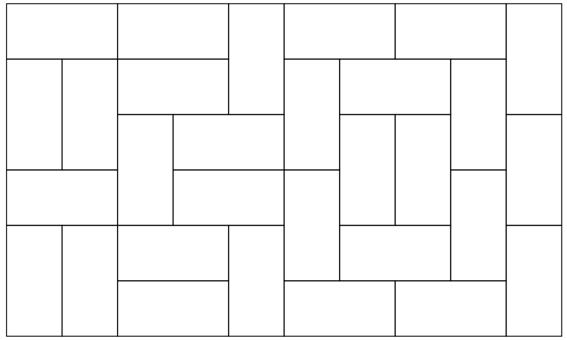
Tilings, Paths, Resultants, Tableaux

Talk given at INRIA on February 25, 2002

Volker Strehl, University of Erlangen-Nürnberg (Germany)



the first values of $k_{m,n}$ = the number of domino tilings of an $(m \times n)$-rectangle

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 \\
1 & 0 & 3 & 0 & 11 & 0 & 41 & 0 & 153 \\
1 & 1 & 5 & 11 & 36 & 95 & 281 & 781 & 2245 \\
1 & 0 & 8 & 0 & 95 & 0 & 1183 & 0 & 14824 \\
1 & 1 & 13 & 41 & 281 & 1183 & 6728 & 31529 & 167089 \\
1 & 0 & 21 & 0 & 781 & 0 & 31529 & 0 & 1292697 \\
1 & 1 & 34 & 153 & 2245 & 14824 & 167089 & 1292697 & 12988816
\end{bmatrix}$$

the KTF-formula

$$k_{m,n} = \prod_{j=1}^{\lceil m/2 \rceil} \frac{c_j^{n+1} - \widehat{c}_j^{n+1}}{2b_j}$$

where $b_j = \sqrt{1 + \cos^2 \frac{j\pi}{m+1}}$, $c_j = b_j + \cos \frac{j\pi}{m+1}$, $\widehat{c}_j = -b_j + \cos \frac{j\pi}{m+1}$,

or else

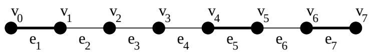
$$k_{2m,2n} = 4^{mn} \prod_{j=1}^{m} \prod_{k=1}^{n} \left(\cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+1} \right)$$

Kasteleyn/Temperley/Fisher (1961)

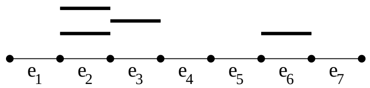
a combinatorial proof of the KTF-formula

- preliminaries on heaps of dominos
- from tilings to GV-systems on $\Gamma_{m,n}$
- extending the graph $\Gamma_{m,n}$
- factorizing the paths in $\overline{\Gamma}_{m,n}$
- dualizing the heaps
- from Binet-Cauchy to Pascal: the resultant appears

