Module 5 : Complex Integration

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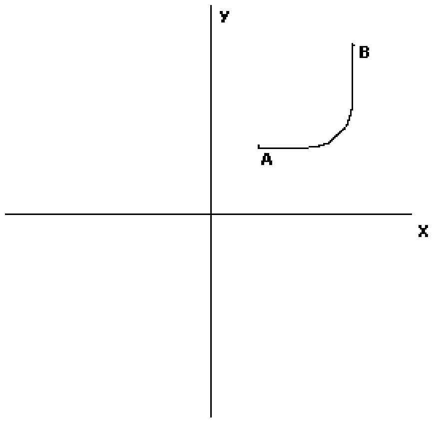
5.1 Introduction:

Complex variable techniques have been used in a wide variety of areas of engineering. This has been particularly true in areas such as electromagnetic field theory, fluid dynamics, aerodynamics and elasticity. With the rapid developments in computer technology and the consequential use of sophisticated algorithms for analysis and design in engineering there has been, in recent years, less emphasis on the use of complex variable techniques and a shift towards numerical techniques applied directly to the underlying full partial differential equations which model the situation. However it is useful to have an analytical solution, possibly for an idealized model in order to develop a better understanding of the solution and to develop confidence in numerical estimates for the solution of more sophisticated models. The design of aerofoil sections for aircraft is an area where the theory was developed using complex variable techniques. Throughout engineering, transforms defined as complex integrals in one form or another play a major role in analysis and design. The use of complex variable techniques allows us to develop criteria for the stability of systems.

5.2 Complex line integral:

Suppose $f(z)$ is a function defined at every point on a given continuous curve C on the argand diagram.



We define $\int_C f(z)\,dz$ as the limit of a sum as follows,

Introduce a very large number of points $z_1, z_2, z_3 \ldots\ldots z_{n-1}$ along the curve C. Suppose A and B represent complex numbers z_0, and z_n, respectively. Put $z_1 - z_0 = \delta z_1$, $z_2 - z_1 = \delta z_2 \ \ldots \ z_n - z_{n-1} = \delta z_n$

Consider the sum

$$f(z_1)\delta z_1 + f(z_2)\delta z_2 + f(z_3)\delta z_3 + f(z_4)\delta z_4 \ldots\ldots\ldots + f(z_n)\delta z_n = \sum_{i=1}^{n} f(z_i)\delta z_i.$$

Let $n \to \infty$ such that each $\delta z_i \to 0$ then we define $\int_C f(z)\,dz = \lim_{\delta z_i \to 0} \sum_{i=1}^{n} f(z_i)\delta z_i$

This integral is called as the line integral of v the complex function f(z).

5.2.1 Properties of line integrals:

1. $\int_A^B f(z)\,dz = -\int_B^A f(z)\,dz$.
2. $\int_A^B f(z)\,dz = \int_A^C f(z)\,dz + \int_C^B f(z)\,dz$.
3. $\int_C k\ f(z)\,dz = k\int_C f(z)\,dz$.
4. $\int_C [f(z) \pm g(z)]dz = \int_C f(z)\,dz \pm \int_C g(z)\,dz$.

Note: Evaluation of a line integral of a complex valued function:

$$\int_C f(z)\,dz ,= \int [u(x,y)+iv(x,y)](dx+idy) = \int_c (udx - vdy) + i\int_c (vdx+udy)\,.$$

Problems:

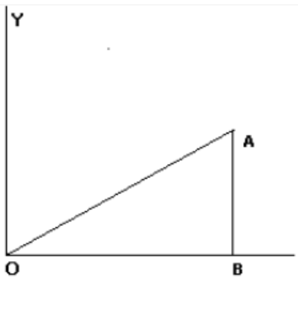
1. Evaluate $\int_c z^2\,dz$ where C is

(i) The line joining the points 0 and 3 + i ,
(ii) The path OBA consisting of two line segments OB and OA, where A and B are z=3+i and z=3 respectively.

Solution:

$z = x+iy \Rightarrow z^2 = x^2 - y^2 + 2ixy$ and $dz = dx + idy$

$$\therefore z^2\,dz = (x^2 - y^2 + 2ixy)(dx+idy) = \left[\left(x^2 - y^2\right)dx - 2xydy\right] + i\left[2xydx + \left(x^2 - y^2\right)dy\right]$$



(i) C is the line segment joining O(0,0) and A(3,1)
Equation of OA is $x - 3y = 0 \Rightarrow x = 3y \Rightarrow dx = 3dy$
Further, y varies from y =0 to y =1.

$$\int_C z^2\,dz = \int_0^1 \left[\left(9y^2 - y^2\right)3dy - 6y^2\right] + i\left[6y^2 .3dy + 8y^2\,dy\right]$$

$$= \int_0^1 18y^2\,dy + i\int_0^1 26y^2\,dy = 6 + \frac{26}{3}i.$$

(ii) C is the path OAB. Thus, we can write,

$$\int_C z^2\,dz = \int_{OBA} z^2\,dz = \int_{OB} z^2\,dz + \int_{BA} z^2\,dz$$

Along OB: $y = 0,\ dy = 0$ and x varies from $x = 0$ to $x = 3$.
Along BA: $x = 0,\ dx = 0$ and y varies from $y = 0$ to $y = 1$.

