Department of Mathematics

Module 2: Vector Integration

- **Line, surface and volume integrals**
- **Green's theorem (without proof) and related problems**
- **Stoke's theorem (without proof) and related problems**
- **Gauss divergence theorem (without proof) and related problems**

2.1 Introduction

Integral calculus of the vector point functions is known as Vector integral calculus. It has applications in fluid flow, design of underwater transmission cables, heat flow in stars, study of satellites etc. Line integrals are useful in the calculation of work done by variable forces along paths in space and the rates at which fluids flow along curves and cross boundaries.

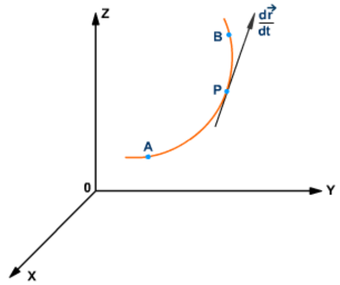
- **Green's theorem** gives a relationship between the line integral of a two-dimensional vector field over a closed path in the plane and the double integral over the region it encloses. The fact that the integral of a (two-dimensional) conservative field over a closed path is zero is a special case of Green's theorem
- **Stokes' Theorem** is identical to Green's Theorem, except one is working with a surface in three dimensions instead of a plane in two dimensions. Stokes' Theorem relates a surface integral to a line integral around the boundary of that surface. Stokes' Theorem can be used to derive several main equations in physics including the Maxwell-Faraday equation, and Ampere's Law.
- **Divergence Theorem** is sometimes called Gauss's Theorem after the great German mathematician Karl Friedrich Gauss (1777–1855). We use the Divergence Theorem to transform the given surface integral into a triple integral

2.2 Integration of Vector functions

Let $\vec{f}$ be a vector field defined over a region R in space and C be a curve in R whose equation is $\vec{r} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$. Let A and B be two points on Curve C corresponding to $t = a$ and $t = b$,

Department of Mathematics

We have $\hat{t} = \dfrac{\dfrac{d\vec{r}}{dt}}{\left|\dfrac{d\vec{r}}{dt}\right|}$



$\int_{AB} \vec{f}\cdot\hat{t}\,ds$. is called the line integral of $\vec{f}$ along curve C between A and B.

Note:

1. The above integral is also represented as $\int_C \vec{f}\cdot\hat{t}\,ds$
2. Since $\hat{t} = \dfrac{d\vec{r}}{ds}$, $\int_C \vec{f}\cdot\hat{t}\,ds = \int_C \vec{f}\cdot d\vec{r}$
3. $\int_C \vec{f}\cdot d\vec{r} = \int_C f_1\,dx + \int_C f_2\,dy + \int_C f_3\,dz$
4. $\int_C \vec{f}\cdot d\vec{r} = \int_a^b \left(f_1 \dfrac{dx}{dt} + f_2 \dfrac{dy}{dt} + f_3 \dfrac{dz}{dt} \right) dt$

2.3 Line Integral

2.3.1 Definition

A line integral of a vector function $F(\vec{r})$ over a curve C is defined by

$$\int_C F(\vec{r}).d\vec{r} = \int_a^b F[r(t)].\frac{dr}{dt}.dt$$

In terms of components with $dr = [dx, dy, dz]$

$$\int_C F(r).d\bar{r} = \int_C (F_1dx + F_2dy + F_3dz)$$

If the path of integration C is a closed curve, then instead of

$\int_C$ we also write $\oint_C$

Here the integral is a scalar, not a vector because we take the dot product.

2.3.2 Physical Meaning

$\int_C \vec{f}\,.d\,\vec{r}$ represents the work done in moving a particle of unit mass along the curve C from A to B.

2.3.3 General properties of the Line Integral

1) $\int_C KF.dr = K\int_C F.dr$

2) $\int_C (F + C_1).dr = \int_C F.dr + C_1.dr$

3) $\int_C F.dr = \int_{C_1} F.dr + \int_{C_2} F.dr$ where $C = C_1 + C_2$.

2.3.4 Path Independence

Let F be a vector function defined on a open region D in space and suppose that for any two points A and B in D the work $\int_A^B F.dr$ done in moving from A to B is the same over all paths joining A to B. Then the integral $\int F.dr$ is path independent in D and the field 'F' is conservative on D.

2.3.5 The Fundamental Theorem of Line Integrals

Let $\vec{F} = M\,\hat{i} + N\,\hat{j} + P\,\hat{k}$ be a vector field whose components are continuous throughout an open connected region 'D' in space. Then there exists a differentiable function $f(\phi)$such that

$$\vec{F} = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

1. If for all points of A and B in D the value of $\int_A^B F.dr$ is independent of path joining A to B in D.
2. If the integral is independent of the path from A to B, its value is

$$\int_A^B F.dr = f(B) - f(A)$$

$F = \nabla f$ implies path independence of the integral.

Proof:

Suppose that A and B are two points in D and C: $r(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$, $a \le t \le b$, is a smooth curve in D joining A and B. Along the curve, f is a differentiable function 't' and

$$\frac{df}{dt} = \frac{\partial f}{\partial x}.\frac{dx}{dt} + \frac{\partial f}{\partial y}.\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$\frac{df}{dt} = \nabla f \cdot .\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right)$$

$$= \nabla f . \frac{dr}{dt}$$

$$= F.\frac{dr}{dt} \because F = \nabla f$$

Now $\int_C F.dr = \int_{t=a}^{t=b} F.\frac{dr}{dt}dt = \int_a^b \frac{df}{dt}.dt$

$$= f\left[g(t),\ h(t),\ k(t)\right]_a^b$$

$$= f(B) - f(A)$$

Thus, the value of the work integral depends only on the values of f at A and B and not on the path in between.