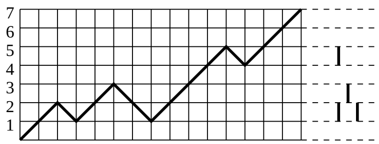
a trivial domino heap



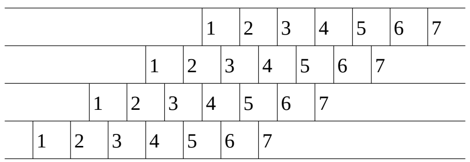
a domino heap



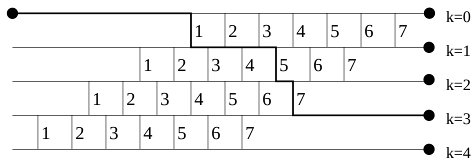
a domino heap seen as a lattice path



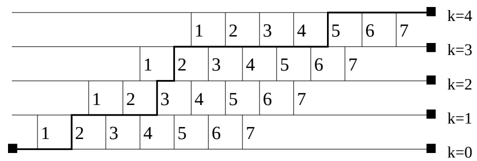
another grid graph



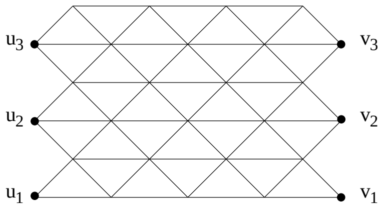
a trivial heap



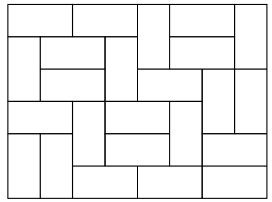
a heap



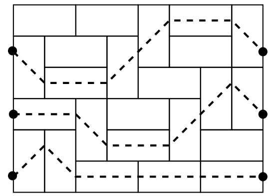
the graph $\Gamma_{3,4}$



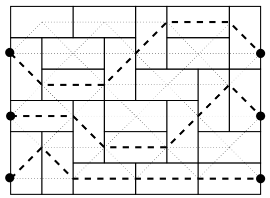
a domino tiling of a 6×8-rectangle



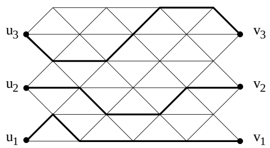
equivalence of domino tilings and path systems



overlay of tiling and path system



the path system equivalent to the tiling

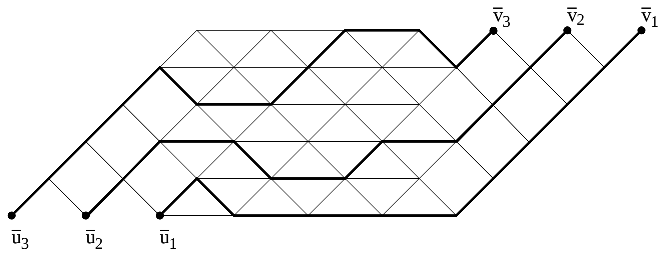


counting tilings of a (6×8)-rectangle by counting Gessel-Viennot-systems on $\Gamma_{3,4}$

$$\mathcal{P}_{3,4} = \begin{bmatrix} 90 & 146 & 69 \\ 146 & 305 & 215 \\ 69 & 215 & 236 \end{bmatrix}$$

$$\det \mathcal{P}_{3,4} = 167089$$

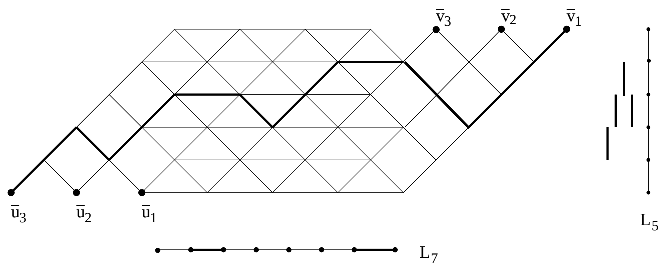
the extended graph $\overline{\Gamma}_{3,4}$ and the extended path system



the numerical picture of this extension

$$\begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 90 & 146 & 69 \\ 146 & 305 & 215 \\ 69 & 215 & 236 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1019 & 3460 & 11470 \\ 284 & 1019 & 3460 \\ 69 & 284 & 1019 \end{bmatrix}$$

factorizing the paths in $\overline{\Gamma}_{3,4}$ into domino heaps and trivial domino heaps



the generating polynomial for trivial heaps of dominos on L_7

$$f_4(t) = t^4 + 7t^3 + 15t^2 + 10t + 1$$

the generating function for domino heaps on L_5

$$\begin{aligned} g(t) &= \frac{1}{1 - 5t + 6t^2 - t^3} \\ &= 1 + 5t + 19t^2 + 66t^3 + 221t^4 + 728t^5 + 2380t^6 + 7753t^7 + \dots \end{aligned}$$

factorizing the GV-matrix for $\overline{\Gamma}_{3,4}$

$$\mathcal{F}_{3,4} = \begin{bmatrix} 1 & 7 & 15 & 10 & 1 & 0 & 0 \\ 0 & 1 & 7 & 15 & 10 & 1 & 0 \\ 0 & 0 & 1 & 7 & 15 & 10 & 1 \end{bmatrix}$$

