(2)} = \frac{1}{20}(25 - 2(1.0025) + 3(-0.9998)) = 0.9998$

Third iteration: $x^{(3)} = \frac{1}{20}(17 - (-0.9998) + 2(0.9998)) = 0.99997 \approx 1$

$y^{(3)} = \frac{1}{20}(-18 - 3(0.99997) + 0.99998) = -1.0000055 \approx -1$

$z^{(3)} = \frac{1}{20}(25 - 2(0.99997) + 3(-1.0000055)) = 1.0000022 \approx 1$

Thus $x = 1, y = -1, z = 1$.

18. Solve the system of equations by Gauss – Seidel method upto three decimal places

$$x + y + 54z = 110; \quad 27x + 6y - z = 85; \quad 6x + 15y + 2z = 72,$$

Solution:

The given system of equations is not diagonally dominant and hence we have to first rearrange the given system of equation as follows

$27x + 6y - z = 85 \quad (|27| > |6| + |-1|)$

$6x + 15y + 2z = 72, \quad (|15| > |6| + |2|)$

$x + y + 54z = 110; (|54| > |1| + |1|)$

Now the given systems of equations are diagonally dominant and the equations are put in the form

$$x = \frac{1}{27}(85 - 6y + z), \quad y = \frac{1}{15}(72 - 6x - 2z), \; z = \frac{1}{54}(110 - x - y) \;,$$

Let us start with trial solution $x = 0, y = 0, z = 0.$

First iteration: $x^{(1)} = \frac{1}{27}(85 - 0 + 0) = 3.14815$

$y^{(1)} = \frac{1}{15}(72 - 6(3.14815) - 0) = 3.54074$

$z^{(1)} = \frac{1}{54}(110 - 3.14815 - 3.54074) = 1.91317$

Second iteration: $x^{(2)} = \frac{1}{27}(85 - 6(3.54074) + 1.91317) = 2.43218$

$y^{(2)} = \frac{1}{15}(72 - 6(2.43218) - 2(1.91317)) = 3.57204$

$$z^{(2)} = \frac{1}{54}(110 - 2.43218 - 3.57204) = 1.92585$$

Third iteration: $$x^{(3)} = \frac{1}{27}(85 - 6(3.57204) + 1.92585) = 2.42569$$

$$y^{(3)} = \frac{1}{15}(72 - 6(2.42569) - 2(1.92585)) = 3.57294$$

$$z^{(3)} = \frac{1}{54}(110 - 2.42569 - 3.57294) = 1.92595$$

Fourth iteration: $$x^{(4)} = \frac{1}{27}(85 - 6(3.57294) + 1.92595) = 2.42549$$

$$y^{(4)} = \frac{1}{15}(72 - 6(2.42549) - 2(1.92595)) = 3.57301$$

$$z^{(4)} = \frac{1}{54}(110 - 2.42549 - 3.57301) = 1.92595$$

Thus $x = 2.426, y = 3.573, z = 1.926$.

19. Solve the system of equations by Gauss – Seidel method

$$5x + 2y + z = 12; \quad x + 4y + 2z = 15; \quad x + 2y + 5z = 0,$$

Solution: The given system of equations are diagonally dominant and the equations are put in the form $x = \frac{1}{5}(12 - 2y - z)$, $y = \frac{1}{4}(15 - x - 2z)$, $z = \frac{1}{5}(-x - 2y)$,

Let us start with trial solution $x = 0, y = 0, z = 0.$

First iteration: $$x^{(1)} = \frac{1}{5}(12 - 0 - 0) = 2.4$$

$$y^{(1)} = \frac{1}{4}(15 - 2.4 - 0) = 3.15$$

$$z^{(1)} = \frac{1}{5}(-2.4 - 2(3.15)) = -1.74$$

Second iteration: $$x^{(2)} = \frac{1}{5}(12 - 2(3.15) - (-1.74)) = 1.488$$

$$y^{(2)} = \frac{1}{4}(15 - 1.458 - 2(-1.74)) = 4.255$$

$$z^{(2)} = \frac{1}{5}(-1.488 - 2(4.255)) = -1.9996$$

Third iteration: $$x^{(3)} = \frac{1}{5}(12 - 2(4.255) - (-1.9996)) = 1.0979$$

$$y^{(3)} = \frac{1}{4}(15 - 1.0979 - 2(-1.9996)) = 4.4753$$

$$z^{(3)} = \frac{1}{5}(-1.0979 - 2(4.4753)) = -2.9997$$

Fourth iteration: $$x^{(4)} = \frac{1}{5}(12 - 2(1.0979) - (-2.9997)) =$$

$$y^{(4)} = \frac{1}{4}(15 - 1.0979 - 2(-1.9996)) = 4.4753$$

$$z^{(4)} = \frac{1}{5}(-1.0979 - 2(4.4753)) = -2.9997$$

Thus $x = 1, y = -1, z = 1$.

Eigen values and Eigen vectors

Let 'A' be a given square matrix of order $n \times n$. Suppose there exists a non-zero column matrix 'X' of order $1 \times n$ and a real or complex number 'λ' such that $AX = \lambda X$ then X is called an **Eigen vector** of A and λ is called the corresponding **Eigen value** of A.

If I is the unit matrix of the same order as that of A, we write

$AX = \lambda X \Rightarrow AX = \lambda(IX) \Rightarrow AX = (\lambda I)X \quad \Rightarrow [A - \lambda I]X = 0$

$(A - \lambda I)X = 0$ Represents a set of homogeneous equations.

A non – trivial solution for this system exists if $|A - \lambda I| = 0$. on expanding, we get a polynomial equation in λ. This is called the characteristic equation of A. The roots of this equation are called the Eigen values or Eigen roots or characteristic roots or latent roots. For each value of λ, there will be an Eigen vector $X \neq O$ which is also called a characteristic vector.

Problems

20. Find all the Eigen values and the corresponding Eigen vectors of the matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

Solution: Let $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$,

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

On expanding, we have $\lambda^3 - 18\lambda^2 + 15\lambda = 0$ solving, we get $\lambda = 0,\ 3,\ 15$ are the Eigen values.