Thus we have, $\displaystyle\int_{OBA} z^2\,dz = \int_0^3 x^2\,dx + \int_0^1 -6ydy + i\left(9 - y^2\right)dy = 6 + \frac{26}{3}i\,.$

2. Evaluate $\int_0^{3+i} z^2\,dz$ **along the line** $3y = x.$

Solution:

$$\int_0^{3+i} z^2\,dz = \int_0^{3+i} (x+iy)^2 (dx + idy)$$

Along the line, $3y = x.$,we have $dx = 3dy$ and y varies from 0 to 1.

$$\int_0^1 (x+iy)^2 (dx + idy) = \int_0^1 (3y+iy)^2 (3dy + idy)$$

$$= (3+i)^3 \int_0^1 y^2\,dy = \frac{(3+i)^3}{3}.$$

3. Evaluate $\int_{(0,3)}^{(2,4)} (2y + x^2)\,dx + (3x - y)\,dy$ **along**

(i) The parabola $x = 2t,\ y = t^2 + 3$

(ii) The straight line from $(0,\ 3)$ **to** $(2,\ 4)$

Solution:

(i). x varies from 0 to 2 and hence

$$\begin{matrix} if\ \ x = 0 \Rightarrow t = 0, \\ if\ \ x = 2, \Rightarrow t = 1, \end{matrix} \Rightarrow t \text{ Varies from 0 to 1}$$

$$I = \int_{(0,3)}^{(2,4)} (2y + x^2)\,dx + (3x - y)\,dy$$

$$= \int_{t=0}^{1} \{2(3+t^2) + 4t^2\}2\,dt + \{3(2t) - (3+t^2)\}2t\,dt$$

$$I = \int_{t=0}^{1} (24t^2 - 2t^3 - 6t + 12)\,dt = 33/2.$$

(ii). Equation of the straight line joining (0,3) and (2,4) is given by $\frac{y-3}{x-0} = \frac{4-3}{2-0}$.

$$\Rightarrow x = 2y - 6.\ Hence\ \ dx = 2dy.$$

Hence the given integral I become,

$$I = \int_{y=3}^{4} \{2y + (2y-6)^2\} 2y\,dy + \{3(2y-6) - y\}\,dy$$

$$= \int_{y=3}^{4} \left(8y^2 - 39y + 54\right) dy = 97/6.$$

4. Evaluate $\int_c (\bar{z})^2\,dz$ around the circle (i) $|z| = 1$, (ii) $|z-1| = 1$.

Solution:

(i) $|z| = 1$, is the circle at (0,0) and radius equal to 1.

On the circle we have $z = re^{i\theta}, dz = ie^{i\theta}d\theta$ further θ varies from 0 to 2π.

$\bar{z} = e^{-i\theta}$ and $(\bar{z})^2 = e^{-2i\theta}$

Now, $\int_C (\bar{z})^2\,dz = \int_0^{2\pi} e^{-2i\theta} ie^{i\theta}\,d\theta = i\int_0^{2\pi} e^{-i\theta}\,d\theta = 0.$

(ii) $|z-1| = 1$ is the circle with centre at (1,0) and radius equal to 1.

On this circle $z - 1 = e^{i\theta}$ $0 \le \theta \le 2\pi$

$\Rightarrow z = 1 + e^{i\theta}$, $dz = ie^{i\theta}d\theta$

Also, $\bar{z} = 1 + e^{-i\theta}$ and $(\bar{z})^2 = 1 + 2e^{-i\theta} + e^{-2i\theta}$

$\therefore \int_c (\bar{z})^2\,dz = \int_0^{2\pi} (1 + 2e^{-i\theta} + e^{-2i\theta}) ie^{i\theta}\,d\theta = 4\pi i.$

5. Show that (i) $\int_c \frac{dz}{z-a} = 2\pi i$ (ii) $\int_c \frac{dz}{(z-a)^n} = 0$, n=2,3,4…, where C is the circle $|z-a| = r$.

Solution:

$|z-a| = r$, is the circle with centre at (a,0) and radius r.

On this circle we have $z - a = re^{i\theta};\ 0 \le \theta \le 2\pi \Rightarrow dz = ire^{i\theta}d\theta.$

(i) $\int_c \frac{dz}{z-a} = \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}}\,d\theta = 2\pi i.$

(ii) Again $\int_c \frac{dz}{(z-a)^n} = \int_0^{2\pi} \frac{ire^{i\theta}}{r^n e^{ni\theta}}\,d\theta$

$$= \frac{1}{r^{n-1}} \int_0^{2\pi} e^{i(1-n)\theta} d\theta = \frac{i}{r^{n-1}} \left[\frac{e^{i(1-n)\theta}}{i(1-n)} \right]_0^{2\pi} = \frac{1}{(1-n)r^{n-1}} \left[e^{i(1-n)2\pi} - 1 \right] = 0$$

Because $e^{i(1-n)2\pi}$ =1.

5.3 Cauchy's Theorem:

Statement: If $f(z)$ **is analytic function and** $f'(z)$ **is continuous at each point within and on a closed curve C, then** $\int_c f(z)\,dz = 0$.

Proof: Let $f(z) = u(x,y) + iv(x,y)$ and $dz = dx + idy$

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (vdx + udy) \qquad \text{...(1)}$$

Since $f'(z)$ is continuous, therefore $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in the region D enclosed by C. Hence by the Greeen's theorem, terms of equation (1) can be as follows.

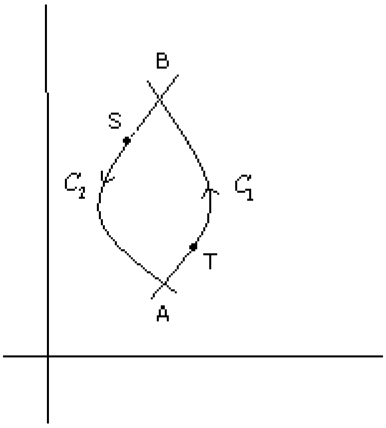
$$\int_C f(z)\,dz = -\int\int_D \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] dxdy + i\int\int_D \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] dxdy \qquad \text{...(2)}$$

Now $f(z)$ being analytic, u and v are necessarily satisfy the C-R equations and thus the integrands of the two double integrals in (2) vanish identically.

Hence $\int_C f(z)\,dz = 0$.