$$\mathcal{G}_{3,4} = \begin{bmatrix} 221 & 66 & 19 & 5 & 1 & 0 & 0 \\ 728 & 221 & 66 & 19 & 5 & 1 & 0 \\ 2380 & 728 & 221 & 66 & 19 & 5 & 1 \end{bmatrix}$$

$$\mathcal{F}_{3,4} \cdot \mathcal{G}^t_{3,4} = \begin{bmatrix} 1019 & 3460 & 11470 \\ 284 & 1019 & 3460 \\ 69 & 284 & 1019 \end{bmatrix} = \mathcal{H}_{3,4}$$

$$\det \mathcal{H}_{3,4} = 167089$$

from Binet-Cauchy to Pascal: the resultant appears

$$\det \begin{bmatrix} 1 & 7 & 15 & 10 & 1 & 0 & 0 \\ 0 & 1 & 7 & 15 & 10 & 1 & 0 \\ 0 & 0 & 1 & 7 & 15 & 10 & 1 \end{bmatrix} \cdot \begin{bmatrix} 221 & 728 & 2380 \\ 66 & 221 & 728 \\ 19 & 66 & 221 \\ 5 & 19 & 66 \\ 1 & 5 & 19 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\det \begin{bmatrix} 1 & 7 & 15 & 10 & 1 & 0 & 0 \\ 0 & 1 & 7 & 15 & 10 & 1 & 0 \\ 0 & 0 & 1 & 7 & 15 & 10 & 1 \\ -1 & 5 & -6 & 1 & 0 & 0 & 0 \\ 0 & -1 & 5 & -6 & 1 & 0 & 0 \\ 0 & 0 & -1 & 5 & -6 & 1 & 0 \\ 0 & 0 & 0 & -1 & 5 & -6 & 1 \end{bmatrix} = \mathrm{resultant}_t(f_4(t), f_3(-t))$$

the general statement

- the matching polynomials $f_n(t)$ for L_{2n-1} are

$$f_n(t) = \sum_{k=0}^{n} \binom{n+k}{n-k} t^k = \sum_{k=0}^{n} f_{n,k}\, t^{n-k}$$

- they satisfy the recurrence

$$\begin{aligned} f_0(t) &= 1 \\ f_1(t) &= 1 + t \\ f_{n+1}(t) &= (t+2) f_n(t) - f_{n-1}(t) \quad (n \geq 1) \end{aligned}$$

- they are essentially Chebychev polynomials of the second kind:

$$f_n(t) = (-1)^n\, U_{2n}(i\sqrt{t})$$

- the result:

$$\boxed{\begin{array}{c} \text{the number of domino tilings of a } (2m \times 2n)\text{-rectangle is} \\ k_{2m,2n} = \mathrm{resultant}_t(f_n(t), f_m(-t)) \end{array}}$$

about the proof

$$\mathcal{F}_{m,n} = \begin{bmatrix} f_{n,0} & f_{n,1} & f_{n,2} & \dots & f_{n,n} & 0 & 0 & \dots & 0 \\ 0 & f_{n,0} & f_{n,1} & \dots & \dots & f_{n,n} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \ddots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & f_{n,0} & f_{n,1} & f_{n,2} & \dots & f_{n,n} \end{bmatrix}$$

$$\mathcal{G}_{m,n} = \begin{bmatrix} g_{m,n} & g_{m,n-1} & \cdots & g_{m,0} & 0 & 0 & \dots & 0 \\ g_{m,n+1} & g_{m,n} & \cdots & g_{m,1} & g_{m,0} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \ddots & \vdots \\ g_{m,m+n-1} & g_{m,m+n-2} & \cdots & g_{m,m-1} & g_{m,m-2} & g_{m,m-3} & \cdots & g_{m,0} \end{bmatrix}$$