Now the system of equations is
$$\begin{aligned} (8-\lambda)x - 6y + 2z &= 0 \\ -6x + (7-\lambda)y - 4z &= 0 \\ 2x - 4y + (3-\lambda)z &= 0 \end{aligned}$$

Case (i): Let $\lambda = 0$ and **t**he system of equations becomes
$$\begin{aligned} 8x - 6y + 2z &= 0 \quad &-(1) \\ -6x + 7y - 4z &= 0 \quad &-(2) \\ 2x - 4y + 3z &= 0 \quad &-(3) \end{aligned}$$

Applying the rule of cross multiplication for (1), (2)

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

ie, $\frac{x}{10} = \frac{y}{20} = \frac{z}{20}$ *or* $\frac{x}{1} = \frac{y}{2} = \frac{z}{2}$

$\therefore (x, y, z)$ are proportional to $(1,\ 2,\ 2)$ and we can write $x = k,\ y = 2k,\ \ z = 2k$ where k is arbitrary $\therefore$ the Eigen vector X_1 for $\lambda = 0$ *is* $\begin{bmatrix} k \\ 2k \\ 2k \end{bmatrix}$

Case (ii): Let $\lambda = 3$, As in the case (i), we have the Eigen vector $X_2 = \begin{bmatrix} 2k \\ k \\ 2k \end{bmatrix}$

Case (iii): Let $\lambda = 15$, We have the Eigen vector $X_3 = \begin{bmatrix} 2k \\ -2k \\ k \end{bmatrix}$. Hence the solution.

21. Find all the Eigen values and the corresponding Eigen vectors of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Solution: The characteristic equation of A is ....

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0 \;..\;\;.$$

On expanding, we obtain $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$, by solving, we get $\lambda = 2,\ 2,\ 8$.

Now the system of equations are
$$(6-\lambda)x - 2y + 2z = 0$$
$$-2x + (3-\lambda)y - 1z = 0$$
$$2x - 1y + (3-\lambda)z = 0$$

Case-(i): Let. $\lambda = 2$., and the corresponding equations are
$$4x - 2y + 2z = 0$$
$$-2x + y - z = 0 \quad .|A - \lambda I| = 0.$$
$$2x - y + z = 0$$

The above sets of equations are all same as we have only one independent equation $2x - y + z = 0$ and hence we can choose two variables arbitrarily.

Let $z = k_1$ and $y = k_2$, $x = \dfrac{(k_2 - k_1)}{2}$

$X_1 = \left(\dfrac{(k_2 - k_1)}{2}, k_2, k_1 \right)$ is the eigen vector corresponding to $\lambda = 2$

Case-(ii): Let $\lambda = 8$ and the corresponding equations are
$$-2x - 2y + 2z = 0$$
$$-2x - 5y - z = 0$$
$$2x - y - 5z = 0$$

$$\Rightarrow \frac{x}{12} = \frac{-y}{6} = \frac{z}{6} \; or \; \frac{x}{2} = \frac{y}{-1} = \frac{z}{1}$$

. $X_2 = (2, -1, 1)$. is the Eigen vector corresponding to $\lambda = 8$.

22. Find all the Eigen values and the corresponding Eigen vectors of the matrix $\begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$

Solution: The characteristic equation of A is $|A - \lambda I| = 0$ $(\lambda - 1)^3 = 0$

$$\begin{bmatrix} -3-\lambda & -7 & -5 \\ 2 & 4-\lambda & 3 \\ 1 & 2 & 2-\lambda \end{bmatrix} = 0$$

On expanding, we obtain ,by solving, we get $\lambda = 1, 1, 1$. All the Eigen values are equal we now form the system of equations,

$$(-3-\lambda)x-7\lambda-5\lambda=0$$
$$2x+(4-\lambda)y+3z=0$$
$$x+2y+(2-\lambda)z=0$$

Putting $\lambda=1$, we obtain,

$$-4x-7y-5z=0$$
$$2x+3y+3z=0 \quad \Rightarrow \frac{x}{-6}=\frac{y}{-1}=\frac{z}{-1}$$
$$x+2y+z=0$$

Thus, $X=(3,\ -1,\ -1)$, is the Eigen vector corresponding to the coincident Eigen value $\lambda=1$.

23. Find all the Eigen values and the corresponding Eigen vectors of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ $|A-\lambda I|=0$

Solution: The characteristic equation of A is

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}=0$$

On expanding, we obtain $\lambda^3-7\lambda^2+36=0$, by solving, we get $\lambda=-2,\ 3,\ 6$

We now form the system of equations,

$$(1-\lambda)x+y+3z=0$$
$$x+(5-\lambda)y+z=0$$
$$3x+y+(1-\lambda)z=0$$

Case (i): Let $\lambda=-2$,

and corresponding equations are
$$3x+y+3z=0$$
$$x+2y+z=0$$
$$3x+y-2z=0$$

$$\Rightarrow \frac{x}{-1}=\frac{-y}{0}=\frac{z}{1}$$

$\therefore$ $(x,y,z)=(1,0,-1)$ is the eigen vector corresponding to $\lambda=-2$

Case (ii): Let $\lambda=3$,

And the corresponding equations are
$$-2x+y3z=0$$
$$x+2y+z=0$$
$$3x+y-2z=0$$

$$\Rightarrow \frac{x}{1}=\frac{-y}{1}=\frac{z}{1}$$

$\therefore$ $(x,y,z)=(1,-1,\ 1)$ is the Eigen vector corresponding to $\lambda=3$

Case (iii): Let $\lambda=6$,

and corresponding equations are
$$-5x+y+3z=0$$
$$x-y+z=0$$
$$3x+y-5y=0$$

$$\Rightarrow \frac{x}{1}=\frac{y}{2}=\frac{z}{1}$$

$\therefore$ $(x,y,z)=(1,\ 2,\ 1)$ is the eigen vector corresponding to $\lambda=6$

24. **Find all the Eigen values and the corresponding Eigen vectors of the matrix** $\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$

Solution: The characteristic equation of A is $|A-\lambda I| = 0$

$$\begin{bmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{bmatrix} = 0$$

On expanding, we obtain $\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$, by solving, we get $\lambda = 3,\ 6,\ 9$ are the eigen values.

