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Module -1: Multivariable Calculus (Integration)

- **Double integrals (Cartesian)**
- **change of order of integration in double integrals**
- **Change of variables (Cartesian to polar)**
- **Applications: area and volumes**
- **Triple integrals(Cartesian)**
- **Orthogonal curvilinear coordinates.**

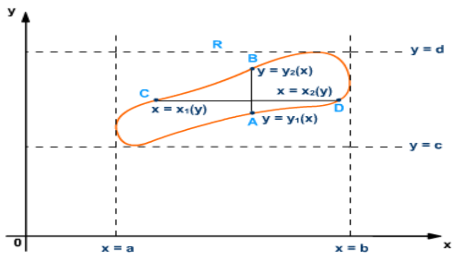
1.1 Introduction

Integration can be traced as far as ancient Egypt 1800 BC, with the Moscow Mathematical Papyrus demonstrating knowledge of a formula for the volume of a Pyramidal frustum. The significant advances in integral calculus did not begin to appear until the 16th century. Steps were made in the early 17th century by Barrow and Torricelli, who provided the first hints of a connection between integration and differentiation

Previous classes had the concept of evaluating indefinite and definite integrals involving a single variable. Now in this chapter, evaluation of integrals involving more than one variable will be seen in detail. Further the topics will consist of change of the order of integration and change of variables and its application to area and volume.

1.2 Double Integrals

Let a $f(x, y)$ function of two variables be defined over a region R in the xy plane and be bounded by a closed curve C. Let $a \le x \le b$ and $c \le y \le d$ belongs to region R.



Consider an arbitrary chord AB parallel to y -axis to incorporate the variation in y and x is fixed, hence y varies from $y = y_1(x)$ to $y = y_2(x)$.

Let $\phi(x) = \int_{y=y_1(x)}^{y=y_2(x)} f(x,y)\,dy$.............(1) which is function of x only

Now let us integrate $\phi(x)$ w.r.t. x between $x = a$ and $x = b$ and denote this by an integral $I = \int_A f(x,y)\,dA$ (which covers the whole of A)

$$I = \int_{x=a}^{x=b} \phi(x)\,dx = \int_{x=a}^{x=b} \left\{ \int_{y=y_1(x)}^{y=y_2(x)} f(x,y)\,dy \right\} dx............(2)$$

Similarly consider an arbitrary chord CD parallel to x -axis. On CD we have y is fixed and x varies from $x = x_1(y)$ to $x = x_2(y)$

Let $\psi(y) = \int_{x=x_1(y)}^{x=x_2(y)} f(x,y)\,dx$.............(3) which is function of y alone

Now let us integrate $\psi(y)$ w.r.t. y between $y = c$ and $y = d$ and denote this by an integral $I = \int_A f(x,y)\,dA$ (which covers the whole of A)

$$I = \int_{y=c}^{y=d} \psi(y)\,dy = \int_{y=c}^{y=d} \left\{ \int_{x=x_1(y)}^{x=x_2(y)} f(x,y)\,dx \right\} dy............(4)$$

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Note 1: $\iint_R f(x,y)xdy = \int_{x=a}^{x=b}\int_{y=y_1(x)}^{y=y_2(x)} f(x,y)dydx = \int_{y=c}^{y=d}\int_{x=x_1(y)}^{x=x_2(y)} f(x,y)dxdy$ even through process is opposite

Note 2: $\iint_R f(x,y)dxdy = \int_A f(x,y)dA$ is called the Double integral.

Problems:

1. **Evaluate** $\int_0^1\int_0^{\sqrt{x}} xy\ dydx.$

Solution:

$$\int_0^1\int_x^{\sqrt{x}} xy\ dy\ dx = \int_0^1\left\{\int_{y=x}^{y=\sqrt{x}} xy\,dy\right\}dx = \int_0^1 \left.\frac{xy^2}{2}\right]_x^{\sqrt{x}} dx = \frac{1}{2}\int_0^1 (x^2 - x^3)\ dx = \frac{1}{24}$$

2. **Evaluate** $\int_0^1\int_0^y xy\,dxdy$

Solution:

$$\int_0^1\int_0^y xy\,dx\ dy = \int_{y=0}^{y}\left\{\int_{x=0}^{x=y} dy\ dx\right\} dy = \int_0^1 y.\left.\frac{x^2}{2}\right]_0^y dy = \int_0^1 \frac{y^3}{2}dy = \frac{1}{2}.\left.\frac{y^4}{4}\right]_0^1 = \frac{1}{8}$$

3. Evaluate $\int_0^1\int_0^1 \frac{dxdy}{(1+x^2)(1+y^2)}$

Solution:

Here the limits of both the integrals are constants and the variables can be separated.

$$\therefore \quad \int_0^1\int_0^1 \frac{dx\,dy}{(1+x^2)\,(1+y^2)} = \left(\int_0^1 \frac{dx}{1+x^2}\right)\left[\int_0^1 \frac{dy}{1+y^2}\right] = \tan^{-1}x\Big]_0^1 \tan^{-1}y\Big]_0^1 = \frac{\pi^2}{16}$$

4. **Show that** $\int_0^\infty\int_0^\infty e^{-x^2(1+y^2)}x\ dxdy=\frac{\pi}{4}$.

Solution:

$$\text{LHS}\quad =\int_0^\infty\int_0^\infty e^{-x^2(1+y^2)}x\ dxdy \quad =\int_0^\infty\left\{\int_0^\infty e^{-x^2}(1+y^2)\,x\,dx\right\}dy$$

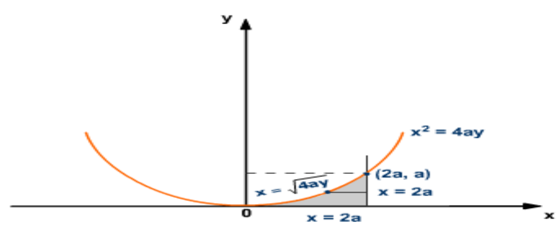
Put $\quad x^2=t \ \Rightarrow 2xdx=dt$ or $xdx=\frac{1}{2}dt$

$$=\frac{1}{2}\int_0^\infty\left\{\int_0^\infty e^{-t(1+y^2)}dt\right\}dy=\frac{1}{2}\int_0^\infty\left[\frac{e^{-t(1+y^2)}}{-(1+y^2)}\right]_0^{-\infty}dy$$

$$=-\frac{1}{2}\int_0^\infty\left[0-\frac{1}{1+y^2}\right]dy=\frac{1}{2}\int_0^\infty\frac{1}{1+y^2}dy=\frac{1}{2}\tan^{-1}y\Big]_0^\infty \quad =\frac{\pi}{4}$$

5. **Evalute** $\iint_A xy\,dxdy$ **where** A **is the region bounded by** x **- axis ordinate** $x=2a$ **and the curve** $\quad x^2=4ay$.

Solution:



$$\iint_A \text{xy dx dy}=\int_{y=0}^{a}\left\{\int_{x=\sqrt{4ay}}^{x=2a}xy\,dx\right\}dy \quad =\int_0^a\left[\frac{x^2y}{2}\right]_{2\sqrt{ay}}^{2a}dy$$

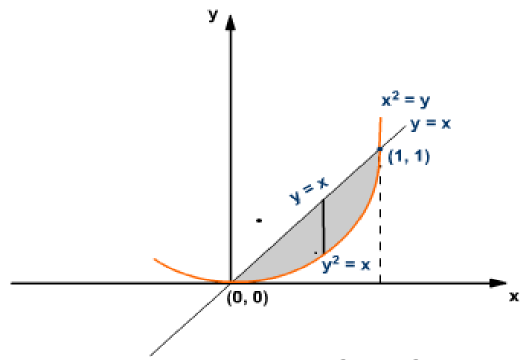
$$=\int_0^a\frac{4a^2y-4ay^2}{2}dy$$

$$=2a\int_0^a(ay-y^2)dy=2a\left\{\frac{ay^2}{2}-\frac{y^3}{3}\right\}_0^a$$

$$=\frac{a^4}{3}$$

6. **Evalute** $\iint_A xy(x+y)\,dxdy$ **where** A **is the region bounded by** $y=x^2$ **and** $y = x$.

Solution:



$$\iint_A xy\,(x+y)\,dx\,dy = \int_0^1 \left\{ \int_{y=x^2}^{y=x} (x^2 y + xy^2)\,dy \right\} dx$$

$$= \int_0^1 \left[\frac{x^2 y^2}{2} + \frac{xy^3}{3} \right]_{x^2}^{x} dx = \int_0^1 \left(\frac{5}{6}x^4 - \frac{1}{2}x^6 - \frac{1}{3}x^7 \right) dx = \frac{3}{56}$$

7. **Evaluate** $\iint y\,dxdy$ **over the region bounded by the first quadrant of the ellipse** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution: From $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ we have $\frac{y^2}{b^2}=1-\frac{x^2}{a^2}=\left(\frac{a^2-x^2}{a^2}\right)$, $y=\frac{b}{a}\sqrt{a^2-x^2}$

Since $y=0$ is the equation of x-axis we can say that y varies from 0 to $\frac{b}{a}\sqrt{a^2-x^2}$

Therefore,

$$\iint y\,dxdy = \int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} y\,dy\,dx = \int_{x=0}^{a} \left[\frac{y^2}{2}\right]_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dx = \frac{b^2}{2a^2}\left[a^2 x - \frac{x^3}{3}\right]_0^a = \left[\frac{ab^2}{3}\right]$$

Note : We can also write $I = \int_{y=0}^{1} \int_{x=y}^{\sqrt{y}} (x^2 y + xy^2)\,dydx$ $\quad I = \frac{3}{56}$

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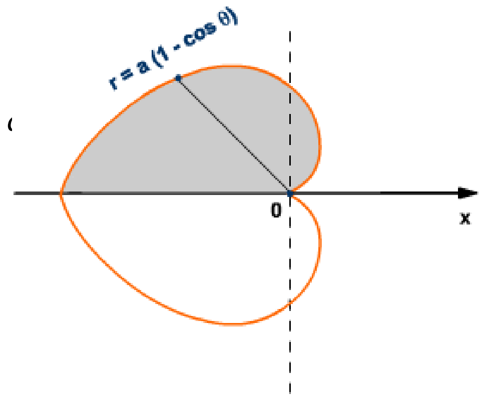
9. Evaluate $\iint r\sin\theta \; dr d\theta$ over the cardiode $r = a(1-\cos\theta)$ above the initial line.

