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Module 3: Laplace Transforms and Inverse Laplace Transform

- **Definition, Laplace transforms of elementary functions,**
- **Properties of Laplace transforms (without proof).**
- **Laplace transform of periodic functions (without proof)**
- **Unit- step function and Dirac 's delta function**
- **Inverse Laplace transform: Definition, transforms of standard functions and properties,**
- **Convolution Theorem (without proof) and evaluation of inverse Laplace transform using convolution theorem**
- **Solving ordinary differential equation using Laplace transform (initial and boundary value problems)**

3.1 Introduction

Laplace transform is an integral transform which is used in physics and engineering for analysis of linear time-invariant systems such as electrical circuits, harmonic oscillators, optical devices, and mechanical systems. It is used to solve differential and integral equations.

The Laplace transform is named after mathematician and astronomer Pierre-Simon Laplace, who used a similar transform (now called Z transform) in his work on probability theory. The current widespread use of the transform came about soon after World War II. In the linear mathematical models for physical systems such as a spring/mass system or a series electrical circuit, the right member or input of the ordinary differential equations $L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{c}=V(t)$, $m\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+kx=f(t)$ is a driving function and represents either an external force $f(t)$ or an external voltage $E(t)$.The driving force may be continuous or discontinuous. For example, the impressed voltage on a circuit could be piecewise continuous and periodic. Solving the differential equation of the circuit in this case is difficult but not impossible. The Laplace transform is powerful tool in solving such problems where the driving force is discontinuous. It is often interpreted as a transformation from the time-domain in which inputs and outputs are functions of time to the frequency-domain, where the same inputs and outputs are functions of complex angular frequency in radians per unit time.

The process of solving linear differential equations using Laplace transforms involves the following steps:

- Apply Laplace transforms to the given differential equation so that it is transformed into an algebraic equation.
- The algebraic equation is solved by purely algebraic manipulations.
- Apply the Laplace transforms to get back solution of given initial value problem.

The main idea which made the Laplace transform, a very powerful technique is that it replaces operations of calculus by operations of algebra. In this chapter we will be looking at how the Laplace transform transforms one class of complicated functions $f(t)$ to produce another class of simpler functions $F(s)$ and also how to use Laplace transforms to solve differential equations.

3.2 Definition of Laplace Transform

Let $f(t)$ be a function of t, $t > 0$. And the Laplace transform of $f(t)$ denoted by $L\{f(t)\} = F(s)$ is defined as $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$, where s is a parameter which may be real or complex.

L is known as Laplace transform operator and it is linear operator with a real argument t $(t \geq 0)$ that transforms $f(t)$ to a function $F(s)$ with real or complex argument s, And the original function given function of $f(t)$ known as determining function depends on while the new function to be determined $F(s)$, called as generating function depends only on s as the improper integral on right hand side of above definition is integrated with respect to t.

3.2.1 Sufficient conditions for the existence of Laplace transform of given function

The Laplace transform of given function $f(t)$ exists if the following conditions are satisfied:

1. $f(t)$ is continuous or piecewise continuous.
2. $\lim\limits_{t\to\infty} \left\{e^{-at} f(t)\right\}$ is finite

Note: $f(t)$ A function $f(t)$ is called piecewise continuous on an interval if the interval can be broken into a finite number of subintervals on which the function is continuous on each open subinterval (i.e. the subinterval without its endpoints) and has a finite limit at the endpoints of each subinterval. The following is the graph of piecewise continuous function.

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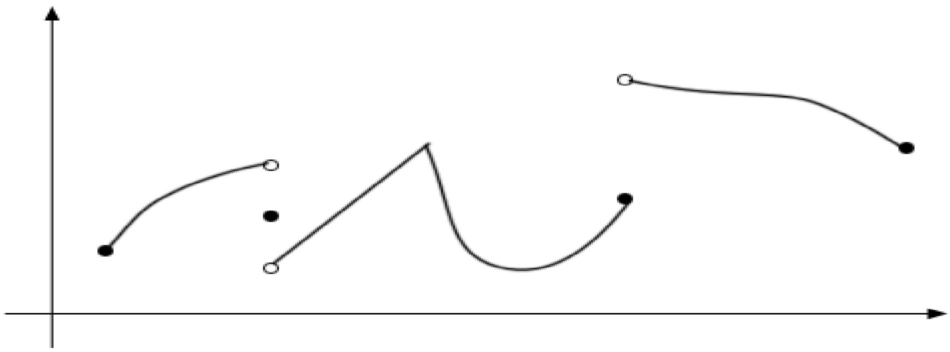

3.2.2 Properties of Laplace Transforms

1. **Linearity property:** If α and β are constants and $f(t)$ and $g(t)$ are functions whose Laplace transforms exist then $L\{\alpha f(t) + \beta g(t)\} = \alpha L\{f(t)\} + \beta L\{g(t)\}$.
2. **Shifting property:** If $L\{f(t)\} = F(s)$ then $L\{e^{at} f(t)\} = F(s-a)$
3. **Change of scale property:** If $L\{f(t)\} = F(s)$ then $L\{f(at)\} = \frac{1}{a} F(s)$.

3.3 Laplace transforms of some Elementary Functions

1. If $f(t) = 1$ then $L\{f(t)\} = \int_0^\infty e^{-st}\, 1.dt = \left.\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}$, $If\ \ s > 0$

$$\therefore L\{1\} = \frac{1}{s},\ s > 0$$

2. If $f(t) = e^{at}$ then $L\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\ dt = \int_0^\infty e^{-(s-a)t}\ dt = \left.\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{1}{s-a}$

$$\therefore L\{e^{at}\} = \frac{1}{s-a},\ s > a$$

3. If $f(t) = \sin at$ then $L\{\sin at\} = \int_0^\infty e^{-st} \sin at\ dt$

$$\int e^{ax} \sin bx\ dx = \frac{e^{ax}}{a^2 + b^2}\left[a \sin bx - b \cos bx\right]$$

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Using this formula $L\{\sin at\} = \int_0^{\infty} e^{-st} \sin at\, dt = \left.\frac{e^{-st}}{s^2+a^2}\left[-s\sin at - a\cos at\right]\right\}_0^{\infty}$

$\therefore L\{\sin at\} = \frac{a}{s^2+a^2}$

4. If $f(t) = \cos at$ then $L\{\cos at\} = \int_0^{\infty} e^{-st}\cos at\, dt$

We have $\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$

Using this formula $L\{\cos at\} = \int_0^{\infty} e^{-st}\cos at\, dt = \left.\frac{e^{-st}}{s^2+a^2}\left[-s\cos at + a\sin at\right]\right\}_0^{\infty}$

$\therefore L\{\cos at\} = \frac{s}{s^2+a^2}$

5. If $f(t) = \sinh at$ then $L\{\sinh at\} = L\left\{\frac{e^{at}-e^{-at}}{2}\right\} = \frac{1}{2}L\{e^{at}\} - \frac{1}{2}L\{e^{-at}\}$

$= \frac{1}{2}.\frac{1}{s-a} - \frac{1}{2}\frac{1}{s+a} = \frac{a}{s^2-a^2}$

$\therefore L\{\sin h\ at\} = \frac{a}{s^2-a^2}\ \ if\ \ s > |a|$

6. If $f(t) = \cosh at$ then $L\{\cos h\ at\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\} = \frac{1}{2}L\{e^{at}\} + \frac{1}{2}L\{e^{-at}\}$

$= \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}.\frac{1}{s+a} = \frac{s}{s^2-a^2}$

$\therefore L\{\cos h\ at\} = \frac{s}{s^2-a^2}\ \ if\ \ s > |a|$

7. If $f(t) = t^n$**,** where n is real number different from non-negative integer then

$L\{t^n\} = \int_0^{\infty} e^{-st}t^n dt \quad put\ st = u \Rightarrow sdt = du = \int_0^{\infty} e^{-u}\left(\frac{u}{s}\right)^n.\frac{du}{s} = \frac{1}{s^{n+1}}\int_0^{\infty} e^{-u}\ u^n\ du$

$\therefore L\{t^n\} = \frac{1}{s^{n+1}}\Gamma(n+1)$

Note: If n is a positive integer $\Gamma(n+1) = n!\ \therefore L\{t^n\} = \frac{n!}{s^{n+1}}$, *if n is a positive integer.*

Problems:

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1. Find the Laplace Transform of $3t^2 - t + 5$.

Solution:

$$L\{3t^2 - t + 5\} = 3L\{t^2\} - L\{t\} + 5L\{1\} = 3\frac{2}{s^3} - \frac{1}{s^2} + 5\frac{1}{s} = \frac{6}{s^3} - \frac{1}{s^2} + \frac{5}{s}$$

2. Find the Laplace Transform of $3e^{-2t} - 2e^{3t}$.

Solution:

$$L\{3e^{-2t} - 2e^{3t}\} = 3L\{e^{-2t}\} - 2L\{e^{3t}\} = 3\frac{1}{s+2} - 2\frac{1}{s-3}$$

3. Find the Laplace Transforms of $\cos^3 2t$.

Solution:

$$L\{\cos^3 2t\} = \frac{1}{4}L\{\cos 6t\} + \frac{3}{4}L\{\cos 2t\}$$

$$= \frac{1}{4}\left\{\frac{s}{s^2+36}\right\} + \frac{3}{4}\left\{\frac{s}{s^2+4}\right\}$$

4. Find the Laplace Transforms of $\sqrt{t}$.

Solution:

$$\mathrm{L}\{\sqrt{\mathrm{t}}\} = L\{t^{1/2}\} = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}} = \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

5. Find the Laplace transform of $f(t) = \begin{cases} e^t & 0 < t < 1 \\ 0 & t > 1 \end{cases}$.

Solution:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st} e^t\,dt + \int_1^\infty e^{-st}(0)\,dt$$

$$= \int_0^1 e^{-(s-1)t}\,dt = \left.\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^1 = \frac{1}{s-1}\left\{1 - e^{-(s-1)}\right\}$$

6. Find the Laplace transform of $f(t) = \begin{cases} t/a & 0 \le t < a \\ 1 & t \ge a \end{cases}$.

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Solution:

$$L\{f(t)\}=\int_0^\infty e^{-st}\ f(t)\ dt=\int_0^a e^{-st}\ \frac{t}{a}\ dt+\int_a^\infty e^{-st}\ (-1)\,dt=\frac{1}{a}\int_0^a e^{-st}\ t\ dt+\int_a^\infty e^{-st}\ dt$$

$$=\frac{1}{a}\left(t\frac{e^{-st}}{-s}-\int\frac{e^{-st}}{-s}\ dt\right)_0^a+\left(\frac{e^{-st}}{-s}\right)_a^\infty=\frac{1}{a}\left[\frac{t\,e^{-st}}{-s}+\frac{1}{s}.\frac{e^{-st}}{-s}\right]_0^a+\left(\frac{e^{-st}}{-s}\right)_a^\infty$$

$$=\frac{1}{a}\left[\frac{ae^{-as}}{-s}-\frac{1}{s^2}e^{-as}+\frac{1}{s^2}\right]+\frac{1}{s}e^{-as}$$

7. Find the Laplace Transforms of $\frac{1}{\sqrt{t}}$.

Solution:

$$\mathrm{L}\left\{\frac{1}{\sqrt{\mathrm{t}}}\right\}=L\left\{t^{-1/2}\right\}=\frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{\frac{-1}{2}+1}}=\frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}=\frac{\sqrt{\pi}}{\sqrt{s}}$$

8. Find the Laplace transform of $f(t)=\begin{cases}\sin 2t & 0<t\le\pi\\ 0 & t>\pi\end{cases}$

Solution:

$$L\{f(t)\}=\int_0^\pi e^{-st}\sin 2t\ dt+\int_\pi^\infty e^{-st}\,(0)\ dt=\frac{e^{-st}}{s^2+4}\{-s\sin 2t-2\cos 2t\}\Big]_0^\pi$$

$$=\frac{1}{s^2+4}\{e^{-\pi s}(-2)-1(-2)\}=\frac{2}{s^2+4}\ (1-e^{-\pi s})$$

9. Find the Laplace transform of $f(t)=\sin 2t\cos 3t$.

Solution:

We have $\sin 2t\cos 3t=\frac{1}{2}[\sin 5t-\sin t]$

$$L\{\sin 2t\cos 3t\}=L\left\{\frac{1}{2}[\sin 5t-\sin t]\right\}=\frac{1}{2}\left[\frac{5}{s^2+25}-\frac{1}{s^2+1}\right]$$

