Appendix C

Power Series and Special Functions

A special function is a function — usually of a single variable — that arises in sufficiently many applications as to warrant its own name, an investigation of its basic properties, and the development of special algorithms for computing its values. The first "special functions" one meets up with are the exponential function, the natural logarithm, and the trigonometric functions. However, since these common functions, along with polynomials, rational functions, algebraic functions (combinations of roots), and the hyperbolic functions appear much earlier in one's mathematical education, they are usually referred to as *elementary functions*. True *special functions*, such as the gamma function, Bessel functions, Legendre functions, Airy functions, hypergeometric functions, and many many more, await a more advanced mathematical training. These special functions play a starring role in more advanced applications in physics, engineering and mathematics. They initially appear when one tries to solve linear partial differential equations in higher dimensions in non-rectangular coordinate systems. Application of the method of separation of variables method reduces the partial differential equation to an ordinary differential equations of a non-elementary type. The solutions to these special ordinary differential equations are the aforementioned special functions.

In this appendix, we collect together the required results about a few of the most important classes of special functions, including a short presentation of the series approach for solving non-elementary ordinary differential equations. We assume that the reader is fluent in the very basics of infinite series, including the definitions of convergence and absolute convergence, and the basic convergence tests, specifically the comparison, ratio and root tests. Anby standard calculus text, e.g., [**9**, **168**, **170**], can be relied on for this. For more advanced treatment of special functions, the reader can consult a variety of sources, including the handbooks [**3**, **145**], and the detailed texts [**136**, **144**].

C.1. Power Series.

By definition, a *power series* has the form

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots = \sum_{k=0}^{\infty} c_k x^k. \tag{C.1}$$

The coefficients c_k nd the variable x can either be real or complex numbers. A power series clearly converges at $x = 0$, since then all terms but the first are zero, and so $f(0) = c_0$. A simple application of the comparison test shows that if the series converges for a given value $x = a$, then is automatically converges, in fact absolutely, for all x with smaller modulus (absolute value): $|\, x\,| < |\, a\,|$.

As a consequence, there are precisely three possibilites for the convergence of a power series:

(*a*) The series converges for all x.

(*b*) There is a positive number $\rho > 0$, called the *radius of convergence*, such that the series converges absolutely whenever $|\, x\,| < \rho$ and diverges whenever $|\, x\,| > \rho$. The series may or may not converge at points on the interface $|\, x\,| = \rho$.

(*c*) The series only converges, trivially, at $x = 0$. An example is the power series $\sum_n n!\, x^n$.

In case (*a*), we say $\rho = \infty$, while in case (*c*), $\rho = 0$. Thus, in the real case, a power series converges on a symmetric interval centered at the origin, including, possibly, one or both endpoints. In the conplex category, the power series converges inside a disk or radius ρ centered at the origin, including possibly some of its boundary points.

A function $f(x)$ represented by a convergent power series whose radius of convergence $\rho > 0$ is called *analytic*.

Applying the Weierstrass M test of Theorem 12.26, we deduce that the power series converges uniformly on any interval (disk) $\{\, \|\, x\,\| \leq r\,\}$ of radius $0 \leq r < \rho$. Thus, Proposition 12.27 implies that the differentiated power series converges uniformly, on the same subset, to the derivative

$$f'(x) = c_1 + 2\,c_2 x + 3\,c_3 x^2 + \;\cdots\; + (n+1) c_{n+1} x^n + \;\cdots\; = \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k. \tag{C.2}$$

In particular, the derivative of an analytic function exists and is itself analytic. Setting $x = 0$ in (C.2) yields $c_1 = f'(0)$. Clearly we can continue to differentiate, and hence every analytic function is infinitely differentiable. Interestingly, the converse is true for complex functions — see Chapter 16 — but false for real functions, as noted in Exercise 2.2.28. A straightforward induction shows that the $n^{\rm th}$ derivative has the power series expansion

$$f^{(n)}(x) = n!\, c_n + (n+1)!\, c_{n+1} x + \tfrac{1}{2}\,(n+2)!\, c_{n+2} x^2 + \;\cdots\; .$$

Again substituting $x = 0$ leads to the following key formula:

$$c_n = \frac{f^{(n)}(0)}{n!}\,, \tag{C.3}$$

for the $n^{\rm th}$ coefficient in the original power series as a multiple of the $n^{\rm th}$ derivative of the function at the origin. Thus, replacing each coefficient by its value, we conclude that

$$f(x) = f(0) + f'(0)\, x + \tfrac{1}{2}\, f''(0)\, x^2 + \;\cdots\; = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\, x^k. \tag{C.4}$$

You should instantly recognize the result as the *Taylor series* of the function $f(x)$. Thus, *a convergent*[†] *power series is the Taylor series for its sum*!

[†] More accurately, a power series with a positive radius of convergence.

The self-same principles immediately extend to series

$$f(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \ \cdots \ = \sum_{k=0}^{\infty} c_k(x - x_0)^k \tag{C.5}$$

in powers of $x - x_0$. Such a power series converges in the interval (disk) $|\,x - x_0\,| < \rho_0$ and diverge for $|\,x - x_0\,| > \rho_0$ for some $0 \le \rho_0 \le \infty$, the radius of convergence. Assuming $\rho > 0$, the sum $f(x)$ is, by definition, analytic at $x = x_0$, and, by the same argument, (C.5) is the Taylor series for $f(x)$ at x_0:

$$f(x) = f(x_0) + f'(x_0)\,(x - x_0) + \tfrac{1}{2}\,f''(x_0)\,(x - x_0)^2 + \cdots \ = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}\,(x - x_0)^k. \tag{C.6}$$

With a little more work, it can be shown that $f(x)$ is, in fact, analytic at all points x_1 such that $|\,x_1 - x_0\,| < \rho_0$. This means that $f(x)$ can be represented, near the point x_1, by a convergent series in powers of $x - x_1$. Mmoreover, the radius of convergence of the resulting iseries is at least as large as $\rho_1 = \rho_0 - |\,x_1 - x_0\,|$, which represents the distance from x_1 to the points (circle) $|\,x - x_0\,| = \rho$ representing the interface between the convergent and non-convergent regions of the original series (C.5). If, in fact, $\rho_1 > \rho_0 - |\,x_1 - x_0\,|$, then the new power series convrges at points not handled by the original power series, and the the result is an *ana;lytic continuation* of the function $f(x)$. The process can often be continued, although may result in multiply-valued complex functions. See Chapter 16 for some additional dicsussion.

The convergence of the Taylor series means that its partial sums will provide polynomial approximations to the (analytic) function. We state this result in a slightly altered notation that emphasizes the

The scalar version of Taylor's theorem with remainder.

Theorem C.1. *Let $u(x) \in \mathrm{C}^n$. Then the n^{th} order Taylor expansion of $u(x + h)$, for small h, is*

$$u(x + h) = u(x) + u'(x)\,h + u''(x)\,\frac{h^2}{2} + \ \cdots \ + u^{(n)}(x)\,\frac{h^n}{n!} + R_n(x, h), \tag{C.7}$$

where the remainder *goes to zero faster than*

$$\frac{R_n(x,h)}{h^n} \ \longrightarrow \ 0 \qquad \text{as} \qquad h \ \longrightarrow \ 0.$$

There are various formulas for the error, of which the most relevant is Cauchy's form

$$R_n(x, h) = u^{(n+1)}(\xi)\,\frac{h^{n+1}}{(n+1)!}, \tag{C.8}$$

where ξ is a point lying between[†] x and $x + h$. Cauchy's version assumes that $u \in \mathrm{C}^{n+1}$. This is a generalization of the well-known Mean Value Theorem

[†] In the complex version, the point ξ lies on the line segment connecting x to $x + h$.

Theorem C.2. *Suppose $f(u)$ is continuously differentiable. Then*

$$f(x) - f(a) = f'(\xi)\,(x - a) \qquad \text{for some } \xi \text{ between } x \text{ and } a. \tag{C.9}$$

Similarly, the first order Taylor expansion takes the form

$$f(x) = f(a) + f'(a)\,(x - a) + \tfrac{1}{2}\, f''(\xi)\,(x - a)^2 \quad \text{for some } \xi \text{ between } x \text{ and } a. \tag{C.10}$$

Example C.3. Consider the function

$$f(x) = (1 + x)^r.$$

Note that

$$f^{(k)}(x) = r(r-1)(r-2)\ \cdots\ (r-k+1)(1+x)^{r-k}.$$

Therefore, its Taylor expansion at $x = 0$ is

$$(1+x)^r = 1 + r\,x + \frac{r(r-1)}{2}\,x^2 + \frac{r(r-1)(r-2)}{6} + \ \cdots\ = \sum_{k=0}^{\infty} \binom{r}{k} x^k. \tag{C.11}$$

The coefficients are called *binomial coefficients*, and denoted

$$\binom{r}{k} = \frac{r\,(r-1)\,(r-2)\ \cdots\ (r-k+1)}{k\,(k-1)\,(k-2)\ \cdots\ 1}. \tag{C.12}$$

A simple application of the ratio test proves that the series (C.11) converges for all x such that $|\,x\,| < 1$, and diverges for $|\,x\,| > 1$, and thus the radisu of copnvergence is $\rho = 1$.

