Unit 1: Matrix Algebra

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Cramers Rule

$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$.

$AX = B$.

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \qquad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$x = \frac{D_x}{D} . \qquad y = \frac{D_y}{D} .$$

① $12x + 3y = 15$

$2x - 3y = 13$.

$$D = \begin{vmatrix} 12 & 3 \\ 2 & -3 \end{vmatrix} = -36 - 6 = -42$$

$$D_x = \begin{vmatrix} 15 & 3 \\ 13 & -3 \end{vmatrix} = 15(-3) - (13)3 = -84.$$

$$D_y = \begin{vmatrix} 12 & 15 \\ 2 & 13 \end{vmatrix} = 156 - 30 = 126$$

$$x = \frac{-84}{-42} = 2 \qquad y = \frac{126}{-42} = -3.$$

(2) $x + y - z = 6$

$3x - 2y + z = -5$

$x + 3y - 2z = 14$

$$D = \begin{vmatrix} 1 & 1 & -1 \\ 3 & -2 & 1 \\ 1 & 3 & -2 \end{vmatrix}$$

$= 1(4-3) - 1(-6-1) - 1(9+2)$

$= 1 + 7 - 11 = -3$

$$D_x = \begin{vmatrix} 1 & -1 & 6 \\ -2 & 1 & -5 \\ 3 & -2 & 14 \end{vmatrix}$$

$= 1(14-10) + 1(-28+15) + 6(4-3)$

$= +4 - 13 + 6 = -3$

$$D_y = \begin{vmatrix} 1 & -1 & 6 \\ 3 & 1 & -5 \\ 1 & -2 & 14 \end{vmatrix} = 1(14-10) + 1(42+5) + 6(-6-1)$$

$= 4 + 47 - 42$

$= 9$

$$D_z = \begin{vmatrix} 1 & 1 & 6 \\ 3 & -2 & -5 \\ 1 & 3 & 14 \end{vmatrix}$$

$= 1(-28+15) - 1(42+5) + 6(9+2)$

$= -13 - 47 + 66 = 6$

$x = \frac{-3}{-3} = 1$ $\quad y = \frac{9}{-3} = -3$ $\quad z = \frac{6}{-3} =$

$x = 1$ $\quad y = -3$ $\quad z = -2$

(3) $x - 3y + 7z = 13$

$x + y + z = 1$

$x - 2y + 3z = 4$

$$D = \begin{vmatrix} 1 & -3 & 7 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 1(3+2) + 3(3-1) + 7(-2-1)$$

$= 5 + 6 - 21 = -10$

$$D_x = \begin{vmatrix} -3 & 7 & 13 \\ 1 & 1 & 1 \\ -2 & 3 & 4 \end{vmatrix} = -3(4-3) - 7(4+2) + 13(3+2)$$

$= -3 - 42 + 65$

$= 20$

$$D_y = \begin{vmatrix} 1 & 7 & 13 \\ 1 & 1 & 1 \\ 1 & 3 & 4 \end{vmatrix} = 1(4-3) - 7(4-1) + 13(3-1)$$

$$= 1 - 21 + 26$$

$$= 6.$$

$$D_z = \begin{vmatrix} 1 & -3 & 13 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{vmatrix} = 1(4+2) + 3(4-1) + 13(-2-1)$$

$$= 6 + 9 - 39$$

$$= -24$$

$$x = \frac{20}{-10} = -2$$

$$y = \frac{6}{-10} = -0.6$$

$$z = \frac{-24}{-10} = 2.4$$

4|2|20

4) $4x + y + z + w = 6$

$3x + 7y - z + w = 1$

$7x + 3y - 5z + 8w = -3$

$x + y + z + 2w = 3$

$$D = \begin{vmatrix} 4 & 1 & 1 & 1 \\ 3 & 7 & -1 & 1 \\ 7 & 3 & -5 & 8 \\ 1 & 1 & 1 & 2 \end{vmatrix}$$

$$= 4\left[7(-10-8) + 1(6-8) + 1(3+5)\right]$$

$$-1\left[3(-10-8) + 1(14-8) + 1(7+5)\right]$$

$$+1\left[3(6-8) - 7(14-8) + 1(7-3)\right]$$

$$-1\left[3(3+5) - 7(7+5) - 1(7-3)\right]$$

$$= -480 + 36 - 44 + 64$$

$$= -424$$

$$D_x = \begin{vmatrix} 1 & 1 & 1 & 6 \\ 7 & -1 & 1 & 1 \\ 3 & -5 & 8 & -3 \\ 1 & 1 & 2 & 3 \end{vmatrix} \Rightarrow \underline{D_x = -424.}$$

$$= 1\begin{vmatrix} -1 & 1 & 1 \\ -5 & 8 & -3 \\ 1 & 2 & 3 \end{vmatrix} - 1\begin{vmatrix} 7 & 1 & 1 \\ 3 & 8 & -3 \\ 1 & 2 & 3 \end{vmatrix} + 1\begin{vmatrix} 7 & -1 & 1 \\ 3 & -5 & -3 \\ 1 & 1 & 3 \end{vmatrix} - 6\begin{vmatrix} 7 & -1 & 1 \\ 3 & -5 & 8 \\ 1 & 1 & 2 \end{vmatrix}$$