where

$$g_m(t) = \frac{1}{f_m^*(-t)} = \sum_{k\geq 0} g_{m,k}\, t^k$$

then

$$\begin{aligned} k_{2m,2n} &= \det \mathcal{P}_{m,n} && \text{the GV-matrix for } \Gamma_{m,n} \\ &= \det \underbrace{\mathcal{L}_m \cdot \mathcal{P}_{m,n} \cdot \mathcal{R}_m}_{\mathcal{H}_{m,n}} && \text{the GV-matrix for } \overline{\Gamma}_{m,n} \\ &= \det \mathcal{F}_{m,n} \cdot \mathcal{G}^t_{m,n} && \text{paths in } \overline{\Gamma}_{m,n} \leftrightarrow \begin{cases} (\text{triv. heap in } L_{2n-1}, \\ \text{heap in } L_{2m-1}) \end{cases} \\ &= \det \begin{bmatrix} \mathcal{F}_{m,n} \\ \mathcal{F}'_{n,m} \end{bmatrix} && \text{dualizing the heaps} \\ &= \det \mathrm{Sylvester}(f_n(t), f_m(-t)) \\ &= \mathrm{resultant}_t(f_n(t), f_m(-t)) \end{aligned}$$

note

$$\begin{aligned} \mathcal{L}_m \cdot \mathcal{P}_{m,n} \cdot \mathcal{R}_m &= \mathcal{H}_{m,n} \\ &= \begin{bmatrix} h_n & h_{n+1} & \dots & h_{n+m-1} \\ h_{n-1} & h_n & \dots & h_{n+m-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n-m+1} & h_{n-m+2} & \dots & h_n \end{bmatrix} \\ &= \mathcal{F}_{m,n} \cdot \mathcal{G}^t_{m,n} \end{aligned}$$

where

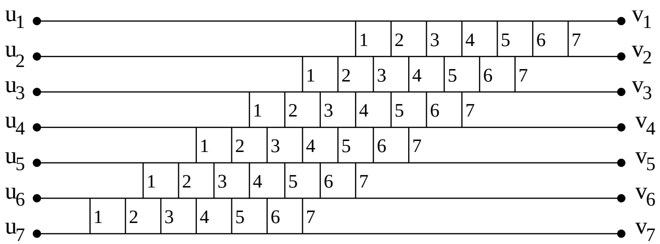
$$\sum_{k\geq 0} h_k\, t^k = \frac{f_n^*(t)}{f_m^*(-t)}$$

from Binet-Cauchy to Laplace: the algebraic picture

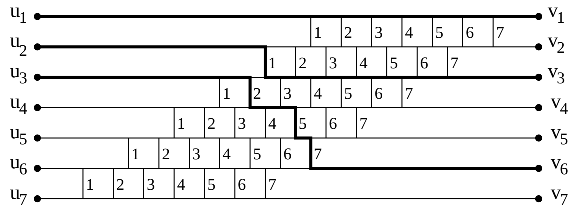
$$\begin{aligned} \det(\mathcal{F}_{m,n} \cdot \mathcal{G}^t_{m,n}) &= \sum_{J \in \binom{[m+n]}{m}} \det \mathcal{F}_{m,n}\langle J\rangle \cdot \det \mathcal{G}_{m,n}\langle J\rangle \\ &= \sum_{J \in \binom{[m+n]}{m}} \det \mathcal{F}_{m,n}\langle J\rangle \cdot \det \mathcal{F}_{n,m}\langle J^c\rangle \\ &= \sum_{J \in \binom{[m+n]}{m}} \mathrm{sign}(J, J^c) \cdot \det \mathcal{F}_{m,n}\langle J\rangle \cdot \det \mathcal{F}'_{n,m}\langle J^c\rangle \\ &= \det \begin{bmatrix} \mathcal{F}_{m,n} \\ \mathcal{F}'_{n,m} \end{bmatrix} \end{aligned}$$

from Binet-Cauchy to Laplace: the combinatorial picture (dualization of path systems)

