CS 4501 : SLGM HW0 Rishi Shah

① a) $E[X] = \sum_{x \in X} x p(x) = -2(\frac{1}{8}) - 1(\frac{1}{4}) + 0(\frac{1}{4}) + (1)\frac{1}{4} + 2(\frac{1}{8}) = \boxed{0}$

$Var(X) = E[X^2] - E[X]^2 = \sum_{x \in X} x^2 p(x) - 0^2$

$= (-2)^2(\frac{1}{8}) + (-1)^2(\frac{1}{4}) + (0^2)(\frac{1}{4}) + (1)^2(\frac{1}{4}) + (2)^2(\frac{1}{8})$

$= \frac{1}{2} + \frac{1}{4} + 0 + \frac{1}{4} + \frac{1}{2} = \underline{\frac{3}{2}}$

$\boxed{E[X] = 0, \quad Var(X) = \frac{3}{2}}$

b) $Y = X^2 + 1$

$E[Y] = E[X^2 + 1] = E[X^2] + 1 = Var(X) + E[X]^2 + 1$

$= \frac{3}{2} + 0 + 1 = \underline{\frac{5}{2}}$

$Var(Y) = Var(X^2 + 1) = Var(X^2) = E[X^4] - E[X^2]^2$

$= \sum_{x \in X} x^4 p(x) - (Var(X) + E[X]^2)^2$

$= (-2)^4(\frac{1}{8}) + (-1)^4(\frac{1}{4}) + 0^4(\frac{1}{4}) + (1)^4(\frac{1}{4}) + (2)^4(\frac{1}{8}) - (\frac{3}{2})^2$

$= 2 + \frac{1}{4} + 0 + \frac{1}{4} + 2 - \frac{9}{4} = \frac{9}{4} + \frac{9}{4} - \frac{9}{4} = \underline{\frac{9}{4}}$

$\boxed{E[Y] = \frac{5}{2}, \quad Var(Y) = \frac{9}{4}}$

② We know by definition $\mathbb{P}\{A_2 | A_1\} = \frac{\mathbb{P}\{A_2 \cap A_1\}}{\mathbb{P}\{A_1\}}$,

which can be rewritten as $\mathbb{P}\{A_2 \cap A_1\} = \mathbb{P}\{A_2 | A_1\}\mathbb{P}\{A_1\}$

Base case: $n=1$ very trivial; $\mathbb{P}\{A_1\} = \mathbb{P}\{A_1\}$

Base case: $n=2$

$$\mathbb{P}\{A_2 \cap A_1\} = \mathbb{P}\{A_2 | A_1\}\mathbb{P}\{A_1\} = \mathbb{P}\{A_1\}\mathbb{P}\{A_2 | A_1\}$$

Inductive step:

Let $A = A_1 \cap A_2 \cap \cdots \cap A_{n-1} = \bigcap_{i=1}^{n-1} A_i$,

where $\mathbb{P}\{A\} = \mathbb{P}\{A_1\}\mathbb{P}\{A_2 | A_1\} \cdots \mathbb{P}\{A_{n-1} | A_1 \cap A_2 \cap \cdots \cap A_{n-2}\}$

We can say $\mathbb{P}\{A_n \cap A\} = \mathbb{P}\{A_n | A\}\mathbb{P}\{A\}$ by the definition stated above. It follows that

$$\mathbb{P}\{A_n \cap A\} = \mathbb{P}\{A_n \cap A_{n-1} \cap \cdots \cap A_1\} = \mathbb{P}\{\bigcap_{i=1}^{n} A_i\}$$

$$= \mathbb{P}\{A_n | A_1 \cap A_2 \cap \cdots \cap A_n\}\mathbb{P}\{A_{n-1} | A_1 \cap A_2 \cap \cdots \cap A_{n-2}\} \cdots \mathbb{P}\{A_2 | A_1\}\mathbb{P}\{A_1\}$$

$$= \underline{\mathbb{P}\{A_1\}\mathbb{P}\{A_2 | A_1\} \cdots \mathbb{P}\{A_n | A_1 \cap A_2 \cap \cdots \cap A_{n-1}\}}$$ ✓

3 a) $E[X] = \sum_{x,y \in X,Y} x \cdot p_{X,Y}(x,y) = \sum_{x \in X} x \sum_{y \in Y} p_{x,y}(x,y)$