Problems:

1. If $F = \left(3x^2 + 6y\right)\hat{i} + 14yz\,\hat{j} + 20xz^2\,\hat{k}$, evaluate $\int_C \vec{F}.d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the following paths C.

a) $x = t, y = t^2, z = t^3$

b) The straight line joining $(0,0,0)$ and $(1,0,0)$

Solution:

We know that $\int_C F.dr = \int_C F_1 dx + F_2 dy + F_3 dz$

$$\int_C \vec{F}.d\vec{r} = \int_C \left((3x^2 + 6y)\hat{i} - 14yz\,\hat{j} + 20xz^2\hat{k}\right).\ (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \int_C (3x^2 + 6y)\,dx - 14yzdy + 20xz^2dz$$

Department of Mathematics

a) If $x = t, y = t^2, z = t^3$, points $(0,0,0)$ and $(1,1,1)$ correspond to $t = 0$ and $t = 1$ respectively, then

$$\int_C \vec{F}.d\vec{r} = \int_{t=0}^{1} \left(9t^2 dt - 28t^6 dt + 60t^9\right)dt = \left[3t^3 - 4t^7 6t^{10}\right]_0^1 = 5 \text{ units}$$

b) Along the straight line from $(0,0,0)$ to $(1,0,0)$, $y = 0, z = 0, dy = 0, dz = 0$ while x varies from 0 to 1, then the integral over this part of the path is

$$\int_C \vec{F}.d\vec{r} = \int_0^1 3x^2\,dx = \int_{r=0}^{1} 3t^2 dt = \left[x^3\right]_0^1 = 1$$

2 If $\vec{F} = xy\hat{i} + yz\,\hat{j} + zx\,\hat{k}$, evaluate $\int_C \vec{F}\cdot d\,\vec{r}$, where C is the curve represented by $x = t, y = t^2, z = t^3, -1 \le t \le 1$.

Solution:

Given $\vec{F} = xy\hat{i} + yz\,\hat{j} + zx\,\hat{k}$ and $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ will give $d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$\therefore \vec{F}\cdot d\vec{r} = xy\,dx + yz\,dy + zx\,dz$

Since $x = t, y = t^2, z = t^3$ by data, we get

$dx = dt, dy = 2t\,dt, dz = 3t^2 dt$

Thus $\vec{F}\cdot d\vec{r} = t^3\,dt + t^5(2t)\,dt + t^4(3t^2)\,dt$

$\therefore \vec{F}\cdot d\vec{r} = (t^3 + 2t^6 + 3t^6)\,dt = \left(t^3 + 5t^6\right)dt$

$$\int_C \vec{F}\cdot d\,\vec{r} = \int_{t=-1}^{1} \left(t^3 + 5t^6\right)dt = \left[\frac{t^4}{4}\right]_{-1}^{1} + 5\left[\frac{t^7}{7}\right]_{-1}^{1}$$

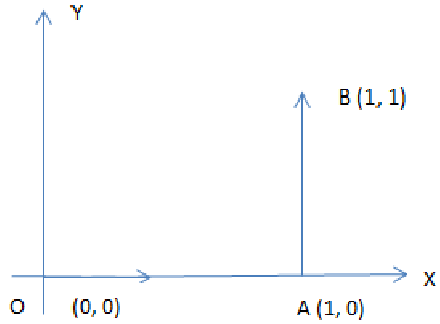
Thus $\int_C \vec{F}\cdot d\,\vec{r} = \frac{10}{7}$ units

3 Evaluate $\int_C \vec{F}.d\vec{r}$ where $\vec{F} = xyi + (x^2 + y^2)j$ along

(i) the path of the straight line from $(0,0)$ to $(1,0)$ and then to $(1,1)$

(ii) the straight line joining the origin and $(1,2)$.

Solution:



Department of Mathematics

$$\int_C \vec{F}.d\vec{r} = \int_C xydx + (x^2 + y^2)dy \qquad \ldots (1)$$

(i) $$\int_C \vec{F}.d\vec{r} = \int_{OA} \vec{F}.d\vec{r} + \int_{AB} \vec{F}.d\vec{r} \qquad \ldots (2)$$

Along OA: $y = 0$

$\therefore$ $dy = 0$ and $0 \le x \le 1$

From (1) $\int_{OA} \vec{F}.d\vec{r} = 0$ .... (3)

Along AB : $x = 1 \therefore dx = 0$ and $0 \le y \le 1$ Again from (1)

$$\int_{OA} \vec{F}.d\vec{r} = \int_{y=0}^{1} 0 + (1+y\ \ )dy = \left[y + \frac{y^3}{3} \right]_{y=0}^{1} = 1 + \frac{1}{3} = \frac{4}{3} \ldots\ldots\ldots (4)$$

Using (3) and (4) in (2) we obtain $\int_{OA} \vec{F}.d\vec{r} = 0 + \frac{4}{3} = \frac{4}{3}$

(ii) C is the straight line joining $(0,0)$ *and* $(1,2)$

The equation of the line given by $\frac{y-0}{x-0} = \frac{2-0}{1-0}$

That is $y = 2x \therefore dy = 2dx$ and $0 \le x \le 1$

Hence from (1)

$$\int_{OA} \vec{F}.d\vec{r} = \int x \cdot 2x\,dx + (x^2 + 4x^2)2xdx \text{ Thus } \int_{OA} \vec{F}.d\vec{r} = \int_{x=0}^{1} 12x^2\,dx = 12\left[\frac{x^3}{3}\right]_{x=0}^{1} = 4$$

4. **Let** $F(x,y) = (3+2xy)\hat{i} + (x^2 - 3y^2)\hat{j}$ **. Find a function** f **such that** $\nabla f = F$ **. Also evaluate** $\int_C \vec{F} \cdot d\vec{r}$ **, where C is the curve given by** $r(t) = e^t \sin t\,\hat{i} + \cos t\,\hat{j}$ **,** $t \in [0, \pi]$**.**

Solution:

As $\nabla f = F$, we have $f_x(x,y) = 3 + 2xy$.