= −36 − 19([illegible] 720 = 424

$$D_y = \begin{vmatrix} 4 & 6 & 1 & 1 \\ 3 & 1 & -1 & 1 \\ 7 & -3 & -5 & 8 \\ 1 & 3 & 1 & 2 \end{vmatrix}$$

$$= 4\begin{vmatrix} 1 & -1 & 1 \\ -3 & -5 & 8 \\ 3 & 1 & 2 \end{vmatrix} - 6\begin{vmatrix} 3 & -1 & 1 \\ 7 & -5 & 8 \\ 1 & 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 & 1 \\ 7 & -3 & 8 \\ 1 & 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 & -1 \\ 7 & -3 & -5 \\ 1 & 3 & 1 \end{vmatrix}$$

$$= 4(-36) - 6(-36) + 1(-72) + 1(0)$$

$$= -144 + 216 - 72 = 0$$

$= Ø$

$$D_z = \begin{vmatrix} 4 & 1 & 6 & 1 \\ 3 & 7 & 1 & 1 \\ 7 & 3 & -3 & 8 \\ 1 & 1 & 3 & 2 \end{vmatrix}$$

$$= 4\begin{vmatrix} 7 & 1 & 1 \\ 3 & -3 & 8 \\ 1 & 3 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 & 1 \\ 7 & -3 & 8 \\ 1 & 3 & 2 \end{vmatrix} + 6\begin{vmatrix} 3 & 7 & 1 \\ 7 & 3 & 8 \\ 1 & 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 7 & 1 \\ 7 & 3 & -3 \\ 1 & 1 & 3 \end{vmatrix}$$

$= -848$

$D_w = 0$

$$x = \frac{-424}{-424} = 1$$

$$y = \frac{0}{-424} = 0$$

$$z = \frac{-848}{-424} = 2$$

$$w = \frac{0}{-424} = 0$$

(2)

$4x + 0y + 3z - 2w = 2$

$3x + 1y + 2z - 1w = 4$

$1x - 6y - 2z + 2w = 0$

$2x + 2y + 0z - 1w = 1$

$x = \frac{52}{39}$ $\quad y = 1$

$\cancel{z} = \frac{52}{39}$ $\quad w = \frac{143}{39}$

$$D = \begin{vmatrix} 4 & 0 & 3 & -2 \\ 3 & 1 & 2 & -1 \\ 1 & -6 & -2 & 2 \\ 2 & 2 & 0 & -1 \end{vmatrix}$$

$$= 4(-6) - 0 + 3(-3) + 2(36)$$

$$= -24 - 9 + 72 = 39$$

$$D_x = \begin{vmatrix} 2 & 0 & 3 & -2 \\ 4 & 1 & 2 & -1 \\ 0 & -6 & -2 & 2 \\ 1 & 2 & 0 & -1 \end{vmatrix}$$

$= 2(-6) - 0 + 3(4) + 2(26)$

$= -12 + 12 + 52 = 52$

$$D_y = \begin{vmatrix} 4 & 2 & 3 & -2 \\ 3 & 4 & 2 & -1 \\ 1 & 0 & -2 & 2 \\ 2 & 1 & 0 & -1 \end{vmatrix}$$

$= 4(10) - 2(12) + 3(13) + 2(-8)$

$= 40 - 24 + 39 - 16$

$= 39$

$$D_z = \begin{vmatrix} 4 & 0 & 2 & -2 \\ 3 & 1 & 4 & -1 \\ 1 & -6 & 0 & 2 \\ 2 & 2 & 1 & -1 \end{vmatrix}$$

$= 4(-4) - 0 + 2(-3) + 2(37)$

$= -16 - 6 + 74 = 52$

$$D_w = \begin{vmatrix} 4 & 0 & 3 & 2 \\ 3 & 1 & 2 & 4 \\ 1 & -6 & -2 & 6 \\ 2 & 2 & 0 & 1 \end{vmatrix}$$

$= 4(26) - 0 + 3(37) - 2(36)$

$= 104 + 111 - 72 = 143$

$x = \frac{52}{39}$ $\quad y = \frac{39}{39}$ $\quad z = \frac{52}{39}$ $\quad w = \frac{143}{39}$

$y = 1$

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Linear Transformation

$Y = AX \rightarrow A^{-1}Y = X$

$y_1 = a_1 x_1 + b_1 x_2$

$y_2 = a_2 x_1 + b_2 x_2$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

→ Consider a system of eq's $y_1 = a_1 x_1 + b_1 x_2$
$y_2 = a_2 x_1 + b_2 x_2$. $\quad Y = AX$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

where A is non-singular matrix or regular matrix

i.e $|A| \neq 0$.