the graph $\Phi_{3,4}$

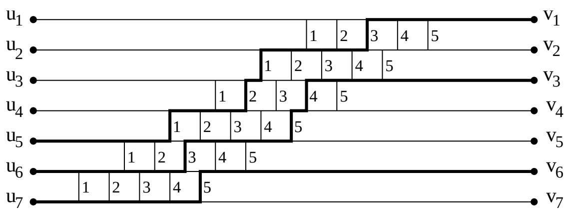


a family of trivial heaps represented in $\Phi^{\downarrow}_{3,4}$



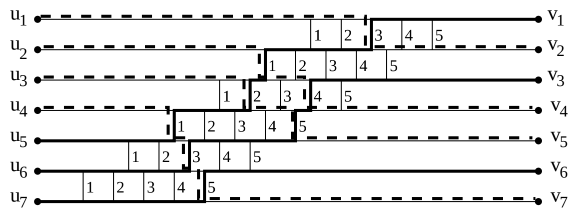
$$\det \mathcal{F}_{m,n}\langle J\rangle = \begin{cases}\text{the number of families of vertex disjoint paths}\\ \pi \in \mathcal{P}^0(\Phi^{\downarrow}_{m,n}; \boldsymbol{u}_{[m]}, \boldsymbol{v}_J)\end{cases}$$

a familiy of heaps represented in $\Phi^{\downarrow}_{4,3}$

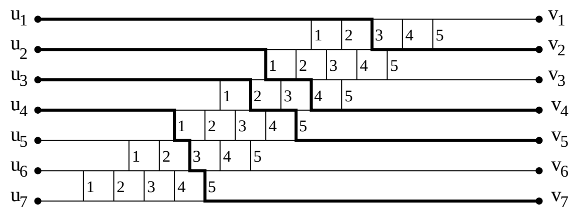


$$\det \mathcal{G}_{m,n}\langle J\rangle = \begin{cases}\text{the number of families of vertex disjoint paths}\\ \gamma \in \mathcal{P}^0(\Phi^{\uparrow}_{n,m}; \boldsymbol{u}_{[n+m\setminus[m]}, \boldsymbol{v}_J)\end{cases}$$

from heap families in $\Phi^{\uparrow}_{4,3}$ to trivial heap families in $\Phi^{\downarrow}_{4,3}$



the complementary path system after dualization



consequences

- the KTF-formula

$$k_{2m,2n} = 4^{mn} \prod_{j=1}^{m} \prod_{k=1}^{n} \left(\cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+1} \right)$$

follows from knowledge about Chebychev polynomials and classical properties of the resultant, indeed:

$$f_n(t) = \prod_{1 \le j \le n} \left(t + 4\cos^2 \frac{j\pi}{2n+1} \right) = \prod_{1<d|2n+1} \prod_{\substack{1 \le j < d \\ j \text{ odd} \\ (j,d)=1}} \left(t + 4\cos^2 \frac{j\pi}{d} \right)$$

(the factorization over $\mathbb{Q}$)

note that the factorization is an immediate consequence of the relation between the $f_n(t)$ and the cyclotomic polynomials

$$f_n^*(t-2)|_{t \leftarrow x + \frac{1}{x}} = \frac{1 + x^{2n+1}}{1+x} = \prod_{1 \ne d|2n+1} \Phi_d(-x)$$

which follows from the recurrence for the $f_n(t)$

a beautiful consequence of the KTF-formula

$$\lim_{n \to \infty} \frac{\log k_{2n,2n}}{|K_{2n,2n}|} = \frac{G}{\pi}$$

where $|K_{2n,2n}|$ is the size of the $(2n \times 2n)$-square and

$$G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} \mp$$

is Catalan's constant

further properties of the numbers $k_{m,n}$ can be deduced from knowledge about Chebychev polynomials, like

$$\gcd\left(f_{(p-1)/2}(t), f_{(q-1)/2}(t)\right) = f_{(\gcd(p,q)-1)/2}(t)$$

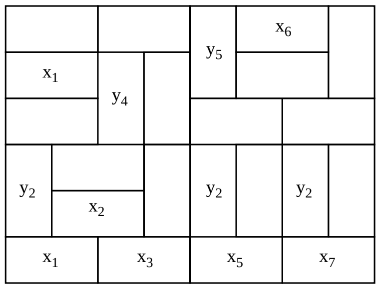
and properties of resultants

extending the KTF-formula

- counting tiles according to their position
  - $c_t(\mathbf{x}, \mathbf{y})$: the content of a tiling t
  - the generating polynomial for the tiling content

$$C_{m,n}(\mathbf{x}, \mathbf{y}) := \sum_{t \text{ is (m,n)-tiling}} c_t(\mathbf{x}, \mathbf{y})$$

a tiling t with content $c_t(\boldsymbol{x}, \boldsymbol{y}) = x_1^2 x_2 x_3 x_5 x_6 x_7 y_2^3 y_4 y_5$