Integrating with respect to x, we get $f(x,y) = 3x + x^2 y + g(y)$,

where $g(y)$ is an integration constant, but it could be a function of y.

Thus $f_y(x,y) = x^2 + g'(y)$ so that $x^2 + g'(y) = x^2 - 3y^2$. That is $g'(y) = -3y^2$.

Integrating $g'(y)$ with respect to y, we obtain $g(y) = -y^3 + k$, where k is a constant. Consequently, $f(x, y) = 3x + x^2 y - y^3 + k$.

Since F is conservative, the line integral $\int_C \vec{F} \cdot d\vec{r}$ is independent of path. In fact, f is a potential function for F. Thus by the fundamental theorem for line integrals, we have $\int_C \vec{F} \cdot d\vec{r} = f(0,-1) - f(0,1) = 2$

5. If $\vec{F} = x^2 i + xy\, j$ evaluate $\int_{OA} \vec{F} . d\vec{r}$ from $(0,0)\ to (1,2)$ along

(i) The line $y = x$

(ii) the parabola $y = \sqrt{x}$

Solution:

(i) Along $y = x \therefore dy = dx$ and $0 \le x \le 1$

$$\vec{F} \cdot d\vec{r} = \int_{x=0}^{1} x^2 dx + \int_{x=0}^{1} x^2 dx = \int_{x=0}^{1} 2x^2 dx = \left[\frac{2x^3}{3}\right]_0^1 = \frac{2}{3}$$

(ii) Along $y = \sqrt{x} \therefore 2y\, dy = dx$ and $0 \le y \le 1$

$$\vec{F} \cdot d\vec{r} = \int_{y=0}^{1} 2y^5\, dy + \int_{y=0}^{1} y^3 dy = \left[\frac{y^6}{3}\right]_0^1 + \left[\frac{y^4}{4}\right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

Note: Work done by a Force

A natural application of the line integral is to define the work done by a force F in moving particle along a curve C from a point 'a' to 'b' as

W = Work done = $\int_a^b \vec{F} . d\vec{r}$

6. Find the work done in moving a particle once around a circle C in the xy plane, if the circle has centre at origin and radius 3 and if the force field is given by $F = (2x - y + z)\hat{i} + (x + y - z^2)\hat{j} + (3x - 2y + 4z)\hat{k}$.

Solution:

In the plane $z = 0$ and $dy = 0$ so that the work done is

$$W = \int_C \vec{F} . d\vec{r} = \int_C [(2x - y)i + (x + y)\, j + (3x - 2y)k].[dxi + dyj]$$

$$= \int_C (2x - y)\,dx + (x + y)\,dy$$

The parametric equations of the circle are $x = 3\cos t, y = 3\sin t$ where 't' varies from 0 to 2π

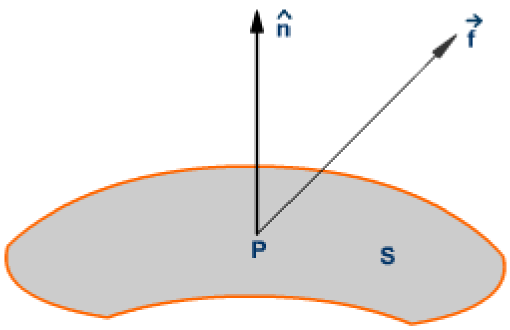
$$\therefore W = \int_0^{2\pi} (6\cos t - 3\sin t)(-3\sin t)\,dt + (3\cos t + 3\sin t)(3\cos t)\,dt$$

$$= \int_0^{2\pi} (9 - 9\sin t\,\cos t)\,dt = \left[9t - \frac{9}{2}\sin^2 t\right]_0^{2\pi} = 18\pi$$

Note: If C in clockwise direction, the value of the integral would be -18π.

2.4 Surface Integral of a vector function

The surface integral of $\vec{f}$ over a surface S is defined as $\int_S \vec{f}\,.\hat{n}\,dS$ where $\hat{n}$ is the unit normal to the surface S and $dS = dxdy$



2.4.1 Physical Meaning:

The surface integral of $\vec{f}$ gives the total normal flux through a surface.

2.4.2 Volume integral of a vector function

The volume integral $\vec{f}$ over a volume V is defined as

$$\int_V \vec{f}\,dV = \left(\int_V f_1 dV\right)\hat{i} + \left(\int_V f_2\,dV\right)\hat{j} + \left(\int_V f_3 dV\right)\hat{k}$$

2.5 Integral Theorems

2.5.1 Green's Theorem (Statement only)

Let $M(x,y)$ and $N(x,y)$ be two functions defined in a region A in the xy plane with a simple closed curve C as its boundary then

$$\int_C (M\ dx + Mdy) = \iint_A \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

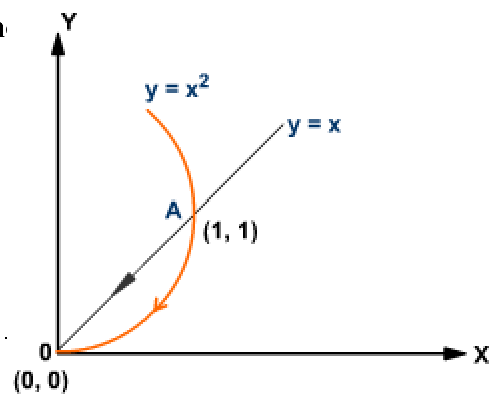
Problems:

7. Verify Green's Theorem in the plane for $\int_c (xy + y^2)dx + x^2 dy$. c is the closed curve of the region bounded by $y = x, y = x^2$.

