① a) $p(\theta|y_1^n) \propto p(y_1^n|\theta)p(\theta)$

$= \prod_{i=1}^{n} \theta(1-\theta)^{y_i-1} \cdot \theta^{\alpha-1}(1-\theta)^{\beta-1}$

$= \theta^{\alpha+n-1}(1-\theta)^{\beta+n\bar{y}-1}$

$\Rightarrow \theta|y_1^n \sim \underline{Beta(\alpha+n, \beta+n\bar{y}-n)}$

b) $E[\theta|y_1^n] = \dfrac{\alpha+n}{\beta+n\bar{y}-n}$

$\lim_{n\to\infty} E[\theta|y_1^n] \approx \frac{1}{\bar{y}} = \hat{\theta}_{MLE}$

c) $Var(\theta|y_1^n) = \dfrac{(\alpha+n)(\beta+n\bar{y}-n)}{(\alpha+\beta+n\bar{y})^2(\alpha+\beta+n\bar{y}-1)}$

d) $\lim_{n\to\infty} Var(\theta|y_1^n) \approx \lim_{n\to\infty} \frac{n^2}{n^3} = 0$

∴ Accuracy of point estimate will increase as uncertainty will decrease.

e) $E[y_{n+1}|y_1^n] = E[E[y_{n+1}|\theta]|y_1^n]$

$= E[\frac{1}{\theta}|y_1^n] = \dfrac{\beta+n\bar{y}-n}{\alpha+n}$

② a) $p(\lambda|x_1^n) \propto p(x_1^n|\lambda)p(\lambda)$

$= \lambda^{a-1}e^{-b\lambda}\prod_{i=1}^{n}\frac{\lambda^{x_i}e^{-\lambda}}{x!} \propto \lambda^{n\bar{x}+a-1}e^{-(n+b)\lambda}$

$\Rightarrow \lambda|x_1^n \sim Gamma(n\bar{x}+a, n+b)$

b) $E[\lambda|x_1^n] = \dfrac{n\bar{x}+a}{n+b}$

c) $Var(\lambda|x_1^n) = \dfrac{n\bar{x}+a}{(n+b)^2}$

d) $E[(\lambda-\hat{\lambda}_{MLE})^2|x_1^n] = E[(\lambda-E[\lambda|x_1^n])^2|x_1^n] + (E[\lambda|x_1^n]-\hat{\lambda}_{MLE})^2$

$= Var(\lambda|x_1^n) + (E[\lambda|x_1^n]-\bar{x})^2 = \dfrac{n\bar{x}+a}{(n+b)^2} + \dfrac{(a-b\bar{x})^2}{(n+b)^2}$

e) $MSE(\hat{\lambda}_B) = Var(\hat{\lambda}_B) + Bias(\hat{\lambda}_B)^2$

$= Var(E[\lambda|x_1^n]) + Bias(E[\lambda|x_1^n])^2 = Var\left(\frac{n\bar{x}+a}{n+b}\right) + Bias\left(\frac{n\bar{x}+a}{n+b}\right)^2$

$= \frac{n^2}{(n+b)^2}Var(\bar{x}) + \left(E\left[\frac{n\bar{x}+a}{n+b}\right]-\lambda\right)^2 = \frac{n\lambda}{(n+b)^2} + \left(\frac{\lambda n+a}{n+b}-\lambda\right)^2$

f) $\left(\frac{\lambda n+a}{n+b}-\lambda\right)^2 + \frac{n\lambda}{(n+b)^2} \le \frac{\lambda}{n}$

If $a=\lambda b$, then bias term is zero.

Let $a=\lambda b \Rightarrow \frac{n\lambda}{(n+b)^2} \le \frac{\lambda}{n} \Rightarrow \frac{n}{(n+b)^2} \le \frac{1}{n}$

Want large value for b. for $MSE(\hat{\lambda}_B)$ to be strictly smaller than $MSE(\hat{\lambda}_{MLE})$ as $n\to\infty$.

③ a)

$$p(\theta_1,\dots,\theta_6 \mid n_1,\dots,n_6) \propto p(n_1,\dots,n_6\mid\theta_1,\dots,\theta_6)\,p(\theta_1,\dots,\theta_6)$$

$$\propto \prod_{i=1}^{6}\theta_i^{\alpha_i-1}\prod_{i=1}^{6}\theta_i^{n_i} = \prod_{i=1}^{6}\theta^{n_i+\alpha_i-1}$$

$$\Rightarrow \quad \theta_1,\dots,\theta_6 \mid n_1,\dots,n_6 \sim \mathrm{Dir}(n_1+\alpha_1,\dots,n_6+\alpha_6)$$

b) $\mathbb{P}\{Y_{n+1}=i \mid Y_i^n\} = \mathbb{P}\{\mathbb{P}\{Y_{n+1}=i\mid\theta_i\}\mid\theta_i\}$

$$= \int \theta_i\, p(\theta\mid Y_i^n)\,d\theta = E[\theta_i \mid Y_i^n] = \frac{n_i+1}{6+\sum_{i=1}^{6} n_i}$$

④ a) $\mathbb{P}\{p\mid x\} \propto \mathbb{P}\{x\mid p\}\mathbb{P}\{p\}$

$$= p^{\alpha-1}(1-p)^{\beta-1}\binom{n}{x}p^{x}(1-p)^{n-x}$$

$$\propto p^{x+\alpha-1}(1-p)^{n-x+\beta}$$

$$\Rightarrow p\mid x \sim \mathrm{Beta}(x+\alpha,\ n-x+\beta)$$

b) $\hat{p} = \dfrac{x+\alpha}{n+\alpha+\beta}$, $s^2 = \dfrac{(x+\alpha)(n-x+\beta)}{(n+\alpha+\beta)^2(n+\alpha+\beta+1)}$

c) $\mathbb{P}\{|\hat{p}-p| \le M \mid x\}$

$$= \mathbb{P}\left\{\left|\frac{\hat{p}-p}{s}\right| \le \frac{M}{s}\,\middle|\, x\right\}$$

$$\le 1-\frac{1}{(M/s)^2} = 1-\frac{s^2}{M^2} = 1-\frac{1}{a^2}$$

$$\Rightarrow \frac{1}{a^2} = \frac{s^2}{M^2} \Rightarrow \underline{M = as}$$

d) $1-\frac{1}{a^2} = .96 \Rightarrow a^2 = 25$

$$M = \sqrt{a^2 s^2}$$

$$= \sqrt{25\cdot\frac{(1125+1)(2500-1125+1)}{(2500+2)^2(2500+3)}}$$

$$M \approx \underline{.04972}$$

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