CS 202: Mathematics for Computer Science II: An Introduction to Mathematical Logic
Lecture 18-19

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Outline

- Relevance Lemma
- Satisfiability and Validity
- Semantic Entailment
- Proof Methods
  - Natural Deduction
- Discussion on Godel’s Completeness Theorem
- Discussion on FOL with Equality
- Discussion on FOL with Peano arithmetic
- Discussion on Incompleteness Theorem
- Final words

Relevance Lemma

Relevance Lemma formalizes the intuitive fact that satisfiability of a formula depends only on the values assigned to the variables which are free in the formula but not on the values assigned to other variables in the environment.

Lemma: Let α be a first order formula, I be an interpretation, and E_1, E_2 two environments such that $E_1(x) = E_2(x)$ for all $x \in FV(\alpha)$. Then $I \vDash_{E_1} \alpha \ iff \ I \vDash_{E_2} \alpha$.

Proof: By induction on the structure of α.

Some examples of Satisfiability and Validity

- **Example 1**

Let L $be\ a\ language\ consisting\ of$

$$variables\ x, y, z;$$
$$function\ symbols\ f^2, g^1;$$
$$predicate\ symbol\ P^2$$

Let $\alpha \overset{\text{def}}{=} P(f(g(x), g(y)), g(z))$

$I: dom(I) = N\ ; f\ is\ addition, g\ is\ squaring, and\ P\ is\ equality$

Then for $E: E(x) = 3, E(y) = 4, E(z) = 5, \alpha \overset{\text{def}}{=} x^2 + y^2 = z^2\ and$

$I \vDash_E \alpha$ If fact, with the same $I, for\ any\ environment$

$where\ x, y, z\ are\ pythagorian\ triplets, \alpha\ is\ satisfied.$

But $\boldsymbol{\alpha\ is\ not\ a\ valid\ formula}$.

Another Example

- Example 2:
- Let L be a language consisting of variables x, y; predicate symbol Q^2

$$I: D = \{1,2\}; Q^I = \emptyset$$
$$E: E(x) = 1\,, E(y\,) = 2$$

Consider $\alpha \stackrel{\text{def}}{=} \left(\exists x \left(\forall y\; Q(x,y)\right)\right)$

Clearly $I \;\not\vDash_E\; \alpha$

Consider $J1$:

$$D = \{1.2\}; Q^I = \{(1,2)\}$$
$$G\!:\; G(x) = 1, G(y) = 2$$
$$J1 \vDash_G \;(\exists x\,(\exists y\; Q(x,y)))$$

Consider $J2$:

$$D = \{1,2\}; Q^I$$
$$= \{(1,2), (2,2)\}$$
$$G: G(x) = 1, G(y) = 2$$
$$J2 \vDash_G \;\left(\forall x\,(\exists y\; Q(x,y)\right))$$

Semantic Entailment

Let Σ be a wff and α is a wff. The interpretation I and environment E, if $I \vDash_E \ \Sigma \ implies \ I \vDash_E \ \alpha$, then we write $\Sigma \vDash \alpha$ (Σ entails α).

If $\emptyset \vDash \alpha$ then we say $\alpha\ is\ valid$. This means α is satisfied by all The interpretation I and environment E.

$If\ \Sigma = \{\alpha_1, \alpha_2, \dots, \alpha_n\}\ and\ \Sigma \vDash \beta$, then we can also say

$$\emptyset \vDash (\alpha_1 \wedge (\alpha_2 \wedge \left(\alpha 3 \wedge \left(\dots \dots \wedge (\alpha_{n-1} \wedge \alpha_n)\right) \dots \right) \Rightarrow \beta$$

In other words $(\alpha_1 \wedge (\alpha_2 \wedge \left(\alpha 3 \wedge \left(\dots \dots \wedge (\alpha_{n-1} \wedge \alpha_n)\right) \dots \right) \Rightarrow \beta$ is a valid wff.

How do you prove semantic entailments?

- In propositional logic it was straight forward albeit exponential complexity – if there were n propositional variables in the formulas, try out 2^n possible models and check if those satisfying $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ also satisfy β.
- In the first order logic, there are immense number of possible interpretations, and environments – it is not possible that way.
- However, we can reason about the models (Interpretations and environments) that would satisfy the formulas using our high school level proof techniques such as proof by contradiction.
- Let’s try some examples.

Semantic Entailment Proofs

Example 1:

Show that $\emptyset \vDash \left(\left(\forall x\ (\alpha \Rightarrow \beta)\right) \Rightarrow ((\forall x\ \alpha) \Rightarrow (\forall x\ \beta))\right)$

Proof by contraction: Suppose the above formula is not valid. That would mean there is a model (an interpretation I, and environment E) such that $I\ \nvDash_E\ \left(\left(\forall x\ (\alpha \Rightarrow \beta)\right) \Rightarrow ((\forall x\ \alpha) \Rightarrow (\forall x\ \beta))\right)$.