Solution:

Let $y = x$ and $y = x^2$

Finding the points of intersection we get $(0,0)$ and $(1,1)$ as th



Let $M = xy + y^2 \Rightarrow \dfrac{\partial M}{\partial y} = x + 2y$, $N = x^2$, $\dfrac{\partial N}{\partial x} = 2x$

In RHS. for the region R, y varies between $y = x^2$ to $y = x$ and x varies from 0 and 1(we are integrating w r t y first)

$$\text{RHS} = \int_{x=0}^{x=1} \int_{y=x^2}^{y=x} (2x - x - 2y)dydx = \int_{x=0}^{x=1} \int_{y=x^2}^{y=x} (x - 2y)dydx = \int_{x=0}^{1} \Big(xy$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{-1}{20}$$

In LHS:C is $AO + \overrightarrow{OA}$

Along $\overline{OA}$ $y = x^2$; $dy = 2x\,dx$ and x varies from 0 to 1.

$$\int_{OA} (M\ dx + Mdy) = \int_0^1 (x.x^2 + x^4)dx + x^2.2xdx = \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1 = \frac{19}{20}$$

Along AO, $y = x$ $dy = dx$ and x varies from 0 to 1.

$$\int_{AO}(M\ dx + Mdy) = \int_1^0 \left[\left(x\cdot x + x^2\right)dx + x^2 dx\right]dx = \int_1^0 3x^2 dx = -1$$

$$\int_C (M\ dx + Mdy) = \int_{OA}(M\ dx + Mdy) + \int_{AO}(M\ dx + Mdy) = \frac{19}{20} - 1 = \frac{-1}{20}$$

$\therefore$ Green's Theorem is verified.

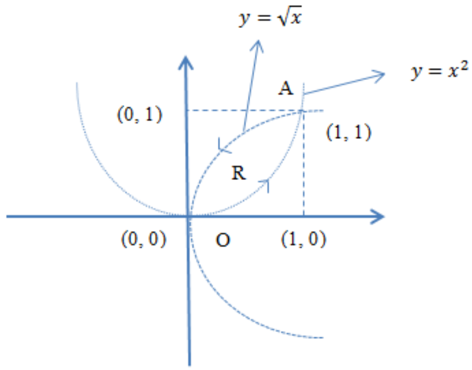
8. **Verify Green's theorem in a plane for** $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ **where C is the boundary of the region enclosed by** $y = \sqrt{x}$ and $y = x^2$.

Solution:

We shall find the points of intersection of the parabola's $y = \sqrt{x}$ and $y = x^2$

i.e., $\sqrt{x} = x^2 \Rightarrow x = x^4$ or $x(x^3 - 1) = 0$

$\therefore$ $x = 0,1$ and hence $y = 0,1$. The points of intersection are $(0,0)$ and $(1\ 1)$



Let $M = 3x^2 - 8y^2, \quad N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y; \quad \frac{\partial M}{\partial x} = -6y$$

We have Green's theorem in a plane

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

$$L.H.S = \oint_C Mdx + Ndy$$

$$= \int_{OA} Mdx + Ndy + \int_{AO} Mdx + Ndy = I_1 + I_2$$

(Say)

Along OA: $Y = x^2, \ dy = 2x\,dx, \ \ x$ varies from 0 to 1.

$$I_1 = \int_{x=0}^{1} (3x^2 - 8x^4)dx + (4x^2 - 6x^3)2x\,dx$$

$$I_1 = \int_{x=0}^{1} (3x^2 + 8x^3 - 20x^4)dx = \left[x^2 + 2x^4 - 4x^5\right]_0^1$$

Along OA: $y = \sqrt{x}\ \ OR\ \ x = y^2 \Rightarrow, dx = 2y\,dy, \ \ y$ varies from 1 to 0.

$$I_2 = \int_{y=1}^{0} (3y^4 - 8y^2)2y\,dy + (4y\ \ - 6y^3)\,dy$$

$$I_2 = \int_{y=1}^{0} (4y - 22y^3 + 6y^5)dy = \left[2y^2 - \frac{11}{2}y^4 + y^6\right]_1^0$$

$$I_2 = 0 - \left(2 - \frac{11}{2} + 1\right) = \frac{5}{2}$$

Hence L.H.S $= I_1 + I_2 = -1 + \frac{5}{2} = \frac{3}{2}$

Also, R.H.S $= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (-6y + 16y)\, dy\, dx$$

$$= \int_{x=0}^{1} 10\left[\frac{y^2}{2}\right]_{y=x^2}^{\sqrt{x}} dx$$

$$= 5\int_{x=0}^{1} (x - x^4)\, dx$$

$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = 5\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{2}$$

We have L.H.S = R.H.S and Theorem is verified.

