SUBJECT CODE : 210241

Strictly as per Revised Syllabus of

SAVITRIBAI PHULE PUNE UNIVERSITY

Choice Based Credit System (CBCS)

S.E. (Computer) Semester - I

DISCRETE MATHEMATICS

(For IN SEM Exam - 30 Marks)

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PREFACE

The importance of **Discrete Mathematics** is well known in various engineering fields. Overwhelming response to our books on various subjects inspired us to write this book. The book is structured to cover the key aspects of the subject **Discrete Mathematics.**

The book uses plain, lucid language to explain fundamentals of this subject. The book provides logical method of explaining various complicated concepts and stepwise methods to explain the important topics. Each chapter is well supported with necessary illustrations, practical examples and solved problems. All the chapters in the book are arranged in a proper sequence that permits each topic to build upon earlier studies. All care has been taken to make students comfortable in understanding the basic concepts of the subject.

The book not only covers the entire scope of the subject but explains the philosophy of the subject. This makes the understanding of this subject more clear and makes it more interesting. The book will be very useful not only to the students but also to the subject teachers. The students have to omit nothing and possibly have to cover nothing more.

We wish to express our profound thanks to all those who helped in making this book a reality. Much needed moral support and encouragement is provided on numerous occasions by our whole family. We wish to thank the **Publisher** and the entire team of **Technical Publications** who have taken immense pain to get this book in time with quality printing.

Any suggestion for the improvement of the book will be acknowledged and well appreciated.

Authors
Dr. H. R. Bhapkar
Dr. Rajesh N. Phursule

Dedicated to the Readers of the Book

SYLLABUS

Discrete Mathematics - (210241)

Credit Scheme	Examination Scheme and Marks
03	Mid_Semester (TH) : 30 Marks

Unit I Set Theory and Logic

Introduction and significance of Discrete Mathematics, **Sets** – Naïve Set Theory (Cantorian Set Theory), Axiomatic Set Theory, Set Operations, Cardinality of set, Principle of incl usion and exclusion, **Types of Sets** - Bounded and Unbounded Sets, Diagonalization Argument, Countable and Uncountable Sets, Finite and Infinite Sets, Countably Infinite and Uncountably Infinite Sets, Power set, **Propositional Logic -** logic, Propositional Equivalences, Application of Propositional Logic - Translating English Sentences, Proof by Mathematical Induction and Strong Mathematical Induction. **(Chapters - 1, 2, 3)**

Unit II Relations and Functions

Relations and their Properties, n-ary relations and their applications, Representing relations, Closures of relations, Equivalence relations, Partial orderings, Partitions, Hasse diagram, Lattices, Chains and Anti-Chains, Transitive closure and Warshall's algorithm. **Functions** - Surjective, Injective and Bijective functions, Identity function, Partial function, Invertible function, Constant function, Inverse functions and Compositions of functions, The Pigeonhole Principle. **(Chapters - 4, 5)**

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Unit - I

1 Theory of Sets

Syllabus

***Introduction** and significance of Discrete Mathematics, **Sets** – Naïve Set Theory (Cantorian Set Theory), Axiomatic Set Theory, Set Operatio ns, Cardinality of set, Principle of inclusion and exclusion, **Types of Sets** - Bounded and Unbounded Sets, Diagonalization Argument, Countable and Uncountable Sets, Finite and Infinite Sets, Countably Infinite and Uncountably Infinite Sets, Power set.*

Contents

1.1 Introduction

The notion of a set is a fundamental concept to all of Mathematics and every branch of mathematics can be considered as a study of sets of objects of one kind or another. A great mathematician Cantor was the founder of the theory of sets. Let us now consider the idea of a set.

1.2 Sets

A set is a collection of well defined objects. An object in the set is called a member or element of the set. The objects themselves can be almost anything. e.g. Books, numbers, cities, countries, animals, etc. In the above definition, the words set and collection for almost all practical purposes are synonymous. Elements of a set are usually denoted by lower case letters i.e. a, b, c, ... while sets are denoted by capital letters i.e. A, B, C, ...

The symbol '$\in$' indicates the membership in a set.

If "x is an element of a set A" then we write $x \in A$.

If "x is not an element of a set A" then we write as $x \notin A$.

Examples :

1) The set of letters forming the word "MATHEMATICS"
2) The set of students in a class SE Computer Engineering
3) s = {1, 2, 3, 4, 5, 6}
4) The set of professors in SPPU university.
5) The set of all telephone numbers in the directory.

1.3 Methods of Describing Sets

The following are the most useful and common methods of describing sets.

1.3.1 Roster Method (Listing Method)

A set may be described by listing all the members of the set between a pair of braces.

In this method the order in which the elements are listed is immaterial and it is used for small sets. e.g. A = {1, 2, 3, 4, 5}, B = {a, b, c, d, e, f, g}

1.3.2 Statement Form

In this form, set is formed especially where the elements have a common characteristic.

e.g. 1) The set of all even integers

2) The set of all Prime Ministers of India

1.3.3 Set Builder Notation

It is not always possible to describe a set by the listing method or statement form.

A more compact or concise way of describing the set is to specify the common property of all elements of the set

e.g. $A = \{x \mid x \text{ is real and } x^2 > 100\}$

$B = \{x \mid x \text{ is real and } x^{20} - x^{10} + 1 = 0\}$

1.4 Some Special Sets

The following sets are important and occur frequently in our discussion.

1) Set of natural numbers = N = {1, 2, 3, ...}

2) Set of all integers = Z = {...–3, –2, –1, 0, 1, 2, 3 ....}

3) Set of all positive integers = Z^+ = {0, 1, 2, 3, ...}

4) Set of all rational numbers = Q = $\left\{\frac{p}{q} \mid p, q \in R, (p, q) = 1\ q \neq 0\right\}$

5) Set of real numbers = $\Re$

6) Set of complex numbers = $\mathbb{C}$

1.5 Subsets

A set A is said to be a subset of the set B if every element of A is also an element of B.

We also say that A is contained in B and denoted by $A \subseteq B$

If A is a subset of B i.e. $A \subseteq B$, then the set

B is called the superset of A.

The set with no elements is called an empty set or null set. It is denoted by ϕ or { }.

If $A \subseteq B$ then there are two possibilities.

i) A = B

ii) $A \neq B$, $A \subseteq B$

1.5.1 Proper Subset

A set A is called proper subset of B iff

i) $A \subseteq B$ i.e. A is a subset of B.

ii) $\exists$ at least one element in B which is not in A.

i.e. B is not subset of A.

e.g. If A = {1, 2, 3} and B = {1, 2, 3, 4, 5}

then A is a proper subset of B.

It is denoted by $A \subset B$ or $A \subset B$

1.5.2 Improper Subsets

Every set is a subset of itself and null set is a subset of every set.

Subsets A and Q are called improper subsets of A.

1.5.3 Equal sets

SPPU : May-18

Two sets A and B are said to be equal sets if $A \subseteq B$ and $B \subseteq A$. We write A = B

e.g. If A = $\{x \mid x^2 = 1\}$ and B = $\{1, -1\}$ then A = B

1.6 Types of Sets

Depending upon some properties there are mainly following types of sets.

1) Universal Set :

A non empty set of which all the sets under consideration are subsets is called universal set.

It is denoted by U. Universal set is not unique.

2) Singleton Set :

A set having only one element is called a singleton set. e.g. A = {5}, B = {ϕ}

3) Finite Set :

A set is said to be finite if it has finite number of elements.

The number of elements in a set is called the cardinality of that set. It is denoted by $|A|$ cardinality of set may be finite or infinite.

e.g. $\{1, 2, 3, 4, 5\} \Rightarrow |A| = 5$

4) Infinite Set :

A set is said to be infinite if it has infinite number of elements

e.g. N = set of natural numbers

$|N| = \infty$

5) Power Set :

The set of all subsets of a set A is called the power set of A.

The power set of A is denoted by P(A)

Hence $P(A) = \{X \mid X \subseteq A\}$

If A has n elements then P(A) has 2^n elements.

e.g.

i) $A = \{a, b\}$, $P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$.

ii) $P(A) = \phi$ iff $A = \phi$

i.e. empty set has only subset ϕ, $\therefore$ $P(\phi) = \{\phi\}$

iii) $A = \{a, b, c\}$, $P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$.

6) Power set of power set of A :

The power set of the power set of A is denoted by P(P(A)).

e.g. If $A = \{a, b\}$, $p(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

$$\begin{aligned} P(P(A)) = \ & \{\phi, p(A), \{\phi\}, \{a\}, \{b\}, \{a, b\}, \{\{a\}, \{b\}\} \{\phi, \{a\}\} \\ & \{\{a\}, \{a, b\}\}, \{\phi, \{b\}\}, \{\{b\}, \{a, b\}\} \\ & \{\phi \{a, b\}\}, \{\phi, \{a\}, \{b\}\}, \{\phi, \{a\}, \{a, b\}\} \\ & \{\phi, \{b\}, \{a, b\}\}, \{\{a\}, \{b\}, \{a, b\}\}\} \end{aligned}$$

If $|A| = 2$ then $|P(A)| = 2^2$, $|P(P(A)| = 2^4 = 16$

7) A set itself can be an element of some set

Hence one should understand the difference between an element of a set and subset of a set.

e.g. If $A = \{x, y\}$, $P(A) = \{\phi, A, \{x\}, \{y\}\}$

Then

i) $x \in A$ is true. But $x \subset A$ is false as x is an element of A not subset of A

ii) $\{x\} \subset A$ is true as $\{x\}$ is a subset of A

iii) $\{x\} \in A$ is false as $\{x\}$ is not an element of A

iv) $\{x\} \in P(A)$ is true.

v) $\phi \in P(A)$, $\phi \subset P(A)$ are true.

vi) $\{x\} \in P(A)$ is true.

vii) $\{\{x\}\} \in P(A)$ is false but $\{\{x\}\} \subset P(P(A))$

viii) $\phi \subseteq \phi$, $\phi \subseteq \{\phi\}$, $\phi \in \{\phi\}$, $\{\phi\} \subseteq \{\phi\}$ are true

ix) $\phi \in \phi$ is false

x) $\{a, b\} \in \{a, b, c, \{a, b, c\}\}$ is false

xi) $\{a, b, c\} \in \{a, b, c, \{a, b, c\}\}$ is true.

xii) $\{a, b\} \subset \{a, b, c, \{a, b, c\}\}$ is true

xiii) $\{a, \phi\} \in \{\phi, \{a, \phi\}\}$

1.7 Venn Diagrams

We often use pictures in mathematics. The relation between sets can be conveniently illustrated by certain diagrams called Venn diagrams.

This representation first time used by the British Logician John Venn.

In Venn diagram

i) Universal set (U) is represented by a large rectangle.

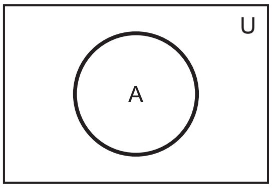
ii) Subsets of U are represented by circles or some closed curve

iii) If $A \subseteq B$ then the circle representing A lies inside of circle representing B.

iv) If A and B are disjoint sets then circles of A and B do not have common area.

v) If A and B are not disjoint sets then circles of A and B have some common area.

1) $A \subset U$



2) $A \subset B$

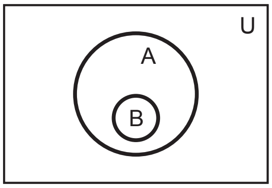


Fig. 1.7.1

1.8 Operations on Sets

To define new sets by combining the given sets, we require set operations. These operations are analogous to the algebraic operations +, –, × of numbers.

1.8.1 Union of Two Sets

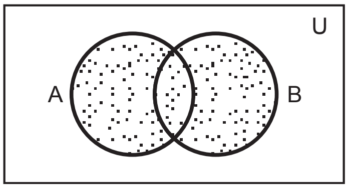
SPPU : May-18

Let A and B be two sets. The union of two sets A and B is the set of all those elements which are either in A or in B or in both sets.

If is denoted by $A \cup B$

$$\therefore \quad A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

By Venn diagram $A \cup B$ is represented as



$A \cup B \equiv$

Fig. 1.8.1

Examples

1) $A = \{1, 2, 3, 4\}$, $B = \{x, y, z\}$

$A \cup B = \{1, 2, 3, 4, x, y, z\}$

2) $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $B = \{2, 3, 5, 7, 11, 13\}$

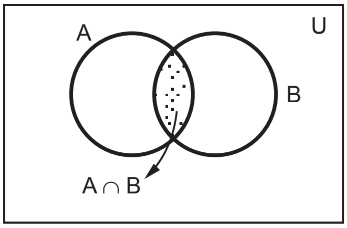
$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13\}$

1.8.2 Intersection of Two Sets

The intersection of two sets A and B is the set of all elements which are in A and also in B. It is denoted by $A \cap B$ -

$\therefore \quad A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

By Venn diagram $A \cap B$ is represented as



$A \cap B$:

Fig. 1.8.2

Examples :

1) $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 4\}$

$\therefore \quad A \cap B = \{1, 4\} = B$

2) $A \cap \phi = \phi$

3) $A = \{a, b, c\}$, $B = \{x, y\}$

$A \cap B = \phi$

1.8.3 Disjoint Sets

I) Two sets A and B are said to be disjoint sets if

$$A \cap B = \phi$$

e.g. $A = \{1, 2\}$, $B = \{x, y\}$ are disjoint sets.

1.8.4 Complement of a Set

Let A be a given set. The complement of A is denoted by $\overline{A}$ or A' and defined as

$$\overline{A} = \{x \mid x \notin A\}.$$

By Venn diagram A' is represented as

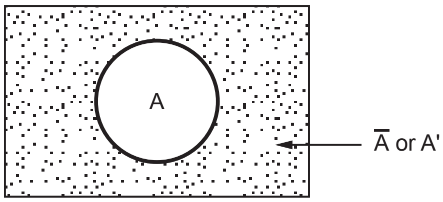


Fig. 1.8.3

Examples :

1) $A = \{\text{Set of even numbers}\}$ and $\cup = \text{IR}$

then $\overline{A}$ or A' = set of odd numbers

2) $U = \{1, 2, 3, 4, 5, ... 15\}$, $A = \{1, 2, 3, 4, 5, 6, 7\}$

$\overline{A} = \{8, 9, 10, 11, 12, 13, 14, 15\}$

Properties of complement of sets

1) $\overline{U} = \phi$ and $\phi = U$

2) $\overline{(\overline{A})} = A$

3) $A \cup \overline{A} = U$, $A \cap \overline{A} = \phi$

4) $A \cup U = U$, $A \cap U = A$

1.8.5 Difference of Sets

Let A and B be two sets. The difference A-B is the set defined as

$$A - B = \{x \mid x \in A \text{ and } x \in B\}$$

It is also known as the relative complement of B in A.

Similarly,

$$B - A = \{x \mid x \in B \text{ and } x \notin A\}$$

By Venn diagram it is represented as

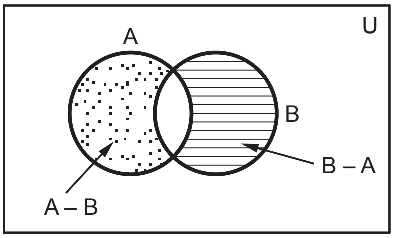


Fig. 1.8.4

Examples :

1) $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6, 7\}$

$\therefore$ $A - B = \{1, 2\}$, $B - A = \{6, 7\}$

2) A = Set of prime numbers

B = Set of odd integers

$A - B = \{2\}$

3) $A = \{a, b, \phi, \{a, c\}\}$

$A - \{a, b\} = \{\{a, c\}, \phi\}$

$\{a, c\} - A = \{c\}$

Properties of difference

Let A and B be two sets. Then

1) $\overline{A} = U - A$

2) $A - A = \phi$, $\phi - \phi = \phi$

3) $A - \overline{A} = A$, $\overline{A} - A = \overline{A}$, $U - \overline{\phi} = \phi$,

4) $A - \phi = A$, $A - \overline{\phi} = U - A$

5) $A - B = A \cap \overline{B}$

6) $A - B = B - A$ iff $A = B$

7) $A - B = A$ iff $A \cap B = \phi$

8) $A - B = \phi$ iff $A \subseteq B$

1.8.6 Symmetric Difference of Sets

The symmetric difference of two sets A and B is denoted by $A \oplus B$ and defined as

$$A \oplus B = \{x \mid x \in A - B \text{ or } x \leftarrow B - A\}$$

i.e. $A \oplus B = (A - B) \cup (B - A)$

If is also denoted by $A \Delta B$.

By Venn diagram it is represented as

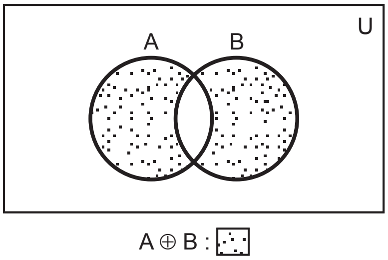


Fig. 1.8.5

Examples :

1) $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{3, 4, 5, 6, 7, 8\}$

$A - B = \{1, 2\}$, $B - A = \{7, 8\}$

$A \oplus B = \{1, 2, 7, 8\}$

2) $A = \phi$, $B = \{x, \phi, \{\phi\}\}$

$A \oplus B = \{x, \{\phi\}\}$

3) $A = \{a\}$, $B = \{a, b\}$

$P(A) \oplus P(B) = \{\{b\}, \{a, b\}\}$

Properties of symmetric difference

1) $A \oplus A = \phi$

2) $A \oplus \phi = A$

3) $A \oplus U = U - A = \overline{A}$

4) $A \oplus \overline{A} = U$

5) $A \oplus B = (A \cup B) - (A \cap B)$

1.9 Algebra of Set Operations

The set operations obey the same rules as those of numbers such as commutativity, associativity and distributivity. In addition there are similar rules to logic such as Idempotent, Absorption De Morgan's Law's and so on.

Theorem : Let A, B, C be any sets then

1) Commutativity with respect to $\cup$ and $\cap$

a) $A \cup B = B \cup A$ b) $A \cap B = B \cap A$

2) Associativity w.r.t. $\cup$ and $\cap$

a) $A \cup (B \cup C) = (A \cup B) \cup C$

b) $A \cap (B \cap C) = (A \cap B) \cap C$

3) Distributivity

a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4) Idempotent Laws

a) $A \cup A = A$ b) $A \cap A = A$

5) Absorption Laws

a) $A \cup (A \cap B) = A$ b) $A \cap (A \cup B) = A$

6) De Morgan's Laws

a) $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ b) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

7) Double Complement : $\overline{(\overline{A})} = A$

Proof : Proofs of 1), 2), 4), 5) are easy, so reader can prove these

3) To Prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Let $x \in A \cup (B \cap C) \Rightarrow x \in A$ or $x \in B \cap C$

$\Rightarrow x \in A$ or $(x \in B$ and $x \in C)$

$\Rightarrow (x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$

$\Rightarrow (x \in A \cup B)$ and $(x \in A \cap C)$

$\Rightarrow x \in (A \cup B) \cap (A \cap C)$

Hence $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Similarly prove that $(A\cup B)\cap(A\cup C)\subseteq A\cup(B\cap C)$

Hence the proof

Similarly reader can prove

$$A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$$

6) Prove that $\overline{A\cup B} = \overline{A}\cap\overline{B}$

Let $x\in\overline{A\cup B}\Rightarrow x\notin A\cup B$

$$\Rightarrow x\notin A \text{ and } x\notin B$$

$$\Rightarrow x\in\overline{A} \text{ and } x\in\overline{B}$$

$$\Rightarrow x\in\overline{A}\cap\overline{B}$$

$$\Rightarrow \overline{A\cup B}\subseteq\overline{A}\cap\overline{B}$$

Similarly $\overline{A}\cap\overline{B} = \overline{A\cup B}$

Hence $\overline{A\cup B} = \overline{A}\cap\overline{B}$

Similarly the reader can prove

$$(\overline{A\cap B}) = \overline{A}\cup\overline{B}$$

7) $\overline{\overline{A}} = \{x\,|\,x\notin\overline{A}\}$

$$= \{x\,|\,x\in A\} = A$$

1.10 Principle of Duality

SPPU : Dec.-06, 08, 11, 12, 13, 14, 15, 17, 19, May-05, 06, 08, 14

The principle of duality states that any proved result involving sets and complements and operations of union and intersection gives a corresponding dual result by replacing $\cup$ by ϕ and $\cup$ by $\cap$ and vice versa.

e.g. The dual of $A\cup\overline{A} = U$ is $A\cap\overline{A} = \phi$

Examples

Example 1.10.1 *If $U = \{x\in Z'|-5<x<5\}$ and $A = \{x\in Z'|-2<x<3\}$ Where Z' = Set of integers. State the elements of the sets.*

$\overline{A}$, $\overline{A}\cap\overline{A}$, $A\cap U$, $A\cup U$, $\overline{A}\cap U$, $A\cap\overline{A}$

Solution :

$$\overline{A} = \{x\in Z'/(-5<x<-2)\cup(3<x<5)\}$$

$$\overline{A} \cap \overline{A} = \overline{A}$$

$$A \cap U = A \quad , \quad A \cup U = U$$

$$\overline{A} \cap U = \overline{A}$$

$$\overline{A} \cap \overline{\overline{A}} = \phi$$

Example 1.10.2 *If $A = \{x, y, \{x, z\} \phi\}$. Determine the following sets*

i) $A - \{x, z\}$ ii) $\{\{x, z\} - A$

iii) $A - \{\{x, y\}\}$ iv) $\{x, z\} - A$

v) $A - P(A)$ vi) $\{x\} - A$

vii) $A - \{x\}$ viii) $A - \phi$

ix) $\phi - A$ x) $\{x, y, \phi\} - A$ **SPPU : Dec.-08, Marks 4**

Solution :

i) $A - \{x, z\} = \{x, y, \phi\}$

ii) $\{\{x, z\}\} - A = \phi$

iii) $A - \{\{x, y\}\} = A$

iv) $\{x, z\} - A = \{z\}$

v) $A - p(A) = \{x, y, \{x, z\}\}$

vi) $\{x\} - A = \phi$

vii) $A - \{x\} = \{y, \{x, z\}, \phi\}$

viii) $A - \phi = \{x, y, \{x, z\}\}$

ix) $\phi - A = \phi$

x) $\{x, y, \phi\} - A = \{\{x, z\}\}$

Example 1.10.3 *If $A = \{\phi, b\}$, construct the following sets $A - \phi$, $\{\phi\} - A$, $A \cup P(A)$*

SPPU : Dec.-19, Marks 03

Solution : We have $A = \{\phi, b\}$

$$\therefore \quad A - \phi = \{b\}$$

$$\{\phi\} - A = \phi \text{ and } P(A) = \{\phi, \{\phi\}, \{b\}, A\}$$

$$A \cup P(A) = \{\phi, \{\phi\}, \{b\}, A\}$$

Example 1.10.4 *Let $U = Z'$ = Set of all integers*

A = Set of even integers

B = Set of odd integers

C = Set of prime numbers

Find $\overline{A}-B$, $\overline{B}-A$, $\overline{C}-A$, $\overline{A}-C$, $\overline{B}-C$

Solution :

$$\overline{A}-B = \overline{B}-B = \phi$$

$$\overline{B}-A = A-A = \phi$$

$\overline{C}-A$ = Set of non prime integers which are not even

$\overline{A}-C$ = $B - C$ = Set of odd integers which are not prime = $\{\pm 1, \pm 9, \pm 15, \ldots\}$

$\overline{B}-C$ = $A - C$ = Set of even integers which are not prime = $\{\pm 4, \pm 6, \pm 8, \ldots\}$

Example 1.10.5 *Let A, B, C be three sets and $A \cap B = A \cap C$*

$\overline{A} \cap B = \overline{A} \cap C$ is it necessary that $B = C$? Justify.

Solution :

Yes, $B = C$.

Consider
$$\begin{aligned} B &= B \cap U = B \cap (A \cup \overline{A}) \\ &= (B \cap A) \cup (B \cap \overline{A}) \\ &= (A \cap B) \cup (\overline{A} \cap B) \\ &= (A \cap C) \cup (\overline{A} \cap C) \\ &= (A \cup \overline{A}) \cap C \\ &= U \cap C \end{aligned}$$

$\therefore$ $\boxed{B = C}$

Example 1.10.6 *Let A, B, C be three sets*

i) Given that $A \cup B = A \cup C$, is it necessary that $B = C$?

ii) Given that $A \cap B = A \cap C$, is it necessary that $B = C$?

Solution :

i) No,

Let $A = \{x, y, z\}$, $B = \{x\}$, $C = \{z\}$

$A \cup B = A$ and $A \cup C = A$

$\therefore$ $A \cup B = A \cup C = A$ But $B \neq C$

ii) No,

Let $A = \{x, y\}$, $B = \{y, z, w\}$, $C = \{y, p, q\}$

$A \cap B = \{y\} = B \cap C$ But $B \neq C$

Example 1.10.7 *If $A \oplus B = A \oplus C$, is $B = C$?*

Solution :

Yes, Let $x \in B \Rightarrow x \in A$ or $x \notin A$

Suppose $x \in A$ then $x \in A \cap B \Rightarrow x \notin A \oplus B$

Hence $x \notin A \oplus C \Rightarrow x \in A \cap C \Rightarrow x \in C$

i.e. if $x \in A$ then $B \subseteq C$

Suppose $x \notin A$ then $x \notin A \cap B$ so that $x \in A \oplus B$

$\Rightarrow$ $x \notin A \oplus C \Rightarrow x \notin A \cup B \Rightarrow x \in C$

Hence $B \subseteq C$

Similarly $C \subseteq B$

Hence $B = C$

Example 1.10.8 *Let A and B are two sets. If $A \subseteq B$, then prove that $P(A) \subseteq P(B)$. where $P(A)$ and $P(B)$ are power sets of A and B sets.* **SPPU : Dec.-17, Marks 6**

Solution : Let A and B be two sets.

The powr set of A is the set of all subsets of a set A.

Let P(A) be the power set of A and P(B) be the power set of B.

If A is a subset of B i.e. $A \subseteq B$ then all elements of A are in B.

If $X \subseteq P(A)$ but $A \subseteq B \Rightarrow X \in P(B)$

Thence $P(A) \subseteq P(B)$ $\forall \times \in P(A)$

Example 1.10.9 *If ϕ is an empty set then find $p(\phi)$, $p(p(\phi))$ $p(p(p(\phi)))$*

Solution :

$$p(\phi) = \{\phi\}$$

$$p(p(\phi)) = \{\phi, \{\phi\}\}$$

$$p(p(p(\phi))) = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$$

Example 1.10.10 *Show that* $[A\cap(B\cup\overline{A})]\cup B = B$

Solution :

$$\begin{aligned}[A\cap(B\cup\overline{A})]\cup B &= [(A\cap B)\cup(A\cap\overline{A})]\cup B\\ &= [(A\cap B)\cup\phi]\cup B\\ &= (A\cap B)\cup B = B\end{aligned}$$

Example 1.10.11 *Prove that* $A\cap(B-C)]\subset A-(B\cap C)$

Solution : Let $x\in A\cap(B-C)$ then

$$\Rightarrow x\in A \text{ and } x\in B-C$$

$$\Rightarrow x\in A \text{ and } (x\in B \text{ and } x\notin C)$$

$$\Rightarrow x\in A \text{ and } x\notin (B\cap C)$$

$$\Rightarrow x\in A-(B\cap C)$$

$\therefore \quad A\cap(B-C) \subset A-(B\cap C)$

Example 1.10.12 *Salad is made with combination of one or more eatables, how many different salads can be prepared from onion, carrot, cabbage and cucumber?*

SPPU : Dec.-13, Marks 4

Solution : The number of different salads can be prepared from onion, carrot, cabbage and cucumber with combination of one or more eatables is $2^4-1 = 16-1 = 15$

Example 1.10.13 *Explain the concepts of countably infinite set with example.*

SPPU : Dec.-14, Marks 4

Solution : A set is said to be countable if its all elements can be labelled as 1, 2, 3, 4, ... A set is said to be countably infinite

if, i) It is countable

ii) It has infinitely many elements i.e. It's cardinality is ∞.

For example

1) The set of natural numbers {1, 2, 3, ...} is countably infinite.
2) The set of integers is countably infinite.
3) The set of real numbers is infinite but not countable.

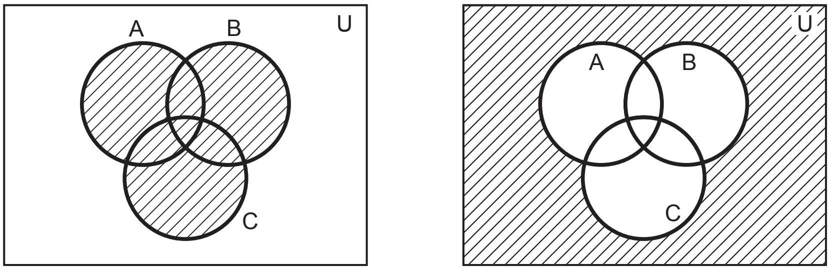
Example 1.10.14 *Draw Venn diagram and prove the expression. Also write the dual of each of the given statements.*

i) $(A \cup B \cup C)^C = (A \cup C)^C \cap (A \cup B)^C$

ii) $(U \cap A) \cup (B \cap A) = A$ **SPPU : Dec.-11, Marks 6**

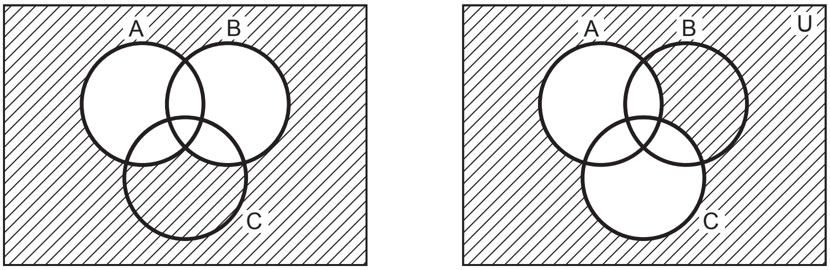
Solution : Consider the following Venn diagrams.

ii) Consider the following Venn diagrams.



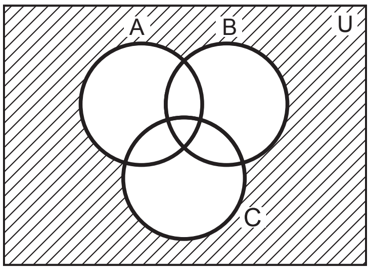
① $A \cup B \cup C =$ ▨ ② $(A \cup B \cup C)^c =$ ▨

Fig. 1.10.1



③ $(A \cup B)^c =$ ▨ ④ $(A \cup C)^c =$ ▨

Fig. 1.10.2



From (2) and (5)
$(A \cup B \cup C)^c = (A \cup C)^c \cap (A \cup B)^c$

⑤ $(A \cup B)^c \cap (A \cup C)^c$

Fig. 1.10.3

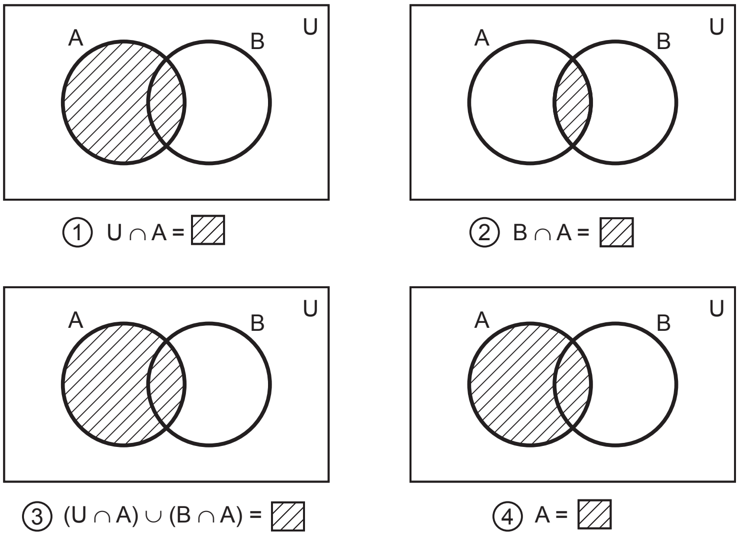


Fig. 1.10.4

From Venn diagrams (3) and (4)

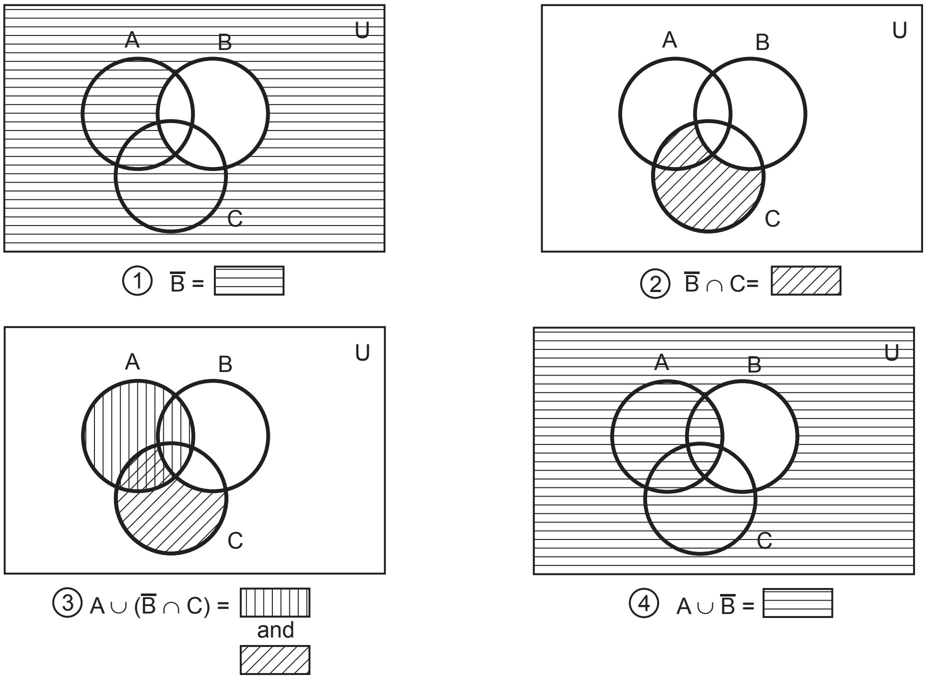
$$(U\cap A)\cup(B\cap A) = A$$

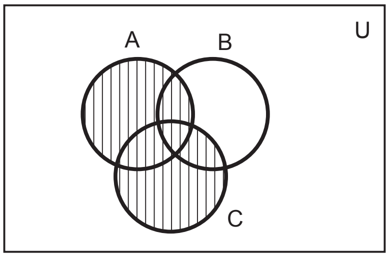
Example 1.10.15 *Using Venn diagram show that :*

$A\cup(\overline{B}\cap C) = (A\cup\overline{B})\cap(A\cup C)$ **SPPU : May-05, Marks 4**

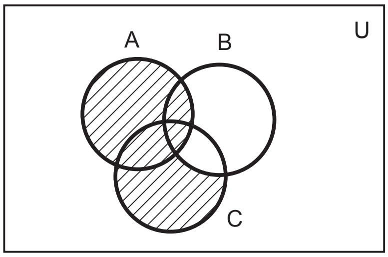
Solution : Consider the following venn diagrams

$$\overline{B} = \{x \,/\, x \notin B\}$$





5 A C =



6 (A $\overline{B}$) (A C) =

From 3 and 6,

$$A \cup (\overline{B} \cap C) = (A \cup \overline{B}) \cap (A \cup C)$$

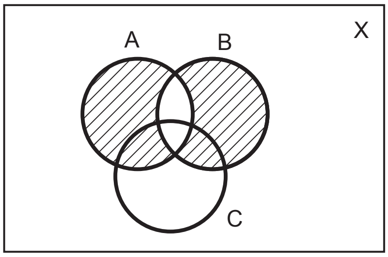
Example 1.10.16 *Using Venn diagram, prove or disprove.*

i) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$

ii) $A \cap B \cap C = A - [(A - B) \cup (A - C)]$ **SPPU : May-06, Marks 4**

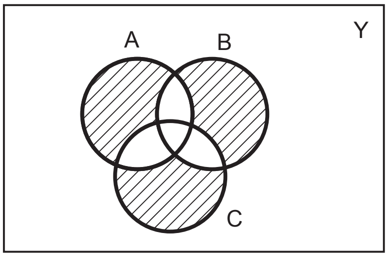
Solution : i) $A \oplus B = (A \cup B) - (A \cap B)$

$A \oplus B$: elements which are either in A or in B but not in both A and B.