That means $I \vDash_E ((\forall x\ (\alpha \Rightarrow \beta))\ but\ I\ \nvDash_E ((\forall x\ \alpha) \Rightarrow (\forall x\ \beta))$

Now $I\ \nvDash_E ((\forall x\ \alpha) \Rightarrow (\forall x\ \beta))$ means $I \vDash_E\ (\forall x\ \alpha)\ but\ I\ \nvDash_E\ (\forall x\ \beta)$

Now by definition of $\forall x$, $I \vDash_{E[x \mapsto a]} (\alpha \Rightarrow \beta)$ for all $a \in dom(I)$

And $\forall x$, $I \vDash_{E[x \mapsto a]} \alpha$ for all $a \in dom(I)$. This apply modus ponens on all $a \in dom(I)$, we get $I \vDash_{E[x \mapsto a]} \beta$ but that means $I \vDash_E (\forall x\ \beta)$ ← contradiction. QED.

Semantic Entailment Proofs

Example 2:

Show that $\{(\forall x\ (\neg\gamma))\} \vDash (\neg\ (\exists x\ \gamma))$

Proof by contradiction. Suppose by way of contradiction that there is a model (an interpretation I, and environment E) such that $I \vDash_E (\forall x\ (\neg\gamma))\ but\ I\ \nvDash_E (\neg(\exists x\ \gamma))$ which means $I \vDash_E\ (\exists x\ \gamma)$

Now from $I \vDash_E (\forall x\ (\neg\gamma))$ we can say that for all $a \in dom(I)$, $I \vDash_{E[x\mapsto a]}\ \neg\gamma$ which means for no $a \in dom(I)$, $I \vDash_{E[x\mapsto a]}\ \gamma$ and that contradicts $I \vDash_E\ (\exists x\ \gamma)$. Thus contradiction achieved. QED.

Semantic Entailment Proofs

- **Example 3**:

Find $wffs\ \alpha\ and\ \beta\ such\ that$

$$\{((\forall x\ \alpha) \Rightarrow (\forall x\ \beta))\} \nvDash (\forall x\ (\alpha \Rightarrow \beta))$$

- Consider I: $dom(I) = \{a, b\}, predicate\ symbol\ P^I\ = \{a\}$
- Suppose $\alpha \stackrel{\text{def}}{=} P(x)\ and\ \ \beta \stackrel{\text{def}}{=} (\neg\ P(x))$
- Then $P(a) = T; P(b) = F; \neg P(a) = F, and\ \neg P(b) = T$.
- So $P(a)\ cannot\ imply\ \neg P(a)$
- $P(b)\ being\ false, it\ can\ imply\ \neg P(b)\ but$
- $we\ cannot\ say\ (\forall x\ \left(P(x) \Rightarrow (\neg P(x))\right))$

Semantic Entailment Examples

Example 4:

For any formula α, and term $t, show\ that$:

$$\emptyset \vDash ((\forall x\ \alpha) \Rightarrow \left(\alpha\left[\frac{t}{x}\right]\right)))$$

Proof by contradiction: Suppose $(\forall x\ \alpha)$ is true, but $\alpha\left[\frac{t}{x}\right]$ is false under an interpretation I, and environment E.

Then $I \vDash_{E[x \mapsto a]} \alpha$ for all $a \in dom(I)$.

We also know that $t^{I,E} \in dom(I)$.

So $\left(\alpha\left[\frac{t}{x}\right]\right)^{I,E} = \alpha^{I,E}\left[\frac{t^{I,E}}{x}\right]$ $cannot\ be\ false$. Contradiction. QED.

Semantic Entailment Examples

Example 5: Let α be any wff without a free occurrence of x. Let I be an interpretation, and E be an environment. Then $\alpha^{I,E} = (\forall x\ \alpha)^{I,E}$.

Proof: (Note that this is intuitive – because if α does not have free occurrence of x, then quantifying over x will not make any difference).

Let $D = dom(I),\ x \notin FV(\alpha), so\ E(y) = E[x \mapsto a](y)\ for\ all\ y \in FV(\alpha),$ for all a $\in D$. Then by **the Relevance lemma,** we can say that

$$I \vDash_E \alpha\ iff$$

$$I \vDash_{E[x \mapsto a]} \alpha \text{ for all } a \in D \ \textit{ iff}$$

$$I \vDash_E (\forall x\ \alpha)$$

QED.

Discussion on Proving semantic entailment

- Unlike in propositional logic, where an exponential number of models can be tried to check validity or satisfiability, here the choices are too many.
- Therefore, we need a proof technique like resolution refutation in proposition logic, but for first order logic.
- One can design multiple proof systems each of which must be separately proven to be sound and complete.
- We focus on Natural Deduction Proof System.
- Natural Deduction proof system is sound and complete but we will not get time to prove that.