9. Verify Green's theorem in a plane for $\oint_C (x^2 + y^2)dx + 3x^2 y dy$ **where C is the Circle by** $x^2 + y^2 = 4$ **traced in the positive sense.**

Solution:

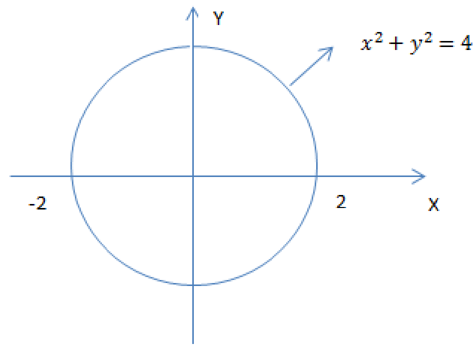
We have Green's theorem

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

L.H.S $= \int_C (x^2 + y^2)dx + 3x^2 ydy$ and the parametric equation of the given circle is

$$x = 2\cos\theta,\ \ y = 2\sin\theta, \qquad 0 \le \theta \le 2\pi$$

L.H.S $= \int_{\theta=0}^{2\pi} 4(-2\sin\theta)\, d\theta + \int_{\theta=0}^{2\pi} 3(4\cos^2\theta)(2\sin\theta)\, d\theta$



$$= 8[\cos\theta]_0^{2\pi} + 48\int_{\theta=0}^{2\pi}\cos^3\theta\sin\theta\, d\theta$$

$$= 8[\cos 2\pi - \cos 0] \; - 48\left[\frac{\cos^4\theta}{4}\right]_0^{2\pi} = 0$$

$$\because \quad \cos 2\pi = 1 = \cos 0$$

Now if $M = x^2 + y^2, \quad N = 3x^2 y$

$$\frac{\partial M}{\partial y} - \frac{\partial M}{\partial x} = (6xy - 2y)$$

Also, R.H.S $= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2y(3x-1)\, dx\, dy$$

$$= \int_{x=0}^{1} 10\left[\frac{y^2}{2}\right]_{y=x^2}^{\sqrt{x}} dx$$

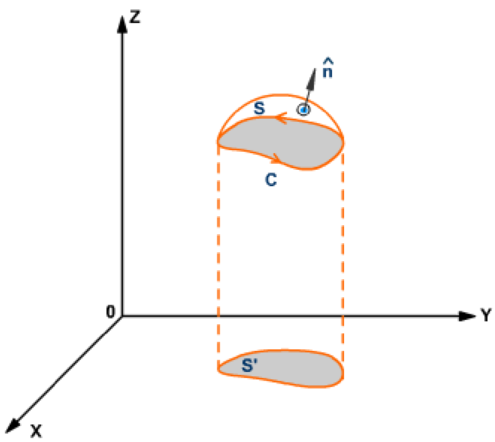
$$= \int_{x=-2}^{2}(3x-1)\left[y^2\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{x=-2}^{2}(3x-1)\left\{(4-x^2) - (4-x^2)\right\} = 0$$

L.H.S = R.H.S and Theorem is verified.

2.5.2 Stokes Theorem (Statement only)

Let S be an open surface bounded by a simple closed curve C for a field $\vec{F}$ defined over a region containing S, then $\int_C \vec{F} \cdot d\vec{r} = \int_S (Curl\ \vec{F}) \cdot \hat{n}\, dS$



10. **Verify Stoke's theorem for** $\vec{A} = (2x - y)\hat{i} - yz^2\hat{j} - y^2 z\hat{k}$, **S is the upper half surface of the sphere** $x^2 + y^2 + z^2 = 1$ **and C is its boundary.**

Solution:

The boundary C of S is a circle in the $x - y$ plane of radius 1 and centre at the origin.

$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

We know $x = \cos t, y = \sin t, z = 0$ are the parametric equations of C. $0 \le t \le 2\pi$

$$\text{LHS} \quad .= \int_C \vec{A}.d\vec{r} = \int_C (2x - y)dx - yz^2 dy - y^2 zdz$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)dt \quad (\because z = 0)$$

$$= \int_0^{2\pi} (-2\sin 2t + \frac{1}{2}(1 - \cos 2t)\, dt \quad (\because z = 0)$$

$$= \pi \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\text{Also, } curl\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= i(-2yz + 2yz) - j(0) + k(0 + 1) = k$$

$$\therefore \overrightarrow{ds} = \hat{n}\, ds = dy\, dz\, i + dz\, dx\, i + dx\, dy\, i$$

$$\text{Hence R.H.S} = \iint_s curl\,\vec{F} \cdot \hat{n}\, ds = \iint dx\, dy = \pi \;\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$\because \iint dx\,dy$ represents the area of the circle $x^2+y^2=1$ which is π

Thus from eq (1) and eq (2) Stoke's theorem is verified.

11. Using Stoke's theorem evaluate $\int_C (x+y)dx+(2x-z)dy+(y+z)dz$ where C is the boundary of the triangle with vertices $(2,0,0),(0,3,0)$ and $(0,0,6)$.

Solution:

Here $\vec{F}=(x+y)\hat{i}+(2x-z)\hat{j}+(y+z)\hat{k}$

$$Curl\,\vec{F}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

Also equation of the plane through A, B, C is

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1 \text{ or } 3x+2y+z=6$$

Vector N normal to this plane is

$$\nabla(3x+2y+z-6)=3\hat{i}+2\hat{j}+\hat{k}$$

$$\therefore \hat{N}=\frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{9+4+1}}=\frac{1}{\sqrt{14}}3\hat{i}+2\hat{j}+\hat{k}$$

Hence $\int_C [(x+y)\,dx+(2x-z)\,dy+(y+z)\,dz]=\int_C \vec{F}\cdot d\vec{r}$

$=\int_S curl\,\vec{F}\cdot\hat{N}\,ds$ where S is the triangle ABC

$$=\int_S \left(2\hat{i}+\hat{k}\right)\cdot\left(\frac{1}{\sqrt{14}}3\hat{i}+2\hat{j}+\hat{k}\right)ds=\frac{1}{\sqrt{14}}(6+1)\int_S ds$$