Solution:



$$\iint r\sin\theta \; dr d\theta = \int_{\theta=0}^{\pi}\left\{\int_{r=0}^{r=a(1-\cos\theta)} r\sin\theta dr\right\}d\theta \;=\int_{0}^{\pi}\sin\theta\left[\frac{r^2}{2}\right]_0^{a(1-\cos\theta)}$$

$$=\frac{a^2}{2}\left\{\int_0^{\pi}\sin\theta d\theta - \int_0^{\pi}\cos^2\theta(-\sin\theta)d\theta - \int_0^{\pi}\sin 2\theta d\theta\right\}$$

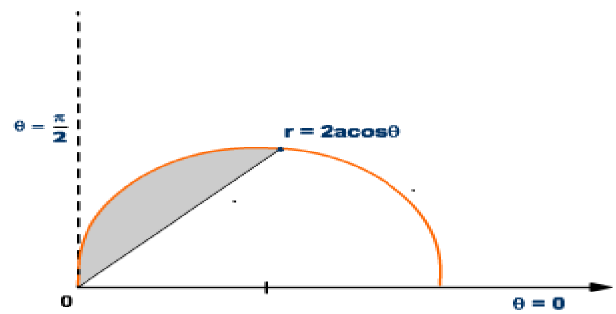
$$=\frac{a^2}{2}\left\{-\cos\theta - \frac{\cos^3\theta}{3} + \frac{\cos 2\theta}{2}\right\}_0^{\pi} = \frac{4\text{a}^2}{3}.$$

10. If R is the region bounded by the circle $r = 2a\cos\theta$ above the initial line show that $\iint_R r^2\sin\theta \, dr d\theta = \frac{2}{3}a^3$

Solution:

The circle $r = 2a\cos\theta$ is $r^2 = 2ar\cos\theta$ in Cartesian form it is equivalent to $x^2 + y^2 - 2ax = 0$

i.e., a circle with centre at $(a,0)$ and passing through the origin.



$$\iint_R r^2\sin\theta \; dr \; d\theta = \int_{\theta=0}^{\pi/2}\left\{\int_{r=0}^{2a\cos\theta}\sin\theta r^2 dr\right\}d\theta$$

$$=\int_{\theta=0}^{\pi/2}\sin\theta\left[\frac{r^3}{3}\right]_0^{2a\cos\theta} d\theta \;= \int_0^{\pi/2}\frac{\sin\theta}{3}\left[8a^3\cos^3\theta\right]d\theta$$

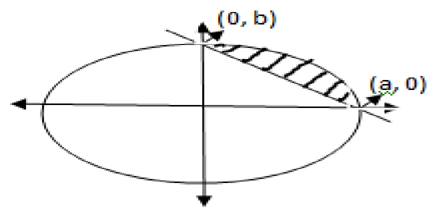
$$=\frac{-8a^3}{3}\int_0^{\pi/2}\cos^3\theta\,.(-\sin\theta)\,d\theta = -\frac{8a^3}{3}\left.\frac{\cos^4\theta}{4}\right]_0^{\pi/2}$$

$$=\frac{2a^3}{3}$$

11. **Evaluate** $\iint xy\,dx\,dy.$ **taken over the region bounded ellipse** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ **and** $\frac{x}{a}+\frac{y}{b}=1$

Solution:

x varies from 0 to a



$\frac{x}{a}+\frac{y}{b}=1$ or $y=\frac{b}{a}(a-x)$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or $y^2=\frac{b^2}{a^2}(a^2-x^2)$ or $y=\frac{b}{a}\sqrt{(a^2-x^2)}$

$$\mathrm{I}=\iint_R xy\,dx\,dy=\int_{x=0}^{a}\int_{y=\frac{b}{a}(a-x)}^{y=\frac{b}{a}\sqrt{(a^2-x^2)}} xy\,dy\,dx$$

$$=\int_{x=0}^{a} x\left[\frac{y^2}{2}\right]_{y=\frac{b}{a}(a-x)}^{y=\frac{b}{a}\sqrt{(a^2-x^2)}} dx=\frac{1}{2}\int_{x=0}^{a} x\left\{\frac{b^2}{a^2}\sqrt{(a^2-x^2)}-\frac{b^2}{a^2}(a-x)^2\right\}dx$$

$$=\frac{b^2}{2a^2}\int_0^a(a^2x-x^3-a^2x+2ax^2-x^3)\,dx=\frac{b^2}{2a^2}\int_0^a 2(ax^2-x^3)\,dx$$

$$=\frac{b^2}{2a^2}\left[a\frac{x^3}{3}-\frac{x^4}{4}\right]_0^a=\frac{a^2b^2}{12}$$

12. **Evaluate** $\iint_R x^2y\,dxdy$ **where** R **is the region bounded by the lines** $y=x, y+x=2$ **&** $y=0$

Solution:

The lines $y=x, y+x=2$ intersect at $(1,1)$

$$I=\iint_R x^2y\,dxdy=\int_{y=0}^{1}\int_{x=y}^{2-y} x^2y\,dxdy=\int_{y=0}^{1} y\left[\frac{x^3}{3}\right]_{x=y}^{2-y} dy$$

$$=\frac{1}{3}\int_{y=0}^{1} y\left\{(2-y)^3-y^3\right\}dy=\frac{1}{3}\int_{y=0}^{1} y(8-12y+6y^2-y^3-y^3)dy$$

$$=\frac{1}{3}\int_{y=0}^{1}(8y-12y^2+6y^3-2y^4)dy=\left[4y^2-4y^3+\frac{3}{2}y^4-2\frac{y^5}{5}\right]_{y=0}^{1}$$

$$=\frac{1}{3}\left(4-4+\frac{3}{2}-\frac{2}{5}\right)=\frac{11}{30}$$

13. **Evaluate** $\iint xy\,dxdy$ **over the positive quadrant of the circle** $x^2+y^2=a^2$

Solution:

$$I=\iint_R xy\,dxdy=\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy\,dxdy$$

$$I=\int_{x=0}^{a} x\left[\frac{y^2}{2}\right]_{y=0}^{\sqrt{a^2-x^2}}dx \;=\frac{1}{2}\int_{x=0}^{a} x(a^2-x^2)dx=\frac{1}{2}\left[a^2\frac{x^2}{2}-\frac{x^4}{4}\right]_{x=0}^{a}=\frac{1}{2}\left[\frac{a^4}{2}-\frac{a^4}{4}\right]=\frac{a^4}{8}$$

1.3 Properties of Double Integral

I. $$\iint_R kf(x,y)dxdy=k\iint f(x,y)dxdy$$

II. $$\iint_R\{f(x,y)\pm g(x,y)\}dxdy=\iint_R f(x,y)dxdy\pm\iint_R g(x,y)dxdy$$

III. If $R = R_1 + R_2$ (R is the union of two non overlapping regions R_1 and R_2)

$$\iint_R f(x,y)dxdy=\iint_{R_1} f(x,y)dxdy+\iint_{R_2} f(x,y)dxdy$$

1.4 Change of order of Integration

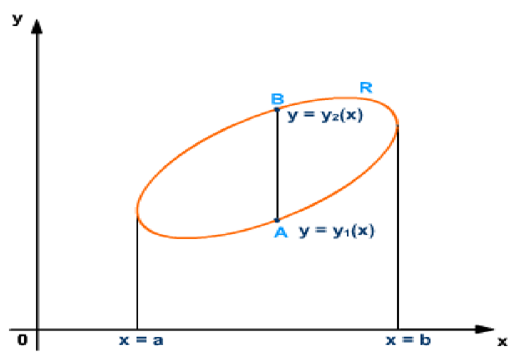
In case the double integral $\int_{x=a}^{x=b}\left\{\int_{y=y_1(x)}^{y=y_2(x)} f(x,y)dy\right\}dx$ is difficult to integrate with respect to y or we prefer to integrate with respect to x first we may change the order of integration by changing the limits of integration of the variables.

1.4.1 Evaluation of double integrals by change of order of integration.

To change the order of integration we follow the following procedure.

Step 1: Given Integral $\int_{x=a}^{x=b}\int_{y=y_1(x)}^{y=y_2(x)} f(x,y)dydx$ Observe that Integral is to be evaluated by integrating with respect to y.

Step 2: Determine the points of intersection by drawing a vertical step



Step 3: To reverse the order of integration draws a horizontal strip in the closed region R.

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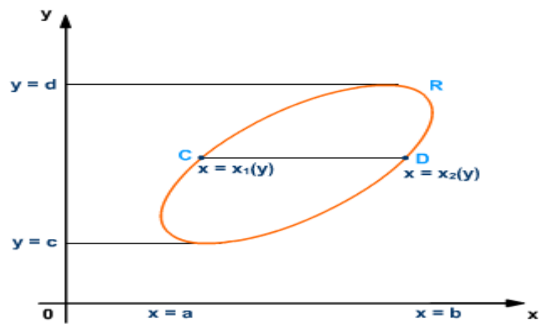


Write the limits of x from $x = x_1(y)$ to $x = x_2(y)$ in the inner integral and limits of y from $y = c$ to $y = d$ in the outer integral.

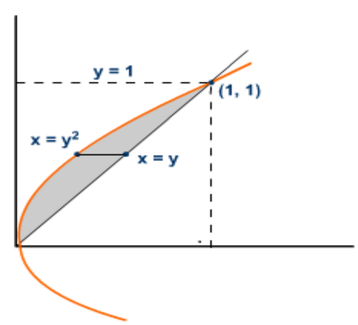
Note: Similar method can be used for changing order of integration for $\int_{y=c}^{y=d} \left\{ \int_{x=x_1(y)}^{x=x_2(y)} f(x,y)dx \right\} dy$

14. Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy\ dydx$ by changing the order of integration.

Solution:

Given $\int_0^1 \left\{ \int_x^{\sqrt{x}} xydy \right\} dx$

The region is bounded by $y = x, y = \sqrt{x}$ and $x = 0, x = 1$. since y is expressed as a function of x .By changing the order of integration we integrate w.r.t. x first and then evaluate the integral by integrating with respect to y .



Consider a horizontal strip in the Region R . In this region

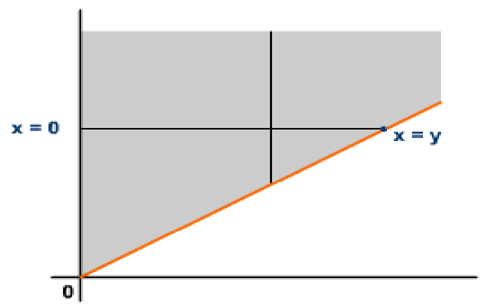
x varies from $x = y^2$ to $x = y$ and y varies from $y = 0$ to $y = 1$.