We now form the system of equations,
$$(7-\lambda)x - 2y + 0z = 0$$
$$-2x + (6-\lambda)y - 2z = 0$$
$$0x - 2y + (5-\lambda)z = 0$$

Case (i): Let $\lambda = 3$, and the corresponding equations are

$$4x - 2y + 0z = 0 \quad ....(1)$$
$$-2x + 3y - 2z = 0 \quad ....(2)$$
$$0x - 2y + 2z = 0 \quad ....(3)$$

From (1) and (2) , $\Rightarrow \frac{x}{1} = \frac{y}{2} = \frac{z}{2}$.

$\therefore \quad X_1 = (1,\ 2,\ 2)$ is the eigen vector corresponding to $\lambda = 3$.

Case (ii): Let $\lambda = 6$ and the corresponding equations are

$$1x - 2y + 0z = 0 \quad ......(4)$$
$$-2x - 0y - 2z = 0 \quad ....(5)$$
$$0x - 2y - 4z = 0 \quad ....(6)$$

From (4) and (5), $\Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$.

$\therefore \quad X_2 = (2,\ 1,\ -2)$ is the eigen vector corresponding to $\lambda = 6$.

Case (iii) : Let $\lambda = 9$ and the corresponding equations are

$$-2x - 2y + 0z = 0 \quad ...(7)$$
$$-2x - 3y - 2z = 0 \quad ...(8)$$
$$0x - 2y - 4z = 0 \quad .....(9)$$

From (7) and (8) $\Rightarrow \frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$.

$\therefore \quad X_3 = (2,\ -2,\ 1)$ is the Eigen vector corresponding to . $\lambda = 9$.

Properties of Eigen values

I. **Any square matrix A and its transpose A' have the same Eigen values.**

Proof:-

We have

$$\left(A - \lambda I\right)' = A' - \lambda I' = A' - \lambda I$$

$$\therefore \left|\left(A - \lambda I\right)'\right| = \left|A' - \lambda I\right|$$

$$\therefore \left|\left(A - \lambda I\right)\right| = \left|A^1 - \lambda I\right|$$

$|A-\lambda I|=0 \Leftrightarrow |A'-\lambda I|=0 \Rightarrow \lambda$ is an Eigen value of $A \Leftrightarrow \lambda$ is an Eigen value of A^1.

II. The Eigen values of a triangular matrix are just the diagonal elements of the matrix.

Proof: Let $A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ .. & .. & & .. \\ .. & .. & & .. \\ 0 & 0 & & a_{nn} \end{bmatrix}$ be a triangular matrix of order n.

Then $|A-\lambda I|=(a_{11}-\lambda)(a_{22}-\lambda)\dots(a_{nn}-\lambda)$.

$\therefore$ Roots of $|A-\lambda I|=0$ are $\lambda=a_{11}, \lambda=a_{22}, \dots\dots, \lambda=a_{nn}$.

$\therefore$ The Eigen values of A are the diagonal elements of A.

III. The Eigen values of an idempotent matrix are either zero or unity.

Proof:- Let A be an idempotent matrix.

$\therefore \quad A^2=A$.

Let λ be an Eigen value of A.

Then there exists a non-zero vector x such that $AX=\lambda X$ …….(1)

$\therefore A(AX)=A(\lambda X)$

$\Rightarrow A^2X=\lambda(AX)$

$\Rightarrow AX=\lambda(AX). \quad [\because A^2=A \text{ and } AX=\lambda X]$

$\Rightarrow AX=\lambda^2 X$ …………(2)

From (1), (2) $\Rightarrow \quad \lambda^2 X=\lambda X \quad \Rightarrow (\lambda^2-\lambda)X=0 \Rightarrow \lambda^2-\lambda=0 \Rightarrow \lambda=0$ or 1.

IV. The sum of the Eigen values of a matrix is the sum of the elements of the principal diagonal.

Proof: Consider the square matrix $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then $|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$

$=-\lambda^3+\lambda^2(a_{11}+a_{22}+a_{33})-\lambda().+()\dots\dots\dots\dots(1)$

If $\lambda_1, \lambda_2, \lambda_3$ are the Eigen values of , then

$|A-\lambda I|=(-1)^3(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3).=-\lambda^3+\lambda^2(\lambda_1+\lambda_2+\lambda_3)-\lambda().+()\dots\dots(2)$

Equating the RHS of (1) and (2) and comparing the coefficients of λ^2,

we get $\lambda_1+\lambda_2+\lambda_3=a_{11}+a_{22}+a_{33}$.

V. The product of the Eigen values of a matrix A is equal to its determinant. (Try yourself).

VI. If λ is an Eigen value of a matrix A, then $1/\lambda$ is the Eigen value of A^{-1}.

Proof:

Let X be the Eigen vector corresponding to λ.Then $AX = \lambda X$..........(1)

Pre multiply both sides by A^{-1}, we get

$A^{-1}AX = A^{-1}\lambda X$

$IX = \lambda A^{-1}X$

$X = \lambda\left(A^{-1}X\right)$

$\therefore A^{-1}X = \left(\dfrac{1}{\lambda}\right)X$........(2)

Comparing (1) and (2), we have $\dfrac{1}{\lambda}$ is the Eigen value of the inverse matrix A^{-1}.

VII. If λ is an Eigen value of an orthogonal matrix then $1/\lambda$ is also its Eigen value. (Try yourself).

VIII. If $\lambda_1, \lambda_2, ...,\lambda_n$ are the Eigen values of a matrix A, then A^m has the Eigen values $\lambda_1{}^m, \lambda_2{}^m, ...\lambda_n{}^m$. Where m is a positive integer.

Proof: Let λ_1 be the Eigen value of A and X_1 be the corresponding Eigen vector.

Then $AXi = \lambda_i X_i$

Now $A^2 X_i = A\left(AX_i\right)$

$= A\left(\lambda_i X_i\right) = \lambda_i\left(AX_i\right) = \lambda_i\left(\lambda_i X_i\right)$

$A^2 X_i = \lambda_i{}^2 X_i$

Similarly, $A^3 X_i = \lambda_i{}^3 X_i$ In general, $A^m X_i = \lambda_i{}^m X_i$.

Hence, $\lambda_i{}^m$ is an Eigen value of A^m .