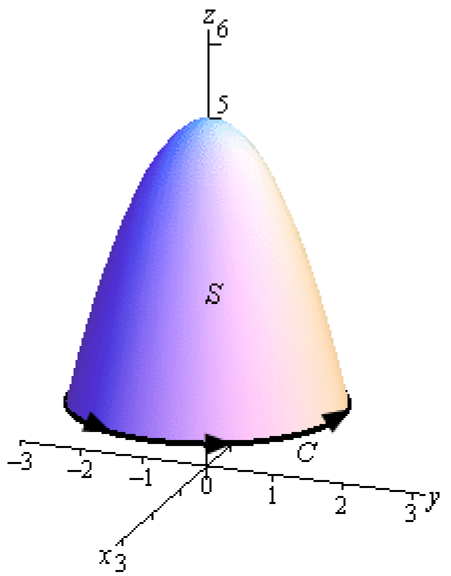
$$=\frac{7}{\sqrt{14}}(\text{Area of the triangle ABC})=\frac{7}{\sqrt{14}}.3\sqrt{14}=21$$

12. Use stokes theorm to evaluate $=\iint_S curl\,\vec{F}\,d\vec{s}$ where $\vec{F}=z^2\,\hat{i}-3xy\,\hat{j}+x^3y\hat{k}$ and *S* is the part of $z=5-x^2-y^2$ above the plane $z=1$. Assume that *S* is oriented upwards.

Solution:

In this case the boundary curve C will be where the surface intersects the plane $z = 1$ and so will be the curve $1 = 5 - x^2 - y^2 \Rightarrow x^2 + y^2 = 4$

So, the boundary curve will be the circle of radius 2 that is in the plane $z = 1$. The parameterization of this curve $\vec{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} + \hat{k}, 0 \le t \le 2\pi$



The first two components give the circle and the third Component makes sure that it is in the plane $z = 1$.

Using Stokes' Theorem we can write the surface integral as the following line integral.

$$\iint_S curl\,\vec{F}\cdot d\vec{s} = \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}\left(\vec{r}(t)\right)\cdot \vec{r}'(t)\,dt$$

So, it looks like we need a couple of quantities before we do this integral. Let's first get the vector field evaluated on the curve. Remember that this is simply plugging the components of the parameterization into the vector field.

$$\vec{F}\left(\vec{r}(t)\right) = (1)^2\,\hat{i} - 3(2\cos t)(2\sin t)\,\hat{j} + (2\cos t)^3(2\sin t)^3\,\hat{k} = \hat{i} - 12\cos t\sin t\,\hat{j} + 64\cos^3 t\sin^3 t\,\hat{k}$$

Next, we need the derivative of the parameterization and the dot product of this and the vector field.

$$\vec{r}'(t) = -2\sin t\,\hat{i} + 2\cos t\,\hat{j}$$

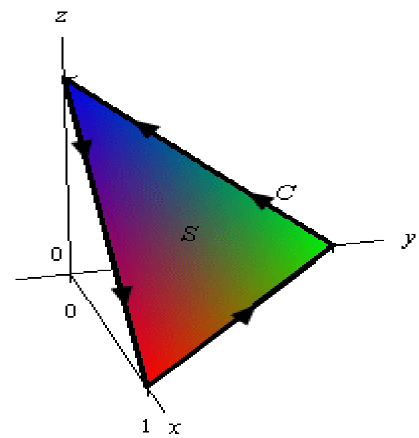
$$\vec{F}\left(\vec{r}(t)\right)\cdot\vec{r}' = -2\sin t - 24\sin t\cos^2 t$$

We can now do the integral

$$\iint_S curl\,\vec{F}\cdot d\vec{s} = \int_0^{2\pi}(-2\sin t - 24\sin t\cos^2 t)\,dt = \left(2\cos t + 8\cos^3 t\right)\Big|_0^{2\pi} = 0$$

13. Use Stokes' Theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = z^2\hat{i} + y^2\hat{j} + x\hat{k}$ and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with counter-clockwise rotation.

Solution:



$$curl\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2 & y^2 & x \end{vmatrix} = 2z\,\hat{j} - \hat{j} = (2z-1)\,\hat{j}$$

Now, all we have is the boundary curve for the surface t surface integral. However, as noted above all we need is a boundary curve. So, let's use the following plane with surface.

Since the plane is oriented upwards this induces the positive direction on C as shown. The equation of this plane is, $x+y+z=1 \Rightarrow g(x,y)=1-x-y$

Now, let's use Stokes' Theorem and get the surface integral set up.

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S curl\,\vec{F}\cdot d\vec{s} = \iint_S (2z-1)\,\hat{j}\cdot d\vec{s} = \iint_D (2z-1)\,\hat{j}\cdot\frac{\nabla f}{\|\nabla f\|}\|\nabla f\|\,dA$$

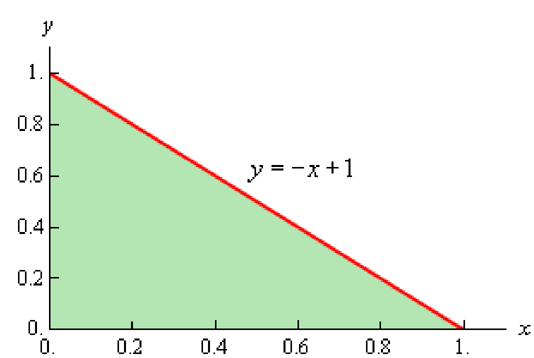
Okay, we now need to find a couple of quantities. First let's get the gradient. Recall that this comes from the function of the surface.

$f(x,y,z) = z - g(x,y) = z - 1 + x + y$

$\nabla f = \hat{i} + \hat{j} + \hat{k}$

Note as well that this also points upwards and so we have the correct direction.

Now, D is the region in the xy-plane shown below,



We get the equation of the line by plugging in $z=0$ into the equation of the plane.