Now the double integral

$$\int_0^1\left\{\int_x^{\sqrt{x}} xy\,dy\right\}dx = \int_{y=0}^{y=1}\left\{\int_{x=y^2}^{x=y} xy\,dx\right\}dy = \int_{y=0}^{y=1}\left[\frac{x^2}{2}y\right]_{y^2}^{y} dy = \int_0^1\left[\frac{y^3}{2}-\frac{y^5}{2}\right]dy = \left[\frac{y^4}{8}-\frac{y^6}{12}\right]_0^1 = \frac{1}{24}$$

15. Change the order of integration and evaluate $\int_0^\infty\int_0^x xe^{-x^2/y}dydx$.

Solution:



Given double integral $\int_{x=0}^{\infty}\left\{\int_{y=0}^{y=x} xe^{-x^2/y}dy\right\}dx$

In the given region vertical strip is consider to change the order of integration of horizontal strip is to be considered.
Now x varies from $x = y$ to $x = \infty$ and y varies from 0 to ∞

$$\therefore \quad \int_{x=0}^{\infty}\left\{\int_{y=0}^{\infty} x\,e^{-x^2/y}\,dy\right\}dx = \int_{y=0}^{\infty}\left\{\int_{x=y}^{\infty} xe^{-x^2/y}\,dx\right\}dy = \int_{y=0}^{\infty}\left\{\int_{x=y}^{\infty}\left(-\frac{y}{2}\right)e^{-x^2/y}\left(-\frac{2x}{y}\right)dx\right\}dy$$

$$= \int_{y=0}^{\infty}\left\{\left(-\frac{y}{2}\right)\left[e^{-x^2/y}\right]_y^{\infty}\right\}dy \quad = \int_{y=0}^{\infty}\left(\frac{-y}{2}\right)\left[0-e^{-y}\right]dy = \frac{1}{2}\int_0^{\infty} y e^{-y}\,dy = \frac{1}{2}\left\{\left[y\frac{e^{-y}}{-1}\right]_0^{\infty} - \left[1.e^{-y}\right]_0^{\infty}\right\}$$

$$= \frac{1}{2}$$

16. Change the order of integration and evaluate $\int_0^1\left\{\int_{x=\sqrt{y}}^{x=2-y} xy\,dx\right\}dy$

Solution:

Given double integral $\int_0^1\left\{\int_{x=\sqrt{y}}^{x=2-y} xy\,dx\right\}dy$

Region is determined by $x^2 = y$ and $x + y = 2$ on $y = 0$ to $y = 1$. The Region R be union of two non over lapping Regions R_1 and R_2. In Region R_1, y varies from $y = 0$ to $y = x^2$ and x varies from $x = 0$ to $x =$ 1.

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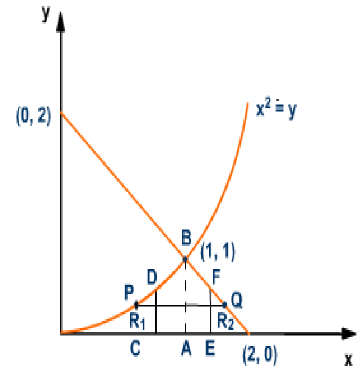


$$\therefore \quad \iint_{R_1} xy\,dx\,dy = \int_{x=0}^{1}\left\{\int_{y=0}^{y=x^2} xy\,dy\right\}dx$$

$$=\int_0^1\left[\frac{xy^2}{2}\right]_0^{x^2} dx = \frac{1}{2}\int_0^1 x^5\,dx = \frac{1}{12}$$

In Region R_2, y varies from $y=0$ to $y=2-x$ and x varies from $x=1$ to $x=2$

$$\therefore \quad \iint_{R_2} xy\,dx\,dy = \int_{x=1}^{x=2}\left\{\int_{y=0}^{y=2-x} xy\,dy\right\}dx = \int_{x=1}^{2}\left\{\frac{xy^2}{2}\right\}_0^{2-x} dx = \int_1^2 \frac{x(2-x)^2}{2}\,dx = \frac{5}{24}$$

$$\therefore \quad \iint_{R} xy\,dx\,dy = \iint_{R_1} xy\,dx\,dy + \iint_{R_2} xy\,dx\,dy = \frac{1}{12}+\frac{5}{24} = \frac{7}{24}$$

17. **Change the order of integration and evaluate** $\int_0^{\infty}\int_x^{\infty}\frac{e^{-y}}{y}dydx$

Solution:

The area of integration is the portion of the first quadrant between $y = x$ and the y-axis.

So, by changing the order of integration.

$$\int_{x=0}^{\infty}\int_{y=x}^{\infty}\frac{e^{-y}}{y}\,dx\,dy = \int_{y=0}^{\infty}\int_{x=0}^{y}\frac{e^{-y}}{y}\,dx\,dy = \int_0^{\infty}\frac{e^{-y}}{y}\left[x\right]_0^y dy = \int_0^{\infty} e^{-y}dy = \left[-e^{-y}\right]_0^{\infty} = 1$$

18. **Change the order of integration and evaluate** $\int_0^{a}\int_0^{2\sqrt{ax}} x^2 dydx$

Solution:

$$y = 2\sqrt{ax} \Rightarrow y^2 = 4ax, \text{ where } x = a \text{ on } y^2 = 4ax,\ y^2 = 4a^2 \Rightarrow y = \pm 2a$$

So, on $y = 2\sqrt{ax}$, $y = 2a$ when $x = a$

The integral is over shaded region.

$$\int_0^a \int_0^{2\sqrt{ax}} x^2 \; dy \; dx = \int_{y=0}^{2a} \int_{x=\frac{y^2}{4a}}^{a} x^2 \; dy \; dx = \int_0^{2a} \left[\frac{x^3}{3} \right]_{\frac{y^2}{4a}}^{a} dy$$

$$= \int_0^{2a} \left(\frac{a^3}{3} - \frac{y^6}{192a^3} \right) dy = \left[\frac{a^3}{3} y - \frac{y^7}{192a^3 * 7} \right]_0^{2a} = \frac{4}{7} a^4$$

19. **Change the order of integration and hence evaluate** $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dydx$

Solution:

$$I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} y^2 \, dydx$$

On changing the order of integration we have from the figure

$$I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} y^2 \, dxdy$$

$$I = \int_{y=0}^{1} y^2 \, [x]_{x=0}^{\sqrt{1-y^2}} \, dy = \int_{y=0}^{1} y^2 \sqrt{1-y^2} \, dy$$

Put $y = \sin\theta \therefore dy = \cos\theta d\theta$ and θ varies from 0 to $\frac{\pi}{2}$

$$I = \int_{\theta=0}^{\pi/2} \sin^2\theta \cos^2\theta d\theta = \frac{(1)(1)}{(4)(2)} \frac{\pi}{2} = \frac{\pi}{16} \text{ (using Reduction formula)}$$

1.5 Change of Variables

The evaluation of a single integral is simplified by making a proper substitution (or change the variable). Similarly to evaluation of double integrals can also be evaluated by changing the variables from (x, y) to (u, v)

Let $x = x(u,v)$ and $y = y(u,v)$, Now $f(x, y) = F(u,v)$ {expressed in terms of u and v } and the region R in xy plane is transformed to R' in the uv plane. With these transformations the double integral will be transformed as

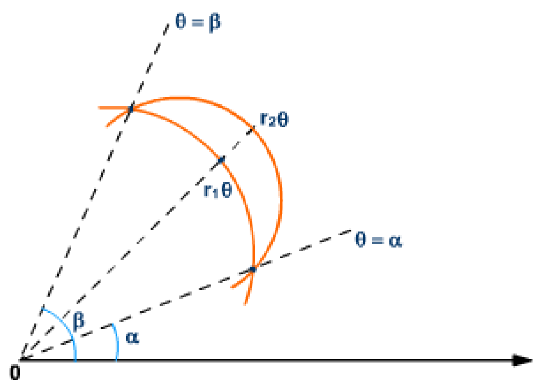
$$\iint_R f(x, y)dxdy = \iint_{R'} F(u,v)Jdudv, \text{ where } J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial(x, y)}{\partial(u,v)}$$

1.5.1 Evaluation of double integrals by changing to Polar form.

In case of polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ and hence $J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

$$\iint_R f(x, y)dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)r\,dr\,d\theta$$

$$\iint_R f(r,\theta)drd\theta = \int_{\theta=\alpha}^{\theta=\beta} \left\{ \int_{r=r_1(\theta)}^{r=r_2(\theta)} f(r_1,\theta)dr \right\} d\theta$$

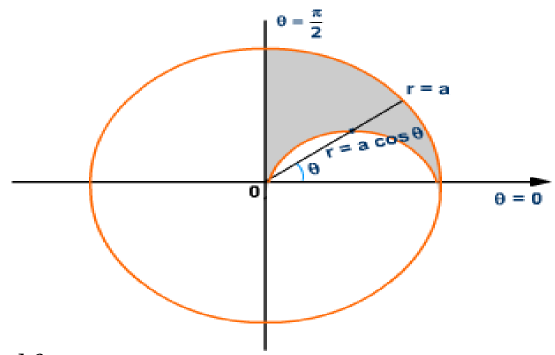


By changing the order of integration

$$\iint_R f(r,\theta)drd\theta = \int_{r=r_1}^{r=r_2} \left\{ \int_{\theta=\theta_1(r_1)}^{\theta=\theta_2(r_2)} f(r,\theta)d\theta \right\} dr$$

20. **Evaluate** $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{xy}{x^2+y^2} e^{-(x^2+y^2)} dxdy$ **by changing to polar form.**

Solution:



Region of Integration is

$y=\sqrt{ax-x^2} \Rightarrow x^2+y^2=ax \Rightarrow r=a\cos\theta$

$y=\sqrt{a^2-x^2} \Rightarrow x^2+y^2=a^2 \Rightarrow r=a$

x varies from 0 to a $\therefore$ θ varies from 0 to $\frac{\pi}{2}$

Now while transforming to polar coordinates $dxdy = rdrd\theta$

$$\therefore \int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{xy}{(x^2+y^2)} e^{-(x^2+y^2)} dxdy = \int_{\theta=0}^{\theta=\pi/2} \int_{r=a\cos\theta}^{r=a} \frac{r^2 \sin\theta\cos\theta}{r^2} e^{-r^2} rdrd\theta$$

$$= \int_{\theta=0}^{\pi/2} \left\{ \frac{1}{2} \int_{r=a\cos\theta}^{r=a} \sin 2\theta\, r\, e^{-r^2} dr \right\} d\theta = \frac{1}{4} \int_{\theta=0}^{\pi/2} \left\{ \int_{r=a\cos\theta}^{r=a} \sin 2\theta\, (2r) e^{-r^2} dr \right\} d\theta = \frac{1}{4} \int_{\theta=0}^{\pi/2} \left[\frac{e^{-r^2}}{-1} \right]_{a\cos\theta}^{a} \sin 2\theta\; d\theta$$

$$= \frac{-1}{4} \int_{\theta=0}^{\pi/2} \sin 2\theta \left[e^{-a^2} - e^{-a^2\cos^2\theta} \right] d\theta = -\frac{1}{4} e^{-a^2} \int_{\theta=0}^{\pi/2} \sin 2\theta\; d\theta + \frac{1}{4} \int_{\theta=0}^{\pi/2} e^{-a^2\cos^2\theta} \sin 2\theta\; .d\theta$$

$$= \frac{1}{8} e^{-a^2} .\cos 2\theta \Big]_0^{\pi/2} + \frac{1}{4a^2} \left[e^{-a^2\cos^2\theta} \right]_0^{\pi/2} = \frac{1}{8} e^{-a^2} [\cos\pi - \cos\theta] + \frac{1}{4a^2} \left[1 - e^{-a^2} \right]$$

$$= \frac{1}{8} e^{-a^2} (-1-1) + \frac{1}{4a^2} (1-e^{-a^2}) = \frac{1}{4a^2} \left\{ 1 - (a^2+1) e^{-a^2} \right\}$$

21. **Evaluate** $\iint_R \frac{(x^2+y^2)}{x^2y^2} dxdy$ **where** R **is common to** $x^2+y^2=ax,\quad x^2+y^2=by,$ $a \neq b,\ b \neq 0,\ a > b > 0.$

Solution:

Convert to polar form using $x = r\ cos\theta$, $y = r\ sin\theta\ dxdy = rdrd\theta$

The region of integration common to $r = a\ cos\theta$ and $r = b\ sin\theta$ is shaded in figure above.