$= 0 \cdot \left(\frac{1}{16} + \frac{1}{8} + \frac{3}{16}\right) + 1 \cdot \left(\frac{1}{16} + \frac{1}{4} + \frac{1}{16}\right) + 2 \cdot \left(\frac{1}{8} + \frac{1}{8} + 0\right)$

$= 1 \cdot \left(\frac{3}{8}\right) + 2\left(\frac{2}{8}\right) = \underline{\frac{7}{8}}$

$E[Y] = \sum_{x,y \in X,Y} y \, p_{x,y}(x,y) = \sum_{y \in Y} y \sum_{x \in X} p_{X,Y}(x,y)$

$= 0 \cdot \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{8}\right) + 1 \cdot \left(\frac{1}{8} + \frac{1}{4} + \frac{1}{8}\right) + 2 \cdot \left(\frac{3}{16} + \frac{1}{16} + 0\right)$

$= 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) = \underline{1}$

$Var(X) = E[X^2] - E[X]^2 = \sum_{x,y \in X,Y} x^2 p_{X,Y}(x,y) - \left(\frac{7}{8}\right)^2$

$= 1^2 \cdot \frac{3}{8} + 2^2 \cdot \frac{2}{8} - \frac{49}{64} = \frac{11}{8} - \frac{49}{64} = \frac{88-49}{64} = \underline{\frac{39}{64}}$

$Var(Y) = E[Y^2] - E[Y]^2 = \sum_{x,y \in X,Y} y^2 p_{x,y}(x,y) - 1^2$

$= 1^2 \cdot \frac{1}{2} + 2^2 \cdot \frac{1}{4} - 1 = \underline{\frac{1}{2}}$

$$\boxed{\begin{array}{ll} E[X] = \frac{7}{8} & Var(X) = \frac{39}{64} \\ E[Y] = 1 & Var(Y) = \frac{1}{2} \end{array}}$$

b) For X & Y to be independent, $\forall x, y \in X, Y$ $p_{X,Y}(x,y) = p_X(x) p_Y(y)$.

From the joint PMF, we can say

$p_Y(0) = \frac{1}{4}$, $p_Y(1) = \frac{1}{2}$, $p_Y(2) = \frac{1}{4}$

$p_X(0) = \frac{3}{8}$, $p_X(1) = \frac{3}{8}$, $p_X(2) = \frac{1}{4}$

we know $p_{X,Y}(0,0) = \frac{1}{16}$ and $p_X(0) p_Y(0) = \left(\frac{1}{4}\right)\left(\frac{3}{8}\right) = \frac{3}{32}$

Since $p_{X,Y}(0,0) \neq p_X(0) p_Y(0)$, X & Y are <u>not independent.</u>

c) $p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)}$ ∴

$p_{Y\|X}(y\|x)$	$x=0$	$x=1$	$x=2$
$y=0$	$\frac{1/16}{3/8}$	$\frac{1/16}{3/8}$	$\frac{1/8}{1/4}$
$y=1$	$\frac{1/8}{3/8}$	$\frac{1/4}{3/8}$	$\frac{1/8}{1/4}$
$y=2$	$\frac{3/16}{3/8}$	$\frac{1/16}{3/8}$	$\frac{0}{1/4}$

⇒

$p_{Y\|X}(y\|x)$	$x=0$	$x=1$	$x=2$
$y=0$	$1/6$	$1/6$	$1/2$
$y=1$	$1/3$	$2/3$	$1/2$
$y=2$	$1/2$	$1/6$	0

d)

$$E[Y|X=x] = \begin{cases} 0\left(\frac{1}{6}\right) + 1\left(\frac{1}{3}\right) + 2\left(\frac{1}{2}\right), & x=0 \\ 0\left(1/6\right) + 1\left(\frac{2}{3}\right) + 2\left(1/6\right), & x=1 \\ 0\left(1/2\right) + 1\left(\frac{1}{2}\right) + 2(0), & x=2 \end{cases}$$

$$= \begin{cases} 4/3, & x=0, & p_X(0) = 3/8 \\ 1, & x=1, & p_X(1) = 3/8 \\ 1/2, & x=2, & p_X(2) = 1/4 \end{cases}$$

$$E[E[Y|X]] = \frac{3}{8}\left(\frac{4}{3}\right) + \frac{3}{8}(1) + \frac{1}{4}\left(\frac{1}{2}\right)$$

$$= \frac{1}{2} + \frac{3}{8} + \frac{1}{8} = \underline{1}$$

$$\boxed{E[E[Y|X]] = 1}$$

(4)