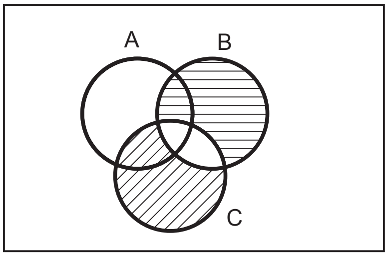
$A \oplus B =$

1



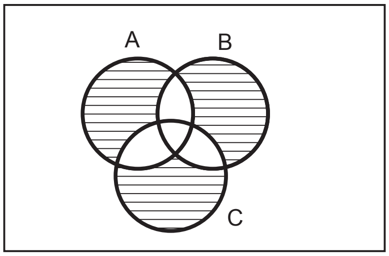
$(A \oplus B) \oplus C$

2



$B \oplus C =$

3 and



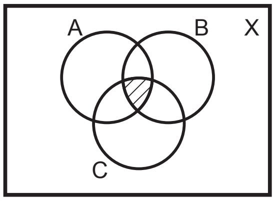
$A \oplus (B \oplus C)$

4

From 2 and 4,

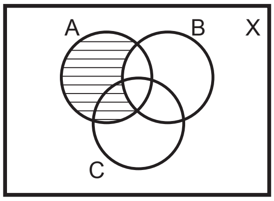
$$(A \oplus B) \oplus C = A \oplus (B \oplus C)$$

ii) $A \cap B \cap C = A - [(A - B) \cup (A - C)]$



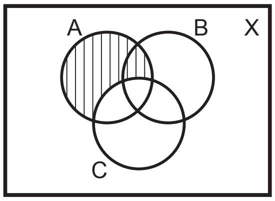
$A \cap B \cap C =$

(1)



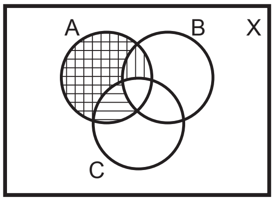
$A - B =$

(2)



$A - C =$

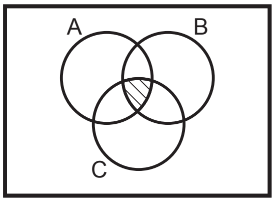
(3)



$(A - B) \cup (A - C) =$

and

(4)



$A - [(A-B) \cup (A-C)] =$

(5)

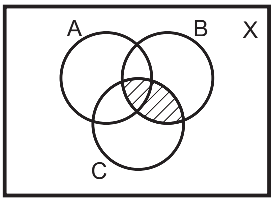
From (1) and (5)

$$A \cap B \cap C = A - [(A - B) \cup (A - C)]$$

Example 1.10.17 *Using Venn diagrams show that*

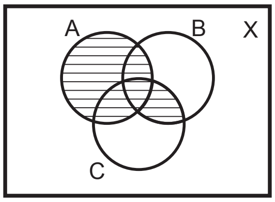
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ **SPPU : May-08, Dec.-12, Marks 3**

Solution :



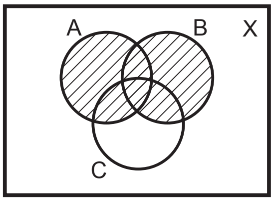
$B \cap C =$

1



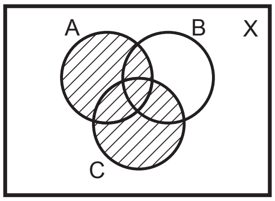
$A \cup (B \cap C) =$

2



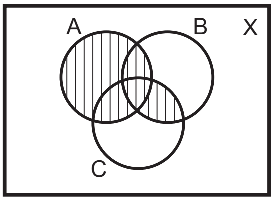
$A \cup B =$

3



$A \cup C =$

4



$(A \cup B) \cap (A \cup C) =$

5

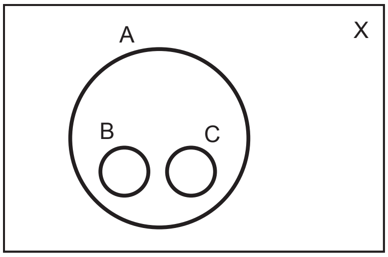
From 2 and 5,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Example 1.10.18 *Let A, B, C be sets. Under what conditions the following statements are true ?*

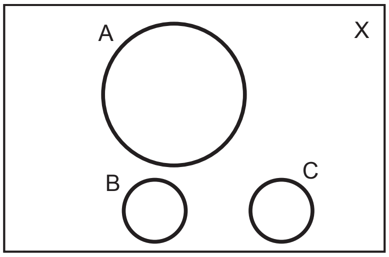
i) $(A - B) \cup (A - C) = A$ *ii)* $(A - B) \cup (A - C) = \varnothing$ **SPPU : Dec.-06, 15, Marks 3**

Solution : i) $(A - B) \cup (A - C) = A$



1

OR



2

If $B \subset A$ and $C \subset A$

then $(A - B) \cup (A - C) = A$

Or A, B, C are disjoint sets.

$$\text{i.e. } A \cap B \cap C = \varnothing$$

$$\text{Then } (A - B) \cup (A - C) = A$$

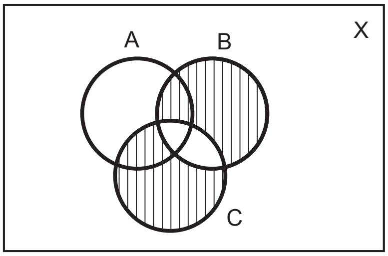
ii) $(A - B) \cup (A - C) = \varnothing$ is true.

If A is empty set i.e. $A = \varnothing$

Example 1.10.19 *Prove the following using Venn diagram.*

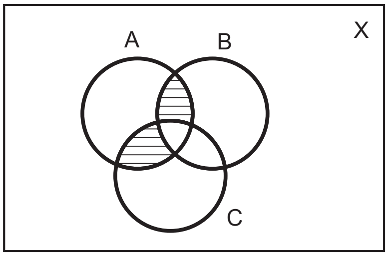
$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$ **SPPU : May-08, 14, Dec.-12, Marks 3**

Solution :



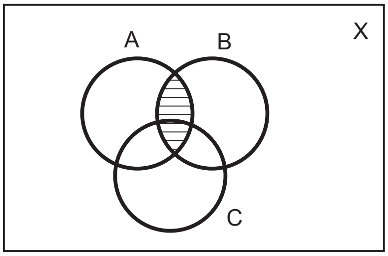
$B \oplus C =$

1



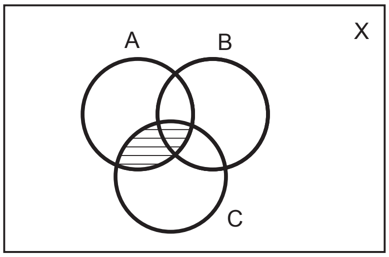
$A \cap (B \oplus C) =$

2



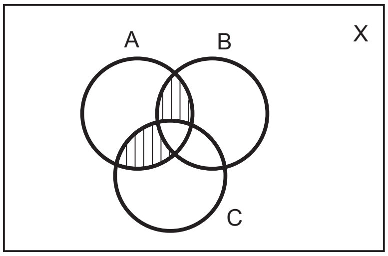
$A \cap B =$

3



$A \cap C =$

4



$(A \cap B) \oplus (A \cap C) =$

5

From (2) and (5)

$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$

1.11 Cardinality of Sets

In the analysis of computer algorithms, we required to count the number of operations executed by various algorithms. This is necessary to estimate the cost effectiveness of a particular algorithm. Hence we require to study the cardinality of finite sets and understand related properties.

Definition :

Let A be any finite set. The number of elements in the set A, is called the cardinality of the set A. It is denoted by $|A|$ or $n(A)$.

e.g. If $A = \{x, y, z, p\}$ then $|A| = 4$

Theorem 1 : If $A = \phi$ then $|A| = 0$

Theorem 2 : If $A \subseteq B$ then $|A| \leq |B|$

Theorem 3 : (Addition Principle)

Let A and B be two finite sets which are disjoint then $|A \cup B| = |A| + |B|$

Proof : If $A = \phi$, $B = \phi$ then proof is obvious

Let us assume that $A \neq \phi$, $B \neq \phi$

Suppose $A = \{a_1, a_2, ... a_n\}$, $B = \{b_1, b_2, ... b_m\}$

$\therefore \quad A \cap B = \phi, \quad |A| = n, \quad |B| = m$

$\therefore \quad A \cup B = \{a_1, a_2, ... a_n, b_1, b_2, ... b_m\}$

$\therefore \quad |A \cup B| = m + n = |A| + |B|$. Hence the proof

Theorem 4 : Let $A_1, A_2, A_3, ... A_n$ be mutually disjoint finite sets then

$$|A_1 \cup A_2 \cup A_3 \cup ... \cup A_n| = |A_1| + |A_2| + |A_3| + ... + |A_n|$$

Theorem 5 : If A is finite set and B is any set then

$$|A - B| = |A| - |A \cap B|$$

Proof : By Venn diagram

$$A = (A - B) \cup (A \cap B)$$

$$(A - B) \cap (A \cap B) = \phi$$

$\therefore$ By addition principle $|A| = |A - B| + |A \cap B|$

$\Rightarrow |A - B| = |A| - |A \cap B|$

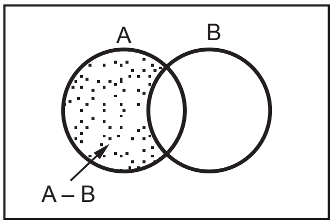


Fig. 1.11.1

1.12 The Principle of Inclusion and Exclusion for Sets

SPPU : Dec.-04, 05, 07, 08, 10, 13, 14, 15, 19
May-05, 06, 07, 08, 14, 15, 17, 19

Theorem 6 : (Principle of Inclusion - Exclusion for 2 sets)

Let A and B be finite sets then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof : By venn diagram

$$A \cup B = (A - B) \cup B$$

As A–B and B are disjoint sets.

$$\begin{aligned} |A \cup B| &= |A - B| + |B| \\ &= |A| - |A \cap B| + |B| \\ |A \cup B| &= |A| + |B| - |A \cap B| \end{aligned}$$

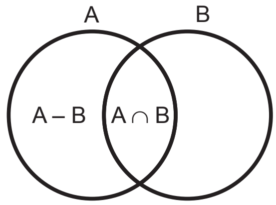


Fig. 1.12.1

Principle of Inclusion-Exclusion for three sets.

Theorem 7 : Let A, B, C be finite sets. Then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$$

Proof Let $D = B \cup C$

$$\begin{aligned} \therefore \quad |A \cup D| &= |A| + |D| = |A \cap D| \\ \text{and} \quad |D| &= |B \cup C| = |B| + |C| - |B \cap C| \\ \therefore \quad |A \cup B \cup C| &= |A \cup D| = |A| + |B| + |C| - |B \cap C| - |A \cap D| \\ &= |A| + |B| + |C| - |B \cap C| - |A \cap (B \cup C)| \\ &= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)| \\ &= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C| \\ &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \end{aligned}$$

Theorem 8 : Let A, B, C, D be finite sets then

$$\begin{aligned} |A \cup B \cup C \cup D| &= |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| \\ &\quad - |B \cap C| - |B \cap D| - |C \cap D| \\ &= |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\ &\quad - |A \cap B \cap C \cap D| \end{aligned}$$

Examples :

Example 1.12.1 *Among the integers 1 to 300 find how many are not divisible by 3, not by 5. Find also how many are divisible by 3 but not by 7.* **SPPU : Dec.-08, Marks 6**

Solution : Let A denotes the set of integers 1 to 300 divisible

by 3, B denotes the set of integers 1 to 300 divisible

by 5, C denotes the set of integers 1 to 300 divisible.

by 7 $|A| = \left[\dfrac{300}{3}\right] = 100$, $|B| = \left[\dfrac{300}{5}\right] = 60$, $|C| = \left[\dfrac{300}{7}\right] = 42$, $|A \cap B| = \left[\dfrac{300}{3 \times 5}\right] = 20$

Find $|\overline{A} \cap B|$ and $|A - C|$

We have $\overline{A} \cap \overline{B} = \overline{A \cup B} = U - (A \cup B)$

$$\therefore \quad |A \cup B| = |A| + |B| - |A \cap B|$$

$$= 100 + 60 - 20 = 140$$

$$\therefore \quad |\overline{A} \cap \overline{B}| = |U| - |A \cup B| = 300 - 140 = \mathbf{160}$$

Hence 160 integers between 1 – 300 are not divisible by 3, not by 5.

$$|A - C| = |A| - |A \cap C|, \qquad |A \cap C| = \left[\frac{300}{3 \times 7}\right] = 14$$

$$|A - C| = 100 - 14 = 86$$

Hence, 86 integers between 1 – 300 are not divisible by 3 but not by 7.

Example 1.12.2 *Out of 30 students in a college, 15 take an art course, 8 take maths course, 6 take physics course. It is know that 3 students take all courses. Show that 7 or more students take none of the courses?*

Solution : Let A be the set of students taking an art course

B be the set of students taking a maths course

C be the set of students taking a physics course

We have $|A| = 15, \quad |B| = 8, \quad |C| = 6,$

$$|A \cap B \cap C| = 3$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 15 + 8 + 6 - |A \cap B| - |A \cap C| - |B \cap C| + 3$$

$$= 32 - |A \cap B| - |A \cap C| - |B \cap C|$$

But $|A \cap B| \geq |A \cap B \cap C|, \qquad |B \cap C| \geq |A \cap B \cap C|$

and $|A \cap C| \geq |A \cap B \cap C|$

$\therefore \quad |A \cap B| \geq 3, \qquad |B \cap C| \geq 3, \qquad |A \cap C| \geq 3$

$\therefore \quad |A \cap B| + |A \cap C| + |B \cap C| \geq 3|A \cap B \cap C|$

$\therefore \quad |A \cup B \cup C| \geq 32 - 3|A \cap B \cap C| = 32 - 3 \times 3 = 23$

Hence $|A \cup B \cup C| \geq 23$

Hence the number of students taking at least one course is ≥ 23. The students taking none of the course is $\leq 30 - 23 = 7$

Thus 7 or less students take none of the courses.

Example 1.12.3 *How many integers between 1 to 2000 are divisible by 2 or 3 or 5 or 7.*

Solution : Suppose set A denotes the number of integers between 1 to 2000 divisible by 2.

Set B is the number of integers between 1 and 2000 divisible by 3.

Set C is the number of integers between 1 and 2000 divisible by 5.

Set D is the number of integers between 1 and 2000 divisible by 7.

$$|A| = \left[\frac{2000}{2}\right] = 1000$$

$$|B| = \left[\frac{2000}{3}\right] = 666$$

$$|C| = \left[\frac{2000}{5}\right] = 400$$

$$|D| = \left[\frac{2000}{7}\right] = 285$$

$$|A \cap B| = \left[\frac{2000}{2 \times 3}\right] = 333$$

$$|A \cap C| = \left[\frac{2000}{2 \times 5}\right] = 200$$

$$|A \cap D| = \left[\frac{2000}{2 \times 7}\right] = 142$$

$$|B \cap C| = \left[\frac{2000}{3 \times 5}\right] = 133$$

$$|B \cap D| = \left[\frac{2000}{3 \times 7}\right] = 95$$

$$|C \cap D| = \left[\frac{2000}{5 \times 7}\right] = 57$$

$$|A \cap B \cap C| = \left[\frac{2000}{2 \times 3 \times 5}\right] = 66$$

$$|A \cap B \cap D| = \left[\frac{2000}{2 \times 3 \times 7}\right] = 47$$

$$|A \cap C \cap D| = \left[\frac{2000}{2 \times 5 \times 7}\right] = 28$$

$$|B \cap C \cap D| = \left[\frac{2000}{3 \times 5 \times 7}\right] = 19$$

$$|A \cap B \cap C \cap D| = \left[\frac{2000}{2 \times 3 \times 5 \times 7}\right] = 9$$

Number of elements divisible by 2 or 3 or 5 or 7 are $|A \cup B \cup C \cup D|$.

From inclusion exclusion principle

$$\begin{aligned} |A \cup B \cup C \cup D| &= |A|+|B|+|C|+|D| \\ &\quad - [|A \cap B|+|B \cap C|+|A \cap C|+|A \cap D|+|B \cap D|+|C \cap D|] \\ &\quad + [|A \cap B \cap C|+|A \cap B \cap D|+|A \cap C \cap D|+|B \cap C \cap D|] \\ &\quad - [|A \cap B \cap C \cap D|] \end{aligned}$$

$$\begin{aligned} \therefore \quad |A \cup B \cup C \cup D| &= 1000 + 666 + 400 + 285 - [333 + 200 + 142 + 133 + 95 + 57] \\ &\quad + [66 + 47 + 28 + 19] - 9 \\ &= 2351 - 960 + 160 - 9 \\ &= \mathbf{1542} \end{aligned}$$

Example 1.12.4 *In the survey of 260 college students, the following data were obtained :*

64 had taken a maths course,

94 had taken a cs course,

58 had taken a business course,

28 had taken both a maths and a business course,

26 had taken both a maths and a cs course,

22 had taken both a cs and a business course,

14 had taken all types of courses.

How many students were - surveyed who had taken none of the three types of courses.

SPPU : May-17, Marks 3

Solution : Let A be the set of students, taken a maths course.

Let B be the set of students, taken a cs course

Let C be the set of students, taken a business course.

$\therefore$ We have U = 260, $|A| = 64$, $|B| = 94$

$|C| = 58$, $A \cap C| = 28$, $|A \cap B| = 26$

$|B \cap C| = 22$, $|A \cap B \cap C| = 14$,

We have,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B|$$

$$- |A \cap C\} - |B \cap C| + |A \cap B \cap C|$$

$$= 64 + 94 + 58 - 28 - 26 - 22 + 14 = 154$$

The total number of students taken none of the three types of courses

$$= |U| - |A \cup B \cup C|$$

$$= 260 - 154 = 106$$

Example 1.12.5 *Among 100 students, 32 study mathematics, 20 study physics , 45 study biology, 15 study mathematics and biology, 7 study mathematics and physics, 10 study physics and biology and 30 do not study any of the three subjects.*

a) Find the number of students studying all three subjects.

b) Find the number of students studying exactly one of the three subjects.

Solution : Let A, B, C denotes the set of students studying mathematics, physics and biology respectively.

And
$$|X| = 100$$
$$|A| = 32$$
$$|B| = 20$$
$$|C| = 45$$
$$|A \cap C| = 15$$
$$|A \cap B| = 7$$
$$|B \cap C| = 10$$
$$|A' \cap B' \cap C'| = 30$$
$$|A' \cap B' \cap C'| = 100 - |A \cup B \cup C|$$

Or
$$|A \cup B \cup C| = 100 - 30$$
$$= 70$$

a)
$$|A \cup B \cup C| = |A| + |B| + |C| - [|A \cap B| + |A \cap C| + |B \cap C|] + |A \cap B \cap C|$$
$$70 = 32 + 20 + 45 - [7 + 15 + 10] + |A \cap B \cap C|$$

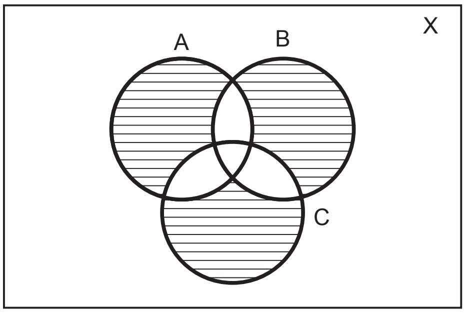
$$70 = 97 - [32] + |A \cap B \cap C|$$

$$70 - 65 = |A \cap B \cap C|$$

$$\Rightarrow |A \cap B \cap C| = 5$$

5 students study all 3 subjects.

b) Number of students studying exactly one subject.



Number of students studying only mathematics is

$$|A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$

$$= 32 - 7 - 15 + 5$$

$$= 15$$

Number of students studying only physics is

$$|B| - |B \cap A| - |B \cap C| + |A \cap B \cap C|$$

$$= 20 - 7 - 10 + 5$$

$$= 8$$

Number of students studying only biology is

$$|C| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 45 - 15 - 10 + 5$$

$$= 25$$

$\therefore$ Number of students studying exactly one subject

$$= 15 + 8 + 25 = 48$$

Example 1.12.6 *A survey was conducted among 1000 people of these 595 are Democrats, 595 wear glasses, and 550 like icecream. 395 of them are Democrats who wear glasses, 350 of them are Democrats who like icecream, and 400 of them wear glasses and like icecream 250 of them are Democrats who wear glasses and like icecream. How many of them are not Democrats do not wear glasses, and do not like icecream ? How many of them are Democrats who do not wear glasses and do not like icecream ?*

Solution : $|X| = 1000$

Let $|A|$ be the number of Democrats.

$$|A| = 595$$

$|B|$ be the number of people who wear glasses.

$$|B| = 595$$

$|C|$ be the number of people who like icecream.

$$|C| = 550$$

$$|A \cap B| = 395$$

$$|A \cap C| = 350$$

$$|B \cap C| = 400$$

$$|A \cap B \cap C| = 250$$

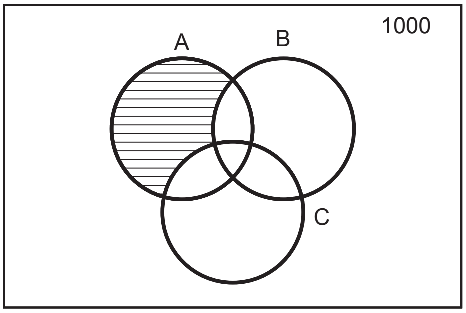
$$|A \cup B \cup C| = |A|+|B|+|C| - [|A \cap B|+|B \cap C|+|A \cap C|]+|A \cap B \cap C|$$

$$= 595 + 595 + 550 - [395 + 350 + 400] + 250$$

$$|A \cup B \cup C| = 845$$

845 people are either Democrats or wear glasses or icecream.

$\Rightarrow$ 1000 – 845 = 155 people are neither Democrats, nor wear glasses nor like icecream.

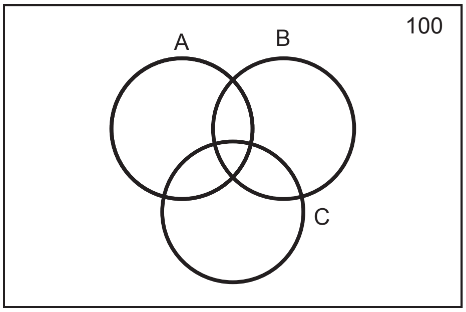


Number of people who are Democrats who do not like icecream and do not wear glasses.

$$= |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$
$$= 595 - [395 + 300] + 250$$
$$= 845 - 695$$
$$= 150$$

Example 1.12.7 *It is known that at the university 60 percent of the professors play tennis, 50 percent of them play bridge. 70 percent jog, 20 percent play tennis and bridge, 30 percent play tennis and jog, and 40 percent play bridge and jog. If some one claimed that 20 percent of the professors jog and play bridge and tennis, would you believe this claim ? Why ?* **SPPU : Dec.-13, Marks 6**

Solution :



Let A, B, C, denotes the number of professors play tennis, bridge and jog respectively.

$$|A| = 60$$
$$|B| = 50$$
$$|C| = 70$$
$$|A \cap B| = 20$$
$$|A \cap C| = 30$$
$$|B \cap C| = 40$$
$$|A \cap B \cap C| = 20$$
$$|A \cup B \cup C| = |A| + |B| + |C| - [|A \cap B| + |A \cap C| + |B \cap C|] + |A \cap B \cap C|$$
$$= 60 + 50 + 70 - [20 + 30 + 40] + 20$$
$$= \mathbf{110}$$

which is not possible as $|A \cup B \cup C| \subset X$ and the number of elements in $|A \cup B \cup C|$ cannot exceed number of elements in the universal set X.

Example 1.12.8 *Among 200 students in a class, 104 students got an 'A', in first examination and 84 students got 'A' in second examination. If 68 students did not get an 'A' in either of the examination.*

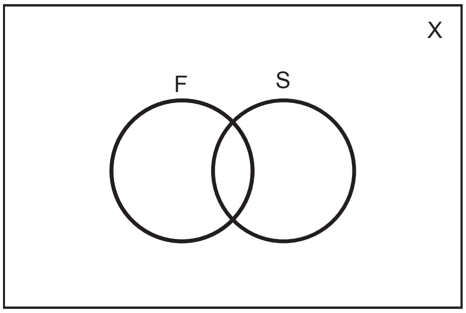
i) How many students got 'A' in both the examination.

ii) If number of students who got an 'A' in the first examination is equal to that who got an 'A' in second examination. If the total number of students who got 'A' in exactly one examination is 160 and if 16 students did not get 'A' in either examination. Determine the number of students who got 'A' in first examination, those who got 'A' in second examination and number of students who got 'A' in both examinations.

SPPU : Dec.-04, Marks 6

Solution : $X \rightarrow$ Universal set

$$|X| = 200$$



Let F denote the set of students who got an 'A' in first examination and S denote the set of students who got an 'A' in second examination.

68 students did not get A in either examination i.e.

$$|(F \cup S)'| = 68$$

$$\Rightarrow \quad |F \cup S| = |X| - 1|(F \cup S)'|$$

$$= 200 - 68$$

$$\Rightarrow \quad |F \cup S| = 132$$

i) Number of students who got A in both the examination will be $|F \cap S|$.

Now $$|F \cup S| = |F| + |S| - |F \cap S|$$

$$132 = 104 + 84 - |F \cap S|$$

$$\therefore \quad |F \cap S| = 56$$

ii) $|F| = |S|$

The total number of students who got A in exactly one examination is $|F-S| \cup |S-F|$ i.e. $F \oplus S$

and $|F \oplus S| = 160$

16 students did not get A in either examination

$\therefore \quad |(F \cup S)'| = 16$

$\Rightarrow \quad |F \cup S| = |X| - |(F \cup S)'|$

$\Rightarrow \quad |F \cup S| = 200 - 16 = 184$

Now $\quad |F \cup S| = |F \oplus S| + |F \cap S|$

As $F \oplus S$ and $F \cap S$ are pairwise disjoint sets.

$\Rightarrow \quad 184 = 160 + |F \cap S|$

$\Rightarrow \quad |F \cap S| = 184 - 160 = 24$

$\therefore \quad |F \cup S| = |F| + |S| - |F \cap S|$

$184 = 2|F| - 24$

$2|F| = 208$

$\Rightarrow \quad |F| = 104$ and $|F| = |S|$

$\Rightarrow \quad |S| = 104$

$\therefore$ Number of students who got A in first examination

$= |F| - |F \cap S|$

$= 104 - 24 = \mathbf{80}$

Similarly number of students who got A in second examination

$= |S| - |F \cap S| = 104 - 24 = \mathbf{80}$

Example 1.12.9 *Consider a set of integers 1 to 500. Find*

i) How many of these numbers are divisible by 3 or 5 or by 11 ? **Dec.-14**

ii) Also indicate how many are divisible by 3 or by 11 but not by all 3, 5 and 11.

iii) How many are divisible by 3 or 11 but not by 5 ? **SPPU : May-05, Marks 6**

Solution : Let A denote numbers divisible by 3.

B denote numbers divisible by 5.

C denote numbers divisible by 11.

$|A|$ denotes cardinality of A similarly $|B|$ and $|C|$ denotes cardinality of B and C.

$$|A| = \left[\frac{500}{3}\right] = 166$$

$$|B| = \left[\frac{500}{5}\right] = 100$$

$$|C| = \left[\frac{500}{11}\right] = 45$$

$$|A \cap B| = \left[\frac{500}{3 \times 5}\right] = 33$$

$$|A \cap C| = \left[\frac{500}{3 \times 11}\right] = 15$$

$$|B \cap C| = \left[\frac{500}{5 \times 11}\right] = 9$$

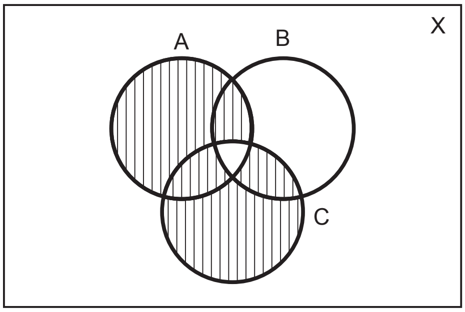
$$|A \cap B \cap C| = \left[\frac{500}{3 \times 5 \times 11}\right] = 3$$

i) $$|A \cup B \cup C| = |A| + |B| + |C| - [|A \cap B| + |A \cap C| + |B \cap C|] + |A \cap B \cap C|$$

$$= 166 + 100 + 45 - [13 + 15 + 9] + 3$$

$$= \mathbf{257}$$

ii)



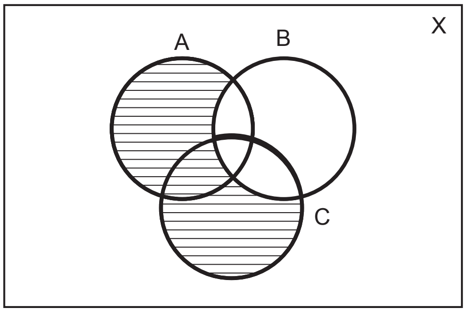
Number of integers divisible by 3 or by 11 but not by all 3, 5 and 11.

$$= |A \cup C| - |A \cap B \cap C|$$

$$= [|A| + |C| - |A \cap C|] - |A \cap B \cap C|$$

$$= 166 + 45 - 15 - 3 = \mathbf{193}$$

iii)



Number of integers divisible by 3 or 11 but not by 5.

$$= |A \cup B \cup C| - |B|$$

$$= 257 - 100 = \mathbf{157}$$

Example 1.12.10 *In the survey of 60 people, it was found that 25 read newsweek magazine, 26 read time, 26 read fortune. Also 9 read both newsweek and fortune, 11 read both newsweek and time, 8 read both time and fortune and 8 read no magazine at all.*

i) Find out the number of people who read all the three magazines.

ii) Fill in the correct numbers in all the regions of the Venn diagram.

iii) Determine number of people who reads exactly one magazine.

SPPU : Dec.-05, 10, 19, Marks 6

Solution : X $\rightarrow$ Universal set

$$|X| = 60$$

Let N denote the number of people who read newsweek magazine.

$$\therefore \quad |N| = 25$$

T denote the number of people who read time magazine.

$$\therefore \quad |T| = 26$$

F denote the number of people who read fortune magazine.

$$\therefore \quad |F| = 26$$

$$|N \cap T| = 11$$

$$|N \cap F| = 9$$

$$|T \cap F| = 8$$

$$|N' \cap T' \cap F'| = 8$$

Using DeMorgan's law

$$|N' \cap T' \cap F'| = |X| - |N \cup T \cup F|$$

$$8 = 60 - |N \cup T \cup F|$$

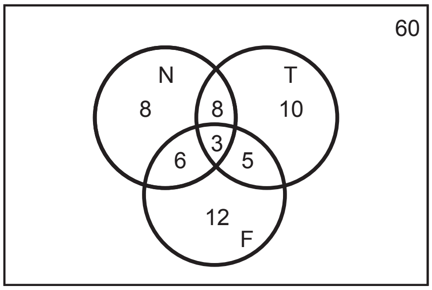
$$\Rightarrow \quad |N \cup T \cup F| = 52$$

i) $|N \cup T \cup F| = |N| + |T| + |F| - [|N \cap T| + |N \cap F| + |T \cap F|] + |N \cap T \cap F|$

$$\Rightarrow \quad 52 = 25 + 26 + 26 - [11 + 9 + 8] + |N \cap T \cap F|$$

$$\Rightarrow \quad |N \cap T \cap F| = 3$$

ii)



Explanation :

As $|N \cap T \cap F| = 3$

and people who read N and T both = 11 which includes people of $|N \cap T \cap F|$ also.

Hence people who read only N and T but not F will be 11 – 3, hence 8 people read only N and T similarly $|T \cap F| = 8$.

Then people read only T and F = 8 – 3 i.e. 5.

and $|N \cap F| = 9$

$\Rightarrow$ People read only N and F = 9 – 3 i.e. 6.

Also,

$$|N| = 25, \text{ which includes } |N \cap T|, |N \cap F| \text{ and } |N \cap T \cap F|.$$

$$\text{People who read only N} = |N| - [|N \cap T| + |N \cap F|] + |N \cap T \cap F|$$

$$= 25 - [11 + 9] + 3$$

$\therefore$ People who read only N = 8 …(1)

$$\text{People who read only T} = |T| - [|N \cap T| + |T \cap F|] + |N \cap T \cap F|$$

$$= 26 - [11 + 8] + 3$$

$\therefore$ People who read only T $= 10$ ... (2)

People who read only F $= |F| - [|N \cap F| + |T \cap F|] + |N \cap T \cap F|$

$= 26 - [9 + 8] + 3$

$\therefore$ People who read only F $= 12$ ... (3)

iii) Number of people who reads exactly one magazine

= Equation [(1) + (2) + (3)] = 8 + 10 + 12 = 30

Example 1.12.11 *Suppose that 100 out of 120 mathematics students at a college take at least one of the languages French, German and Russian. Also suppose*

65 study French, 20 study French and German

45 study German, 25 study French and Russian

42 study Russian, 15 study German and Russian

i) Find the number of students who study all the three languages.

ii) Fill in correct number of students in each region of Venn diagram.

iii) Determine the number K of students who study

a) Exactly one language.

b) Exactly two languages. **SPPU : Dec.-05, Marks 6**

Solution : X $\rightarrow$ Universal set

$$|X| = 120$$

Let F be the number of students who study French. G be the number of students who study German and R be the number of students who study Russian.

$$|F \cup G \cup R| = 100$$

$$|F| = 65, \quad |F \cap G| = 20$$

$$|G| = 45, \quad |F \cap R| = 25$$

$$|R| = 42, \quad |G \cap R| = 15$$

i) $|F \cup G \cup R| = |F| + |G| + |R| - [|F \cap G| + |F \cap R| + |G \cap R|] + |F \cap G \cap R|$

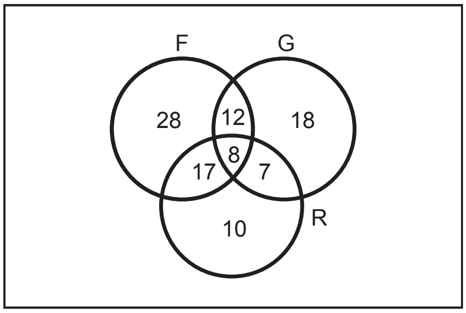
$$100 = 65 + 45 + 42 - [20 + 25 + 15] + |F \cap G \cap R|$$

$$100 = 92 + |F \cap G \cap R|$$

$\Rightarrow$ $|F \cap G \cap R| = 8$

Hence 8 students study all three languages.

ii)



Explanation

As $|F \cap G \cap R| = 8$

Students who study F and G but not all 3 will be

$$|F \cap G| - |F \cap G \cap R|$$

$$= 20 - 8 = 12 \qquad \dots (1)$$

Students who study F and R but not all 3 will be

$$|F \cap R| - |F \cap G \cap R|$$

$$= 25 - 8 = 17 \qquad \dots (2)$$

Students who study G and R but not all 3 will be

$$|G \cap R| - |F \cap G \cap R|$$

$$= 15 - 8 = 7 \qquad \dots (3)$$

Students who study only F but not G and not R.

$$= |F| - [|F \cap G| + |F \cap R|] + |F \cap G \cap R|$$

$$= 65 - [20 + 25] + 8 = 28 \qquad \dots (4)$$

Students who study only G but not F and not R

$$= |G| - [|F \cap G| + |G \cap R|] + |F \cap G \cap R|$$

$$= 45 - [20 + 15] + 8 = 18 \qquad \dots (5)$$

Students who study only R but not F and not G.

$$= |R| - [|F \cap R| + |G \cap R|] + |F \cap G \cap R|$$

$$= 42 - [25 + 15] + 8 = 10 \qquad \dots (6)$$

iii) a) Number of students who study exactly one language

$$= \text{Equation (4)} + \text{Equation (5)} + \text{Equation (6)}$$

$$= 28 + 18 + 10 = 56$$

b) Number of students who study exactly two languages.

$$= \text{Equation (1)} + \text{Equation (2)} + \text{Equation (3)}$$

$$= 12 + 17 + 7 = 36$$

Example 1.12.12 *Out of the integers 1 to 1000.*

i) How many of them are not divisible by 3, nor by 5, nor by 7 ?

ii) How many are not divisible by 5 and 7 but divisible by 3 ?