We have $\tan\theta = \frac{a}{b} \Rightarrow \theta = \tan^{-1}\frac{a}{b}$

The Region R is divided into R_1 and R_2

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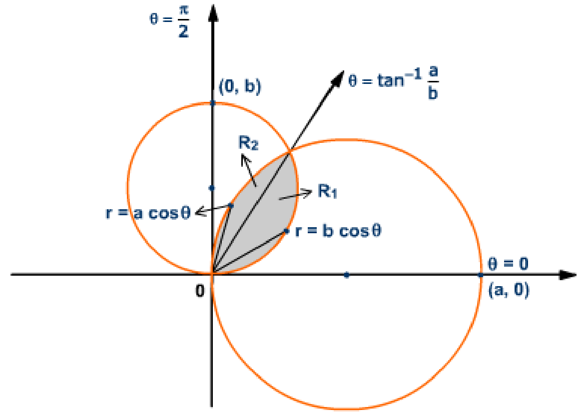

In R_1 r varies from 0 to $r = b\ sin\theta$ and

θ varies from 0 to $\theta = \tan^{-1}\dfrac{a}{b}$

Similarly in R_2 varies, r varies from 0 to $r = a\ cos\theta$

and θ varies from $\tan^{-1}\dfrac{a}{b}$ to $\dfrac{\pi}{2}$.

$$\iint_R \frac{(x^2+y^2)}{x^2y^2}\,dx\,dy = \iint_{R_1}\frac{(x^2+y^2)^2}{x^2y^2}\,dx\,dy + \iint_{R_2}\frac{(x^2+y^2)^2}{x^2y^2}\,dx\,dy$$

$$\iint_{R_1}\frac{(x^2+y^2)^2}{x^2y^2}\,dx\,dy = \int_{\theta=0}^{\tan^{-1}a/b}\int_{r=0}^{b\sin\theta}\frac{(r^2)^2}{r^2+\sin^2\theta\cos^2\theta}\,dr\,d\theta$$

$$= \int_{\theta=0}^{\tan^{-1}a/b}\int_{r=0}^{b\sin\theta} 4.r\frac{\cos e^2 2\theta}{4}\,dr\,d\theta = \int_{\theta=0}^{\tan^{-1}b\ a/b}\left\{\frac{2\ \cos ec^2\ 2\theta}{\theta}\ r^2\right\}_0^{b\sin\theta} d\theta$$

$$= \int_{\theta=0}^{\tan^{-1}a/b}\frac{2}{8}\left[b^2\sin^2\theta : \frac{1}{4\sin^2\theta\cos^2\theta}\right]d\theta = \frac{1}{2}b^2\tan\theta\Big]_{\theta=0}^{\tan^{-1}\frac{a}{b}} = \frac{a}{2b} = \frac{ab}{2}$$

$$\iint_{R_2}\frac{(x^2+y^2)^2}{x^2y^2}\,dx\,dy = \int_{\theta=\tan^{-1}a/b}^{\pi/2}\int_{r=0}^{r=a\cos\theta}\frac{(r^2)^4}{r^4\sin^2\theta\cos^2\theta}\,r\,dr\,d\theta$$

$$= \int_{\theta=\tan^{-1}a/b}^{\pi/2}\left\{\int_{r=0}^{a\cos\theta}\frac{r^2}{2\sin^2\theta\cos^2\theta}\right\}dr\ d\theta = \int_{\theta=\tan^{-1}a/b}^{\pi/2}\left[\frac{r^2}{2\sin^2\theta\ \cos^2\theta}\right]_0^{a\cos\theta} d\theta$$

$$= \int_{\theta=\tan^{-1}a/b}^{\pi/2}\frac{a^2\ \cos^2\theta}{2\sin^2\theta\cos^2\theta}\ d\theta = \frac{a^2}{2}\int_{\theta=\tan^{-1}a/b}^{\pi/2}\cos ec^2\theta\ d\theta = \frac{a^2}{2}\left[-\cot\theta\right]_{\theta=\tan^{-1}\frac{a}{b}}^{\pi/2}$$

$$= \frac{a^2}{2}\left[0+\cot\left(\tan^{-1}\frac{a}{b}\right)\right] = \frac{a^2}{2}\cot\left(\cot^{-1}\frac{b}{a}\right) = \frac{a^2}{2}\cdot\frac{b}{a} = \frac{ab}{2}$$

$$\therefore \iint_R\frac{(x^2+y^2)}{x^2y^2}\,dx\,dy = \frac{ab}{2}+\frac{ab}{2} = ab.$$

22. Evaluate $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\, dxdy$ by changing into polar coordinates.

Solution: In polars we have $x = r\cos\theta$, $y = r\sin\theta$

$x^2 + y^2 = r^2$ and $dxdy = rdrd\theta$, Since x, y varies from 0 to ∞, r also varies from 0 to ∞.

In the first quadrant θ varies from 0 to $\Pi/2$. Thus I = $\int_{\theta=0}^{\Pi/2}\int_{r=0}^{\infty} e^{-(r^2)}\, rdrd\theta$

Put $r^2 = t \therefore r\ dr = \dfrac{dt}{2}$ t also varies from 0 to ∞

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-t}\,\frac{dt}{2}\, d\theta = \frac{1}{2}\int_{\theta=0}^{\pi/2}\left[-e^{-t}\right]_0^\infty d\theta = \frac{1}{2}\int_{\theta=0}^{\pi/2}(0-1)d\theta = \frac{1}{2}\int_{\theta=0}^{\pi/2} d\theta = \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4}$$

23. Evaluate $\int_0^a\int_0^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\, dx\, dy$ by changing into polar form.

Solution:

$$\text{I} = \int_0^a\int_0^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\, dx\, dy$$

We have $x = \sqrt{a^2 - y^2}$ or $x^2 + y^2 = a^2$ is a circle with centre origin and radius a.

Since, y varies from 0 to a the region of integration is the first quadrant of the circle.

In polars, we have $x = r\cos\theta$, $y = \sin\theta$

$\therefore\ x^2 + y^2 = r^2$ i.e., $r^2 = a^2 \Rightarrow r = a$

Also $x = 0, y = 0$ will give r =0 and hence we can say that r varies from 0 to a. In the first quadrant θ varies from 0 to $\pi/2$, we know that $dxdy = r\, drd\theta$

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$$\text{I} = \int_{r=0}^{a}\int_{\theta=0}^{\pi/2} r\ \sin\theta\ r\ \ r\,drd\theta = \int_{r=0}^{a}\int_{\theta=0}^{\pi/2} r^3 \sin\theta\, drd\theta = \int_{r=0}^{a} r^3\left(-\cos\theta\right)_0^{\pi/2} dr$$

$$= \int_0^a -r^3\left(0-1\right)dr = \left[\frac{r^4}{4}\right]_0^a = \frac{a^4}{4}$$

24. **Change the integral** $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\,dydx$ **into polar coordinates and hence evaluate the same.**

Solution:

Clearly θ varies from 0 to π

If $x = r\cos\theta,\ y = \sin\theta,\ x^2+y^2 = r^2\ i.e., a^2 = r^2 \Rightarrow r = a$

Thus r varies from 0 to a and $dxdy = rdrd\theta$

$$\therefore I = \int_{\theta=0}^{\pi}\int_{r=0}^{a} rr\ \ drd\theta$$

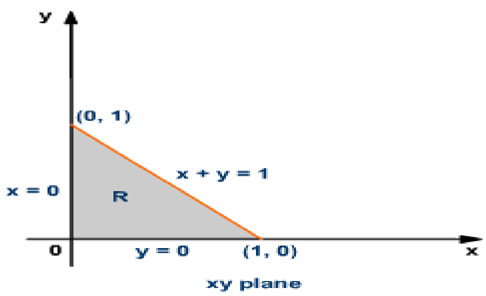
$$\therefore I = \int_{\theta=0}^{\pi}\left[\frac{r^3}{3}\right]_0^a d\theta \therefore = \frac{a^3}{3}\left[\theta\right]_0^{\pi} = \frac{a^3}{3}(\pi-0) = \frac{\pi a^3}{3} \Rightarrow I = \frac{\pi a^3}{3}$$

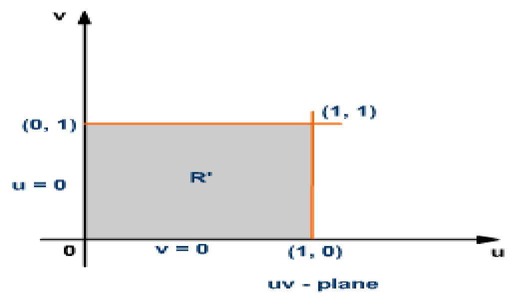
25. **Using the transformation** $x+y=u$ **and** $y=uv$**, Evaluate** $\int_0^1\int_0^{1-x} e^{y/(x+y)}dxdy$

Solution:

$I = \iint_R e^{y/(x+y)}\ dy$ Region R is bounded by x -axis, $x+y=1$ and y varies from $y=0$ to $y=1$.

Solving form x and y we get $x = u\left(1-v\right)$ and $y = uv$





$$\therefore \qquad J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u$$

To determine the region R'.

(i) The x-axis ($y=0$) gives $y=0$, $v=0$ and since $y>0 \Rightarrow u>0$ and $v>0$

(ii) The y-axis, ($x=0$) give $u=0$, $v=1$, since $x>0 \Rightarrow u(1-v)>0 \Rightarrow u>0$ and $v<1$

(iii) The line $x+y=1$ gives $u=1$.

The region R' is bounded by $u=0$, $v=0$, $v=1$ and $u=1$

$$\therefore \iint_R e^{y/x+y}dxdy = \int_{v=0}^{v=1}\left\{\int_{u=0}^{u=1} ue^{uv/u}du\right\}dv = \int_{v=0}^{v=1}\left[\frac{u^2}{2}e^v\right]_0^1 dv = \int_0^1 e^v\left[\frac{1}{2}\right]dv = \frac{1}{2}e^v\Big]_0^1 = \frac{(e-1)}{2}$$

1.6 Double Integral as an Area

(i) We $\iint_A f(x,y)dxdy = \iint_A f(x,y)dA$ for $f(x,\ y)=1$

$$\iint dxdy = \int_A dA = A \text{ (total over } A\text{) of the Region } R$$

(ii) Put $x = rcos\theta,\ y = sin\theta$ then $J = r$

$$\therefore \iint_R dxdy = \iint_{R'} rdrd\theta = A \text{ (total area of the Region } R\text{)}$$

26. Find the area enclosed by the parabolas $x^2 = 4ay$ and $y^2 = 4ax$ by double integration.

Solution:

Let $x^2 = 4ay$ …(1)

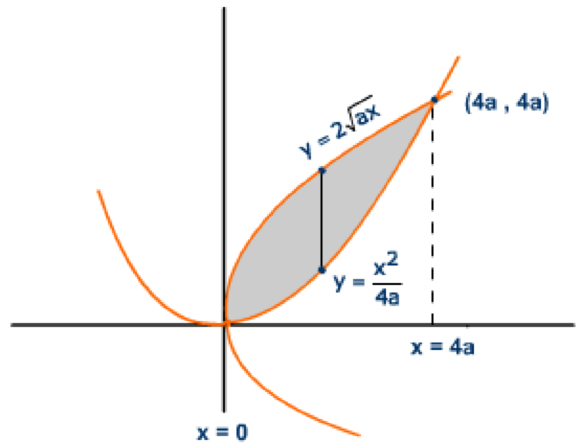
$y^2 = 4ax$ …(2)