→ S.T the transformation is regular and write down the inverse transformation

$y_1 = 2x_1 + x_2 + x_3$

$y_2 = x_1 + x_2 + 2x_3$

$y_3 = x_1 - 2x_3$

$$\begin{matrix} -2 & +4 & 1 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{matrix}$$

$Y = AX$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{vmatrix} = -1 \neq 0$$

Regular.

$$A^{-1} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

$X = A^{-1} Y$

$$X = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$x_1 = 2y_1 - 2y_2 - y_3$

$x_2 = -4y_1 + 5y_2 + 3y_3$

$x_3 = y_1 - y_2 - y_3$.

→ $u = x\cos\alpha - y\sin\alpha$

$v = x\sin\alpha + y\cos\alpha$

Write the matrix A of transformation & P.T. $A^{-1} = A'$ & hence write the inverse transformation

Sol:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$|A| = \begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix}$$

$= \cos^2\alpha + \sin^2\alpha = 1$

$$A^{-1} = \frac{\begin{vmatrix} \cos\alpha & +\sin\alpha \\ -\sin\alpha & \cos\alpha \end{vmatrix}}{1}$$

$$A^{-1} = \begin{vmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{vmatrix}$$

$$|A^{-1}| = \cos^2\alpha + \sin^2\alpha$$

$= 1$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$x = \cos\alpha\, u + \sin\alpha\, v$

$y = -\sin\alpha\, u + \cos\alpha\, v$

→ A transformation from the variables x_1, x_2, x_3 to y_1, y_2, y_3 is given by $Y = AX$ & another transformation from y_1, y_2, y_3 to z_1, z_2, z_3 is given by $Z = BY$ obtain the transformation from x_1, x_2, x_3 to z_1, z_2, z_3.

z to x

Sol:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix} \qquad Y = A.X$$

$$Z = BY$$

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix} \qquad Z = (BA)X$$

$$AB = \begin{pmatrix} 3 & 4 & 3 \\ -1 & -4 & -9 \\ 2 & 6 & 6 \end{pmatrix} \qquad B.A = \begin{pmatrix} 1 & 4 & -1 \\ 1 & 5 & -3 \\ -3 & 14 & -1 \end{pmatrix}$$

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -1 \\ +1 & +5 & -3 \\ -3 & 14 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$z_1 = 1x_1 + 4x_2 - 1x_3$

$z_2 = +x_1 + 5x_2 - 3x_3$

$z_3 = -3x_1 + 14x_2 - 1x_3.$

→ Given.

$y_1 = 5x_1 + 3x_2 + 3x_3$

$y_2 = 3x_1 + 2x_2 - 2x_3$ &

$y_3 = 2x_1 - x_2 + 2x_3$

$z_1 = 4x_1 + 2x_3$

$z_2 = x_2 + 4x_3$

$z_3 = 5x_3.$

L.T z from y

$Y = AX \qquad Z = BX$

$X = A^{-1}Y \qquad Z = (BA^{-1})Y$

$$A = \begin{bmatrix} 5 & 3 & 3 \\ 3 & 2 & -2 \\ 2 & -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2/41 & 9/41 & 12/41 \\ 10/41 & -4/41 & -19/41 \\ 7/41 & -11/41 & -1/41 \end{bmatrix}$$

$$A^{-1} = \frac{1}{41}\begin{bmatrix} -2 & 9 & 12 \\ 10 & -4 & -19 \\ 7 & -11 & -1 \end{bmatrix}$$

$$BA^{-1} = \begin{bmatrix} 0.1463 & 0.3414 & 1.1219 \\ 0.9268 & -1.17 & -0.56 \\ 0.8536 & -1.341 & -0.121 \end{bmatrix}$$

$$Z = (BA')Y$$

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0.41 & 0.34 & 1.12 \\ 0.92 & -1.17 & -0.56 \\ 0.85 & -1.34 & -0.12 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$z_1 = 0.41y_1 + 0.34y_2 + 1.12y_3$$

$$z_2 = 0.92y_1 - 1.17y_2 - 0.56y_3$$

$$z_3 = 0.85y_1 - 1.34y_2 - 0.12y_3.$$

Properties of determinant

1) If two rows (or) two cols are interchanged in a determinant then the value of determinant is

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= -\Delta$$

2) If two rows (or) two cols are identical then determinant will be zero.