SPPU : May-06, 08, 14, Marks 6, May-19, Marks 13

Solution : i) Let A denote numbers divisible by 3.

B denote numbers divisible by 5.

and C denote numbers divisible by 7.

$$|A| = \left[\frac{1000}{3}\right] = 333$$

$$|B| = \left[\frac{1000}{5}\right] = 200$$

$$|C| = \left[\frac{1000}{7}\right] = 142$$

$$|A \cap B| = \left[\frac{1000}{3\times5}\right] = 66$$

$$|A \cap C| = \left[\frac{1000}{3\times7}\right] = 47$$

$$|B \cap C| = \left[\frac{1000}{5\times7}\right] = 28$$

$$|A \cap B \cap C| = \left[\frac{1000}{3\times5\times7}\right] = 9$$

$$|A \cup B \cup C| = |A|+|B|+|C|-[|A \cap B|+|A \cap C|+|B \cap C|] + |A \cap B \cap C|$$

$$= 333 + 200 + 142 - [66 + 47 + 28] + 9$$

$$= 543$$

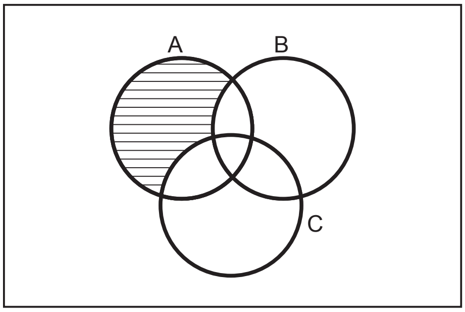
This show 543 numbers are divisible by 3 or 5 or 7. Hence numbers which are not divisible by 3, nor by 5, nor by 7.

$$= |A' \cap B' \cap C'|$$

$$= |X| - |A \cup B \cup C|$$

$$= 1000 - 543 = \mathbf{457}$$

ii)



Number of integers divisible by 3 but not by 5 and not by 7 is $|A \cap B' \cap C'|$.

$$|A \cap B' \cap C'| = |A \cap (B \cup C)'|$$

$$= |A| - [|A \cap B| + |A \cap C|] + |A \cap B \cap C|$$

$$= 333 - [66 + 47] + 9 = \mathbf{229}$$

Example 1.12.13 *In the class of 55 students the number of studying different subjects are as given below : Maths 23, Physics 24, Chemistry 19, Maths + Physics 12, Maths + Chemistry 9, Physics + Chemistry 7, all three subjects 4.*

Find the number of students who have taken :

i) At least one subject ii) Exactly one subject iii) Exactly two subjects.

SPPU : May-07, Marks 6

Solution : Let X denote universal set.

Let M denote set of students studying mathematics.

P denote set of students studying physics

and C denote set of students studying chemistry.

Then $|X| = 55, |M| = 23, |P| = 24, |C| = 19, |M \cap P| = 12,$

$$|M \cap C| = 9, |P \cap C| = 7$$

$$|M \cap P \cap C| = 4$$

i) $|M \cup P \cup C| = |M| + |P| + |C| - [|M \cap P| + |M \cap C| + |P \cap C|] + |M \cap P \cap C|$

$$= 23 + 24 + 19 - [12 + 9 + 7] + 4 = \mathbf{42}$$

ii) Number of students studying mathematics but not physics and not chemistry.

$$= |M| - [|M \cap P| + |M \cap C|] + |M \cap P \cap C|$$

$$= 23 - [12 + 9] + 4 = \mathbf{6} \qquad \text{... (1)}$$

Number of students studying physics but not mathematics and not chemistry.

$$= |P| - [|M \cap P| + |P \cap C|] + |M \cap P \cap C|$$

$$= 24 - [12 + 7] + 4 = \mathbf{9} \qquad \text{... (2)}$$

Number of students studying chemistry but not mathematics and not physics.

$$= |C| - [|M \cap C| + |P \cap C|] + |M \cap P \cap C|$$

$$= 19 - [9 + 7] + 4 = \mathbf{7} \qquad \text{... (3)}$$

Hence number of students studying exactly one subject = 6 + 9 + 7 = 22

iii) Number of students studying mathematics and physics but not chemistry

$$= |M \cap P| - |M \cap P \cap C|$$

$$= 12 - 4 = \mathbf{8} \qquad \text{... (4)}$$

Number of students studying physics and chemistry but not mathematics

$$= |P \cap C| - |M \cap P \cap C|$$

$$= 7 - 4 = \mathbf{3} \qquad \text{... (5)}$$

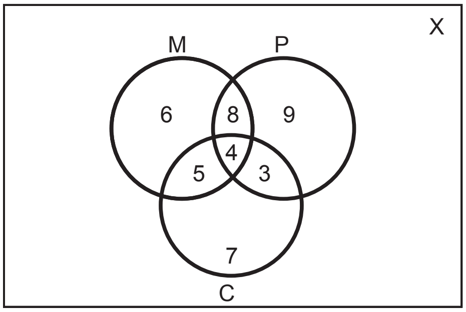
Number of students studying mathematics and chemistry but not physics

$$= |M \cap C| - |M \cap P \cap C|$$

$$= 9 - 4 = \mathbf{5} \qquad \text{... (6)}$$

Hence number of students studying exactly two subjects

$$= 8 + 3 + 5 = \mathbf{16}$$



Example 1.12.14 *100 of the 120 engineering students in a college take part in atleast one of the activity group discussion, debate and quiz.*

Also : 65 participate in group discussion, 45 participate in debate, 42 participate in quiz, 20 participate in group discussion and debate 25 participate in group discussion and quiz 15 participate in debate and quiz. Find the number of students who participate in : i) All the three activities ii) Exactly one of the activities. **SPPU : Dec.-07, Marks 4**

Solution : Let X denote universal set

$$|X| = 120$$

Let A denote number of students taking part in group discussion.

B denote number of students taking part in debate

and C denote number of students taking part in quiz.

$|A \cup B \cup C| = 100, \quad |A| = 65, \quad |B| = 45, \quad |C| = 42,$

$|A \cap B| = 20, |A \cap C| = 25, \ |B \cap C| = 15.$

i) $|A \cup B \cup C| = |A| + |B| + |C| - [|A \cap B| + |A \cap C| + |B \cap C|] + |A \cap B \cap C|$

$$100 = 65 + 45 + 42 - [20 + 25 + 15] + |A \cap B \cap C|$$

$\Rightarrow |A \cap B \cap C| = \mathbf{8}$

ii) Number of students taking part in group discussion but not in debate and not in quiz.

$$= |A| - [|A \cap B| + |A \cap C|] + |A \cap B \cap C|$$

$$= 65 - [20 + 25] + 8 = \mathbf{28} \qquad \text{... (1)}$$

Number of students taking part in debate but not in group discussion and not in quiz.

$$= |B| - [|A \cap B| + |B \cap C|] + |A \cap B \cap C|$$

$$= 45 - [20 + 15] + 8 = \mathbf{18} \qquad \text{... (2)}$$

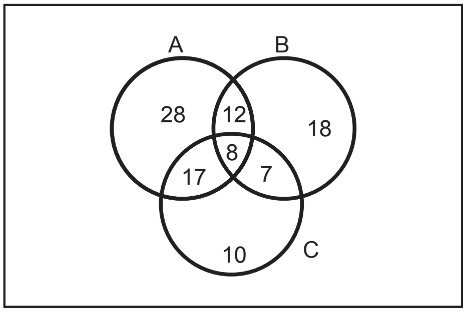
Number of students taking part in quiz but not in group discussion and not in debate.

$$= |C| - [|A \cap C| + |B \cap C| + |A \cap B \cap C|]$$

$$= 42 - [25 + 15] + 8 = \mathbf{10} \qquad \text{... (3)}$$

Hence number of students doing exactly one activity

$$= 28 + 18 + 10 = \mathbf{56}$$



Example 1.12.15 *It was found that in the first year computer science class of 80 students, 50 knew COBOL, 55 'C' and 46 PASCAL. It was also know that 37 knew 'C' and COBOL, 28 'C' and PASCAL and 25 PASCAL and COBOL. 7 students however knew none of the languages. Find*

i) How many knew all the three languages ?

ii) How many knew exactly two languages ?

iii) How many knew exactly one language ? **SPPU : May-15, Dec.-15, Marks 4**

Solution : Let A denote the set of students who know COBOL.

B denote the set of students who know 'C'.

and C denote the set of students who know PASCAL.

and X denote universal set.

Then $|X| = 80, \ |A| = 50, \ |B| = 55, \ |C| = 46$

$|A \cap B| = 37, |B \cap C| = 28, |A \cap C| = 25$

$|A' \cap B' \cap C'| = 7$

Hence $|(A \cup B \cup C)'| = 7$

Also $|A \cup B \cup C| = |X| - |(A \cup B \cup C)'|$

Hence $|A \cup B \cup C| = 80 - 7 = 73$

i) $|A \cup B \cup C| = |A| + |B| + |C| - [|A \cap B| + |A \cap C| + |B \cap C|] + |A \cap B \cap C|$

$73 = 50 + 55 + 46 - [37 + 28 + 25] + |A \cap B \cap C|$

$\Rightarrow \quad |A \cap B \cap C| = \mathbf{12}$

ii) Number of students who know only COBOL and 'C' but not PASCAL.

$= |A \cap B| - |A \cap B \cap C|$

$= 37 - 12 = \mathbf{25}$ ... (1)

Number of students who know only COBOL and PASCAL but not 'C'.

$$= |A \cap C| - |A \cap B \cap C|$$

$$= 25 - 12 = \mathbf{13} \qquad \text{... (2)}$$

Number of students who know only 'C' and PASCAL but not COBOL.

$$= |B \cap C| - |A \cap B \cap C|$$

$$= 28 - 12 = \mathbf{16} \qquad \text{... (3)}$$

Hence number of students who know exactly two languages.

$$= 25 + 13 + 16 = \mathbf{54}$$

iii) Number of students who know only COBOL but not 'C' and not PASCAL.

$$= |A| - [|A \cap B| + |A \cap C|] + |A \cap B \cap C|$$

$$= 50 - [37 + 25] + 12 = \mathbf{0} \qquad \text{... (4)}$$

Number of students who know only 'C' but not COBOL and not PASCAL.

$$= |B| - [|A \cap B| + |B \cap C|] + |A \cap B \cap C|$$

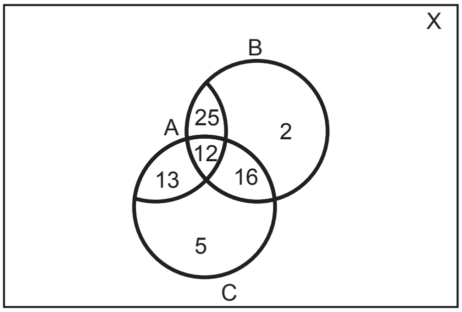
$$= 55 - [37 + 28] + 12 = \mathbf{2} \qquad \text{... (5)}$$

Number of students who know only PASCAL but not COBOL and not 'C'.

$$= |C| - [|A \cap C| + |B \cap C|] + |A \cap B \cap C|$$

$$= 46 - [25 + 28] + 12 = \mathbf{5} \qquad \text{... (6)}$$

Hence number of students who know only one language = 0 + 2 + 5 = **7**



Example 1.12.16 *In a survey of 500 television watchers produced the following information 285 watch football, 195 watch hockey, 115 watch basketball, 45 watch football and basketball, 70 watch football and hockey, 50 watch hockey and basketball and 50 do not watch any of the three games.*

i) How many people in the survey watch all three games ?

ii) How many people watch exactly one game ? **SPPU : May-08, Marks 6**

Solution : Let F denote the set of people who watch football.

H denote the set of people who watch hockey and B denote the set of people who watch basketball.

$X \rightarrow$ Denote universal set

Then $|X| = 500, |F| = 285, |H| = 195, |B| = 115.$

$|F \cap B| = 45, |F \cap H| = 70, |H \cap B| = 50$

$$|F' \cap B' \cap H'| = 50$$

i.e. $$|(F \cup B \cup H)'| = 50$$

$$\Rightarrow \quad |F \cup B \cup H| = |X| - |(F \cup B \cup H)'|$$

$$= 500 - 50 = \mathbf{450}$$

i) Number of people who watch all the three games $= |F \cap H \cap B|$.

$$|F \cup H \cup B| = |F| + |H| + |B| - [|F \cap H| + |F \cap B| + |H \cap B|] + |F \cap H \cap B|$$

$$450 = 285 + 195 + 115 - [70 + 45 + 50] + |F \cap H \cap B|$$

$$|F \cap H \cap B| = 20$$

ii) Number of people who watch only football.

$$= |F| - [|F \cap B| + |F \cap H|] + |F \cap B \cap H|$$

$$= 285 - [45 + 70] + 20 = \mathbf{190} \quad \text{... (1)}$$

Number of people who watch only hockey.

$$= |H| - [|H \cap B| + |H \cap F|] + |F \cap B \cap H|$$

$$= 195 - [50 + 70] + 20 = \mathbf{95} \quad \text{... (2)}$$

Number of people who watch only basketball.

$$= |B| - [|B \cap H| + |B \cap F|] + |F \cap B \cap H|$$

$$= 115 - [50 + 45] + 20 = \mathbf{40} \quad \text{... (3)}$$

$\therefore$ Number of people who watch exactly one game.

$$= 190 + 95 + 45 = \mathbf{325}$$

1.13 Multiset

SPPU : Dec.-13, May-19

We know that the set is the collection of well defined distinct objects. But there are so many practical situations in which objects are not distinct. For example 1) The collection of alphabets in the word "Missicippi", collection = [m, i, i, i, i, s, s, c, p, p]

2) Birth month of students studying in S.E. Computer Engineering and so on.

Multiset is the generalization of set.

Definition :

A collection of objects that are not necessarily distinct is called a multiset (or Msets). To distinguish set and multiset we denote multiset by enclosing elements in a square brackets e.g. [a, a, b]

1.13.1 Multiplicity of an Element

Let S be a multiset and $x \in S$. The multiplicity of x is defined as the number of times the element x appears in the multiset S. It is denoted by $\mu(x)$.

For example i) $S = [a, b, c, d, d, d, e, e]$

$$\mu(a) = 1, \mu(b) = 1, \mu(c) = 1, \mu(d) = 3, \mu(e) = 2$$

1.13.2 Equality of Multisets

Let A and B be two multisets. A and B are said to be equal multisets iff $\mu_A(x) = \mu_B(x), \ \forall x \in A \text{ or } B$

e.g. $[a, b, a, b] = [a, a, b, b]$ but $[a, a] \neq [a]$

Subset of Multisets

A multiset 'A' is said to be the multiset of B if multiplicity of each element in A is less or equal to it's multiplicity in B.

e.g. $[a] \subseteq [a, a]$, $[a, b, a, b, a] \subseteq [a, a, a, b, b, b]$.

1.13.3 Union and Intersection of Multisets

If A and B are multisets then $A \cup B$ and $A \cap B$ are also multisets.

The multiplicity of an element $x \in A \cup B$ is equal to the maximum of the multiplicity of x in A and in B.

The multiplicity of an element $x \in A \cap B$ is equal to the minimum of the multiplicity of x in A and in B.

Example :

$$A = [a, b, c, a, b, c, a, a, a], \qquad B = [a, a, b, b, b, b]$$

$$A \cup B = [a, a, a, a, a, b, b, b, b, c, c]$$

$$A \cap B = [a, a, b]$$

1.13.4 Difference of Multisets

Let A and B be two multisets. The difference A – B is a multiset such that $x \in A - B$

if $(\mu_A(x) - \mu_B(x)) \geq 1$

e.g. 1) A = [a, a, b, a], B = [a, b]

$\therefore$ A – B = [a, a]

2) A = [1, 2, 3, 4, 2, 2, 3, 3], B = [1, 1, 1, 2, 2, 2, 3, 3, 3, 3, 4]

A – B = [ ] or ϕ

1.13.5 Sum of Multisets

Let A and B be two multisets. The sum of A and B is denoted by A + B and defined as for each

$x \in A + B,\ \mu(x) = \mu_A(x) + \mu_B(x)$

e.g. A = [a, b, c, c], B = [a, a, b, b, c, c]

A + B = [a, a, a, b, b, b, c, c, c, c]

Examples :

Example 1.13.1 *Find the multiset to solve equation*

$A \cap [1, 2, 2, 3, 4] = [1, 2, 3, 4]$

$A \cup [1, 2, 2, 3] = [1, 1, 2, 2, 3, 3, 4]$

Solution : Maximum multiplicity of each element is as follows

$$\mu(1) = 2, \quad \mu(2) = 2, \quad \mu(3) = 2, \quad \mu(4) = 1$$

and minimum multiplicity of each element in A is as follows

$$\mu(1) = 1, \quad \mu(2) = 1, \quad \mu(3) = 1, \quad \mu(4) = 1$$

Therefore,

$$A = [1, 2, 3, 4] \quad \text{or} \quad A = [1, 1, 2, 2, 3, 3, 4]$$

Example 1.13.2 *Explain examples of multisets with its significance.* **SPPU : Dec.-13, Marks 4**

Solution : 1) Multisets are used to denote roots of the polynomial.

$\therefore \quad x^3 + 3x^2 + 3x + 1 = 0$

Roots are –1, –1, –1

$\therefore \quad A = [-1, -1, -1]$

2) Multisets are used to denote prime factors of every non-negative integer.

e.g. prime factors of 72 are $2 \times 2 \times 2 \times 3 \times 3$

$\therefore \quad A = [2, 2, 2, 3, 3]$

3) Multisets are used to denote zeros or poles of analytic functions.

4) In Computer Science, multisets are applied in a variety of search and sort procedure.

Example 1.13.3 *If $P = \{a, a, a, c, d, d\}$, $Q = \{a, a, b, c, c\}$. Find union, intersection and difference of P and Q.* **SPPU : May-19, Marks 3**

Solution : We have

$$P \cup Q = \{a, a, a, b, c, c, d, d\}$$

$$P \cap Q = \{a, a, c\}$$

$$P - Q = \{a, d, d\}$$

❑❑❑

Unit - I

2 Propositional Calculus

Syllabus

Propositional Logic - *logic, Propositional Equivalences, Application of Propositional Logic - Translating English Sentences.*

Contents

2.1 Introduction

A discrete structure is defined by a set of axioms. The properties of structure are derived from the axioms as theorems. Such theorems are proved using valid rules of reasoning. The propositional calculus (Mathematical logic) is concerned with all kinds of reasoning. It has two aspects.

1) It is analytic theory of art of reasoning whose goal is systematic and codify the principles of valid reasoning.
2) It is inter-related with problems relating the foundation of mathematics.

A great mathematician Frege G. (1884-1925) developed the idea regarding a mathematical theory as applied branch of logic.

Every student of engineering should learn logic because principles of logic are valuable to problem analysis, programming, logic designing, code designing and many more.

2.2 Statements or Propositions

A statement is a declarative sentence which is either true or false but not both. The truth or falsity of a statement is called its truth value. The truth value of a true statement is denoted by 'T' and the false statement is denoted by 'F'. They are also denoted by 1 or 0.

Statements are usually denoted by A, B, C, ... or a, b, c.

Examples :

1) There are 5 days in a week
2) $2 + 5 = 7$
3) $y + 3 = 8$
4) It will rain tomorrow
5) There are 12 months in a year

Examples (2) and (5) are true statements

Example (1) is false statement

In example (3), it's truth value depends upon the value of y. If y is 5 then sentence is true and if $y \neq 5$ then sentence is false. Therefore (4) is not a statement.

In example (4), it's truth value cannot be predicted at this moment but it can be definitely determined tomorrow. Hence it is a statement.

2.3 Laws of Formal Logic

There are two laws of formal logic

2.3.1 Law of Contradiction

It states that the same statement cannot be both true and false

2.3.2 Law of Excluded Middle

It p is a statement then either p is true or p is false and there cannot be middle ground. If a student appear for the examination of discrete structure then student will be either pass or fail in that exam. There is no middle stage.

2.4 Connectives and Compound Statements **SPPU : Dec.-06,18, May-05,07,08**

A statement which cannot be further split into simple sentences, is called primary or primitive or atomic statements.

In day to day life, we use the words NOT, AND, OR, IF-Then, as well as, BUT, WHILE to connect two or more sentences. But these connectives are flexible in their meanings, and lead to inexact and ambiguous interpretations. However, mathematics is a very precise language and every symbol of mathematics has the unique meaning or interpretation in mathematical ocean.

Hence we take some special connectives with precise meaning to suit our purpose. Following are the common connectives with symbols or their rotations.

Sr. No.	Name of connectives	Symbol
1	Negation	~ or $\neg$
2	And (Meet)	^
3	Or (Join)	v
4	If .... then	$\rightarrow$
5	If an only if (iff)	$\leftrightarrow$

2.4.1 Compound Statement

A statement which is formed from primary statements by using logical connectives is called a compound statement.

e.g. If p : I am studying in SE computer class

q : I am learning discrete structure subject

The compound statement is

I am studying in SE computer class and I am learning discrete structure subject.

2.4.2 Truth Table

A table showing the truth value of a statement formula is called truth table.

2.4.3 Negation

Let p be any simple statement, then the negation of p is formed by writing "It is false that" before p or introducing the word "not" at the proper place. The negation of p is also obtained by writing "p is false". The negation of p is denoted by ~ p or ¬ p or $\bar{p}$. If the statement p is true, then ~ p is false and if p is false then ~ p is true.

The truth table for negation of p is given below

p	~ p
T	F
F	T

Examples :

Sr. No.	Statement p	~ p
1.	p : Atharva is intelligent boy	~ p : Atharva is not intelligent boy OR It is false that Atharva is intelligent boy
2.	p : Delhi is the capital of India	~ p : Delhi is not the capital of India OR It is false that Delhi is the capital of India OR It is not the case that Delhi is the capital of India

Note : ~ p is the **unary** connective as only one statement is required to form negation.

2.4.4 Conjunction

A compound statement which is obtained by combining two primary statements by using the connective "and" is called conjunction i.e. the conjunction of two statements p and q is the statement $p \wedge q$. It is read as "p and q" or "p meet q".

The statement $p \wedge q$ has the truth value T whenever both p and q have the truth value T, otherwise $p \wedge q$ has the truth value F.

It's tabular representation is as follows :

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 2.4.1

Sr. No.	*p*	*q*	*Conjunction*	$p \wedge q$
1.	*3 > 2 (T)*	*5 is prime (T)*	*3 > 2 and 5 is prime*	*T*
2.	*Delhi is in India (T)*	*3 + 30 = 30 (F)*	*Delhi is in India and 3 + 30 = 30*	*F*

3. Construct a truth table for the conjunction of $n < 20$ *and* $n > 5$, $n \in N$

Solution : When $n < 20$ and $n > 5$ are true then the conjunction "$n < 20$ and $n > 5$" is true.

The truth table is as follows :

$p : n < 20$	$q : n > 5$	$p \wedge q : n < 20$ and $n > 5$ i.e. $5 < n < 20$
T (n = 15)	T (n = 15)	T ($\because 5 < 15 < 20$)
T (n = 2)	F (n = 2)	F ($\because n < 5$)
F (n = 30)	T (n = 30)	F ($\because n > 20$)
F (n = 50)	F ---	F (There does not exist any number s.t. $n > 20$ and $n < 5$)

2.4.5 Disjunction

A compound statement obtained by combining two simple statements by using the connective "or" is called the disjunction.

i.e. If p and q are simple statements then the compound statement "p or q" is the disjunction of p and q. It is denoted by $p \vee q$. $p \vee q$ is read as "p or q" or "p join q".

If p is false and q is false then $p \vee q$ is false. Otherwise $p \vee q$ is true. The truth table of $p \vee q$ is as follows :

p	q	$p \vee q$
T	T	T

T	F	F
F	T	T
F	F	F

Example 2.4.2 *1) If p : The data is wrong, q : There is an error in the program.*

Then $p \vee q$ *: The data is wrong or there is an error in the program.*

The connective 'v' is used in the inclusive sense i.e. at least one possibility exists or even both possibilities exist.

2)

Sr. No.	*p*	*q*	*Disjunction*	$p \wedge q$
1.	*6 < 2 (F)*	*5 + 3 = 4 (F)*	*6 > 2 or 5 + 3 = 4*	*F (∵ p and q both are false)*
2.	*Delhi is in India (T)*	*5.3 = 10 (F)*	*Delhi is in India or 5.3 = 10*	*T (∵ p is true)*
3.	*6 – 3 = 3 (T)*	*5 is prime (T)*	*6 – 3 = 3 or 5 is prime*	*T (∵ p and q are true)*

3) Construct a truth table for disjunction of p : n is prime, q : n > 10 for $n \in N$*.*

Solution : The truth table for $p \vee q$ is as follows :

p : n is prime	**q : n > 10**	$p \vee q$ **: n is prime or n > 10**
T (n = 13)	T (13 > 10)	T (∵ n = 13 > 10 or 13 is prime)
T (n = 5)	F (5 < 10)	T (∵ n = 5 is prime)
F (n = 12)	T (12 > 10)	T (∵ n = 12 > 10)
F (n = 6)	F (6 < 10)	F (∵ Both p and q are false)

2.4.6 Conditional Statement (If ..... then ....)

If p and q are two statements, then the statement $p \rightarrow q$ which is read as "if p then q" is called as conditional statement.

The conditional statement $p \rightarrow q$ can also be read as i) p only if q ii) p implies q iii) q if p iv) p is sufficient for q

The statement p is called the hypothesis or antecedent and q is called conclusion or consequent.

If p is true and q is false then $p \rightarrow q$ is false. Otherwise $p \rightarrow q$ is true.

The truth table of $p \rightarrow q$ is as follows :

p	**q**	$p \rightarrow q$
T	T	T

T	F	F
F	T	T
F	F	T

Examples :

1) Let p : Atharva is a graduate

  q : 3 + 5 = 8

$\therefore$ $p \to q$: If Atharva is a graduate then 3 + 5 = 8

Let N be the set of natural numbers and Q be the set of rational numbers. We know that every natural number is a rational number.

Let p : n is a natural number

  q : n is a rational number

$\mathbf{p : n \in N}$	$\mathbf{q : n \in Q}$	$\mathbf{p \to q}$ **: if** $\mathbf{n \in N}$ **then** $\mathbf{n \in Q}$
T (n = 2)	T ($n \in Q$)	T ($\because$ n = 2 is rational)
T (n = 5)	F (Impossible)	F ($\because$ Every natural number is rational)
F $\left(n = \frac{3}{2}\right)$	T $\left(\frac{3}{2} \in Q\right)$	T ($\because$ $\exists$ number which is not natural but it is rational)
F $(n = \sqrt{2})$	F $(\sqrt{2} \notin Q)$	T ($\because$ $\exists$ number which is neither natural nor rational)

Note : If p then q

p only if q

q if p

p is sufficient for q

q is necessary for p

Above all are equivalent to $p \to q$

2.4.7 Biconditional (If and only if)

If p and q are two statements, then the compound statement "p if and only if q" is called a biconditional statement. It is denoted by $P \underset{\leftarrow}{\to}$ or $p \leftrightarrow q$.

The biconditional statement is also read as

"p if and only if q" or "p iff q" or "p implies and implied by q" or "p is necessary and sufficient for q".

If p and q have the same truth value then $p \leftrightarrow q$ is true. In other cases $p \leftrightarrow$ is false.

The truth table for $p \leftrightarrow q$ is as follows

p	**q**	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	p	T	F	F
F	F	T	T	T	T

By above table

$p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$ are equivalent.

Examples

1) Let p : x = 4 q : x + 9 = 13

∴ $p \leftrightarrow q$: x = 4 if and only if x + 9 = 13

2) 5 + 6 = 11 iff 11 + 6 = 17

3) We know that an integer n is an even iff n is divisible by 2.

p : n is even	**q : n is divisible by 2**	**$p \leftrightarrow q$: n is even iff n is divisible by 2**
T (n = 10)	T (2 divides 10)	T (∵ n = 10 is even and divisible by 2)
T (n = 10)	F (Impossible)	F (∵ There does not exist even integer which is not divisible by 2)
F (n = 5)	T (Impossible)	F (∵ There does not exist odd integer which is divisible by 2)
F (n = 5)	F (5 is not divisible by 2)	T (∵ Integer is not even iff it is not divisible by 2)

2.4.8 Special Propositions

If $p \to q$ is a conditional statement, then

i) $q \to p$ is called its converse

ii) $\sim p \to \sim q$ is called its inverse

iii) $\sim q \to \sim p$ is called its contrapositive

The truth table of these propositions are as follows

p	q	$p \rightarrow q$	$q \rightarrow p$	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Example 2.4.3 *Let p denote the statement, "The material is interesting'. q denote the statement, "The exercises are challenging", and r denote the statement, "The course is enjoyable".*

Write the following statements in symbolic form :

i) The material is interesting and exercises are challenging.

ii) The material is interesting means the exercises are challenging and conversely.

iii) Either the material is interesting or the exercises are not challenging but not both.

iv) If the material is not interesting and exercises are not challenging, then the course is not enjoyable.

v) The material is uninteresting, the exercises are not challenging and the course is not enjoyable. **SPPU : Dec.-06, May-08, Marks 6**

Solution :

i) $p \wedge q$

ii) $(p \rightarrow q) \wedge (q \rightarrow p)$

iii) $p \oplus \sim q$

iv) $(\sim p \wedge \sim q) \rightarrow \sim r$

v) $\sim p \wedge \sim q \wedge \sim r$

Example 2.4.4 *Express following statement in propositional form :*

i) There are many clouds in the sky but it did not rain.

ii) I will get first class if and only if I study well and score above 80 in mathematics.

iii) Computers are cheap but softwares are costly.

iv) It is very hot and humid or Ramesh is having heart problem.

v) In small restaurants the food is good and service is poor.

vi) If I finish my submission before 5.00 in the evening and it is not very hot I will go and play a game of hockey. **SPPU : May-05, Marks 6**

Solution :

i) p : There are many clouds in the sky

q : It rain

$\therefore$ $p \wedge \sim q$

ii) p : I will get first class

q : I study well

r : Score above 80 in mathematics

$\therefore$ $p \leftrightarrow (q \wedge r)$

iii) p : Computers are cheap

q : Softwares are costly

$\therefore$ $p \wedge q$

iv) p : It is very hot

q : It is very humid

r : Ramesh is having heart problem

$\therefore$ $(p \wedge q) \vee r$

v) p : In small restaurant food in good

q : Service is poor

$\therefore$ $p \wedge q$

vi) p : I finish my submission before 5:00 p.m.

q : It is very hot

r : I will go

s : I will play a game of hockey

$\therefore$ $(p \wedge \sim q) \rightarrow (r \wedge s)$

Example 2.4.5 *Use p : I will study discrete mathematics, q : I will go to a movie, r : I am in a good mood. Write the following in English sentence :*

a) $\sim r \rightarrow q$ *b)* $\sim q \wedge p$ *c)* $q \rightarrow \sim p$ *d)* $\sim p \rightarrow \sim r$ **SPPU : Dec.-18, Marks 04**

Solution :

a) $\sim r \rightarrow q$ means, If I am not in a good mood, then I will go to a movie

b) $\sim q \wedge p$ means, I will not go to movie and I will study discrete mathematics.

c) $q \rightarrow \sim p$ means, If I will go to a movie then I will not study discrete maths.

d) $\sim p \rightarrow \sim r$ means, If I will not study discrete mathematics then I am not in a good mood.

Example 2.4.6 *Express the contrapositive, converse and inverse forms of the following statement if 3 < b and 1 + 1 = 2, then* $\sin\frac{\pi}{3} = \frac{1}{2}$. **SPPU : May-07, Marks 6**

Solution :

$$p \ : \ 3 < b$$
$$q \ : \ 1 + 1 = 2$$
$$r \ : \ \sin\frac{\pi}{3} = \frac{1}{2}$$

Symbolic form : $p \wedge q \rightarrow r$

Contrapositive : $(\sim r \rightarrow \sim (p \wedge q))$

i.e. $\sim r \rightarrow (\sim p \vee \sim q)$

i.e. if $\sin\frac{\pi}{3} \neq \frac{1}{2}$ then $3 \geq b$ or $1+1 \neq 2$

Converse : $r \rightarrow (p \wedge q)$

i.e. if $\sin\frac{\pi}{3} = \frac{1}{2}$ then $3 < b$ and $1 + 1 = 2$

Inverse : $\sim(p \wedge q) \rightarrow \sim r$

i.e. $(\sim p \vee \sim q) \rightarrow \sim r$

i.e. if $3 \geq b$ or $1+1 \neq 2$ then $\sin\frac{\pi}{3} \neq \frac{1}{2}$

Example 2.4.7 *Express the contrapositive, converse, inverse and negation forms of the conditional statement given below.*
'If x is rational, then x is real'. **SPPU : Dec.-06, Marks 4**

Solution :

Let p : x is rational

q : is real

Symbolic form : $p \rightarrow q$

Contrapositive : $(\sim q \rightarrow \sim p)$

If x is not real, then x is not rational

Converse : $(q \rightarrow p)$

If x is real then x is rational

Inverse - $(\sim p \rightarrow q)$

If x is not rational, then x is not real

Negation : $\sim (p \rightarrow q)$

$$\equiv \sim (p \vee \sim q)$$
$$\equiv \sim p \wedge \sim\sim q$$
$$\equiv \sim p \wedge q$$

2.5 Propositional or Statement Formula

A statement formula contains one or more simple statements and some logical connectives.

A statement formula is a string consisting of variables, parentheses and logical connective symbols.