Similarity of Matrices and Diagonalization of matrices

Two square matrices A and B of the same order are said to be similar if there exists a non-singular matrix P such that $B = P^{-1}AP$.

Here B is said to be similar to A.

Diagonalization of a square matrix

Property: If A is a square matrix of order n linearly independent eigen vectors then there exists an n^{th} order square matrix P such that $P^{-1}AP$ is a diagonal matrix.

We shall establish this result by considering a third order square matrix to make an important and interesting observation.

Let A be a third order square matrix having eigen values λ_1, λ_2, λ_3 and the corresponding eigen vectors.

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, \quad X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

Let the square matrix P be equal to $\left[X_1, \quad X_2 \quad X3\right]$

i.e., $P = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$

Now $AP = A\begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} = \begin{bmatrix} AX_1 & AX_2 & AX_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 X_1 & \lambda_2 X_2 & \lambda_3 X_3 \end{bmatrix}$

Or $AP = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

ie., $AP = PD$ where D is the diagonal matrix represented by

$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

Consider $AP = PD$

Pre multiplying by P^{-1} we have

$P^{-1}AP = P^{-1}PD = (P^{-1}P)D = ID = D$

$P^{-1}AP = D$

It is important that. $P^{-1}AP$ .is a diagonal matrix having the eigen values of $A(\lambda_1 \quad \lambda_2 \quad \lambda_3)$ in its principal diagonal. We say that the matrix P diagonalizes A where P is constituted by the eigen vectors of A.

Note:

1. The transformation of a square matrix A to $P^{-1}AP$ is known as Similarity Transformation.
2. The matrix P which diagonalizes A called the modal matrix of A and the resulting diagonal matrix is called the Spectral matrix of A.

Computation of powers of square matrix

Diagonalization of a square matrix A also helps us to find the powers of $A: A^2 \quad A^3 \quad A^4$,........ etc.

We have $D = P^{-1}AP$

$\therefore D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A(PP^{-1})AP = P^{-1}A[AP] = P^{-1}A^2P$.

i.e., $D^2 = P^{-1}A^2P$

Pre multiplying by P and post multiplying by P^{-1} we have

$PD^2P^{-1} = (PP^{-1}) = A^2(PP^{-1}) = IA^2I = A^2$

i.e., $A^2 = PD^2P^{-1}$.

Thus in general, $A^n = PD^nP^{-1}$ where $D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$

Working procedure for diagonalization of a square matrix A of order 3:

- We find Eigen values $\lambda_1 \quad \lambda_2 \quad \lambda_3$.
- We find the Eigen vectors $X_1 \quad X_2 \quad X_3$ corresponding to the eigen values $\lambda_1 \quad \lambda_2 \quad \lambda_3$.
- We form the modal matrix $P = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$
- We compute $P^{-1} = \dfrac{1}{|P|}(AdjP)$.

  The diagonalization of A is given by $D = P^{-1}AP$ where $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$.

Problems

25. **Reduce the matrix** $A=\begin{bmatrix}-1 & 3\\ -2 & 4\end{bmatrix}$ **to the diagonal form and hence find** A^4**.**

Solution:

The characteristic equation of A is $|A-\lambda I|=0$

$$\begin{vmatrix}(-1-\lambda & 3\\ -2 & 4-\lambda\end{vmatrix}=0$$

$(-1-\lambda)(4-\lambda)+6=0$

$\lambda^2-3\lambda+2=0$

$\therefore \lambda=1,2$ are the eigen values of A.

Now consider $[A-\lambda I][X]=[0]$

$$\begin{bmatrix}(-1-\lambda) & 3\\ -2 & (4-\lambda)\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$(-1-\lambda)x+3y=0$

$-2x+(4-\lambda)y=0$

Case-(i): Let $\lambda=1$,

We get $-2x+3y=0$ or $2x=3y$ or $\dfrac{x}{3}=\dfrac{y}{2}$.

$\therefore X_1=(3\quad 2)'$ is the eigen vector corresponding to $\lambda=1$.

Case-(ii): Let $\lambda=2$,

We get $-3x+3y=0$ or $x=y$ or $\dfrac{x}{1}=\dfrac{y}{1}$

$\therefore X_2=(1\quad 1)'$ is the eigen vector corresponding to $\lambda=2$.

Modal matrix $P=[X_1\quad X_2]=\begin{bmatrix}3 & 2\\ 2 & 1\end{bmatrix}$

We have $|P|=1$ and $P^{-1}=\dfrac{1}{|P|}(AdjP)$

$$P^{-1}=\begin{bmatrix}1 & -1\\ -2 & 3\end{bmatrix}$$

Now $P^{-1}AP=\begin{bmatrix}1 & -1\\ -2 & 3\end{bmatrix}\begin{bmatrix}-1 & 3\\ -2 & 4\end{bmatrix}\begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}=\begin{bmatrix}1 & -1\\ -2 & 3\end{bmatrix}\begin{bmatrix}3 & 2\\ 3 & 2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 2 & 2\end{bmatrix}$

Thus $P^{-1}AP = D = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$ is the diagonal matrix.

Or $P^{-1}AP = Diag(1 \quad 2)$.

Also we have $A^n = PD^nP^{-1}$

$$A^4 = PD^4P^{-1} \quad \text{where } D^4 = \begin{bmatrix} 1^4 & 0 \\ 0 & 2^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -29 & 45 \\ -30 & 46 \end{bmatrix}$$

Thus $A^4 = \begin{bmatrix} -29 & 45 \\ -30 & 46 \end{bmatrix}$.