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = 0$$

3) If we multiply the elem of one row or one col is multiplied with the same no. then ~~then~~ the

$$\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ \lambda b_1 & \lambda b_2 & \lambda b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \lambda\Delta.$$

4)
$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ \lambda a_1 & \lambda a_2 & \lambda a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \lambda(0)$$

5) If any ele of a row (or) col is the sum of two numbers then the determinant could be considered as sum of other two determinants.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1+d_1 & b_2+d_2 & b_3+d_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ d_1 & d_2 & d_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

6) If we add to the ele of a row (or) col the corresponding element of another row (or) col, multiplied by a same number then the determinant doesn't change.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1+\lambda a_1 & b_2+\lambda a_2 & b_3+\lambda a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Characterisation of Invertible matrix:

If the argumented matrix $[A|I]$ is transformed into a matrix of the form $[I|B]$ then the matrix A is invertible & the inverse matrix of A^{-1} is given by B.

→ $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 3 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 = R_2 - 2R_1$

$$= \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -3 & 4 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$R_1 = R_1 + R_2$

$R_3 = 3R_3 + R_2$

$C_1 \rightarrow \cancel{C_1 + C_3}$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & -3 & 4 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 3 \end{array}\right]$$

$R_1 = R_1 - 2R_3$

$R_2 = R_2 - 4R_3$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -1 & -6 \\ 0 & -3 & 0 & 6 & -3 & -12 \\ 0 & 0 & 1 & -2 & 1 & 3 \end{array}\right]$$

$R_2 = \frac{R_2}{-3}$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -1 & -6 \\ 0 & 1 & 0 & -2 & 1 & 4 \\ 0 & 0 & 1 & -2 & 1 & 3 \end{array}\right]$$

$A^{-1} = B$

$= [I|B]$

→ $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & -3 & 2 \\ 3 & 6 & -2 \end{bmatrix}$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & -3 & 2 & 0 & 1 & 0 \\ 3 & 6 & -2 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \rightarrow R_3 - 3R_1$ | $R_2 \rightarrow R_2 + R_1$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -3 & 4 & 1 & 1 & 0 \\ 0 & 6 & -8 & -3 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -3 & 4 & 1 & 1 & 0 \\ 0 & 6 & -8 & -3 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 + 2R_2$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -3 & 4 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 1 \end{array}\right]$$

∴ It is not invertable matrix

Rank of a matrix

- Echelon form (Row operation)
- Normal form (Row & column operation)

$$\begin{bmatrix} 1 & 4 & 5 \\ 6 & 2 & 7 \\ 3 & 4 & 3 \end{bmatrix}$$

→ Use the elimentary transformation to reduce the matrix into Echelon form:

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$

$R_3 = R_3 - R_1$

$$= \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$$

$R_3 = R_3 - 2R_2$

$$= \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 = \frac{R_2}{-7}$

$$= \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -8/7 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$.

→ $$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$

$R_3 = R_3 - R_1$

$$= \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

$R_3 = R_3 + R_2$

$$= \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 = \frac{R_2}{-1}$

$$= \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$.

$$A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$= \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$$

$R_3 = R_3 - 2R_1$

$$= \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{bmatrix}$$

$R_3 = R_3 - R_2$

$$= \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 = \frac{R_1}{2}$; $R_2 = \frac{R_2}{2}$

$$= \begin{bmatrix} 1 & 3/2 & 5/2 & 2 \\ 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$.

Normal form

$$A = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \\ 2 & -2 & 3 & 7 \end{bmatrix}$$

$R_2 = R_2 - R_1$
$R_3 = R_3 - 3R_1$
$R_4 = R_4 - 2R_1$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \\ 0 & -4 & 1 & -5 \end{bmatrix}$$

$C_2 = C_2 - C_1$
$C_3 = C_3 - C_1$
$C_4 = C_4 - 6C_1$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \\ 0 & -4 & 1 & -5 \end{bmatrix}$$

$R_3 = R_3 - R_2$
$R_4 = R_4 - 2R_2$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

$C_3 = 2C_3 + C_2$
$C_4 = 2C_4 - C_2$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -6 & -8 \\ 0 & 0 & -2 & -6 \end{bmatrix}$$

$R_4 = 3R_4 - R_3$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -6 & -18 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 = C_4 - 3C_3$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 = \frac{R_2}{-2}$ $\quad R_3 = \frac{R_3}{-6}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \rho(A) = 3$$

$$\rightarrow \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$R_2 = 2R_2 + R_1$

