Rishi Shah CS 4501 - SLGM HW2

(1) $p(x_i;\lambda) = \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$ $\quad l(\lambda) = \ln\left(\prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}\right)$

$= \sum_{i=1}^{n} \ln \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \sum_{i=1}^{n}\left(x_i \ln(\lambda) - \lambda - \ln(x_i!)\right) = -n\lambda + \ln(\lambda)\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \ln(x_i!)$

$\frac{dl(\lambda)}{d\lambda} = -n + \frac{1}{\lambda}\sum_{i=1}^{n} x_i \Rightarrow 0 = \frac{dl(\lambda)}{d\lambda} \Rightarrow \hat{\lambda}_{MLE} = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$

$\text{Bias}(\lambda_{MLE}) = E[\lambda_{MLE}] - \lambda = E[\bar{x}] - \lambda = \frac{1}{n}E\left[\sum_{i=1}^{n} x_i\right] - \lambda = \frac{1}{n}\sum_{i=1}^{n} E[x_i] - \lambda$

$= \frac{n\lambda}{n} - \lambda = 0$

$\text{Var}(\lambda_{MLE}) = \text{Var}(\bar{x}) = \text{Var}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) = \frac{1}{n^2}\text{Var}\left(\sum_{i=1}^{n} x_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}\text{Var}(x_i)$

$= \frac{1}{n^2} \cdot n\lambda = \frac{\lambda}{n}$

$\text{MSE}(\lambda_{MLE}) = \text{Var}(\lambda_{MLE}) - \text{Bias}(\lambda_{MLE})^2 = \frac{\lambda}{n} - 0^2 = \frac{\lambda}{n}$

$\boxed{\lambda_{MLE} = \bar{x},\ \text{Bias} = 0,\ \text{Variance} = \frac{\lambda}{n},\ \text{MSE} = \frac{\lambda}{n}}$

(2)

a) $l(\theta) = \ln\left(\prod_{i=1}^{n} \theta(1-\theta)^{y_i - 1}\right) = \sum_{i=1}^{n} \ln\left(\theta(1-\theta)^{y_i-1}\right) = \sum_{i=1}^{n}\left(\ln\theta + (y_i - 1)\ln(1-\theta)\right)$

$= n\ln(\theta) - n\ln(1-\theta) + \sum_{i=1}^{n} y_i \ln(1-\theta)$

$\frac{dl(\theta)}{d\theta} = \frac{n}{\theta} + \frac{n}{1-\theta} - \frac{1}{1-\theta}\sum_{i=1}^{n} y_i = 0 \Rightarrow \frac{1}{1-\theta}\left(\left(\sum_{i=1}^{n} y_i\right) - n\right) = \frac{n}{\theta}$

$\theta\sum_{i=1}^{n} y_i - \theta n = n - n\theta \Rightarrow \theta = \frac{1}{\sum_{i=1}^{n} y_i} \cdot n = \frac{1}{\bar{y}} \Rightarrow \boxed{\theta_{MLE} = \frac{1}{\bar{y}}}$

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b) $p_1=\theta,\ p_2=\theta,\ p_3=2\theta,\ p_4=1-4\theta,$

$$p(n_1,n_2,n_3,n_4;p_1,p_2,p_3,p_4)=\frac{n!}{n_1!n_2!n_3!n_4!}\ \theta^{n_1+n_2+n_3}\ 2^{n_3}\ (1-4\theta)^{n_4}$$

$$l(\theta)=\ln\left(\frac{n!}{n_1!n_2!n_3!n_4!}\ \theta^{n_1+n_2+n_3}\ 2^{n_3}\ (1-4\theta)^{n_4}\right)$$
$$=\ln\left(\frac{n!}{n_1!n_2!n_3!n_4!}\right)+(n_1+n_2+n_3)\ln(\theta)+n_3\ln(2)+n_4\ln(1-4\theta)$$

$$\frac{dl(\theta)}{d\theta}=\frac{n_1+n_2+n_3}{\theta}-\frac{4n_4}{1-4\theta}=0 \Rightarrow 4\theta n_4=(n_1+n_2+n_3)-4\theta(n_1+n_2+n_3)$$
$$\Rightarrow \theta=\frac{n_1+n_2+n_3}{4(n_1+n_2+n_3+n_4)} \qquad \boxed{\hat{\theta}_{MLE}=\frac{n_1+n_2+n_3}{4(n_1+n_2+n_3+n_4)}}$$