10. Find the Laplace transform of $f(t)=(5e^{3t}-1)^2$

Solution:

$$L\left\{(5e^{3t}-1)^2\right\}=L\{25e^{6t}-10e^{3t}+1\}=\frac{25}{s-6}-\frac{10}{s-3}+\frac{1}{s}$$

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11. Find the Laplace transform of $f(t)=(2t+3)^3+6^t$**.**

Solution:

$$L\left[(2t+3)^3+6^t\right]=L\left[8t^3+27+12t^2+18t+e^{t\log 6}\right]$$
$$=8\frac{3!}{s^4}+\frac{27}{s}+12\frac{2!}{s^3}+\frac{18}{s^2}+\frac{1}{s-\log 6}$$
$$=\frac{48}{s^4}+\frac{27}{s}+\frac{24}{s^3}+\frac{18}{s^3}+\frac{1}{s-\log 6}$$

12. Find the Laplace transform of $f(t)=t^{-\frac{3}{2}}+t^{\frac{3}{2}}$**.**

Solution:

$$L\left[t^{\frac{3}{2}}+t^{-\frac{3}{2}}\right]=\frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}}+\frac{\Gamma\left(-\frac{1}{2}\right)}{s^{-\frac{1}{2}}}$$
$$=\frac{\frac{3}{4}\sqrt{\pi}}{s^{\frac{5}{2}}}\frac{-2\sqrt{\pi}}{s^{-\frac{1}{2}}}$$

13. Find the Laplace transform of $f(t)=\cos(2t+3)+\cos 7t\cos 3t$**.**

Solution:

We have $\cos(2t+3)=$ $\cos 2t\cos 3-\sin 2t\sin 3$, $\cos 7t\cos 3t=+\frac{1}{2}\{\cos 10t+\cos 4t\}$

$$L\left[\cos(2t+3)+\cos 7t\cos 3t\right]=(\cos 3)L[\cos 2t]-(\sin 3)L[\sin 2t]+\frac{1}{2}L\{\cos 10t+\cos 4t\}$$
$$=\frac{s\cos 3}{s^2+4}-\frac{2\sin 3}{s^2+4}+\frac{1}{2}\left\{\frac{s}{s^2+100}+\frac{s}{s^2+16}\right\}$$

14. Find the Laplace transform of $f(t)=1+\cos 2t$

Solution:

$$L[1+\cos 2t]=L[1]+L[\cos 2t]=\frac{1}{s}+\frac{s}{s^2+4}$$

15. Find the Laplace transform of $f(t)=t\sqrt{t}+15t^3+7^t$

Solution:

$$L\left[t\sqrt{t}+15t^3+7^t\right]=L\left[t^{\frac{3}{2}}\right]+15L\left[t^3\right]+L\left[e^{t\log 7}\right]$$

$$=\frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}}+\frac{90}{s^4}+\frac{1}{s-\log 7}$$

$$=\frac{3\sqrt{\pi}}{4s^{\frac{5}{2}}}+\frac{90}{s^4}+\frac{1}{s-\log 7}$$

3.4 Multiplication by t

If $L\{f(t)\}=F(s)$ **then** $L\{t^n f(t)\}=(-1)^n F^{(n)}(s)$

Proof:

$$L\{f(t)\}=\int_0^\infty \mathrm{e}^{-st}\ f(t)dt \ \Rightarrow F(s)=\int_0^\infty e^{-st}\ f(t)dt$$

Differentiate with respect to s, both sides $F'(s)=\frac{d}{ds}\left\{\int_0^\infty e^{-st}\ f(t)dt\right\}$

$$=\int_0^\infty \frac{\partial}{\partial s}\left(e^{-st}\ f(t)\right)dt=\int_0^\infty -t\ e^{-st}\ f(t)dt=-\int_0^\infty e^{-st}t\ f(t)dt$$

$F'(s)=-L\{tf(t)\}$ Therefore true for n = 1.

Assume that the result is true for n=m

$$(-1)^m F^{(m)}(s)=L\{t^m f(t)\}\Rightarrow(-1)^m\ F^{(m)}(s)=\int_0^\infty e^{-st}\ t^m\ f(t)dt$$

Differentiate with respect to s, both sides

$$(-1)^m\ F^{(m+1)}(s)=\frac{d}{ds}\left\{\int_0^\infty e^{-st}t^m\ f(t)\ dt\right\}=\int_0^\infty\frac{\partial}{\partial s}\left\{e^{-st}t^m\ f(t)\right\}dt$$

$$=\int_0^\infty -t\ e^{-st}.\,t^m\ f(t)\ dt=\int_0^\infty e^{-st}.\,t^{m+1}\ f(t)\ dt$$

$$\therefore(-1)^{m+1}F^{(m+1)}(s)=\int_0^\infty e^{-st}\ t^{m+1}\ f(t)dt=L\{t^{m+1}f(t)\}$$

Therefore true for n = m + 1 and hence true for all +ve integers.

3.4 Division by t

If $L\{f(t)\}=F(s)$ **then** $L\left\{\frac{f(t)}{t}\right\}=\int_s^\infty F(s)ds$

Proof:

We have $F(s) = \int_0^\infty e^{-st} f(t)\, dt$

Integrating w r t s

$$\int_s^\infty F(s)\, ds = \int_s^\infty \left\{ \int_0^\infty e^{st} f(t)\, dt \right\} ds = \int_0^\infty f(t) \left\{ \int_s^\infty e^{-st} ds \right\} dt = \int_0^\infty f(t) \left\{ \frac{e^{-st}}{-t} \right\}_s^\infty dt$$

$$= \int_0^\infty f(t) \left\{ 0 - \frac{e^{-st}}{-t} \right\} dt = \int_0^\infty e^{-st} \frac{f(t)}{t}\, dt = L\left[\frac{f(t)}{t} \right]$$

$$L\left\{ \frac{f(t)}{t} \right\} = \int_s^\infty F(s)\, ds$$

3.5 Transform of Integrals

If $L\{f(t)\} = F(s)$ **then** $L\left\{ \int_0^t f(u)\, du \right\} = \frac{1}{s} F(s)$

Proof:

Let $g(t) = \int_0^t f(u)\, du \quad \therefore g(0) = 0$ and $g'(t) = f(t)$

$$L\{g'(t)\} = l\{f(t)\} \Rightarrow sL\{g(t)\} - g(0) = L\{f(t)\} \Rightarrow s\, L\left\{ \int_0^t f(u)\, du \right\} - 0 = F(s)$$

$$\therefore L\left\{ \int_0^t f(u)\, du \right\} = \frac{F(s)}{s}$$

Problems:

16. **Find** $L\{e^{-t} t^2\}$

Solution:

We have $L\{t^2\} = \frac{2!}{s^3} \quad \therefore L\{e^{-t} t^2\} = \frac{2!}{(s+1)^3}$ using shifting property

17. **Find** $L\{e^{2t} \sin^2 t\}$.

Solution:

$$L\{\sin^2 t\} = L\left\{ \frac{1-\cos 2t}{2} \right\} = \frac{1}{2} L\{1\} - \frac{1}{2} L\{\cos 2t\} = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s}{s^2+4}$$

$$\therefore L\{e^{2t} \sin^2 t\} = \frac{1}{2(s-2)} - \frac{1}{2}\frac{(s-2)}{(s-2)^2+4}$$ using shifting property

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18. **Find** $L\{t^2 \sin at\}$.

Solution:

$$\{L \sin at\} = \frac{a}{s^2+a^2} = F(s) \Rightarrow L\{t^2 \sin at\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{d}{ds}\left(\frac{-a}{(s^2+a^2)^2} \cdot 2s\right) = \frac{-2a(a^2-3s^2)}{(s^2+a^2)^3}$$

19. **Find** $L\{t^2 \cos t\}$.

Solution:

$$L\{\cos t\} = \frac{s}{s^2+1} = F(S) \Rightarrow L\{t^2 \cos t\} = (-1)^2 F^{//}(s)$$

$$F'(s) = \frac{(s^2+1)-s(2s)}{(s^2+1)^2} = \frac{s^2+1-2s^2}{(s^2+1)^2} = \frac{1-s^2}{(s^2+1)^2}$$

$$F^{//}(s) = \frac{(s^2+1)^2(-2s)-2(1-s^2)(s^2+1)2s}{(s^2+1)^4} = \frac{2s(s^2-3)}{(s^2+1)^3}$$

$$\therefore L\{t^2 \cos t\} = \frac{2s(s^2-3)}{(s^2+1)^3}$$

20. **Find** $L\left\{\frac{\cos at - \cos bt}{t}\right\}$.

Solution:

$$L\left\{\frac{\cos at - \cos bt}{t}\right\}. = L\left\{\frac{\cos at}{t}\right\} - L\left\{\frac{\cos bt}{t}\right\}$$

$$L\{\cos at\} = \frac{s}{s^2+a^2} \Rightarrow L\left\{\frac{\cos at}{t}\right\} = \int_s^\infty \frac{s}{s^2+a^2}\, ds$$

$$= \lim_{k\to\infty} \int_s^k \frac{s}{s^2+a^2}\, ds = \lim_{k\to\infty} \frac{\log(s^2+a^2)}{2}\Bigg]_s^k = \frac{1}{2}\lim_{k\to\infty} \log\frac{k^2+a^2}{s^2+a^2}$$

Similarly $L\left\{\frac{\cos bt}{t}\right\} = \frac{1}{2}\lim_{k\to\infty} \log\frac{k^2+b^2}{s^2+b^2}$

$$\therefore L\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2}\lim_{k\to\infty} \log\frac{k^2+a^2}{k^2+b^2} - \frac{1}{2}\log\frac{(s^2+a^2)}{s^2+b^2} = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

21. **Find** $L\left\{\frac{1-\cos t}{t^2}\right\}$.

Solution:

$$L\{1-\cos t\} = \frac{1}{s} - \frac{s}{s^2+1} \Rightarrow L\left\{\frac{1-\cos t}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$$

$$= \lim_{k\to\infty} \int_s^k \frac{1}{s} - \frac{s}{s^2+1}\, ds = \lim_{k\to\infty} \left[\log s - \log\sqrt{s^2+1}\right]_s^k = \lim_{k\to\infty}\left[\log\frac{k}{\sqrt{k^2+1}} - \log\frac{s}{\sqrt{s^2+1}}\right]$$

$$= -\log\frac{s}{\sqrt{s^2+1}} = \log\frac{\sqrt{s^2+1}}{s}$$

$$\text{Now, } L\left\{\frac{1-\cos t}{t^2}\right\} = \lim_{k\to\infty}\int_s^k \log\frac{\sqrt{s^2+1}}{s}\, ds$$

$$= \lim_{k\to\infty}\int_s^k \log\sqrt{s^2+1}\, ds - \lim_{k\to\infty}\int_s^k \log s\, ds = \lim_{k\to\infty}\left[\log\sqrt{s^2+1}\,.\, s - \int \frac{s}{\sqrt{s^2+1}}\,\frac{2s}{2\sqrt{s^2+1}}\, ds\right]_s^k$$

$$-\lim_{k\to\infty}\left[\log s\,.\,s - \int s\frac{1}{s}\, ds\right]_s^k = \lim_{k\to\infty}\left[s\log\sqrt{s^2+1} - \frac{2}{2}\int\frac{s^2}{s^2+1}\, ds\right]_s^k - \lim_{k\to\infty}[s\log s - s]_s^k$$

$$= \lim_{k\to\infty}\left[s\log\sqrt{s^2+1} - s + \tan^{-1} s\right]_s^k - \lim_{k\to\infty}[s\log s - s]_s^k$$

$$= -k + \frac{\pi}{2} - s\log\sqrt{s^2+1} + s - \tan^{-1} s + k + s\log s - s$$

$$= \frac{\pi}{2} - s\log\sqrt{s^2+1} + s\log s - \tan^{-1} s = \cot^{-1} s - s\log\left\{\frac{\sqrt{s^2+1}}{s}\right\}$$

22. **Find** $L\left\{\int_0^t u\cos au\ du\right\}$.

Solution:

$$L\{\cos at\} = \frac{s}{s^2+a^2} = F(s)$$

$$\therefore L\{t\cos at\} = (-1)\ F'(s) = \frac{s^2-a^2}{(s^2+a^2)^2}$$

$$\therefore L\left\{\int_0^t u\cos au\ du\right\} = \frac{1}{s}\,\frac{s^2-a^2}{(s^2+a^2)^2}$$

23. **Find** $L\left\{\int_0^t \frac{\sin u}{u}\, du\right\}$.