$R_3 = 2R_3 + R_1$

$$= \begin{bmatrix} -2 & -1 & -3 & -1 \\ 0 & 3 & 3 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$C_2 = 2C_2 - C_1$

$C_3 = 2C_3 - 3C_1$

$C_4 = 2C_4 - C_1$

$$= \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 6 & 6 & 6 \\ 0 & -2 & -2 & 2 \\ 0 & 2 & 2 & 2 \end{bmatrix}$$

$R_3 = 3R_3 + R_2$

$R_4 = 3R_4 - R_2$

$$= \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 6 & 6 & 2 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$C_3 = C_3 - C_1$

$C_4 = 3C_4 - C_1$

$$= \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

$R_3 = R_3 - 2R_4$

$$= \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

$R_1 = R_1/-2$

$R_2 = R_2/6$

$R_4 = R_4/12$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \rho(A) = 3$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 6 & 8 \\ 4 & 8 & 12 & 16 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$

$R_3 = R_3 - 4R_1$

$R_4 = R_4 - R_1$

$$= \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$C_2 = C_2 - 2C_1$

$C_3 = C_3 - 4C_1$

$C_4 = C_4 - 3C_1$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

~~$R_3 = R_3 - 2R_2$~~

~~$R_4 = 2R_4 - R_2$~~

$R_3 = R_3 - 2R_2$

$R_4 = 2R_4 - R_2$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_2 \leftrightarrow C_3$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_4 = C_4 + C_2$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 = \frac{R_2}{-2}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \rho(A) = 2$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -2 & 2 & 5 \\ 3 & 1 & 1 & 8 \\ 2 & -2 & 3 & 7 \end{bmatrix}$$

$R_2 = R_2 - R_1$

$R_3 = R_3 - 3R_1$

$R_4 = R_4 - 2R_1$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \\ 0 & -4 & -3 & -5 \end{bmatrix}$$

$C_2 = C_2 - C_1$

$C_3 = C_3 - C_1$

$C_4 = C_4 - 6C_1$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \\ 0 & -4 & -3 & -5 \end{bmatrix}$$

$R_3' = R_3 + R_2$

$R_4 = R_4 + 2R_2$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -1 & -11 \\ 0 & 0 & -1 & -7 \end{bmatrix}$$

$C_3 = 2C_3 + C_2$

$C_4 = 2C_4 - C_2$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & -22 \\ 0 & 0 & -2 & -14 \end{bmatrix}$$

$$R_4 = R_4 - R_3 \quad = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & -22 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$$C_4 = C_4' - 11C_3 \quad = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$R_2 = R_2/_{-2}$, $R_3 = R_3/_{-2}$, $R_4 = R_4/8$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\rho(A) = 4$

Leontief Input Output Method.

→ An economy consists of 2 co-independence industries steel & Lumber. it takes 0.1 units of steel & 0.5 units of Lumber to make each unit of steel. It takes 0.2 units of steel & 0.0 units of Lumber to make each unit of Lumber. The economy will export 16 units of steel & 8 unit of Lumber next month. how many units of steel & Lumber will they need to make this external demand.

Sol:

Steel - X

Lumber - Y

Total Demand = Internal + External.

$$X = 0.1X + 0.2Y + 16 \quad (1)$$

$$Y = 0.5X + 0.0Y + 8 \quad (2)$$

$$P = \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$\underbrace{P}_{\text{production}} = \underbrace{A}_{\text{Technology matrix}} P + \underbrace{d}_{\text{demand external}}$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 \\ 0.5 & 0.0 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} 16 \\ 8 \end{bmatrix}$$

$P = AP + d$

$P - AP = d$,

$(I-A)P = d$.

$I - A$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.2 \\ 0.5 & 0 \end{bmatrix} = \begin{bmatrix} 0.9 & -0.2 \\ -0.5 & 1 \end{bmatrix}$$

Set up an augmented matrix to solve for P.

$$\Rightarrow \left[\begin{array}{cc|c} 0.9 & -0.2 & 16 \\ -0.5 & 1 & 8 \end{array}\right] \times 10$$

$$= \left[\begin{array}{cc|c} 9 & -2 & 160 \\ -5 & 10 & 80 \end{array}\right]$$

$R_2 = 9R_2 + 5R_1$

$$= \left[\begin{array}{cc|c} 9 & -2 & 160 \\ 0 & 80 & 1520 \end{array}\right]$$

$R_1 = 40R_1 + R_2$

$$= \left[\begin{array}{cc|c} 360 & 0 & \cancel{160}\ 7920 \\ 0 & 80 & 1520 \end{array}\right]$$