If $r = n$ is a non-negative integer, then the Taylor expansion terminates, and reduces to the well-known *Binomial Formula*

$$(1+x)^n = 1 + n\,x + \frac{n\,(n-1)}{2}\,x^2 + \ \cdots\ + n\,x^{n-1} + x^n = \sum_{k=0}^{n} \binom{n}{k}\, x^k. \tag{C.13}$$

In the vector-valued case, the first order Taylor expansion of a vector-valued function at a point $\mathbf{u}^\star$ takes the form

$$F(\mathbf{u}) = F(\mathbf{u}^\star) + \nabla F(\mathbf{u}^\star)\cdot(\mathbf{u} - \mathbf{u}^\star) + R(\mathbf{u} - \mathbf{u}^\star). \tag{C.14}$$

The remainder term depends quadratically on the distance from $\mathbf{u}$ to $\mathbf{u}^\star$, meaning that there is a positive constant C (depending on $\varepsilon > 0$) such that[†]

$$|\,R(\mathbf{u} - \mathbf{u}^\star)\,| \le C\,\|\,\mathbf{u} - \mathbf{u}^\star\,\|^2 \qquad \text{whenever} \qquad \|\,\mathbf{u} - \mathbf{u}^\star\,\| < \varepsilon. \tag{C.15}$$

We will also have occasion to use the second order expansion

$$F(\mathbf{u}) = F(\mathbf{u}^\star) + \nabla F(\mathbf{u}^\star)\cdot(\mathbf{u} - \mathbf{u}^\star) + \tfrac{1}{2}\,(\mathbf{u} - \mathbf{u}^\star)^T\nabla^2 F(\mathbf{u}^\star)\,(\mathbf{u} - \mathbf{u}^\star) + S(\mathbf{u} - \mathbf{u}^\star), \tag{C.16}$$

[†] One can use any convenient norm here.

where

$$| S(\mathbf{u} - \mathbf{u}^\star) | \le C_3 \, \| \mathbf{u} - \mathbf{u}^\star \|^3 \qquad \text{whenever} \qquad \| \mathbf{u} - \mathbf{u}^\star \| < \varepsilon. \tag{C.17}$$

Proposition C.4. *A convergent power series*

$$u(x_1, \ldots, x_n) = \sum_J \frac{c_J}{J\,!} \, \mathbf{x}^J$$

is the Taylor series for the analytic function $u(\mathbf{x})$ at the origin $\mathbf{x}_0 = \mathbf{0}$, and so its coefficients $c_J = \partial^J u / \partial x^J(\mathbf{0})$ are the partial derivatives of u at the origin.

The proof is the same as the one-dimensional version Proposition C.4: differentiate and substitute.

C.2. Special Functions.

Very few differential equations can be solved explictly in closed form. Even for linear ordinary differential equations, once one tries to move beyond the simplest constant coefficient equations, there are not very many examples with explicit solutions. One important example that has been solved are the Euler equations (3.84) But many other fairly simple second order equations, including the Bessel and Legendre equations that arose as a result of our separation of variables solution to partial differential equations, do not have elementary functions as solutions. These and other equations that appear in a number of key applications lead to new types of "special functions" that occur over and over again in applications.

Just as the student learned to become familiar with exponential and trigonometric functions, thus, at a more advanced level, applications in physics, engineering and mathematics require gaining some familiarity with the properties of these functions. The purpose of this section is to introduce the student to some basic properties of the most important special functions, including the gamma function, the Airy functions, the Legendre functions and, finally the Bessel functions. Lack of space will prevent us from introducing additional important special functions, such as hypergeometric functions, confluent hypergeometric functions, parabolic cylinder functions, the zeta function, elliptic functions, and many others. The interested reader can consult more advanced texts, such as [**144, 190**], and the handbook [**3**], as well as the soon to appear update [**145**] and its web site, for the latest information on this fascinating and very active field of mathematics and applications. We should remark that there is no precise definition of the term "special function" — it merely designates a function that plays a distinguished role in a range of applications and whose properties and evaluation are therefore of particular interest.

Most special functions arise most naturally as solutions to second order linear ordinary differential equations with variable coefficients. One method of gaining analytical insight into their properties is to formulate them as power series. Therefore, we will learn how to construct power series solutions to differential equations when closed form solutions are not available. As we shall see, although computationally messy at times, the power series method is straightforward to implement in practice. When we are at a regular point for the differential equation, the solutions can be obtained as ordinary power series. At so-called regular singular points, a more general type of series known as a Frobenius expansion is required. More general singular points require more advanced techniques, and will not be discussed here.

The Gamma Function

The first special function that we shall treat does not, in fact arise as the solution to a differential equation. Rather, it forms a generalization of the factorial function from integers to arbitrary real and complex numbers. As such, it will often appear in power series solutions to differential equations when parameters take on non-integral values.

First recall that the factorial of a non-negative integer n is defined inductively by the iterative formula

$$n! = n \cdot (n-1)!, \qquad \text{starting with} \qquad 0! = 1. \tag{C.18}$$

Thus, if n is a non-negative integer, the iteration based on the second formula terminates, and yields the familiar expression

$$n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1. \tag{C.19}$$

If n is not a non-negative integer, then the iteration will not terminate, and we cannot use it to compute the factorial. Our goal is to circumvent this difficulty, and introduce a function $f(x)$ that is defined for *all* values of x, and will play the role of such a factorial. The function should satisfy the functional equation

$$f(x) = x\, f(x-1) \tag{C.20}$$

where defined. If, in addition, $f(0) = 1$, then we know $f(n) = n!$ whenever n is a non-negative integer, and hence such a function will extend the definition of the factorial to more general real and complex numbers.

A moment's thought should convince the reader that there are many possible ways to construct such a function. The most important method relies on an integral formula, and leads to the definition of the gamma function, originally discovered by Euler.

Definition C.5. The *gamma function* is defined by

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt. \tag{C.21}$$

The first fact is that the gamma function integral converges whenever $\operatorname{Re} z > 0$; otherwise the singularity of t^{z-1} is too severe to permit convergence of the improper integral at $t = 0$. The key property that turns the gamma function into a substitute for the factorial function relies on an elementary integration by parts:

$$\Gamma(z+1) = \int_0^\infty e^{-t}\, t^z\, dt = \; - \, e^{-t}\, t^z \, \Big|_{t=0}^{\infty} \; + \; z \int_0^\infty e^{-t}\, t^{z-1}\, dt.$$

The boundary terms vanish whenever $\operatorname{Re} z > 0$, while the final integral is merely $\Gamma(z)$. Therefore, the gamma function satisfies the recurrence relation

$$\Gamma(z+1) = z\, \Gamma(z) \qquad \text{provided} \qquad \operatorname{Re} z > 0. \tag{C.22}$$

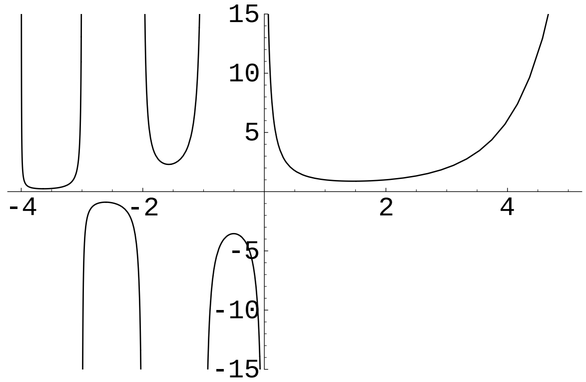


Figure C.1. The Gamma Function.

If we set $f(x) = \Gamma(x+1)$, then (C.22) is the same as (C.20). Moreover, by direct integration

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1.$$

Combining this with the recurrence relation (C.22), we deduce that

$$\Gamma(n+1) = n! \tag{C.23}$$

whenever $n \geq 0$ is a non-negative integer. Therefore, we can identify $x!$ with the value $\Gamma(x+1)$ whenever $x > -1$ is *any* real number.

Remark: The reader may legitimately ask why not replace t^{z-1} by t^z in the definition of $\Gamma(z)$, which would avoid the $n-1$ in (C.23). There is no simple answer; we are merely following a well-established precedent set originally by Euler.