Solution:

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$$L\{\sin t\} = \frac{1}{s^2+1} \quad \Rightarrow \quad L\left\{\frac{\sin t}{t}\right\} = \lim_{k\to\infty}\int_s^k \frac{1}{s^2+1}\,ds \quad = \lim_{k\to\infty} \tan^{-1} s\Big]_s^k \quad = \lim_{k\to\infty} \tan^{-1} k - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s \quad L\left\{\frac{\sin t}{t}\right\} = \cot^{-1} s \;\therefore\; L\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \frac{1}{s}\cot^{-1}(s)$$

24. **Find** $L\{e^{-t}\sin t\cos 2t\}$.

Solution:

$$L\{\sin\ \cos 2t\} = L\left\{\frac{\sin 3t - \sin t}{2}\right\} = \frac{1}{2}L\{\sin 3t\} - \frac{1}{2}L\{\sin t\} = \frac{1}{2}\frac{3}{s^2+9} - \frac{1}{2}\frac{1}{s^2+1}$$

$$L\{e^{-t}\sin t\cos 2t\} = \frac{3}{2}\frac{1}{(s+1)^2+9} - \frac{1}{2}\frac{1}{(s+1)^2+1} \text{ using shifting property}$$

25. **Find** $L\{e^{-2t}\sin 4t\}$

Solution:

We have $L\{\sin 4t\} = \dfrac{4}{s^2+16} \;\therefore\; L\{e^{-2t}\sin 4t\} = \dfrac{4}{(s^2+2)^2+16} = F(s)$

$$\Rightarrow L\{t\,e^{-2t}\sin 4t\} = (-1)'\,F'(s) = -1\frac{d}{ds}\left(\frac{4}{(s+2)^2+16}\right) = \frac{4\,.\,2(s+2)}{((s+2)^2+16)^2}$$

$$= \frac{8(s+2)}{[(s+2)^2+16]^2}$$

26. **Find** $L\left\{\dfrac{e^{-at}-e^{-bt}}{t}\right\}$.

Solution:

$$L\{e^{-at}-e^{-bt}\} = \frac{1}{s+a} - \frac{1}{s+b} = \frac{s+b-s-a}{(s+a)(s+b)} = \frac{b-a}{(s+a)(s+b)}$$

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \lim_{k\to\infty}\int_s^k \frac{1}{s+a} - \frac{1}{s+b}\,ds = \lim_{k\to\infty} \log(s+a) - \log(s+b)\Big]_s^k$$

$$= \lim_{k\to\infty} \log(k+a) - \log(k+b) - \log(s+a) + \log(s+b) = \log\left(\frac{s+b}{s+a}\right)$$

27. **Find** $L\left\{\int_0^t e^{-u}\cos u\,du\right\}$.

Solution:

$$L\{\cos t\} = \frac{s}{s^2+1}$$

$$\Rightarrow L\{e^{-t}\cos t\} = \frac{s+1}{(s+1)^2+1} \Rightarrow L\left\{\int_0^t e^{-u}\cos u\ du\right\} = \frac{1}{s}\,\frac{s+1}{\{(s+1)^2+1\}}$$

28. **Show that** $\int_0^\infty e^{-t}\, t\sin t\ dt = \frac{1}{2}$.

Solution:

$$L\{\sin t\} = \frac{1}{s^2+1} \Rightarrow L\{t\sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{1}{(s^2+1)^2}\,.\,2s$$

$$L\{t\sin t\} = \int_0^\infty e^{-St}\, t\sin t\ dt \qquad \therefore \int_0^\infty e^{-t}\, t\sin t\ dt = \frac{1}{(1+1)^2}\,.\,2(1) = \frac{1}{2}$$

29. **Find the value of** $\int_0^\infty e^{-2t}\sin^3 t\ dt$.

Solution:

$$\int_0^\infty e^{-st}\sin^3 t\ dt = L\{\sin^3 t\} = \frac{3}{4}L\{\sin t\} - \frac{1}{4}L\{\sin 3t\} \because \sin 3t = 3\sin t - 4\sin^3 t$$

$$\int_0^\infty e^{-st}\sin^3 t\ dt = \frac{3}{4}\,\frac{1}{s^2+1} - \frac{1}{4}\,\frac{3}{s^2+9}$$

Put $s = 2$, $\int_0^\infty e^{-2t}\sin^3 t\ dt = \frac{6}{65}$

30. **Evaluate** $L\left[e^{-4t}\,\frac{\sin 3t}{t}\right]$.

Solution:

$$L[\sin 3t] = \frac{3}{s^2+9}$$

$$L\left[\frac{\sin 3t}{t}\right] = \int_s^\infty \frac{3}{s^2+9}\,ds = \frac{3}{3}\tan^{-1}\frac{s}{3}\Big|_s^\infty = \frac{\pi}{2} - \tan^{-1}\frac{s}{3} = \cot^{-1}\frac{s}{3}$$

$$\therefore L\left[e^{-4t}\,\frac{\sin 3t}{t}\right] = \cot^{-1}\frac{s+4}{3}.$$

31. **Evaluate** $L[\cos at\sinh at]$.

Solution:

$$L[\cos at\sinh at] = L\left[\cos at\left(\frac{e^{at}-e^{-at}}{2}\right)\right]$$

$$\frac{1}{2}L\left[e^{at}\cos at - e^{-at}\cos at\right] = \frac{1}{2}\left\{\frac{s-a}{(s-a)^2+a^2} - \frac{s-a}{(s+a)^2+a^2}\right\}$$

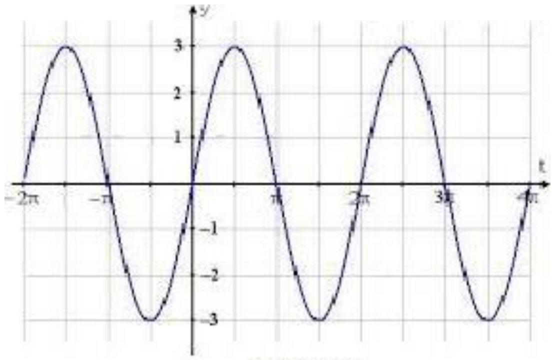
3.6 Laplace Transform of Periodic Functions

Periodic function:

A function $f(t)$ is said to be periodic function of period T>0

if $f(t+nT)=f(t)$ where n=1, 2, 3….

A periodic function has regular repetitive behavior.



sine function graph

Example: $\sin t, \cos t$ are periodic functions of period 2π

Theorem: If $f(t)$ is a periodic function with period T, then $L\{f(t)\} = \dfrac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$.

Proof:

Given $f(t)$ is periodic function of period T i.e. $f(t+T)=f(t)$

$$L\{f(t)\}=\int_0^\infty e^{-st} f(t)dt$$

$$L\{f(t)\}=\int_0^T e^{-st} f(t)dt+\int_T^\infty e^{-st} f(t)dt$$

Let $t=u+T \Rightarrow dt=du$

When $t=T, u=0$ and $t=\infty, u=\infty$

$$L\{f(t)\}=\int_0^T e^{-st} f(t)dt+\int_0^\infty e^{-s(u+T)} f(u+T)du$$

$$\Rightarrow L\{f(t)\}=\int_0^T e^{-st} f(t)dt+\int_0^\infty e^{-su}e^{-sT} f(u)du$$

$$\Rightarrow L\{f(t)\}=\int_0^T e^{-st} f(t)dt+e^{-sT}\int_0^\infty e^{-su} f(u)du$$

$$\Rightarrow L\{f(t)\}=\int_0^T e^{-st} f(t)dt+e^{-sT}L\{f(t)\} \Rightarrow L\{f(t)\}(1-e^{-sT})=\int_0^T e^{-st} f(t)dt$$

Department of Mathematics

$$\Rightarrow L\{f(t)\} = \frac{1}{(1-e^{-sT})}\int_0^T e^{-st} f(t)dt$$

Problems:

32. **Find the Laplace transform of the periodic function** $f(t) = t^2,\ 0 < t \le 2$ **with period T = 2.**

Solution:

$$L\{f(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st}\ t^2\ dt = \frac{1}{1-e^{-2s}}\left[\left[t^2\frac{e^{-st}}{-s}\right]_0^2 + \frac{2}{s}\int_0^2 e^{-st}\ t\ dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{4e^{-2s}}{-s} + \frac{2}{s}\left(\frac{te^{-st}}{-s}\right)_0^2 + \frac{1}{s}\int_0^2 e^{-st}\ dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{-4e^{-2s}}{+s} + \frac{2}{s}\left\{\frac{2e^{-2s}}{-s} + \frac{e^{-2s}}{-s^2} + \frac{1}{s^2}\right\}\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{-4e^{-2s}}{s} - \frac{4e^{-2s}}{s^2} - \frac{2e^{-2s}}{s^3} + \frac{2}{s^3}\right]$$

$$= \frac{+2}{s^3(1-e^{-2s})}\left[1-(1+2s+2s^2)e^{-2s}\right].$$

33. **Find the Laplace transform of** $f(t) = \begin{cases} E\sin\omega t, & 0 < t \le \pi/\omega \\ 0\quad , & \dfrac{\pi}{\omega} < t \le \dfrac{2\pi}{\omega} \end{cases}$

Solution:

We have the period $T = \dfrac{2\pi}{\omega}$

$$\therefore L\{f(t)\} = \frac{1}{1-e^{-\frac{2\pi}{\omega}s}}\int_0^{\pi/\omega} E\sin\omega t\ e^{-st}\ dt \quad = \frac{E}{1-e^{-\frac{2\pi}{\omega}s}}\left[\frac{e^{-st}}{s^2+\omega^2}[-s\sin\omega t - \omega\cos\omega t\right]_0^{\pi/\omega}$$

$$= \frac{E}{1-E^{-\frac{2\pi s}{\omega}}}\ \frac{1}{s^2+\omega^2}\left[E^{-\pi s/\omega}(\omega) - 1(-\omega)\right] = \frac{E\omega}{1-E^{-2\pi s/\omega}(s^2+\omega^2)}\ \left[E^{-\pi s/\omega}+1\right]$$

$$= \frac{E\omega\left[E^{-\pi s/\omega}+1\right]}{(1+E^{-\pi s/\omega})\ (1-E^{-\pi s/\omega})\ (s^2+\omega^2)} = \frac{E\omega}{(1-E^{-\pi s/\omega})(s^2+\omega^2)}$$

Department of Mathematics

Half wave rectifier wave form:

34. **If** $f(t)=\begin{cases} E & 0\le t\le a \\ -E & a<t\le 2a \end{cases}$ **prove that** $L\{f(t)\}=\frac{E}{s}\tan h\left(\frac{as}{2}\right)$

Solution:

We have the period $T=2a$

$$L\{f(t)\}=\frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st}\ f(t)\ dt=\frac{1}{1-e^{-2as}}\left[\int_0^{a} e^{-st}\ E\ dt+\int_a^{2a} e^{-st}\ (-E)\ dt\right]$$

$$=\frac{E}{1-e^{-2as}}\left[\int_0^{a} e^{-st}\ dt-\int_a^{2a} e^{-st}\ dt\right]=\frac{E}{1-e^{-2as}}\left\{\frac{e^{-as}}{-s}+\frac{1}{s}-\left[\frac{e^{-2as}}{-s}+\frac{e^{-as}}{s}\right]\right\}$$

$$=\frac{E}{s(1-e^{-2as})}\left[-e^{-as}+1+e^{-2as}-e^{-as}\right]=\frac{E}{s(1-e^{-2as})}\left[1+e^{-2as}-2e^{-as}\right]$$

$$=\frac{E}{s(1-e^{-2as})}\left(1-e^{-as}\right)^2=\frac{E}{s}\frac{1}{(1+e^{as})(1-e^{-as})}(1-e^{-as})^2=\frac{E}{s}\frac{1-e^{-as}}{1+e^{-as}}$$

$$=\frac{E}{s}\frac{e^{as/2}-e^{-as/2}}{e^{as/2}+e^{-as/2}}=\frac{E}{s}\tan h\left(\frac{as}{2}\right)$$

35. **Find the Laplace transform of** $f(t)=\begin{cases} t & 0\le t\le a \\ 2a-t & t>a \end{cases}$ **.with period** $T=2a$**.**