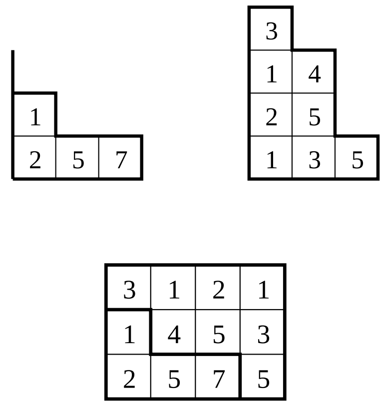
the generalized matching polynomials $f_n(\mathbf{x}; t)$,
where $\mathbf{x} = (x_1, x_2, \ldots, x_{2n-1})$, are defined by the recursion

$$\begin{aligned}
f_0(-; t) &= 1 \\
f_1(x_1; t) &= t + x_1 \\
f_{n+1}(x_1, \ldots, x_{2n+1}; t) &= (t + x_{2n} + x_{2n+1}) f_n(x_1, \ldots, x_{2n-1}; t) \\
&\quad - x_{2n-1} x_{2n} f_{n-1}(x_1, \ldots, x_{2n-3}; t)
\end{aligned}$$

statement of the general result

> the generating polynomial for tilings of a $2m \times 2n$ rectangle by content is
>
> $$C_{m,n}(\mathbf{x}, \mathbf{y}) = \mathrm{resultant}_t(f_n(\mathbf{x}; t), f_m(\mathbf{y}; -t))$$

complementary path systems seen as complementary 2-tableaux



interpreting complementary GV-systems in terms of 2-tableaux

- a 2-tableau of shape λ is a mapping $\tau : \lambda \to \mathbb{N}_+$ which is
  - 2-increasing along rows: $\tau_{i,j} \geq \tau_{i,j+1} + 2$
  - (-1)-increasing along columns: $\tau_{i+1,j} \geq \tau_{i,j} - 1$

  (whenever defined)
- the content of a 2-tableau τ of shape λ is

$$c_\tau(\mathbf{x}) = \prod_{(i,j)\in\lambda} x_{\tau_{i,j}}$$

- For m, n fixed and any shape λ that fits into a $m \times n$-rectangle,

$$t_\lambda(\mathbf{x}) = \sum_{\substack{\tau:\lambda\to[1..2n-1] \\ 2-\text{tableau}}} c_\tau(\mathbf{x})$$

- For m, n fixed and any shape λ that fits into a $m \times n$-rectangle, $\tilde{\lambda}$ denotes the complementary shape
- The tableau-interpretation of the resultant yields

$$\mathrm{resultant}_t(f_n(\mathbf{x}; t), f_m(\mathbf{y}, -t)) = \sum_{\lambda\subseteq m\times n} t_\lambda(\mathbf{x})\, t_{\tilde{\lambda}}(\mathbf{y})$$

the extended KTF-formula contains the "dual Cauchy-identity" for Schur functions

- specializing $x_2 = x_4 = \ldots = x_{2n-2} = 0$ gives

$$f_n(x_1, 0, x_3, 0, \ldots, 0, x_{2n-1}; t) = \prod_{1 \le j \le n} (t + x_{2j+1})$$

$$t_\lambda(x_1, 0, x_3, 0, \ldots, 0, x_{2n-1}) = s_\lambda(x_1, x_3, \ldots, x_{2n-1})$$

- specializing the y_i in the same way yields

$$\prod_{1 \le j \le n} \prod_{1 \le k \le m} (x_{2j+1} + y_{2k+1}) = \sum_{\lambda \subseteq m \times n} s_\lambda(x_1, x_3, \ldots, x_{2n-1})\, s_{\tilde{\lambda}}(y_1, y_3, \ldots, x_{2m-1})$$