Thus, at integer values of z, the gamma function agrees with the elementary factorial. A few other values can be computed exactly. One important case is when $z = \frac{1}{2}$. Using the substitution $t = x^2$, with $dt = 2\,x\,dx$, we find

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-t}\, t^{-1/2}\, dt = \int_0^\infty 2\, e^{-x^2}\, dx = \sqrt{\pi}, \tag{C.24}$$

where the final integral was evaluated earlier, (Gaussint■). Thus, using the identification with the factorial function, we identify this value with $\left(-\frac{1}{2}\right)! = \sqrt{\pi}$. The recurrence relation (C.22) will then fix the value of the gamma function at all half-integers $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$. For example,

$$\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\sqrt{\pi}, \tag{C.25}$$

and hence $\frac{1}{2}! = \frac{1}{2}\sqrt{\pi}$. Further properties of the gamma function are outlined in the exercises. A graph of the gamma function appear in Figure C.1. Note the appearance of singularities at negative integer values of $x = -1, -2, \ldots$.

One of the most useful formulas involving the gamma function is *Stirling's Formula*,

$$\Gamma(n+1) = n! \;\sim\; \sqrt{2\pi n}\,\frac{n^n}{e^n}\,, \qquad\qquad n \longrightarrow \infty, \tag{C.26}$$

which gives the asympotitic values of the factorial function for large n. A proof is outlined in the exercises.

C.3. Series Solutions of Ordinary Differential Equations.

When confronted with a novel differential equation, there are a few standard options for making progress in solving and understanding the solutions. One of these is the "look-up" method, that relies on published collections of differential equations and their solutions. One of the most useful references that collects together many solved differential equations is the classic German compendium written by Kamke, [**112**]. Two more recent English-language handbooks are [**196**, **198**].

Of course, numerical integration — see Chapter 20 for a presentation of basic methods — is always an option for approximating the solution. Numerical methods do, however, have their limitations, and are best accompanied by some understanding of the underlying theory, coupled with qualitative or quantitative expectations of how the solutions should behave. Furthermore, numerical methods provide less than adequate insight into the nature of the special functions that appear as solutions of the particular differential equations arising in separation of variables. A numerical approximation cannot, in itself, be used to establish rigorous mathematical properties of the solutions of the differential equation.

A more classical means of constructing and approximating the solutions of differential equations is based on their power series or Taylor series expansions. The Taylor expansion of a solution at a point x_0 is found by substituting a general power series into the differential equation and equating coefficients of the various powers of $x-x_0$. The initial conditions at x_0 serve to uniquely determine the coefficients and hence the derivatives of the solution at the initial point. The Taylor expansion of a special function can be used to deduce many of the key properties of the solution, as well as provide reasonable numerical approximations to its values within the radius of convergence of the series. (Howeer, serious numerical computations more often rely on non-convergent asymptotic expansions, [**144**].)

Example C.6. Before developing the general computational machinery, a naïve computation of a simple equation will be enlightening. Consider the initial value problem

$$\frac{d^2x}{du^2} + u = 0, \qquad u(0) = 1, \qquad u'(0) = 0.$$

Let's see how to construct a soloution in the form of a power series

$$u(x) = u_0 + u_1\,x + u_2\,x^2 + u_3\,x^3 + \;\cdots\; = \sum_{n=0}^{\infty} u_n\,x^n$$

for the solution Term-by-term differentiation yields the series expansions†

$$u''(x) = u_2 + 6\,u_3\,x + 12\,u_4\,x^2 + 20\,u_5\,x^3 + \;\cdots\; = \sum_{n=0}^{\infty} (n+1)(n+2)\,u_{n+2}\,x^n, \qquad (C.27)$$

for its derivatives. Substituting into the equation, and then equating the various powers of x to 0 leads to the following recurrence relations relating the coefficients of our power series.

† If we choose to work with the series in summation form, we need to re-index appropriately in order to display the term of degree n.

The left column indicates the power of x, while the right column tells us the recurrence relation:

$$
\begin{array}{cr}
1 & 2\,u_2 + u_0 = 0, \\
x & 6\,u_3 + u_1 = 0, \\
x^2 & 12\,u_4 + u_2 = 0, \\
x^3 & 20\,u_5 + u_3 = 0, \\
x^4 & 30\,u_6 + u_4 = 0, \\
\vdots & \vdots \\
x^n & (n+1)(n+2)\,u_{n+2} + u_n = 0.
\end{array}
$$

The initial conditions serve to prescribe the first two coefficients;

$$u_0 = u(0) = 1, \qquad u_1 = u'(0) = 0.$$

We solve the recurrence relations in order. The first equation determines $u_2 = -\frac{1}{2}\,u_0 = -\frac{1}{2}$. The second prescribes $u_3 = -\frac{1}{6}\,u_0 = 0$. Next we find $u_4 = -\frac{1}{12}\,u_2 = \frac{1}{24}$. Next, $u_5 = -\frac{1}{20}\,u_3 = 0$. Then $u_6 = -\frac{1}{30}\,u_4 = -\frac{1}{720}$. In general, it is not hard to see that

$$u_{2k} = \frac{(-1)^k}{(2\,k)!}\,, \qquad u_{2k+1} = 0,$$

and hence the resulting series solution is

$$u(x) = 1 - \tfrac{1}{2}\,x^2 + \tfrac{1}{24}\,x^3 - \tfrac{1}{720}\,x^6 + \;\cdots\; = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\,x^{2\,k},$$

which is merely the well-known Taylor series for $\cos x$, the solution to the initial value problem. Note that the generation of the Taylor series does not rely on any a priori knowledge of trigonometric functions or alternatuve solution methods for ordinary differential equations.

In this section, we provide a brief introduction to the basic series solution techniques for ordinary differential equations, concentrating on second order linear differential equations, since these form by far the most important class of examples arising in applications. When x_0 is a regular point, the method will construct a standard Taylor expansion for the solution, while so-called regular singular points require a slightly more general series expansion. Generalizations to higher order equations, nonlinear equations, and even (nonlinear) systems are left to other more detailed texts, including [**107**].

Regular Points

We shall concentrate on solving a homogeneous linear differential equation of the form

$$p(x)\,\frac{d^2u}{dx^2} + q(x)\,\frac{du}{dx} + r(x)\,u = 0. \tag{C.28}$$

The coefficients $p(x), q(x), r(x)$ are assumed to be analytic functions where defined. This means that, at a point x_0, they admit convergent power series expansions

$$\begin{aligned} p(x) &= p_0 + p_1\,(x - x_0) + p_2\,(x - x_0)^2 + \cdots\,, \\ q(x) &= q_0 + q_1\,(x - x_0) + q_2\,(x - x_0)^2 + \cdots\,, \\ r(x) &= r_0 + r_1\,(x - x_0) + r_2\,(x - x_0)^2 + \cdots\,. \end{aligned} \tag{C.29}$$

We expect that the solutions to the differential equation will also be analytic functions. This expectation is justified provided that the equation is *regular* at the point x_0, in the following sense.

Definition C.7. A point $x = x_0$ is a *regular point* of a second order linear ordinary differential equation (C.28) provided the leading coefficient does not vanish there:

$$p_0 = p(x_0) \neq 0.$$

A point where $p(x_0) = 0$ is known as a *singular point*.

In short, a regular point is where the second order derivative terms does not disappear, and so the equation is "genuinely" second order.

Remark: The definition of a singular point assumes that the other coefficients do not both vanish there, i.e., either $q(x_0) \neq 0$ or $r(x_0) \neq 0$. If all three functions happen to vanish at x_0, we would factor out a common factor $(x - x_0)^k$, and hence, without loss of generality, can assume at least one of the coefficients is nonzero at x_0.

The basic existence theorem for differential equations at regular points follows. See, e.g., [**100**, **107**] for a proof.

Theorem C.8. *Let x_0 be a regular point for the second order homogeneous linear ordinary differential equation* (C.28)*. Then there exists a unique solution $u(x)$ to the initial value problem*

$$u(x_0) = a, \qquad u'(x_0) = b. \tag{C.30}$$

Moreover, the solution $u(x)$ is an analytic function for x sufficiently close to x_0.

Remark: It can be proved, [**100**], that the radius of convergence of any analytic solution $u(x)$ is equal to the distance from the regular point to the nearest singular point in the complex plane.

At any regular point, the second order differential equation (C.28) admits two linearly independent analytic solutions, which we denote by $u(x)$ and $\widetilde{u}(x)$. The general solution can be written as a linear combination of the two basis solutions:

$$u(x) = a\,u(x) + b\,\widetilde{u}(x). \tag{C.31}$$

A standard choice for the two basis solutions is to take the first to satisfy the initial conditions

$$u(x_0) = 1, \qquad u'(x_0) = 0, \tag{C.32}$$

and the second to satisfy

$$\widetilde{u}(x_0) = 0, \qquad\qquad \widetilde{u}'(x_0) = 1, \tag{C.33}$$

although other choices may be used depending upon particular circumstances. With this choice, the linear combination automatically satisfies the initial conditions (C.30).