Solving (1) and (2) we get the points of intersection $(0,0)$ and $(4a,4a)$

$$\text{Required Area} = \iint_A dxdy = \int_{x=0}^{4a}\left\{\int_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}} dy\right\}dx$$

$$= \int_{x=0}^{4a} y\Big]_{\frac{x^2}{4a}}^{2\sqrt{ax}} dx = \int_0^{4a}\left[2\sqrt{ax}\ 0\ \frac{x^2}{4a}\right]dx$$

$$= 2(ax)^{3/2}\frac{2}{3}\frac{1}{a} - \frac{x^3}{12a}\Big]_0^{4a} = \frac{16a^2}{3}$$



27. Find the area common to the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = 2ax$.

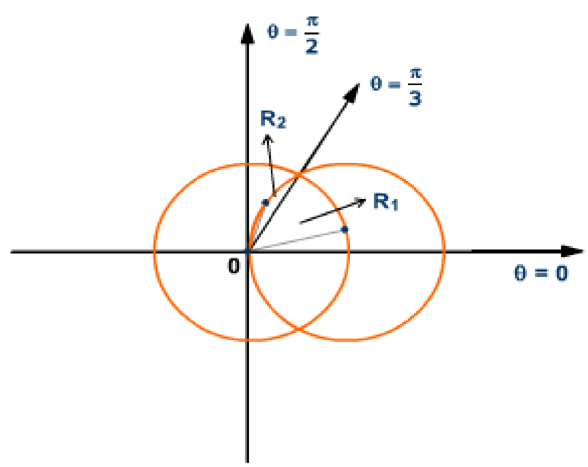
Solution:

Required area is shown in figure.

We have

$x^2 + y^2 = a^2 \quad \Rightarrow \quad r = a$ … (1)

$x^2 + y^2 = 2ax \quad \Rightarrow \quad r = 2a\cos\theta$ … (2)



Solving (1) and (2) $\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

In R_1, r varies from 0 to a and θ varies from 0 to $\frac{\pi}{3}$

In R_2, $= 2\{R_1 + R_2\}$ varies from 0 to $2a\cos\theta$ and θ varies from $\frac{\pi}{3}$ and $\frac{\pi}{2}$

Required Area = $2\{R_1 + R_2\}$

$$=2\int_{\theta=0}^{\pi/3}\left\{\int_{r=0}^{a} r dr\right\} d\theta + 2\int_{\theta=\pi/3}^{\pi/2}\left\{\int_{r=0}^{2a\cos\theta} r\,dr\right\} d\theta = 2.\int_0^{\pi/3} a^2\,d\theta + 2\frac{1}{2}\int_{\pi/3}^{\pi/2} 4a^2\cos^2\theta\,d\theta$$

$$=2a^2.\frac{\pi}{3} + 2a^2\int_{\pi/3}^{\pi/2}(1+\cos 2\theta)\,d\theta = \frac{2a^2\pi}{3} + 2a^2.\theta + \left.\frac{\sin 2\theta}{2}\right]_{\pi/3}^{\pi/2}$$

$$=\frac{2a^2\pi}{3} + 2a^2\left[\frac{\pi}{2} + 0 - \frac{\pi}{3} - \frac{\sqrt{3}}{4}\right] = \frac{2a^2\pi}{3} + \pi a^2 - \frac{2\pi a^2}{3} - \frac{\sqrt{3}a^2}{2} = \pi a^2 - \frac{\sqrt{3}a}{2}$$

28. Find by double integration the area enclosed by th curve $r = a(1+\cos\theta)$ between $\theta = 0$ and $\theta = \pi$

Solution:

Area $A = \iint r\,dr\,d\theta$ where r varies from 0 to $a(1+\cos\theta)$ and θ from 0 to π

$$A = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\;dr\,d\theta = \int_{\theta=0}^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta = \frac{1}{2}\int_{\theta=0}^{\theta=\pi} a^2(1+\cos\theta)^2\,d\theta$$

$$A = \frac{a^2}{2}\int_{\theta=0}^{\pi}\{2\cos^2(\theta/2)\}^2\,d\theta = 2a^2\int_0^{\pi}\cos^4(\theta/2)\,d\theta$$

Put $\theta/2 = \phi$, $d\theta = 2d\phi$ and ϕ varies from 0 to $\pi/2$

$$A = 2a^2 \int_0^{\pi/2} \cos^4 \phi . 2\, d\phi = 4a^2 \int_0^{\pi/2} \cos^4 \phi\, d\phi = 4a^2 . \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2}$$

Thus the required area $A = 3\pi a^2 / 4$ **Sq. units**

1.7 Evaluation of Triple Integrals

On the same lines as we defined the double integral in two dimensions we can define a triple integral over a region R in 3-D and evaluate as follows.

$$\iiint_R f(x,y,z)dv = \iiint_R (x,y,z)dxdydz = \int_{x=a}^{x=b} \int_{y=y_1(x)}^{y=y_2(x)} \int_{z=z_1(x,y)}^{z=z_2(x,y)} f(x,y,z)dzdydx$$

29. **Evaluate** $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$ **.**

Solution:

$$I = \int_{x=0}^{a} \int_{y=0}^{x} \int_{z=0}^{x+y} e^{x+y+z} dzdydx = \int_{x=0}^{a} \int_{y=0}^{x} e^{x+y} \left[e^z\right]_0^{x+y} dy\, dx = \int_{x=0}^{a} \int_{y=0}^{x} e^{2x} . e^{2y} \left[e^{x+y} - 1\right] dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} (e^{2x} . e^{2y} - e^x . e^y) dy\, dx = \int_{x=0}^{a} \left\{ e^{2x} \left[\frac{e^{2y}}{2}\right]_{y=0}^{x} - e^x \left[e^y\right]_{y=0}^{x} \right\} dx = \left[\frac{e^{4x}}{8} - \frac{3e^{2x}}{4} + e^x\right]_0^a$$

$$I = \frac{1}{8}(e^{4a} - 6e^{2a} + 8e^a - 3)$$

30. **Evaluate** $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dy\, dx\, dz$ **.**

Solution :

$$I = \int_{z=0}^{4} \int_{x=0}^{2\sqrt{z}} \int_{y=0}^{\sqrt{4z-x^2}} dydxdz = \int_{z=0}^{4} \int_{x=0}^{2\sqrt{z}} \left[y\right]_0^{\sqrt{4z-x^2}} dx\, dz = \int_{z=0}^{4} \int_{x=0}^{2\sqrt{z}} \sqrt{4z - x^2}\, dx\, dz$$

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Let $4z = a^2$ (for Convinience) so that $2\sqrt{z} = a$

$$= \int_{z=0}^{4}\int_{x=0}^{a}\sqrt{a^2-x^2}\,dx\,dz \quad = \int_{z=0}^{4}\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_{x=0}^{a} dz \quad = \int_{z=0}^{4} 0+\frac{a^2}{2}(\sin^{-1}1-\sin^{-1}0)\,dz$$

$$= \frac{\pi}{2}\int_{z=0}^{4} 2z\,dz \Rightarrow I = 8\pi$$

31. **Evaluate** $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)dzdydx$

Solution:

Since all the limits are constants we can integrate in the given order.

$$\int_{-c}^{c}\int_{-b}^{b}\left(x^2x+y^2z+\frac{z^3}{3}\right)_{-a}^{a} dydx = \int_{-c}^{c}\int_{-b}^{b}\left[x^2(2a)+y^2(2a)+\frac{2a^3}{3}\right]dydx$$

$$= \int_{-c}^{c} 2ax^2y+2a\frac{y^3}{3}+\frac{2a^3}{3}y\Big]_{-b}^{b} dx = \int_{-c}^{c}\left[4ab\,x^2+4a\frac{b^3}{3}+\frac{4a^3b}{3}\right]dx$$

$$= 4ab\ \frac{x^3}{3}+\frac{4ab^3}{3}x+\frac{4a^3bx}{3}\Big]_{-c}^{c} = \frac{8abc^3}{3}+\frac{8ab^3c}{3}+\frac{8a^3bc}{3} = \frac{8abc}{3}(a^2+b^2+c^2)$$

32. **Evaluate** $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)dxdydz$

Solution:

$$\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)dydxdz = \int_{-1}^{1}\int_{0}^{z}\left[xy+\frac{y^2}{2}+zy\right]_{x-z}^{x+z} dxdz \quad = \int_{-1}^{1}\int_{0}^{z}(4xz+2z^2)dxdz$$

$$= \int_{-1}^{1}(2z^3+2z^3)dz = \int_{-1}^{1}4z^3dz = z^4\Big]_{-1}^{1} = 0$$

33. **Evaluate** $\int_{2}^{3}\int_{-1}^{4}\int_{1}^{0}(4x^2y-z^3)dzdydx$

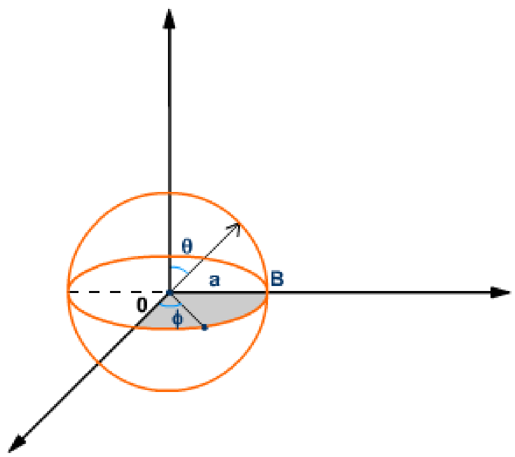
Solution:

$$\int_{2}^{3}\int_{-1}^{4}\int_{1}^{0}(4x^2y-z^3)\,dz\,dy\,dx=\int_{2}^{3}\int_{-1}^{4}\left(4x^2yz-\frac{1}{4}z^4\right)_1^0 dy\,dx=\int_{2}^{3}\int_{-1}^{4}\left(\frac{1}{4}-4x^2y\right)dy\,dx$$

$$=\int_{2}^{3}\left(\frac{1}{4}-4\frac{x^2y^2}{2}\right)_{-1}^{4}dx=\int_{2}^{3}\left(\frac{5}{4}-30x^2\right)dx=\left(\frac{5}{4}x-10x^3\right)_2^3=\frac{-755}{4}$$

34. **Find by triple integration the volume of the sphere of radius a.**

Solution:



$$\iiint_V dx\,dy\,dz\quad=8\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}}dz\,dy\,dx$$