Rules of Natural Deduction

Name	*Proof Notation* $(\vdash)$	*Deduction Notation*
$\wedge\ introduction\ (\wedge_i)$	If $\Sigma \vdash \alpha\ and\ \Sigma \vdash \beta\ then$ $\Sigma \vdash (\alpha \wedge \beta)$	$\dfrac{\alpha \quad \beta}{(\alpha \wedge \beta)}$
$\wedge\ elimination\ (\wedge_e)$	$if\ \Sigma \vdash (\alpha \wedge \beta)\ then$ $\Sigma \vdash \alpha\ \ and\ \Sigma \vdash \beta$	$\dfrac{(\alpha \wedge \beta)}{\alpha} \quad \dfrac{(\alpha \wedge \beta)}{\beta}$
$\vee\ introduction\ (\vee_i)$	$If\ \Sigma \vdash \alpha\ \ then$ $\Sigma \vdash (\alpha \vee \beta)$	$\dfrac{\alpha}{(\alpha \vee \beta\}} \qquad \dfrac{\alpha}{(\beta \vee \alpha)}$
$\vee\ elimination\ (\vee_e)$	$If\ \Sigma, \alpha \vdash \gamma\ \ and\ \Sigma, \beta \vdash \gamma\ then$ $\Sigma, (\alpha \vee \beta) \vdash \gamma$	$\dfrac{(\alpha \vee \beta) \quad \boxed{\begin{array}{c}\alpha\\ \vdots\\ \gamma\end{array}} \quad \boxed{\begin{array}{c}\beta\\ \vdots\\ \gamma\end{array}}}{\gamma}$
$\Rightarrow\ introduction\ (\Rightarrow_i)$	$If\ \Sigma, \alpha \vdash \beta\ then\ \Sigma \vdash (\alpha \Rightarrow \beta)$	$\dfrac{\boxed{\begin{array}{c}\alpha\\ \vdots\\ \beta\end{array}}}{(\alpha \Rightarrow \beta)}$

Name	Proof notation ($\vdash$)	Deduction Notation
$\Rightarrow$ *elimination* ($\Rightarrow_e$)	*if* $\Sigma \vdash (\alpha \Rightarrow \beta)$ *and* $\Sigma \vdash \alpha$ *then* $\Sigma \vdash \beta$	$\dfrac{(\alpha \Rightarrow \beta) \quad \alpha}{\beta}$
Reflexivity or Premise	$\Sigma, \alpha \vdash \alpha$	$\dfrac{\alpha}{\alpha}$
$\bot$ *introduction* ($\bot_i$) $\neg$ *elimination* ($\neg_e$)	$\Sigma, \alpha, (\neg\alpha) \vdash \bot$	$\dfrac{\alpha \quad (\neg\alpha)}{\bot}$
$\neg$ *introduction* ($\neg_i$)	*If* $\Sigma, \alpha \vdash \bot$ *then* $\Sigma \vdash (\neg\alpha)$	$\dfrac{\boxed{\begin{matrix}\alpha \\ \vdots \\ \bot\end{matrix}}}{(\neg\alpha)}$
$\neg\neg$ *elimination* ($\neg\neg_e$)	$\Sigma \vdash (\neg(\neg\alpha))$ *then* $\Sigma \vdash \alpha$	$\dfrac{(\neg(\neg\alpha))}{\alpha}$
Contradiction elimination ($\bot_e$)	$\Sigma \vdash \bot$ *then* $\Sigma \vdash \alpha$ for any α	$\dfrac{\bot}{\alpha}$

Name	*Proof notation* ($\vdash$)	*Deduction Notation*
$\forall$ *elimination* ($\forall_e$)	*If* $\Sigma \vdash (\forall x\ \alpha)$ *then* $\Sigma \vdash \alpha[\frac{t}{x}]$	$\dfrac{(\forall x\ \alpha)}{\alpha\left[\frac{t}{x}\right]}$
$\exists$ *introduction* ($\exists_e$)	*If* $\Sigma \vdash \alpha\left[\frac{t}{x}\right]$ *then* $\Sigma \vdash (\exists x\ \alpha)$	$\dfrac{\alpha\left[\frac{t}{x}\right]}{(\exists x\ \alpha)}$
$\forall$ *introduction* ($\forall_i$)	*if* $\Sigma \vdash \alpha\left[\frac{y}{x}\right]$ *where* y *is a fresh varia* *and* y *not free in* Σ, α *the* $\Sigma \vdash (\forall x\ \alpha)$	$\dfrac{\boxed{\begin{array}{c} y \quad fresh \\ \vdots \\ \alpha[y/x] \end{array}}}{(\forall x \alpha)}$
$\exists$ *elimination* ($\exists_e$)	*If* $\alpha\left[\frac{u}{x}\right] \vdash \beta$ *with fresh* u, *then* $\Sigma, (\exists x\ \alpha) \vdash \beta$	$\dfrac{(\exists x \alpha) \quad \boxed{\begin{array}{c} \alpha[u/x] \\ \vdots \\ \beta \end{array}}}{\beta}$

Example Proofs using Natural Deduction

Example 1:

Prove $\{p, q\} \vdash p$

1. $p \quad premise$
2. $q \quad premise$
3. $p \quad reflexivity$ 1.