Therefore, every solution to an analytic differential equation at a regular point can be expanded in an ordinary power series

$$u(x) = u_0 + u_1\,(x - x_0) + u_2\,(x - x_0)^2 + \ \cdots \ = \sum_{n=0}^{\infty} u_n\,(x - x_0)^n \tag{C.34}$$

at the point. Since the power series coincides with the Taylor series for $u(x)$, its the coefficients

$$u_n = \frac{u^{(n)}(x_0)}{n!}$$

are multiples of the derivatives of the function at the point x_0. (Some authors prefer to keep the $n!$'s in the power series; this is purely a matter of taste.) In particular, the first two coefficients

$$u_0 = u(x_0) = a, \qquad\qquad u_1 = u'(x_0) = b. \tag{C.35}$$

are prescribed by the initial conditions. Once the initial conditions have been specified, the remaining coefficients must be uniquely prescribed since there is only one solution to the initial value problem.

The basic computational technique for constructing the power series solution to the initial value problem is quite straightforward. One substitutes the known power series (C.29) for the coefficient functions and the unknown power series (C.34) for the solution into the differential equation (C.28). Multiplying out the formulae will result in a (complicated) power series that must be equated to zero. At this point, one analyzes the individual coefficients. We rely on the basic observation that

Two power series are equal if and only if their individual coefficients are equal,

generalizing the standard test for equality of polynomials. In particular, a power series represents the zero function if and only if all its coefficients are 0.

Thus, the power series solution method continues by equating, in order, the coefficients of the resulting power series to zero, starting with the lowest order (constant) and working upwards. The lowest order terms are multiples of $(x - x_0)^0 = 1$, i.e., the constant terms in the differential equation, lead to a linear recurrence relation

$$u_2 = R_2(u_0, u_1) = R_2(a, b)$$

that prescribes the coefficient u_2 in terms of the initial data. The coefficients of $(x - x_0)$ lead to a linear recurrence relation

$$u_3 = R_3(u_0, u_1, u_2) = R_3(a, b, R_2(a, b))$$

that prescribes the coefficient u_3 in terms of the initial data and the prviously computed coefficient u_2. And so on. At the n^{th} stage of the procedure, the coefficients of $(x - x_0)^n$ lead to the n^{th} linear *recurrence relation*

$$u_{n+2} = R_n(u_0, u_1, \ldots, u_{n+1}), \qquad\qquad n = 0, 1, 2, \ldots\,, \tag{C.36}$$

that will prescribe the $(n+2)^{\text{nd}}$ order coefficient in terms of the previous ones. Once the coefficients u_0 and u_1 have been specified by the initial conditions, the remaining coefficients $u_2, u_3, u_4, \ldots$ are successively fixed by the recurrence relations (C.36). In this fashion, we easily deduce the existence of a formal power series solution to the differential equation at a regular point. The one remaining issue is whether the resulting power series actually converges. This can be proved with a detailed analysis, [**107**], and will serve to complete the proof of the general existence Theorem C.8.

Rather than continue in generality, the best way to learn the method is to investigate simple examples.

The Airy Equation

A particularly easy case to analyze is the *Airy equation*

$$u'' = x\,u. \tag{C.37}$$

This second order ordinary differential equation arises in optics, dispersive waves, caustics (focusing of light waves as with a magnifying glass) and diffraction, and rainbows. It was first derived by the English mathematician George Airy in 1839, [**5**]. In Exercise 15.2.8, we saw how it arises in a separation of variables solution to the Tricomi equation arising in supersonic fluid motion.

The solutions to the Airy equation are known as *Airy functions*. While Airy functions cannot be written in terms of the standard elementary functions, it is relatively straightforward to determine their power series expansion. Since the leading coefficient $p(x) \equiv 1$ is constant, and every point x_0 is a regular point of the Airy equation. For simplicity, we only treat the case $x_0 = 0$, and therefore consider a power series

$$u(x) = u_0 + u_1\,x + u_2\,x^2 + u_3\,x^3 + \ \cdots \ = \sum_{n=0}^{\infty} u_n\,x^n$$

for the solution. Term-by-term differentiation yields the series expansions†

$$\begin{aligned} u'(x) &= u_1 + 2\,u_2\,x + 3\,u_3\,x^2 + 4\,u_4\,x^3 + \ \cdots \ = \sum_{n=0}^{\infty} (n+1)\,u_{n+1}\,x^n, \\ u''(x) &= u_2 + 6\,u_3\,x + 12\,u_4\,x^2 + 20\,u_5\,x^3 + \ \cdots \ = \sum_{n=0}^{\infty} (n+1)(n+2)\,u_{n+2}\,x^n, \end{aligned} \tag{C.38}$$

† If we choose to work with the series in summation form, we need to re-index appropriately in order to display the term of degree n.

for its derivatives. On the other hand,

$$x\,u(x) = u_0\,x + u_1\,x^2 + u_2\,x^3 + \;\cdots\; = \sum_{n=1}^{\infty} u_{n-1}\,x^n. \tag{C.39}$$

Equating this power series to that of $u''(x)$ leads to the following recurrence relations relating the coefficients of our power series. The left column indicates the power of x, while the right column displays the recurrence relation:

$$\begin{array}{cr}
1 & u_2 = 0, \\
x & 6\,u_3 = u_0, \\
x^2 & 12\,u_4 = u_1, \\
x^3 & 20\,u_5 = u_2, \\
x^4 & 30\,u_6 = u_3, \\
\vdots & \vdots \\
x^n & (n+1)(n+2)\,u_{n+2} = u_{n-1}.
\end{array}$$

We solve the recurrence relations in order. The first equation determines u_2. The second prescribes $u_3 = \frac{1}{6}\,u_0$ in terms of u_0. Next we find $u_4 = \frac{1}{12}\,u_1$ in terms of u_1. Next, $u_5 = \frac{1}{20}u_2 = 0$. Then $u_6 = \frac{1}{30}\,u_3 = \frac{1}{180}\,u_0$ is first given in terms of u_3, but we already know the latter in terms of u_0. And so on. At the n^{th} stage of the recursion, stage we determine u_{n+2} using our previously tabulated formula for u_{n-1}.

The only coefficients that are not determined by this procedure are the first two, u_0 and u_1. These correspond to the value of the solution and its derivative at the initial point $x_0 = 0$, as in (C.30).

Let us construct the two basis solutions. The first uses the initial conditions

$$u_0 = u(0) = 1, \qquad\qquad u_1 = u'(0) = 0.$$

The recurrence relations then show that the only nonvanishing coefficients c_n are when $n = 3\,k$ is a multiple of 3; all others are zero. Moreover,

$$c_{3k} = \frac{c_{3k-3}}{3\,k(3\,k-1)}$$

A straightforward induction proves that

$$c_{3k} = \frac{1}{3\,k(3\,k-1)(3\,k-3)(3\,k-4)\cdots 6\cdot 5\cdot 3\cdot 2}\,.$$

The resulting solution is

$$u_1(x) = 1 + \tfrac{1}{6}\,x^3 + \tfrac{1}{180}\,x^6 + \cdots = \sum_{k=1}^{\infty} \frac{x^{3k}}{3\,k(3\,k-1)(3\,k-3)(3\,k-4)\cdots 6\cdot 5\cdot 3\cdot 2}\,. \tag{C.40}$$

Note that the denominator is similar to a factorial, except every third term is omitted.

Similarly, starting with the initial conditions

$$u_0 = u(0) = 0, \qquad\qquad u_1 = u'(0) = 1,$$

we find that the only nonvanishing coefficients c_n are when $n = 3k+1$ leaves a remainder of 1 when divided by 3. The recurrence relation

$$c_{3k+1} = \frac{c_{3k-2}}{(3k+1)(3k)} \qquad \text{yields} \qquad c_{3k+1} = \frac{1}{(3k+1)(3k)(3k-2)(3k-3)\cdots 7\cdot 6\cdot 4\cdot 3}\,.$$

The resulting solution is

$$u_2(x) = x + \tfrac{1}{12}\,x^4 + \tfrac{1}{504}\,x^7 + \cdots = \sum_{k=1}^{\infty} \frac{x^{3k+1}}{(3k+1)(3k)(3k-2)(3k-3)\cdots 7\cdot 6\cdot 4\cdot 3}\,. \tag{C.41}$$

Again, the denominator skips every third term in the product. Every solution to the Airy equation can be written as a linear combination

$$u(x) = a\,u_1(x) + b\,u_2(x), \qquad \text{where} \qquad a = u(0), \qquad b = u'(0)$$

correspond to the initial conditions of $u(x)$ at $x = 0$. The power series (C.40, 41), converge quite rapidly for all values of x, and so the first few terms provide a reasonable approximation to the two solutions for moderate values of x.