Corollary 1*:* If $f(z)$ **is analytic in the domain D and A & B are two points in D, then** $\int_A^B f(z)\,dz$ **is independent of the path joining A & B.**

Proof: Let $C_1 = ATB$ & $C_2 = ASB$ be two simple curves joining the points A & B such that c_1 & c_2 both lies in the domain D.



Clearly ATBSA is a closed contour in D. Thus by Cauchy's theorem we have

$$\int_{ATBSA} f(z)\,dz = 0$$

$$\Rightarrow \int_{ATB} f(z)\,dz + \int_{BSA} f(z)\,dz = 0 \Rightarrow \int_{ATB} f(z)\,dz - \int_{ASB} f(z)\,dz = 0$$

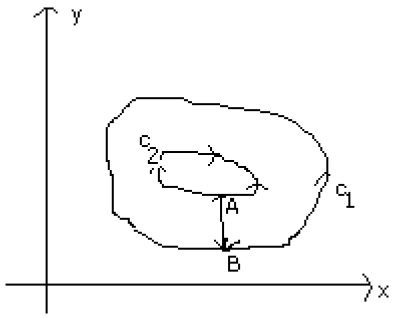
$$\Rightarrow \int_{ATB_1} f(z)\,dz = \int_{BSA} f(z)\,dz \Rightarrow \int_{C_1} f(z)\,dz = \int_{C_2}{}_2\, f(z)\,dz$$

$\Rightarrow \int_A^B f(z)\,dz$ is independent of the path.

Corollary 2: If C_{-1} and C_2 are two simple curves, with C_1 enclosing C_2 and f(z) be analytic inside and on the boundary of the annular region between C_1 and C_2 then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz\,.$$

Proof: Let the curve C_1 and C_2 both be described in the positive direction



We shall introduce a cross cut AB from a point A on C_2 to a point B on C_1.

i) line AB

ii) arc BB along C_1

iii) line BA

iv) arc AA along C_2 in clock wise direction

Thus by Cauchy's theorem, we have.

$$\int_C f(z)\,dz = 0$$

$$\int_{AB} f(z)\,dz + \int_{C_1} f(z)\,dz + \int_{BA} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0$$

$$\int_{AB} f(z)\,dz + \int_{C_1} f(z)\,dz - \int_{AB} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

Hence the proof.

Corollary 3: If C is a simple closed curve enclosing non over lapping simple closed curves $C_1, C_2, C3, \ldots, C_n$ and if $f(z)$ is analytic in the annular region between C and these curves then $\int_C f(z)\,dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)dz, \ldots, \int_{C_n} f(z)\,dz$.

5.4 Cauchy's integral Formula:

Statement: If $f(z)$ is analytic within and on a closed curve C and if 'a' is any point within C, then $f(a)=\frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}dz.$

Proof: Consider the function $\frac{f(z)}{z-a}$ which is analytic at all points with in C except at z=a. With the point α as centre and radius r , draw a small circle C_1 lying entirely within C. Now $\frac{f(z)}{z-a}$ being analytic in the region enclosed by C and C_1, we have Cauchy's theorem,

$$\int_c \frac{f(z)}{z-a}dz=\int_{C_1}\frac{f(z)}{z-a}dz=\int_{C_1}\frac{f(a+re^{i\theta})}{re^{i\theta}}irre^{i\theta}d\theta$$

$$=i\int_{C_1} f(a+re^{i\theta})d\theta$$

In the limiting form, as the circle C_1, shrinks to the point 'a' , i.e as 'r' tends to '0' then the above integral approach to

$$=i\int_{C_1} f(a)d\theta \;=if(a)\int_0^{2\pi} d\theta=2\pi i f(a)$$

$$f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz\ .$$

5.5 Generalized Cauchy's integral Formula:

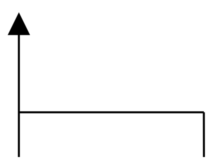
Statement: If $f(z)$ is analytic within and on a closed curve C and if 'a' is any point within C, then $f^{n}(a)=\frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)^{n+1}}dz.$

Problems:

1. Verify Cauchy's theorem for the function $f(z)=z^2$ where C is the square having vertices (0,0) (1,0) (1,1) (0,1).

Solution:

C is the square OABC and we have by Cauchy's theorem $\int_c f(z)dz=0$. Therefore we



have to show that, $\int_{OA} z^2\,dz + \int_{AB} z^2\,dz + \int_{BC} z^2\,dz + \int_{CO} z^2\,dz = 0$

C(0,1) B(1,1)

O(0,0) A(1,0)

Along OA, $y = 0$ & $dy = 0$; $0 \le x \le 1$

$$z^2\,dz = (x+iy)^2(dx+idy) = x^2\,dx.$$

$$\int_{OA} z^2\,dz = \int_{x=0}^{1} x^2\,dx = 1/3. \qquad \text{....(1)}$$

Along AB, $x = 1$ & $dx = 0$; $0 \le y \le 1$

$$z^2\,dz = (x+iy)^2(dx+idy) = (1+iy)^2\,idy.$$

$$\int_{AB} z^2\,dz = i\int_{y=0}^{1}(1+iy)^2\,dy = -1 + 2i/3.$$

...(2)

Along BC, $y = 1$ & $dy = 0$; $1 \le x \le 0$

$$z^2\,dz = (x+iy)^2(dx+idy) = (x+i)^2\,dx.$$

$$\int_{bc} z^2\,dz = \int_{x=1}^{0}(x^2 + 2ix - 1)\,dx = 2/3 - i. \qquad \text{...(3)}$$

Along CO, $x = 0$ & $dx = 0$; $1 \le y \le 0$

$$z^2\,dz = (x+iy)^2(dx+idy) = -iy^2\,idy.$$

$$\int_{CO} z^2\,dz = i\int_{y=1}^{0} -iy^2\,dy = i/3. \qquad \text{...(4)}$$

By adding (1),(2),(3) &(4), we get

$$\int_{OA} z^2\,dz + \int_{AB} z^2\,dz + \int_{BC} z^2\,dz + \int_{CO} z^2\,dz = 1/3 - 1 + 2i/3 + 2/3 - i + i/3 = 0.$$

Hence Cauchy’s theorem is verified.