$R_1 = R_1/360$

$R_2 = R_2/80$

$$= \left[\begin{array}{cc|c} 1 & 0 & 22 \\ 0 & 1 & 19 \end{array}\right]$$

$\Rightarrow X = 22$

$Y = 19$

2^{nd} method

$$(I-A)P = d$$

$$P = (I-A)^{-1} d$$

$$= \begin{bmatrix} 0.9 & -0.2 \\ -0.5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 16 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1.25 & 0.25 \\ 0.625 & 1.125 \end{bmatrix} \begin{bmatrix} 16 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1.25 \times 16 + 0.25 \times 8 \\ 0.625 \times 16 + 1.125 \times 8 \end{bmatrix}$$

$$= \begin{bmatrix} 22 \\ 19 \end{bmatrix}.$$

→ We have 2 industries electric company [E] & water company [W]. output for both companies is measured in dollars. suppose that production of each dollar worth of electricity requires \$0.3 worth of electricity & \$0.1 worth of water & the production of each dollars worth of water requires \$0.2 of electricity & \$0.4 of water the final demand of outside sector of economy \$12M of electricity & \$8M for water.

how much E & W should be produce to meet this demand

$$E = 0.3E + 0.2W + 12$$

$$W = 0.1E + 0.4W + 8$$

$$P = \begin{bmatrix} E \\ W \end{bmatrix} \qquad d = \begin{bmatrix} 12 \\ 8 \end{bmatrix}$$

$$P = AP + d$$

$$A = \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.4 \end{bmatrix}$$

$$(P - AP) = d$$

$$(I - A)P = d$$

$$P = (I-A)^{-1} d.$$

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.7 & -0.2 \\ -0.1 & 0.6 \end{bmatrix}$$

$$(I-A)^{-1} = \begin{bmatrix} 1.5 & 0.5 \\ 0.25 & 1.75 \end{bmatrix}$$

$$P = \begin{bmatrix} 1.5 & 0.5 \\ 0.25 & 1.75 \end{bmatrix} \begin{bmatrix} 12 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5 \times 12 + 0.5 \times 8 \\ 0.25 \times 12 + 1.75 \times 8 \end{bmatrix} = \begin{bmatrix} 22 \\ 17 \end{bmatrix}.$$

$$E = 22, \quad W = 17.$$

→

	X	Y	Z	Demand
X	0.2	0.4	0	20
Y	0	0.2	0.4	10
Z	0.1	0.1	0.3	30

$$X = 0.2X + 0.4Y + 20$$

$$Y = 0.2Y + 0.4Z + 10$$

$$Z = 0.1X + 0.1Y + 0.3Z + 30.$$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0.2 & 0.4 & 0 \\ 0 & 0.2 & 0.4 \\ 0.1 & 0.1 & 0.3 \end{bmatrix} + \begin{bmatrix} 20 \\ 10 \\ 30 \end{bmatrix}$$

$$P = (I-A)^{-1} d$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$(I.A)$

$$I - A = \begin{bmatrix} 0.8 & -0.4 & 0 \\ 0 & 0.8 & -0.4 \\ -0.1 & -0.1 & 0.7 \end{bmatrix}$$

$$(I-A)^{-1} = \begin{bmatrix} 1.3 & 0.7 & 0.4 \\ 0.1 & 1.4 & 0.8 \\ 0.2 & 0.3 & 1.6 \end{bmatrix}.$$

$$P = (I-A)^{-1} d = \begin{bmatrix} 1.3 & 0.7 & 0.4 \\ 0.1 & 1.4 & 0.8 \\ 0.2 & 0.3 & 1.6 \end{bmatrix} \begin{bmatrix} 20 \\ 10 \\ 30 \end{bmatrix}$$

$$= \begin{bmatrix} 1.3 \times 20 + 0.7 \times 10 + 0.4 \times 30 \\ 0.1 \times 20 + 1.4 \times 10 + 0.8 \times 30 \\ 0.2 \times 20 + 0.3 \times 10 + 1.6 \times 30 \end{bmatrix}.$$

$$= \begin{bmatrix} 45 \\ 40 \\ 55 \end{bmatrix}.$$

→

	X	Y	Z	Demand
X	0.2	0	0.1	20
Y	0.4	0.2	0.1	10
Z	0	0.4	0.3	30

$$X = 0.2X + 0.1Z + 20$$

$$Y = 0.4X + 0.2Y + 0.1Z + 10$$

$$Z = 0.4Y + 0.3Z + 30.$$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0.2 & 0 & 0.1 \\ 0.4 & 0.2 & 0.1 \\ 0 & 0.4 & 0.3 \end{bmatrix} + \begin{bmatrix} 20 \\ 10 \\ 30 \end{bmatrix}.$$

$$I - A = \begin{bmatrix} 0.8 & 0 & -0.1 \\ -0.4 & 0.8 & -0.1 \\ 0 & -0.4 & 0.7 \end{bmatrix}$$

$$(I-A)^{-1} d = \begin{bmatrix} 1.3 & 0.1 & 0.2 \\ 0.7 & 1.4 & 0.3 \\ 0.4 & 0.8 & 1.6 \end{bmatrix} \begin{bmatrix} 20 \\ 10 \\ 30 \end{bmatrix}$$