26. Reduce the matrix $A = \begin{bmatrix} 11 & -4 & -7 \\ & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$ **into a diagonal matrix. Also find** A^5 **.**

Solution: The characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{bmatrix} 11-\lambda & -4 & -7 \\ 7 & (-2-\lambda) & -5 \\ 10 & -4 & (-6-\lambda) \end{bmatrix} = 0$$

$$(11-\lambda)\,[(-2-\lambda)(-6-\lambda) - 20] + 4[7(-6-\lambda) + 50] - 7[-28 - 10(-2-\lambda)] = 0$$

$$(11-\lambda)[\lambda^2 + 8\lambda - 8] + 4[8 - 7\lambda] - 7[10\lambda - 8] = 0$$

$$11\lambda^2 + 88\lambda - 88 - \lambda^3 - 8\lambda^2 + 8\lambda + 32 - 28\lambda - 70\lambda + 56 = 0$$

$$\lambda^3 - 3\lambda^2 - 2\lambda = 0$$

$$\lambda = 0, \quad 1 \quad 2.$$

Now consider $[A - \lambda I]\ [X] = [0]$.

$$(11-\lambda)x - 4y - 7z = 0$$
$$7x + (-2-\lambda)y - 5z = 0$$
$$10x - 4y + (-6-\lambda)z = 0$$

Case- (i): Let $\lambda = 0$ and the corresponding equations are

$$11x - 4y - 7z = 0$$
$$7x - 2y - 5z = 0$$
$$10x - 4y - 6z = 0$$

$$\frac{x}{6} = \frac{-y}{-6} = \frac{z}{6} \quad \text{or} \quad \frac{x}{1} = \frac{y}{1} = \frac{z}{1}$$

$X_1 = (1 \quad 1 \quad 1)'$ is the eigen vector corresponding to $\lambda = 0$.

Case (ii): Let $\lambda = 1$ and the corresponding equations are

$$10x - 4y - 7z = 0$$
$$7x - 3y - 5z = 0$$
$$10x - 4y - 7z = 0$$

$$\frac{x}{-1} = \frac{-y}{-1} = \frac{z}{-2} \text{ or } \frac{x}{1} = \frac{y}{-1} = \frac{z}{2}$$

$X_1 = \begin{pmatrix}1 & -1 & 2\end{pmatrix}'$ is the eigen vector corresponding to $\lambda = 1$.

Case (iii): Let $\lambda = 2$ and the corresponding equations are

$$9x - 4y - 7z = 0$$
$$7x - 4y - 5z = 0$$
$$10x - 4y - 8z = 0$$

$$\frac{x}{-8} = \frac{-y}{4} = \frac{z}{-8} \text{ or } \frac{x}{2} = \frac{y}{1} = \frac{z}{2}$$

$X_1 = \begin{pmatrix}2 & 1 & 2\end{pmatrix}'$ is the Eigen vector corresponding to $\lambda = 2$.

Hence the modal matrix $P = \lfloor X_1 \quad X_2 \quad X_3 \rfloor = \begin{bmatrix}1 & 1 & 2\\ 1 & -1 & 1\\ 1 & 2 & 2\end{bmatrix}$

We have $|P| = 1(-2-2) - 1(2-1) + 2(2+1) = 1$

$$AdjP = \begin{bmatrix}-4 & 2 & 3\\ -1 & 0 & 1\\ 3 & -1 & -2\end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|}(AdjP) = \begin{bmatrix}-4 & 2 & 3\\ -1 & 0 & 1\\ 3 & -1 & -2\end{bmatrix}$$

Diagonalization of A is given by $P^{-1}AP$

Now $P^{-1}AP = \begin{bmatrix}-4 & 2 & 3\\ -1 & 0 & 1\\ 3 & -1 & -2\end{bmatrix}\begin{bmatrix}11 & -4 & -7\\ 7 & -2 & -5\\ 10 & -4 & -6\end{bmatrix}\begin{bmatrix}1 & 1 & 2\\ 1 & -1 & 1\\ 1 & 2 & 2\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix} = D$

$P^{-1}AP = D = Diag(0 \quad 1 \quad 2)$.

Further we have $A^n = PD^n P^{-1}$.

$A^5 = PD^5 P^{-1}$ and $D^5 = Diag(0^5 \quad 1^5 \quad 2^5) = Diag(0 \quad 1 \quad 32)$.

Hence $A^5 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 191 & -64 & -127 \\ 97 & -32 & -65 \\ 190 & -64 & -126 \end{bmatrix}$.

27. Determine the Eigen values and the corresponding Eigen values of the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Solution: The Characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} (-2-\lambda) & 2 & -3 \\ 2 & (1-\lambda) & -6 \\ -1 & -2 & (-\lambda) \end{vmatrix} = 0$$

$(-2-\lambda)[-\lambda(1-\lambda)-12]-2[-2\lambda-6]-3[-4+1-\lambda]=0$

$(-2-\lambda)[-\lambda+\lambda^2-12)+(4\lambda+12)+(9+3\lambda)=0$

$(-2-\lambda)(\lambda+3)(\lambda-4)+4(\lambda+3)+3(\lambda+3)=0$

$(\lambda+3)(-\lambda^2+2\lambda+15)=0$

$(\lambda+3)(\lambda+3)(\lambda-5)=0$

$\lambda = -3 \quad -3 \quad 5$.

We now form the system of equations.

$(-2-\lambda)+2y-3z=0$

$2x+(1-\lambda)-6z=0$

$-1x-2y-\lambda z=0$.

Case (i): Let $\lambda = -3$ and the corresponding equations are

$x+2y-3z=0$

$2x+4y-6z=0$

$-x+2y+3z=0$.

It should be observed that the equations are all same and we have only one independent equation $x+2y-3z=0$ (In case the rule of cross multiplication is applied, we get $x=y=z=0$ which is a trivial solution.

Two variables can be arbitrary.

Let $z = k_1$, $y = k_2$ $\quad \therefore\ x = 3k_1 - 2k_2$

The eigen vector corresponding to the coincident eigen value $\lambda = -3$ be denoted by $X_{1,2}$ and we have $X_{1,2} = (3k_1 - 2K_2 \quad k_2 \quad k_1)'$ where k_1, k_2 are arbitray. We choose convenient values for k_1 and k_2 to obtain two distinct eigen vectors.

(i) Let $k_1 = 1,\ k_2 = 1 \ \therefore\ X_1 = (1 \quad 1 \quad 1)'$

(ii) Let $k_1 = 1,\ k_2 = 0 \ \therefore\ X_2 = (3 \quad 0 \quad 1)'$

Case (ii): Let $\lambda = 5$ and the corresponding equations are

$$-7x + 2y - 3z = 0 \qquad \ldots\ldots (1)$$
$$2x - 4y - 6z = 0 \qquad \ldots\ldots. (2)$$
$$-x - 2y - 5z = 0$$

Solving (1) and (2) , $\dfrac{x}{-12-12} = \dfrac{-y}{42+6} = \dfrac{z}{28-4}$

$$\frac{x}{-24} = \frac{-y}{48} = \frac{z}{24} \quad \text{or} \quad \frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$$

$X_3 = (1 \quad 2 \quad -1)'$ is the eigen vector corresponding to $\lambda = 5$.