$$\mathbb{P}\{\text{server works} \mid \text{first fail}\} = \frac{\mathbb{P}\{\text{first fail} \mid \text{server works}\}\,\mathbb{P}\{\text{server works}\}}{\left(\mathbb{P}\{\text{first fail} \mid \text{server works}\}\,\mathbb{P}\{\text{server works}\} + \mathbb{P}\{\text{first fail} \mid \text{server does not work}\}\,\mathbb{P}\{\text{server does not work}\}\right)}$$

a)

$$= \frac{(.1)(.75)}{(.1)(.75) + 1(.25)}$$

$$= \frac{.1(3)}{.1(3)+1} = \frac{.3}{1.3} = \frac{3}{13}$$

$$\boxed{\mathbb{P}\{\text{server works} \mid \text{first fail}\} = \frac{3}{13}}$$

b)

$$\mathbb{P}\{\text{second fail} \mid \text{first fail}\} = \frac{\mathbb{P}\{\text{second fail} \wedge \text{first fail}\}}{\mathbb{P}\{\text{first fail}\}}$$

$$= \frac{\mathbb{P}\{\text{second fail} \wedge \text{first fail} \wedge \text{server works}\} + \mathbb{P}\{\text{second fail} \wedge \text{first fail} \wedge \text{server broken}\}}{\mathbb{P}\{\text{first fail} \wedge \text{second fail}\} + \mathbb{P}\{\text{first fail} \wedge \text{second success}\}}$$

$$= \frac{(.75)(.1)(.1) + (.25)(1)(1)}{\mathbb{P}\{\text{first fail} \wedge \text{second fail} \wedge \text{server works}\} + \mathbb{P}\{\text{first fail} \wedge \text{second fail} \wedge \text{server broken}\} + \mathbb{P}\{\text{first fail} \wedge \text{second success} \wedge \text{server works}\} + \mathbb{P}\{\text{first fail} \wedge \text{second success} \wedge \text{server broken}\}}$$

$$= \frac{(.75)(.1)(.1) + (.25)(1)(1)}{(.75)(.1)(.1) + (.25)(1)(1) + (.75)(.9)(.1) + (.25)(1)(0)} = \frac{.2575}{.2575 + .0675}$$

$$= \frac{2575}{3250} = \frac{103}{130} \Rightarrow \boxed{\mathbb{P}\{\text{second failure} \mid \text{first failure}\} = \frac{103}{130}}$$

5 a) $f_1(x) = A_1 \binom{n}{x} \left(\frac{p}{1-p}\right)^x$

$$\sum_{x=0}^{n} f_1(x) = 1 \Rightarrow \sum_{x=0}^{n} A_1 \binom{n}{x} \left(\frac{p}{1-p}\right)^x = A_1 \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{-x}$$

$$= A_1 \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^x \left(\frac{1-p}{1-p}\right)^n = \frac{A_1}{(1-p)^n} \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \frac{A_1}{(1-p)^n} \Rightarrow \frac{A_1}{(1-p)^n} = 1 \Rightarrow A_1 = (1-p)^n$$

So $f_1(x) = \binom{n}{x} p^x (1-p)^{n-x}$, following Binomial Distribution

b) $f_2(p) = A_2 p^x (1-p)^{n-x}$, $\int_0^1 f_2(p)\,dp = 1$

$$A_2 \int_0^1 p^x (1-p)^{n-x} dp = 1 = A_2 \int_0^1 p^{(x-1)+1} (1-p)^{(n-x-1)+1} dp = 1$$

$$= A_2 \, B(x+1, n-x+1) = 1 \Rightarrow A_2 = \frac{1}{B(x+1, n-x+1)}$$

So $f_2(p) = \frac{p^x (1-p)^{n-x}}{B(x+1, n-x+1)}$, following Beta Distribution

c) $g_1(x) = A_3 \frac{\lambda^x}{x!}$, $x \in \mathbb{N}_0$ $\quad \sum_{x=0}^{n} g_1(x) = 1$

$$A_3 \sum_{x=0}^{n} \frac{\lambda^x}{x!} = A_3 e^{\lambda} = 1 \Rightarrow A = e^{-\lambda}$$

$g_1(x) = \frac{e^{-\lambda} \lambda^x}{x!}$, $x \in \mathbb{N}_0$, following Poisson Distribution