Changing to spherical coordinate system

$$dxdydz=r^2\sin\theta drd\theta d\varphi$$

$$=8\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2}\left.\frac{r^3}{3}\sin\theta\right]_0^a d\phi\,d\theta$$

$$=\frac{8a^3}{3}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2}\sin\theta\,d\phi\,d\theta$$

$$=\frac{8a^3}{3}\int_{\theta=0}^{\pi/2}\left.\sin\theta\,.\,\phi\right]_0^{\pi/2}d\theta$$

$$=\frac{8a^3}{3}\int_{\theta=0}^{\pi/2}\frac{\pi}{2}\sin\theta\,d\theta\quad=\frac{4\pi a^3}{3}\int_0^{\pi/2}\sin\theta\,d\theta=4\frac{\pi a^3}{3}\left[-\cos\theta\right]_0^{\pi/2}=\frac{4\pi a^3}{3}$$

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35. Find the volume of the tetrahedron $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ **and the Coordinate planes.**

Solution:

Volume $$V=\int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)}\int^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dzdydx=\int_{x0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)}[z]_0^{1-\frac{x}{a}-\frac{y}{b}}dydx=\int_{x=0}^{a}\left\{\int_{y=0}^{b\left(1-\frac{x}{a}\right)}\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\right\}dx$$

$$=\int_{x=0}^{a}\left[y-\frac{xy}{a}-\frac{y^2}{2b}\right]_0^{b\left(1-\frac{x}{a}\right)}dx=\int_0^a\left[b\left(1-\frac{x}{a}\right)-\frac{x\left(1-\frac{x}{a}\right)}{a}-\frac{b^2\left(1-\frac{x}{a}\right)^2}{2b}\right]dx=\frac{\text{abc}}{6}$$

36. Find the volume generated by the revolution of the cardiode $r=a(1+\cos\theta)$ **about the initial line.**

Solution:

Volume of the solid of revolution in polars is given by $V=\iint_A 2\pi r^2\sin\theta\, dr\, d\theta$

Recollecting the nature and shape of the cardioide we have

$$V=\int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)}2\pi r^2\; dr\, d\theta \quad =\int_{\theta=0}^{\pi}2\pi\left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)}\sin\theta\, d\theta$$

$$=\frac{2\pi}{3}\int_{\theta=0}^{\pi}a^3(1+\cos\theta)^3\sin\theta\, d\theta$$

Put $1+\cos\theta=t\quad\therefore -\sin\theta\, d\theta=dt$

If $\theta=0,\ t=2;\ \ \theta=\pi,\ t=0$

$$V=\frac{2\pi a^3}{3}\int_2^0 t^3(-dt)=\frac{2\pi a^3}{3}\left[\frac{t^4}{4}\right]_0^2=\frac{8\pi a^3}{3}$$

Thus the required volume $V=8\pi a^3/3$ **Cubic. Units**

37. A pyramid is bounded by three coordinate planes and plane $x+2y+3z=6$. Compute the volume by double integration.

Solution:

$$V=\iint z\,dx\,dy$$

Consider $x+2y+3z=6$. or $\frac{x}{6}+\frac{y}{3}+\frac{z}{2}=1$

We have $z=2[1-(x/6)-(y/3)]$

If $z=0,\ (x/6)+(y/3)=1 \quad \Rightarrow \quad y=3[1-(x/6)]$

If $z=0,\ y=0,\quad then \quad x=6$

$$V=\int_{x=0}^{6}\int_{y=0}^{3[1-(x/6)]} 2[1-(x/6)-(y/3]\,dy\ dx=6$$

Thus the required volume (V)= 6 cubic units

1.8 Orthogonal curvilinear Coordinates

1.8.1 Definitions

Curvilinear coordinates, curvilinear coordinate surfaces and Curvilinear coordinate curves.

Let the coordinates of any point P in space be (x,y,z) in the Cartesian system. Suppose x , y , z are expressible in terms of new coordinates (u_1,u_2,u_3) ,we can say that x , y , z are functions of u_1,u_2,u_3. Let us suppose that we are also in a position to express u_1,u_2,u_3 in terms of x , y , z by solving/eliminating. Then the coordinates (u_1,u_2,u_3) are known as curvilinear coordinates of the point P, where it is assumed that the correspondence between (x,y,z) and (u_1,u_2,u_3) is unique.

The surfaces $u_1=c_1$ and $u_2=c_2$, $u_3=c_3$, c_1,c_2,c_3 being constants are called curvilinear coordinate surfaces and the intersection of each pair of these surfaces give rise to curves called curvilinear coordinate curves.

Note: A system of curvilinear coordinates is said to be orthogonal if at each point the tangents to the coordinate curves are mutually perpendicular.

1.8.2 Unit Vectors, scale factors and orthogonality conditions

Suppose $\vec{r} = xi + yj + zk$ be the position vector of a point in space, we have $\vec{r} = \vec{r}(u_1, u_2, u_3)$.

$\frac{\partial \vec{r}}{\partial u_1}, \frac{\partial \vec{r}}{\partial u_2}, \frac{\partial \vec{r}}{\partial u_3}$ are called the tangent vectors to the coordinate curves and the unit tangent vectors in the same direction are respectively $\hat{e}_1 = \frac{\partial \vec{r}}{\partial u_1} \Big/ \left|\frac{\partial \vec{r}}{\partial u_1}\right|$, $\hat{e}_2 = \frac{\partial \vec{r}}{\partial u_2} \Big/ \left|\frac{\partial \vec{r}}{\partial u_2}\right|$, $\hat{e}_3 = \frac{\partial \vec{r}}{\partial u_3} \Big/ \left|\frac{\partial \vec{r}}{\partial u_3}\right|$.

The quantities $h_1 = \left|\frac{\partial \vec{r}}{\partial u_1}\right|$, $h_2 = \left|\frac{\partial \vec{r}}{\partial u_2}\right|$, $h_3 = \left|\frac{\partial \vec{r}}{\partial u_3}\right|$ are called scale factors

For the orthogonality of the curvilinear coordinate system we must have $\hat{e}_1.\hat{e}_2 = 0$, $\hat{e}_2.\hat{e}_3 = 0$, $\hat{e}_3.\hat{e}_1 = 0$.
These are the analogous to the property of basic unit vectors in the Cartesian system
$\hat{i} \cdot \hat{j} = 0, \hat{j} \cdot \hat{k} = 0, \hat{k} \cdot \hat{i} = 0$. We have $\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$, $\hat{e}_2 \times \hat{e}_3 = \hat{e}_1$, $\hat{e}_3 \times \hat{e}_1 = \hat{e}_2$

Thus $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$, form a right handed system of vectors. If $\vec{A}$ is any vectors in the orthogonal curvilinear coordinates system then $\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$ where A_1, A_2, A_3, are scalar functions of u_1, u_2, u_3.

In addition to the well acquainted rectangular Cartesian coordinates (x, y, z)
We introduce two new set of coordinates.

i. **Cylindrical polar coordinates** (ρ, ϕ, z) given by the transformation:
   $x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z$
ii. **Spherical polar coordinates** given by the transformation
   $x = r\sin\theta\,\cos\phi,\ y = r\sin\theta\,\sin\phi,\ z = r\cos\theta$

We are familiar with the vector differential operator $\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$ and the Laplacian operator

$\nabla^2 = \nabla.\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ operated on scalar and vector point functions

If ψ is a scalar function and $\vec{A}$ is a vector function of (x, y, z) we know that : $\nabla\psi = grad\psi$, $\nabla.\vec{A} = div\ \vec{A}$, $\nabla\times\vec{A} = curl\ \vec{A}$, Laplacian of $\psi = \nabla^2\vec{A}$

1.8.3 Scale factors of the cylindrical and spherical systems

I Cylindrical system

The cylindrical polar coordinates (ρ, ϕ, z) is regarded as a particular case of the general orthogonal curvilinear coordinates (u_1, u_2, u_3) by setting $(u_1, u_2, u_3) = (\rho, \phi, z)$ and are related to the Cartesian coordinates (x, y, z) by the transformation $x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z$

Thus $\vec{r} = xi + yj + zk$ becomes

$$\vec{r} = \rho\cos\phi i + \rho\sin\phi j + zk$$

We have by the definition of scale factors,

$$h_1 = \left|\frac{\partial\vec{r}}{\partial p}\right| = |\cos\phi i + \sin\phi j + 0k| = \sqrt{\cos^2\phi + \sin^2\phi} = 1$$

$$h_2 = \left|\frac{\partial\vec{r}}{\partial\phi}\right| = |-\rho\sin\phi i + \rho\cos\phi j + 0k| = \sqrt{\rho^2(\cos^2\phi + \sin^2\phi)} = \rho^2$$

$$h_3 = \left|\frac{\partial\vec{r}}{\partial z}\right| = |0i + 0j + zk| = \sqrt{0^2 + 0^2 + 1} = 1$$

Thus

$h_1 = 1,\ h_2 = \rho,\ h_3 = 1,$ for the cylindrical system.

II Spherical system

We have $(u_1, u_2, u_3) = (r, \theta, \phi)$ and by the transformation

$$x = r\sin\theta\ \cos\phi,\ y = r\sin\theta\ \sin\phi,\ z = r\cos\theta$$

We have $\vec{r} = r\sin\theta\cos\phi i + r\sin\theta\sin\phi j + r\cos\theta k$

$$h_1 = \left|\frac{\partial\vec{r}}{\partial r}\right| = |\sin\theta\cos\phi i + \sin\theta\sin\phi j + \cos\theta k|$$

$$= \sqrt{\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta} = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$h_2 = \left|\frac{\partial \vec{r}}{\partial \theta}\right| = \left|r\cos\theta\cos\phi i + r\cos\theta\sin\phi j - r\sin\theta k\right|$$

$$= \sqrt{r^2\cos^2\theta(\cos^2\phi + \sin^2\phi) + r^2\sin^2\theta} = \sqrt{r^2(\cos^2\theta + \sin^2\theta)} = r$$

$$h_3 = \left|\frac{\partial \vec{r}}{\partial \phi}\right| = \left|-r\sin\theta\sin\phi i + r\sin\theta\cos\phi j + 0k\right|$$

$$= \sqrt{r^2\sin^2\theta(\cos^2\phi + \sin^2\phi)} = r\sin\theta$$

Thus, $h_1 = 1,\ \ h_2 = r,\ \ h_3 = r\sin\theta,$ for the spherical system.

III Orthogonality of the cylindrical system

We have for the cylindrical system $\vec{r} = \rho\cos\phi i + \rho\sin\phi j + zk$

Let $\hat{e}_\rho, \hat{e}_\phi, \hat{e}_z,$ be the basic unit vectors of this system

They are given by

$$\hat{e}_\rho = \frac{\partial \vec{r}}{\partial \rho}\Big/\left|\frac{\partial \vec{r}}{\partial \rho}\right| = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial \rho} = (\cos\phi + \sin\phi + ok), \qquad \text{since } h_1 = 1,$$

$$\hat{e}_\phi = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial \phi} = \frac{1}{\rho}(-\rho\sin\phi i + \rho\cos\phi j + ok), \qquad \text{since } h_2 = 1,$$

i.e $\hat{e}_\phi = -\sin\phi\, i + \cos\phi\, j + 0k)$

$$\hat{e}_z = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial z} = \frac{1}{1}(0i + 0j + 1k), \qquad \text{since } h_3 = 1,$$

Now $\hat{e}_\rho.\hat{e}_\phi = -\cos\phi\sin\phi + \sin\phi\cos\phi = 0; \quad \hat{e}_\phi.\hat{e}_z = 0, \quad \hat{e}_z.\hat{e}_\rho = 0$

Thus the cylindrical system is orthogonal.