So based on this the ranges that define D are,

$0 \le x \le 1$, $0 \le y \le -x+1$

The integral is then,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D (2z-1)\,\hat{j} \cdot \left(\hat{i} + \hat{j} + \hat{k}\right) dA$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \int_0^{-x+1} 2(1-x-y) - 1\, dy\, dx$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \int_0^{-x+1} 2(1-x-y) - 1\, dy\, dx$$

Don't forget to plug in for z since we are doing the surface integral on the plane. Finishing this out gives,

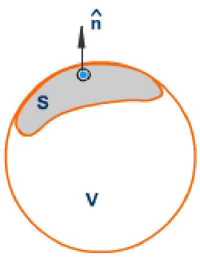
$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \left(y - 2xy - y^2\right)\Big|_0^{-x+1} dx$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \left(x^2 - x\right) dx = \left(\frac{1}{3}x^3 - \frac{1}{2}x^2\right)\Bigg|_0^1 = -\frac{1}{6}$$

2.5.3 Gauss Divergence theorem (Statement only)

Let S be the closed boundary surface of a region of Volume V. Then for a vector field $\vec{F}$ defined in V and on S

i.e., in Cartesian form $\iint_S F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy = \iiint_V \left\{ \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right\} dx\, dy\, dz$

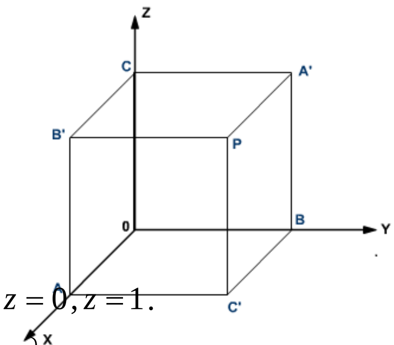


Note: For any vector field $\vec{F}$, and any closed curve S $\int_S Curl\ \vec{F} \cdot \hat{n}\ dS = 0$

Problems:

14. **Verify Gauss divergence theorem** $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ **and S is the surface of the cube bounded by** $x=0, x=1, y=0, y=1, z=0, z=1$**.**

Solution:



Divergence theorem $\iiint_V \nabla.\vec{F}\,dv = \iint_S \vec{F}.\hat{n}\,ds$

In LHS., $\nabla.\vec{F} = 4z - 2y + y = 4z - y$

For the volume V, the limits are, $x=0, x=1, y=0, y=1, z=0, z=1$.

$$\iiint_V \nabla.\vec{F}\,dv = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(4z-y)dxdydz = \int_{x=0}^{1}\int_{y=0}^{1}\left\{\int_{z=0}^{1}(4z-y)dz\right\}dydx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\left(2z^2 - yz\right)_0^1 dydx = \int_{x=0}^{1}\left\{\int_{y=0}^{1}(2-y)dy\right\}dx = \int_{x=0}^{1}\left(2y - \frac{y^2}{2}\right)_0^1 dx = \int_0^1 \frac{3}{2}dx = \frac{3}{2}.$$

For the surface AC'PB' $\hat{n} = \hat{i}, ds = dxdy$. $x=1$, y varies from 0 to 1, z varies from 0 to 1

$$\iint_{AC'PB'} \vec{F}.\hat{n}\,ds = \int_{y=0}^{1}\int_{z=0}^{1}4zdydz = \int_{y=0}^{1}\left\{\int_{z=0}^{1}4zdz\right\}dy = 4\int_{y=0}^{1}\left(\frac{z^2}{2}\right)_0^1 dy = 4\int_{y=0}^{1}\frac{1}{2}dy = \left(4\frac{1}{2}y\right)_0^1 = 2$$

For the surface OBA'C $\hat{n} = -\hat{i}, ds = dxdz$ $y=1$ x varies from 0 to 1, z varies from 0 to 1.

$$\iint_{OBA'C'} \vec{F}.\hat{n}\,ds = \int_0^1\int_0^1 0dydz = 0$$

For the surface BA'PC' $\hat{n} = \hat{j}$, $y=1$ x varies from 0 to 1, z varies from 0 to 1.

$$\iint_{BA'PC} \vec{F}.\hat{n}\,ds = \int_0^1\int_0^1 -y^2dxdz = \int_0^1\int_0^1 -1dxdz = -\int_0^1 [z]_0^1 dx = -\int_0^1 1dx = 1$$

For the surface OCB'A $y=0$, $\hat{n} = -\hat{j}$ x varies from 0 to 1, z varies from 0 to 1

$$\iint_{OCB'A} \vec{F}.\hat{n}\,ds = \int_0^1\int_0^1 0\,dxdz = 0$$

For the surface OAC'B $z=0$, $\hat{n}=-\hat{k}$, x varies from 0 to 1, y varies from 0 to 1

$$\iint_{OAC'B} \vec{F}.\hat{n}\,ds = \int_{x=0}^{1}\int_{y=0}^{1} 0\,dxdy = 0$$

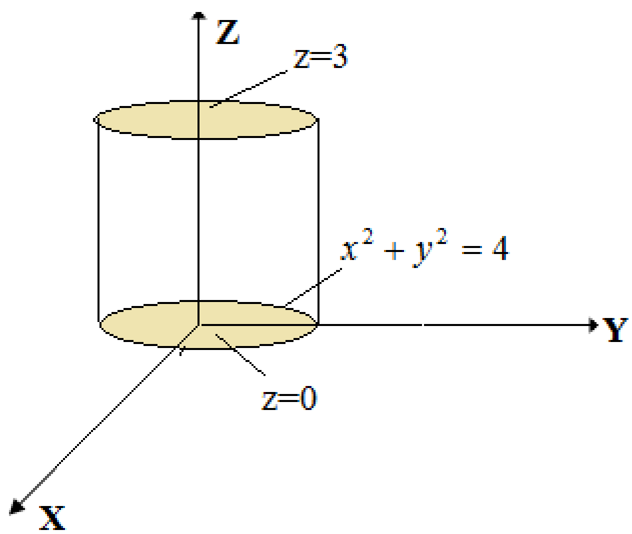
For the surface CB'PA' $z=1$ $\hat{n}=\hat{k}$, x varies from 0 to 1, y varies from 0 to 1

$$\iint_{CB'PA'} \vec{F}.\hat{n}\,ds = \int_{x=0}^{1}\int_{y=0}^{1} y\,dxdy = \int_{x=0}^{1}\left\{\frac{y^2}{2}\right\}_0^1 dx = \int_0^1 \frac{1}{2}dx = \left.\frac{1}{2}x\right)_0^1 = \frac{1}{2}$$