2. Evaluate $\int_c \frac{1}{z(z-1)}\,dz$ where C is the circle $|z|=3$.

Solution:

Given C: $|z|=3$ is the circle with centre at the origin and radius 3.

$\phi(z)=\frac{1}{z(z-1)}$ is not analytic at $z=0$ and at $z=1$.

These points $z=0$ and at $z=1$ lie inside the circle $|z|=3$.

Now $\frac{1}{z(z-1)} = -\frac{1}{z}+\frac{1}{z-1}$

$$\int_c \frac{1}{z(z-1)}\,dz = \int_c \frac{1}{(z-1)}\,dz - \int_c \frac{1}{z}\,dz$$

Taking $f(z)=1$ which is analytic in and on C, we have

$$\int_c \frac{1}{z(z-1)}\,dz = \int_c \frac{f(z)}{(z-1)}\,dz - \int_c \frac{f(z)}{z}\,dz$$

By applying Cauchy's integral formula by taking z_0 =1 in first integral z_0 =0 in the second integral, we have

$$\int_c \frac{1}{z(z-1)}\,dz = 2\pi i f(1) - 2\pi i f(0) = 0 .$$

3. Evaluate $\int_c \frac{z+4}{z^2+2z+5}\,dz$ where C is the circle $|z+1+i|=2$.

Solution:

Given $|z+1+i|=2 \Rightarrow |z-(-1-i)|=2$

This is a circle with centre at $z=-1-i$ and radius equal to 2.

Now $\phi(z)=\frac{z+4}{z^2+2z+5}$ will cease to be analytic where z^2+2z+5 =0.

Now z^2+2z+5 =0. $\Rightarrow z=-1\pm 2i$

i.e. $z^2+2z+5 = (z+1-2i)(z+1+2i)$

Thus $\phi(z)$ is not analytic at $z=-1-2i$ but $z=-1+2i$ lies outside C.

Thus $\int_c \frac{z+4}{z^2+2z+5}\,dz = \int_c \frac{z+4}{(z+1-2i)(z-1-2i)}\,dz$

$$= \int_c \frac{\left[\frac{z+4}{(z+1-2i)}\right]}{z-(-1-2i)}\,dz$$

Taking $f(z) = \dfrac{z+4}{z-(-1+2i)}$ and $z_0 = -1-2i$, applying Cauchy's integral formula, we get

$$\int_c \frac{z+4}{z^2+2z+5}\,dz = 2\pi i f(-1-2i) = 2\pi i\left(\frac{3-2i}{-4i}\right) = \frac{\pi}{2}(2i-3).$$

4. Evaluate $\int_c \dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$ where C is the circle $|z| = 3$.

Solution:

The circle C: $|z| = 3$ is the circle with centre at $z = 0$ and radius 3.

$\phi(z) = \int_c \dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$ is analytic every where except at z=1 and at z=2. These points are inside the circle C.

Now $\dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$

$$\therefore \int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz = \int_c \frac{f(z)}{(z-1)}\,dz - \int_c \frac{f(z)}{(z-2)}\,dz \qquad \text{...(1)}$$

Taking $f(z) = \sin \pi z^2 + \cos \pi z^2$ and $z_0 = 1$, applying Cauchy's integral formula we

formula, $\int_c \dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz = 2\pi i f(1) = 2\pi i(-1) = -2\pi i$

Again by taking $f(z) = \sin \pi z^2 + \cos \pi z^2$ and $z = 2\pi$, we have

$$\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz = 2\pi i f(2) = 2\pi i$$

(1) Becomes $\int_c \dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz = -4\pi i$.

EXERCISES:

1. Evaluate $\int_c (z^2+z)dz$ along the line joining $(1-i)$ & $(2+3i)$.
2. Evaluate $\int_c (\bar{z})^2 dz$ where (i) C is the circle $|z-1|=1$. (ii) C is the circle $|z-2|=1$.
3. Verify Cauchy's theorem for the function $f(z)=z^2$ over the square formed by the points (0,0), (2,0), (2,2) and (0,2).
4. Evaluate $\int_c \frac{(z-1)}{(z+1)^2(z-2)}dz$ where C is the circle $|z-i|=2$.
5. Evaluate $\int_c \frac{e^z}{(z^2+\pi^2)^2}dz$ where C is the circle $|z|=4$.

ANSWERS:

1. $\frac{-1}{6}(103-64i)$
2. (i). $4\pi i$ (ii). $8\pi i$
4. $\frac{-2\pi i}{9}$
5. $\frac{i}{\pi}$

5.6 Taylor's series:

Statement: If $f(z)$ is analytic inside a circle C with centre at a, then for z inside C,

$$f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+\dots+\frac{f^n(a)}{n!}(z-a)^n+\dots$$

Note: If $a=0$ in the Taylor's series we obtain the series in the form

$f(z)=f(0)+z\,f'(0)+z^2\frac{f''(a)}{2!}+\dots\dots$ This is known as Maclaurin's series for $f(z)$.

5.7 Laurent's Series:

Statement: If $f(z)$ is analytic in the ring shaped region R bounded by two concentric circles C_1 and C_2 of radii r and r_1 and with centre at 'a', then for all z in R.

$$f(z)=a_0+a_1(z-a)+a_2(z-a)^2+\dots a_{-1}(z-a)^{-1}+a_{-2}(z-a)^{-2}+\dots$$

Where $a_n=\frac{1}{2\pi i}\int_{C_1}\frac{f(t)}{(t-a)^{n+1}}dt$ ***and*** $a_{-n}=\frac{1}{2\pi i}\int_{C_2}\frac{f(t)}{(t-a)^{-n+1}}dt$.