The solutions (C.40, 41), while easiest to derive using power series techniques, are not the most useful for applications in mathematics and physics. The most important solution is the *Airy function of the first kind*

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(t\,x + \tfrac{1}{3}\,t^3\right) dt. \tag{C.42}$$

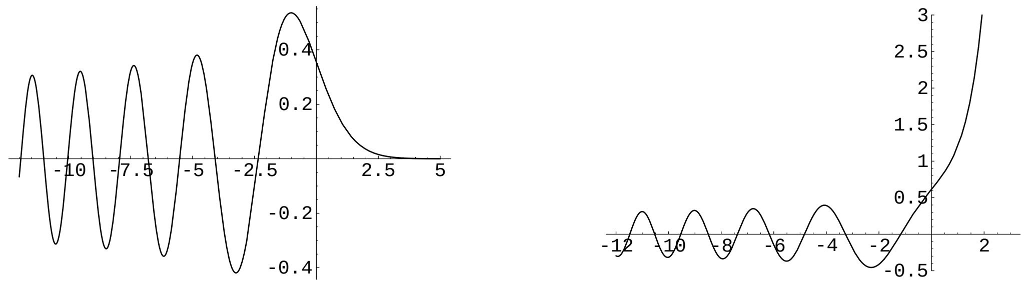
An independent solution is provided by the *Airy function of the second kind*

$$\mathrm{Bi}(x) = \frac{1}{\pi}\int_0^\infty \left[\exp\left(t\,x - \tfrac{1}{3}\,t^3\right) + \sin\left(t\,x + \tfrac{1}{3}\,t^3\right)\right] dt. \tag{C.43}$$

They have the initial values

$$\begin{aligned} \mathrm{Ai}(0) &= \frac{1}{3^{2/3}\Gamma\left(\frac{2}{3}\right)} & \qquad \mathrm{Bi}(0) &= \frac{1}{3^{1/6}\Gamma\left(\frac{2}{3}\right)} \\ \mathrm{Ai}'(0) &= -\,\frac{1}{3^{1/3}\Gamma\left(\frac{1}{3}\right)}\,, & \qquad \mathrm{Bi}'(0) &= -\,\frac{3^{1/6}}{\Gamma\left(\frac{1}{3}\right)}\,. \end{aligned} \tag{C.44}$$

Graphs of the two Airy functions appear in Figure C.2. Both functions oscillate for negative values of x, with a slowly decreasing amplitude. An intuitive explanation is that when $x < 0$ the Airy equation (C.37) corresponds to a constant coefficient differential equation of the form $u'' = -k^2\,u$, which has oscillatory trigonometric solutions. On the other hand, when $x > 0$, the Airy equation is more like a constant coefficient differential equation of the form $u'' = +k^2\,u$, whose basis solutions $e^{k\,x}$ and $e^{-\,k\,x}$ are, respectively, exponentially growing and exponentially decaying. Indeed, as $x \to \infty$, the first Airy



Ai(x)　　　　Bi(x)

Figure C.2. The Airy Functions.

function Ai(x) decays very rapidly, whereas the second Bi(x) grows even more dramatically. Actually, the growth/decay rates are faster than exponential. It can be shown that

$$\operatorname{Ai}(x) \sim \begin{cases} \dfrac{e^{-2x^{3/2}/3}}{2\sqrt{\pi}\, x^{1/4}}, & x \to +\infty, \\[2ex] \dfrac{\sin\left(\dfrac{2}{3}(-x)^{3/2} + \dfrac{\pi}{4}\right)}{\sqrt{\pi}\,(-x)^{1/4}}, & x \to -\infty, \end{cases}$$

$$\operatorname{Bi}(x) \sim \begin{cases} \dfrac{e^{2x^{3/2}/3}}{2\sqrt{\pi}\, x^{1/4}}, & x \to +\infty, \\[2ex] \dfrac{\cos\left(\dfrac{2}{3}(-x)^{3/2} + \dfrac{\pi}{4}\right)}{\sqrt{\pi}\,(-x)^{1/4}}, & x \to -\infty. \end{cases}$$

Every solution to the Airy equation can be written as a linear combination

$$u(x) = a\ \operatorname{Ai}(x) + b\ \operatorname{Bi}(x).$$

Detailed investigations into the properties and numerical computation of the Airy functions can be found in [**3**, **145**, **144**].

The Legendre Equation

A particularly important example is the *Legendre equation*

$$(1-t^2)^2\, \frac{d^2P}{dt^2} - 2\,t\,(1-t^2)\, \frac{dP}{dt} + \left[\, \lambda\,(1-t^2) - m^2 \,\right] P = 0. \tag{C.45}$$

The parameter m governs the *order* of the Legendre equation, which in the cases of interest to us, is an integer, while λ plays the role of an eigenvalue. As we learned in the preceding sections, this differential equation arises in the solutions to a wide variety of partial differential equations in spherical coordinates. The boundary conditions that serve to specify the eigenvalues are that the solution remain bounded at the two singular points $t = \pm 1$, leading to

$$|\,P(-1)\,| < \infty, \qquad\qquad |\,P(+1)\,| < \infty. \tag{C.46}$$

The point $t = 0$ is a regular point of the Legendre equation. Indeed, the only singular points are the boundary points $t = \pm 1$. Therefore, we can determine the solutions to the Legendre equation by the method of power series based at $t_0 = 0$. However, the general recurrence relations are rather complicated to solve in closed form, and we use some tricks to get a handle on the solutions.

Consider first the case $m = 0$. The Legendre equation of order 0 is

$$(1 - t^2)\,\frac{d^2 P}{dt^2} - 2\,t\,\frac{dP}{dt} + \lambda\,P = 0. \tag{C.47}$$

As we noted above, the eigenfunctions are the *Legendre polynomials*

$$P_n(t) = \frac{1}{2^n\,n!}\,\frac{d^n}{dt^n}\,(t^2 - 1)^n.$$

They clearly satisfy the boundary conditions (C.46). To verify that they are indeed solutions to the differential equation (C.47), we let

$$q_n(t) = (t^2 - 1)^n.$$

By the chain rule, the derivative of $q_n(t)$ is

$$q_n' = 2\,n\,t\,(t^2 - 1)^{n-1} \qquad \text{and hence} \qquad (t^2 - 1)q_n' = 2\,n\,t\,(t^2 - 1)^n = 2\,n\,t\,q_n.$$

Differentiating the latter formula,

$$(t^2 - 1)q_n'' + 2\,t\,q_n' = 2\,n\,t\,q_n' + 2\,n\,q_n, \qquad \text{or} \qquad (t^2 - 1)q_n'' = 2(n-1)t\,q_n' + 2\,n\,q_n.$$

A simple induction proves that the k^{th} order derivative $q_n^{(k)}(t) = d^k q_n / dt^k$ satisfies

$$\begin{aligned}(t^2 - 1)\,q_n^{(k+2)} &= 2(n-k-1)\,t\,q_n^{(k+1)} + 2[\,n + (n-1) + \cdots + (n-k)\,]q_n^{(k)} \\ &= 2(n-k-1)\,t\,q_n^{(k+1)} + (k+1)(2n-k)\,q_n^{(k)}.\end{aligned} \tag{C.48}$$

In particular, when $k = n$, this reduces to

$$(t^2 - 1)q_n^{(n+2)} = -\,2\,t\,q_n^{(n+1)} + n(n+1)\,q_n^{(n)} = 0,$$

and so $v_n = q_n^{(n)}$ satisfies

$$(1 - t^2)v_n'' - 2\,t\,v_n' + n(n+1)\,v_n = 0,$$

which is precisely the order 0 Legendre equation (C.47) with eigenvalue parameter $\lambda = n(n+1)$. The Legendre polynomial P_n is a constant multiple of v_n, and hence it too satisfies the order 0 Legendre equation and hence forms an eigenfunction for the Legendre boundary value problem (C.45–46). While it is not immediately apparent that the Legendre polynomials form a complete system of eigenfunctions, this is the case. This is the result of a general theory of eigenfunctions of Sturm–Liouville boundary value problems, [**42**], or, more particularly, the theory of orthogonal polynomials, [**145**]. Indeed, the orthogonality of the Legendre polynomials that was noted in Chapter 5 is, in fact, a consequence of the fact that they are eigenfunctions for this self-adjoint boundary value problem.