Solution:

$$L\{f(t)\}=\frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st}\ f(t)\ dt=\frac{1}{1-e^{-2as}}\left\{\int_0^{a} e^{-st}\ t\ dt+\int_a^{2a} e^{-st}(2a-t)\ dt\right\}$$

$$=\frac{1}{1-e^{-2as}}\left\{\left[\frac{te^{-st}}{-s}+\frac{1}{s}\int e^{-st}\right]_0^a+\left[(2a-t)\frac{e^{-st}}{-s}+\frac{1}{s}\int e^{-st}(-1)\,dt\right]_a^{2a}\right\}$$

$$=\frac{1}{1-e^{-2as}}\left\{\left[\frac{-te^{-st}}{s}-\frac{e^{-st}}{s^2}\right]_0^a-\left((2a-t)\frac{e^{-st}}{s}-\frac{e^{-st}}{s^2}\right)_a^{2a}\right\}$$

$$=\frac{1}{1-e^{-2as}}\,\frac{1}{s^2}\left\{1-e^{-as}\left[as+1-e^{-as}-as^2+1\right]\right\}$$

$$=\frac{1}{s^2\left(1-e^{-2as}\right)}\left(e^{-2as}-2e^{-as}+1\right)=\frac{(1-e^{-as})^2}{s^2(1-e^{-as})(1+e^{-as})}$$

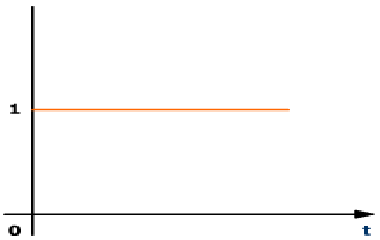
$$=\frac{1}{s^2}\frac{1-e^{-as}}{1+e^{-as}}=\frac{1}{s^2}\frac{e^{as/2}-e^{-as/2}}{e^{as/2}+e^{-as/2}}=\frac{1}{s^2}\tan h\left(\frac{as}{2}\right)$$

3.7 Unit Step Function and Unit Impulse Function

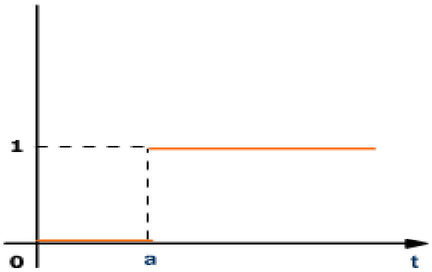
The use of the unit step function and unit impulse function make the method of solving differential equations by using Laplace transforms powerful for problems with inputs that have discontinuities or represent short impulses or complicated periodic functions.

3.7.2 Unit Step function (Heaviside's function)

The Unit step function also called Heavy side's unit step function $H(t)$ is defined as $H(t)=\begin{cases}0 & t<0\\ 1 & t\ge 0\end{cases}$



The displaced unit step function $H(t-a)$ is defined as $H(t-a)=\begin{cases}0 & t<a\\ 1 & t\ge a\end{cases}$



3.7.3 Heavy side shift theorem (Second shifting theorem)

Theorem: If $L\{f(t)\}=F(s)$ then $L\{f(t-a)H(t-a)\}=e^{-as}F(s)$

Proof:

$$L\{f(t-a)\,H(t-a)\}=\int_0^\infty e^{-st}f(t-a)\,H(t-a)\,dt$$

$$=\int_0^a e^{-st}\,f(t-a)H(t-a)dt+\int_a^\infty e^{-st}\,f(t-a)H(t-a)\,dt$$

Department of Mathematics

$$= \int_0^a e^{-st} f(t-a)\, 0\, dt + \int_a^\infty e^{-st} f(t-a)\, 1.\, dt$$

Put $t-a=u \Rightarrow dt=du$ when $t=a \Rightarrow u=0$ as $t \to \infty \Rightarrow u \to \infty$

$$L\{f(t-a)\ H(t-a)\} = \int_0^\infty e^{-s(u+a)}\ f(u)\,du = e^{-as}\int_0^\infty e^{-su}\ f(u)\,du = e^{-as}F(s)$$

Note: It is possible to express discontinuous function $f(t)$ in terms of unit step

I. If $f(t) = \begin{cases} f_1(t), & 0 \le t < a \\ f_2(t), & a \le t < b \\ f_3(t), & t \ge b \end{cases}$, then

$$f(t) = f_1(t) + \{f_2(t) - f_1(t)\} H(t-a) + \{f_3(t) - f_2(t)\} H(t-b).$$

II. If $f(t) = \begin{cases} f_1(t), & t \le a \\ f_2(t), & t > a \end{cases}$ then $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a)$

Problems:

36. Find $L\{e^{-t} H(t-1)\}$.

Solution:

We write $e^{-t} = e^{-t+1-1} = e^{-1} e^{-(t-1)}$

$$= \frac{1}{e} L\{e^{-(t-1)} H(t-1)\}$$

$$= \frac{1}{e} e^{-s} \frac{1}{s+1} = \frac{1}{e^{s+1}(s+1)} = \frac{e^{-(s+1)}}{s+1}$$

37. If $f(t) = \begin{cases} 2t & 0 < t < \pi \\ 1 & t \ge \pi \end{cases}$. **Express** $f(t)$ **in terms of unit step function and hence find its Laplace transform.**

Solution:

$$L\{f(t)\} = L\{2t\} + L\{(1-2t) H(t-\pi)\}$$

Take $1-2t = 1-2t-2\pi+2\pi = 1-2\pi-2(t-\pi)$

Department of Mathematics

$$\therefore L\{f(t)\} = L\{2t\} + L\{(1-2t)\,H(t-\pi)\} = \frac{2}{s^2} + \frac{e^{-\pi s}}{s} - 2e^{-\pi s}\frac{1}{s^2} - 2\pi e^{-\pi s}\frac{1}{s}$$

38. **If** $f(t) = \begin{cases} t^2 & 1 \le t < 2 \\ 4t & t \ge 2 \end{cases}$. **Express** $f(t)$ **in terms of unit step function and hence find its Laplace transform.**

Solution:

We can take $f(t) = \begin{cases} 0 & 0 < t < 1 \\ t^2 & 1 \le t < 2 \\ 4t & t \ge 2 \end{cases}$

$$\therefore f(t) = 0 + (t^2 - 0)\,H(t-1) + (4t - t^2)\,H(t-2)$$

$$L\{f(t)\} = L\{t^2\,H(t-1)\} + L\{(4t-t^2)\,H(t-2)\}$$
$$= L\left\{\left[(t-1)^2 + 2t - 2\right]H(t-1)\right\} + L\left\{\left[4-(t-2)^2\right]H(t-2)\right\}$$
$$= e^{-s}\frac{2!}{s^3} + \frac{2e^{-s}}{s^2} + \frac{e^{-s}}{s} + \frac{4e^{-2s}}{s} - \frac{e^{-2s}\,2!}{s^3}$$

39. **Find the Laplace transform of** $t^2H(t-4)$

Solution:

Let $f(t-3) = t^2 \Rightarrow f(t) = (t+4)^2 = t^2 + 8t + 16$

$$L\{f(t)\} = \frac{2}{s^3} + 8\frac{1}{s^2} + \frac{16}{s} = F(s)$$

$$\therefore L[f(t-3)H(t-3)] = e^{-3s}\left[\frac{2}{s^3} + \frac{8}{s^2} + \frac{16}{s}\right]$$

40. **If** $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \cos 2t & \pi \le t < 2\pi \\ \cos 3t & t \ge 2\pi \end{cases}$. **Express** $f(t)$ **in terms of unit step function and hence find its Laplace transform.**

Solution:

$f(t) = \cos t + \{\cos 2t - \cos t\} H(t-\pi) + \{\cos 3t - \cos 2t\} H(t-2\pi)$

We have

$$\cos 2(t-\pi) = \cos(2t-2\pi) = \cos(2\pi - 2t)$$
$$\cos 2(t-\pi) = \cos 2t$$
$$\cos(t-\pi) = \cos(\pi - t) = -\cos t$$
$$\cos t = -\cos(t-\pi)$$
$$\cos 3(t-2\pi) = \cos(3t-6\pi) = \cos(6\pi - 3t)$$
$$\cos 3(t-2\pi) = \cos 3t,$$
$$\cos 2(t-2\pi) = \cos(2t-4\pi) = \cos(4\pi - 2t),\ \cos 2(t-2\pi) = \cos 2t$$
$$\therefore\ f(t) = \cos t + \{\cos 2(t-\pi) + \cos(t-\pi)\} H(t-\pi) + \{\cos 3(t-2\pi) - \cos 2(t-2\pi)\} H(t-2\pi)$$
$$\therefore\ L\{f(t)\} = \frac{s}{s^2+1} + e^{-\pi s}\left\{\frac{s}{s^2+4} + \frac{s}{s^2+1}\right\} + e^{-2\pi s}\left\{\frac{3s}{s^2+9} - \frac{2s}{s^2+4}\right\}$$

41. Find the Laplace transform of $f(t) = \begin{cases} 0 & t < 0 \\ E & 0 \le t < a \\ 0 & t \ge a \end{cases}$.

Solution:

We have $f(t) = 0 + (E-0) H(t-a) + (0-E) H(t-a) = E\,H(t) - E\,H(t-a)$

$$\therefore\ L\{f(t)\} = \frac{E}{s} - \frac{Ee^{-as}}{s} = \frac{E}{s}(1 - e^{-as})$$

42. If $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ 1 & \pi \le t < 2\pi \\ \sin t & t \ge 2\pi \end{cases}$. **Find the Laplace Transform of** $f(t)$

Solution:

We have $f(t) = \cos t + (1 - \cos t) H(t-\pi) + (\sin t - 1) H(t-2\pi)$

$\cos(t-\pi) = -\cos t,\qquad \sin(t-2\pi) = -\sin(2\pi - t) = \sin t$

$$\therefore\ f(t) = \cos t + (1 + \cos(t-\pi)) H(t-\pi) + \{\sin(t-2\pi) - 1\} H(t-2\pi)$$
$$\therefore\ L\{f(t)\} = \frac{s}{s^2+1} + \frac{e^{-\pi s}}{s} + e^{-\pi s}\frac{s}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1} - \frac{e^{-2\pi s}}{s}$$

Department of Mathematics

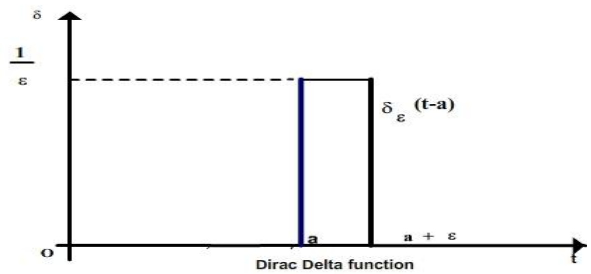
3.8 Laplace Transforms of Unit Impulse function (Dirac–Delta function):

We come across many phenomena of impulsive nature, such as the action of forces or voltages over short intervals of time, for example if a mechanical system is hit by a hammer blow, an airplane makes a hard landing, a ship hit by a single high wave and so on. These phenomena can be mathematically modeled by using Dirac Delta function or unit impulse function and can be solved efficiently by using Laplace Transforms.