It is called a well formed if it can be generated by the following rules :

i) A statement variable p standing alone is a well formed formula.

ii) If p is well formed formula then $\sim p$ is also well formed formula.

iii) If p and q are well formed formulas then $p \wedge q$, $p \vee q$, $p \rightarrow q$ or $q \rightarrow p$ and $p \leftrightarrow q$ are well formed formulas.

iv) A string of symbols is well formed formula iff it is obtained by finitely many applications of rules i), ii) and iii)

$\therefore$ A statement formula is not a statement and has no truth values. But if put definite statements in place of variables in a given formula we get a statement. It's truth value depends upon the truth values of variables.

2.6 Tautology

A statement formula that is true for all possible values of it's propositional variables is called a Tautology.

e.g. $p \vee \sim p$ is a tautology

2.7 Contradiction

A statement formula that is always false for all possible values of variables is called a contradiction or absurdity.

Example : $p \wedge \sim p$ is a contradiction

2.8 Contingency

SPPU : Dec.-10, 12

A statement formula which is neither tautology nor contradiction is called a contingency.

Example 2.8.1 *Prove that $p \rightarrow p$ is a tautology.*

Solution : We construct truth table for $p \rightarrow p$

p	p	$p \rightarrow p$
T	T	T
F	F	T

As $p \rightarrow p$ is always true. Hence $p \rightarrow p$ is a tautology.

Example 2.8.2 *Prove that $p \vee \sim p$ is a tautology and $\sim(p \vee \sim p)$ is a contradiction.*

Solution : Let us construct truth table for the statement $\sim(p \vee \sim p)$

p	~ p	$p \vee \sim p$	$\sim(p \vee \sim p)$
T	F	T	F
F	T	T	F

As $p \vee \sim p$ is always true. Hence $p \vee \sim p$ is a tautology and $\sim(p \vee \sim p)$ is always false. Hence $\sim(p \vee \sim p)$ is a contradiction.

Example 2.8.3 *Show that $p \wedge \sim p$ is a contradiction and $\sim(p \wedge \sim p)$ is tautology.*

Solution : We construct truth table for $\sim(p \wedge \sim p)$

p	~ p	$p \wedge \sim p$	$\sim(p \wedge \sim p)$
T	F	F	T
F	T	F	T

As $p \wedge \sim p$ is always false. Hence $p \wedge \sim p$ is a contradiction. As $\sim(p \wedge \sim p)$ is always true, Hence $\sim(p \wedge \sim p)$ is a tautology.

Example 2.8.4 *Determine whether each of the following statement formula is a tautology, contradiction or contingency.*

i) $(p \wedge q) \wedge \sim(p \vee q)$

ii) $(p \rightarrow q) \leftrightarrow (q \vee \sim p)$

iii) $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$

iv) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ **SPPU : Dec.-12, Marks 6**

Solution : i) Consider the truth table

p	q	$p \wedge q$	$p \vee q$	$\sim(p \vee q)$	$(p \wedge q) \wedge \sim(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Hence $(p \wedge q) \wedge \sim(p \vee q)$ is a contradiction

ii) Consider the truth table

p	q	$\sim p$	$p \to q$	$q \vee \sim p$	$(p \to q) \leftrightarrow (q \vee \sim p)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

As $(p \to q) \leftrightarrow (q \vee \sim p)$ is always true. Hence it is a tautology.

iii) Consider the truth table

p	q	r	$p \to q$	$p \to r$	$q \to r$	$p \to (q \to r)$	$(p \to q) \to (p \to r)$	$[p \to (q \to r)] \to [(p \to q) \to (p \to r)]$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	T	T	T	T
T	F	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

Hence $[(p \to (q \to r)] \to [(p \to q) \to (p \to r)]$ is a tautology.

iv) Consider the truth table

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Hence given statement formula is a tautology.

Example 2.8.5 *Show that $(p \to q) \wedge \sim q \to \sim p$ is a tautology without using truth table.*

Solution : We know that $p \to q$ is true if p is true and q is also true.

$\therefore$ We need only to show that $p \to q$ and ~ q both are true imply ~ p is true.

As the truth value of ~ q is T, the truth value of q is F. And as $p \to q$ is true, this means that p is false ($\because$ F $\to$ F is true)

$\therefore$ The truth value of p is T. Hence the proof.

Example 2.8.6 *Prove that $[(p \to q) \wedge (r \to s) \wedge (p \vee r)] \to (q \vee s)$ is a tautology.*

SPPU : Dec.-10, Marks 4

Solution : Consider truth table

p	q	r	s	$p \to q$	$r \to s$	$p \vee r$	$(p \to q) \wedge (r \to s) \wedge (p \vee r)$ (I)	$q \vee s$	$I \to (q \vee s)$
T	T	T	T	T	T	T	T	T	T
T	T	T	F	T	F	T	F	T	T
T	T	F	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T	T	T
T	F	T	T	F	T	T	F	T	T
T	F	T	F	F	F	T	F	F	T
T	F	F	T	F	T	T	F	T	T
T	F	F	F	F	T	T	F	F	T
F	T	T	T	T	T	T	T	T	T
F	T	T	F	T	F	T	F	T	T
F	T	F	T	T	T	F	F	T	T
F	T	F	F	T	T	F	F	T	T
F	F	T	T	T	T	T	T	T	T
F	F	T	F	T	F	T	F	F	T
F	F	F	T	T	T	F	F	T	T
F	F	F	F	T	T	F	F	F	T

Hence given statement formula is a tautology.

Example 2.8.7 *Prove by truth table $p \to (Q \vee R) \equiv (P \to Q) \vee (P \to R)$.* **SPPU : Dec.-12**

Solution : Consider the truth table

				I			II
P	**Q**	**R**	$\mathbf{Q \vee R}$	$\mathbf{P \rightarrow (Q \vee R)}$	$\mathbf{P \rightarrow Q}$	$\mathbf{P \rightarrow R}$	$\mathbf{(P \rightarrow Q) \vee (P \rightarrow R)}$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

From truth table

$$p \rightarrow (Q \vee R) \equiv (P \rightarrow Q) \vee (P \rightarrow R)$$

2.9 Precedence Rule

The order of preference in which the connectives are applied in a formula of propositions that has no bracket is

i) ~ ii) $\wedge$ iii) $\vee$ and $\oplus$ iv) $\rightarrow$ and $\leftrightarrow$

Remark :

1) '$\vee$' includes 'exclusive or' and 'inclusive or'

2) If $\vee$ and $\oplus$ are present in a statement then first apply the left most one. The same rule is applicable to $\rightarrow$ and $\leftrightarrow$

2.10 Logical Equivalence

SPPU : Dec.-12

In real life, we come across several similar things with respect to different aspects. e.g. Two cars are similar with respect to average, two students are similar with respect to F.E. marks.

Likewise in logic, we can say that two propositions are similar with respect to their truth values.

Definition : Two propositions A and B are logically equivalent iff they have the same truth value for all choices of the truth values of simple propositions involved in it.

Two propositions (formulas) are equivalent even if they have different variables. Two statement formulas P and Q are equivalent

Iff $P \leftrightarrow Q$ is a tautology.

If p is equivalent to Q then it can be represented as $P \Leftrightarrow Q$ or $P \equiv Q$.

The symbol $\Leftrightarrow$ is not a connective

Example 2.10.1 *Prove by constructing the truth table* $p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$.

SPPU : Dec.-12, Marks 3

Solution : Consider truth table

p	q	r	$q \vee r$	$p \to (q \vee r)$	$p \to q$	$p \to r$	$(p \to q) \vee (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

In the columns of $p \to (q \vee r)$ and $(p \to q) \vee (p \to r)$, truth values are same for all possible choices of truth values of p, q and r. Hence

$$p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$$

Example 2.10.2 *Prove that*

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p) \equiv (\sim p \vee q) \wedge (\sim q \vee p).$$

Solution : Consider the truth table

p	q	~ p	~ q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$	$p \leftrightarrow q$	$\sim p \vee q$	$\sim q \vee p$	$(\sim p \vee q) \wedge (\sim q \vee p)$
T	T	F	F	T	T	T	T	T	T	T
T	F	F	T	F	T	F	F	F	T	F
F	T	T	F	T	F	F	F	T	F	F
F	F	T	T	T	T	T	T	T	T	T

From above table

$$(p \leftrightarrow q) \equiv (p \to q) \wedge (q \to p) \equiv (\sim p \vee q) \wedge (\sim q \vee p)$$

2.11 Logical Identities

Sr. No.	Name of identity	Identity
1.	Idempotence of $\vee$	$p \equiv p \vee p$
2.	Idempotence of $\wedge$	$p \equiv p \wedge p$
3.	Commutativity of $\vee$	$p \vee q \equiv q \vee p$
4.	Commutativity of $\wedge$	$p \wedge q \equiv q \wedge p$
5.	Associativity of $\vee$	$p \vee (q \vee r) \equiv (p \vee q) \vee r$
6.	Associativity of $\wedge$	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
7.	Distributivity of $\wedge$ over $\vee$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
8.	Distributivity of $\vee$ over $\wedge$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
9.	Double negation	$p \equiv \sim(\sim p)$
10.	De Morgan's laws	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
11.	De Morgan's laws	$\sim(p \wedge p) \equiv \sim p \wedge \sim q$
12.	Tautology	$p \vee \sim q \equiv$ Tautology
13.	Contradiction	$p \wedge \sim p \equiv$ Contradiction
14.	Absorption laws	$p \vee (p \wedge q) \equiv p$
15.	Absorption laws	$p \wedge (p \vee q) \equiv p$

Example 2.11.1 *De Morgan's laws*

i) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

ii) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

Solution : i) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

Consider the truth table

p	**q**	**$\sim p$**	**$\sim q$**	**$p \vee q$**	**$\sim(p \vee q)$**	**$\sim p \wedge \sim q$**
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

From the table, truth values of $\sim(p \vee q)$ and $\sim p \wedge \sim q$ are same for each choice of p and q

Hence $\sim(p \vee q) \equiv \sim p \wedge \sim q$

ii) $\sim(p \vee q) \equiv \sim p \vee \sim q$

Consider the table

p	q	~ p	~ q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

From the table, truth values of $\sim(p \wedge q)$ and $(\sim p \vee \sim q)$ are same for each choice of p and q.

Hence $\sim(p \wedge q) \equiv (\sim p \vee \sim q)$

Example 2.11.2 *Absorption laws*

i) $p \vee (p \wedge q) \equiv p$ *ii)* $p \wedge (p \vee q) \equiv p$

Solution :

i) $p \vee (p \wedge q) \equiv p$

Consider the table

p	q	$p \wedge q$	$p \vee (p \wedge q)$	p
T	T	T	T	T
T	F	F	T	T
F	T	F	F	F
F	F	F	F	F

From the table, in last two columns truth tables of $p \vee (p \wedge q)$ and p are same for each choice of p and q.

Hence $p \vee (p \wedge q) \equiv p$

ii) $p \wedge (p \vee q) \equiv p$

Consider the table

p	q	$p \vee q$	$p \wedge (p \vee q)$	p
T	T	T	T	T
T	F	T	T	T
F	T	T	F	F
F	F	F	F	F

From the table, last two columns are identical hence $p \wedge (p \vee q) \equiv p$

Example 2.11.3 *$p \to q$ and $\sim p \vee q$ are logically equivalent.*

Solution : Consider the table

p	q	~ p	~ p∨q	p → q
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Hence $\sim p \vee q$ and $p \to q$ are logically equivalent.

2.12 The Duality Principle

Two statement formulas p and p* are said to be duals of each other if either one can be obtained from the other by replacing $\wedge, \vee, T, F$ by $\vee, \wedge, F, T$ respectively.

Example 2.12.1 *Write the duals of*

i) $\sim(p \vee q)$ ii) $p \wedge (q \vee r)$ iii) $p \vee T$

Solution : The duals are i) $\sim(p \wedge q)$ ii) $p \vee (q \wedge r)$ iii) $p \wedge F$

2.13 Logical Implication

A proposition p is said to logically imply a proposition Q iff one of the following condition holds :

1) $\sim p \vee Q$ is a Tautology

2) $p \wedge Q$ is a Contradiction

3) $p \to Q$ is a Tautology

It is denoted by $p \Rightarrow Q$

e.g. 1) $(p \wedge q) \wedge \sim (q \vee p)$ is a contradiction

Hence $p \wedge q \Rightarrow q \vee p$

The relation $p \Rightarrow Q$ is reflexive, antisymmetric and transitive

Note : The symbols $\to$ & $\Rightarrow$ are not same

$\Rightarrow$ is not connective

2.14 Important Connectives

I) NAND :

The word NAND is a combination of "NOT" and "AND" where "NOT" stands for negation and "AND" stands for the conjunction. It is denoted by the symbol $\uparrow$.

If P and Q are two formulas then NAND connective is defined as

$$P \uparrow Q \leftrightarrow \sim (P \wedge Q)$$

The connective $\uparrow$ has the following equivalence

1) $P \uparrow P \leftrightarrow \sim (P \wedge P) \leftrightarrow \sim P \vee \sim P \Leftrightarrow \sim P$

$\therefore$ $P \uparrow P \equiv \sim P$

2) $(P \uparrow Q) \uparrow (P \uparrow Q) \leftrightarrow \sim (P \uparrow Q) \leftrightarrow \sim \sim (P \wedge Q) \leftrightarrow P \wedge Q$

3) $(P \uparrow P) \uparrow (Q \uparrow Q) \leftrightarrow \sim P \uparrow \sim Q \leftrightarrow \sim(\sim P \wedge \sim Q) \leftrightarrow P \vee Q$

Note : NAND connective is commutative but not associative

II) NOR :

The connective "NOR" is a combination of "NOT" and "OR" where "NOT" stands for negation and "OR" stands for the disjunction. It is denoted by the symbol $\downarrow$ and defined as

$$P \downarrow Q \leftrightarrow \sim (P \vee Q) \;\therefore\; P \downarrow Q \equiv \sim (P \vee Q)$$

The connective $\downarrow$ has the following equivalences

i) $P \downarrow P \leftrightarrow \sim (P \vee P) \leftrightarrow \sim P \wedge \sim P \leftrightarrow \sim P$

ii) $(P \downarrow Q) \downarrow (P \downarrow Q) \leftrightarrow \sim (P \downarrow Q) \leftrightarrow P \vee Q$

iii) $(P \downarrow P) \downarrow (Q \downarrow Q) \leftrightarrow \sim P \downarrow \sim Q \leftrightarrow \sim(\sim P \vee \sim Q) \leftrightarrow P \wedge Q$

iv) The connective $\downarrow$ is commutative but not associative

2.15 Normal Forms

SPPU : Dec.-04, 06, 08, 10, 12, 14, May-17

If a given statement formula involves n atomic variables, then we have 2 possible combinations of truth values of statements replacing the variables. The construction of the truth table involves finite number of steps but it may not be practical if number of variables are more. Therefore we reduce the given statement formula to normal form and find whether a given statement formula is a tautology or contradiction or at least satisfiable.

A formula which is a conjunction or product of the variables and their $p \wedge \sim q$ negations is called an elementary product. If p and q are statements then p, $\sim$ p, $\sim p \wedge q$, $p \wedge \sim q$, $\sim p \wedge \sim q$ are some examples of elementary products or fundamental conjunctions. A formula which is a disjunction or sum of the variables and their negations is called an elementary sum or fundamental disjunctions. p, $\sim$ p, $\sim p \vee q$, $p \vee \sim q$, $\sim p \vee p \vee \sim q$ are some examples of an elementary sum or fundamental disjunctions.

2.15.1 Disjunctive Normal Form (dnf)

A statement formula which consists of a disjunction ($\vee$) of fundamental conjunctions or elementary product ($\wedge$). It is abbreviated as dnf.

A disjunctive normal form of a given formula is constructed as follows :

1) Replace '$\rightarrow$' or '$\leftrightarrow$' by using logical connectives $\wedge, \vee \,\&\, \sim$. $p \rightarrow q \equiv \sim p \vee q$, $p \leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p)$
2) Use De Morgan's laws to eliminate '~' before sums or products.
3) Apply distributive laws repeatedly and eliminate product of variables to obtain the required normal form.

Examples :

1) $(p \wedge q) \vee (q) \vee (\sim q \wedge p)$
2) $(p \wedge q \wedge r) \vee (p \wedge r) \vee (p \wedge q) \vee (p \wedge r)$
3) $(p \wedge r) \vee (p \wedge q)$
4) $(\sim p \wedge r) \vee (\sim q \wedge r) \vee (\sim r)$

All above examples are in disjunctive normal form.

2.15.2 Conjunctive Normal Form (cnf)

A statement formula which consists of a conjunction of the fundamental disjunctions ($\vee$). It is denoted as cnf.

Examples :

1) $p \wedge (p \vee q)$
2) $(p \vee q) \wedge (\sim p \vee q) \wedge (\sim q)$
3) $(p \vee q \vee r) \wedge (\sim p \vee r) \wedge (p \vee \sim q \vee r)$

All above examples are in conjunctive normal form.

2.15.3 Principal Normal Form

Let p and q be two statement variables, then $p \wedge q$, $p \wedge \sim q$, $\sim p \wedge q$, $\sim p \wedge \sim q$ are called minterms of p and q. They are also called Boolean conjunctives of p and q. The number of minterms with n variables is 2^n. None of the minterms should contain both a variable and it's negation. $\therefore$ $p \wedge \sim p$ is not minterm.

The dual of minterm is called a maxterm.

$\therefore$ For two statement variables p and q, maxterms are $p \vee q$, $\sim p \vee q$, $p \vee \sim q$ and $\sim p \vee \sim q$.

I) Principal Disjunctive Normal Form

A statement formula which consists of a disjunction of minterms only is called the principal disjunctive normal form.

e.g.

1) $(p \vee q) \wedge (\sim p \wedge \sim q)$, $(p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r)$ are principal cnf.
2) $p \vee (p \wedge q)$, $(p \wedge q) \vee (\sim p \wedge q) \vee (\sim q)$, $(p \wedge q \wedge r) \vee (p \wedge q)$ are not principal dnf.

II) Principal Conjuctive Normal Form

A statement formula which consists of a conjunction of maxterms only is called the principle conjunctive normal form.

e.g.

1) $(p \vee q) \wedge (\sim p \vee q)$, $(p \vee \sim q) \wedge (\sim p \vee \sim q)$ are principal cnf.
2) $(p \vee q) \wedge (\sim p)$ is cnf but not principal cnf.

Example 2.15.1 *Obtain the conjunctive normal form and disjunctive normal form of the following formulae given below :*
i) $p \wedge (p \rightarrow q)$ *ii)* $\sim (p \vee q) \rightleftarrows (p \wedge q)$ **SPPU : Dec.-12, Marks 4**

Solution :

i)
$$\begin{aligned} p \wedge (p \rightarrow q) &\equiv p \wedge (\sim p \vee q) \sim \text{cnf} \\ &\equiv (p \wedge \sim p) \vee (p \wedge q) \\ &\equiv F \vee (p \vee q) \\ &\equiv (p \wedge q) \qquad \text{... Definition a single conjunctive} \end{aligned}$$

ii)
$$\begin{aligned} \sim (p \vee q) \rightleftarrows (p \wedge q) &\equiv (\sim\sim (p \vee q) \vee (p \wedge q)) \wedge (\sim (p \wedge q) \vee \sim (p \vee q)) \\ &\equiv ((p \vee q) \vee (p \wedge q) \wedge ((\sim p \vee \sim q) \vee (\sim p \wedge \sim q) \\ &\equiv (p \vee q) \wedge ((\sim p \vee \sim q \vee \sim p) \wedge (\sim p \vee \sim q \vee \sim q)) \\ &\equiv (p \vee q) \wedge (\sim p \vee \sim q) \wedge (\sim p \vee \sim q) \\ &\equiv (p \vee q) \wedge (\sim p \vee \sim q) \qquad \text{... cnf} \end{aligned}$$

Further

$$\begin{aligned} (p \vee q) \wedge (\sim p \vee \sim q) &\equiv ((p \vee q) \wedge \sim p) \vee ((p \vee q) \wedge \sim q) \\ &\equiv (p \wedge \sim q) \vee (q \wedge \sim p) \vee ((p \wedge \sim q) \vee (q \wedge \sim q) \\ &\equiv F \vee (q \wedge \sim p) \vee (p \wedge \sim q) \vee F \\ &\equiv (p \wedge \sim p) \vee (p \wedge \sim q) \qquad \text{... Definition} \end{aligned}$$

Example 2.15.2 *Find the conjunctive normal form and disjunctive normal form for the following :*

i) $(p \vee \bar{q}) \rightarrow q$ *ii)* $p \leftrightarrow (\bar{p} \vee \bar{q})$ **SPPU : Dec.-06, 14, May-17, Marks 3**

Solution :

i) $(p \vee \bar{q}) \rightarrow q \equiv \overline{(p \vee \bar{q})} \vee q$

$\equiv (\bar{p} \wedge \bar{\bar{q}}) \vee q$

$\equiv (\bar{p} \wedge q) \vee q$ ... Definition

$(\bar{p} \wedge q) \vee q \equiv (\bar{p} \vee q) \wedge (q \vee q)$

$\equiv (\sim p \vee q) \wedge q$ ... cnf

$p \times (\bar{p} \vee \bar{q}) \equiv (\bar{p} \vee (\bar{p} \vee \bar{q})) \wedge ((\overline{\bar{p} \vee \bar{q}}) \vee p)$

$\equiv (\bar{p} \vee \bar{p} \vee \bar{q}) \wedge ((\bar{\bar{p}} \wedge \bar{\bar{q}}) \vee p)$

$\equiv (\bar{p} \vee \bar{q}) \wedge (p \vee p) \wedge (q \vee p)$

$\equiv (\bar{p} \vee \bar{q}) \wedge p \wedge (q \vee p)$ ... cnf

$\equiv ((\bar{p} \wedge p) \vee (\bar{q} \wedge p)) \wedge (q \vee p)$

$\equiv (F \vee (\bar{q} \wedge p)) \wedge (q \vee p)$

$\equiv (\bar{q} \wedge p) \wedge (q \vee p)$

$\equiv (\bar{q} \wedge p \wedge q) \vee (\bar{q} \wedge p \wedge p)$

$\equiv (F \wedge p) \vee (\bar{q} \wedge p)$

$\equiv F \vee (\bar{q} \wedge p)$

$\equiv (\bar{q} \wedge p)$ ... Definition

Example 2.15.3 *Find the conjunctive and disjunctive normal forms for the following without using truth table*

i) $(p \rightarrow q) \wedge (q \rightarrow p)$ *ii)* $((p \wedge (p \rightarrow q)) \rightarrow q)$ **SPPU : Dec.-04, 10, 14, Marks 4**

Solution :

i) $(p \rightarrow q) \wedge (q \rightarrow p) \equiv (\sim p \vee q) \wedge (\sim q \vee p)$

Further, using the distributive law on the above cnf we have

$((\sim p \vee q) \wedge \sim q) \vee ((\sim p \vee q) \wedge p) = (\sim p \wedge \sim q) \vee (q \wedge \sim q) \vee (\sim p \wedge p) \vee (q \wedge p)$

$= (\sim p \wedge \sim q) \vee (q \wedge p)$ ... dnf

$(\because \; p \wedge \sim p \equiv q \wedge \sim q)$

ii) $(p \wedge (p \to q)) \to q \equiv \sim(p \wedge (\sim p \vee q)) \vee q$

$\equiv \sim p \vee \sim(\sim p \vee q) \vee q$

$\equiv \sim p \vee (p \wedge \sim q) \vee q$

Example 2.15.4 *Obtain cnf of each of the following*
i) $p \wedge (p \to q)$ ii) $q \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q)$ **SPPU : Dec.-08, Marks 4**

Solution :

i) $p \wedge (p \to q) \equiv (p \wedge (\sim p \vee q))$

$\equiv (p \wedge \sim p) \vee (p \wedge q)$

$\equiv \vee (p \wedge q)$

$\equiv (p \wedge q)$ ... cnf

ii) $q \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q) \equiv ((q \vee p) \wedge (q \vee \sim q) \vee (\sim p \vee \sim q)$

$\equiv (q \vee p) \wedge T \vee (\sim p \wedge \sim q)$

$\equiv (q \vee p) \vee (\sim p \wedge \sim q)$

$\equiv (q \vee p \vee \sim p) \wedge (q \wedge p \vee \sim q)$

$\equiv (q \vee T) \wedge (p \vee q \vee \sim q)$

$\equiv T \wedge (p \vee T) \equiv T \wedge T$

$\equiv T \equiv (p \vee \sim p)$ which is the required cnf (single disjunct)

Example 2.15.5 *Find DNF of $((p \to q) \wedge (q \to p)) \vee p$*
SPPU : Dec.-14, Marks 4, May-17, Marks 3

Solution :

$((p \to q) \wedge (q \to p) \vee p \equiv [(\sim p \vee q) \wedge (\sim q \vee p)] \vee p$ $(\because p \to q \equiv - \sim p \vee q)$

$\equiv [p \vee (\sim p \vee q)] \wedge [p \vee (\sim q \vee p)]$

$\equiv [(p \vee \sim p) \vee q] \wedge [p \vee p \vee \sim p]$ $(\because p \vee p = p$
$p \vee \sim p = T)$

$\equiv (T \vee q) \wedge (p \vee \sim q)$

$\equiv T \wedge (p \vee \sim q)$

$\equiv p \vee \sim q$ which is the required DNF

Example 2.15.6 *Obtain the cnf : $(\sim p \wedge q \wedge r) \vee (p \wedge q)$* **SPPU : Dec.-12, Marks 4**

Solution :

$(\sim p \wedge q \wedge r) \vee (p \wedge q) \equiv (p \wedge q) \vee (\sim p \wedge q \wedge r)$

$$\equiv (p \vee (\sim p \wedge q \wedge r)] \wedge [q \vee (\sim p \wedge q \wedge r)]$$

$$\equiv [(p \vee \sim p) \wedge (p \vee q)] \wedge [p \vee r)] \wedge [(q \vee \sim p) \wedge (q \vee q) \wedge (q \vee r)]$$

$$\equiv [T \wedge (p \vee q) \wedge (p \vee r)] \wedge [(q \vee \sim p) \wedge q \wedge (q \vee r)]$$

$$\equiv (p \vee q) \wedge (p \vee r) \wedge (q \vee \sim p) \wedge q \qquad \text{which is the required cnf}$$

Example 2.15.7 *Find dnf by using truth table*
$(p \leftrightarrow (q \vee r)) \to \sim p$

Solution : Consider the truth table

p	q	r	~p	$q \vee r$	$p \leftrightarrow (q \vee r)$	$[p \leftrightarrow (q \vee r)] \to \sim p$
T	T	T	F	T	T	F
T	T	F	F	T	T	F
T	F	T	F	T	T	F
T	F	F	F	F	F	T ← 1
F	T	T	T	T	F	T ← 2
F	T	F	T	T	F	T ← 3
F	F	T	T	T	F	T ← 4
F	F	F	T	F	T	T ← 5

Consider only 'T' from last column and choose corresponding values of 'T' from p, q and r. For the first marked row 1 corresponding p is T, q is F and r is F. So take $p \wedge \sim q \wedge \sim r$ or $p \wedge q' \wedge r'$

For 2^{nd} $T \to \sim p \wedge q \wedge r$, 3^{rd} $T \to \sim p \wedge q \wedge \sim r$,

4^{th} $T \to \sim p \wedge q \wedge r$, and 5^{th} $T \to \sim p \wedge \sim q \wedge \sim r$

Hence the logically equivalent dnf form is

$$[p \leftrightarrow (q \vee r)] \to \sim p \equiv (p \wedge \sim q \wedge r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r)$$
$$\vee (\sim p \wedge \sim q \wedge r) \vee (\sim p \wedge \sim q \wedge \sim r)$$

Example 2.15.8 *Obtain the principal dnf of* $(\sim p \vee \sim q) \to (\sim p \wedge r)$

Solution :

$$(\sim p \vee \sim q) \to (\sim p \wedge \sim r) \equiv \sim (\sim p \vee \sim q) \vee (\sim p \wedge r)$$

$$\equiv (\sim (\sim p) \wedge \sim (\sim q)] \vee (\sim p \wedge r)$$

$$\equiv (p \wedge q) \vee (\sim p \wedge r) \qquad \text{which is dnf}$$

$$\equiv (p \wedge q \wedge (r \vee \sim r)) \vee (\sim p \wedge r \wedge (q \vee \sim q))$$

$$\equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge r \wedge q) \vee (\sim p \wedge r \wedge \sim q)$$

which is the required principal dnf.

Example 2.15.9 *Find the principal cnf of* $(p \wedge q) \vee (\sim p \wedge r)$

Solution :

$$\begin{aligned}
[(p \wedge q) \vee (\sim p \wedge r) &\equiv [(p \wedge q) \vee \sim p] \wedge [(p \wedge q) \vee r] \\
&\equiv (p \vee \sim p) \wedge (q \vee \sim p) \wedge (p \vee r) \wedge (q \vee r) \\
&\equiv (q \vee \sim p \vee (r \wedge \sim r) \wedge (p \vee r \vee (q \wedge \sim q)) \wedge (q \vee r \vee (p \wedge \sim p)) \\
&\equiv (q \vee \sim p \vee r) \wedge (q \vee \sim p \vee \sim r) \wedge (p \vee r \vee q) \ \wedge (p \vee r \vee \sim q)) \\
&\quad \wedge (q \vee r \vee p) \wedge (q \vee r \vee \sim p) \\
&\equiv (p \vee q \vee r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge (p \vee \sim q \vee r)
\end{aligned}$$

which is the required principle cnf.

Example 2.15.10 *Find the principal dnf of* $\sim p \vee q$

Solution :

$$\begin{aligned}
\sim p \vee q &\equiv [\sim p \wedge (q \vee \sim q)] \vee [q \wedge (p \vee \sim p)] \\
&\equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (q \wedge p) \vee (q \wedge \sim p) \\
&\equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (p \wedge q)
\end{aligned}$$

which is the required Principle dnf.

2.16 Methods of Proof

SPPU : Dec.-09, May-06, 10

We have learned statements or propositions and their truth values. Now, we will discuss ways by which statements can be linked to form a logically valid argument.

Whenever an assertion is made, which is claimed to be true, one has to state an argument which produces the truth of the assertion. To construct a proof we need to derive new assertions from existing ones by different ways. This is also done by using valid of inference.

Valid argument : A valid argument is a finite sequence of statements p_1, p_2, ... p_n called as premises (or assumptions or hypothesis) together with a statement C, called the conclusion such that $p_1 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n \rightarrow C$ is a tautology.

As mathematical proof is a logical argument that verifies the truth of the theorem. There are several ways of proving a theorem which are based on one or more rules of inference.

There are following most commonly used rules of inference.

2.16.1 Law of Detachment (or Modus Ponens)

Whenever the statements p and $p \rightarrow q$ are accepted as true, then we must accept the statement q as true. This rule is represented in the following form

$$\begin{array}{l} p \rightarrow q \\ \underline{\quad p \quad} \\ \therefore\ q \end{array}$$

The assertions above the horizontal line are called premises OT hypothesis. And the assertion below the line is called the conclusion.

This rule constitutes a valid argument as $(p \rightarrow q) \wedge p \rightarrow q$ is a tautology.

The truth table is as follows

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$[(p \rightarrow q) \wedge p] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

This form of valid argument is called the law of detachment as conclusion q is detached from premise $p \rightarrow q$ and p.

It is also called as the law of direct inference.

Example

If Sushma gets a first class with distinction in B.E. then she will get a good job easily.

Let p : Sushma gets a first class with distinction in B.E.

q : She will get a good job easily.

The inference rule is $$\begin{array}{l} p \rightarrow q \\ \underline{\quad p \quad} \\ \therefore\ q \end{array}$$

Hence this form of argument is valid.

2.16.2 Modus Tollen (Law of Contrapositive)

Modus Tollen is a rule of denying. It can be stated as "If $p \to q$ is true and q is false, then p is false. This is represented in the following form.

$$\begin{array}{l} p \to q \\ \underline{\ \ \sim p\ \ } \\ \therefore \sim q \end{array}$$

Above argument is valid as $(p \to q) \wedge \sim q \to \sim p$ is a tautology. In above example is $b \to q$ and $\sim q$: Sushma will not get a good job easily then $\sim p$: She has not a first class with distinction in B.E.

2.16.3 Disjunctive Syllogism

This rule states that "If $p \vee q$ is true and p is false then q is true.

It is represented in the following form as

$$\begin{array}{l} p \vee q \\ \underline{\ \ \sim p\ \ } \\ \therefore \quad q \end{array}$$

This argument is valid as $(p \vee q) \wedge \sim p \to q$ is a tautology.

2.16.4 Hypothetical Syllogism

It is also known as the transitive rule.

It can be stated as follows

"If $(p \to q)$ and $(q \to r)$ are true then $p \to r$ is true.

This rule is presented in the following form.

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore \ p \to r \end{array}$$

This argument is valid as $(p \to q) \wedge (q \to r) \to (p \to r)$ is a tautology.

Example 2.16.1 *Determine whether the argument is valid or not. If I try hard and I have talent then I will become a musician. If I become a musician, then I will be happy. Therefore if I will not be happy then I did not try hard or I do not have talent.*

SPPU : May-10, Marks 4

Solution :

Let p : I try hard

q : I have talent

r : I will become musician

s : I will be happy

It's symbolic form is as follows

$$\begin{array}{l} s_1 : (p \wedge q) \rightarrow r \\ \underline{s_2 : \quad r \rightarrow s} \\ s_3 : \sim s \rightarrow \sim p \vee \sim q \end{array}$$

To check validity of this statement, one way is to use truth table or prove logically.

Suppose assignment is invalid. This means that for some assignment of truth values s is T, s_2 is T but s_3 is F. s_3 will have truth value F if ~ s is T and $\sim p \vee \sim q$ is F

i.e. s is F and ~ p is F and ~ q is F

i.e. s is F, and p is T and q is T

As s_2 is T, the truth values of r and s both are F. Since s, is T, r is F implies either p or q is F. This is contradiction since by assumption both p and q are true.

Hence given statement is valid.

Example 2.16.2 *Test the validity of the argument. If a person is poor, he is unhappy. If a person is unhappy, he dies young. Therefore poor person dies young.* **SPPU : Dec.-09, Marks 4**

Solution :

Let p : Person is poor

q : Person is unhappy

r : Person dies young

In symbolic form argument is

$$\begin{array}{l} s_1 : p \rightarrow q \\ \underline{s_2 : q \rightarrow r} \\ s : p \rightarrow r \end{array}$$

The above argument is the rule of hypothetical syllogism. Hence it is valid.

Example 2.16.3 *Determine the validity of the following argument.*

s_1 : If I like discrete structure then I will study

s_2 : Either I will study or I will fail

---

s : If I fail then I do not like discrete structure

Solution :

Let p : I like discrete structure

q : I will study

r : I will fail

In symbolic form

$$\begin{array}{l} s_1 : p \rightarrow q \\ \underline{s_2 : q \vee r} \\ s : r \rightarrow \sim p \end{array}$$

We know that for the validity of argument $s_1 \wedge s_2$ should logically imply s.

Assign the truth values T, T, T, to p, q, r respectively. Then s_1 is T and s_2 is also T

But s is $T \rightarrow F$ is F

Hence argument is invalid.

Example 2.16.4 *Determine the validity of the argument*

s_1 : All my friends are musicians

s_2 : John is my friend

s_3 : None of my neighbours are musicians

S : John is not my neighbour

SPPU : May-06

Solution :

Let p : All my friends are musicians

q : John is my friend

r : My neighbours are musicians

s : John is my neighbour

$\therefore$ s_1 : p

s_2 : q

s_3 : $\sim r$

In symbolic form

$$\begin{array}{c} p \\ q \\ \underline{\sim r} \\ \sim s \end{array}$$

As all my friends are musicians and John is my friend $\Rightarrow$ John is musician.