IV Orthogonality of the spherical system

We have for the spherical system $\vec{r} = r\sin\theta\cos\phi i + r\sin\theta\sin\phi j + r\cos\theta\, k$

And let $\hat{e}_r,\ \hat{e}_\theta,\ \hat{e}_\phi,$ be the basic unit vector of this system

Further we have $h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta,$

Now, $\hat{e}_r = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial r} = \sin\theta\cos\phi\, i + \sin\theta\sin\phi\, j + \cos\theta\, k),$

$$\hat{e}_\theta = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial \theta} = \frac{1}{r}(r\cos\theta\cos\phi\, i + r\cos\theta\sin\phi\, j - r\sin\theta\, k),$$

i.e. ., $\hat{e}_\theta = \cos\theta\cos\phi\, i + \cos\theta\sin\phi\, j - \sin\theta\, k),$

$$\hat{e}_\phi = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial \phi} = \frac{1}{r\sin\theta}(-r\sin\theta\sin\phi i + r\sin\theta\cos\phi\, j + 0k),$$

i.e. ., $\hat{e}_\phi = -\sin\phi i + \cos\phi\, j + 0k$

Now $\hat{e}_r.\hat{e}_\theta = \sin\theta\cos\theta(\cos^2\phi + \sin^2\phi) - \sin\theta\cos\theta = 0$

$$\hat{e}_\theta.\hat{e}_\phi = -\cos\theta\cos\phi\sin\phi + \cos\theta\sin\phi\cos\phi = 0$$

$$\hat{e}_\phi.\hat{e}_r = -\sin\theta\cos\phi\sin\phi + \sin\theta\sin\phi\cos\phi = 0$$

Thus the spherical system is orthogonal

V Arc length and volume element in the orthogonal curvilinear coordinate system

We have $\vec{r} = \vec{r}(u_1, u_2, u_3)$ $d\vec{r} = \frac{\partial \vec{r}}{\partial u_1}du_1 + \frac{\partial \vec{r}}{\partial u_2}du_2 + \frac{\partial \vec{r}}{\partial u_3}du_3$ (total derivative)

$$d\vec{r} = h_1 du_1\hat{e}_1 + h_2 du_2\hat{e}_2 + h_3 du_3\hat{e}_3$$

For a curve in space through the point P the arc length ds is given by the relation

$ds = |\vec{dr}|$ i.e $ds = \sqrt{h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2}$

The vector $d\vec{r}$ along the u_1 curve for which u_2 and u_3 are constants given by

$d\vec{r} = h_1 du_1\hat{e}_1$ since $du_2 = 0 = du_3$

Similarly along the u_2 curve u_3 curve we have $h_2 du_2\hat{e}_2$ and $h_3 du_3\hat{e}_3$ respectively

The volume of the rectangular parallelepiped formed by these is called the volume element dV at P in the orthogonal curvilinear coordinate system. Using the geometrical meaning of the scalar triple product of vectors we have,

$$dV = h_1 du_1\hat{e}_1.(h_2 du_2\hat{e}_2 \times h_3 du_3\hat{e}_3) = h_1\, h_2\, h_3\, du_1\, du_2\, du_3 \{\hat{e}_1.(\hat{e}_2 \times \hat{e}_3)\}$$

But $(\hat{e}_2 \times \hat{e}_3) = \hat{e}_1$ and $\hat{e}_1.\hat{e}_1 = 1$ $\therefore$ $dV = h_1\, h_2\, h_3\, du_1\, du_2\, du_3$

Thus $ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$ and $dV = h_1\, h_2\, h_3\, du_1\, du_2\, du_3$

Corollary:

i) In the cylindrical system :

$(u_1, u_2, u_3) = (\rho, \phi, z)$ and $h_1 = 1,\ h_2 = r,\ h_3 = 1,$

$\therefore\ ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2\ ;\ dV = \rho\ d\rho\ d\phi\ dz$

ii) In the spherical system :

$(u_1, u_2, u_3) = (r, \theta, \phi)$ and $h_1 = 1,\ h_2 = \rho,\ h_3 = r\sin\theta,$

$\therefore\ ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi,^2 \quad dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$

1.8.4 Expression for $\nabla\psi$ in orthogonal curvilinear coordinates

Consider a scalar point function $\psi(u_{1,} u_2, u_3)$

Let $\nabla\psi = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$ ....(1)

Where $a_{1,}\ a_{2,}\ a_{3,}$ are to be determined.

We also have $\overrightarrow{r} = \overrightarrow{r}(u_{1,} u_2, u_3)$ and as a total derivative

$$d\vec{r} = \frac{\partial\vec{r}}{\partial u_1}du_1 + \frac{\partial\vec{r}}{\partial u_2}du_2 + \frac{\partial\vec{r}}{\partial u_3}du_3 \ \therefore\ d\vec{r} = h_1 du_1\hat{e}_1 + h_2 du_2\hat{e}_2 + h_3 du_3\hat{e}_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

We have the fact that x, y, z are the functions of $u_{1,}\ u_{2,}\ u_3$ and vice versa

We are also familiar with the result $d\psi = d\vec{r}.\Delta\psi$,

hence $d\psi = a_1h_1du_1 + a_2h_2du_2 + a_3h_3du_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$

But as a total derivative we also have form $\psi = \psi(u_{1,} u_2, u_3)$

$$d\psi = \frac{\partial\psi}{\partial u_1}du_1 + \frac{\partial\psi}{\partial u_2}du_2 + \frac{\partial\psi}{\partial u_3}du_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Equating the R.H.S of (3) and (4) we have

$$a_1h_1 = \frac{\partial\psi}{\partial u_1},\quad a_2h_2 = \frac{\partial\psi}{\partial u_2},\quad a_3h_3 = \frac{\partial\psi}{\partial u_3},\qquad \therefore\ a_1 = \frac{1}{h_1}\frac{\partial\psi}{\partial u_1},\quad a_2 = \frac{1}{h_2}\frac{\partial\psi}{\partial u_1},\quad a_3 = \frac{1}{h_3}\frac{\partial\psi}{\partial u_3},$$

Substituiting these values in (1) we obtain,

$$\nabla\psi = \frac{1}{h_1}\frac{\partial\psi}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial\psi}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial\psi}{\partial u_3}\hat{e}_3 = \sum\frac{1}{h_1}\frac{\partial\psi}{\partial u_1}\hat{e}_1$$

Corollary:

i) In the cylindrical system

$(u_1, u_2, u_3) = (\rho, \phi, z)$ and $h_1 = 1,\ h_2 = r,\ h_3 = 1,$

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{e}_\phi + \frac{\partial\psi}{\partial z}\hat{e}_z$$

ii) In the spherical system:

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{e}_\phi$$

1.8.5 Expression for $div\,\vec{A}$ in orthogonal curvilinear coordinates

Note: The following vector identities are useful for simplification

- $\nabla\times(\nabla\phi) = \vec{0}$
- $\nabla\cdot(\nabla\times\vec{F}) = 0$
- $\nabla\cdot(\phi\vec{F}) = \phi(\nabla\cdot\vec{F}) + (\nabla\phi\cdot\vec{F})$
- $\nabla\times(\phi\vec{F}) = \phi(\nabla\times\vec{F}) + (\nabla\phi\times\vec{F})$
- $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$

Let $\vec{A}_1 = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$

$$div\,\vec{A} = \nabla\cdot\vec{A} = \nabla\cdot(A_1\hat{e}_1) + \nabla\cdot(A_2\hat{e}_2) + \nabla\cdot(A_3\hat{e}_3) \text{ ....(1)}$$

We have $\nabla\psi = \sum \frac{1}{h_1}\frac{\partial\psi}{\partial u_1}\hat{e}_1$ ....(2)

$$\therefore \quad \nabla u_1 = \frac{1}{h_1}\hat{e}_1 + 0 + 0 \text{ i.e } \nabla u_1 = \frac{\hat{e}_1}{h_1},\ \nabla u_2 = \frac{\hat{e}_2}{h_2},\ \nabla u_3 = \frac{\hat{e}_3}{h_3},$$

$$\hat{e}_1 = (\hat{e}_2\times\hat{e}_3) = (h_2\nabla u_2)\times(h_3\nabla u_3) \text{ Or } \hat{e}_1 = h_2h_3(\nabla u_2\times\nabla u_3) \text{ ....(3)}$$

Let us consider only the first term in R.H.S of (1) and proceed as follows

$$\nabla\cdot(A_1\hat{e}_1) = \nabla\cdot\{A_1h_2h_3(\nabla u_2\times\nabla u_3)\}, \text{ by using (3)}$$

$$\text{i.e.,} \quad = \nabla\cdot(\phi\vec{a}) \text{ where } \phi = A_1h_2h_3,\ \vec{a} = (\nabla u_2\times\nabla u_3)$$

$$= \phi(\nabla\cdot\vec{a}) + \vec{a}.\nabla\phi = A_1h_1h_2\{\nabla\cdot(\nabla u_2\times\nabla u_3)\} + (\nabla u_2\times\nabla u_3)\cdot\nabla(A_1h_2h_3)$$

$$= A_1 h_1 h_2 \left\{ \nabla u_3 \cdot \nabla \times (\nabla u_2) - \nabla u_2 \cdot \nabla \times \nabla u_3 \right\} + \frac{\hat{e}_1}{h_1 h_3} \cdot \nabla (A_1 h_2 h_3) = 0 + \frac{\hat{e}_1}{h_1 h_3} . \sum \frac{1}{h_1} \frac{\partial}{\partial u_1} (A_1 h_1 h_2) \hat{e}_1$$

By expanding R.H.S and taking dot product we get,

$$\nabla \cdot (A_1 \hat{e}_1) = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_1} \left(A_1 h_2 h_3 \right)$$

$$\hat{e}_1 . \hat{e}_1 = 1,\ \hat{e}_1 . \hat{e}_2 = 0,\ \hat{e}_1 . \hat{e}_3 = 0$$

Similarly $\nabla \cdot (A_2 \hat{e}_2) = \frac{1}{h_2 h_3 h_1} \frac{\partial}{\partial u_2} \left(A_2 h_3 h_1 \right)$

$$\nabla \cdot (A_3 \hat{e}_3) = \frac{1}{h_3 h_1 h_2} \frac{\partial}{\partial u_3} \left(A_3 h_1 h_2 \right)$$

Adding these results we have,

$$\nabla \cdot (A_1 \hat{e}_1) + \nabla \cdot (A_2 \hat{e}_2) + \nabla \cdot (A_3 \hat{e}_3) = \frac{1}{h_1 h_2 h_3} \sum \frac{\partial}{\partial u_1} \left(A_1 h_2 h_3 \right)$$

Using (1) for the L.H.S. we have

$$\nabla \cdot \vec{A} = div \vec{A} = \frac{1}{h_1 h_2 h_3} \sum \frac{\partial}{\partial u_1} \left(A_1 h_2 h_3 \right)$$

Corollary:

We deduce expression for $\nabla \cdot \vec{A}$ in the cylindrical and spherical system by using the expression for the same in the expanded form