We have modal matrix

$$P = [X_1 \quad X_2 \quad X_3] = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

$$|P| = 1(-2) - 3(-3) + 1(1) = 8.$$

$$AdjP = \begin{bmatrix} +(0-2) & -(-3-1) & +(6-0) \\ -(-1-2) & +(-1-1) & -(2-1) \\ +(1-0) & -(1-3) & +(0-3) \end{bmatrix}$$

Diagonalization of A is given by $P^{-1}AP$,

$$= \frac{1}{8}\begin{bmatrix} -2 & 4 & 6 \\ 3 & -2 & -1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = D$$

Thus $P^{-1}AP = D = Diag(-3 \quad -3 \quad 5)$.

28. Show that the following matrix is not diagnosable $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Solution: The characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)^3 = 0$$
$$\lambda = 2 \quad 2 \quad 2.$$

The Eigen vector corresponding to $\lambda = 2$ has to be obtained by solving the system of equations.

$$(2-2)x + 1y + 0Z = 0$$
$$0x + (2-2)y + 1z = 0$$
$$0x + 0y + (2-2)z = 0$$

$y = 0,\ z = 0$; x can be arbitrary.

$\therefore\ x = k,\ y = 0,\ z = 0$ is the eigen vector corresponding to the coincident eigen value $\lambda = 2$.
It is evident that we cannot obtain three linearly independent eigen vectors.

Thus we conclude that the matrix A is not diagonizable.

Convert an n^{th} order differential equation to a system of equations

An n^{th} order D.E. $y^{(n)} = F\left(t, y, y', y'', \ldots\ldots, y^{(n-1)}\right)$ .... (1)
can always be reduced to a system of 'n' first order D.E.'s simply by setting
$y_1 = y, y_2 = y', y_3 = y'', \ldots\ldots, y_n = y^{(n-1)}$ .... (2)
We then immediately obtain the first order system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\ldots\ldots \\ &\ldots\ldots \\ y_{n-1}' &= y_n \\ y_n' &= F(t, y_1, y_2, y_3, \ldots\ldots, y_n) \end{aligned} \qquad \ldots\ldots\ldots (3)$$

The first (n-1) of these equations follows immediately from (2) by differentiation. Also from (2), $y_n' = y^{(n)}$, so that the last equation in (3) follows from the given D.E. (1).

- Converting n^{th} order D.E. to a system of 'n' first order D.E.'s is practically & theoretically important.
- Practically, it permits the study & solution of single equations by methods for systems.
- Theoretically, it gives a possibility of including the theory of higher order equations into that of first order systems.

29. Write the following 2^{nd} order differential equation as a system of first order, linear

Differential equations: $\dfrac{d^2x}{dt^2} + 3a\dfrac{dx}{dt} - 4a^2x = 0$.

Solution: In this problem we'll need 2 new functions.

$$\begin{aligned} &Put\ x' = y \\ &\Rightarrow x'' = y' \\ &Now\ the\ given D.E.\ reduces\ to\ y' = -3ay + 4a^2x \\ &Hence the\ system of\ \ first\ order\ linear\ D.E.'s\ is \end{aligned}$$

$$x' = y$$
$$y' = -3ay + 4a^2x$$

30. Write the following 4th order differential equation as a system of first order, linear differentialequations: $y^{(4)}+3y''-\sin(t)y'+8y=t^2$, $y(0)=1, y'(0)=2, y''(0)=3, y'''(0)=4$.

Solution:

In this problem we'll need 4 new functions.

$x_1 = y \Rightarrow x_1' = y' = x_2$

$x_2 = y' \Rightarrow x_2' = y'' = x_3$

$x_3 = y'' \Rightarrow x_3' = y''' = x_4$

$x_4 = y''' \Rightarrow x_4' = y^{(4)} = -3y''+\sin(t)y'-8y+t^2 = -8x_1+\sin(t)x_2-3x_3+t^2$

The system along with the initial conditions is then,

$$\begin{array}{ll} x_1' = x_2 & x_1(0)=1 \\ x_2' = x_3 & x_2(0)=2 \\ x_3' = x_4 & x_3(0)=3 \\ x_4' = -8x_1+\sin(t)x_2-3x_3+t^2 & x_4(0)=4 \end{array}$$

Solution of simultaneous differential equation

Here we employ the elementary technique of solving a system of linear algebraic equations in cancelling either of the dependent variables (x or y).

Problems

31. Solve $\frac{dx}{dt}+2y=-\sin t, \quad \frac{dy}{dt}-2x=\cos t$.

Solution: Taking $D=\frac{d}{dt}$ we have the system of equations.

$Dx+2y=-\sin t$

$-2x+Dy=\cos t$

Multiplying (1) by 2 and operate (2) by D.

i.e.

$$\begin{array}{l} 2Dx+4y=-2\sin t \\ \underline{-2Dx+D^2y=D(\cos t)} \end{array}$$

On adding we get, $(D^2+4)y=-3\sin 3t$

A.E. is $m^2+4=0 \quad \Rightarrow m=\pm 2i$

$\phi_c = c_1\cos 2t+c_2\sin 2t$, $\phi_p=\frac{-3\sin 3t}{D^2+4}=\frac{-3\sin 3t}{-1^2+4}=-\sin t$.

By considering $\frac{dy}{dt}-2x=\cos t, we\ \ get.$

$$x=\frac{1}{2}\left[\frac{dy}{dt}-\cos t\right]$$

$$x=\frac{1}{2}\left[\left\{\frac{d}{dt}(c_1\cos 2t+c_2\sin 2t-\sin t)-\cos t\right\}\right]$$

By using (3)

$$=\frac{1}{2}\left[-2c_1\sin 2t+2c_2\cos 2t-\cos t-\cos t\right] \Rightarrow x=-c_1\sin 2t+c_2\cos 2t-\cos t.$$

32. **Solve** $\frac{dy}{dx}+y=z+e^x, \quad \frac{dz}{dx}+z=y+e^x$

Solution: We have

$$(D+1)y-z=e^x$$

$$-y+(D+1)z=e^x$$

Operating (1) by (D+1) we have,

$$(D^2+1)^2y-(D+1)z=(D+1)e^x=2e^x$$

$$\underline{-y+(D+1)z=e^x}$$

On adding we get, $\left[(D+1)^2-1\right]y=3e^x$, i.e. $(D^2+2D)y=3e^x$.