More generally, if we substitute $k = m + n$ in (C.48), we have

$$(1-t^2)\, w_n'' - 2(m+1)\, t\, w_n' + (m+n+1)(n-m)\, w_n = 0, \tag{C.49}$$

where $w_n = q_n^{(m+n)}$. This is *not* the order m Legendre equation, but can be converted into it by setting

$$w_n = (1-t^2)^{-m/2}\, z_n.$$

Differentiating, we find

$$\begin{aligned} w_n' &= (1-t^2)^{-m/2}\, z_n' - m\, t\, (1-t^2)^{-m/2-1}\, z_n, \\ w_n'' &= (1-t^2)^{-m/2}\, z_n'' - 2\, m\, t\, (1-t^2)^{-m/2-1}\, z_n' + (m + m(m+1)t^2)\, (1-t^2)^{-m/2-2}\, z_n. \end{aligned}$$

Therefore, after a little algebra, equation (C.49) takes the alternative form

$$(1-t^2)^{-m/2+1}\, z_n'' - 2\, t\, (1-t^2)^{-m/2}\, z_n' + (n(n+1)(1-t^2) - m^2)\, (1-t^2)^{-m/2-1}\, z_n = 0,$$

which, when multiplied by $(1-t^2)^{m/2+1}$, is precisely the order m Legendre equation (C.45) with eigenvalue parameter $\lambda = n(n+1)$. We conclude that

$$z_n(t) = (1-t^2)^{m/2} w_n(t) = (1-t^2)^{m/2}\, \frac{d^{n+m}}{dt^{n+m}}\, (t^2-1)^n$$

is a solution to the order m Legendre equation. Moreover, $z_n(\pm 1) = 0$, and hence $z_n(t)$ is an eigenfunction for the order m Legendre boundary value problem. Indeed, $z_n(t)$ is a constant multiple of the associated Legendre function $P_n^m(t)$, as defined in (18.28). With some more work, it can be proved that the associated Legendre functions form a complete system of eigenfunctions for the the order m Legendre boundary value problem.

Regular Singular Points

In a large range of applications, one is articularly interested in the behavior of solutions to a differential equation near a singular point. Usually, a power series expansion (C.34) fails to produce a solution at a singular point. As before, we write the differential equation as

$$p(x)\, \frac{d^2u}{dx^2} + q(x)\, \frac{du}{dx} + r(x)\, u = 0. \tag{C.50}$$

Here, we assume that the functions p, q, r are analytic at x_0, where now we assume that $p(x_0) = 0$, but at least one of $q(x_0)$, $r(x_0)$ is non-zero. If the singular point is not too "wild", one can construct solutions using a relatively simple modification of the basic power series.

In order to formulate the key definition, we rewrite the differential equation in solved form

$$\frac{d^2u}{dx^2} = g(x)\, \frac{du}{dx} + h(x)\, u \qquad \text{where} \qquad g(x) = -\,\frac{q(x)}{p(x)}\,, \qquad h(x) = -\,\frac{r(x)}{p(x)}\,.$$

If $p(x_0) = 0$, then, typically, the functions $g(x), h(x)$ will have singularities at $x = x_0$, and we need to ensure that these singularities are not too bad.

Definition C.9. A singular point x_0 is called a *regular singular point* if

$$g(x) = \frac{k(x)}{x - x_0}\,, \qquad\qquad h(x) = \frac{\ell(x)}{(x - x_0)^2}\,, \tag{C.51}$$

where $k(x)$ and $\ell(x)$ are analytic at $x = x_0$.

Thus, in the language of complex analysis, the point x_0 is a regular singular point provided $g(x)$ has a pole of order at most 1, while $h(x)$ has a pole of order at most 2 at $x = x_0$. In terms of the original coefficients, the regularity conditions (C.51) require that we can write the differential equation in the form

$$(x - x_0)^2\, a(x)\, \frac{d^2 u}{dx^2} + (x - x_0)\, b(x)\, \frac{du}{dx} + c(x)\, u = 0, \tag{C.52}$$

where $a(x), b(x)$ and $c(x)$ are analytic at $x = x_0$ and, moreover, $a(x_0) \neq 0$.

Fortunately, almost all ordinary differential equations arising in applications have only regular singular points. The irregular singular points are much harder to deal with, and must be relegated to an advanced treatment, e.g., [**100**, **107**].

The simplest example of an equation with a regular singular point is the *Euler equation*

$$a\, x^2\, u'' + b\, x\, u' + c\, u = 0, \tag{C.53}$$

where $a \neq 0, b, c$ are constants. The point $x = 0$ is a regular singular point; indeed, the solved form of the Euler equation is

$$u'' = -\,\frac{b}{a\,x}\, u' - \frac{c}{a\,x^2}\, u,$$

and hence satisfies (C.51). All other points $x_0 \neq 0$ are regular points for the Euler equation.

As discussed in Example 7.35, Euler equations are solved by substituting the power ansatz $u(x) = x^r$ into the equation. As a result, the exponent r is determined by the associated characteristic equation (7.53), namely

$$a\, r\, (r - 1) + b\, r + c = 0.$$

If this quadratic equation has two distinct roots $r_1 \neq r_2$, we obtain two linearly independent (possibly complex) solutions $u(x) = x^{r_1}$ and $\widetilde{u}(x) = x^{r_2}$. The general solution $u(x) = c_1\, x^{r_1} + c_2\, x^{r_2}$ is a linear combination of these two basis solutions. Note that unless r_1 and r_2 are non-negative integers, the solutions have a singularity — either a pole or branch point — at the singular point $x = 0$. A repeated root, $r_1 = r_2$, requires an additional logarithmic term, $\widetilde{u}(x) = x^{r_1} \log x$, in the second solution, and the general solution has the form $u(x) = c_1\, x^{r_1} + c_2\, x^{r_1} \log x$.

The series solution method at more general regular singular points is modeled on the simple example of the Euler equation. One now seeks a solution that has a series expansion of the form

$$u(x) = (x - x_0)^r \sum_{n=0}^{\infty} u_n (x - x_0)^n = u_0 (x - x_0)^r + u_1 (x - x_0)^{r+1} + u_2 (x - x_0)^{r+2} + \cdots. \tag{C.54}$$

The full theory was established by the German mathematician Georg Frobenius in the late 1800's, and the series are sometimes known as *Frobenius expansions*. The exponent r is known as the *index* of the expansion.

Remark: If the index $r = -n$ is a negative integer, then (C.54) has the form of a Laurent series expansion, [**4**] But r can be non-integral, or even complex, and the resulting expansion is known in complex analysis as a *Puiseux expansion*, [**95**].

We can assume, without any loss of generality, that the leading coefficient $u_0 \neq 0$. Indeed, if $u_k \neq 0$ is the first non-zero coefficient, then the series begins with $u_k(x-x_0)^{r+k}$, and we replace r by $r+k$ to write it in the preceding form. Moreover, since any scalar multiple of a solution is a solution, we can divide by u_0 and assume that $u_0 = 1$ or any other convenient non-zero value, as desired.

Warning: Unlike ordinary power series expansions, the coefficients u_0 and u_1 are *not* prescribed by the initial conditions at the point x_0. Indeed, as we learned in our study of the Bessel and Legendre equations, one cannot typically impose specific initial values for the solutions at a singular point. Often, mere boundedness will suffice to distinguish a solution. Here, the solution is usually completely detremined by the index r and the leading coefficient u_0.

The Frobenius solution method proceeds by substituting the series (C.54) into the differential equation (C.52). Since

$$\begin{aligned} u(x) &= (x-x_0)^r + u_1(x-x_0)^{r+1} + \cdots\,, \\ (x-x_0)\,u'(x) &= r\,(x-x_0)^r + (r+1)u_1(x-x_0)^{r+1} + \cdots\,, \\ (x-x_0)\,u''(x) &= r\,(r-1)\,(x-x_0)^r + (r+1)\,r\,u_1(x-x_0)^{r+1} + \cdots\,, \end{aligned}$$

the lowest order terms are multiples of $(x-x_0)^r$. Equating this particular coefficient to zero leads to a quadratic equation of the form

$$s_0\,r\,(r-1) + t_0\,r + r_0 = 0, \tag{C.55}$$

where

$$s_0 = s(x_0) = \tfrac{1}{2}\,p''(x_0), \qquad t_0 = t(x_0) = q'(x_0), \qquad r_0 = r(x_0),$$

are the leading coefficients in the power series expansions of the coefficients of the differential equation. The quadratic equation (C.55) is known as the *indicial equation*, since it determines the possible indices r in the Frobenius expansion of a solution.

Therefore, just as in the Euler equation, it turns out that (typically) there are two allowable indices, say r_1 and r_2, which are the roots of the quadratic indicial equation. If the indices are distinct, then one expects to find two different Frobenius expansions. Usually, this assumption is valid, but there is an important exception, which occurs when the roots differ by an integer. The general result is summarized in the following list.

(*i*) If $r_2 - r_1$ is not an integer, then there are two linearly independent solutions $u(x)$ and $\widetilde{u}(x)$, each having a convergent Frobenius expansions of the form (C.54).

(*ii*) If $r_1 = r_2$, then there is only one solution with a convergent Frobenius expansion.

(*iii*) Finally, if $r_2 = r_1 + k$, where $k > 0$ is a positive integer, then there is a solution with a convergent Frobenius expansion corresponding to the smaller index r_1. The solution associated with the larger index r_2 may or may not have a convergent Frobenius expansion.