Examples:

Problems:

1. Find the Taylor's expansion of $f(z)=\dfrac{1}{(z+1)^2}$ **about the point** $z=-i$.

Solution:

To expand $f(z)$ about $z=-i$ i,e in powers of $z+i$, put $z+i$ =t , then

$$f(z)=\frac{1}{(t-i+1)^2}=(1-i)^{-2}[1+t/(1-i)]^{-2}=\frac{i}{2}\left[1-\frac{2t}{1-i}+\frac{3t^2}{(1-i)^2}\ldots\ldots\right]$$

$$=\frac{i}{2}\left[1+\sum_{n=1}^{\infty}(-1)^n\frac{(n+1)(z+i)^n}{(1-i)^n}\right].$$

2. Expand $f(z)=\dfrac{z-1}{(z-2)(z-3)}$ **in non negative powers of z.**

Solution:

we shall resolve into partial fractions.

$$\frac{z-1}{(z-2)(z-3)}=\frac{A}{z-2}+\frac{B}{z-3}=\frac{-1}{z-2}+\frac{2}{z-3} \qquad \text{....(1)}$$

$$f(z)=\frac{-1}{z-2}+\frac{2}{z-3}=\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}-\frac{2}{3}\left(1-\frac{z}{3}\right)^{-1}$$

The expansion is valid for $\left|\frac{z}{2}\right|<1$ and $\left|\frac{z}{3}\right|<1$. i.e., $|z|<2$ and $|z|<3$. Since $|z|<2$ implies $|z|<3$ also, we can say that the expansion is valid when $|z|<2$.

Using binomial expansion:

$$f(z)=\frac{1}{2}\left\{1+\frac{z}{2}+\frac{z^2}{4}+\ldots\right\}-\frac{2}{3}\left\{1+\frac{z}{3}+\frac{z^2}{9}+\ldots\right\}.$$

3. Find the Laurent's expansion of $f(z)=\dfrac{3z^2-6z+2}{z^3-3z^2+2z}$ **in the region** $\begin{matrix} i) & 1<|z|<2 \\ ii) & |z|>2 \end{matrix}$.

Solution:

By resolving into partial fraction,

$$f(z)=\frac{3z^2-6z+2}{z^3-3z^2+2z}=\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z-2}. \qquad \ldots(1)$$

Case i: $1<|z|<2 \Rightarrow \frac{1}{|z|}<1$ *and* $\frac{|z|}{2}<1$.

Hence we have from (i)

$$f(z)=\frac{1}{z}+\frac{1}{z\left(1-\frac{1}{z}\right)}+\frac{1}{-2\left(1-\frac{z}{2}\right)}=\frac{1}{z}+\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$=\frac{1}{z}+\left\{1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\_\_\right\}-\frac{1}{2}\left\{1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\_\_\right\}$$

$$f(z)=\frac{1}{2}+\frac{2}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\_\_-\frac{z}{4}-\frac{z^2}{8}-\frac{z^3}{16}\_\_$$

Case ii: $|z|>2 \Rightarrow |z|>1 \quad \therefore \frac{2}{|z|}<1, \quad \frac{1}{|z|}<1$

Hence from equation (1) we have

$$f(z)=\frac{1}{z}\left[1+\left(1-\frac{1}{z}\right)^{-1}+\left(1-\frac{2}{z}\right)^{-1}\right]$$

$$=\frac{1}{z}\left[1+\left\{1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\_\_\right\}+\left\{1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\_\_\right\}\right]$$

$$=\frac{1}{z}\left[3+\frac{3}{z}+\frac{5}{z^2}+\frac{9}{z^3}+\_\_\right]$$

4. Find the Laurent's expansion of $f(z)=\frac{7z-2}{(z+1)z(z-2)}$ **in the region** $1<z+1<3$.

Solution:

put $u=z+1$, we have

$$f(z)=\frac{7(u-1)-2}{u(u-1)(u-1-2)}=\frac{7u-9}{u(u-1)(u-3)}$$

$$=-\frac{3}{u}+\frac{1}{u-1}+\frac{2}{u-3}=-\frac{3}{u}+\frac{1}{u}\left(1-\frac{1}{u}\right)^{-1}-\frac{2}{3}\left(1-\frac{u}{3}\right)^{-1}$$

$$f(z)=\frac{-3}{u}+\frac{1}{u}\left(1+\frac{1}{u}+\frac{1}{u^2}+\ldots\right)-\frac{2}{3}\left(1+\frac{u}{3}+\frac{u^2}{3^2}+\ldots\right)$$

$$f(z)=\frac{-2}{z+1}+\frac{1}{(z+1)^2}-\frac{1}{(z+1)^3}+\_\_-\frac{2}{3}\left[1+\frac{z+1}{3}+\frac{(z+1)^2}{3^2}+\_\_\right].$$

5. Expand $f(z)=\dfrac{z}{(z-1)(2-z)}$ **in Laurent's series valid for** $|z-1|<1$.

Solution:

put $u=z+1\ or\ z-1=u$, we have

$$f(z)=\frac{(u+1)}{u(2-u-1)}=\frac{u+1}{u(1-u)}=g(u).$$

$$Let\frac{(u+1)}{u(1-u)}=\frac{A}{u}+\frac{B}{1-u}=\frac{1}{u}+\frac{2}{1-u}. \qquad ......(i)$$