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} \left(A_1 h_2 h_3 \right) + \frac{\partial}{\partial u_2} \left(A_2 h_3 h_1 \right) + \frac{\partial}{\partial u_3} \left(A_3 h_1 h_2 \right) \right\}$$

We have $\left(u_1, u_2, u_3 \right) = (\rho, \phi, z)$ and $h_1 = 1,\quad h_2 = \rho,\quad h_3 = 1$ for the cylindrical system. Hence we obtain

$$\nabla \cdot \vec{A} = \frac{1}{\rho} \left\{ \frac{\partial}{\partial \rho} \left(\rho A_1 \right) + \frac{\partial}{\partial \phi} \left(A_2 \right) + \frac{\partial}{\partial z} \left(\rho A_3 \right) \right\} \qquad \text{(Cylindrical system)}$$

$\left(u_1, u_2, u_3 \right) = (r, \theta, \phi)$ and $h_1 = 1,\quad h_2 = r,\quad h_3 = r \sin\theta$ for the spherical system. Hence we obtain

$$\nabla \cdot \vec{A} = \frac{1}{r^2 \sin\theta} \left\{ \frac{\partial}{\partial r} \left(r^2 \sin\theta A_1 \right) + \frac{\partial}{\partial \theta} \left(r \sin\theta A_2 \right) + \frac{\partial}{\partial \phi} \left(r A_3 \right) \right\} \qquad \text{(spherical system)}$$

1.8.6 Expression for $curl\vec{A}$ in orthogonal curvilinear coordinates

Let $\vec{A}_1 = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$

$curl\vec{A}_1 = \nabla\times\vec{A} = \nabla\times(A_1\hat{e}_1) + \nabla\times(A_2\hat{e}_2) + \nabla\times(A_3\hat{e}_3)$ ....(1)

We have, $\nabla\psi = \sum \frac{1}{h_1}\frac{\partial\psi}{\partial u_1}\hat{e}_1$ $\quad \therefore\ \nabla u_1 = \frac{1}{h_1}\hat{e}_1$ i.e $\hat{e}_1 = h_1\nabla u_1$ ....(2)

we shall consider only the first term in R.H.S of (1) and proceed as follows

$\nabla\times(A_1\hat{e}_1) = \nabla\times(A_1h_1\nabla u_1),$ by using (2)

i.e., $= \nabla\times(\phi\vec{a})$ where $\phi = A_1h_1,\quad \vec{a} = \nabla u_1$

$$= \phi(\nabla\times\vec{a}) + \nabla\phi\times\vec{a} = A_1h_1\{\nabla\times(\nabla u_1)\} + \nabla(A_1h_1)\times\nabla u_1 = 0 + \nabla(A_1h_1)\times\nabla u_1$$

$$= \left\{\frac{1}{h_1}\frac{\partial}{\partial u_1}(A_1h_1)\hat{e}_1 + \frac{1}{h_2}\frac{\partial}{\partial u_2}(A_1h_1)\hat{e}_2 + \frac{1}{h_3}\frac{\partial}{\partial u_3}(A_1h_1)\hat{e}_3\right\}\times\frac{\hat{e}_1}{h_1}$$

Where we have used the expression format of $\nabla\psi$ in the expanded form and (2)

Also, using the fact that $\hat{e}_1\times\hat{e}_1 = 0,\quad \hat{e}_2\times\hat{e}_1 = -\hat{e}_3,\ \hat{e}_3\times\hat{e}_1 = \hat{e}_2$ we have

$$\nabla\times(A_1\hat{e}_1) = \frac{-\hat{e}_3}{h_1h_2}\frac{\partial}{\partial u_2}(A_1h_1) + \frac{\hat{e}_2}{h_1h_3}\frac{\partial}{\partial u_3}(A_1h_1)$$

Similarly by symmetry,

$$\nabla\times(A_2\hat{e}_2) = \frac{-\hat{e}_1}{h_2h_3}\frac{\partial}{\partial u_3}(A_2h_2) + \frac{\hat{e}_3}{h_2h_1}\frac{\partial}{\partial u_1}(A_2h_2)$$

$$\nabla\times(A_3\hat{e}_3) = \frac{-\hat{e}_2}{h_3h_1}\frac{\partial}{\partial u_1}(A_3h_3) + \frac{\hat{e}_3}{h_3h_2}\frac{\partial}{\partial u_2}(A_3h_3)$$

Adding these results, L.H.S becomes $\nabla\times\vec{A}$ according to (1) and R.H.S can be put in the determinant form as follows

Thus $$\nabla\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1h_1 & A_2h_2 & A_3h_3 \end{vmatrix}$$

Corollary:

i) In the cylindrical system: $(u_1, u_2, u_3) = (\rho, \phi, z)$ and $h_1 = 1, \quad h_2 = \rho, \quad h_3 = 1$

and the basic unit vectors are denoted by $\hat{e}_\rho, \ \hat{e}_{\phi,} \ \hat{e}_z$

$$\therefore \quad \nabla \times \vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{e}_\rho & \rho\hat{e}_\phi & \hat{e}_z \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_1 & \rho A_2 & A_3 \end{vmatrix}$$

ii) In the spherical system: $(u_1, u_2, u_3) = (r, \theta, \phi)$ and $h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta$

and the basic unit vectors are denoted by $\hat{e}_r, \ \hat{e}_{\theta,} \ \hat{e}_\phi$

$$\therefore \quad \nabla \times \vec{A} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_1 & rA_2 & r\sin\theta A_3 \end{vmatrix}$$

1.8.7 Expression for $\nabla^2\psi$ (Laplacian of ψ) in orthogonal curvilinear coordinates

We know that $\nabla^2\psi = \nabla.\nabla\psi$ and we have $\nabla\psi = \sum \frac{1}{h_1}\frac{\partial \psi}{\partial u_1}\hat{e}_1$

Also if $\vec{A}_1 = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$ we have $\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3}\left\{\sum \frac{\partial}{\partial u_1}(A_1 h_2 h_3)\right\}$

We need to substitute (1) in (2). That is by taking $\vec{A}_1 = \nabla\psi$ which is equivalent to taking $A_1 = \frac{1}{h_1}\frac{\partial \psi}{\partial u_1}$ since

$\vec{A}_1 = \sum A_1\hat{e}_1$

$$\nabla^2 = \frac{1}{h_1 h_2 h_3}\left\{\sum \frac{\partial}{\partial u_1}\left(\frac{1}{h_1}\frac{\partial \psi}{\partial u_1} h_2 h_3\right)\right\}$$

$$\nabla^2 = \frac{1}{h_1 h_2 h_3}\left\{\sum \frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial u_1}\right)\right\}$$

Thus $\nabla^2 = \dfrac{1}{h_1h_2h_3}\left[\dfrac{\partial}{\partial u_1}\left(\dfrac{h_2h_3}{h_1}\dfrac{\partial\psi}{\partial u_1}\right)+\dfrac{\partial}{\partial u_2}\left(\dfrac{h_3h_1}{h_2}\dfrac{\partial\psi}{\partial u_2}\right)+\dfrac{\partial}{\partial u_3}\left(\dfrac{h_1h_2}{h_3}\dfrac{\partial\psi}{\partial u_3}\right)\right]$

Corollary:

i) $\nabla^2\psi$ in the cylindrical system

we have $(u_1,u_2,u_3)=(\rho,\phi,z)$ and $h_1=1,\quad h_2=\rho,\quad h_3=1$.

$$\nabla^2\psi=\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)+\frac{\partial}{\partial\phi}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\right)+\frac{\partial}{\partial z}\left(\rho\frac{\partial\psi}{\partial z}\right)\right]$$

$$\nabla^2\psi=\frac{1}{\rho}\left[\rho\frac{\partial^2\psi}{\partial\rho^2}+\frac{\partial\psi}{\partial\rho}+\frac{1}{\rho}\frac{\partial^2\psi}{\partial\phi^2}+\rho\frac{\partial^2\psi}{\partial z^2}\right]$$

Thus $\nabla^2\psi=\dfrac{\partial^2\psi}{\partial\rho^2}+\dfrac{1}{\rho}\dfrac{\partial\psi}{\partial\rho}+\dfrac{1}{\rho^2}\dfrac{\partial^2\psi}{\partial\phi^2}+\dfrac{\partial^2\psi}{\partial z^2}$

ii) $\nabla^2\psi$ in the spherical system

In the spherical system we have
$(u_1,u_2,u_3)=(r,\theta,\phi)$ and $h_1=1,\quad h_2=r,\quad h_3=r\sin\theta$.

Substituting in the general expression $\nabla^2\psi$ we get

$$\nabla^2\psi=\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\psi}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\phi}\right)\right]$$

$$\nabla^2\psi=\frac{1}{r^2\sin\theta}\left[\sin\theta\left(r^2\frac{\partial^2\psi}{\partial r^2}+2r\frac{\partial\psi}{\partial r}\right)+\sin\theta\frac{\partial^2\psi}{\partial\theta^2}+\cos\theta\frac{\partial\psi}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]$$

$$\nabla^2\psi=\frac{\partial^2\psi}{\partial r^2}+\frac{2}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

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Exercise:

SI.NO	Questions	Answers
1.	Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx$	$I = \frac{1}{24}$
2.	Evaluate $\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y\, dx\, dy$	$I = \frac{1}{24}$
3.	Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) dy\, dx$	$I = \frac{3}{35}$
4	Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\, dy\, dx$.	$I = \frac{\pi^2}{8}$
5	Evaluate $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z}\, dz\, dy\, dx$.	$I = \frac{\log 256}{3} - \frac{19}{9}$
6	Evaluate. $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dy\, dx\, dz$	$I = 8\pi$
7	Evaluate $\iint_R x\, y\; dx\, dy$ where R is the region bounded by the x-axes and the line $x + y = 1$.	$I = \frac{1}{24}$
8	Evaluate $\iint_R x^2 y\; dx\, dy$ where R is the region bounded by the line $y = x$. $y + x = 2$ and $y = 0$	$\frac{11}{30}$
9	Evaluate $\int_0^1 \int_{x^2}^{2-x} xy\; dy\, dx$ by changing the order of integration.	$I = \frac{3}{8}$
10	Evaluate $\int_{-2}^{2} \int_0^{\sqrt{4-x^2}} (2-x)\, dy\, dx$ by changing the order of integration.	$I = 4\pi$
11	Evaluate $\int_1^2 \int_1^{x^2} (x^2 + y^2)\, dy\, dx$ by changing the order of integration.	$I = \frac{1006}{105}$

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12	Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ by double integration	$I=\pi ab$
13.	Find the area of a plate in the form of a quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$I=\frac{\pi ab}{4}$
14	A pyramid is bounded by three co-ordinate planes and the plane $x+2y+3z=6$. Compute the volume by double integration.	$V=6$ cubic unit
15.	Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$.	$I=16\pi$
16.	Prove that the Spherical system is Orthogonal	
17	Derive the expression for $\nabla\psi$ in orthogonal Curvilinear coordinates	$\sum\frac{1}{h_1}\frac{\partial\psi}{\partial u_1}e_1$
18	Derive the expression for $div\vec{A}$ in orthogonal Curvilinear coordinates	$\frac{1}{h_1h_2h_3}\sum\frac{\partial}{\partial u_1}(A_1h_2h_3)$