Adding all $2+0-1+0+0+\frac{1}{2}=\frac{3}{2}$

LHS = RHS $\therefore$ Gauss divergence theorem is verified.

15. Evaluate $\int_S \vec{F}\cdot ds$ where $\vec{F}=4x\hat{i}-2y^2\hat{j}+z^2\hat{k}$ and S is the surface bounding the region $x^2+y^2=4, z=0$ and $z=3$.



Solution

$$\int_S \vec{F}\cdot ds = \int_V div\,\vec{F}\,dv$$

$$= \int_V \left[\frac{\partial}{\partial x}(4x)+\frac{\partial}{\partial y}\left(-2y^2\right)+\frac{\partial}{\partial z}\left(z^2\right)\right] d$$

$$= \iiint_V (4-4y \_ 2z)\,dz\,dy\,dx$$

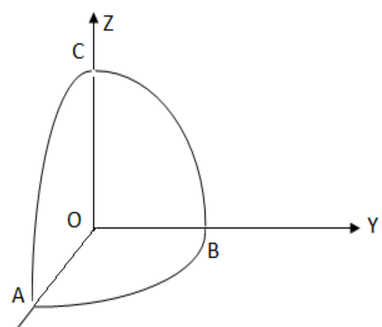
$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^3 (4-4y \_ 2z)\,dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12-12y+9)\,dy\,dx = 42\int_{-2}^{2}\sqrt{4-x^2}\,dx$$

$$= 84\left|\frac{x\sqrt{4-x^2}}{2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right|_0^2 = 84\pi$$

16. Evaluate $\int_S \vec{F}\cdot ds$ where $\vec{F}=yz\hat{i}+zx\hat{j}+xy\hat{k}$ and S is the surface of the sphere $x^2+y^2+z^2=a^2$ in the first octant.

Solution :



The surface of the region V:OABC is piecewise

smooth and is comprised of 4 surfaces

(i) S_1-circular quadrant OBC in the yz-plane
(ii) S_2-circular quadrant OCA in the zx-plane
(iii) S_3-circular quadrant OAB in the xy-plane
(iv) S-Surface ABC of the sphere in the first octant-plane

Also $F = yz\,I + zx\,J + xy\,K$

By divergence theorem,

$$\Rightarrow \int_V div\,\vec{F}\,dv = \int_{S_1} \vec{F}\cdot ds + \int_{S_2} \vec{F}\cdot ds + \int_{S_3} \vec{F}\cdot ds + \int_S \vec{F}\cdot ds \quad \text{..(1)}$$

Now $div\,\vec{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy) = 0.\ For\ s_{1,}x = 0$

$$\int_{S_1} \vec{F}\cdot ds = \int_0^a \int_0^{\sqrt{a^2-y^2}} \left(yz\,\hat{i}\right)\cdot\left(-dydz\,\hat{i}\right) = -\int_0^a \int_0^{\sqrt{a^2-y^2}} (yz)\,dydz = -\frac{a^4}{8}$$

Thus equation (1) becomes

$$\Rightarrow 0 = -\frac{3a^4}{8} + \int_s F.ds$$

Hence,

$$\Rightarrow \int_S \vec{F}\cdot ds = \frac{3a^4}{8}$$

Exercise:

SI.No.	Questions	Answers
1.	Evaluate $\int_c \vec{F}d\vec{r}$ where $\vec{F} = (x^2 - y^2)\hat{i} + xy\hat{j}$ where C is the arc of the	$(824/21)$

Department of Mathematics

	curves $y = x^3$ from $(0,0)$ to $(2,8)$	
2.	Find the total work done in moving a particle in a force field $\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve: $x = t^2 + 1, y = 2t^2, z = t^3$ from $t = 1$ to $t = 2$	303
3.	Verify Greens theorem for $\int_c [(x^2 - \cosh y)dx + (y + \sin x)dy]$ where C is the rectangle with vertices $(0,0),(\pi,0),(\pi,1),(0,1)$	
4	Using Green's Theorem find the area between the parabolas $y^2 = 4x$ and $x^2 = 4y$.	16/3
5	Employ Greens theorem in a plane to show that the area enclosed by the plane curve C is $\frac{1}{2}\oint_C xdy - ydx$ and hence find the area of the Ellipse	πab
6	Verify Stokes theorem for the vector $\vec{F} = (x^2 + y^2)\hat{i} - 2xy\hat{j}$ taken round the rectangle bounded by $x = 0, x = a, y = 0, y = b$	$-2ab^2$
7	Verify Stokes theorem for the vector $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ where S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary	$-\pi$
8	Evaluate $\int_C xydx + xy^2dy$ by stokes theorem where C is the square in the xy plane with vertices $(1,0)$ $(-1,0)$ $(0,1)$ $(0,-1)$	$4/3$
9	Verify Gauss Divergence Theorem for $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ over the unit cube	0
10	Evaluate $\iint_S \vec{F}.\vec{n}ds$ given $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ over the sphere $x^2 + y^2 + z^2 = a^2$	$4\pi a^3$