$Now|z-1|<1 is equivalent to|u|<1.$

Hence (i) can be written in the form

$$g(u)=\frac{1}{u}+2(1-u)^{-1}$$

$$=\frac{1}{u}+2(1+u+u^2+u^3+\ldots)$$

$$i.e.,\ F(z)=\frac{1}{z-1}+2+2(z-1)+2(z-1)^2+\ldots$$

6. Expand $f(z)=\dfrac{z-1}{(z-2)(z-3)^2}$ **as a Laurent's series valid for**

$(i)|Z|>3, \quad (ii)2<|Z|<3.$

Solution:

we shall first resolve $f(z)$ in to partial fractions,

$$\frac{z-1}{(z-2)(z-3)^2}=\frac{A}{(z-2)}+\frac{B}{(z-3)}+\frac{C}{(z-3)^2}$$

$$f(z)=\frac{1}{(z-2)}-\frac{1}{(z-3)}+\frac{2}{(z-3)^2} \qquad \ldots(i)$$

$case(i): |z|>3. \text{ This implies } |z|>2 \text{ also.}$

$i.e., \frac{3}{|z|}<1, \frac{2}{|z|}<1$

We have to write (i) in the form

$$f(z)=\frac{1}{z\left(1-\frac{2}{z}\right)}-\frac{1}{z\left(1-\frac{3}{z}\right)}+\frac{2}{\left\{z\left(1-\frac{3}{z}\right)\right\}^2}$$

$$=\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}-\frac{1}{z}\left(1-\frac{3}{z}\right)^{-1}+\frac{2}{z^2}\left(1-\frac{3}{z}\right)^{-2}$$

$$=\frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\ldots\right)-\frac{1}{z}\left(1+\frac{3}{z}+\frac{9}{z^2}+\frac{27}{z^3}+\ldots\right)+\frac{2}{z^2}\left(1+\frac{6}{z}+\frac{27}{z^2}+\frac{108}{z^3}+\ldots\right)$$

$$f(z)=\frac{1}{z^2}+\frac{7}{z^3}+\frac{35}{z^4}+\ldots$$

$case(ii): 2<|z|<3.$

$i.e., 2<|z| \Rightarrow \frac{2}{|z|}<1; |z|<3 \Rightarrow \frac{|z|}{3}<1$

$\therefore$ $f(z)$ as given by (i) has to be written in the form

$$f(z)=\frac{1}{z\left(1-\frac{2}{z}\right)}-\frac{1}{(-3)\left(1-\frac{z}{3}\right)}+\frac{2}{\left\{(-3)\left(1-\frac{z}{3}\right)\right\}^2}$$

$$=\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}+\frac{1}{3}\left(1-\frac{z}{3}\right)^{-1}+\frac{2}{9}\left(1-\frac{z}{3}\right)^{-2}$$

$$=\frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\ldots\right)+\frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+\ldots\right)+\frac{2}{9}\left(1+\frac{2z}{3}+\frac{z^2}{3}+\frac{4z^3}{27}+\ldots\right)$$

$$f(z)=\frac{1}{z}+\frac{2}{z^2}+\frac{4}{z^3}+\frac{8}{z^4}+\ldots+\frac{5}{9}+\frac{7}{27}z+\frac{1}{9}z^2+\frac{11}{243}z^3+\ldots$$

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EXERCISES:

I. Obtain the Taylor's Series expansion for the following functions of

a. $\frac{z-1}{z^2}$ in powers of $z-1$

b. $\frac{z-1}{z+1}$ in powers of $z=1$

c. $\cos z$ about the point $z=\pi/2$

II. Find the Laurent's expansion for the following functions

a. $\frac{e^z}{(z-1)^2}$ about $z=1$

b. $\frac{e^{2z}}{(z-1)^3}$ about the singularity $z=1$

c. $\frac{z-1}{z^2}$ for $|z-1|>1$

d. $\frac{1-\cos z}{z^3}$ about z=0

e. $\frac{z^2-1}{z^2+5z+6}$ about $z=0$ in the region $2<|z|<3$

ANSWERS:

I.

a) $\sum_{n=1}^{\infty}(-1)^{n+1}n(z-1)^n$

b) $\frac{1}{2}(z-1)-\frac{1}{2^2}(z-1)^2+\frac{1}{2^3}(z-1)^3-.....$

c) $-\left(z-\frac{\pi}{2}\right)+\frac{\left(z-\frac{\pi}{2}\right)^3}{3!}-\frac{\left(z-\frac{\pi}{2}\right)^5}{5!}+.....$

II.

a) $e\left[(z-1)^{-2}+(z-1)^{-1}+\frac{1}{2!}+\frac{1}{3!}(z-1)+\frac{1}{4!}(z-1)^2+...\right]$

b) $e^2(z-1)^{-3}+2e^2(z-1)^{-2}+2e^2(z-1)^{-1}+\frac{4e^2}{3}+\frac{2e^2}{3}(z-1)+\ldots$

c) $\sum_{n=1}^{\infty}(-1)^{n-1}n(z-1)^{-n}$ for $|z-1|>1$

d) $-\sum_{n=2}^{\infty}\frac{z^{2n-5}}{2(n-1)!}$

e) $1+\frac{3}{z}\left(1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+\ldots\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}+\ldots\right)$

5.8 Zeros of an analytic function:

A zero of an analytic function $f(z)$ is that value of z for which $f(z)=0$.

5.8.1 Singularity: Singular point of a function is a point at which the function ceases to be analytic.

Types of singularities:

1. **Isolated Singularity:** If $z=a$ is a singularity of $f(z)$ such that $f(z)$ is analytic at each point in its neighborhood then $z=a$ is called an isolated singularity.
2. **Removable Singularity:** If all the negative powers of $z-a$ in Laurent's series are zero, then such singularity is said to be removable singularity.
3. **Poles:** If all the negative powers of $z-a$ in Laurent's series after the nth term are missing, then the $z=a$ is called a pole of order n. Pole of order one is called **Simple pole.**
4. **Essential Singularity:** If the number of negative powers of $z-a$ in Laurent's is infinite then $z=a$ is called an Essential Singularity.