3.8.1 Definition:

The Dirac Delta function or the unit impulse function is defined as

$$\delta_\varepsilon(t-a) = \begin{cases} \dfrac{1}{\varepsilon} & a \le t \le a+\varepsilon \\ 0 & otherwise \end{cases}$$



Dirac Delta function

As $\varepsilon \to o$ we observe that $\delta \to \infty$ such that area is always 1 **i.e** $\int_0^\infty \delta(t-a)\,dt = 1$

3.8.2 Laplace Transform of Dirac Delta function

Now $\int_0^\infty f(t)\delta(t-a)\,dt = \int_a^{a+\varepsilon} f(t)\frac{1}{\varepsilon}\,dt = f(\eta)$ where $a < \eta < a+\varepsilon$

As $\varepsilon \to o$ we get $\int_0^\infty f(t)\delta(t-a)\,dt = f(a)$

If $f(t) = e^{-st}$ **then** $\int_0^\infty e^{-st}\delta(t-a)\,dt = e^{-as}$ i.e $L[\delta(t-a)] = \int_0^\infty e^{-st}\delta(t-a)\,dt = e^{-as}$

Problems:

43. **Find** $L\left[\frac{1}{t}\delta(t-a)\right]$

Solution:

We know $L[\delta(t-a)] = e^{-as}$

$$\therefore L\left[\frac{1}{t}\delta(t-a)\right] = \int_s^\infty e^{-as}ds = \frac{1}{a}e^{-as}$$

44. **Find** $L\left[\cosh 3t\delta(t-2)\right]$

Solution:

$$\cosh 3t\delta(t-2) = \frac{1}{2}\left(e^{3t} + e^{-3t}\right)\delta(t-2)$$

$$= \frac{1}{2}\left\{L\left[e^{3t}\delta(t-2) + e^{-3t}\delta(t-2)\right]\right\} = \frac{1}{2}\left\{L[\delta(t-2)]_{s\to s-3} + L[\delta(t-2)]_{s\to s+3}\right\}$$

$$= \frac{1}{2}\left[\left(e^{-2s}\right)_{s\to s-3} + \left(e^{-2s}\right)_{s\to s+3}\right] = \frac{1}{2}\left[e^{-2(s-3)} + e^{-2(s+3)}\right] = \frac{e^{-2s}}{2}\left[e^{6} + e^{-6}\right]$$

45. **Find** $L\left[\dfrac{2\delta(t-1)+6\delta(t-2)}{t}\right]$

Solution:

$$L[2\delta(t-1)+6\delta(t-2)] = 2e^{-s} + 6e^{-2s}.$$

$$L\left[\frac{2\delta(t-1)+6\delta(t-2)}{t}\right] = \int_s^\infty (2e^{-s} + 6e^{-2s})ds$$

$$= 2e^{-s} + 3e^{-2s}$$

3.9 Inverse Laplace Transforms

3.9.1 Introduction

So far the Laplace Transform has been studied with an expression for $F(s)$ given by $\int_0^\infty e^{-st} f(t)\,dt = F(s)$ where $f(t)$ is the given original function and $F(s)$ is its Laplace Transform. Now, to find the original function $f(t)$, the inverse of a given function $F(s)$ has been found. This is slightly different process than taking Laplace Transform, which was quite straightforward.

Department of Mathematics

3.9.2 Definition: Inverse Laplace Transform:

The solution of the operator equation $L[f(t)] = F(s)$ whenever it exists is called the inverse Laplace transform of $F(s)$ and is denoted by $L^{-1}[F(s)]$.

Thus $L^{-1}[F(s)] = f(t)$ L^{-1} is known as the inverse Laplace Transform operator and is such that $L^{-1}L = LL^{-1} = 1$. From the table of Laplace transforms of elementary functions by using definition and Linearity property we can obtain a table of inverse Laplace transforms.

Sl No	Laplace Transform	Inverse Laplace Transform
1	$L\{1\} = \dfrac{1}{s}$	$\therefore L^{-1}\left\{\dfrac{1}{s}\right\} = 1$
2	$L\{e^{at}\} = \dfrac{1}{s-a}$	$\therefore L^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$
3	$L\{\sin at\} = \dfrac{a}{s^2+a^2}$	$\therefore L^{-1}\left\{\dfrac{1}{s^2+a^2}\right\} = \dfrac{1}{a}\sin at$
4	$L\{\cos at\} = \dfrac{s}{s^2+a^2}$	$\therefore L^{-1}\left\{\dfrac{s}{s^2+a^2}\right\} = \cos at$
5	$L\{\sin h\,at\} = \dfrac{a}{s^2-a^2}$	$\therefore L^{-1}\left\{\dfrac{1}{s^2-a^2}\right\} = \dfrac{1}{a}\sinh at$
6	$L\{\cos h\,at\} = \dfrac{s}{s^2-a^2}$	$\therefore L^{-1}\left\{\dfrac{s}{s^2-a^2}\right\} = \cosh at$
7	$L\{t^n\} = \dfrac{\Gamma(n+1)}{s^{n+1}}$	$\therefore L^{-1}\left\{\dfrac{1}{s^n}\right\} = \dfrac{t^{n-1}}{\Gamma(n)}$
8	$L\{f(t-a)H(t-a)\} = e^{-as}F(s)$	$\therefore L^{-1}\{e^{-as}F(s)\} = f(t-a)H(t-a)$
9	$L\{t\,f(t)\} = -F'(s)$	$\therefore L^{-1}\{F(s)\} = -t\,f(t)$

10	$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\ ds$	$\therefore L^{-1}\left\{\int_s^\infty F(s)\ ds\right\} = \frac{f(t)}{t}$
11	$L\left\{\int_0^t f(u)\ du\right\} = \frac{F(s)}{s}$	$\therefore L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u)\ du$

3.9.3 Use of Partial Fraction:

Now we use some of the above companion formulas to find the inverse Laplace transform. The most general method of finding the inverse transform of any function is to decompose the given function into partial fractions. Using linearity of L^{-1}, we write the inverse transform of each fraction involved.

Problems:

46. Find the inverse Laplace transform of the function $\frac{1}{2s^2+9}$.

Solution:

$$\frac{1}{2s^2+9} = \frac{1}{2\left(s^2+\frac{9}{2}\right)} = \frac{1}{2\left(s^2+\left(\frac{3}{\sqrt{2}}\right)^2\right)}$$

$$\therefore L^{-1}\left\{\frac{1}{2s^2+9}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s^2+\left(\frac{3}{\sqrt{2}}\right)^2}\right\} = \frac{1}{2}\sin\left(\frac{3}{\sqrt{2}}t\right)\frac{\sqrt{2}}{3} = \frac{1}{3\sqrt{2}}\sin\left(\frac{3}{\sqrt{2}}t\right)$$

47. Find the inverse Laplace transform of the function $\frac{s+1}{s^2+s+1}$.

Solution:

$$\frac{s+1}{s^2+s+1}. = \frac{s+1}{s^2+s+1+\frac{1}{4}-\frac{1}{4}} = \frac{s+1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{s+\frac{3}{2}+\frac{3}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

Department of Mathematics

$$\therefore L^{-1}\left\{\frac{s+1}{s^2+s+1}\right\} = e^{-t/2}\cos\frac{\sqrt{3}}{2}t + \frac{1}{2}e^{-t/2}\sin\frac{\sqrt{3}}{2}t\frac{2}{\sqrt{3}}$$

$$\therefore L^{-1}\left\{\frac{s+1}{s^2+s+1}\right\} = e^{-t/2}\left[\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right]$$

48. Find the inverse Laplace transform of the function $\frac{s}{s^4+4a^4}$.

Solution:

$$\frac{s}{s^4+4a^4}. = \frac{s}{(s^2)^2+(2a^2)^2} = \frac{s}{(s^2+2a^2)-2(s^2)(2a^2)} = \frac{s}{(s^2+2a^2)-(2as)^2}$$

$$= \frac{s}{(s^2+2a^2+2as)(s^2+2a^2-2as)} = \frac{s}{(s^2+2a^2)-(2as)^2}$$

$$= \frac{1}{4a}\left\{\frac{1}{s^2+2a^2-2as} - \frac{1}{s^2+2a^2+2as}\right\} = \frac{1}{4a}\left\{\frac{1}{(s-a)^2+a^2} - \frac{1}{(s+a)^2+a^2}\right\}$$

$$\therefore L^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = \frac{1}{4a}\left\{\frac{e^{at}\sin at}{a} - \frac{e^{at}\sin at}{a}\right\} = \frac{\sin at}{4a^2}\left\{e^{at}-e^{-at}\right\} = \frac{\sin at}{2a^2}\sinh at$$

49. Find the inverse Laplace transform of the function $\frac{4s+5}{(s-1)^2(s+2)}$

Solution:

Let $\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2}$.

$$\frac{4s+5}{(s-1)^2(s+2)} = \frac{1/3}{s-1} + \frac{3}{(s-1)^2} + \frac{-1/3}{s+2}$$

$$\therefore L^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\} = \frac{1}{3}e^t + 3e^t t - \frac{1}{3}e^{-2t}$$

50. Find the inverse Laplace transform of the function $\frac{5s+3}{(s-1)(s^2+2s+5)}$

Solution:

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}.$$

Let $\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}$.

$\Rightarrow A(s^2+2s+5)+(Bs+C)(s-1) = 5s+3$

This becomes

$$\frac{5s+3}{(s-1)(s^2+2s+5)}=\frac{1}{s-1}+\frac{2-s}{s^2+2s+5}$$

$$\therefore L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}=e^t-L^{-1}\left\{\frac{s-2}{(s+1)^2+2^2}\right\}$$

$$=e^t-L^{-1}\left\{\frac{(s+1)-3}{(s+1)^2+2^2}\right\}=e^t-L^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\}+3L^{-1}\left\{\frac{3}{(s+1)^2+2^2}\right\}$$

$$=e^t-e^{-t}\cos 2t+3e^{-t}\frac{\sin 2t}{2}$$

51. Find the inverse Laplace transform of the function $\frac{3s+5\sqrt{2}}{s^2+8}$.

Solution:

$$\frac{3s+5\sqrt{2}}{s^2+8}.=3\frac{s}{s^2+(2\sqrt{2})^2}+5\sqrt{2}\frac{1}{s^2+(2\sqrt{2})^2}$$

$$\therefore L^{-1}\left\{\frac{3s+5\sqrt{2}}{s^2+8}\right\}=3\cos 2\sqrt{2}\,t+\frac{5}{2}\sin\left(2\sqrt{2}\,t\right)$$

52. Find the inverse Laplace transform of the function $\frac{s+2}{s^2-4s+13}$.

Solution:

$$\frac{s+2}{s^2-4s+13}=\frac{s+2}{s^2-4s+4+9}=\frac{s-2+4}{(s-2)^2+3^2}=\frac{s-2}{(s-2)^2+3^2}+4\frac{1}{(s-2)^2+3^2}$$

$$\therefore L^{-1}\left\{\frac{s+2}{s^2-4s+13}\right\}=e^{2t}\cos 3t+4e^{2t}\frac{\sin 3t}{3}$$

53. Find $L^{-1}\left\{\frac{s}{\left(s^2+a^2\right)^2}\right\}$

Solution:

Let $F(s)=\frac{1}{s^2+a^2}\Rightarrow f(t)=\frac{1}{a}\sin at$

$$F'(s)=-\frac{(2s)}{\left(s^2+a^2\right)^2}\quad\Rightarrow L^{-1}\{F'(s)\}=-2L^{-1}\left\{\frac{s}{\left(s^2+a^2\right)^2}\right\}\Rightarrow -tf(t)=-2L^{-1}\left\{\frac{s}{\left(s^2+a^2\right)^2}\right\}$$

Department of Mathematics

$$\therefore L^{-1}\left\{\frac{s}{\left(s^2+a^2\right)^2}\right\}=\frac{t}{2}f(t)=\frac{t}{2a}\sin at$$

54. Find the inverse transform of $\dfrac{1+2s}{(s+2)^2(s-1)^2}$

Solution:

Let $\dfrac{1+2s}{(s+2)^2(s-1)^2}=\dfrac{A}{s+2}+\dfrac{B}{(s+2)^2}+\dfrac{C}{s-1}+\dfrac{D}{(s-1)^2}$

$$\frac{1+2s}{(s+2)^2(s-1)^2}=\frac{-1/3}{(s+2)^2}+\frac{1/3}{(s-1)^2}$$

$$L^{-1}\left[\frac{1+2s}{(s+2)^2(s-1)^2}\right]=-\frac{1}{3}e^{-2t}t+\frac{1}{3}e^{t}t$$

3.9.4 Shifting Property:

If $L^{-1}[F(s)]=f(t)$ then $L^{-1}[F(s-a)]=e^{at}f(t)=e^{at}L^{-1}[F(s)]$

55. Find the inverse transform of $\dfrac{3s+1}{(s+1)^4}$

Solution:

$$L^{-1}\left[\frac{3s+1}{(s+1)^4}\right]=e^{-t}L^{-1}\left[\frac{3(s+1)+1}{(s+1)^4}\right]=e^{-t}\left\{3L^{-1}\left[\frac{1}{s^3}\right]-2L^{-1}\left[\frac{1}{s^4}\right]\right\}=e^{-t}\left[3\frac{t^2}{2!}-\frac{t^3}{3}\right]$$

56. Find the inverse transform of $\dfrac{s+1}{\left(s^2+6s+25\right)}$

Solution:

$$L^{-1}\left[\frac{s+1}{\left(s^2+6s+25\right)}\right]=L^{-1}\left[\frac{s+1}{\left((s+3)^2+16\right)}\right]$$

Using shifting property,

$$=e^{-3t}L^{-1}\left[\frac{s-3+1}{\left(s^2+16\right)}\right]=e^{-3t}L^{-1}\left[\frac{s-2}{\left(s^2+16\right)}\right]=e^{-3t}\left\{\cos 4t-\frac{1}{2}\sin 4t\right\}$$

3.9.5 Transform of integrals:

If $L^{-1}[F(s)] = f(t)$, then $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)dt$. Also $L^{-1}\left[\frac{F(s)}{s^2}\right] = \int_0^t \left[\int_0^t f(t)dt\right]dt$ and

$$L^{-1}\left[\frac{F(s)}{s^3}\right] = \int_0^t \left\{\int_0^t \left[\int_0^t f(t)dt\right]dt\right\}dt$$

57. Find $L^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\}$.

Solution:

We have $L^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\} = \frac{1}{a^2}\{1-\cos at\}$

$$\therefore L^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\} = \int_0^t \frac{1-\cos au}{a^2}.du = \frac{1}{a^2}\left(u - \frac{\sin au}{a}\right)\Bigg|_0^t = \frac{1}{a^2}\left(t - \frac{\sin at}{a}\right)$$

58. Find $L^{-1}\left\{\frac{1}{s(s+a)^3}\right\}$

Solution: Let $F(s) = \frac{1}{(s+a)^3} \Rightarrow f(t) = e^{-at}\frac{t^2}{2!}$

$$\therefore L^{-1}\left\{\frac{1}{s(s+a)^3}\right\} = \int_0^t e^{-au}\frac{u^2}{2}du = \frac{t^2e^{-at}}{-2a} - \frac{te^{-at}}{a^2} - \frac{e^{-at}}{a^3} + \frac{1}{a^3}$$

3.9.6 Multiplication by t:

If $L^{-1}[F(s)] = f(t)$ then $L^{-1}\left\{\frac{d^n[F(s)]}{ds^n}\right\} = (-1)^n.t^n.f(t)$, n = 1,2,3,………..