33

$$P = (I-A)^{-1} d$$

$$= \begin{bmatrix} 33 \\ 37 \\ 64 \end{bmatrix}.$$

· MATRIX FACTORIZATION (LU Decomposition)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}.$$

$A = LU$.

Given $AX = B$

$(LU)X = B$.

$L(UX) = B$

$\boxed{LY = B}$ And Y

$UX = Y$.

--> Use LU to solve the following system of equations.

$2x_1 + x_2 + 4x_3 = 12$

$4x_1 + 11x_2 - x_3 = 33$

$8x_1 - 3x_2 + 7x_3 = 20$

$$\begin{bmatrix} 2 & 1 & 4 \\ 4 & 11 & -1 \\ 8 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 33 \\ 20 \end{bmatrix}$$

· $AX = B$ (1)

$A = LU$

$$\begin{bmatrix} 2 & 1 & 4 \\ 4 & 11 & -1 \\ 8 & -3 & 2 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{22} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}$$

$l_{11} = 2$

$l_{21} = 4$

$l_{31} = 8$

$l_{11}u_{12} = 1$

$u_{12} = \frac{1}{l_{11}} = \frac{1}{2}$

$u_{12} = 1/2$

$l_{21}u_{12} + l_{22} = 11$

$4 \times \frac{1}{2} + l_{22} = 11$

$l_{22} = 9$

$l_{31}u_{12} + l_{32} = -3$

$8 \times \frac{1}{2} + l_{32} = -3$

$l_{32} = -7$

$u_{13} = 2$

$u_{23} = -1$

$l_{33} = -21$.

$$\rightarrow L = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 9 & 0 \\ 8 & -7 & -21 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1/2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

① →

$AX = B$

$(LU)X = B$

$L(UX) = B$

$LY = B$, where $Y = UX$.

consider. $LY = B$.

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & 9 & 0 \\ 8 & -7 & -21 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 33 \\ 20 \end{bmatrix}$$

$2y_1 = 12 \Rightarrow y_1 = 6$

$4y_1 + 9y_2 = 13 \Rightarrow 24 + 9y_2 = 33$

$9y_2 = 9 \Rightarrow y_2 = 1$

$8y_1 - 7y_2 - 21y_3 = 20$

$8(6) - 7(1) - 21y_3 = 20$

$21y_3 = 48 - 7 - 20$

$21y_3 = 21$

$y_3 = 1$

consider $Y = UX$

$$\begin{bmatrix} 6 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$6 = x_1 + \frac{x_2}{2} + 2x_3$

$1 = x_2 - x_3$

$\boxed{1 = x_3}$

$x_2 = 1 + 1$

$\boxed{x_2 = 2}$

$6 = x_1 + 1 + 2$

$\boxed{x_1 = 3}$

→ $4x_1 + x_2 + x_3 = 4$

$x_1 + 4x_2 - 2x_3 = 4$

$3x_1 + 2x_2 - 4x_3 = 6$

$$\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & -2 \\ 3 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$$

$AX = B$ — (1)

$A = LU$

$$\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & -2 \\ 3 & 2 & -4 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

$l_{11} = 4$

$l_{21} = 1$

$l_{31} = 3$

$l_{11}u_{12} = 1$

$4 \times u_{12} = 1$

$u_{12} = 1/4$

$l_{21}u_{12} + l_{22} = 4$

$1 \times 1/4 + l_{22} = 4$

$l_{22} = 4 - 1/4 = \frac{15}{4}$.

$l_{11}u_{13} = 1$

$\cancel{4x}\ 4 \times u_{13} = 1$

$u_{13} = 1/4$.

$l_{31}u_{12} + l_{32} = 2$

$3 \times \frac{1}{4} + l_{32} = 2$.

$l_{32} = 2 - \frac{3}{4} = 5/4$.

$l_{21}u_{13} + l_{22}u_{23} = -2$.

$1 \times \frac{1}{4} + \frac{15}{4}u_{23} = -2$.