(d) $g_2(\lambda) = A_4 \lambda^x e^{-\lambda}$, $\lambda \geq 0$, $\int_0^\infty g_2(\lambda)\, d\lambda = 1$

$$A_4 \int_0^\infty \lambda^x e^{-\lambda}\, d\lambda = A_4 \int_0^\infty \lambda^{(x+1)-1} e^{-\lambda}\, d\lambda = A_4 \Gamma(x+1) = 1$$

$$\Rightarrow A_4 = \frac{1}{\Gamma(x+1)}$$

So $g_2(\lambda) = \dfrac{\lambda^x e^{-\lambda}}{\Gamma(x+1)}$, following Gamma $(x+1, 1)$ Distribution

$\propto \lambda^{(x+1)-1} e^{-1\cdot\lambda}$

(e) $h_1(x) = A_5 p(1-p)^{x-1}$, $x \in \mathbb{N}$, $\sum_{x=1}^{\infty} h_1(x) = 1$

$$A_5 \sum_{x=1}^{\infty} p(1-p)^{x-1} = A_5 \left(\frac{p}{1-p}\right) \sum_{x=1}^{\infty} (1-p)^x = A_5 \left(\frac{p}{1-p}\right)\left(\frac{1-p}{p}\right) = A_5 = 1$$

$$\Rightarrow A_5 = 1$$

So $h_1(x) = p(1-p)^{x-1}$, following Geometric Distribution

$\sim \text{Geom}(x)$

(f) $h_2(p) = A_6\, p(1-p)^{x-1}$, $0 \leq p \leq 1$, $\int_0^1 h_2(p)\, dp = 1$

$$A_6 \int_0^1 p(1-p)^{x-1} dp = A_6 \left(\left[\frac{-(1-p)^x p}{x}\right]_0^1 - \left(\int_0^1 \frac{(1-p)^x}{x}\, dp\right)\right)$$

$$= A_6 \left(\left[\frac{-(1-p)^x p}{x}\right]_0^1 - \left(\left[\frac{(1-p)^{x+1}}{x(x+1)}\right]_0^1\right)\right) = A_6 \left((0-0) - \left(0 - \frac{1}{x(x+1)}\right)\right)$$

$$= \frac{A_6}{x^2+x} = 1 \quad \Rightarrow \quad A_6 = x^2 + x$$

So $h_2(p) = (x^2+x)\, p(1-p)^{x-1}$ following Beta $(2, x)$ Distribution

$\propto p^{2-1}(1-p)^{x-1}$

(6) $E[Y] = E\left[\frac{1}{N}\sum_{i=1}^{N} x_i\right] = \frac{1}{N}\sum_{i=1}^{N} E[x_i] = \frac{1}{N}\sum_{i=1}^{N}\mu = \frac{N\mu}{N} = \mu$

$Var(Y) = Var\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right) = \frac{1}{N^2}\sum_{i=1}^{N} Var(x_i) = \frac{1}{N^2}\sum_{i=1}^{N}\sigma^2 = \frac{N\sigma^2}{N^2} = \frac{\sigma^2}{N}$

$\boxed{E[Y] = \mu, \quad Var(Y) = \frac{\sigma^2}{N}}$

(7) $A \sim Pois(4)$, $B \sim Pois(5)$ & A, B are independent

Let $Bank = A + B \Rightarrow Bank \sim Pois(9)$

We know this because for two independent Poisson variables $X \sim Pois(\lambda_1)$ and $Y \sim Pois(\lambda_2)$, $X + Y \sim Pois(\lambda_1 + \lambda_2)$.

$\mathbb{P}\{Bank = 4\} = \frac{9^4 e^{-9}}{4!} = \frac{729}{24e^9} = \frac{243}{8e^9} \approx .0337$

$\boxed{\mathbb{P}\{Bank = 4\} = .0337}$

(8) $Cov(X,Y) = E[XY] - E[X]E[Y]$ Given

$Cov(aX+b, cY) = E[(aX+b)(cY)] - E[aX+b]\,E[cY]$

$= E[acXY + bcY] - (E[aX] + E[b])\,E[cY])$

$= ac\,E[XY] + bc\,E[Y] - ac\,E[X]E[Y] - bc\,E[Y]$

$= ac\,E[XY] - ac\,E[X]E[Y]$

$= ac\,(E[XY] - E[X]E[Y])$

$= \underline{ac\,Cov(X,Y)}$ ✓

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