Thus, in every case the differential equation has at least one solution with a Frobenius expansion. When the leading coefficient $u_0 = 1$ is fixed, then the remaining coefficients $u_1, u_2, \ldots$ are uniquely prescribed by the recurrence relations stemming from substitution of the expansion into the differential equation. If the second solution does not have a Frobenius expansion, then it has an additional logarithmic term, as with the Euler equation, of a well-prescribed form. Details appear in the exercises. Rather than try to develop the theory in any more detail here, we suffice with consideration of some particular examples.

Example C.10. Consider the second order ordinary differential equation

$$u'' + \left(\frac{1}{x} + \frac{x}{2}\right) u' + u = 0 \tag{C.56}$$

that we needed to solve for finding the fundamental solution to the heat equation; see (C.39). We look for series solutions based at $x = 0$. Since the coefficient of u' has a simple pole, the point $x = 0$ is a regular singular point, and thus we can work with a Frobenius expansion as in (C.61). Substituting into the differential equation, we find that the coefficients of x^r lead to the indicial equation

$$r^2 = 0.$$

There is only one root, $r = 0$, and hence even though we are at a singular point, we are dealing with an ordinary power series. The next term tells us that $u_1 = 0$. Since $r = 0$, the general recurrence relation is

$$(n+2)^2 u_{n+2} + \tfrac{1}{2}(n+2) u_n = 0,$$

and hence

$$u_{n+2} = -\,\frac{u_n}{2(n+2)}\,.$$

Therefore, the odd coefficients $u_{2k+1} = 0$ are all zero, while the even ones are

$$u_{2k} = -\,\frac{u_{2k-2}}{4k} = \frac{u_{2k-4}}{4k(4k-4)} = -\,\frac{u_{2k-6}}{4k(4k-4)(4k-8)} = \cdots = \frac{(-1)^k}{4^k\,k\,!} \qquad \text{since} \qquad u_0 = 1.$$

The resulting power series takes a familiar form:

$$u(x) = \sum_{k=1}^{\infty} u_{2k} x^{2k} = \sum_{k=1}^{\infty} \frac{1}{k\,!}\left(-\,\frac{x^2}{4}\right)^k = e^{-x^2/4}.$$

The second solution will require a logarithmic term. However, it can be found directly by a general reduction method. Once we know one solution to a second order ordinary differential equation, the second solution can be found by substituting the ansatz

$$\widetilde{u}(x) = u(x)\,v(x) = e^{-x^2/4}\,v(x)$$

into the equation. Thus,

$$\widetilde{u}'' + \left(\frac{1}{x} + \frac{x}{2}\right)\widetilde{u}' + \widetilde{u} = \left[u'' + \left(\frac{1}{x} + \frac{x}{2}\right)u' + u\right] v + u\,v'' + 2\,u'\,v' + \left(\frac{1}{x} + \frac{x}{2}\right)u\,v'$$
$$= e^{-x^2/4}\left(v'' + \frac{1}{x}\,v'\right).$$

Therefore, v' satisfies a linear first order ordinary differential equation:

$$v'' + \frac{v'}{x} = 0, \qquad \text{and hence} \qquad v' = c\,\frac{1}{x}, \qquad v = c\,\log x + d.$$

The general solution to the original differential equation is

$$\widetilde{u}(x) = u(x)\,v(x) = e^{-x^2/4}\,(c\,\log x + d).$$

Bessel's Equation

Perhaps the most important "non-elementary" ordinary differential equation is

$$x^2\,u'' + x\,u' + (x^2 - m^2)\,u = 0, \tag{C.57}$$

known as Bessel's equation of order m. We assume here that the order $m \geq 0$ is a non-negative real number; see Exercise C.3.61 for the Bessel equation of imaginary order. As we have seen, the Bessel equation arises from separation of variables in a remarkable number of partial differential equations, including the Laplace, heat and wave equations on a disk, a cylinder, and a spherical ball. Interestingly, the solutions to the Bessel equation were first discovered by the German mathematician Bessel in a completely different context: the study of celestial mechanics, i.e., the Newtonian theory of planets orbiting around a central sun; see (19.16).

The Bessel equation cannot (except in a few particular instances) be solved in terms of elementary functions, and so the use of power series is natural. The leading coefficient $p(x) = x^2$ is nonzero *except* when $x = 0$, and so all points except the origin are regular points. Therefore, at all nonzero points $x_0 \neq 0$, the standard power series construction can be used to produce the appropriate power series solutions of the Bessel equation. However, the recurrence relations for the coefficients are not particularly easy to solve in clsoed form. Moreover, applications tend to demand understanding the behavior of the solutions to the Bessel equation at the singular point $x_0 = 0$. Writing the Bessel equation in solved form

$$u'' = -\,\frac{1}{x}\,u' + \left(\frac{m^2}{x^2} - 1\right)u,$$

we immediately see that $x = 0$ satisfies the conditions to qualify as a regular singular point. Consequently, we are led to seek a solution in the form of a Frobenius expansion. We first compute the expressions for the first two derivatives

$$\begin{aligned} u(x) &= x^r + u_1 x^{r+1} + u_2 x^{r+2} + \cdots \\ u'(x) &= r x^{r-1} + (r+1) u_1 x^r + (r+2) u_2 x^{r+1} + \cdots \\ u''(x) &= r(r-1) x^{r-2} + (r+1) r u_1 x^{r-1} + (r+2)(r+1) u_2 x^r + \cdots\ , \end{aligned} \tag{C.58}$$

of our purported solution. Substituting these expressions into (C.57), we find

$$\begin{aligned}&\big[\, r(r-1)\, x^r + (r+1) r\, u_1\, x^{r+1} + (r+2)(r+1)\, u_2\, x^{r+2} + \cdots \,\big] + \\ &+ \big[\, r\, x^r + (r+1)\, u_1\, x^{r+1} + (r+2)\, u_2\, x^{r+2} + \cdots \,\big] + \\ &+ \big[\, x^{r+2} + u_1\, x^{r+3} + u_2\, x^{r+4} + \cdots \,\big] - \big[\, m^2\, x^r + m^2\, u_1\, x^{r+1} + m^2\, u_2\, x^{r+2} + \cdots \,\big] = 0,\end{aligned}$$

We equate the coefficients of the various powers of x to zero. The coefficient of the lowest order power, x^r, is the indicial equation

$$r(r-1) + r - m^2 = r^2 - m^2 = 0.$$

There are two solutions to the indicial equation, $r = \pm m$, unless $m = 0$ in which case there is only one possible index $r = 0$.

The higher powers of x lead to recurrence relations for the successive coefficients u_n. If we replace m^2 by r^2, we find the following constraints:

$$\begin{aligned}&x^{r+1}: && \big[\,(r+1)^2 - r^2\,\big] u_1 = (2r+1) u_1 = 0, && u_1 = 0, \\ &x^{r+2}: && \big[\,(r+2)^2 - r^2\,\big] u_2 + 1 = (4r+4) u_2 + 1 = 0, && u_2 = -\frac{1}{4r+4}, \\ &x^{r+3}: && \big[\,(r+3)^2 - r^2\,\big] u_3 + u_1 = (6r+9) u_3 + u_1 = 0, && u_3 = -\frac{u_1}{6r+9} = 0,\end{aligned}$$

and, in general,

$$x^{r+n}: \qquad \big[\,(r+n)^2 - r^2\,\big] u_n + u_{n-2} = n(2r+n) u_n + u_{n-2} = 0.$$

Thus, the basic recurrence relation is

$$u_n = -\frac{1}{n(2r+n)}\, u_{n-2}, \qquad n = 2, 3, 4, \ldots\,. \tag{C.59}$$

Starting with $u_0 = 1$, $u_1 = 0$, it is easy to deduce that all $u_n = 0$ for all odd $n = 2k+1$, while for even $n = 2k$,

$$\begin{aligned}u_{2k} &= -\frac{u_{2k-2}}{4k(k+r)} = \frac{u_{2k-4}}{16k(k-1)(r+k)(r+k-1)} = \cdots \\ &= \frac{(-1)^k}{2^{2k}\, k(k-1)\cdots 3\cdot 2\,(r+k)(r+k-1)\cdots(r+2)(r+1)}\,.\end{aligned}$$

Therefore, the series solution is

$$u(x) = \sum_{k=0}^{\infty} u_{2k}\, x^{m+2k} = \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{m+2k}}{2^{2k}\, k(k-1)\cdots 3\cdot 2\,(r+k)(r+k-1)\cdots(r+2)(r+1)}\,. \tag{C.60}$$

So far, we not paid attention to the precise values of the indices $r = \pm m$, or whether our solution to the recurrence relations is valid. In order to continue the recurrence, we need to ensure that the recurrence relation (C.59) is legitimate, meaning that the denominator is never 0. Since $n > 0$, this will *not* be the case if and only if $2r + n = 0$, which requires that

$r = -\frac{1}{2}n$ be either a negative integer $-1, -2, -3, \ldots$, or half-integer, $-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$. These cases occur when the order $m = -r = \frac{1}{2}n$ is either an integer or a half-integer. Indeed, these cases are precisely the cases when the two indices, namely $r_1 = -m$ and $r_2 = m$, differ by an integer, $r_2 - r_1 = n$, and so we are in the tricky case (*iii*) of the Frobenius method.