3)
a)
$$l(\theta)=\ln\left(\prod_{i=1}^{n}\frac{1}{\theta}e^{-t/\theta}\right)=\sum_{i=1}^{n}\ln\left(\frac{1}{\theta}e^{-t/\theta}\right)=\sum_{i=1}^{n}\left(-\ln(\theta)-\frac{t}{\theta}\right)=-n\ln\theta-\frac{1}{\theta}\sum_{i=1}^{n}t_i$$

$$\frac{dl(\theta)}{d\theta}=\frac{-n}{\theta}+\frac{1}{\theta^2}\sum_{i=1}^{n}t_i=0\Rightarrow\frac{n}{\theta}=\frac{1}{\theta^2}\sum_{i=1}^{n}t_i\Rightarrow\theta=\frac{1}{n}\sum_{i=1}^{n}t_i\Rightarrow\boxed{\hat{\theta}_{MLE}=\bar{t}}$$

b)
$$MSE(\hat{\theta}_{MLE})=Var(\hat{\theta}_{MLE})+Bias(\hat{\theta}_{MLE})^2=\frac{1}{n^2}Var\left(\sum_{i=1}^{n}t_i\right)+\left(E\left[\frac{1}{n}\sum_{i=1}^{n}t_i\right]-\theta\right)^2$$
$$=\frac{1}{n^2}\sum_{i=1}^{n}Var(t_i)+\left(\frac{1}{n}\sum_{i=1}^{n}E[t_i]-\theta\right)^2=\frac{1}{n^2}\sum_{i=1}^{n}\frac{1}{(1/\theta)^2}+\left(\frac{\theta n}{n}-\theta\right)^2$$
$$=\frac{\theta^2 n}{n^2}+0^2=\frac{\theta^2}{n}\qquad\Rightarrow\boxed{MSE(\hat{\theta}_{MLE})=\frac{\theta^2}{n}}$$

4) a) If we repeat the poll n times with random samples, we can expect the true proportion p to be in CI $(\hat{p}-.05,\ \hat{p}+.05)$ in $n(1-\alpha)$ polls wher α is significance level.

b)
$$l(p)=\ln\left(\binom{n}{x}p^x(1-p)^{n-x}\right)=\ln\binom{n}{x}+x\ln(p)+(n-x)\ln(1-p)$$

$$\frac{dl(p)}{dp}=\frac{x}{p}-\frac{n-x}{1-p}=0\Rightarrow x(1-p)=p(n-x)\Rightarrow x-xp=pn-xp$$
$$\Rightarrow p=\frac{x}{n}\qquad\Rightarrow\boxed{\hat{p}_{MLE}=\frac{x}{n}}$$

d) $\mathbb{P}\left\{ |\hat{p}_{MLE} - p| \ge a\sqrt{\frac{p(1-p)}{n}} \right\}$

$$= \mathbb{P}\left\{ \frac{|\hat{p}_{MLE} - p|}{\sqrt{\frac{p(1-p)}{n}}} \ge a \right\} = \mathbb{P}\left\{ \left|\frac{\hat{p} - \hat{p}}{\sigma}\right| \ge a \right\} \le \frac{1}{a^2}$$

e) $\mathbb{P}\left\{ |\hat{p} - p| \ge \frac{a}{2\sqrt{n}} \right\} = \mathbb{P}\left\{ |x - np| \ge \frac{a\sqrt{n}}{2} \right\} = \mathbb{P}\left\{ \frac{|x-np|}{\sqrt{np(1-p)}} \ge \frac{a}{2\sqrt{p(1-p)}} \right\}$

$$\le \frac{1}{\left(\frac{a}{2\sqrt{p(1-p)}}\right)} = \frac{4p(1-p)}{a^2} \le \frac{1}{a^2} \quad \text{since } p \in [0,1]$$

f) $1 - \frac{1}{a^2} = .96 \Rightarrow .04 = \frac{1}{a^2} \Rightarrow a^2 = \frac{1}{.04} = 25$

$a = 5 \Rightarrow \frac{a}{2\sqrt{n}} = .05 = \frac{5}{2\sqrt{n}} \Rightarrow \sqrt{n} = 50 \Rightarrow n = 2500$

$$\boxed{\begin{array}{l} a = 5 \\ n = 2500 \end{array}}$$