A.E. is $m^2+2m=0 \quad \Rightarrow m(m+2)=0 \quad \Rightarrow m=0,-2.$

$$\phi_c=c_1+c_2e^{-2x},\ \phi_p=\frac{3e^x}{D^2+2D}=\frac{3e^x}{1^2+2.1}=e^x.$$

Thus $y=c_1+c_2e^{-2x}+e^x$.

Let us now consider $\frac{dy}{dx}+y=z+e^x$.

$$z=\frac{dy}{dx}+y-e^{-x}$$

$$=\frac{d}{dx}[c_1+c_2e^{-2x}+e^x]+(c_1+c_2e^{-2x}+e^x)-e^x$$

$$z=c_1-c_2e^{-2x}+e^x.$$

Solve the system of linear differential equations by diagonalization method and discuss the stability of the system

Solution by Diagonalization method

This is an alternative method for solving a homogeneous system $X'=AX$ of linear first order D.E.'s. This method is applicable to such a system whenever the coefficient matrix A is diagonalizable.

Coupled Systems: A homogeneous linear system $X'=AX$ **:**

$$\begin{pmatrix} x_1' \\ x_2' \\ .. \\ x_n' \end{pmatrix}=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{pmatrix} \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1)$$

In which each x_i' is expressed as a linear combination of $x_1, x_2, x_3, \dots\dots\dots, x_n$ is said to be **coupled**. If the coefficient matrix A is diagonalizable, then the system can be uncoupled in that each x_i' can be expressed solely in terms of x_i.

If the matrix A has n linearly independent eigen vectors then we have learnt that we can find a matrix P such that $P^{-1}AP = D,$ where D is a diagonal matrix. If we make the substitution $X = PY$ in the system $X' = AX$, then $PY' = APY \Rightarrow Y' = P^{-1}APY \Rightarrow Y' = DY$ ……….(2)

The last equation in (2) is the same as

$$\begin{pmatrix} y_1' \\ y_2' \\ .. \\ y_n' \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ .. \\ y_n \end{pmatrix} \qquad \text{………………………………… (3)}$$

Since D is diagonal, an inspection of (3) reveals that this new system is uncoupled: each D.E. in the system is of the form $y_i' = \lambda_i y_i,\ i = 1,2,.........,n$. The solution of each of these linear equations is $y_i = c_i e^{\lambda_i t},\ i = 1,2,.........,n$. Hence the general solution of (3) can be written as the column vector

$$Y = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \\ c_n e^{\lambda_n t} \end{pmatrix}$$

Since we know now Y and since the matrix P can be constructed from the eigen vectors of A, the general solution of the original system $X' = AX$ is obtained from $X = PY$.

Eigenvalue and Stability of the system

The table below gives a complete overview of the stability corresponding to each type of eigenvalue.

Eigen values	Stability
All real & positive	Unstable
All real & negative	Stable
Mixed positive & negative real	Unstable
a+bi	Unstable
-a+bi	Stable
0+bi	Unstable
Repeated values	Depends upon orthogonality of eigen vectors

33. **Solve** $X' = \begin{pmatrix} -2 & -1 & 8 \\ 0 & -3 & 8 \\ 0 & -4 & 9 \end{pmatrix} X$ **by diagonalization.**

Solution: First find eigen values & corresponding eigen vectors.

$|A - \lambda I| = 0 \Rightarrow -(\lambda + 2)(\lambda - 1)(\lambda - 5) = 0 \Rightarrow \lambda = -2,1,5$.

Since eigen values are distinct then eigen vectors are linearly independent.

$Solving\ (A - \lambda_i I)K = 0,\ for\ i = 1,2,3$

$$K_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ K_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix},\ K_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Thus a matrix that diagonalizes the coefficient matrix is $P = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

The entries on the main diagonal of D are the eigen values of A corresponding to order in which the eigen vectors appear in P.

$$D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

As we know that the substitution $X = PY \text{ in } X' = AX$ gives the uncoupled system $Y' = DY$. The general solution of this last system is immediate:

$$Y = \begin{pmatrix} c_1 e^{-2t} \\ c_2 e^{t} \\ c_3 e^{5t} \end{pmatrix}$$

Hence the solution of the given system is

$$X = PY = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{-2t} \\ c_2 e^{t} \\ c_3 e^{5t} \end{pmatrix} = \begin{pmatrix} c_1 e^{-2t} + 2c_2 e^{t} + c_3 e^{5t} \\ 2c_2 e^{t} + c_3 e^{5t} \\ c_2 e^{t} + c_3 e^{5t} \end{pmatrix}$$

This can also be written in usual manner as

$$X = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{5t}$$

Stability: Since at least one eigen value is real & positive, hence the system is unstable.

Note: Solution by diagonalization will always work provided we can find 'n' linearly independent eigen vectors of $n \times n$ matrix A, the eigen values of A could be real & distinct, complex or repeated. The method fails when A has repeated eigen values &'n' linearly independent eigen vectors cannot be found. In this situation A is not diagonalizable.