59. Find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$.

Solution:

Let $F(s) = \frac{1}{s^2+a^2} \Rightarrow f(t) = \frac{1}{a}\sin at$

$$F'(s) = \frac{-1}{(s^2+a^2)^2}(2s) \Rightarrow L^{-1}\{F'(s)\} = -2L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$$

$$\Rightarrow -t.f(t) = -2L^{-1}\left\{\frac{s}{\left(s^2+a^2\right)^2}\right\} \therefore L^{-1}\left\{\frac{s}{\left(s^2+a^2\right)}\right\} = \frac{t}{2}f(t) = \frac{t}{2a}\sin at$$

60. **Find** $L^{-1}\left\{\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right\}$

Solution:

Let $F(s) = \frac{1}{2}\log\left(s^2+a^2\right) \Rightarrow F'(s) = \frac{s}{s^2+b^2} - \frac{s}{s^2+a^2}$

We have $L^{-1}\{F'(s)\} = \cos bt - \cos at$

$$\Rightarrow -t.f(t) = \cos bt - \cos at \quad \therefore f(t) = \frac{\cos at - \cos bt}{t}$$

61. **Find** $L^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\}$.

Solution:

$$L^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\}. = L^{-1}\left\{\cot^{-1}\left(\frac{s^2}{2}\right)\right\}$$

Let $F(s) = \cot^{-1}\left(\frac{s^2}{2}\right) \Rightarrow F'(s) = \frac{-1}{1+\left(\frac{s^2}{2}\right)}\frac{1}{2}2s = \frac{-4s}{2^2+\left(s^2\right)^2}$

$$= -4s\frac{1}{4}\left\{\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right\}$$

$$F'(s) = \frac{-s}{(s-1)^2+1} + \frac{s}{(s+1)^2+1}$$

$$\therefore L^{-1}\{F'(s)\} = -e^{t}\sin t + e^{-t}\sin t \Rightarrow -t.f(t) = \sin t\left(e^{-t} - e^{t}\right)$$

$$\Rightarrow f(t) = \frac{\sin t}{t}\left(e^{t} - e^{-t}\right)$$

3.9.7 Division by t:

If $L^{-1}[F(s)] = f(t)$ then $L^{-1}\left[\int_s^\infty F(s).ds\right] = \frac{f(t)}{t}$

62. **Find** $L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]$

Solution:

Let $f(t)=L^{-1}\left[\dfrac{s}{\left(s^2+a^2\right)^2}\right]$

By using the above property,

$$L\left[\frac{f(t)}{t}\right]=\int_s^\infty \frac{s}{\left(s^2+a^2\right)^2}ds=\frac{1}{2}\int_s^\infty \frac{2s}{\left(s^2+a^2\right)^2}ds=-\frac{1}{2}\left(\frac{1}{s^2+a^2}\right)_s^\infty$$

$$L\left[\frac{f(t)}{t}\right]=\frac{1}{2}\frac{1}{s^2+a^2}\Rightarrow\frac{f(t)}{t}=\frac{1}{2}L^{-1}\left(\frac{1}{s^2+a^2}\right)=\frac{\sin at}{2a}$$

Hence $f(t)=\dfrac{1}{2a}t.\sin at$

3.9.8 Computation of the inverse of e^{-as} F(s):

We have $L^{-1}\left[f(t-a)u(t-a)\right]=e^{-as}F(s)$

$\therefore L^{-1}\left[e^{-as}F(s)\right]=f(t-a)u(t-a)\Rightarrow L^{-1}\left[e^{-as}F(s)\right]=f(t-a)H(t-a)$

62. **Find** $L^{-1}\left\{e^{-\pi s}\dfrac{s}{s^2+1}\right\}$.

Solution:

$$L^{-1}\left\{e^{-\pi s}\frac{s}{s^2+1}\right\}=\cos(t-\pi)H(t-\pi)$$

63. **Find** $L^{-1}\left\{e^{-s}\dfrac{1}{s^3}\right\}$.

Solution:

$$L^{-1}\left\{e^{-s}\frac{1}{s^3}\right\}=\frac{(t-1)}{2!}H(t-1)$$

3.9.9 Miscellaneous Problems:

64. **Find** $L^{-1}\left[\dfrac{2s-1}{s^2+4s+29}\right]$

Solution:

$$L^{-1}\left[\frac{2s-1}{s^2+4s+29}\right]=L^{-1}\left[\frac{2s-1}{(s+2)^2+25}\right]=L^{-1}\left[\frac{2(s+2)-5}{(s+2)^2+25}\right]$$

$$=e^{-2t}L^{-1}\left[\frac{2s-5}{s^2+5^2}\right]=e^{-2t}\left\{2L^{-1}\left[\frac{s}{s^2+5^2}\right]-L^{-1}\left[\frac{5}{s^2+5^2}\right]\right\}$$

Thus $L^{-1}\left[\dfrac{2s-1}{s^2+4s+29}\right]=e^{-2t}\left(2\cos 5t-\sin 5t\right)$

64. **Find** $L^{-1}\left\{\dfrac{e^{-3s}}{(s+4)^{\frac{5}{2}}}\right\}$.

Solution:

Let $F(s)=\dfrac{1}{(s+4)^{\frac{5}{2}}}$

we have $L\{t^n\}=\dfrac{\Gamma(n+1)}{s^{n+1}}\Rightarrow L^{-1}\left\{\dfrac{1}{s^n}\right\}=\dfrac{t^{n-1}}{\Gamma(n)}$

$\therefore L^{-1}\{F(s)\}=e^{-4t}\dfrac{(t+4)^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)}\Rightarrow L^{-1}\{e^{-3s}F(s)\}=e^{-4(t-3)}\dfrac{(t-3+4)^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)}u(t-3)$

65. **Find** $L^{-1}\left\{\dfrac{1}{s(s^2+a^2)}\right\}$

Solution:

Let $F(s)=\dfrac{1}{s^2+a^2}\Rightarrow f(t)=\dfrac{1}{a}\sin at$

We have $L^{-1}\left\{\dfrac{1}{s}F(s)\right\}=\int_0^t\dfrac{1}{a}\sin au.du=\dfrac{1}{a}-\dfrac{\cos au}{a}\Big|_0^t=\dfrac{1}{a^2}[1-\cos at]$

$\Rightarrow L^{-1}\left\{\dfrac{1}{s(s^2+a^2)}\right\}=\dfrac{1}{a^2}[1-\cos at]$

66. **Find** $L^{-1}\left\{\dfrac{1}{(s^2+a^2)^2}\right\}$.

Solution:

$$L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}=L^{-1}\left\{\frac{1}{s}\frac{s}{(s^2+a^2)}\right\}$$

We know that

$$L^{-1}\left\{\frac{s}{(s^2+a^2)}\right\}=\frac{t}{2a}\sin a$$

$$\therefore L^{-1}\left\{\frac{1}{s}\frac{s}{\left(s^2+a^2\right)^2}\right\}=\int_0^t \frac{u}{2a}\sin au.du=\frac{1}{2a}\int_0^t u\sin au.du$$

$$=\frac{1}{2a}\left(\left.\frac{-u\cos au}{a}\right|_0^t+\int_0^t\frac{\cos au}{a}du\right)=\frac{-t\cos at}{2a^2}+\frac{t}{a}+\left.\frac{1}{2a}\frac{\sin au}{a^2}\right|_0^t$$

$$=\frac{-t\cos at}{2a^2}+\frac{t}{a}+\frac{\sin at}{2a^3}$$

67. **Find** $L^{-1}\left\{\frac{1}{s}\sin\left(\frac{1}{s}\right)\right\}$.

Solution:

We have the series expansion of $\sin x$ as

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+.............$$

$$\therefore \sin\left(\frac{1}{s}\right)=\frac{1}{s}-\frac{1}{3!}\frac{1}{s^3}+\frac{1}{5!}\frac{1}{s^5}-\frac{1}{7!}\frac{1}{s^7}+......$$

$$\Rightarrow \frac{1}{s}\sin\left(\frac{1}{s}\right)=\frac{1}{s^2}-\frac{1}{3!}\frac{1}{s^4}+\frac{1}{5!}\frac{1}{s^6}-\frac{1}{7!}\frac{1}{s^8}+.....$$

$$\therefore L^{-1}\left\{\frac{1}{s}\sin\left(\frac{1}{s}\right)\right\}=\frac{t}{1!}-\frac{1}{3!}\frac{t^3}{3!}+\frac{1}{5!}\frac{t^5}{5!}-.................$$

68. **Find** $L^{-1}\left\{\log\left(\frac{1+s}{s}\right)\right\}$.

Solution:

We have

$$F(s)=\log(1+s)-\log s$$

$$F(s)=\frac{1}{s+1}-\frac{1}{s}\Rightarrow L^{-1}\{F'(s)\}=e^{-t}-1$$

$$-t.f(t)=e^{-t}-1\Rightarrow f(t)=\frac{1-e^t}{t}$$

69. **Find** $L^{-1}\left\{\log\frac{(s+a)}{(s+b)}\right\}$

Solution:

Let $F(s)=\log(s+a)-\log(s+b)$

$$\therefore F'(s) = \frac{1}{s+a} - \frac{1}{s+b} \Rightarrow L^{-1}\{F'(s)\} = e^{-at} - e^{-bt}$$

$$-t.f(t) = e^{-at} - e^{-bt} \Rightarrow f(t) = \frac{e^{-bt} - e^{-at}}{t}$$

70. **Find** $L^{-1}\left\{\frac{e^{-3s}}{(s-4)^2}\right\}$

Solution:

$$L^{-1}\left\{\frac{e^{-3s}}{(s-4)^2}\right\} = e^{-4(t-3)}(t-3)H(t-3)$$

71. **Find** $L^{-1}\left\{\frac{se^{\frac{-s}{2}} + \pi e^{-s}}{s^2+\pi^2}\right\}$.

Solution:

$$L^{-1}\left\{\frac{se^{-s/2} + \pi e^{-s}}{s^2+\pi^2}\right\} = L^{-1}\left\{\frac{se^{\frac{-s}{2}}}{s^2+\pi^2}\right\} + L^{-1}\left\{\frac{e^{-s}\pi}{s^2+\pi^2}\right\}$$

$$= \cos\pi\left(t-\frac{1}{2}\right)H\left(t-\frac{1}{2}\right) + \sin\pi(t-1)H(t-1)$$

72. **Find** $L^{-1}\left\{\frac{1}{s+e^{-s}}\right\}$.