$\therefore$ $p \wedge q \rightarrow$ John is musician

$p \wedge q \wedge \sim r \rightarrow$ John is musician and my neighbours are not musicians

$\therefore$ $p \wedge q \wedge \sim r \rightarrow$ John is not my neighbour

$p \wedge q \wedge \sim r \rightarrow \sim s$ is true.

Therefore given argument is valid.

Example 2.16.5 *I am happy if my program runs. A necessary condition for the program to run is it should be error free. I am not happy. Therefore the program is not error free.*

Solution :

Let p : I am happy

q : My program runs

r : Program should be error free

In symbolic form

$$\begin{array}{ll} & q \rightarrow p \\ & q \rightarrow r \\ & \sim p \\ \hline \therefore & \sim r \end{array}$$

To check validity, consider the truth table

p	q	r	$q \rightarrow p$	$q \rightarrow r$	~ p	$(q \rightarrow p) \wedge (q \rightarrow r) \wedge \sim p$ (I)	(I) $\rightarrow$ ~ r
F	F	T	T	T	T	T	F

Last column is not T

$\therefore$ Given argument is not tautology.

$\therefore$ Given argument is invalid.

2.17 Quantifiers

SPPU : Dec.-08, 09,10, 11, 15, May-15

In grammar a predicate is the word in a sentence which expresses what is said about object. i.e. properties of an object or relation among objects. For example "is a good teacher", "is a clever student" are predicates. In logic predicate has a broaden role than in grammar. Predicate is presented by using a variable x in place of holder. e.g. x is a prime number.

An assertion that contains one or more variables is called a predicate. It's truth value is predicated after assigning truth values to its variables.

A predicate p containing n variables $x_1, x_2, \ldots x_n$ is called an n-place predicate and denoted by $p(x_1, x_2, x_3, \ldots x_n)$. Each variable x_i is called as argument.

The values which the variables may assume constitute a collection is called the universe of discourse or domain of discourse.

There are two types of quantifiers.

2.17.1 Universal Quantifiers

If p(x) is a predicate with x as an argument then the universal quantifier for p(x) is the statement.

"For all values of x, p(x) is true". We denote the phrase "For all" by $\forall$

$\forall$ means for all or for each or for every.

If p(x) is true for all values of x then

$\forall x\, p(x)$ is true.

For example - p(x) : $x \geq 0$, where x is any positive integer.

$\therefore$ The proposition $\forall x\, p(x)$ is true.

However, if x is an integer $\forall x\, p(x)$ is false.

2.17.2 Existential Quantifier

In some situations, we only require that there is at least one value for the predicate is true. Suppose for the predicate p(x), $\forall x\, p(x)$ is false, but there exists at least one value of x for which p(x) is true, then we say that in this proposition, x is bound by existential quantification.

The phrase "there exists an x" is called an existential quantifier.

The symbol "$\exists$" is used to denote the logical quantifier "there exists" or "there exists an x" or "there is x" or "for some x" or "for at least one x".

The existential quantifier for p(x) is denoted by $\exists x\, p(x)$.

2.17.3 Negation of Quantified Statement

Consider the statement $\forall x\, p(x)$. It's negation is "It is not the case that for all x, p(x) is true". This means that for some x = a, p(a) is not true or $\exists$ x s.t. ~ p(x) is true.

Hence the negation of $\forall x\, p(x)$ is logically equivalent to $\exists x\, [\sim p(x)]$.

Sr. No.	Statement	Negation
1.	$\forall x\, p(x)$	$\exists x\, [\sim p(x)]$
2.	$\exists x\, [\sim p(x)]$	$\forall x\, p(x)$
3.	$\forall x [\sim p(x)]$	$\exists x\, p(x)$
4.	$\exists x\, p(x)$	$\forall x [\sim p(x)]$

I) Equivalence involving quantifiers

1) Distributivity of $\exists$ over $\vee$

$$\exists x\, [p(x) \vee Q(x)] \equiv \exists x\, p(x) \vee \exists x\, Q(x)$$

$$\exists x\,[p \vee Q(x)] \equiv p \vee (\exists x\, Q(x))$$

2) Distributivity of $\forall$ over $\wedge$

$$\forall x\,[p(x) \wedge Q(x)] \equiv \forall x\, p(x) \wedge \forall x\, Q(x)$$

$$\forall x\,[p \wedge Q(x)] \equiv p \wedge (\forall x\, Q(x))$$

3) $\exists x\,[p \wedge Q(x)] \equiv p \wedge [\exists x\, Q(x)]$

4) $\forall x\,[p \vee Q(x)] \equiv p \wedge [\forall x\, Q(x)]$

5) $\sim [\exists x p(x)] \equiv \forall x\,[\sim p(x)]$

6) $\sim [\forall x\, p(x)] \equiv \exists x\,[\sim p(x)]$

7) $\forall x\, p(x) \Rightarrow \exists x\, p(x)$

8) $\forall x\, p(x) \;\vee\; \forall x\, Q(x) \Rightarrow \forall x\,(p(x) \vee Q(x)$

9) $\exists x(p(x) \wedge Q(x)) \Rightarrow \exists x\, p(x) \wedge \exists x\, Q(x)$

II) Rules of Inference for addition and deletion of quantifiers

1) Rule 1 : Universal Instantiation

$\dfrac{\forall x\, p(x)}{\therefore\;\; p(k)}$, k is some element of the universe

2) Rule 2 : Existential Instantiation

$\dfrac{\exists x\, p(x)}{\therefore\;\; p(k)}$, k is some element for which p(k) is true.

3) Rule 3 : Universal Generalization

$$\frac{px}{\forall x\, p(x)}$$

4) Rule 4 : Existential Generalization

$$\frac{p(k)}{\therefore\;\; \exists x\, p(x)}$$

k is some element of the universe

III)

Sr. No.	Quantifiers	Expression
1.	$\exists x \forall y\, p(x,y)$	There exists a value of x such that for all values of y, p(x, y) is true.
2.	$\forall y\, \exists x\, p(x,y)$	For each value of y, there exists x such that p(x, y) is true.
3.	$\exists x \exists y\, p(x,y)$	There exist value of x and value of y such that p(x, y) is true.
4.	$\forall x\, \forall y\, p(x,y)$	For all values of x and y. p(x, y) is true.

Example 2.17.1 *Represent the arguments using quantifiers and find its correctness.*

All students in this class understand logic. Ganesh is a student in this class. Therefore Ganesh understands logic. **SPPU : Dec.-11, Marks 4**

Solution :

Let C(x) : x is a student in this class

L(x) : x understands logic

In symbolic form

$$\begin{array}{ll} & \forall x\,(C(x) \rightarrow L(x)) \\ & C(a) \\ \hline \therefore & L(a) \end{array}$$

Here a means Ganesh

This is Modus Ponen

Therefore this argument is valid.

Example 2.17.2 *Let p(x) : x is even*

Q(x) : x is a prime number

R(x, y) : x + y is even

a) Using the information given above write the following sentences in symbolic form.

i) Every integer is an odd integer

ii) Every integer is even or prime

iii) The sum of any two integers is an odd integer. **SPPU : Dec.-10, Marks 4**

Solution :

i) $\forall x\,[\sim p(x)]$

ii) $\forall x\,[p(x) \vee Q(x)]$

iii) $\forall x\,\forall y\,[\sim R(x, y)]$

Example 2.17.3 *Using information in example 2.17.2 write an english sentence for each of the symbolic statement given below*

i) $\forall x\,(\sim Q(x))$

ii) $\exists y(\sim p(y))$

iii) $\sim[\exists x\,(p(x) \wedge Q(x))]$ **SPPU : Dec.-10, Marks 4**

Solution :

i) All integers are not prime numbers

ii) At least one integer is not even.

iii) It is not the case that there exists an integer which is even and prime.

Example 2.17.4 *Determine the validity of the following argument*

s_1 : All my friends are musician

s_2 : John is my friend

s_3 : Name of my neighbours are musician

s : John is not my neighbour.

Solution : Let the universe of discourse be the set of people.

Let F(x) : x is my friend

M(x) : x is a musician

N(x) : x is my neighbour

It's symbolic form is

$$\begin{array}{l} s_1 : \ \forall x\,[F(x) \rightarrow M(x)] \\ s_2 : \ F(a)\ (a = \text{John}] \\ \underline{s_3 : \ \forall x\,[N(x) \rightarrow \sim M(x)]} \\ \qquad \therefore \quad s : \sim N(a) \end{array}$$

Suppose ~N(a) has value F.

$\therefore$ N(a) is T. Since s_3 is T, we must have ~M(a) is T or M(a) is F. But s_1 is T. Hence we must have F(a) to be false but this is contradiction. Hence if s is false either of s_1 or s_3 should be false. Hence argument is valid.

Example 2.17.5 *For the universe of all integers. Let*

p(x) : x > 0

Q(x) : x is even

R(x) : x is a perfect square

S(x) : x is divisible by 4

T(x) : x is divisible by 7

Write the following statement in symbolic form

i) At least one integer is even

ii) There exists a positive integer that is even

iii) If x is even then x is not divisible by 7

iv) No even integer is divisible by 7

v) There exists an even integer divisible by 7

vi) If x is even and x is perfect square then x is divisible by 4.

Solution :

i) $\exists x\, Q(x)$

ii) $\exists x\,[p(x) \wedge Q(x)]$

iii) $\forall x\,[Q(x) \rightarrow \sim T(x)]$

iv) $\forall x [Q(x) \rightarrow \sim T(x)]$

v) $\exists x [Q(x) \wedge T(x)]$

vi) $\forall x [Q(x) \wedge R(x) \rightarrow s(x)]$

Example 2.17.6 *Rewrite the following statements using quantifier variables and predicate symbols.*

i) All birds can fly

ii) Not all birds can fly

iii) Some men are genius

iv) Some numbers are not rational

v) There is a student who likes Maths but not Hindi

vi) Each integer is either even or odd **SPPU : Dec.-08, Marks 4**

Solution :

i) Let $B(x)$: x is a bird

$F(x)$: x can fly

Then the statement can be written as

$\forall x [B(x) \rightarrow F(x)]$

ii) $\exists x [B(x) \wedge \sim F(x)]$

iii) Let $M(x)$: x is a man

$G(x)$: x is a genius

The statement in symbolic form as $\exists x [M(x) \wedge G(x)]$

iv) Let $N(x)$: x is a number

$R(x)$: x is rational

The statement in symbolic form as $\exists x [N(x) \wedge \sim R(x)]$ or $\sim [\forall x (N(x) \rightarrow R(x)]$

v) Let $S(x)$: x is a student

$M(x)$: x likes Maths

$H(x)$: x likes Hindi

$\therefore$ The statement in symbolic form as $\exists x [S(x) \wedge M(x) \wedge \sim H(x)]$

vi) Let $I(x)$: x is an integer

$E(x)$: x is even

$O(x)$: x is odd

The statement in symbolic form as $\forall x [I(x) \rightarrow E(x) \vee O(x)]$

Example 2.17.7 *Negate each of the following statements*

i) $\forall x, |x| = x$

ii) $\exists x, x^2 = x$ **SPPU : Dec.-09, 15, May-15, Marks 4**

Solution :

i) $\exists x, |x| \neq x$ ii) $\forall x, x^2 \neq x$

Example 2.17.8 *Negate the following*

i) If there is a riot, then someone is killed.

ii) It is day light and all the people are arisen. **SPPU : May-15, Dec.-15, Marks 4**

Solution :

i) It is not the case that if there is a riot then someone is killed.

ii) It is not the case that it is day light and all the people are arisen.

OR

i) Let p : There is a riot

q : Someone is killed

Given statement is $p \rightarrow q$

Hence $\sim(p \rightarrow q) \equiv \sim(\sim p \vee q) \equiv p \wedge \sim q$

$\equiv$ There is a riot and someone is not killed.

ii) Let p : It is a day light

q : All the people are arisen

Given statement is $p \wedge q$

Hence $\sim(p \wedge q) = \sim p \vee \sim q$

Hence either it is not a day light or all the people are not arisen.

❑❑❑

Unit - I

3 Mathematical Induction

Syllabus

Proof by Mathematical Induction and Strong Mathematical Induction.

Contents

3.1 Introduction

Mathematical induction is a powerful technique in applied mathematics especially in number theory, where many properties of natural numbers are proved by this method.

In day to day life, we are often required to generalise a particular pattern for the prediction purpose. The generalisation is achieved by using a statement involving a variable as natural number.

Mathematical induction is very useful technique or tool for the programmers to check whether a program statement is loop invariant or not.

There are two principles of mathematical induction :

1) First principle of mathematical induction.
2) Second principle of mathematical induction.

3.2 First Principle of Mathematical Induction Statement

Let P(n) be a statement involving a natural number $n \geq n_0$ such that,

1) If P(n) is true for $n = n_0$ where $n_0 \leq N$ and
2) Assume that P(k) is true for $k = \geq n_0$

We prove P(k+1) is also true,

Then P(n) is true for all natural numbers $n = n_0$.

Step 1 is called as the basis of induction.

Step 2 is called as the induction step.

3.3 Second Principle of Mathematical Induction Statement (Strong Mathematical Induction)

SPPU : Dec.-04, 05, 06, 10, 11, 12, 13, 14, 15, 18, May-05, 06, 07, 08, 14, 15, 17, 18, 19

Let P(n) be a statement involving a natural number $n \geq n_0$ such that,

1) If P(n) is true for $n = n_0$ where $n_0 \leq N$ and
2) Assume that P (n) is true for $n_0 < n \leq k$ i.e. $P(n_0+1)(n_0+2)$............P(k) are true.

we prove that P(k + 1) is true,

Then P(n) is true for all natural numbers $n \geq n_0$.

Example 3.3.1 *Prove that :* $1 + 2 + 3 + 4 \ldots + n = \dfrac{n(n+1)}{2}$

SPPU : May-17, Dec.-18, Marks 4

Solution : Let, P(n) be the given statement

Consider the following steps,

Step 1 : Basis of induction

For $n = 1$, L.H.S. = 1

$$\text{R.H.S.} = \frac{1(1+1)}{2} = 1$$

$\therefore$ For n = 1, L.H.S. = R.H.S.

$\therefore$ P(1) is true.

Step 2 : Assume that P(k) is true

i.e. $1 + 2 + 3 + \ldots\ldots + k = \dfrac{k(k+1)}{2}$ … (1)

Consider, $1 + 2 + 3 + \ldots\ldots + k + (k+ 1)$

$$= \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)+(k+2)}{2}$$

Hence P(k + 1) is true

$\therefore$ By the principle of mathematical induction P(n) is true for all n.

Example 3.3.2 *Prove by mathematical induction for $n \geq 1$.*

$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

SPPU : May-05, Marks 6

Solution : Let P(n) the given statement

1. Basis of induction

For $n_0 = 1$ L.H.S. = 1.2 = 2

$$\text{R.H.S.} = \frac{1(2)(3)}{3} = 2 \Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

Hence P(1) is true.

2. Induction step

Assume that, P(k) is true

i.e. $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + k(k+1) = \dfrac{k(k+1)(k+2)}{3}$ ... (1)

Then we have

$[1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + k(k+1)] + (k+1)(k+2)$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \quad \text{… (Using 1)}$$

$$= (k+1)(k+2)\left[\frac{k}{3}+1\right] = \frac{(k+1)(k+2)(k+3)}{3}$$

Hence assuming P(k) is true, P(k + 1), is also true. Therefore by mathematical induction P(n) is true for all $n \geq 1$.

Example 3.3.3 *Show by induction that* , $n \geq 1$

$$1^2 + 3^2 + 5^2 + ..(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

SPPU : May-14, Marks 4

Solution : Let P(n) be the given statement,

1. Basis of induction :

For n = 1, L.H.S. $= 1^2 = 1$,

$$\text{R.H.S.} = \frac{1(1)(3)}{3} = 1$$

$$\Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

Hence P(1) is true

2. Induction step : Assume that P(k) is true.

i.e. $$1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3} \quad \ldots (1)$$

Hence

$$[1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2] + (2k+1)^2 = \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2 \quad \ldots \text{(Using 1)}$$

$$= \frac{(2k+1)}{3}[2k^2 - k + 3(2k+1)]$$

$$= \frac{(2k+1)}{3}[2k^2 + 5k + 3]$$

$$= \frac{(2k+1)}{3}[2k^2 + 2k + 3k + 3]$$

$$= \frac{(2k+1)}{3}[(2k+3)(k+1)]$$

$$= \frac{(k+1)(2k+1)(2k+3)}{3}$$

$$= \frac{(k+1)[2(k+1)-1][2(k+1)+1]}{3}$$

Hence assuming P(k) is true P(k + 1) is also true. Therefore by mathematical induction P(n) in true for all $n \geq 1$.

Example 3.3.4 *Show that* $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4} = (1+2+3+\ldots+n)^2$

SPPU : Dec.-12, May-18, Marks 4

Solution : Let P(n) be the given statement,

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2 (n+1)^2}{4}$$

1. Basis of induction :

For n = 1, L.H.S. = 1,

$$\text{R.H.S.} = \frac{1(1+1)^2}{4} = 1$$

$\Rightarrow$ L.H.S. = R.H.S.

Hence P(1) is true.

2. Induction step : Assume that P(k) is true.

i.e. $$1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^2 (k+1)^2}{4} \quad \ldots (1)$$

Then we have

$$\begin{aligned}(1^3 + 2^3 + 3^3 + \ldots + k^3) + (k+1)^3 &= (1+2+3+\ldots+k)^2 + (k+1)^3 \quad \text{(Using 1)} \\ &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\ &= (k+1)^2 \left[\frac{k^2}{4} + k + 1\right] \\ &= (k+1)^2 \left[\frac{k^2 + 4k + 4}{4}\right] \\ &= \frac{(k+1)^2 (k+2)^2}{4} \\ &= \left(\frac{(k+1)(k+2)}{2}\right)^2 = \frac{(k+1)^2 (k+2)^2}{4}\end{aligned}$$

Hence assuming P(k) is true, P(k + 1) is also true. Therefore by mathematical induction P(n) is true for all n ≥ 1.

Example 3.3.5 *Show that* $\frac{1^2}{1 \cdot 3} + \frac{2^2}{3 \cdot 5} + \ldots + \frac{n^2}{(2n-1)(2n+1)} = \frac{n(n+1)}{2(2n+1)}$

Solution : Let P(n) be the given statement,

1. Basis of induction : For n = 1

We have, $\text{L.H.S.} = \frac{1}{1.3} = \frac{1}{3}$

$\text{R.H.S.} = \frac{1\,(2)}{1(3)} = \frac{1}{3}$

$\Rightarrow$ L.H.S. = R.H.S.

Hence P(1) is true.

2. Induction step : Assume that P(k) is true

i.e.
$$\frac{1^2}{1\cdot 3}+\frac{2^2}{3\cdot 5}+\ldots\frac{k^2}{(2k-1)\,(2k+1)} = \frac{k\,(k+1)}{2\,(2k+1)} \qquad \ldots (1)$$

Then we have,

$$\left[\frac{1^2}{1\cdot 3}+\frac{2^2}{3\cdot 5}+\ldots+\frac{k^2}{(2k-1)\,(2k+1)}\right]+\frac{(k+1)^2}{[2\,(k+1)-1][2\,(k+1)+1]}$$

$$= \frac{k\,(k+1)}{2\,(2k+1)}+\frac{(k+1)^2}{(2k+1)\,(2k+3)} \qquad \ldots \text{(Using 1)}$$

$$= \frac{(k+1)}{(2k+1)}\left[\frac{k(2k+3)+2\,(k+1)}{2\,(2k+3)}\right]$$

$$= \frac{(k+1)}{2k+1}\left[\frac{2k^2+5k+2}{2\,(2k+3)}\right]$$

$$= \frac{(k+1)}{(2k+1)}\left[\frac{2k^2+4k+k+2}{2\,(2k+3)}\right]$$

$$= \frac{(k+1)}{(2k+1)}\left[\frac{2k\,(k+2)+1\,(k+2)}{2\,(2k+3)}\right]$$

$$= \frac{(k+1)\,(k+2)}{2\,(2k+3)}$$

$$= \frac{(k+1)\,[(k+1)+1]}{2\,[2(k+1)+1]}$$

$\therefore$ P(k + 1) is true.

Therefore by mathematical induction P(n) is true for all $n \geq 1$.

Example 3.3.6 *Show that a)* $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n\,(n+1)} = \frac{n}{n+1}$

b) Show that $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

c) Show that $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\ldots+\frac{1}{(3n-1)(3n+1)} = \frac{n}{3n+1}$

SPPU : Dec.-05, Marks 6

Solution : Let P(n) be the given statement,

a) 1. Basis of induction :

For n = 1 L.H.S. $= \frac{1}{1.2} = \frac{1}{2}$

R.H.S. $= \frac{1}{1+1} = \frac{1}{2}$

L.H.S. = R.H.S.

Hence P(1) is true.

2. Induction step : Assume that P(k) is true.

i.e.
$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k\,(k+1)} = \frac{k}{k+1} \qquad \ldots (1)$$

Then we have

$$\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k\,(k+1)}\right]+\frac{1}{(k+1)\,(k+2)} = \frac{k}{k+1}+\frac{1}{(k+1)(k+2)} \qquad \ldots \text{(Using 1)}$$

$$= \frac{k\,(k+2)+1}{(k+1)\,(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)\,(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)\,(k+2)}$$

$$= \frac{k+1}{k+2} = \frac{(k+1)}{(k+1)+1}$$

Hence assuming P(k) is true, P(k + 1) is also true. Therefore P(n) in true for all n ≥ 1.

b) Let P(n) : $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2\,n-1)(2n+1)}=\frac{n}{2n+1}$

1. Basis of induction : For n = 1

$$\text{L.H.S.} = \frac{1}{1.3}=\frac{1}{3},\ \text{R.H.S.}=\frac{1}{3}$$

$$\frac{1}{3} = \frac{1}{2.1+1}$$

$$\Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

Hence P(1) is true.

2. Induction step : Assume that P(k) is true

i.e. $$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots.+\frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1} \quad \ldots (1)$$

Then we have,

$$\left[\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ldots+\frac{1}{(2k-1)(2k+1)}\right]+\frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2+3k+1}{(2k+1)\,(2k+3)}$$

$$= \frac{(2k+1)\,(k+1)}{(2k+1)(2k+3)}$$

$$= \frac{k+1}{2k+3}$$

$$= \frac{k+1}{2(k+1)+1}$$

Hence assuming P(k) is true. P(k + 1) is also true. Therefore P(n) is true for all n ≥ 1.

c) Let P(n) : $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$

1. Basis of induction

For n = 1, $$\text{L.H.S.} = \frac{1}{1.4}=\frac{1}{4},\ \text{R.H.S.}=\frac{1}{4}$$

$$\frac{1}{1.4} = \frac{1}{3.1+1}$$

$\Rightarrow$ L.H.S. = R.H.S.

Hence P(1) is true.

2. Induction step : Assume that P(k) is true.

i.e. $$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\ldots+\frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1} \quad \ldots (1)$$

Then we have,

$$\left[\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\ldots+\frac{1}{(3k-2)(3k+1)}\right]+\frac{1}{(3k+1)\,(3k+4)} \quad \ldots \text{(Using 1)}$$

$$= \frac{k}{(3k+1)}+\frac{1}{(3k+1)(3k+4)}$$

$$= \frac{3k^2+4k+1}{(3k+1)\,(3k+4)}$$

$$= \frac{(3k+1)\,(k+1)}{(3k+1)(3(k+1)+1)}$$

$$= \frac{k+1}{3(k+1)+1}$$

Hence assuming P(k) is true. P(k + 1) is also true. Therefore P(n) is true for all n $\geq$ 1.

Example 3.3.7 *Prove by induction for $n \geq 0$. $1+a+a^2+\ldots+a^n = \dfrac{1-a^{n+1}}{1-a}$.*

SPPU : Dec.-10, Marks 4

Solution : Let P(n) be the given statement,

1. Basis of induction

For n = 0, L.H.S. = 1, R.H.S. = $\dfrac{1-a}{1-a}$ = 1

For n = 1, L.H.S. = 1 + a, R.H.S. = $\dfrac{1-a^2}{1-a}$ = 1 + a

$\therefore$ For n = 0, 1, L.H.S. = R.H.S.

Hence P(0), P(1) are true.

2. Induction step : Assume that P(k) is true

$$\therefore \quad 1+a+a^2+\ldots+a^k = \frac{1-a^{k+1}}{1-a} \quad \ldots (1)$$

Consider,

$$1+a+a^2+....+a^k+a^{k+1} = \frac{1-a^{k+1}}{1-a}+a^{k+1} \qquad \text{... (Using 1)}$$

$$= \frac{1-a^{k+1}+(1-a)\,a^{k+1}}{1-a}$$

$$= \frac{1-a^{k+1}+a^{k+1}-a^{k+2}}{1-a}$$

$$= \frac{1-a^{k+2}}{1-a}$$

Hence P(k + 1) is true.

Therefore by the mathematical induction P(n) is true for all $n \geq 0$.

Example 3.3.8 *Use mathematical induction to show that $n(n^2-1)$ is divisible by 24. Where n is any odd positive number.* **SPPU : Dec.-14, Marks 4**

Solution : If $n(n^2-1) = n^3-n$ is divisible by 24.

Then $n^3-n = 24\,(m)$ where m is any positive integral.

Let P(n) be the given statement,

1. Induction step : For n = 1,

$$n(n^2-1) = 0 \text{ which is divisible by 24.}$$

For n = 3, $n\left(n^2-1\right) = 24$ which is divisible by 24.

∴ P(1) and P(3) is true.

2. Induction step :

Assume that P(k) is true.

i.e. $k(k^2-1) = k^3-k$ is divisible by 24.

∴ $k(k^2-1) = k^3-k = 24\,(m_0),\ m_0 \leftarrow z$ ... (1)

Consider

$$(k+1)\,[(k+1)^2-1] = (k+1)^3-(k+1)$$

$$= k^3+3k^2+3k+1-k-1$$

$$= k^3+3k^2+2k$$

$$= (k^3-k)+3k^2+3k \qquad \text{... (Using 1)}$$

$$= 24\,m_0 + 3k\,(k+1) \quad \text{(As k(k + 1) is multiple of 8 for k odd positive integer and } k \geq 3)$$

$$= 24m_0 + 3(8\,m_1)$$

$$= 24\,(m_0 + m_1)$$

$$= 24\,m_2 \;\; (\because m_0 + m_1 = m_2)$$

$\therefore$ P(k + 1) is true.

$\therefore$ By mathematical induction P(n) is true for all n odd positive number.

Example 3.3.9 *Show that $n^4 - 4\,n^2$ is divisible by 3 for all $n \geq 2$.* **SPPU : Dec.-15, Marks 4**

Solution : Let P(n) be the given statement,

1. Basis of induction

For $n = 2$

$$2^4 - 4\,(2^2) = 16 - 16$$

$= 0$ is divisible by 3 as 0 is divisible by every number

$\therefore$ P(2) is true.

2. Induction step : Assume that P(k) is true

i.e. $k^4 - 4k^2$ is divisible by 3

Then we have,

$$(k+1)^4 - 4(k+1)^2 = k^4 + 4k^3 + 6k^2 + 4k + 1 - 4\,(k^2 + 2k + 1)$$

$$= (k^4 - 4k^2) + 4\,(k^3 + 2k) + 6k^2 + 12k - 3$$

$k^4 - 4\,k^2$ is divisible by 3.

$k^3 + 2k$ is divisible by 3

Also $6\,k^2 + 12\,k - 3 = 3\,(2k^2 + 4k - 1)$ is divisible by 3.

Hence $(k+1)^4 - 4\,(k+1)^2$ is divisible by 3.

Hence assuming P(k) is true.

P(k + 1) in also true. Therefore P(n) is true for $n \geq 2$.

Example 3.3.10 *Prove that : $7^{2n} + 2^{3n-3} \cdot 3^{n-1}$ is divisible by 25 $\forall n$.*

SPPU : May-19, Marks 3

Solution : Let P(n) be the given statement.

1) Basis of induction : For n = 1,

$7^2 + 2^0 \cdot 3^0$ = 49 + 1 = 50, which is divisible by 25.

$\therefore$ P(1) is true.

2) Induction step :

Assume that P(k) is true.

i.e. $7^{2k} + 2^{3k-3}\, 3^{k-1} = 25\ (m)$

Consider

$$\begin{aligned}
7^{2(k+1)} + 2^{3(k+1)-3} \cdot 3^{k+1-1} &= 7^{2k+2} + 2^{3k}\, 3^k \\
&= 49 7^{2k} + 2^{3k} \cdot 3k \\
&= 49[25\,m - 2^{3k-3} \cdot 3^{k-1}] + 2^{3k} \cdot 3^k \\
&= 25\ (49m) - 49\ 2^{3k-3} \cdot 3^{k-1} + 2^{3k-3} \cdot 3^{k-1}(8 \times 3) \\
&= 25\ (49m) + 2^{3k-3} \cdot 3^{k-1}\ (-49 + 24) \\
&= 25(49\,m) + 2^{3k-3} 3^{k-1}(-25) \\
&= 25\ [49\,m - 2^{3k-3} \cdot 3^{k-1}] \\
&= 25\ (P) : P \in N
\end{aligned}$$

$\therefore$ P(k+1) is true.

$\therefore$ By mathematical induction, P(n) is true for all n.

Example 3.3.11 *Using mathematical induction, prove that*

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n-1} n^2 = (-1)^{n-1} \frac{n(n+1)}{2}$$

SPPU : Dec.-04, Marks 6

Solution : Let P(n) be the given statement,

1. Basis of induction

For n = 1 L.H.S = 1, R.H.S. = 1

$\Rightarrow$ L.H.S. = R.H.S.

Hence P(1) is true.

2. Induction step : Assume that P(k) in true.

i.e. $1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k-1} k^2 = (-1)^{k-1} \dfrac{k(k+1)}{2}$ ... (1)

Then we have,

$[1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k-1} k^2] + (-1)^k (k+1)^2$ ... (Using 1)

$$= (-1)^{k-1}\frac{k(k+1)}{2} + (-1)^k(k+1)^2$$

$$= (-1)^k\ (k+1)\left[-\frac{k}{2} + (k+1)\right]$$

$$= (-1)^k\ (k+1)\left[\frac{-k+2k+2}{2}\right]$$

$$= (-1)^k\frac{(k+1)(k+2)}{2}$$

Hence assuming P(k) is true, P(k + 1) is also true. Therefore P(n) in true for all $n \geq 1$.

Example 3.3.12 *Prove by mathematical induction that for $n \geq 1$:*
$1\cdot 1! + 2\cdot 2! + 3\cdot 3\,! + \ldots + n\cdot n\,! = (n+1)! - 1.$ **SPPU : May-08, 15, Marks 6**

Solution : Let P(n) be the given statement,

1. Basis of induction

For n = 1, L.H.S = 1, R.H.S. = 1

$\Rightarrow$ L.H.S. = R.H.S.

Hence P(1) is true.

2. Induction step : Assume that, P(k) is true.

i.e. $1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \ldots + k\cdot k!$
$= (k + 1)! - 1$ ... (1)

Then we have,

$[1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \ldots + k\cdot k!] + (k + 1)\cdot(k + 1)!$ ... (Using 1)

$$= [(k + 1)! - 1] + (k + 1)\cdot(k + 1)!$$

$$= (k + 1)! + (k + 1)\cdot(k + 1)! - 1$$

$$= (k + 1)!\ [k + 1 + 1] - 1$$

$$= (k + 2)\ (k + 1)! - 1$$

$$= (k + 2)! - 1$$

Hence assuming P(k) is true, P(k + 1) is also true. Therefore P(n) is true for $n \geq 1$.

Example 3.3.13 *Prove that for any positive integer n the number $n^5 - n$ is divisible by 5.* **SPPU : Dec.-08, Marks 6**

Solution : Let P(n) be the given statement,

1. Basis of induction :

For n = 1, $1^5 - 1 = 0$ is divisible by 5.

As 0 is divisible by every number.

Hence P(1) is true.

2. Induction step : Assume that, P(k) is true.

i.e. $k^5 - k$ is divisible by 5

Then we have

$$(k+1)^5 - (k+1) = (k^5 + {}^5C_1k^4 + {}^5C_2k^3 + {}^5C_3k^2 + {}^5C_4k + {}^5C_5) - (k+1)$$
$$= k^5 + 5\,k^4 + 10\,k^3 + 10\,k^2 + 5k + 1 - k - 1$$
$$= (k^5 - k) + 5\,[k^4 + 2k^3 + 2k^2 + k]$$

$k^5 - k$ is divisible by 5.

and $5\,(k^4 + 2k^3 + 2k^2 + k)$ is divisible by 5.

Hence $(k+1)^5 - (k+1)$ is divisible by 5.

Hence assuming P(k) is true. P(k + 1) is also true. Therefore P(n) is true for n ≥ 1.

Example 3.3.14 *Prove that $8^n - 3^n$ is a multiple of 5 by mathematical induction for $n \geq 1$.*

SPPU : May-06, 07, Dec.-13, Marks 6

Solution : Let P(n) be the given statement,

1. Basis of induction :

For n = 1 $\quad 8^1 - 3^1 = 5$

$$= 5 \cdot 1$$

Obviously a multiple of 5.

∴ P(1) is true.

2. Induction step : Assume that, P(k) in true.

i.e. $8^k - 3^k$ is multiple of 5 say 5 r

i.e. $$8^k - 3^k = 5\,r \qquad \text{... (1)}$$

where r is an integer

Then we have,

$$8^{k+1} - 3^{k+1} = 8^k \cdot 8 - 3^k \cdot 3$$
$$= 8^k \cdot (5+3) - 3^k \cdot 3$$
$$= 8^k \cdot 5 + (8^k \cdot 3 - 3^k \cdot 3)$$
$$= 8^k \cdot 5 + 3(8^k - 3^k)$$

Obviously $8^k \cdot 5$ is multiple of 5 and also $8^k - 3^k$ is multiple of 5.

Therefore, $8^{k+1} - 3^{k+1}$ is multiple of 5.

Hence assuming P(k) is true, P(k + 1) is also true. Therefore P(n) is true for all n ≥ 1.

Example 3.3.15 *Show that the sum of the cubes of three consecutive natural number is divisible by 9.* **SPPU : Dec.-06, Marks 6**

Solution : Let n, n + 1, n + 2 be three consecutive natural numbers.

We have to show that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

Let P(n) be the above statement,

1. Basis of induction : For n = 1

$$1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36 \quad \text{which is divisible by 9.}$$

$\therefore$ P(1) is true.

2. Induction step : Assume that P(k) is true.

i.e. $k^3 + (k+1)^3 + (k+2)^3$ in divisible by 9.

Then we have,

$$\begin{aligned}(k+1)^3 + (k+2)^3 + (k+3)^3 &= [(k+1)^3 + (k+2)^3] + [k^3 + {}^3C_1 k^2 (3) + {}^3C_2 k (3)^2 + {}^3C_3 (3^3)] \\ &= (k+1)^3 + (k+2)^3 + k^3 + [9k^2 + 27k + 27] \\ &= [k^3 + (k+1)^3 + (k+2)^3] + 9[k^2 + 3k + 3]\end{aligned}$$

$k^3 + (k+1)^3 + (k+2)^3$ is divisible by 9 and $9(k^3 + 3k + 3)$ is divisible by 9.

$\Rightarrow (k+1)^3 + (k+2)^3 + (k+3)^3$ is divisible by 9.

Hence assuming P(k) in true, P(k + 1) is also true. Therefore P(n) is true for all n ≥ 1.