$\frac{15u_{23}}{4} = -2 - \frac{1}{4} = \frac{-9}{4}$

$u_{23} = -\frac{9}{4} \times \frac{4}{15} = -3/5$

$l_{31}u_{13} + l_{32}u_{23} + l_{33} = -4$

$3 \times \frac{1}{4} + \frac{5}{4} \times -\frac{3}{5} + l_{33} = -4$

$\frac{3}{4} - \frac{3}{4} + l_{33} = -4$

$\boxed{l_{33} = -4}$

$$L = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 15/4 & 0 \\ 3 & 5/4 & -4 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1/4 & 1/4 \\ 0 & 1 & -3/5 \\ 0 & 0 & 1 \end{bmatrix}$$

$AX = B$.

$(LU)X = B$

$L(UX) = B$

$LY = B$

$$\begin{bmatrix} 4 & 0 & 0 \\ 1 & 15/4 & 0 \\ 3 & 5/4 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$$

$4y_1 = 4$

$y_1 + \frac{15}{4}y_2 = 4$

$3y_1 + 5/4\, y_2 - 4y_3 = 6$

$y_1 = 1$

$\frac{15}{4}y_2 = 3$

$y_2 = \frac{\cancel{3} \times 4}{\cancel{15}\ 5} = 4/5$

$3(1) + \frac{5}{4}\left(\frac{4}{5}\right) - 4y_3 = 6$

$-4y_3 = 2$

$\cancel{-4y_3 = 3}$

$\cancel{y_3 = 3/4}$

$y_3 = -1/2$

$Y = UX$

$$\begin{bmatrix} 1 \\ 4/5 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 1/4 & 1/4 \\ 0 & 1 & -3/5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$x_1 + \frac{x_2}{4} + \frac{x_3}{4} = 1$

$x_2 - \frac{3}{5}x_3 = 4/5$

$x_3 = -1/2 \Rightarrow x_2 = 1/2$

$x_1 = 1$

$\rightarrow$ $x + 5y + z = 14$ 14

$2x + y + 3z = 13$ $5/3$

$3x + y + 4z = 17.$ 3

$Aa = B$

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}$$

1
2
3

$A = LU$

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

$l_{11} = 1$ $\quad l_{11}u_{12} = 5$ $\quad l_{21}u_{12} + l_{22} = 1$ $\quad l_{11}u_{13} = 1$

$l_{21} = 2$ $\quad 1 \times u_{12} = 5$ $\quad 2 \times 5 + l_{22} = 1$ $\quad 1 \times u_{13} = 1$

$l_{31} = 3$ $\quad u_{12} = 5$ $\quad l_{22} = -9$ $\quad u_{13} =$

$l_{31}u_{12} + l_{32} = 1$ $\quad \cancel{l_{21}u_{12} + l_{22} = 1}$ $\quad l_{21}u_{13} + l_{22}u_{23} = 3.$

$3 \times 5 + l_{32} = 1$ $\quad \cancel{2 \times 5 + l_{22} = 1}$ $\quad 2 \times 1 + (-9)u_{23} = 3$

$l_{32} = -14$ $\quad \cancel{l_{22}}$ $\quad -9u_{23} = 1$

$u_{23} = -1/9.$

$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 4$

$(3)(1) + (-14)(-\frac{1}{9}) + l_{33} = 4$

$\frac{14}{9} + l_{33} = 1$

$l_{33} = 1 - \frac{14}{9} = \frac{9-14}{9} = -5/9.$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -9 & 0 \\ 3 & -14 & -5/9 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 1 & -1/9 \\ 0 & 0 & 1 \end{bmatrix}$$

$Aa = B$

$LUa = B$

$L(Ua) = B$

$LY = B,$ where $Ua = Y.$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -9 & 0 \\ 3 & -14 & -5/9 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}.$$

$y_1 = 14.$

$2y_1 - 9y_2 = 13$

$28 - 9y_2 = 13.$

$9y_2 = 15$

$y_2 = 15/9 = 5/3$

$3y_1 - 14y_2 - \frac{5}{9}y_3 = 17$

$3(14) - 14(5/3) - \frac{5}{9}y_3 = 17$

$y_3 = 3$

$Ua = Y$

$$\begin{bmatrix} 1 & 5 & 1 \\ 0 & 1 & -1/9 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 5/3 \\ 3 \end{bmatrix}.$$

$x + 5y + z = 14.$

$y - z/9 = 5/3$

$y - \frac{1}{3} = 5/3$

$\cancel{y = 5/3 + 3}$

$\boxed{y = 6/3 = 2}$

$x + 5(2) + 3 = 14$

$x + 13 = 14$

$\boxed{x = 1}$