Example: Find the nature and location of singularity of the function $\frac{z-\sin z}{z^2}$

Solution: Here $z=0$ is a singularity.

$$\frac{z-\sin z}{z^2}=\frac{1}{z^2}\left\{z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\ldots\right)\right\}=\frac{z}{3!}-\frac{z^3}{5!}+\frac{z^5}{7!}-\ldots$$

Therefore $z=0$ is removable singularity.

5.8.2 Residues:

The co-efficient of $(z-a)^{-1}$ in the expansion of $f(z)$ is called the residue of $f(z)$ at the pole $z=a$.

If $z=a$ is a pole of order m of $f(z)$ then the residue of $f(z)$ at $z=a$ is denoted by $R[m,a]$ and is given by $R[m,a] = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}$.

Note:

1. If $f(z)$ has a simple pole at $z=a$ then $\operatorname{Res} f(a) = \lim_{z\to a}[(z-a)f(z)]$

2. If $f(z)$ has a pole of order at $z=a$ then

$$\operatorname{Res} f(a) = \frac{1}{(n-1)!}\lim_{z\to a}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}.$$

5.9 Cauchy's Residue Theorem:

Statement: If $f(z)$ is analytic inside and on the boundary of a simple closed curve C except for a finite number of poles $a, b, c.....,$ then the integral of $f(z)$ over C is equal to $2\pi i$ times the sum of the residues at the poles inside C.

$$\int_C f(z)dz = 2\pi i(a_{-1}+b_{-1}+c_{-1}+...)$$

i.e.,

$$\int_C f(z)dz = 2\pi i\sum R, \text{ where } \sum R \text{ denote the sum of the residues at the poles lying in } C.$$

Problems:

1. Find the residues of the function $f(z) = \frac{z}{(z+1)(z-2)^2}$ at $i).\ z=-1$ & $ii)\ z=2$

Solution:

$z=-1$ is a pole of order 1(simple pole) and $z=2$ is a pole of order 2.

The residue of $f(z)$ for a pole of order m at $z=a$ is given by

$R[m,a] = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}$ and in particular for a simple pole (m=1),

$R[1,a] = \lim_{z\to a}\{(z-a)\ f(z)\}$

Case (i): Residue at $z=a=-1$ is given by

$$\lim_{z\to -1}(z+1)\frac{z}{(z+1)(z-2)^2}=\lim_{z\to -1}\frac{z}{(z-2)^2}=\frac{-1}{9}.$$

Case (ii): Residue at $z=a=2$ where m=2 is given by

$$\lim_{z\to 2}\frac{1}{1}\frac{d}{dz}\left\{(z-2)^2\frac{z}{(z+1)(z-2)^2}\right\}=\lim_{z\to 2}\frac{d}{dz}\left\{\frac{z}{(z+1)}\right\}$$

$$=\lim_{z\to 2}\left\{\frac{(z+1)^2-z}{(z+1)^2}\right\}=\lim_{z\to 2}\frac{1}{(z+1)^2}=\frac{1}{9}.$$

Thus the required residues are -1/9 and 1/9.

2. **Evaluate** $\int_C \frac{(z^2+5)}{(z-2)\ (z-3)}dz$ using residue theorem, $C:|z|=4$

Solution:

The poles of the function $f(z)=\frac{(z^2+5)}{(z-2)\ (z-3)}$ are $z=2,\ z=3$ and both the poles lie within the circle $C:|z|=4$.

Therefore, residue at $z=a=2,$ which is a simple pole is given by

$$\lim_{z\to 2}(z-2)\ f(z)=\lim_{z\to 2}(z-2)\frac{(z^2+5)}{(z-2)\ (z-3)}$$

$$=\lim_{z\to 2}\left\{\frac{z^2+5}{z-3}\right\}=-9=R_1.$$

Similarly , residue at $z=a=3,$ which is given by

$$\lim_{z\to 3}(z-3)\ f(z)=\lim_{z\to 3}(z-3)\frac{(z^2+5)}{(z-2)\ (z-3)}$$

$$=\lim_{z\to 3}\left\{\frac{z^2+5}{z-2}\right\}=14=R_2.$$

We have by Cauchy's theorem,

$$\int_C f(z)dz=2\pi i(R_1+R_2)$$

$$Thus,\ \int_C \frac{(z^2+5)}{(z-2)\ (z-3)}dz=2\pi i(-9+14)=10\pi i.$$

2. **Evaluate** $\int_C \frac{3z^3+2}{(z-1)\left(z^2+9\right)}dz$ where C is the circle $i)\ |z-2|=2$ $ii)\ |z|=4$

Solution:

let $f(z) = \frac{3z^3+2}{(z-1)\left(z^2+9\right)} = \frac{3z^3+2}{(z-1)\ (z+3i)(z-3i)}$

The poles of $f(z),\ z=1,\ z=3i,\ z=-3i$ (simple poles) are represented by the points (1,0) (0,3) (0,-3) respectively.

Case (i): $C:|z-2|=2.$ this is a circle with center (2,0) and radius 2.

The point (1,0) i.e., $z=1$ only lies within C.