Solution:

$$L^{-1}\left\{\frac{1}{s+e^{-s}}\right\} = L^{-1}\left\{\frac{1}{s\left[1+\frac{e^{-s}}{s}\right]}\right\} = L^{-1}\left\{\left[1+\frac{e^{-s}}{s}\right]^{-1}\frac{1}{s}\right\}$$

$$= L^{-1}\left\{\frac{1}{s}\left[1-\frac{e^{-s}}{s}+\frac{e^{-2s}}{s^2}-\frac{e^{-3s}}{s^3}+\frac{e^{-4s}}{s^4}-\dots\dots\dots\dots\right]\right\}$$

$$= L^{-1}\left\{\frac{1}{s} - \frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^3} - \frac{e^{-3s}}{s^4} + \frac{e^{-4s}}{s^5}\right\}$$

$$= 1 - (t-1)H(t-1) + \frac{(t-2)^2}{2!}H(t-2) - \frac{(t-3)}{3!}H(t-3) + \ldots\ldots\ldots\ldots$$

3.9.10 Convolution Theorem

Definition:

The convolution of $f(t)$ and $g(t)$ denoted by $f(t)*g(t)$ i.e.,

(Note: * is commutative**)**

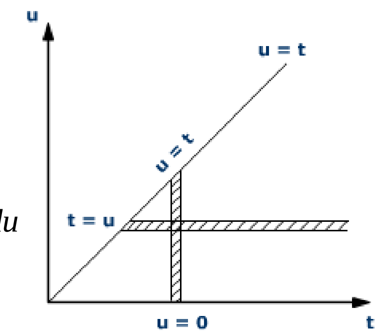
$$f(t)*g(t) = \int_0^t f(u).g(t-u).du$$

Theorem

If $L\{f(t)\} = F(s)$ and $L\{g(t)\} = G(s)$ then $L\{f(t)*g(t)\} = F(s)G(s)$

Proof:

$$L\{f(t)*g(t)\} = L\left\{\int_0^t f(u)g(t-u).du\right\} = \int_{t=0}^{\infty} e^{-st}\left\{\int_{u=0}^{u=t} f(u)g(t-u).du\right\}.dt$$



By changing the order of integration

$$L\{f(t)*g(t)\} = \int_{u=0}^{\infty}\int_{t=u}^{\infty} e^{-st} f(u)g(t-u).dtdu = \int_{u=0}^{\infty} f(u)\int_{t=u}^{\infty} e^{-st} g(t-u)dtdu$$

Put $t-u=v \Rightarrow dt=dv$ when $t=u \Rightarrow v=0$ as $t\to\infty, v\to\infty$

$$= \left\{\int_{u=0}^{\infty} f(u)e^{-su}\right\}\left\{\int_{v=o}^{\infty} g(v)\,e^{-sv}\right\} = F(s)G(s)$$

$$\therefore L\{f(t)*g(t)\} = F(s)G(s)$$

Note: $L^{-1}\{F(s).G(s)\} = f(t)*g(t) = \int_0^t f(u)g(t-u).du$

Department of Mathematics

73. Verify convolution theorem for the functions t and e^t.

Solution:

We have $F(s)=\frac{1}{s^2}$ and $G(s)=\frac{1}{s-1}$

Now $F(s)G(s)=\frac{1}{s^2(s-1)}$,

$$\therefore L\{f(t)*g(t)\}=F(s)G(s)=\frac{1}{s-1}-\frac{1}{s^2}-\frac{1}{s}=\frac{1}{s^2(s-1)}=F(s)G(s)$$

$$f(t)*g(t)=\int_0^t(t-u)e^u du=[(t-u)e^u-(-1)e^u]_0^t=e^t-t-1$$

74. Verify convolution theorem for the function $f(t)=t$ and $g(t)=\cos t$

Solution:

We get e $F(s)=\frac{1}{s^2}$ and $G(s)=\frac{s}{s^2+1}\Rightarrow \mathrm{F}(s)G(s)=\frac{1}{s(s^2+1)}$

Now $f(t)*g(t)=\int_0^t(t-u)\cos u du=[(t-u)\sin u-(-1)(-\cos u)]_0^t=1-\cos t$

$$L\{f(t)*g(t)\}=\frac{1}{s}-\frac{s}{s^2+1}-\frac{1}{s}=\frac{1}{s(s^2+1)}=F(s)G(s)$$

75. Find the inverse Laplace transform of the function $\frac{s}{(s+1)^2(s^2+1)}$ using convolution theorem.

Solution:

We use $L^{-1}\{F(s).G(s)\}=f(t)*g(t)=\int_0^t f(u)g(t-u)du$

$$F(s)=\frac{s}{s^2+1} and G(s)=\frac{1}{(s+1)^2}\Rightarrow f(t)=\cos t \text{ and } g(t)=e^{-t}t$$

$$L^{-1}\{F(s)G(s)\}=\int_0^t\cos(t-u)e^{-u}udu=\int u\{\cos(t-u)e^{-u}\}du$$

$$=u\frac{e^{-u}}{1+1}[-1\cos(t-u)+(-1)\sin(t-u)]_0^t+\int_{t=0}^{t}\frac{e^{-u}}{2}(\cos(t-u)+\sin(t-u))du$$

$$=\frac{te^{-t}}{2}[-1+0]+\frac{1}{2}\int_0^t e^{-u}\cos(t-u).du+\int_0^t \sin(t-u)e^{-u}.du$$

$$=\frac{-te^{-t}}{2}+\frac{1}{2}\left\{\cos(t-u)\frac{e^{-u}}{-1}+\int e^{-u}.du\right\}$$

$$=\frac{-te^{-t}}{2}+\frac{1}{2}\frac{e^{-u}}{2}\left[-\cos(t-u)-\sin(t-u)\right]_0^t+\frac{1}{2}\frac{e^{-u}}{2}\left[-\sin(t-u)+\cos(t-u)\right]_0^t$$

$$=\frac{-te^{-t}}{2}+\frac{e^{-t}}{4}[-1]+\frac{\cos t+\sin t}{4}+\frac{e^{-t}}{4}[1]+\frac{\sin t-\cos t}{4}$$

$$=\frac{-te^{-t}}{2}+\frac{2\sin t}{4}\Rightarrow L^{-1}\left[\frac{s}{(s+1)^2(s^2+1)}\right]=\frac{\sin t-te^{-t}}{2}$$

76. **Find the inverse Laplace transform of the function** $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ **using convolution theorem**.

Solution:

Let $F(s)=\frac{s}{s^2+a^2}$ and and $g(t)=\cos bt$

$$L^{-1}\{F(s)G(s)\}=\int_0^t \cos au.\cos b(t-u).du$$

$$=\frac{1}{2}\int_0^t \cos\{au+bt-bu\}+\cos\{au-bt+bu\}.du$$

$$=\frac{1}{2}\int_0^t \cos(au+bt-bu).du+\frac{1}{2}\int_0^t \cos(au-bt+bu).du$$

$$=\frac{1}{2}\frac{\sin(au+bt-bu)}{a-b}\Big|_0^t-\frac{1}{2}\frac{\sin(au-bt+bu)}{a+b}\Big|_0^t=\frac{1}{2}\frac{\sin at}{a-b}-\frac{1}{2}\frac{\sin bt}{a-b}+\frac{1}{2}\frac{\sin at}{a+b}+\frac{1}{2}\frac{\sin bt}{a+b}$$

$$=\frac{\sin at}{2}\frac{2a}{a^2-b^2}+\frac{\sin bt}{2}\frac{-2b}{a^2-b^2}\Rightarrow L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]=\frac{a\sin at-b\sin bt}{a^2-b^2}$$

77. Find the inverse Laplace transform of the function $\frac{1}{\left(s^2+a^2\right)^2}$ using convolution theorem.

Solution:

Let $F(s)=\frac{1}{s^2+a^2}$ and $G(s)=\frac{1}{s^2+a^2} \Rightarrow f(t)=\frac{\sin at}{a}$ and $g(t)=\frac{\sin at}{a}$

$$L^{-1}\{F(s)G(s)\}=\frac{1}{a^2}\int_0^t \sin au.\sin(t-u)a.du=\frac{1}{2a^2}\int_0^t \cos(au-at+au)-\cos(au+at-au).du$$

$$=\frac{1}{2a^2}\frac{\sin(2au-at)}{2a}\Bigg|_0^t-\frac{1}{2a^2}\cos at.u\Bigg|_0^t$$

$$=\frac{1}{4a^3}(\sin at+\sin at)-\frac{1}{2a^2}\cos at.t=\frac{\sin at}{2a^3}-\frac{t\cos at}{2^2}$$

78. Find the inverse Laplace transform of the function $\frac{1}{(s+1)\left(s^2+1\right)}$ using convolution theorem.

Solution:

Let $F(s)=\frac{1}{s+1}$ and $G(s)=\frac{1}{s^2+1} \Rightarrow f(t)=e^{-t}$ and $g(t)=\sin t$

$$L^{-1}\{F(s)G(s)\}=\frac{1}{a}\int_0^t e^{-u}\sin(t-u)du=\frac{e^{-u}}{1+1}\left[-\sin(t-u)+\cos(t-u)\right]_0^t$$

$$=\frac{1}{2}e^{-t}(1)-\frac{1}{2}\left[-\sin u+\cos t\right]=\frac{e^{-t}+\sin t-\cos t}{2}$$

79. Find the inverse Laplace transform of the function $\frac{s}{\left(s^2+a^2\right)^2}$ using convolution theorem.

Solution:

Let $F(s)=\frac{s}{s^2+a^2}$ and $G(s)=\frac{1}{s^2+a^2} \Rightarrow f(t)=\cos at$ and $g(t)=\frac{\sin at}{a}$

$$L^{-1}\{F(s)G(s)\}=\frac{1}{a}\int_0^t \sin au\cos(t-u).du=\frac{1}{2a}\int_0^t \sin(au+at-au)+\sin(au-at+au).du$$

$$=\frac{1}{2a}\int_0^t \sin at.du+\frac{1}{2a}\int_0^t \sin(2au-at).du=\frac{\sin at}{2a}u\Bigg|_0^t+\frac{1}{2a}\frac{-\cos(2au-at)}{2a}\Bigg|_0^t$$

$$= \frac{\sin at}{2a} t - \frac{1}{4a^2} [\cos at - \cos at]$$

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{t}{2a} \sin at$$

80. Find the inverse Laplace transform of the function $\frac{s^2}{(s^2+a^2)^2}$ using convolution theorem.

Solution:

Let $F(s) = \frac{s}{(s^2+a^2)^2}$ and $G(s) = \frac{s}{s^2+a^2} \Rightarrow f(t) = \cos at$ and $g(t) = \cos at$

$$L^{-1}\{F(s)G(s)\} = \int_0^t \cos au \cos a(t-u).du = \frac{1}{2}\int_0^t \cos(at-au+au) + \cos(at-au-au).du$$

$$= \frac{1}{2}\int_0^t \cos(at-au+au) + \cos(at-au-au).du = \frac{1}{2}\int_0^t \cos at.du + \frac{1}{2}\int_0^t \cos(at-2au).du$$

$$= \frac{1}{2}\cos at.u\Big|_0^t + \frac{1}{2}\frac{\sin(at-2au)}{-2a}\Big|_0^t = \frac{t}{a}\cos at - \frac{1}{4a}[-\sin at - \sin at]$$

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{t}{2}\cos at + \frac{1}{2a}\sin at$$

3.9.11 Solving ordinary differential equations (ODEs) using Laplace Transforms.

Procedure:

- Laplace transform method of solving differential equations yields particular solutions without the necessity of first finding the general solution and then evaluating the arbitrary constant and in non homogeneous ODEs without first solving the corresponding homogeneous ODE.
- More importantly, the use of the unit step function and Dirac's delta function make the method particularly powerful for problems with inputs (driving forces) that have discontinuities.

- For instance, in the linear mathematical models for physical systems such as spring mass system ,the governing ODE is $m\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+kx=f(t)$
  Where , $f(t)$ represents external force and is a discontinuous function.

  Solving this DE is difficult but not impossible. This can be solved by using Laplace transform.

3.9.12 Derivative of Laplace Transform:

1. If $f'(t)$ be continuous and $L[f(t)]=f(s)$,then $L[f'(t)]=sF(s)-f(0)$.

Proof:

$$L[f'(t)]=\int_0^\infty e^{-st}f'(t).dt=\left[e^{-st}f(t)\right]_0^\infty-\int_0^\infty(-s)e^{-st}f(t).dt$$

Now assuming $f(t)$to be such that $\underset{t\to\infty}{lt}\, e^{-st}f(t)=0$ When this condition is satisfied, $f(t)$ is said to be of exponential of order s.