34. Solve $X' = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} X + \begin{pmatrix} 3e^{t} \\ e^{t} \end{pmatrix}$ **by diagonalization and hence discuss the stability of the system.**

Solution: Eigen values & eigen vectors of the coefficient matrix are found to be

$$\lambda = 0, 5,\ K_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix},\ K_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\therefore P = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \ \& \ P^{-1} = \begin{pmatrix} 1/5 & -2/5 \\ 2/5 & 1/5 \end{pmatrix}$$

Using substitution $X = PY$ and

$$P^{-1}F = \begin{pmatrix} 1/5 & -2/5 \\ 2/5 & 1/5 \end{pmatrix} \begin{pmatrix} 3e^{t} \\ e^{t} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} e^{t} \\ \frac{7}{5} e^{t} \end{pmatrix}$$

The uncoupled system is $Y' = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix} Y + \begin{pmatrix} \frac{1}{3} e^{t} \\ \frac{7}{5} e^{t} \end{pmatrix}$

The solutions of the two D.E.'s $y_1' = \frac{1}{5}e^t$ & $y_2' = 5y_2 + \frac{7}{5}e^t$ are

$y_1 = \frac{1}{5}e^t + c_1$ & $y_2 = -\frac{7}{20}e^t + c_2 e^{5t}$ *respectivdy.*

Hence the solution of the original system is

$$X = PY = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{5}e^t + c_1 \\ -\frac{7}{20}e^t + c_2 e^{5t} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}e^t + c_1 + 2c_2 e^{5t} \\ -\frac{3}{4}e^t - 2c_1 + c_2 e^{5t} \end{pmatrix}$$

Writing in usual manner

$$X = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t} - \begin{pmatrix} 1/2 \\ 3/4 \end{pmatrix} e^t$$

Stability: Since all the eigen values are real, distinct & positive, hence the system is unstable.

Question Bank

Write the following matrices in the Row echelon form and hence find their rank:	
1	$\begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \end{bmatrix}$
2	$\begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & -2 \end{bmatrix}$
3	$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 3 & 3 & 5 & 2 \\ 2 & -1 & 1 & -3 \end{bmatrix}$
4	$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 4 \\ 2 & 1 & 5 \end{bmatrix}$
5	$\begin{bmatrix} -2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$
Test for consistency and solve the following system of equations.	
6	$2x+6y+11=0, 6y-18z+1=0 \;\&\; 6x+20y-6z+3=0$,
7	$5x+3y+7z=4,\ 3x+26y+2z=9\ \&$ $7x+2y+10z=5$
8	$x-4y+7z=14,\ 3x+8y-2z=13\ \&\ 7x-8y+26z=5$
9	$5x+3y+7z=5,\ \ 3x+26y+2z=9\ \ \&\ \ 7x+2y+10z=5$
10	$4x-5y+z=-3, 2x+3y-z=3, 3x-y+2z=5\ \&\ x+2y-5z=-9$
11	Find the values of k for which the system of equations $(3k-8)x+3y+3z=0,\ 3x+(3k-8)y+3z=0,\ 3x+3y+(3k-8)z=0$ has a non-trivial solution.
12	Investigate the values of $'a'$ and $'b'$ so that the equations $x+y+z=6,\ x+2y+3z=10,\ x+2y+az=b$, may have i] no solution, ii] a unique solution and iii] an infinite no. of solutions.
Apply Gauss elimination method & Gauss Siedel to solve the equations	
13	$x+y+z=9,\ x-2y+3z=8,\ 2x+y-z=3$

14	$5x+y+z+t=4,\ x+7y+z+t=12,\ x+y+6z+t=-5\ \&\ x+y+z+4t=-6$
15	$2x+3y-z=5,\ 4x+4y-3z=3,\ 2x-3y+2z=2$
16	$3x+4y+5z=18,\ 2x-y+8z=13,\ 5x-2y+7z=20$

Find the Eigen values and Eigen vectors of $\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$
Find the Eigen values and Eigen vectors of $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$
Find the Eigen values and Eigen vectors of $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$
Find the Eigen values and Eigen vectors of $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$
Find all the Eigen values and the corresponding Eigen vectors of the matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$
Convert the following second order differential equation as a system of first order linear differential equation and solve by elimination technique. $y''+5y'-10y=\sin t,\ y(0)=2\ ,\ y'(0)=4$
Convert the following second order differential equation as a system of first order linear differential equation and solve by elimination technique. $y''-4y'+13y=8\ \sin 3x,\ y(0)=1\ ,\ y'(0)=2$
Convert the differential equation $x''+5x'+6x=0,\ x(0)=0\ ,\ x'(0)=15$ into system of linear differential equation and hence solve them by diagonalization. Also discuss the stability of the system
Convert the differential equation $2y''+3y'-2y=14x^2-4x-11,\ \ y(0)=0\ ,\ y'(0)=0$, into system of linear differential equation and hence solve them by elimination technique.
Solve by elimination technique $\dfrac{dy}{dx}+y=z+e^x\ ,\ \dfrac{dz}{dx}+z=y+e^x$
Solve the simultaneous equations: $\dfrac{dx}{dt}+5x-2y=t\ ,\ \dfrac{dy}{dt}+2x+y=0$ being given $x=y=0$ when $t=0$
Solve the following system of differential equations by elimination method. $\dfrac{dx}{dt}+4y=e^t\ ,\ \dfrac{dy}{dt}-9x=t,$ $x(0)=2,\ y(0)=1$
Solve by elimination technique $\dfrac{dx}{dt}+2y=-\sin t\ ,\ \dfrac{dy}{dt}-2x=\cos t$
Solve the following system of differential equations by elimination method. $\dfrac{dx}{dt}=5x+y\ ,\ \dfrac{dy}{dt}=y-4x\ ,\ x(0)=0,\ y(0)=1$
Reduce the matrix $A=\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$ to the diagonal form
Reduce the matrix $A=\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ to the diagonal form

Diagonalize the matrix $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
Solve $X' = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} X$ by diagonalization and discuss the stability of the system.
Solve $X' = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} X + \begin{bmatrix} 3e^t \\ e^t \end{bmatrix}$ by diagonalization and hence discuss the stability of the system.
Solve $X' = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} X$ by diagonalization and discuss the stability of the system.
Solve $X' = \begin{bmatrix} -2 & -1 & 8 \\ 0 & -3 & 8 \\ 0 & -4 & 9 \end{bmatrix} X$ by diagonalization. and hence discuss the stability of the system.
Solve $X' = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} X + \begin{bmatrix} 3t \\ e^{-t} \end{bmatrix}$ by diagonalization & hence discuss the stability of the system
Solve $X' = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} X + \begin{bmatrix} 2t \\ 8 \end{bmatrix}$ by diagonalization and hence discuss the stability of the system.
Solve the following initial value problem using matrix method: $X' = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix} X$; $X(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $X' = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}$, $X = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$
Find a matrix P which diagonalizes the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$. Verify that $P^{-1}AP = D$, where D is the diagonal matrix.