There is, in fact, a key distinction between the integral and the half integral cases. Recall that the odd coefficients $u_{2k+1} = 0$ in the Frobenius series automatically vanish, and so we only have to worry about the recurrence relation (C.59) for *even* values of n. Thus, for even $n = 2k$, the factor $2r + n = 2(r + k) = 0$ vanishes only when $r = -k$ is a negative integer; the half integral values do not, in fact cause problems. Therefore, if the order $m \geq 0$ is *not* a non-negative integer, then the Bessel equation of order m admits two linearly independent Frobenius solutions, given by the expansions (C.60) with exponents $r = +m$ and $r = -m$. If m is an integer, however, there is only one Frobeius solution, namely the expansion (C.60) with $r = +m$ given by the positive exponent. The second independent solution has an additional logarithmic term in its formula; details appear in Exercise C.3.77.

By convention, the standard *Bessel function* of order m is obtained by multiplying this solution by

$$\frac{1}{2^m\, m!} \qquad \text{or, rather,} \qquad \frac{1}{2^m\, \Gamma(m+1)}\,, \tag{C.61}$$

where the first factorial form can be used if m is a non-negative integer, while the more general gamma function expression must be employed for non-integral values of m. The result is

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{m+2k}}{2^{2k+m}\, k!\, \Gamma(m+k+1)}\,, \tag{C.62}$$

The series is well-defined for all[†] m except when $m = -1, -2, -3, \ldots$ is a negative integer. We conclude that

Theorem C.11. *If $m > 0$ is* not *an integer, then the two linearly independent solutions to the Bessel equation of order m are the Bessel functions $J_m(x)$ and $J_{-m}(x)$. If $m = 0, 1, 2, 3, \ldots$ is an integer, the the Bessel function $J_m(x)$ is a solution to the Bessel equation. The second solution, traditionally denoted $Y_m(x)$, can be found as a limiting case*

$$Y_m(x) = \lim_{\nu \to m} Y_\nu(x) = \lim_{\nu \to m} \frac{\cos \nu\pi\, J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi} \tag{C.63}$$

of a certain linear combination of Bessel functions of non-integral order ν.

The justification of the last statement of the theorem can be found in Exercise C.3.77. We note that for $\nu \neq m$, the linear combination of Bessel functions in the limiting expression is a solution to the Bessel equation of order ν which is indepedent from $J_\nu(x)$. It can

[†] Actually, if m is a negative integer, the first $2m + 1$ terms in the series vanish because $\Gamma(-n) = \infty$ at negative integer values. The series $J_{-m}(x) = J_m(x)$ then actually coincides with its positive sibling.

be proved that this continues to hold in the limit. The series formula for $Y_m(x)$ is quite complicated, [**144, 186**], and its derivation is left to a more advanced course.

Example C.12. Consider the particular case when $m = \frac{1}{2}$. There are two indices, $r = \pm\frac{1}{2}$, for the Bessel equation of order $m = \frac{1}{2}$, leading to two solutions $J_{1/2}(x)$ and $J_{-1/2}(x)$ obtained by the Frobenius method. For the first, with $r = \frac{1}{2}$, the recurrence relation (C.59) takes the form

$$u_n = -\frac{1}{(n+1)n}\, u_{n-2}\,.$$

Starting with $u_0 = 1$ and $u_1 = 0$, the general formula is easily found to be

$$u_n = \begin{cases} \dfrac{(-1)^k}{(n+1)!}\,, & n = 2k \text{ even}, \\ 0 & n = 2k+1 \text{ odd}. \end{cases}$$

Therefore, the resulting solution is

$$u(x) = \sqrt{x}\,\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k} = \frac{1}{\sqrt{x}}\,\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1} = \frac{\sin x}{\sqrt{x}}\,.$$

According to (C.61), the Bessel function of order $\frac{1}{2}$ is obtained by dividing this function by

$$\sqrt{2}\,\Gamma\left(\tfrac{3}{2}\right) = \sqrt{\frac{\pi}{2}}\,,$$

where we used (C.25) to evaluate the gamma function at $\frac{3}{2}$. Therefore,

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\sin x\,. \tag{C.64}$$

Similarly, for the other index $r = -\frac{1}{2}$, the recurrence relation

$$u_n = -\frac{1}{n(n-1)}\, u_{n-2}$$

leads to the formula

$$u_n = \begin{cases} \dfrac{(-1)^k}{n!}\,, & n = 2k \text{ even}, \\ 0 & n = 2k+1 \text{ odd}, \end{cases}$$

for the coefficients, corresponding to the solution

$$u(x) = x^{-1/2}\,\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, x^{2k} = \frac{\cos x}{\sqrt{x}}\,.$$

Therefore, using (C.61) and (C.24), the Bessel function of order $-\frac{1}{2}$ is

$$J_{-1/2}(x) = \frac{\sqrt{2}}{\Gamma\left(\frac{1}{2}\right)}\,\frac{\cos x}{\sqrt{x}} = \sqrt{\frac{2}{\pi x}}\,\cos x\,. \tag{C.65}$$

Remark: If we now substitute (C.64) into the defining formula (18.90) for the spherical Bessel functions, we prove our earlier elementary formula (18.91) for the spherical Bessel function of order 0.

Finally, we demonstrate how Bessel functions of different orders are related by an important recurrence relation.

Proposition C.13. *The Bessel functions are interconnected by the following recurrence formulae:*

$$\frac{dJ_m}{dx} + \frac{m}{x}\, J_m(x) = J_{m-1}(x), \qquad\qquad - \frac{dJ_m}{dx} + \frac{m}{x}\, J_m(x) = J_{m+1}(x). \tag{C.66}$$

Proof: Let us differentiate the power series

$$x^m\, J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{2m+2k}}{2^{2k+m}\, k!\,(m+k)!}.$$

We find

$$\begin{aligned} \frac{d}{dx}\left[\, x^m\, J_m(x)\,\right] &= \sum_{k=0}^{\infty} \frac{(-1)^k\, 2\,(m+k) x^{2m+2k-1}}{2^{2k+m}\, k!\,(m+k)!} \\ &= x^m \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{m-1+2k}}{2^{2k+m-1}\, k!\,(m-1+k)!} = x^m\, J_{m-1}(x). \end{aligned}$$

Expansion of the left hand side of this formula leads to

$$x^m\, \frac{dJ_m}{dx} + m\, x^{m-1}\, J_m(x) = \frac{d}{dx}\left[\, x^m\, J_m(x)\,\right] = x^m\, J_{m-1}(x),$$

which proves the first recurrence formula (C.66). The second formula is proved by a similar manipulation involving differentiation of $x^{-m}\, J_m(x)$. *Q.E.D.*

Example C.14. For instance, we can use (C.66) to find the corresponding recurrence formulae for the spherical Bessel functions

$$S_n(x) = \sqrt{\frac{\pi}{2\,x}}\; J_{n+1/2}(x).$$

Differentiating and using the second recurrence relation, we find

$$\begin{aligned} \frac{dS_n}{dx} &= \sqrt{\frac{\pi}{2\,x}}\; \frac{dJ_{n+1/2}}{dx} - \frac{1}{2}\sqrt{\frac{\pi}{2}}\; \frac{1}{x^{3/2}}\; J_{n+1/2}(x) \\ &= -\sqrt{\frac{\pi}{2\,x}}\; \left(J_{n+3/2}(x) + \frac{n+\frac{1}{2}}{x}\; J_{n+1/2}(x) \right) - \frac{1}{2}\sqrt{\frac{\pi}{2}}\; \frac{1}{x^{3/2}}\; J_{n+1/2}(x) \\ &= -\sqrt{\frac{\pi}{2\,x}}\; J_{n+3/2}(x) + \frac{n}{x}\, \sqrt{\frac{\pi}{2\,x}}\; J_{n+1/2}(x) = -\, S_{n+1}(x) + \frac{n}{x}\, S_n(x). \end{aligned}$$

This completes the proof of the spherical Bessel recurrence formula (18.92).

With this, we conclude our brief introduction to the method of Frobenius and the theory of Bessel functions. The reader interested in further delving into either the general method, or the host of additional properties of Bessel functions is encouraged to consult the texts [**190**, **144**, **107**, **186**].