Thus, $L[f'(t)]=f(0)+s\int_0^\infty e^{-st}f(t).dt=sF(s)-f(0)$.

2. If $f'(t)$and its first $(n-1)$derivatives be continuous, then

$$L\left[f^n(t)\right]=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-..........-f^{n-1}(0)$$

Proof:

$$L\left[f^n(t)\right]=\int_0^\infty e^{-st}f^n(t).dt$$

$$=-f^{n-1}(0)-sf^{n-2}(0)-s^2f^{n-3}(0)-......-s^{n-1}f(0)+s^n.\int_0^\infty e^{-st}f(t).dt$$

$$\Rightarrow L\left[f^n(t)\right]=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-.......-f^{n-1}(0)$$

Note:

Let $Y=L\{y(t)\}$ then we have

$L\{y'(t)\}=sY-y(0)$

$L\{y''(t)\}=s^2Y-sy(0)-y'(0)$

$L\{y'''(t)\}=s^3Y-s^2y(0)-sy'(0)-y''(0)$

81. **Solve** $y''+4y'+3y=e^{t}, y(0)=1=y'(0)$.

Solution:

Let $L\{y(t)\}=Y(s) \Rightarrow L\{y'(t)\}=sY(s)-y(0)=sY(s)-1$

$L\{y''(t)\}=s^2Y(s)-sy(0)-y'(0)=s^2Y(s)-s-1$

We have $y''+4y'+3y=e^t$

$$\therefore L\{y''\}+4L\{y'\}+3L\{y\}=\frac{1}{s-1}$$

$$s^2Y(s)-s-1+4[sY(s)-s-1]+3Y(s)=\frac{1}{s-1} \text{ Or}$$

$$Y=\frac{s^2+4s-4}{(s-1)^2(s^2+4s+3)}=\frac{A}{s-1}+\frac{B}{s+3}+\frac{C}{s+1}=\frac{1}{8(s-1)}-\frac{7}{8(s+3)}+\frac{7}{4(s+1)}$$

Taking inverse Laplace transform

$$y(t)=\frac{e^t}{8}-\frac{7}{8}e^{-3t}+\frac{7}{4}e^{-t}$$

82. **Solve** $y''+4y'+3y=e^{-t}, y(0)=1=y'(0)$

Solution:

Let $L\{y(t)\}=Y(s)=Y$

Given $y''+4y'+3y=e^{-t}, y(0)=1=y'(0)$

Taking transform on both sides

$$L\{y''(t)\}+4L\{y'(t)\}+3L\{y(t)\}=\frac{1}{s+1}$$

$$s^2Y-sy(0)-y'(0)+4\{sY-y(0)\}+3Y=\frac{1}{s+1}$$

$$Y=\frac{(s^2+6s+6)}{(s+1)(s+1)(s+3)}=\frac{(s^2+6s+6)}{(s+1)^2(s+3)}$$

Resolving into partial fractions

$$Y=\frac{7/4}{s+1}+\frac{1/2}{(s+1)^2}+\frac{-3/4}{(s+3)}$$

Taking inverse Laplace transform

$$y(t)=\frac{7}{4}L^{-1}\left\{\frac{1}{s+1}\right\}+\frac{1}{2}L^{-1}\left\{\frac{1}{(s+1)^2}\right\}-\frac{3}{4}L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$y(t)=\frac{7}{4}e^{-t}+\frac{1}{2}te^{-t}-\frac{3}{4}e^{-3t}$$

83. **Solve** $y''+2y'-3y=\sin t, y(0)=y'(0)=0$

Solution:

Given $y''+2y'-3y=\sin t, y(0)=y'(0)=0$

$$\therefore L\{y''\}+2L\{y'\}-3L\{y\}=\frac{1}{s^2+1}$$

$$s^2Y-sy(0)-y'(0)+2\{sY-y(0)\}-3y=\frac{1}{s^2+1}$$

$$Y=\frac{1}{(s^2+2s-3)(s^2+1)}=\frac{1}{(s+3)(s-1)(s^2+1)}$$

Let $Y=\frac{1}{(s+3)(s-1)(s^2+1)}=\frac{A}{s+3}+\frac{B}{s-1}+\frac{Cs+D}{s^2+1}$

After resolving into partial fractions and taking inverse transforms

$$y(t)=\frac{-1}{40}e^{-3t}+\frac{1}{8}e^{t}-\frac{1}{10}\cos t-\frac{1}{5}\sin t$$

84. **Solve** $y''+y=H(t-1), y(0)=0, y'(0)=1.$

Solution:

Given $y''+y=H(t-1), y(0)=0, y'(0)=1.$

$$\therefore L\{y''\}+L\{y\}=e^{-s}\frac{1}{s}$$

$$s^2Y-sy(0)-y'(0)+Y=\frac{e^{-s}}{s}\Rightarrow s^2Y-1+Y=\frac{e^{-s}}{s}\Rightarrow Y=\frac{1}{s^2+1}+\frac{e^{-s}}{s(s^2+1)}$$

Now $L^{-1}\left\{\frac{1}{s^2+1}\right\}=\sin t\Rightarrow L^{-1}\left\{\frac{1}{s}\frac{1}{s^2+1}\right\}=\int_0^t \sin u.du=-\cos\Big|_0^t=1-\cos t$

$$L^{-1}\left\{\frac{e^{-s}}{s(s^2+1)}\right\}=\{1-\cos(t-1)\}H(t-1)\Rightarrow y(t)=\sin t+[1-\cos(t-1)]H(t-1)$$

Department of Mathematics

85. **Solve** $y' + 2y + \int_0^t y.du = \sin t, y(0) = 1$

Solution:

Given $y' + 2y + \int_0^t y.du = \sin t, y(0) = 1$

$$L\{y'\} + 2L\{y\} + L\int_0^t y.du = \frac{1}{s^2+1} \quad sY - y(0) + 2Y + \frac{1}{s}Y = \frac{1}{s^2+1}$$

$$Y = \frac{s^3+s+s}{s^2+1}\frac{1}{(s+1)^2} = \frac{s^3+2s}{(s+1)^2(s^2+1)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+1}$$

After resolving into partial fractions

$$Y = \frac{1}{s+1} + \frac{-3/2}{(s+1)^2} + \frac{1/2}{s^2+1}$$

Taking inverse transforms

$$y(t) = e^{-t} - \frac{3}{2}te^{-t} + \frac{1}{2}\sin t$$

86. **Solve** $(D^2 + n^2)x = a\sin(nt + a)$ **given** $x = Dx = 0$ **when** $x = 0$

Solution:

Given $x'' + n^2x = a\{\sin nt\cos\alpha + \cos nt\sin\alpha\}$

Taking transform on both sides

$$L\{x''\} + n^2x = a\cos\alpha\frac{n}{s^2+n^2} + a\sin\alpha\frac{s}{s^2+n^2}$$

$$s^2X - sx(0) - x'(0) + n^2X = a\cos\alpha\frac{n}{s^2+n^2} + a\sin\alpha\frac{s}{s^2+n^2}$$

$$(s^2+n^2)X = a\cos\alpha\frac{n}{s^2+n^2} + a\sin\alpha\frac{s}{s^2+n^2}$$

$$X = a\cos\alpha\frac{n}{(s^2+n^2)^2} + a\sin\alpha\frac{s}{(s^2+n^2)^2}$$

$$x(t) = a\cos\alpha L^{-1}\left\{\frac{n}{(s^2+n^2)^2}\right\} + a\sin\alpha L^{-1}\left\{\frac{s}{(s^2+n^2)^2}\right\}$$

Department of Mathematics

$$x(t) = a\cos\alpha \frac{1}{2n^3}\left[\sin nt - nt\cos nt\right] + a\sin\alpha \frac{t}{2n}\sin nt .$$

3.10 Exercise

Sl.No	Questions	Answers
1.	Find $L\left[(3t+4)^3 + 5^t\right]$	$\frac{162}{s^4}+\frac{216}{s^3}+\frac{144}{s^2}+\frac{64}{s}+\frac{1}{s-\log 5}$
	Find $L[\sin 5t \cdot \cos 2t]$	$\frac{1}{2}\left[\frac{7}{s^2+49}+\frac{3}{s^2+9}\right]$
	Find $L\left[\sqrt{t}-\frac{1}{\sqrt{t}}\right]^3$	$\sqrt{\pi}\left[\frac{3}{4s^2\sqrt{s}}+2\sqrt{s}-\frac{3}{2s\sqrt{s}}+\frac{3}{\sqrt{s}}\right]$
	Find $L\left[e^{-t}\cos^2 3t\right]$	$\frac{1}{2}\left[\frac{1}{s+1}+\frac{(s+1)}{(s+1)^2+36}\right]$
	Find $L\left[t^3\cosh t\right]$	$3\left[\frac{1}{(s-1)^2}+\frac{1}{(s+1)^2}\right]$
	Find $L\left[\frac{\sin^2 t}{t}\right]$	$\frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)$
2	Evaluate $\int_0^\infty \frac{e^{-t}\sin t}{t}dt$	$\frac{\pi}{4}$
3.	Evaluate $\int_0^\infty \frac{\cos 6t-\cos 4t}{t}dt$	$\log(2/3)$
4	Find Laplace Transform of $f(t)=\begin{cases}3t & 0\le t\le 2\\ 6 & 2\le t\le 4\end{cases}$,where $f(t)$ is a periodic function of period 4	$\frac{1}{1-e^{-4s}}\left\{\frac{3-3e^{-2s}-6e^{-4s}}{s^2}\right\}$
5	Find Laplace Transform of $f(t)=\begin{cases}3t & 0\le t\le 2\\ 6 & 2\le t\le 4\end{cases}$,where $f(t)$ is a periodic function of period 2π	$\frac{1}{1-e^{-\pi s}(s^2+1)}$

Department of Mathematics

6	Express the function in terms of Heaviside unit function and hence find its Laplace Transform $f(t)=\begin{cases} e^{2t} & 0\le t\le 1 \\ 2 & t\ge 1 \end{cases}$	$\frac{1}{s-2}+e^{-s}\left(\frac{2}{s}-\frac{e^2}{s-2}\right)$
7	Express the function in terms of Heaviside unit function and hence find its Laplace Transform $f(t)=\begin{cases} t & 0< t< 4 \\ 5 & t> 4 \end{cases}$	$\frac{1}{s^2}+e^{-4s}\left(\frac{1}{s}-\frac{1}{s^2}\right)$
8	Find $L\left[t^4\delta(t-3)\right]$	$81e^{-3s}$
9.	Find $L^{-1}\left[\frac{s+5}{s^2-6s+13}\right]$	$e^{3t}(\cos 2t+4\sin 2t)$
10	Find $L^{-1}\left[\frac{s+2}{s^2(s+3)}\right]$	$\frac{1}{9}\left(1+6t-e^{-3t}\right)$
11	Find $L^{-1}\left[\cot^{-1}(s/a)\right]$	$\frac{\sin at}{t}$
12	Find $L^{-1}\left[\log\left[\frac{s^2+4}{s(s+4)(s-4)}\right]\right]$	$\frac{e^{-bt}-e^{-at}}{t}$
13	Find $L^{-1}\left[\frac{s}{(s^2-a^2)^2}\right]$	$\frac{t\sinh at}{2a}$
14	Find $L^{-1}\left[\frac{3e^{-3s}}{s}-\frac{e^{-s}}{s^2}\right]$	$3.u(t-3)-(t-1)u(t-1)$
15	Verify Convolution for the functions $f(t)=\sin t$ and $g(t)=e^{-t}$	$\frac{1}{(s^2+1)(s+1)}$
16.	Verify Convolution for the functions $f(t)=\cos at$ and $g(t)=\cos bt$	$\frac{s^2}{(s^2+a^2)(s^2+b^2)}$
17	Find $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$,Using Convolution Theorem	$\frac{1}{a^2}\left(1-\cos at\right)$
18	Find $L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right]$,Using Convolution Theorem	$\frac{1}{2}\left(e^t-\sin t-\cos t\right)$
19	Solve the initial Value Problem by using Laplace transform $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+4y=e^{-t}$; $y(0)=0=y'(0)$	$e^{-t}-(t+1)e^{-2t}$

Department of Mathematics

20	Solve the initial Value Problem by using Laplace transform $\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=5e^{2t}$; $y(0)=2, y'(0)=1$	$\frac{1}{4}e^{2t}+\frac{23}{4}e^{-2t}-4e^{-3t}$