The residue of $f(z)$ at $z=1$ (simple pole) is given by

$$\lim_{z\to 1}(z-1)\,f(z) = \lim_{z\to 1}(z-1)\frac{3z^3+2}{(z-1)\left(z^2+9\right)}$$

$$= \lim_{z\to 3}\left\{\frac{3z^3+2}{z^2+9}\right\} = 1/2.$$

By Cauchy's theorem,

$$\int_C f(z)dz = 2\pi i(R_1) = 2\pi i\frac{1}{2} = \pi i.$$

$$Thus,\ \int_C \frac{3z^3+2}{(z-1)\left(z^2+9\right)}dz = \pi i.$$

Case (ii): $C:|z|=4.$ this is a circle with center (0,0) and radius 4. All the three poles $z=1,\ z=3i,\ z=-3i$ lies within C. Let $R_1,\ R_2\ R_3,$ denotes the residues at the simple poles $z=1,\ z=3i,\ z=-3i$ respectively.

$$Now\ R_1 = \lim_{z\to 1}(z-1)\,f(z) = \lim_{z\to 1}(z-1)\frac{3z^3+2}{(z-1)\left(z^2+9\right)}$$

$$= \lim_{z\to 3}\left\{\frac{3z^3+2}{z^2+9}\right\} = 1/2.$$

$$R_2 = \lim_{z \to 3i} (z-3i)\, f(z) = \lim_{z \to 3i} (z-3i) \frac{3z^3+2}{(z-1)(z+3i)(z-3i)}$$

$$= \lim_{z \to 3i} \frac{3z^3+2}{(z-1)(z+3i)} = \frac{15+49i}{12}.$$

$$R_3 = \lim_{z \to -3i} (z+3i)\, f(z) = \lim_{z \to -3i} (z+3i) \frac{3z^3+2}{(z-1)(z+3i)(z-3i)}$$

$$= \lim_{z \to -3i} \frac{3z^3+2}{(z-1)(z-3i)} = \frac{15-49i}{12}.$$

By Cauchy's theorem,

$$\int_C f(z)dz = 2\pi i(R_1+R_2+R_3) = 2\pi i\left(\frac{1}{2}+\frac{15+49i}{12}+\frac{15-49i}{12}\right) = 6\pi i.$$

$$Thus,\ \int_C \frac{3z^3+2}{(z-1)\left(z^2+9\right)}dz = 6\pi i.$$

3. Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)}dz$ **, where C is the circle** $|z| = 3$

Solution:

$f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)}$ is analytic within the circle $|z| = 3$ excepting the poles $z = 1$ and $z = 2$

$$\text{Res } f(1) = \frac{1}{(1)!} \lim_{z \to 1} \left\{\frac{d}{dz}\left[(z-1)^2 f(z)\right]\right\} = 2\pi + 1$$

$$\text{Res } f(2) = \lim_{z \to 2} \{(z-2) f(z)\} = \lim_{z \to 2} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2} = 1$$

By residue theorem,

$$\int_C f(z)dz = 2\pi i[\text{Res } f(1) + \text{Res } f(2)] = 2\pi i(2\pi+1+1) = 4\pi(\pi+1)i$$

EXERCISES:

I. Find the nature and location of the singularities of the following functions

a. $\frac{1}{z(2-z)}$

b. $\sin\left(\frac{1}{z}\right)$

c. $\tan\left(\dfrac{1}{z}\right)$

e. $\dfrac{e^z}{(z-1)^4}$

d. $\dfrac{z^2-1}{(z-1)^3}$

f. $\dfrac{\cot \pi z}{(z-a)^2}$

II. Determine the poles of the following and the residue at the each pole

a. $\dfrac{z^2+1}{z^2-2z}$

c. $\dfrac{2z+4}{(z^2+1)(z+1)}$

b. $\dfrac{z^2-2z}{(z+1)^2(z^2+1)}$

III. Find the residue of the following functions at each pole

a. $\dfrac{1-e^{2z}}{z^4}$

b. $\dfrac{ze^{iz}}{(z^2+1)}$

c. $\cot z$

IV. Evaluate the following integrals

a. $\int_c \dfrac{1-2z}{z(z-1)(z-2)}\,dz \qquad C:|z|=1.5$

e. $\int_C \dfrac{z+4}{z^2+2z+5}\,dz \qquad C:|z+1-i|=2$

b. $\int_C \dfrac{4z^2-4z+1}{(z-2)(z^2+4)}\,dz \qquad C:|z|=1$

f. $\int_C \dfrac{z}{(z-1)(z-2)^2}\,dz \qquad C:|z-2|=0.5$

c. $\int_C \dfrac{3z^2+z+1}{(z+3)(z^2-1)}\,dz \qquad C:|z|=2$

g. $\int_C \dfrac{3z^2+2}{(z-1)(z^2+9)}\,dz \qquad C:|z-2|=2$

d. $\int_C \dfrac{1+2z}{(2z-1)^2}\,dz \qquad C:|z|=1$

h. $\int_C \dfrac{1}{(z^2+4)^2}\,dz \qquad C:|z-i|=2$

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Answers:

I.

a) $z=0,\ z=2$ are the isolated singularities.
b) $z=0$ are the isolated essential singularities.
c) $z=0$ are the non isolated essential singularities.
d) $z=1$ is a pole of order 2.
e) $z=1$ is a pole of order 4.
f) $z=a$ is a double pole and $z=0, \pm 1, \pm 2, \ldots$ are simple poles.

II.

a) $\operatorname{Re} sf(0) = -1/2,\ \operatorname{Re} sf(2) = 2\frac{1}{2}$.

b) $\operatorname{Re} sf(-1) = 0,\ \operatorname{Re} sf(i) = \frac{1+2i}{2(1-i)},\ \operatorname{Re} sf(-i) = \frac{-1+2i}{2(-1+i)}$,

c) $\operatorname{Re} sf(-1) = 1,\ \operatorname{Re} sf(i) = \frac{i+2}{(-1+i)},\ \operatorname{Re} sf(-i) = \frac{i-2}{(1+i)}$.

III.

a) $\operatorname{Re} sf(0) = \frac{-4}{3}$.

b) $\operatorname{Re} sf(i) = \frac{1}{2}e^{-1},\ \operatorname{Re} sf(-i) = \frac{1}{2}e.$

c) $\frac{-i\pi}{4},\ \operatorname{Re} sf(n\pi) = 1,$ n an integer.

IV.

a) $3\pi i$
b) 0
c) $\frac{\pi i}{4}$
d) πi
e) 0
f) $-2\pi i$
g) πi
h) $\frac{\pi}{16}$.