QED

Example 2:

Prove $\{(p \wedge q)\} \vdash (q \wedge p)$

1. $(p \wedge q) \quad premise$
2. $q \quad \wedge_e \quad on\ 1.$
3. $p \quad \wedge_e \quad on\ 1.$
4. $(q \wedge p) \quad \wedge_i \quad on\ 2, 3$

QED

Example Proofs using Natural Deduction

- Example 3: Prove

$$\{(p \Rightarrow q), (q \Rightarrow r)\} \vdash (p \Rightarrow r)$$

1.	$(p \Rightarrow q)$	$Premise$
2.	$(q \Rightarrow r)$	$Premise$
3.	p	$Assumption$
4.	q	$\Rightarrow_e$ 1,3 $(modus\ ponens)$
5.	r	$\Rightarrow_e$ 2,4 $(modus\ ponens)$
6.	$(p \Rightarrow r)$	$\Rightarrow_i$ $3 - 5$

Example Proofs using Natural Deduction

- Example 4: Prove
  - $\{(p \vee q)\} \vdash \left((p \Rightarrow q) \vee (q \Rightarrow p)\right)$

1. $(p \vee q)$ $\quad$ *Premise*
2. p $\quad$ *Asssumption*
3. q $\quad$ *Assumption*
4. p $\quad$ *Reflexivity* 2
5. $(q \Rightarrow p)$ $\quad$ $\Rightarrow_i$ 3,4
6. $\left((p \Rightarrow q) \vee (q \Rightarrow p)\right)$ $\quad$ $\vee_i$ 5.
7. q $\quad$ *Assumption*
8. p $\quad$ *Assumption*
9. q $\quad$ *Reflexivity* 7
10. $(p \Rightarrow q)$ $\quad$ $\Rightarrow_i$ $8-9$
11. $\left((p \Rightarrow q) \vee (q \Rightarrow p)\right)$ $\quad$ $\vee_i$ 10
12. $\left((p \Rightarrow q) \vee (q \Rightarrow p)\right)$ $\quad$ $\vee_e$ $1,2-6,7-11$

Example 5: Prove

$$\{(\alpha \Rightarrow (\neg\alpha))\} \vdash (\neg\alpha)$$

1. $(\alpha \Rightarrow (\neg\alpha))$ $\quad$ *Premise*
2. α $\quad$ *Assumption*
3. $(\neg\alpha)$ $\quad$ $\Rightarrow_e$ $1,2$
4. $\perp$ $\quad$ $\neg_e$ $2,3$
5. $(\neg\alpha)$ $\quad$ $\neg_i$ $2-4$

Natural Deduction for Predicate Logic Examples

- Example 1: Prove

$$\{(\forall x\ P(x))\} \vdash (\exists x\ P(x))$$

1.	$(\forall x\ P(x))$	$Premise$
2.	$P(u)$	$\forall_e\ 1$
3.	$(\exists x\ P(x))$	$\exists_i\ 2$

More Examples

- Example 2: Prove

$$\left\{P(t), \left(\forall x \left(P(x) \Rightarrow (\neg Q(x))\right)\right)\right\} \vdash (\neg Q(t))$$

1. $P(t)$ $\qquad$ $Premise$
2. $\left(\forall x \left(P(x) \Rightarrow (\neg Q(x))\right)\right)$ $\qquad$ $Premise$
3. $\left(P(t) \Rightarrow (\neg Q(t))\right)$ $\qquad$ $\forall_e\ 2$
4. $(\neg Q(t))$ $\qquad$ $\Rightarrow_e\ 1-3$
5. QED

Examples

Example 3: Show that

$$\{(\neg P(y))\} \vdash (\exists x\,(P(x) \Rightarrow Q(y)))$$

1. $(\neg P(y))$ $\quad$ $premise$
2. $P(y)$ $\quad$ $Assumption$
3. $\bot$ $\quad$ $\neg_e\ 2{,}1$
4. $Q(y)$ $\quad$ $\bot_e\ 3$
5. $(P(y) \Rightarrow Q(y))$ $\quad$ $\Rightarrow_i\ 2-4$
6. $(\exists x(P(x) \Rightarrow Q(y)))$ $\quad$ $\exists_i\ 5$