Example 3.3.16 *Using mathematical induction prove that* $3 + 3.5 + 3.5^2 + \ldots. + 3.5^n \cdot \left(\frac{5^{n+1} - 1}{4}\right)$.

For non-negative number n. **SPPU : Dec.-11, Marks 6**

Solution : Cancelling 3 from the both sides of given.

Statement, We get

$$1 + 5 + 5^2 + \ldots + 5^n = \frac{5^{n+1} - 1}{5 - 1} \qquad \ldots (1)$$

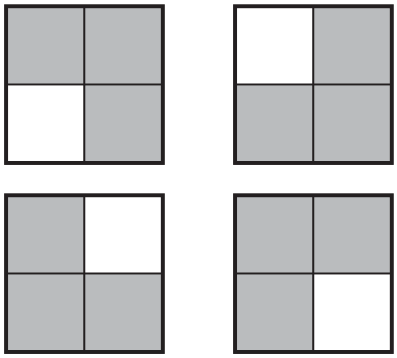
Let P(n) be the above statement.

To prove this refer example 3.3.7 for a = 5.

Example 3.3.17 *Let n be a positive integer. Show that any $2^n \times 2^n$ chessboard with one square removed can be covered by L-shaped pieces, where each piece covers three squares at a time.*

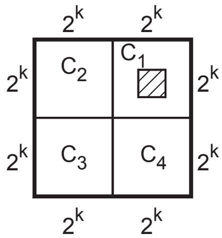
Solution : Let P(n) be the proposition that any $2^n \times 2^n$ chessboard with one square removed can be covered using L-shaped pieces.

Basis of induction : For n = 1, P(1) implies that any 2×2 chessboard with one square removed can be covered using L shaped pieces. P(1) is true, as seen below.

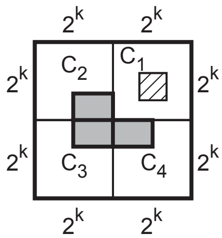


Induction step : Assume that, P(k) is true i.e. any $2^k \times 2^k$ chessboard with one square removed can be covered using L-shaped pieces.

Then, we have to show that P(k + 1) is true. For this consider, a $2^{k+1} \times 2^{k+1}$ chesshoard with one square removed. Divide the chessboard into four equal halves of size $2^k \times 2^k$, as shown below.



The square which has been removed, would have been removed from one of the four chessboards, say C_1. Then by induction hypothesis, C_1 can be covered using L-shaped pieces. Now, from each of the remaining chessboards, remove that particular piece (or tile), lying at the centre of the large chessboards.



Then by induction hypothesis, each of these $2^k \times 2^k$ chessboards with a piece (or tile) removed can be covered by the L-shaped pieces. Also the three tiles removed from the centre can be covered by one L-shaped piece. Hence the chessboard of $2^{k+1} \times 2^{k+1}$ can be covered by L-shaped pieces.

Hence proved.

Example 3.3.18 *Suppose we have unlimited stamps of two different denominations, 3 rupees and 5 rupees. We want to show that it is possible to make up exactly any postage of 8 rupees or more using stamps of these two denominations.*

Solution : For k = 8, we have one 5 rupees stamp and one 3 rupees stamp.

For k = 9, replace 5 rupees stamp by two 3 rupees stamp, similarly for k = 10, replace all 3, 3 rupees stamps by, two 5 rupees stamp an so on.

Hence let us assume that, it is possible to make up k rupees stamp using 3 rupees and 5 rupees stamps (for $k \geq 8$).

Now we have to show that it is also possible to make up (k + 1) rupees stamps using 3 rupees and 5 rupees stamps.

We examine two cases :

1) Suppose we make up stamps of k rupees using at least one 5 rupees stamp. Replacing a 5 rupees stamp by two 3 rupees stamp, we can make up k + 1 rupees stamps.
2) Suppose we make up a stamp of k rupees using 3 rupees only. Since $k \geq 8$ we must have at least 3, 3 rupees stamps. Replacing these 3, 3 rupees stamps by two five rupees stamps. We can make up stamps of k + 1 rupees.

Hence proved.

Example 3.3.19 *The king summoned the best mathematicians in the kingdom to the palace to find out how smart they were. The king told them*
"I have placed white hats on some of you and black hats on the others. You may look at, but not talk, to, one another. I will leave now and will come back every hour on the hour. Every time I return, I want those of you who have determined that you are wearing white hats to come up and tell me immediately."
As it turned out, at the n^{th} hour every one of the n mathematician who were given white hats informed the king that she knew that she was wearing a white hat. Why ?

Solution :

1. Basis of induction : For n = 1, there is only one mathematician wearing a white hat. Since the king said that white hats were placed on some one of the mathematician (king never lie). The mathematician who saw that all other mathematicians had on black hats would realize immediately that she was wearing a white hat. Consequently she would inform the king on the first hour. (when the king returned for the first time) that she was wearing a white hat.

Now let n = 2, i.e. two mathematicians wearing white hats.

Consider, one of these two mathematicians. She saw that one of her colleagues was wearing a white hat. She reasoned that if she were wearing a black hat, her colleague would be the only one wearing a white hat. In that case, her colleague would have figured out the situation and informed the king on the first hour. That did not happen shows that she was also wearing a white hat. Conesquently she told the king on the second hour (and so did the other mathematician with a white hat, since all the mathematicians are smart).

2. Induction step : Assume that, if there were k mathematicians wearing white hats, then they would have figured out that they were wearing white hats and informed the king so on the k^{th} hour. Now, suppose that there were k + 1 mathematicians wearing white hats.

Every mathematician wearing a white hat saw that k of her colleagues were wearing white hats. However, that her k colleagures did not inform the king of their findings on the k^{th} hour can only imply that there were more then k. People wearing white hats. Consequently she knew that, she must be wearing a white hat also on the $(k+1)^{th}$ hour. She (together with all other mathematicians wearing white hats) would inform the king their conclusion.

❑❑❑

Unit - II

4 Relations

Syllabus

***Relations** and their Properties, n-ary relations and their applications, Representing relations, Closures of relations, Equivalence relations, Partial orderings, Partitions, Hasse diagram, Lattices, Chains and Anti-Chains, Transitive closure and Warshall's algorithm.*

Contents

4.1 Introduction

In the chapter 1, we dealt with sets, elements, different types and general properties of sets. There may exists various relationships among elements of sets. We will study relations and functions with various properties and examples.

Relation may involves equality or inequality of elements. In mathematics the expressions like " is equal to " is similar to ", " is greater than", "is parallel to " are some relations.

4.2 Cartesian Product

Let A and B be two non empty sets. The cartesian product of A and B is denoted by A × B and defined as

$$A \times B = \{(a, b) / a \in A \text{ and } b \in B\}$$

e.g. If $A = \{1, 2, 3, 4\}$, $B = \{a, b\}$ then

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a)\ (3, b)\ (4, a)\ (4, b)\}$$

$$B \times A = \{(a, 1), (a, 2), (a, 3), (a, 4), (b, 1), (b, 2), (b, 3), (b, 4)\}$$

$$(a, b) \notin A \times B \text{ or } B \times A$$

and $(1, a) \neq (a, 1)$

Hence $A \times B \neq B \times A$

So the cartesian product is not commutative.

Similarly,

$$A \times B \times C = \{(a, b, c) / a \in A \text{ and } b \in B \text{ and } c \in C\}$$

In general

$$A_1 \times A_2 \times \ldots\ldots \times A_n = \{a_{1,}\ a_2, a_3 \ldots\ldots\ldots\ldots, a_n) / a_1 \in A_1 \text{ and } a_2 \in A_2, \ldots. \text{ and } a_n \in A_n)\}$$

Theorem 1 : If $|A| = m$ and $|B| = n$ then $|A \times B| = |B \times A| = mn$

Proof : The proof is trivial by set theory.

Theorem 2 : If A, B, C are non empty sets then $A \subseteq B \Rightarrow A \times C \subseteq B \times C$

Proof : Let (x,y) be any element in $A \times C$, then

$\Rightarrow$ $x \in B$ and $y \in C$

$\Rightarrow$ $(x, y) \in B \times C$

Hence $A \times C \subseteq B \times C$

Theorem 3 : If A, B, C, are sets then

i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

ii) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

iii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

iv) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Proof :

i)
$$\begin{aligned} A \times (B \cap C) &= \{(x, y) / x \in A \text{ and } (y \in B \cap C)\} \\ &= \{(x, y) / x \in A \text{ and } / (y \notin B \text{ and } y \in C)\} \\ &= \{(x, y) / x \in A, y \in B \text{ and } y \in A, y \in C\} \\ &= \{(x, y) / (x, y) \in A \times B \text{ and } (x, y) \in A \times C\} \\ &= \{(x, y) / (x, y) \in (A \times B) \cap (A \times C)\} \\ A \times (B \cap C) &= (A \times B) \cap (A \times C) \end{aligned}$$

Similarly students can prove remaining results.

Note : If $\Re$ is the set of real numbers then

$$\Re \times \Re = \Re^2 \rightarrow \text{Euclidean plane}$$

$$\Re \times \Re \times \Re = \Re^3 \rightarrow \text{3D space}$$

4.3 Relation

Let A and B be two non empty sets. A relation from A to B is any subset of $A \times B$. It is denoted by $R : A \rightarrow B$

e.g. Let $A = \{x,y,z\}, \ B = \{1, 2, 3)$

then $A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3), (z, 1), (z, 2) (z, 3)\}$

$R_1 = \{(x,1), (y,2), (z,1),)\}, \ R_2 = \{(z,3)\}, \ R_3 = \{\phi\}$

are relations from A to B

But $R_4 = \{(1, x), (2, x)\}$is not relation from A to B. R_4 is the relation from B to A.

Important Results :

1) If $(a, b) \in R$ then it is denoted by a R b.
2) If R is a relation from A to B then $R \subseteq A \times B$
3) If $R \subseteq A \times B$ then R is a relation from A to A and R is called a relation on A.
4) If R is a relation from A to B, then the set of all first elements of the ordered pairs $(a,b) \in R$ is called the domain of R. It is denoted by $D(R) = \{a/(a, b) \in R\}$.

The range of R is the set of all second co-ordinates of the ordered pairs (a,b) ∈ R. It is denoted by R (R) = {(b/(a, b) ∈ R}.

5) The null set is the subset of $A \times B$.

∴ ϕ is a relation called null relation or empty relation.

Example :

Let A = {1, 2, 3}, B = {x, y}

then R_1 = {1, x) (1, y), (3, x)} is a relation from A to B

$$\therefore \quad D(R_1) = \{1,3\}$$

$$R(R_1) = \{x, y\}$$

4.4 Matrix Representation of a Relation

Let $A = \{a_1, a_2, a_3 \ldots. a_n\}$, $B = \{b_1, b_2, b_3, \ldots. b_m\}$ and $R \subseteq A \times B$. Then the relation matrix of R is denoted by $M_R = [m_{ij}]_{n \times m}$ and defined by

$$m_{ij} = \begin{cases} 0 & \text{if } (a_i, b_j) \notin R \text{ i.e. } a_i \not R b_j \\ 1 & \text{if } (a_i, b_j) \in R \text{ i.e } a_i R b_j \end{cases}$$

Example 4.4.1 *Let A = {1, 2, 3, 4} and R = {(x, y)/ x < y } then find* M_R.

Solution : R = {(1, 2) (1, 3) (2, 3) (1, 4) (2, 4) (3, 4) }

$$\therefore \quad M_R = [M_{ij}]_{4 \times 4} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Example 4.4.2 *If A = {1, 2, 3, 4, 5, 6} and a R b iff a divides b for a, b ∈ A. Find relation matrix.*

Solution : R = {(1, 2} (1, 3), (1, 4), (1, 5), (1, 6), (2, 4) (2, 6), (3, 6), (1, 1) (2, 2), (3, 3), (4, 4), (5, 5) (6, 6)}

$$\therefore \quad \text{Relation Matrix} = M_R = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{6 \times 6} \end{array}$$

Example 4.4.3 *A = {x, y, z} Find the relation matrices of the following relations.*
i) R_1 = {(x, x) (y, y) (z, z)} ii) R_2 = {(x, y) (y, x), (y, z), (z, y)} iii) R_3 = {(x, x) (x, y), (x, z) (y, x) (y, y) (y, z), (z, x) (z, y) (z, z)

Solution : (5,5) (6,6)

i) $$M_{R_1} = \begin{array}{c} \begin{matrix} x & y & z \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \end{array}$$

ii) $$M_{R_2} = \begin{array}{c} \begin{matrix} x & y & z \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

iii) $$M_{R_3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

4.4.1 Relation Matrix Operations

We know that relation matrix is a boolean matrix as all entries are either 0 or 1

I) Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ be two relation matrices then

$A + B = [a_{ij} + b_{ij}] = [c_{ij}]_{m\times n}$

where c_{ij} = 1 if $a_{ij} = 1$, or $b_{ij} = 1$

= 0 if $a_{ij} = 0$, and $b_{ij} = 0$

II) Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times k}$ then $A \cdot B = [a_{ij}; b_{jk}] = [d_{jk}]$

where d_{jk} = 1 if $a_{ij} = 1$ and $b_{jk} = 1$

= 0 if $a_{ij} = 0$ or $b_{jk} = 0$

e.g. If $$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

then $$A + B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and $$A \cdot B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

4.4.2 Properties of Relation Matrix

Let R_1 be a relation matrix from A to B and R_2 be a relation matrix from B to C then relation matrices satisfy the following properties :

i) $M_{R_1 \cdot R_2} = M_{R_1} \cdot M_{R_2}$

ii) $M_{R_1^{-1}} =$ Transpose of M_{R_1}

iii) $M_{(R_1 R_2)^{-1}} = M_{R_2^{-1}} \; M_{R_1^{-1}}$

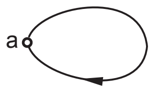
4.5 Diagraphs

A relation can be represented pictorially by drawing its graph. Let A be any non empty set and R be a relation on A. R can be represented by graphically by the following procedure.

i) The elements of a set A are represented by small circles or point i.e. (∘) or ... These elements are called as vertices or nodes.

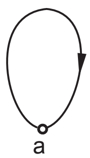
ii) If (a, b) ∈ R or a R b then vertices a and b are joined by a continuous arc with an arrow from a to b. These arcs are known as edges of the graph i.e. if aRb then

iii) If aRa then the vertex a is joined to itself by a loop around a. e.g. if aRa then



This graphical representation of the relation is called as diagraph or directed graph of R. Let us consider the following examples.

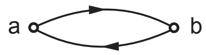
1) aRa



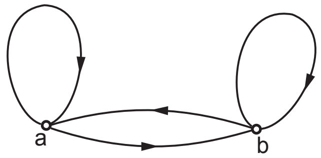
2) aRb

3) bRa

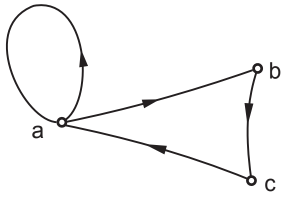
4) aRb ∧ bRa



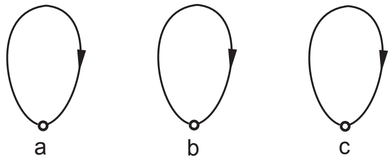
5) aRa $\wedge$ aRb $\wedge$ bRa $\wedge$ bRb



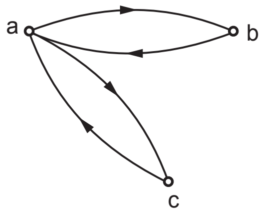
6) aRb $\wedge$ bRc $\wedge$ CRa $\wedge$ aRa



7) aRa $\wedge$ bRb $\wedge$ cRc



8) aRb $\wedge$ bRa $\wedge$ aRc $\wedge$ cRa



Examples :

Example 4.5.1 *A = {1,2,3,4,5,6} and a R_b iff a/b (a divides b). Draw diagraph of R.*

Solution : We have

$$R = \left\{ \begin{array}{c} (1,1)(2,2)(3,3)(4,4)(5,5)(6,6) \\ (1,2)(1,3)(1,4)(1,5)(1,6) \\ (2,4)(2,6)(3,6) \end{array} \right\}$$

The diagraph of R is as follows :

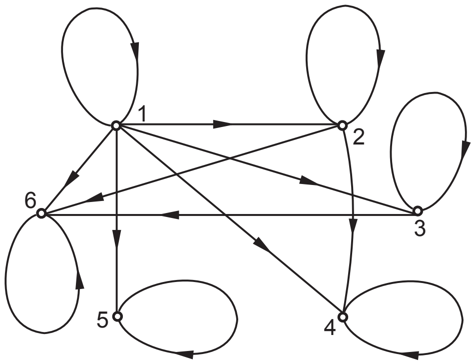


Fig. 4.5.1

Example 4.5.2

Let A = {a, b, c, d} and $M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ *Draw diagraph of R.*

Solution : We have

R = {(a,a) (a,d) (b,b) b,d) (c,a) (c,b) (d,a)}

The digraph of R is as follows :

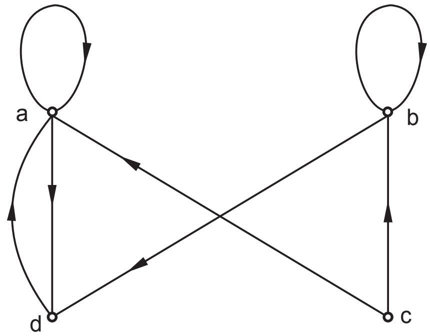


Fig. 4.5.2

Example 4.5.3 *Find the relation from the diagraph.*

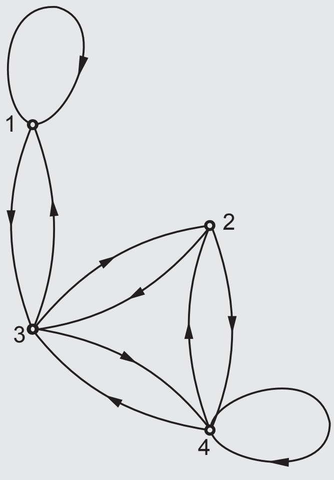


Fig. 4.5.3

Solution : From the given diagraph, the corresponding relation on A = {1, 2, 3, 4} is

R = {(1, 1) (1, 3) (3, 1) (2, 3) (3, 2), (2, 4) (4, 2) (4, 4) (3, 4) (4, 3)}

4.6 Special Types of Relations

4.6.1 Inverse Relation (OR Converse Relation)

Let R be a relation from A to B. The inverse relation of R is denoted by R^{-1} is a relation from B to A defined as

$$R^{-1} = \{(y, x) \mid y \in B, x \in A \text{ and } (x,y) \in R\}$$

i.e. if x R y then $yR^{-1}x$

It is also denoted by R^c

It is also known as the converse relation.

Example :

1) If R = {(1,2) (2,3) (3,1) (4,5) } then R^{-1} or R^c= {(2,1) (3,2) (1,3) (5,4)}

2) If $M_R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ then $M_{R^{-1}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = [M_R]^t$

3) If the diagraph R is given by

$\therefore$ R = {(1,1) (1,2) (2,3) (4,2) (4,1) (4,3) (3,4) }

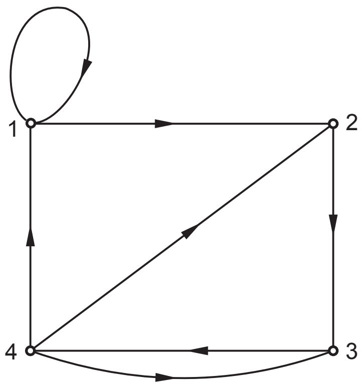


Fig. 4.6.1

then the diagraph of R^{-1} is obtained by changing the direction of arrow only which is given below.

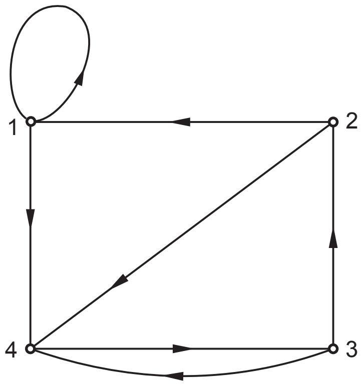


Fig. 4.6.2

Theorem 1 : If R, R_1, R_2 are relations from A to B then

i) $(R^{-1})^{-1} = R$

ii) $(R_1 \cup R_2)^{-1} = R_2^{-1} \cup R_1^{-1} = R_1^{-1} \cup R_2^{-1}$

iii) $(R_1 \cap R_2)^{-1} = R_1^{-1} \cap R_2^{-1}$

4.6.2 Complement of a Relation

Let R be a relation from A to B. The complement of R is denoted by $\overline{R}$ or $R' : A \to B$ and defined as $\overline{R} = \{(x, y) | (x, y) \notin R, x \in A, y \in B\}$

i.e. $x\,\overline{R}\,y$ iff $x \not{R}\, y$

Note $\overline{R}$ is the complement ary set of R w.r.t. universal domain $A \times B$.

Examples :

Example 4.6.1 *Let A = {1, 2, 3, 4}, B = { a, b},*

$A \times B$ = {(1, a) (1, b) (2, a) (2, b) (3, a) (3, b) (4, a) (4, b)},

R = {(1, a) (2, a) (3, a) (4, a)},

S = {4, a) (4, b) (3, a) (3, b)}.

Find $\overline{R}, \overline{S}, \overline{R} \cap \overline{S}, \overline{R} \cup \overline{S}$.

Solution :

$$\overline{R} = \{(1,b)\ (2,b)\ (3,b)\ (4,b)\}$$

$$\overline{S} = \{(1, a)\ (1, b)\ (2, a)\ (2, b)\}$$

$$\overline{R} \cap \overline{S} = \{(1, b)\ (2, b)\}$$

$$\overline{R} \cup \overline{S} = \{(1, a)\ (1, b)\ (2, a)\ (2, b)\ (3, b)\ (4, b)\}$$

Example 4.6.2 *If A = {a, b, c, d} and R = {(a, b) (c,d) (c,c) (d,a) (a,a) (b,b) (d,d) } is a relation on A. Draw diagraph of* $\overline{R}$.

Solution :

$$\overline{R} = \{(a,c)\,(a,d)\,(b,a)\,(b,c)\,(b,d)\,(c,a)\,(c,b)\,(d,b)\,(d,c)\}$$

$\therefore$ The diagraph of $\overline{R}$ is as follows :

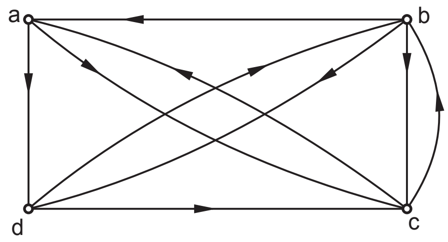


Fig. 4.6.3

Example 4.6.3 *Obtain the matrix of* $\overline{R}$ *if* $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Solution : The matrix of $\overline{R}$ is obtained from M_R be replacing 0 and 1 by 1 and 0 respectively

$$M_{\overline{R}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note : 1) $\overline{(R_1 \cup R_2)} = \overline{R_1} \cap \overline{R_2}$

2) $\overline{(R_1 \cap R_2)} = \overline{R_1} \cup \overline{R_2}$

4.6.3 Composite Relation

Let $R_1 : A \to B$ and $R_2 : B \to C$ be two relations.

The composition of R_1 and R_2 is denoted by

$R_1 \circ R_2$ or $R_1 R_2 : A \to C$ defined as

$R_1 \cdot R_2 = \{(x,z) \mid x R_1 y,\ y R_2 z$ i.e. $(x, y) \in R_1$ and $(y, z) \in R_2$ for $x \in A,\ z \in C\}$

Examples :

Example 4.6.4 *Let A = (x, y, z, w) and* $R_1 = \{(x, y)\,(x, x)\,(y, z)\,(z, w)\}$, $R_2 = \{(x, x)\,(y, x)\,(y, y)\,(w, w)\}$, *Find* R_1, R_2 *and diagraph of* $R_1 \cdot R_2$.

Solution :

In R_1	In R_2	$R_1 \cdot R_2$
1) (x, y)	(y, x) (y, y)	(x, x) (x, y)

i.e. $x \to y$	$y \to x$ $y \to y$	$x \to x$ $x \to y$
2) (x, x)	(x, x)	(x, x)
3) (y, z)	-	-
4) (z, w)	(w, w)	(z, w)

$\therefore \quad R_1 \cdot R_2 = \{(x, x)\}, (x, y), (z, w)\}$

The diagraph of $R_1 \cdot R_2$ is,

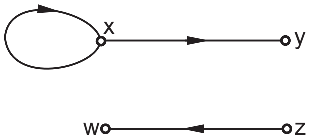


Fig. 4.6.4

Example 4.6.5 *Let $A = (1, 2, 3, 4)$. Let $R_1 = \{(x, y)/\ x+y = 5\}$ and $R_2 = (x, y)\ /\ (y - x) = 1\}$ verify $(R_1 \cdot R_2)^C = R_2^C \cdot R_1^C$*

Solution : We have A = {1, 2, 3, 4}

$$R_1 = \{(1,4)\ (2,3),\ (3,2)\ (4,1)\}$$

$$R_2 = \{(1,2)\ (2,3)\ (3,4)\}$$

$$R_1 \cdot R_2 = \{(2,4)\ (3,3)\ (4,2)\}$$

$$(R_1 \cdot R_2)^C = \{(4,2)\ (3,3)\ (2,4)\}$$

$$R_1^C = \{(4,1),\ (3,2),\ (2,3),\ (1,4)\}$$

$$R_2^C = \{(2,1),\ (3,2)\ (4,3)\}$$

$$R_2^C \cdot R_1^C = \{(2,4),\ (3,3)\ (4,2)\} = (R_1 \cdot R_2)$$

Hence $(R_1 \cdot R_2)^C = R_2^C \cdot R_1^C$

4.7 Types of Relations on Set

SPPU : Dec.-10, 11, 12, 14, 18, 19, May-14, 17, 18, 19

Let R be a relation from A to A. Then we can say that R is a relation on set A.

$\therefore\ R \subseteq A \times A$.

$A \times A$ is the universal set for R. If $R = A \times A$ then R is called the Universal Relation on A.

4.7.1 Identity Relation

Let A be any set. The relation on A is called Identity relation if

$$I_A = \{(a, a) \mid \text{for } a \in A\}$$

e.g. If A = {1, 2, 3, 4} then

R_1 = {(1, 1) (2, 2)}, R_2 = {(4, 4)} are identity relations.

The diagraph of the identity relation is the graph with loops only and it's relation matrix is the diagonal matrix with 0 and 1.

4.7.2 Reflexive Relation

1) Let R be a relation on set A. A relation R is said to be reflexive relation if $(a, a) \in R, \forall a \in A$.
2) A relation R is said to be not reflexive if $\exists$ at least one element $a \in A$ such that $(a, a) \notin R$ i.e. $a \not R a$.
3) A relation R is said to be irreflexive if for every $a \in A$, $(a, a) \notin R$ i.e. $a \not R a$

Notes :

1) If R is a reflexive relation on set A then
i) The relation matrix (M_R) is the identity matrix.
ii) Diagraph of reflexive relation will have loop for every element of A.
2) The relation matrix of irreflexive relation has all diagonal elements zero. It's diagraph is free from loops.

Examples

Example 1 :

Let A = {1,2, 3,4}. Then

i) R_1 = {(1, 1) (2, 2) (3, 3) (4, 4)} is the reflexive relation. It is known as the smallest reflexive relation on set A.

ii) R_2 = {(1, 1) (2, 2) (3, 3) (4, 4) (2, 4) (1, 2) (3, 2)} is reflexive relation.

iii) R_3 = { (1, 1) (2, 2) (3, 4)} is not reflexive relation as $3 \in A$ but $3 \, R^3_3$.

iv) R_4 = {(2, 3) (3, 4)} is irreflexive relation A as $\forall a \in A$, $a \not R_4 a$.

Example 2 :

Let A = Set of all straight lines in a plane. x R y iff x is parallel to y.

For any line $x \in A$, x is parallel to x.

$\therefore$ x Rx $\forall x \in A$

$\therefore$ R is a reflexive relation.

4.7.3 Symmetric Relation

Let R be a relation on set A. R is said to be symmetric relation if whenever a R b then b Ra for a, b $\in$ A.

Notes :

1) A relation is symmetric iff $R = R^{-1}$

2) The relation matrix of symmetric relation is a symmetric matrix.

Example : Let A = {1,2, 3, 4}. then

i) R_1 = {(1, 1) (2, 2) (3, 4) (4, 3)} is a the symmetric relation.

ii) R_2 = {(1, 1) (2, 2) (3, 3) (4, 4)} is a symmetric as well as reflexive relation.

iii) R_3 = { (1, 1) (2, 2) (3, 3) (4, 4) (3, 4)} is reflexive but not symmetric relation.

iv) R_4 = {(a, b)} $a \neq b$ is the smallest non- symmetric relation.

Example 4.7.1 *Let A = Set of all straight lines in a plane. x R y iff x || y i.e. x is parallel to y. S. T. R is a symmetric relation.*

Solution : R = {x R y | x is parallel to y}

Suppose x R y then x is parallel to y.

iff y is parallel to x.

iff y R x

i.e. whenever x R y, then y R x.

$\therefore$ R is a symmetric relation.

4.7.4 Compatible Relation

A relation R on a set A is said to be compatible relation if it is reflexive and symmetric relation. The relation matrix of a compatible relation is a symmetric matrix with the diagonal elements 1.

e.g. If A = {x, y, z} then

R = {(x, x) (y, y) (z, z) (x, y) (y, x)} is a compatible relation.

Note :

1) If R is a compatible relation on set A S.T. |A| = n then |R| $\geq$ n

4.7.4.1 Asymmetric Relation

A relation R on a set A is said to be asymmetric relation if whenever aRb then $b \not R a$.

Hence R is not asymmetric relation if for some a and b in A, aRb and bRa.

e.g. If A = {1, 2, 3} R = {(1, 2) (2, 3) (3, 1)} is a symmetric relation. But R_1 = {(1, 2) (2, 3) (3, 2)} is not asymmetric relation as (2, 3) and (3, 2) $\in R_1$.

4.7.5 Antisymmetric Relation

A relation R on a set A is said to be antisymmetric relation if whenever aRb and bRa then a = b. A relation R is not antisymmetric if $\exists$ a, b $\in$ A such that a $\neq$ b but aRb and bRa. The relation matrix of antisymmetric relation is never symmetric matrix.

Examples :

1) Let A = R and aRb iff $a \leq b$.

Suppose aRb and bRa then $a \leq b$ and $b \leq a$.

$\Rightarrow a = b$.

$\therefore$ R is antisymmetric relation. It is not symmetric relation.

2) Let A = {1, 2, 3} then

i) R_1 = {(1, 1) (2, 2)} is symmetric, antisymmetric relation.

ii) R_2 = {(1, 2) (2, 1)} is symmetric but not antisymmetric relation.

iii) R_3 = {(1, 1) (1, 2) (2, 2)} is antisymmetric but not asymmetric relation.

4.7.6 Transitive Relation

Let R be a relation on set A. R is said to be transitive relation if aRb and bRc, then aRc for a,b,c, in R.

$\therefore$ A relation R is not transitive if $\exists$ a, b and c in A.

Such that aRb, bRc but $a \not R c$

Examples :

1) Let A = R, R be the relation " $\leq$ ".

Then R is transitive as $a \leq b$, $b \leq c \Rightarrow a \leq c$

2) If A = {x, y, z w} then,

i) R_1 = {(x, y) (y, z)} is not transitive relation as x Ry, y Rz but $x \not R z$.

ii) R_2 = {(x, y) (y, z) (z, x) (z, y) (x, z) (y, x) (x, x) (y, y) (z, z) } is transitive relation.

iii) R_3 = {(x, y) (z, y)} is a transitive relation as there are no terms of the form x Ry and y R z,

Diagraph of transitive relation is as follows :

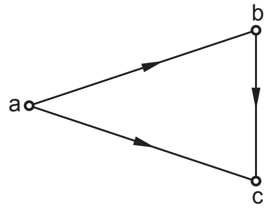


Fig. 4.7.1

4.7.7 Equivalence Relation

Let R be a relation on a set A. Then R is said to be an equivalence relation on A iff R is reflexive symmetric and transitive relations. It is denoted by 'R' or '~'.

Example :

Example 4.7.2 *Let A = R, set of all straight lines in a plane. $R_1 = \{x \, R_1 \, y \mid x \text{ is parallel to } y\}$, $R_2 = \{x \, R_2 \, y \mid x \text{ is perpendicular to } y\}$ check whether R_1 and R_2 are equivalence relations or not.*

Solution : i) For relation R_1 : we know that if x, y, z are straight lines in a plane, then x is parallel to x. $\therefore$ R_1 is reflexive.

ii) If $x \, R_1 \, y \Rightarrow x$ is $\parallel$ to $y \Rightarrow y$ is parallel to x

$\Rightarrow y \, R_1 \, x \Rightarrow R_1$ is symmetric.

iii) If $x \, R_1 \, y, \; y \, R_1 \, z \Rightarrow x \parallel y, \; y \parallel z \Rightarrow x \parallel z \Rightarrow x \, R_1 \, z$.

$\Rightarrow$ R_1 is transitive relation.

$\therefore$ R_1 is an equivalence relation

ii) For relation R_2 :

Any line is not perpendicular to itself.

$\therefore$ $x \not{R}_2 x, \; \forall \, x \in A$

$\therefore$ R_2 is not reflexive.

$\therefore$ R_2 is not equivalence relation.

Example 4.7.3 *If A = {x, y, z } and R = {(x, x) (y, y) (z, z)}. It is an equivalence relation ?*

Solution : The given relation R is reflexive, symmetric and transitive.

$\therefore$ R is an equivalence relation.

4.7.8 Properties of Equivalence Relations

1) If R_1 and R_2 are equivalence relations on a set A, then $R_1 \cap R_2$ is an equivalence relation.

$\Rightarrow$ If R_1 and R_2 are reflexive, symmetric and transitive then $R_1 \cap R_2$ are also reflexive, symmetric and transitive.

$\therefore R_1 \cap R_2$ is an equivalence relation.

2) If R_1 and R_2 are equivalence relations on a set A, then it is not necessary that $R_1 \cup R_2$ is an equivalence relation.

Let $A = \{a, b, c)$

And $R_1 = \{(a, a) (b, b) (c, c) (a, b) (b, a)\}$

$R_2 = \{ (a, a) (b, b) (c, c) (a, c) (c, a)\}$ are equivalence relation.

$R_1 \cup R_2 = \{(a, a) (b, b) (c, c) (a, b) (b, a) (a, c) (c, a)\}$

(b, a) and $(a, c) \in R_1 \cup R_2$ but $(b, c) \notin R_1 \cup R_2$.

$\therefore$ $R_1 \cup R_2$ is not an equivalence relation.

3) If R_1 and R_2 are equivalence relations then $R_1 \cup R_2$ is an equivalence relations iff $R_1 \subseteq R_2$ or $R_2 \subseteq R_1$.

4.7.9 Equivalence Classes

Let R be an equivalence relation on a set A. The equivalence class of $a \in A$ is denoted by $[a]_R$ and defined as

$[a]_R = \{x \in A \mid x R a\}$

= The set of those elements of A which are related to a.

$\therefore$ $a \in [a]_R$

Example 1 :

Let $A = \{x, y, z\}$

$R = A \times A = \{(x, x) (y, y) (z, z) (x, y) (x, z) (y, z) (y, x) (z, x) (z, y)\}$

is an equivalence relation.

$\therefore$ The equivalence class of x is

$[x]_R = \{x, y, z\} = [y]_R = [z]_R$

All these three equivalence classes are identical.

Example 4.7.4 *If A = z, R = {(x, y) | x + y = even, x, y ∈ z }. Is R equivalence relation ? if yes, find equivalence class of 1 and 2.*

Solution : $x + x$ = even number for any $x \in z$.

$\therefore x R x \Rightarrow$ R is reflexive relation.

Suppose $x R y \Rightarrow x + y$ = even

$\Rightarrow$ y + x = even

$\Rightarrow$ yRx

$\therefore$ R is symmetric relation.

Suppose xRy and yRz $\Rightarrow$ x + y = even, y + z = even

$\Rightarrow$ x + y + y + z = even

$\Rightarrow$ x + z = even – 2y = even

$\Rightarrow$ x + z = even

$\Rightarrow$ xRz

$\therefore$ R is a transitive relation.

$\therefore$ R is an equivalence relation.

$\therefore$ The equivalence class of $1 \in z$ is

$$\begin{aligned}
[1]_R &= \{xR_1 / x+1 = \text{even}\} \\
&= \{xR_1 / x = \text{even} - 1 = \text{odd}\} \\
&= \{xR_1 / x \text{ is odd}\} \\
[1]_R &= \text{Set of all odd integers } = \{\ldots -3, -1, 1, 3, 5, \ldots\} \\
[2]_R &= \{xR_2 \mid x+2 = \text{even}\} \\
&= \{xR_2 \mid x = \text{even}\} \\
&= \text{Set of all even integers} \\
[2]_R &= \{\ldots -4, -2, 0, 2, 4 \ldots]
\end{aligned}$$

Theorem 1 : Any two equivalence classes are either identical or disjoint.

Theorem 2 : For any $x \in A$, $x \in A$, $x \in [x]_R$

Theorem 3 : $A = \bigcup_{a \in A} [a]$

Example 4.7.5 *Which of the following are relations from A to B where A = {1, 2, 3, 4} ; B = {x, y, z}*

a) $R_1 = \{(1, x), (1, y), (1, z), (4, x)\}$

(b) $R_2 = \{(x, 1), (y, 1), (z, 1), (x, 4)\}$

(c) $R_3 = \{(1, x), (2, y), (3, z), (4, w)\}$

(d) $R_4 = \{(1, 1), (2, 2)\}$

Solution : Given that R : A $\to$ B

$$\begin{aligned} \text{Where } A &= \text{Domain set} \\ \text{and } B &= \text{Co-domain set} \\ A \times B &= \{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z), (3, x), \\ &\quad (3, y), (3, z), (4, x), (4, y), (4, z)\} \end{aligned}$$

(a) R is a relation from A to B because $R_1 \subset A \times B$

(b) R is not a relation from A to B as $R \not\subset A \times B$

(c) R is not a relation from A to B as $(4, w) \in R$ but $w \not\subset B$

(d) R is not a relation from A to B as $R \not\subset A \times B$

Example 4.7.6 *If A = {1,2,3,4}, B = {1, 4, 6, 8, 9} R is a relation from A to B such that aRb iff $b = a^2$. Find domain, range, relation matrix and diagraph.* **SPPU : Dec.-18, Marks 6**

Solution : We have

$$R = \{(1, 1)\ (2, 4)\ (3, 9)\}$$

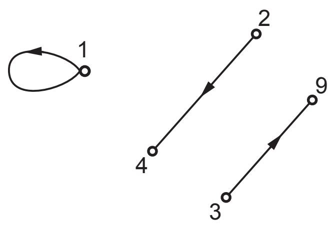
i) Domain of R is {1, 2, 3}

ii) Range of R is {1, 4, 9}

iii) Relation matrix is

$$M(R) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 4 & 6 & 8 & 9 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

iv) Diagraph



Example 4.7.7 *If R is a relation from A to B where A = {1, 2, 8}, B = {1, 2, 3, 5} and aRb iff a < b. Find : a) R in Roster form b) Domain and Range c) Diagraph.*

SPPU : May-18, Marks 4

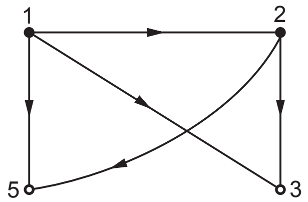
Solution : Given that A= {1, 2, 8}, B = {1, 2, 3, 5}

a) R in Roster form is R = {(1, 2) (1, 3), (1, 5), (2, 3), (2,5)}

b) Domain of R is {1, 2}

Range of R is {2, 3, 5}

c) Diagraph :



Example 4.7.8 *If R is a relation on set A = {1, 2, 3, 4, 5}, R = {(1, 1) (1, 2) (2, 1) (3, 1) (4, 1) (5, 2)}. Find domain set, codomain set, range. Draw its diagraph and find relation matrix.*

Solution : Given that A = {1, 2, 3, 4, 5}

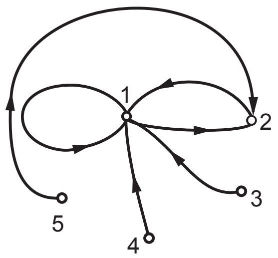
and R is a relation on set A.

Domain set = A = Co-domain set

R = {(1, 1) (1, 2) (2, 1) (3, 1) (4, 1) (5, 2)}

Range set = $\{1, 2\} \subset A$

Its diagraph is as follows :



The matrix form of R is

$$M(R) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}_{5\times 5} \end{array}$$

Example 4.7.9 *Let A = {1, 2, 3, 4, 5} and aRb if a < b. Find i) R in Roster form ii) Domain and Range of R iii) Diagraph of R.* **SPPU : Dec.-19, Marks 3**

Solution : We have

$$A = \{1, 2, 3, 4, 5\}$$

i) By considering the given conditions,

R in Roster form is

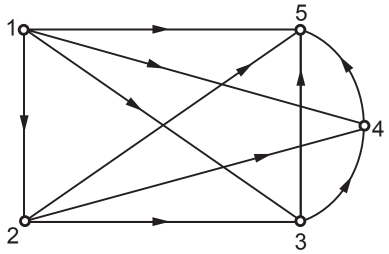
R = {(1, 2), (1, 3), (1, 4), (1, 5), (2, 3) (2, 4), (2, 5) (3, 4) (3, 5) (4, 5)}

ii) Domain of R is = {1, 2, 3, 4}

Range of R is = {2, 3, 4, 5}

iii) The vertex set of R is {1, 2, 3, 4, 5}

It's diagraph is as follows :



Example 4.7.10 *Let A = {1, 2, 3, 4, 5, 6} aRb or (a, b) ∈ R iff a is a multiple of b. Find*

(i) Range set and R (6), R (3).

(ii) Find relation matrix.

(iii) Draw diagraph

(iv) Find in and out degree of each vertex.

Solution : By considering given condition.

$$R = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 3), (4, 1), (4, 2), (4, 4), (5, 1), (5, 5), (6, 1), (6, 2), (6, 3), (6, 6)\}$$

(i) The range set of R is {1, 2, 3, 4, 5, 6}

R (3) = {1, 3} as (3, 1) and (3, 3) ∈ R

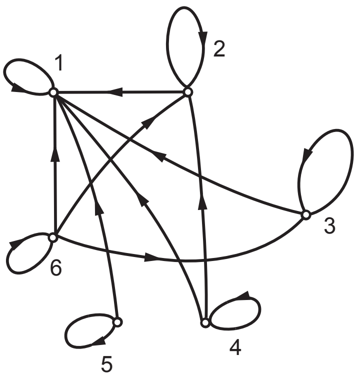
and R (6) = {1, 2, 3, 6} because (6, 1), (6, 2), (6, 3), (6, 6) ∈ R

(ii) The relational matrix is

$$M(R) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(iii) The diagraph of given relation is

vertex set = {1, 2, 3, 4, 5, 6}



(iv) The indegree of a vertex in diagraph is the number of edges coming towards that vertex. The outdegree of a vertex is defined as the number of edges going out from a vertex. Consider the following table.

Vertex	Indegee	Outdegree	Degree
1	6	1	7
2	3	2	5
3	2	2	4
4	3	1	4
5	2	1	3
6	4	1	5

Example 4.7.11 *Let R be a relation on set of natural numbers such that $R = \{(x, y) \mid 2x + 3y \text{ and } x, y \in N\}$. Find (i) The domain and codomain of R. ; (ii) R^{-1}*

Solution : Given that $x, y \in N$

$\therefore$ The smallest values of x and y are 1, 1 respectively.

$$\therefore \quad 2x + 3y = 2(1) + 3(1) = 5$$

which is the smallest value of co-domain

$$x = 1, \quad y = 2 \Rightarrow 2x + 3y = 8$$
$$x = 2, \quad y = 1 \Rightarrow 2x + 3y = 7$$
$$x = 2, \quad y = 2 \Rightarrow 2x + 3y = 10$$
$$x = 2, \quad y = 3 \Rightarrow 2x + 3y = 13$$
$$x = 3, \quad y = 2 \Rightarrow 2x + 3y = 12$$

$\therefore$ The domain of R is = $\{x \mid x \in N\}$

and the codomain of R is = $\{y \mid y \in N\}$

(ii) The inverse relation is

$$R^{-1} = \{(y, x) / x, y \in N\}$$

Example 4.7.12 *Let A = {1, 2, 3, 4, 6} and let R be defined on A as xRy iff x | y (i.e. x divides y)*

(i) Write R as a set of ordered pairs

(ii) Find R^{-1} and describe R^{-1} in words

Solution : Given that,

$$A = \{1, 2, 3, 4, 6\} \text{ and } x\,R\,y \text{ iff } x \text{ divides } y.$$

(i) $\therefore$ R = {(1, 2), (1, 3), (1, 4), (1, 6), (2, 4), (2, 6), (3, 6), (1, 1), (2, 2), (3, 3), (4, 4), (6, 6)} ... [Here 1|2, 1|1, 2|2, ...]

(ii) The inverse relation R^{-1} is

R^{-1} = {(2, 1), (3, 1), (4, 1), (6, 1), (4, 2), (6, 2), (6, 3), (1, 1), (2, 2), (3, 3), (4, 4), (6,6)}

Now R-1 is defined as : $xR^{-1}y$ iff x is divisible by y.

Example 4.7.13 *Let A = {1, 2, 3) R is the relation on A whose matrix is :*

$$M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

show that R is transitive. **SPPU : May-17, Marks 3**

Solution : Given that : A = {1, 2, 3} and $R = \{(1, 1), (1, 2), (1, 3), (2, 3), (3, 3)\}$

In R, $(1, 1), (1, 2) \in R \Rightarrow (1, 2) \in R$

$(1, 1), (1, 3) \in R \Rightarrow (1, 3) \in R$

$(1, 2), (2, 3) \in R \Rightarrow (1, 3) \in R$

$(1, 3), (3, 3) \in R \Rightarrow (1, 3) \in R$

$(2, 3), (3, 3) \in R \Rightarrow (2, 3) \in R$

Hence for any $(a, b), (b, c) \in R \Rightarrow (a, c) \in R$

Thus R is a transitive relation on A.

Example 4.7.14 *Let A = Set of all students in SPPU*
xRy iff x and y belongs to same class of SPPU ∀, y ∈ A.
Is R an equivalence relation ? Find the equivalence class of Atharva in A.

Solution : Given that A = {Set of all students in SPPU}

and xRy iff x and y belong to same class of SPPU.

(i) Any student x and x belong to same class

$\therefore$ xRx , $\forall\, x \in A \Rightarrow$ R is reflexive relation.

(ii) Let xRy $\Rightarrow$ x and y belong to same class of SPPU.

$\Rightarrow$ y and x belong to same class of SPPU.

$\Rightarrow$ yRx

i.e. xRy $\Rightarrow$ yRx, $\forall\, x, y \in R$.

Thus R is symmetric relation.

(iii) Let xRy and yRz

$\Rightarrow$ x and y belong to same class and y and z belong to same class of SPPU

$\Rightarrow$ x, y, z belong to same class of SPPU

$\Rightarrow$ x R z

$\Rightarrow$ R is transitive relation.

Thus R is an equivalence relation.

Suppose there is one student in SPPU whose name is Atharva.

$\therefore$ The equivalence class of Atharva is the set of all students who are studying in the class of Atharva.

Example 4.7.15 *Let A = {Set of all lines in a plane} and x, y ∈ A. Define xRy iff x is perpendicular to y. Is R transitive, reflexive and symmetric ?*

Solution : Given that xRy iff x perpendicular y.

(i) There is no any line which is perpendicular to itself.

$\therefore \nexists$ any x such that xRx.

$\Rightarrow$ R is not reflexive.

(ii) Suppose xRy $\Rightarrow$ x is perpendicular to y.

$\Rightarrow$ y is perpendicular to x.

$\Rightarrow$ yRx

i.e. xRy $\Rightarrow$ yRx i.e. R is symmetric relation.

(iii) Suppose xRy and yRz

$\Rightarrow$ x is perpendicular to y and y is perpendicular to z.

$\Rightarrow$ x and z are parallel lines.

$\Rightarrow$ xRz

$\Rightarrow$ R is not transitive relation.

Example 4.7.16 *Let A = {set of lines in a plane} and x, y $\in$ A. xRy iff x is parallel to y. Is R equivalence relation ? Find the equivalence class of any line 'L'.*

Solution :

(i) We know that every line is parallel to itself.

$\therefore$ xRx , $\forall$ x $\in$ A

$\Rightarrow$ R is reflexive relation.

(ii) Suppose xRy

$\Rightarrow$ x is parallel to y.

$\Rightarrow$ y is parallel to x.

$\Rightarrow$ yRx $\Rightarrow$ R is symmetric relation.

(iii) Suppose xRy and yRz.

$\Rightarrow$ x is parallel to y and y is parallel to z.

$\Rightarrow$ x is parallel to z.

$\Rightarrow$ xRz $\Rightarrow$ R is transitive relation.

Thus R is an equivalence relation.

Let L be any line in A.

Equivalence class of L = [L] = {xRL | x is parallel to L}

$$
\begin{aligned}
[L] &= \{x \mid x \text{ is parallel to } L\} \\
&= \text{Set of all lines which are parallel to } L.
\end{aligned}
$$

Example 4.7.17 *Let A = {a, b, c} and M (R) =* $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Determine whether R is an equivalence relation ? Find the equivalence class of b in A.

Solution : From relation matrix, the relation R is

$$R = \{(a, a)\ (b, b)\ (c, c)\ (b, c)\ (c, b)\}$$

(i) Here a, b, c $\in$ A and (a, a) (b, b) and (c, c) $\in$ R

$\therefore$ R is reflexive relation.

(ii) As (b, c) $\in$ R $\Rightarrow$ (c, b) $\in$ R

R is symmetric relation.

(iii) In relation R, we have,

(b, b), (b, c) $\in$ R $\Rightarrow$ (b, c) $\in$ R

(b, c), (c, b) $\in$ R $\Rightarrow$ (b, b) $\in$ R

(b, c), (c, c) $\in$ R $\Rightarrow$ (b, c) $\in$ R

(c, c), (c, b) $\in$ R $\Rightarrow$ (c, b) $\in$ R

(c, b), (b, b) $\in$ R $\Rightarrow$ (c, b) $\in$ R

(c, b), (b, c) $\in$ R $\Rightarrow$ (c, c) $\in$ R

So R is transitive relation. Thus R is an equivalence relation.

Now, the equivalence class of b in A is,

$$[b] = \{x \mid (x, b) \text{ or } (b, x) \in R\}$$

$$[b] = \{b, c\}$$

Example 4.7.18 *If R = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5) (3, 5), (5, 3), (1, 3), (3, 1)} is an equivalence relation on X = {1, 2, 3, 4, 5}. Find equivalence classes.*

SPPU : May-19, Marks 3

Solution : If R is an equivalence relation on X and a $\in$ X then $[a] = \bar{a} = \{x \mid x \sim a\}$.

Therefore, equivalence classes are as follows

$$[1] = \bar{1} = \{1, 3\}$$

$$[2] = \bar{2} = \{2\},\ [3] = \bar{3} = \{1, 3, 5\}$$

$$[4] = \bar{4} = \{4\}$$

$$[5] = \bar{5} = \{3, 5\}$$

Example 4.7.19 *Let A be the set of integers and let R be defined as xRy iff x $\leq$ y. Is R equivalence relation ?*

Soltuion : Given that xRy iff $x \leq y, \forall x, y \in A$

(i) We have, $x \leq x, \forall x \in A$

$\Rightarrow xRx, \forall x \in A$

$\Rightarrow$ R is reflexive relation.

(ii) If $xRy \Rightarrow x \leq y$

then $y \not\leq x \Rightarrow y \not R x$

$\Rightarrow$ R is not symmetric relation.

$\therefore$ R is not an equivalence relation.

Example 4.7.20 *Let $A = Z^+$ the set of positive integers, and let*

$R = \{(a, b) \in A \times A \mid a \text{ divides } b\}$

Is R symmetric, asymmetric or antisymmetric. **SPPU : May-17, Marks 3**

Ans. : Given that **:** $A = Z^+$ = The set of positive integers

and $R = \{(a, b)/a \text{ divides } b\}$

i) If $(a, b) \in R$ then a divides b

$\therefore$ $b = aK, K \in Z^+$

$\Rightarrow$ $a = \frac{1}{K}b$ but if $K \neq 1$ then $\frac{1}{K}$ is not an integer

$\therefore$ a does not divides b if

b is not multiples of K

$\therefore$ $(a, b) \in R \Rightarrow (b, a)$ may not be present in R.

$\therefore$ R is not symmetric relation.

ii) If $(a, b) \in R$ then a/b

$\therefore$ $b = aK, K \in Z^+$

$\Rightarrow$ $a = \frac{1}{K}b$, As $K \in Z^+$

b is not divide a

i.e. if $(a, b) \in R$ then $(b, a) \notin R$.

$\therefore$ R is asymmetric relation.

iii) As $(a, b) \in R$, $(b, a) \in R$

then $b = K_1 a$, $a = K_2 b$, $K_1, K_2 \notin Z^+$

$b = K_1$, $a = (K_1 K_2)b$

$\Rightarrow K_1 K_2 = 1 \Rightarrow K_1 = K_2 = 1$

$\therefore$ $a = b$

Thus R is antisymmetric relation.

Example 4.7.21 *Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and let N be the relation on $A \times A$ defined by $(a, b) \sim (c, d)$ iff $a + d = b + c$.*

(i) Prove that $\sim$ is an equivalence relation.

(ii) Find equivalence class of (2, 5) **SPPU : Dec.-11**

Solution : Given that $(a, b) \sim (c, d)$ iff $a + d = b + c$, $\forall$ a, b, c, d $\in$ A

(i) (a) We have $a + b = b + a \Rightarrow (a, b) \sim (a, b)$

$\Rightarrow$ '~' is reflexive relation.

(b) If $(a, b) \sim (c, d)$ then $a + d = b + c$.

$\Rightarrow b + c = a + d \Rightarrow c + b = d + a$

$\Rightarrow (c, d) \sim (a, b)$ by definition

$\Rightarrow$ '~' is symmetric relation.

(c) Suppose $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$

then $a + d = b + c$

and $c + f = d + e$

$\Rightarrow \quad a + \cancel{d} + \cancel{c} + f = b + \cancel{c} + \cancel{d} + e$

$a + f = b + e$

$\Rightarrow \quad (a, b) \sim (e, f)$

$\therefore$ ~ is a transitive relation.

Thus '~' is reflexive, symmetric and transitive.

$\Rightarrow$ '~' is an equivalence relation.

(ii) We have $(2, 5) \in A \times A$

The equivalence class of (2, 5) is the set of elements of $A \times A$ which are equivalent to (2, 5)

$$\therefore \quad [(2, 5)] = \{(x, y) \mid (x, y) \sim (2, 5),\ x, y \in A\}$$

$$= \{(x, y) \mid x + 5 = y + 2\}$$

$$= \{(x, y) \mid x - y = -3 \text{ or } y = x + 3\ ;\ x, y \in A\}$$

Hence, $[(2, 5)] = \{(1, 3)\ (2, 5)\ \ (3, 6)\ \ (4, 7)\ \ (5, 8)\ \ (6, 9)\}$ ... $[\because (7, 10) \notin A \times A]$

Example 4.7.22 *Prove that in the set $N \times N$, the relation R defined by (a, b) R (c, d) iff $ad = bc$ is an equivalence relation.*

Solution : We know that relation R is an equivalence relation if R is reflexive, symmetric and transitive.

(i) We have, ab = ba

$\Rightarrow$ (a, b) R (a, b)

$\therefore$ R is reflexive relation.

(ii) Suppose (a, b) R (c, d)

$\Rightarrow$ ad = bc

$\Rightarrow$ bc = ad

$\Rightarrow$ cb = da

$\Rightarrow$ (c, d) R (a, b)

$\therefore$ R is symmetric relation.

(iii) Let (a, b) R (c, d) and (c, d) R (e, f)

$\Rightarrow$ ad = bc and cf = ed

$\Rightarrow$ adcf = bced

$\Rightarrow$ af = be $\Rightarrow$ (a, b) R (e, f)

$\therefore$ R is transitive relation.

Thus, R is an equivalence relation.

Example 4.7.23 *Let A = R × R (R is set of real numbers) and define the following relation on A. (a, b) R (c, d) iff* $a^2+b^2 = c^2+d^2$

(i) Show that (A, R) is an equivalence relation.

(ii) Find equivalence class of (3, 2)

Solution : Given that (a, b) R (c, d) iff $a^2+b^2 = c^2+d^2$

(i) (a) We have, $a^2+b^2 = a^2+b^2$

$\Rightarrow$ (a, b) R (a, b)

R is a reflexive relation.

(b) If (a, b) R (c, d) then $a^2+b^2 = c^2+d^2$

$\Rightarrow$ $c^2+d^2 = a^2+b^2$

$\Rightarrow$ (c, d) R (a, b)

R is symmetric relation.

(c) Suppose (a, b) R (c, d) and (c, d) R (e, f)

$\Rightarrow$ $a^2+b^2 = c^2+d^2$ and $c^2+d^2 = e^2+f^2$

$\Rightarrow$ $a^2+b^2 = e^2+f^2$

$\Rightarrow \quad (a, b) \ R \ (e, f)$

$\therefore$ R is transitive relation.

Thus, R is an equivalence relation.

(ii) An equivalence class of (3, 2) is the set of elements of A which are equivalent to (3, 2)

$$\therefore \quad [(3, 2)] = \{(x, y) \mid (x, y) \ R \ (3, 2) \ ; \ x, y \in R\}$$

$$= \{(x, y) \mid x^2 + y^2 = 9 + 4 = 13 \ ; \ x, y \in \mathfrak{R}\}$$

$$[(3, 2)] = \text{The set of points on circle } x^2 + y^2 = 13.$$

Example 4.7.24 *Let R be the binary relation defined as R = $\{(a, b) \in \mathfrak{R}^2 / (a - b) \leq 3\}$*
(i) Determine whether R is reflexive, symmetric, transitive and antisymmetric relation.
(ii) How many binary relations are there on the finite set

Solution : Given that,

$$R = \{(a, b) \in R^2 \mid (a - b) \leq 3\}$$

(i) (a) We have, $a - a = 0 \leq 3$

$(a, a) \in R \Rightarrow R$ is reflexive.

(b) Suppose $(a, b) \in R \Rightarrow a - b \leq 3$

$$-(b - a) \geq 3$$

$$b - a \leq -3$$

$(b - a) \leq -3$ does not imply $b - a \leq 3$

e.g. $\quad a = -100 \quad , \quad b = 2$

$$a - b = -100 - 2 = -102 < 3$$

$$(-100, 2) \in R$$

But $\quad b - a = 2 - (-100) = 2 + 100 = 102 \not\leq 3$

Thus R is not symmetric relation.

(c) If $(a, b), (b, c) \in R$

then $\quad a - b \leq 3 , \quad b - c \leq 3$

$$a - b + b - c \leq 6$$

$$a - c \leq 6$$

$\Rightarrow a - c$ may or may not be ≤ 3.

R is not tansitive relation.

(d) Suppose aRb and bRa

$\Rightarrow \quad a - b \leq 3$ and $b - a \leq 3$

e.g. $\quad a = 1, \; b = 3,$

$a - b = -2 < 3$ and $b - a = 2 < 3$

but $a \neq b$.

$\therefore$ R is not antisymmetric relation.

(ii) Let n be the number of elements in a finite set A, then they are n^{n^2} binary relations on set A.

Example 4.7.25 *Let S be the set of points in a plane. Let R be a relation such that xRy iff y is within two centimeter from x. Is R equivalence relation ?*

Solution : Given that

xRy iff $|x - y| < 2, \forall x, y \in A$

(i) We have, $|x - x| = 0 < 2$

$\therefore$ xRx, $\forall \; x \in A$

$\Rightarrow$ R is reflexive relation

(ii) If $\quad xRy \Rightarrow |x - y| < 2$

$\Rightarrow |y - x| < 2$

$\Rightarrow$ yRx $\therefore$ R is symmetric relation.

(iii) Suppose xRy and yRz

$\Rightarrow \quad |x - y| < 2$ and $|y - z| < 2$

$\Rightarrow \quad |x - y| + |y - z| < 2 + 2 = 4$

$\Rightarrow \quad |x - z| < 4$

$\Rightarrow \quad |x - z| \not< 2$

$\therefore$ R is not transitive relation.

So R is not an equivalence relation.

Example 4.7.26 *Let R be a relation on the set of positive integers such that*

$R = \{(x, y) | x - y$ is an odd positive integer$\}$

Is R reflexive, symmetric, transitive, antisymmetric, an equivalence relation ?

Solution : Given that, $R = \{(x, y) \mid x - y \text{ is an odd positive integer}\}$

(i) For any positive integer x, $x - x = 0$ which is even integer.

$\therefore (x, x) \notin R$

R is not reflexive relation.

(ii) Suppose $(x, y) \in R$

$$\begin{aligned} \text{then} \quad x - y &= \text{Odd positive integer} \\ \Rightarrow \quad y - x &= \text{Odd negative integer} \\ (y, x) &\notin R \end{aligned}$$

$\therefore$ R is not symmetric relation.

(iii) Suppose $(x, y), (y, z) \in R$

$$\begin{aligned} \therefore \quad x - y &= \text{Odd positive integer} = p \\ y - z &= \text{Odd positive integer} = q \\ x - y + y - z &= p + q \\ x - z &= p + q \quad \text{which is even positive integer.} \end{aligned}$$

{e.g. $(5, 4) \in R$, $(4, 1) \in R$. But $(5, 1) \notin R$ as $5 - 1 = 4$}

$\therefore (x, z) \notin R$

$\therefore$ R is not transitive relation.

(iv) Suppose (x, y) and $(y, x) \in R$

$$\begin{aligned} \text{then} \quad x - y &= \text{Odd positive integer} = p \\ y - x &= \text{Odd positive integer} = q \\ \Rightarrow \quad x &\geq y \quad \text{and} \quad y \geq x \\ \Rightarrow \quad x &= y \end{aligned}$$

R is antisymmetric relation.

Example 4.7.27 *Show that $R = \{(a, b) / a \equiv b \pmod{m}\}$ is an equivalence relation on $\mathbb{Z}$. Show that $x_1 \equiv y_1$ and $x_2 \equiv y_2$ then $x_1 + x_2 \equiv y_1 + y_2$*

Solution : Given that

$$R = \{(a, b) \mid a \equiv b \pmod{m}\}$$

We know that $a \equiv b \pmod{m}$ iff $m \mid (a - b)$

$$a - b = mk, \quad k \in \mathbb{Z}$$

(i) Let $a \in \mathbb{Z}$, $a - a = 0 = m\,(0)$

$\therefore m \mid (a - a) \Rightarrow a \equiv a \pmod{m}$

$(a, a) \in R \Rightarrow R$ is reflexive relation.

(ii) Let $(a, b) \in R$

$$\Rightarrow \quad a \equiv b \pmod{m}$$

$$\Rightarrow \quad a - b = mk$$

$$\Rightarrow \quad b - a = m\,(-k)$$

$$\Rightarrow \quad m \mid b - a$$

$$\Rightarrow \quad b \equiv a \pmod{m}$$

$$\Rightarrow \quad (b, a) \in R$$

$\therefore$ R is symmetric relation.

(iii) Suppose (a, b) $(b, c) \in R$

then $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$

$$\Rightarrow \quad a - b = mk_1 \quad \text{and} \quad b - c = mk_2$$

$$\Rightarrow \quad a - b + b - c = mk_1 + mk_2$$

$$\Rightarrow \quad a - c = m(k_1 + k_2) = mk$$

$$\Rightarrow \quad m \mid (a - c) \Rightarrow a \equiv c \pmod{m}$$

$\therefore$ $(a, c) \in R$

$\therefore$ R is transitive relation.

$\therefore$ R is reflexive, symmetric and transitive.

Thus R is an equivalence relation,

Now, $x_1 \equiv y_1 \Rightarrow x_1 \equiv y_1 \pmod{m}$

$x_2 \equiv y_2 \Rightarrow x_2 \equiv y_2 \pmod{m}$

$x_1 - y_1 = mk_1$ and $x_2 - y_2 = mk_2$

$$\Rightarrow \quad x_1 - y_1 + x_2 - y_2 = mk_1 + mk_2$$

$$\Rightarrow \quad (x_1 + x_2) - (y_1 + y_2) = m\,(k_1 + k_2)$$

$$\Rightarrow \quad m \mid (x_1 + x_2) - (y_1 + y_2)$$

$$\Rightarrow \quad (x_1 + x_2) \equiv (y_1 + y_2) \pmod{m}$$

$$\Rightarrow \quad x_1 + x_2 \equiv y_1 + y_2 \pmod{m}$$

$$\Rightarrow \quad x_1 + x_2 \equiv y_1 + y_2$$

Example 4.7.28 *For each of these relations on set A = {1, 2, 3, 4} decide whether it is reflexive,*
symmetric, transitive or antisymmetric.
$R_1 = \{(1, 1)\ (2, 2)\ (3, 3)\ (4, 4)\}$
$R_2 = \{(1,1)(1, 2)\ (2, 2)\ (2,1)\ (3, 3)\ (4, 4)\}$, $R_3 = \{(1, 3)\ (1, 4)\ (2, 3)\ (2, 4)\ (3, 1)\ (3,4,)\}$.

SPPU : Dec.-10

Solution : i) For R_1 :

As $\forall\, a \in A,\ (a, a) \in R_1$

$\therefore R_1$ is Reflexive, symmetric relation.

$\nexists\, (a, b)$ and $(b, c) \in R_1$ $\therefore R_1$ is transitive relation.

$\nexists\, (a, b)$ and $(b, a) \in R_1$ $\therefore R_1$ is antisymmetric relation.

$\therefore R_1$ reflexive, symmetric, transitive and anti-symmetric relation.

ii) For R_2 :

As $\forall\, a \in A\ (a, a) \in R_2$ and $aR_2b \Rightarrow bR_1a$, for $a,b \in A$

$\therefore R_2$ reflexive and symmetric relation.

For any aR_2b and $bR_2C \Rightarrow aR_2C$

$\therefore R_2$ is transitive realtion

$\therefore R_2$ is an equivalence relation

But (1, 2) and (2, 1) $\in R_2$ and $1 \neq 2$

$\therefore R_2$ is not antisymmetric relation.

iii) For R_3 :

As $2 \in R_3$ but $(2, 2) \notin R_3$

$\therefore R_3$ is not reflexive relation.

As $(1, 4) \in R_3$ but $(4, 1) \notin R_3$

$\therefore R_3$ is not symmetric relation.

As (2, 3) and $(3, 1) \in R_3$ but $(2, 1) \notin R_3$.

$\therefore R_3$ is not transitive relation.

As (1, 3) and $(3, 1) \notin R_3$ but $1 \neq 3$.

$\therefore R_3$ is not transitive relation

As (1, 3) and $(3, 1) \in R_3$ but $1 \neq 3$

$\therefore R_3$ is not antisymmetric relation

Example 4.7.29 *Consider the relation on A = { 1, 2, 3, 4, 5, 6} . R = { (i, j) | i – j| = 2}. Is R reflexive ? Is R symmetric ? Is R transitive ?*

SPPU : Dec.-14

Solution : Given that A = {1, 2, 3, 4, 5, 6}

R = { (1, 3) (3, 1) (2, 4) (4, 2) (3, 5) (5, 3) (4, 6) (6, 4)}

As $2 \in A$ but $(2, 2) \notin R$ ∴ R is not reflexive.

For any $(a, b) \in R \Rightarrow (b,a) \in R$ ∴ R is symmetric.

As (1, 3) (3, 1) ∈ R but (1, 1) ∉ R ∴ R is not transitive.

Example 4.7.30 *Let A = { 1, 2, 3, 4, 5, 6, 7 } and R = {(x, y) | x – y is divisible by 3}. Show that R is an equivalence relation ? Draw graph of R.*

SPPU : May-14, Dec.-12, 19, Marks 6

Solution : We have R = {x, y) | x – y is divisible by 3}

We know that x – x = 0 is divisible by 3

x R x, $\forall x \in A \Rightarrow$ R is reflexive relation.

As x R y ⇒ x – y is divisible by 3

⇒ y – x is also divisible by 3

⇒ y – z is divisible by 3

⇒ y R x for x, y ∈ A

∴ R is a symmetric relation.

As,

x R y and y R ⇒ x – y and y – z are divisible by 3

⇒ (x – y) + (y – z) is also divisible by 3

⇒ x – z is divisible by 3

⇒ x R z

∴ R is a transitive relation.

∴ R is an equivalence relation.

and R = {(1, 1), (2, 2) (3, 3) (4, 4) (5, 5) (6, 6) (7, 7) (1, 4) (4, 1) (1, 7) (7, 1)
(2, 5), (5, 2) (3, 6) (6, 3)}

Its graph is as follows

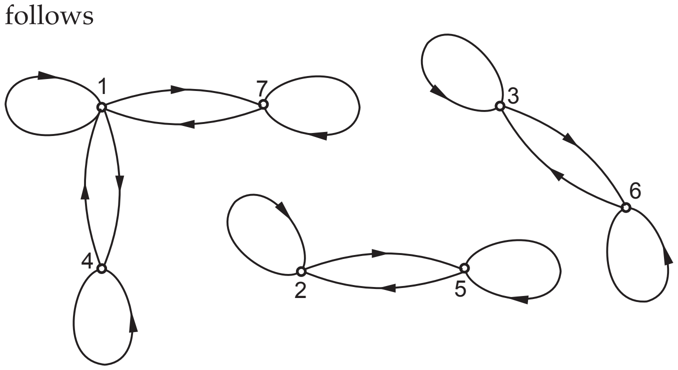


Fig. 4.7.2

4.8 Partitions of a Set

SPPU : Dec.-11, 12, May-17

We shall now discuss the concept of partitions of a set which is similar to equivalence class of set.

Definition :

Let A be any non empty set. A set

$P = \{A_1, A_2, A_3, ... A_n\}$ of non empty subsets of A is called a partition of set A if

i) $A_1 \cup A_2 \cup A_3 \cup .... \cup A_n = A = \bigcup_{i=1}^{n} A_i$

i.e. Set A is the union of the sets $A_1, A_2, ... A_n$.

ii) $A_i \cap A_j = \phi$ for $i \neq j$ i.e. All sets A_i are mutually disjoints.

The partition of a set A is denoted by π.

An element of a partition set is called a block. The rank of a partition is called as the number of blocks of that partitions. It is denoted by $r(\pi)$. For any non empty set, its partitions are not unique. There are different partitions of the same set.

e.g.

1) Let $A = Z$ = Set of integers

A_1 = Set of even integers

A_2 = Set of odd integers

$A_3 = \{1, 2, 3, 4, ...\}$,

$A_4 = \{0\}$

$A_5 = \{-1, -2, -3, -4, ...\}$

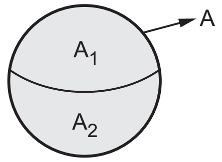


Fig. 4.8.1

$P_1 = \{A_1, A_2\}$ and $P_2 = \{A_3, A_4, A_5\}$ are different partitions of set A.

2) If $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and it's subsets

$A_1 = \{1, 4, 9\}$ $A_2 = \{2, 6, 8, 10\}$, $A_3 = \{3, 5, 7\}$,

The set $P = \{A_1, A_2, A_3\}$ is such that

i) A_1, A_2, A_3 are non empty sets

ii) $A = A_1 \cup A_2 \cup A_3$

iii) $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$, $A_2 \cap A_3 = \phi$

Hence $\{A_1, A_2, A_3\}$ form a partition for set A.

4.8.1 Relation Induced by a Partition

Let P be the partition of a non empty set A. We can define a relation R on a set A as

x Ry iff x and y belong to the same block of the partition P. This relation R is called as the relation induced by the partition.

e.g. i) Let $A = \mathbb{Z}$ and $P = \{A_1, A_2\}$

where A_1 = Set of even integers and

A_2 = Set of odd integers

$\therefore$ P is a partition for $A = \mathbb{Z}$

Define x Ry Iff x and y belong to the same partition of $\mathbb{Z}$

i.e. x and y belong to either A_1 or A_2

i.e. x and y are either even or odd

This relation is an equivalence relation.

Theorem 1 : Let R be an equivalence relation on a set A then the set of equivalence classes $\{[a]_R / a \in A\}$.

Theorem 2 : Let A be any non empty set and let π be a partition of A. Then π induces an equivalence relation on set A.

Example 4.8.1 *Let $A = \{x, y, z, u, v\}$, $\pi = \{\{x, y\}, \{z\}, \{u, v\}\}$. Find the equivalence relation induced by π.*

Solution :

We have $A = \{x, y, z, u, v\}$ and Define

x Ry iff x and y belongs to the same block of the partition of A.

$\pi = \{\{x, y\}, \{z\}, \{u, v\}\}$ has 3 blocks.

The first block $\{x, y\} \rightarrow x, y \in$ same block

$\therefore$ x Rx, xRy, yRx, yRy

The second block, $\{z\} \Rightarrow$ zRz

The third block, $\{u, v\} \Rightarrow$ uRu, uRv, vRu, vRv

$\therefore$ The required relation is

$$R = \{(x,x)\ (x,y), (y,x), (y,y), (z,z), (u,u), (u,v), (v,u), (v,v)\}$$

The relation R is reflexive, symmetric and transitive

$\therefore$ R is an equivalence relation.

Example 4.8.2 *Let A = (1, 2, 3, 4) consider partition.*

P = {{1, 2, 3}, {4}}.

of A find the equivalence relation R on A determined by P. **SPPU : May-17, Marks 3**

Solution : Given that : A = {1, 2, 3, 4} and the partition P = {{1, 2, 3}, {4}}

We know that equivalence classes form a partition for the corresponding set.

P_1 = {1, 2, 3} and P_2 = {4} are two equivalence classes.

∴ In each equivalence class, every element is related to all elements of that class.

∴ Due to P_1 = (1, 1) (1, 2), (1, 3), (2, 2), (2, 3), (2, 1), (3, 3), (3, 1), (3, 2) and for P_2, (4, 4)

Hence R = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 4)}

is an equivalence relation with respect to the given partition P.

Example 4.8.3 *Let A = {1, 2, 3, 4}, π = {{1}, {2, 3}, {4}}. Find the equivalence relation induced by π.*

Solution : Given that

$$\pi = \{\{1\}, \{2, 3\}, \{4\}\} \text{ has 3 blocks}$$

The first block, $\{1\} \Rightarrow iR_1$

The second block, $\{2, 3\} \Rightarrow 2R_2, 2R_3, 3R_2, 3R_3$

The third block, $\{4\} \Rightarrow 4R_4$

∴ The required relation is

$$R = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 4)\}$$

R is reflexive symmetric and transitive

∴ R is an equivalence relation

Example 4.8.4 *Define partition of a set. Let x = {1, 2, 3, 4, 5, 6, 7, 8, 9}. Determine whether or not each of the following is a partition of x :*

A = {{2, 4, 5, 8}, {1, 9}, {3, 6, 7}}

B = {{1, 3, 6}, {2, 8}, {5, 7, 9}} **SPPU : Dec.-11**

Solution : Please refer section 4.8 for definition.

i) For set A :

The set A has 3 blocks,

$$A_1 = \{2, 4, 5, 8\}, \qquad A_2 = \{1, 9\} \qquad A_3 = \{3, 6, 7\}$$

- The union of all these blocks is a set X

$$A_1 \cup A_2 \cup A_3 = X$$

- These blocks are mutually disjoints.

$\therefore$ The set A forms a partition for the set X

ii) For set B :

$\therefore$ The set B has 3 blocks.

$$B_1 = \{1, 3, 6\}, \qquad B_2 = \{2, 8\} \qquad B_3 = \{5, 7, 9\}$$

As $B_1 \cup B_2 \cup B_3 \neq X$,

B is not a partition of set X.

Example 4.8.5 *If S = {1, 2, 3, 4, 5, 6, 7, 8, 9}. Determine whether or not each of the following is a partition of S.*

i) A = {{1, 3, 5}, {2, 6}, {4, 8, 9}}

ii) B = {{1, 3, 5}, {2, 4, 6, 8}, {7, 9}}

iii) C = {{1, 3, 5}}, {2, 4, 6, 8}, {5, 7, 9}}

iv) D = {{5}} **SPPU : Dec.-12**

Solution : Given that

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

i) A is not a partition of S because S is not the union of all blocks of A.

i.e. $S \neq A_1 \cup A_2 \cup A_3$

ii) B is the partition of S as $B_1 \cup B_2 \cup B_3 = S$ and B_1, B_2, B_3 are mutually disjoints.

iii) As blocks of set C are not disjoints. $\therefore$ The set C is not a partition of S.

iv) D = {{s}} is a partition of S, called as trivial partition.

4.8.2 Refinement of Partitions

Let π and π' be partitions of a non empty set A.

Then π' is called refinement of π if every block (or an element) of π' is contained in a block of π.

e.g. Let $A = \{1, 2, 3\}$, $\pi = \{\{1\}, \{2, 3\}\}$, $\pi' = \{\{1\}, \{2\}, \{3\}\}$

$\{1\}, \{2\}, \{3\} \notin \pi$ but $\{1\}, \{2, 3\} \in \pi'$

$\therefore \pi'$ is a refinement of π

Theorem : Let π and π' be partitions of a non empty set A and R_1, R_2 be the equivalence relations induced by π and π' respectively. Then π' refines π iff $R_2 \subseteq R_1$.

4.8.3 Product and Sum of Partitions

I) Let π_1 and π_2 be two partitions of a non empty set A. The product of π_1 and π_2 is denoted by $\pi_1 - \pi_2 = \pi$ is a partition π of A such that

1) π refines both π_1 and π_2

2) If π' refines both π_1 and π_2 then π' refines π.

II) Let π_1 and π_2 be the partitions of a non empty set A. The sum of π_1 and π_2 is denoted by $\pi = \pi_1 + \pi_2$ is a partition of A such that

1) Both π_1 and π_2 refine π

2) If π_1 and π_2 refine π' then π refines π'

4.8.4 Quotient Set

Let A be any non empty set and R be an equivalence relation on A. The set of mutually disjoint equivalence classes in which A is partitioned relatively to the equivalence relation R is called the quotient set of A for relation R. If is denoted by A/R or $\overline{A}$.

e.g. The quotient set of *Zl*, for an equivalence relation

$$R = \{(a, b)/3|\{(a-b),\ a, b \in \mathbb{Z}\ \} \text{ is}$$
$$\mathbb{Z}/R = \{[0], [1], [2]\}.$$

Theorem 1 : Let R_1 and R_2 be an equivalence relations induced by partitions π_1 and π_2 of a non empty set A respectively, then

i) The relation $R = R_1 \cap R_2$ induces the product partition $\pi_1 \cdot \pi_2$

ii) The relation $R = (R_1 \cup R_2)^*$ {i.e. The transitive closure of $R_1 \cup R_2$} is an equivalence relation on A and induces the partition $\pi_1 + \pi_2$.

4.9 Closure of a Relation

SPPU : May-06, 07, 08, 15, 18, Dec.-05, 07, 12, 13, 14, 15, 16

Depending upon the nature of relations, there are mainly three types of closures of relations.

4.9.1 Reflexive Closure

Let R be a relation on a set which is not reflexive relation. A relation $R_1 = R \cup \Delta$ is called the reflexive closure of R if $R \cup \Delta$ is the smallest reflexive relation containing R.

If $A = \{a, b, c, d\}$ then $\Delta = \{(a, a), (b, b), (c, c), (d, d)\}$

Example 4.9.1 *Let $A = \{1, 2, 3\}$. R_1, R_2 and R_3 are relations on set A. Find the reflexive closures of R_1, R_2 and R_3.*
Where $R_1 = \{(1, 1) (2, 1)\}$, $R_2 = \{(1, 1) (2, 2), (3, 3)\}$,
$R_3 = \{(3, 1) (1, 3), (2, 3)\}$.

Solution :

We have $A = \{1, 2, 3\}$ $\therefore$ $\Delta = \{(1, 1), (2, 2), (3, 3)\}$

Then

i) The reflexive closure of R_1 is $R = R_1 \cup \Delta$

$\therefore$ $R = \{(1, 1) (2, 2) (3, 3) (2, 1)\}$

ii) The reflexive closure of R_2 is $R = R_2 \cup \Delta = R_2$

iii) The reflexive closure of R_3 is $R = R_3 \cup \Delta$

$\therefore$ $R = \{(1, 1) (2, 2) (3, 3) (3, 1) (1, 3), (2, 3)\}$

4.9.2 Symmetric Closure

Let R be a relation on a set A and R is not symmetric relation.

A relation $R_1 = R \cup R^{-1}$ is called the symmetric closure of R if $R \cup R^{-1}$ is the smallest symmetric relation containing R.

Example 4.9.2 *Find the symmetric closure of the following relations. On $A = \{1, 2, 3\}$.*

$R_1 = \{(1, 1) (2, 1)\}$

$R_2 = \{(1, 2) (2, 1) (3, 2) (2, 2)\}$

$R_3 = \{(1, 1) (2, 2) (3, 3)\}$ **SPPU : Dec.-12**

Solution : Given that

We have $A = \{1, 2, 3\}$

i) $R_1^{-1} = \{(1, 1) (1, 2)\}$

$\therefore$ $R = R_1 \cup R_1^{-1} = \{(1, 1) (1, 2) (2, 1)\}$ is the symmetric closure of R_1

ii) $R_2^{-1} = \{(2, 1) (1, 2) (2, 3) (2, 2)\}$

$\therefore$ $R = R_2 \cup R_2^{-1} = \{(1, 2) (2, 1) (3, 2) (2, 3) (2, 2)\}$ is the symmetric closure of R_2

iii) R_3 is the symmetric relation.

$\therefore$ R_3 itself is the symmetric closure.

4.9.3 Transitive Closure

Let R be a relation on a set A which is not transitive relation. The transitive closure of a relation R is the smallest transitive relation containing R. It is denoted by R^*.

The following theorem gives a procedure to find the transitive closure of a given relation.

Theorem : Let A be any non empty set and $|A| = n$. Let R be a relation on A. Then the transitive closure of R is $R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$.

Example 4.9.3 *If A = {1, 2, 3, 4, 5} and R = {(1, 2), (3, 4), (4, 5), (4, 1), (1, 1)}. Find it's transitive closure.*

Solution : Let R^* be the transitive closure of given relation R.

$$\therefore \quad R^* = R \cup R^2 \cup R^3 \cup R^4 \cup R^5$$

Now
$$R^2 = R \cdot R = \{(3, 5)\ (3, 1)\ (4, 2)\ (4, 1)\ (1, 2)\ (1, 1)\}$$

$$R^3 = R \cdot R^2 = \{(3, 2)\ (3, 1)\ (4, 2)\ (4, 1)\ (1, 1)\ (1, 2)\}$$

$$R^4 = R \cdot R^3 = \{(3, 1)\ (3, 2)\ (4, 1)\ (4, 2)\ (1, 1)\ (1, 2)\} = R^3$$

$$R^5 = R \cdot R^4 = \{(3, 1)\ (3, 2)\ (4, 1)\ (4, 2)\ (1, 1)\ (1, 2)\} = R^4 = R^3$$

$$R^* = \{(1, 2)\ (3, 4)\ (4, 5)\ (4, 1)\ (1, 1)\ (3, 5)\ (3, 1)\ (4, 2)\ (3, 2)\}$$

Which is the transitive closure of R.

4.9.4 Warshall's Algorithm to Find Transitive Closure

To find the transitive closure of a relation by computing various powers of R or product of the relation matrix is quite impractical for large relations. Warshall's algorithm gives an alternate method for finding transitive closure of R. Warshall's algorithm is practical and efficient method.

Consider the following steps to find transitive closure of the relation R on a set A.

Step 1 : We have $|A| = n$

$\therefore$ We require $W_o, W_1, W_2, \ldots W_n$. Warshall sets

$$W_o = \text{Relation Matrix of } R = M_R.$$

Step 2 : To find the transitive closure of relation R on set A, with $|A| = n$

Procedure to compute W_k from W_{k-1} is as follows

i) Copy 1 to all entries in W_k from W_{k-1}, where there is a 1 in W_{k-1}.

ii) Find the row numbers $p_1, p_2, p_3 \ldots$ for which there is 1 in column k in W_{k-1} and the column numbers $q_1, q_2, q_3 \ldots$ for which there is 1 in row k of W_{k-1}.

iii) Mark entries in W_k as 1 for (p_i, q_i). If there are not already 1.

Step 3 : Stop the procedure when W_n is obtained and it is the required transitive closure of R.

Examples

Example 4.9.4 *Find the transitive closure of R by Warshall's algorithm.*
Where A = {1, 2, 3, 4, 5, 6} and R = {(x, y)/(x – y) = 2} **SPPU : Dec.-05, 12, 13, 14**

Solution :

Step 1 : We have |A| = 6,

$$R = \{(1, 3), (3, 1), (2, 4), (4, 2), (4, 6), (6, 4), (3, 5), (5, 3)\}$$

Thus we have to find Warshall's sets, W_o, W_1, W_2, W_3, W_4, W_5 and W_6.

The first set W_o is same as M_R. Which is shown below

$$W_0 = M_R = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Step 2 : To find W_1 :

To find W_1 from W_o, we consider the first column and first row.

In a column C_1, 1 is present at R_3

In a row R_1, 1 is present at C_3

Thus add new entry in W_1, at (R_3, C_3) which is given below

$$W_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 3 : To find W_2 :

To find W_2 from W_1, we consider the second column C_2 and second row R_2.

In C_2, 1 is present at R_4

In R_2, 1 is present at C_4

Thus add new entry in W_2 at (R_4, C_4), which is given below

$$W_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 4 : To find W_3 :

To find W_3 from W_2, we consider the third column and third row.

In C_3, 1 is present at R_1, R_3, R_5

In R_3, 1 is present at C_1, C_3, C_5

Thus add new entries in W_3 at (R_1, C_1), (R_1, C_3), (R_1, C_5) (R_2, C_1), (R_2, C_3), (R_2, C_5) (R_3, C_1), (R_3, C_3), (R_3, C_5) which is given below

$$W_3 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 5 : To find W_4 :

To find W_4 from W_3, we consider the fourth column and fourth row.

In C_4, 1 is present at R_2, R_4, R_6

In R_4, 1 is present at C_2, C_4, C_6

Thus add new entries in W_4 at (R_2, C_2), (R_2, C_4), (R_2, C_6), (R_4, C_2), (R_4, C_4), (R_4, C_6), (R_6, C_2), (R_6, C_4), (R_6, C_6) which is given below

$$W_4 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Step 6 : To find W_5 :

To find W_5 from W_4, we consider the 5^{th} column and 5^{th} row.

In C_5, 1 is present at R_1, R_3, R_5

In R_5, 1 is present at C_1, C_3, C_5

Thus add new entries in W_5 at (R_1, C_1), (R_1, C_3), (R_1, C_5), (R_3, C_1), (R_3, C_3), (R_3, C_5), (R_5, C_1), (R_5, C_3), (R_5, C_5) which is given below

$$W_5 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} = W_4$$

Step 7 : To find W_6 :

To find W_6 from W_5 we consider the 6^{th} column and 6^{th} row.

In C_6, 1 is present at R_2, R_4, R_6

In R_6, 1 is present at C_2, C_4, C_6

Thus add new entries in W_6 at (R_2, C_2), (R_2, C_4), (R_2, C_6), (R_4, C_2), (R_4, C_4), (R_4, C_6), (R_6, C_2), (R_6, C_4), (R_6, C_6) which is given below

$$W_6 = W_5 = W_4$$

Hence W_6 is the relation matrix of R^*

$$\therefore \quad R^* = \begin{Bmatrix} (1, 1), (1, 3)\ (1, 5)\ (2, 2)\ (2, 4)\ (2, 6)\ (3, 1)\ (3, 3)\ (3, 5) \\ (4, 2)\ (4, 4)\ (4, 6)\ (5, 1)\ (5, 3)\ (5, 5)\ (6, 2)\ (6, 4)\ (6, 6) \end{Bmatrix}$$

Example 4.9.5 *Let R = {(a, d), (b, a), (b, d), (c, b), (c, d), (d, c)}*

Use Warshall's algorithm to find the matrix of transitive closure where A = {a, b, c, d}

SPPU : Dec.-15

Solution : Step 1 : We have

$$A = \{a, b, c, d\} \qquad \therefore \quad |A| = 4$$

$$R = \{(a, d)\ (b, a), (b, d)\ (c, b), (c, d), (d, c)\}$$

Thus we have to find Warshall's sets W_o, W_1, W_2, W_3, W_4

The first set $\quad W_o = M_R$

$$W_0 = M_R = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 0 & 0 & 1 \\ b & 1 & 0 & 0 & 1 \\ c & 0 & 1 & 0 & 1 \\ d & 0 & 0 & 1 & 0 \end{array}$$

Step 2 : To find W_1 :

To find W_1 from W_o, we consider the first column and first row.

In C_1, 1 is present at R_2

In R_1, 1 is present at C_4

Thus add new entry in W_1 at (R_2, C_4)

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Step 3 : To find W_2

To find W_2, from W_1, we consider the second column and second row.

In C_2, 1 is present at R_3

In R_2, 1 is present at C_1 and C_4

Thus add new entries in W_2 at (R_3, C_1), (R_3, C_4) which is given below.

$$W_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Step 4 : To find W_3 :

To find W_3 from W_2, we consider the 3^{rd} column and 3^{rd} row.

In C_3, 1 is present at R_4

In R_3, 1 is present at C_1, C_2, C_4

Thus add news entries in W_3 at (R_4, C_1), (R_4, C_2), (R_4, C_4)

Which is given below

$$W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Step 5 : To find W_4 :

To find W_4 from W_3, we consider the 4^{th} column and 4^{th} row.

In C_4, 1 is present at R_1, R_2, R_3, R_4

In R_4, 1 is present at C_1, C_2, C_3, C_4

Thus we add new entries in W_4 at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_1, C_4), (R_2, C_1), (R_2, C_2), (R_2, C_3), (R_2, C_4), (R_4, C_3), (R_3, C_1), (R_3, C_2), (R_3, C_4), (R_4, C_1), (R_4, C_2), (R_4, C_3), (R_4, C_4)

Which is given below

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Hence W_4 is the relation matrix of R^*.

and $$R^* = \begin{Bmatrix} (a, a), & (a, b) & (a, c) & (a, d) & (b, a) & (b, b) & (b, c) & (b, d) \\ (c, a), & (c, b) & (c, c) & (c, d) & (d, a) & (d, b) & (d, c) & (d, d) \end{Bmatrix}$$

Example 4.9.6 *Find the transitive closure of the relation R on*

A = {1, 2, 3, 4} defined by

R = {(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (3, 4), (3, 2), (4, 2), (4, 3)}

SPPU : Dec.-07, May-15, 18, Marks 4

Solution : Step 1 :

We have |A| = 4, Thus we have to find

Warshall's sets, W_0, W_1, W_2, W_3, W_4

The first set $W_0 = M_R$

$$W_0 = M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Step 2 : To find W_1 :

To find W_1 from W_0, we consider the first column and first row.

In C_1, 1 is present at R_2

In R_1, 1 is present at C_2, C_3, C_4

Thus add new entries in W_1 at (R_2, C_2), (R_2, C_3), (R_2, C_4) which is given below

$$W_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Step 3 : To find W_2 :

To find W_2 from W_1, we consider the 2^{nd} column and 2^{nd} row.

In C_2, 1 is present at R_1, R_2, R_3, R_4

In R_2, 1 is present at C_1, C_2, C_3, C_4

Thus we add new entries in W_2 at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_1, C_4), (R_2, C_1), (R_2, C_2), (R_2, C_3), (R_2, C_4), (R_3, C_1), (R_3, C_2), (R_3, C_2), (R_3, C_4), (R_4, C_1), (R_4, C_2), (R_4, C_3), (R_4, C_4)

Which is given below

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

All entries in W_2 are 1

Hence W_2 is the relation matrix of transitive closure of R.

and $$R^* = \begin{Bmatrix} (1, 1), & (1, 2) & (1, 3) & (1, 4) & (2, 1) & (2, 2) & (2, 3) & (2, 4) \\ (3, 1), & (3, 2) & (3, 3) & (3, 4) & (4, 1) & (4, 2) & (4, 3) & (4, 4) \end{Bmatrix}$$

Example 4.9.7 *If R = {(a, b), (b, a), (b, c), (c, d), (d, a)} be a relation on the set A = {a, b, c, d}. Find the transitive closer of R using Warshall's algorithm.* **SPPU : Dec.-16, Marks 6**

Solution :

Step 1 : We have A = {a, b, c, d}

R = {(a, b), (b, a), (b, c), (c, d), (d, a)}

Thus we have to find Warshall's sets W_0, W_1, W_2, W_3, W_4.

The first set $W_0 = M_R$

$$\therefore \quad W_0 = M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Step 2 : To find W_1

To find W_1 from W_0, we consider the first column and first row. In C_1, 1 is present R_2 and R_4

In R_1, 1 is present at C_2

Thus add new entries in W_1 at $(R_2, C_2)(R_4, C_2)$

$$\therefore \qquad W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Step 3 : To find W_2

To find W_2 from W_1, we consider second column and second row of W_1.

In C_2, 1 is present at R_1, R_2, R_3

In R_2, 1 is present at C_1, C_2, C_3

$\therefore$ Add new entries in W_2 at (R_1, C_1), (R_1, C_2), (R_1, C_3) (R_2, C_1), (R_2, C_2), (R_2, C_3) (R_3, C_1), (R_3, C_2), (R_3, C_3)

$$\therefore \qquad W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Step 4 : To find W_3

To find W_3 from W_2, we consider the IIIrd row and IIIrd column of W_2.

In C_3, 1 is present at R_1, R_2, R_3

In R_3, 1 is present at C_1, C_2, C_3, C_4

$\therefore$ Add new entries in W_2 at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_1, C_4), (R_2, C_1), (R_2, C_2), (R_3, C_3), (R_2, C_4), (R_3, C_1), (R_3, C_2), (R_3, C_3), (R_3, C_4)

$$\therefore \qquad W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Step 5 : To find W_4

To find W_4 from W_3, we consider 4^{th} row and 4^{th} column of W_3.

In C_4, 1 is present at R_1, R_2, R_3

In R_4, 1 is present at C_1, C_2, C_3

$\therefore$ Add new entries in W_3 at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_2, C_1), (R_2, C_2) ,(R_2, C_3), (R_3, C_1), (R_3, C_2), (R_3, C_3)

$$\therefore \qquad W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Hence the W_4 is the relation metrix of R^*

$$\text{And} \qquad R^* = \begin{Bmatrix} (a, a)\ (a, b)\ (a, c)\ (a, d)\ (b, a)\ (b, b)\ (b, c) \\ (b, d)\ (c, a)\ (c, b)\ (c, c)\ (c, d)\ (d, a)\ (d, b) \end{Bmatrix}$$

Example 4.9.8 *Find the transitive closure of R by Warshall's algorithm. A = {Set of positive integers ≤ 10}.*

R = {(a, b) / a divides b} **SPPU : May-07**

Solution :

Step 1 : We have |A| = 10. Thus we have to find

Warshall's sets W_0, W_1, W_2, ... W_{10}

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8), (1, 9), (1, 10), (2, 2), (2, 4), (2, 6), (2, 8), (2, 10), (3, 3), (3, 6), (3, 9), (4, 4), (4, 8), (5, 5), (5, 10), (6, 6), (7, 7), (8, 8), (9, 9), (10, 10)\}$$

The first set $W_0 = M_R$

$$\therefore \qquad W_0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The relation R itself is a transitive relation on the set of positive integers. Hence $R = R^*$ and

$$W_0 = M_R \text{ is the relation matrix of } R^*$$

Example 4.9.9 *Use Warshall's Algorithm to find transitive closure of R where*

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } A = \{1, 2, 3\}$$

SPPU : May-06, May-08

Solution :

Step 1 : We have $|A| = 3$. Thus we have to find Warshall's sets W_0, W_1, W_2, W_3

The first set is

$$W_0 = M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Step 2 : To find W_1 :

To find W_1 from W_0, we consider the first column and the first row.

In C_1, 1 is present at R_1, R_3

In R_1, 1 is present at C_1 and C_3

Thus add new entries in W_1 at (R_1, C_1), (R_1, C_3), (R_3, C_1), (R_3, C_3)

Which is given below

$$W_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Step 3 : To find W_2 :

To find W_2 from W_1, we consider the 2^{nd} column and 2^{nd} row.

In C_2, 1 is present at R_2, R_3

In R_2, 1 is present at C_2

Thus add new entries in W_2 at (R_2, C_2), (R_3, C_2) which is given below

$$W_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = W_1$$

Step 4 : To find W_3

To find W_3 from W_2, we consider the 3^{rd} column and 3^{rd} row.

In C_3, 1 is present at R_1, R_3

In R_3, 1 is present at C_1, C_2, C_3

Thus add new entries in W_3 at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_3, C_1), (R_3, C_2), (R_3, C_3) which is given below

$$W_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Hence W_3 is the relation matrix of R^*

and $R^* = \{(1, 1)\ (1, 2)\ (1, 3)\ (2, 2)\ (3, 1)\ (3, 2)\ (3, 3)\}$

4.10 Partially Ordered Set

SPPU : Dec.-06

A relation R on a set A is called a partially ordered relation iff R is reflexive, anti-symmetric and transitive relation.

The set A together with partially ordered relation is called a partially ordered set or POSET.

It is denoted by (A, R) or $(A, \leq)$ where $\leq$ is a partially ordered relation.

Examples :

1) $(\Re, \leq)$ $(N, \leq)$ are Posets.

where '$\leq$' is reflexive, antisymmetric and transitive relation.

2) If A = P(S) where S = (a, b, c) and for X, Y $\in$ A, Define $X \leq Y$ or XRY iff $X \subseteq Y$.

As $X \leq X \Rightarrow X \leq X$. $\therefore$ ' $\leq$ ' is reflexive.

If $X \leq Y$, $Y \leq Z \Rightarrow X \subseteq Y$ and $Y \subseteq X \Rightarrow X = Y$

$\therefore$ '$\leq$' is antisymmetric relation.

If $X \leq Y, Y \leq Z \Rightarrow X \subseteq Y$ and $Y \subseteq Z \Rightarrow X \subseteq Z \Rightarrow X \leq Z$

$\Rightarrow$ '$\leq$' is transitive relation.

$\therefore$ $(P(S), \subseteq)$ or $(P(S), \leq)$ is a poset.

I) Comparable elements : Let $(A, \leq)$ be a poset. Two elements a, b in A are said to be comparable elements if $a \leq b$ or $b \leq a$. Two elements a and b of a set A are said to be non-comparable if neither $a \leq b$ nor $b \leq a$.

In above example (2),

The comparable elements are

$$\{a\} = \{a, b\}, \{b\} = \{a, b, c\}\ \{b, c\} \subseteq \{a, b, c\}$$

Non comparable elements are

$$\{a\} \not\subseteq \{b\} \quad \{a\} \not\subseteq \{a, c\}$$

II) Totally ordered set : Let A be any nonempty set. The set A is called linearly ordered set or totally ordered set if every pair of elements in A are comparable.

i.e. for any a, b $\in$ A either $a \leq b$ or $b \leq a$.

4.10.1 Hasse Diagram

It is useful tool, which completely describes the associated partially ordered relation. It is also known as ordering diagram.

A diagram of graph which is drawn by considering comparable and non-comparable elements is called Hasse diagram of that relation. Therefore while drawing Hasse diagram following points must be followed.

1) The elements of a relation R are called vertices and denoted by points.
2) All loops are omitted as relation is reflexive on poset.
3) If aRb or $a \leq b$ then join a to b by a straight line called an edge the vertex b appears above the level of vertex a. Therefore the arrows may be omitted from the edges in Hasse diagram.
4) If $a \not\leq b$ and $b \not\leq a$ i.e. a and b are non - comparable elements, then they lie on same level and there is no edge between a and b.
5) If $a \leq b$ and $b \leq c$ then $a \leq c$. So there is a path $a \to b \to c$. Therefore do not join a to c directly i.e. delete all edges that are implied by transitive relation.

Example 4.10.1 *Draw Hasse diagram of a poset (P(s), $\subseteq$) where S = {a, b, c}.*

Solution : P (s) = { ϕ, {a}, {b}, {c}, {a, b} {a, c} {b, c} {a, b, c}}

Now find the comparable and non comparable elements.

$\phi \subseteq \{a\}$, $\phi \subseteq \{b\}$, $\phi \subseteq \{c\}$, $\therefore$ {a}, {b}, {c} lie above the level of ϕ.

$\{a\} \subseteq \{a, b\}, \{b\} \subseteq \{a, b\}, \{c\} \subseteq \{a, c\}$ $\therefore$ {a, b}, {b, c} {a,c} lies above the level of {a}, {b}, {c}.

{a, b} $\subseteq$ S, {b, c} $\subseteq$ S, {a, c} $\subseteq$ S $\therefore$ S lies above the level of {a, b}, {a, c}, {b, c}

But {a}, {b} {c{ are non comparable $\therefore$ {a}, {b}, {c} lie on same level.

{a, b}, {a, c}, {b, c} are non comparable $\therefore$ lie on same level.

By considering the above observations, the Hasse diagram is as follows :

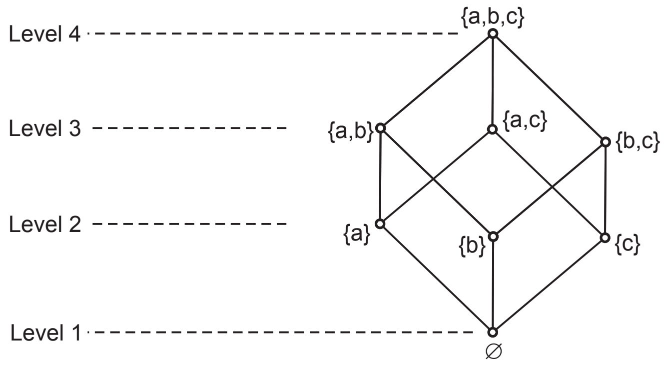


Fig. 4.10.1

4.10.2 Chains and Antichains

Let (A, ≤) be a poset. A subset of A is called a chain if every pair of elements in the subset are related.

A subset of A is called antichain if no two distinct elements in a subset are related. e.g. In above example (4.10.1)

1) The chains are $\{\phi, \{a\}, \{a, b\}, \{a, b, c\}\}$, $\{\{a\}, \{a, c\}, \{a, b, c\}\}$, $\{\{b, c\}, \{a, b, c\}\}$
2) Antichains are {{a}, {b}, {c}}

Note : 1) The number of elements in the chain is called the length of chain.

2) If the length of chain is n in a poset (A, ≤) then the elements in A can be partitioned into n disjoint antichains.

4.10.3 Elements of Poset

1) Let $(A, \leq)$ be a poset. An element $a \in A$ is called a **maximal element** of A if there is no element $c \in A$ such that $a \leq c$.
2) An element b ∈ A is called a **minimal element** of A if there is no element $c \in A$ such that $c \leq b$
3) Greatest element : An element $x \in A$ is called a **greatest element** of A if for all $a \in A$, $a \in x$. It is denoted by 1 and is called the unit element.
4) Least element : An element $y \in A$ is called a **least element** of A if for all $a \in A$, $y \leq a$.

It is denoted by 0 and is called as **zero element**.

5) Least upper bound (*l*ub) : Let $(A, \leq)$ be a poset. For a, b, c ∈ A, an element C is called **upper bound** of a and b if $a \leq c$ and $b \leq c$ C is called as least upper bound of a and b in A if C is an upper bound a and b there is no upper bound d of a and b such that $d \leq c$. It is also known as **supremum.**

6) Greatest lower bound (glb) : Let $(A, \leq)$ be a poset. for a, b, $l \in A$, an element l is called the **lower bound** of a and b if $l \leq a$ and $l \leq b$.

An element l called the **greatest lower bound** of a and b if l is the lower bound of a and b and there is no lower bound f of a and b such that • $l \leq f$.

glb is also called as **infimum.**

Example 4.10.2 *Determine the greatest and least elements of the poset whose Hasse diagrams are shown below.*

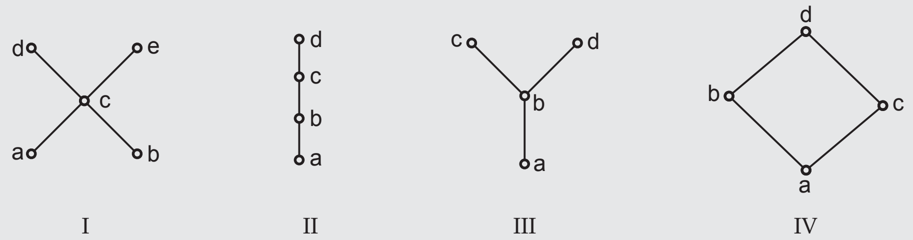


Fig. 4.10.2

Solution : The Poset shown in Fig. 4.10.2 (I) has neither greatest not least element.

The Poset shown in Fig. 4.10.2 (II), has greatest and a as least element.

The Poset shown in Fig. 4.10.2 (III), has no greatest element but a is the least element.

The Poset shown in Fig. 4.10.2 (IV), has greatest and a as least element.

Example 4.10.3 *Find glb, lwb, ub, lb, maximal, minimal, of the poset (A, R), Here aRb if a / b where. A = { 2, 3, 5, 6, 10, 15, 30, 45}*

Solution : We have A = {2, 3, 5, 6, 10, 15, 30, 45} and aRb iff a|b.

Hasse diagram is as follows :

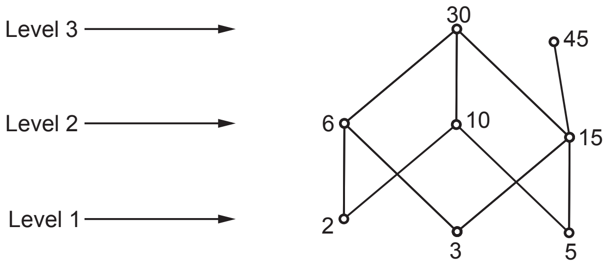


Fig. 4.10.3

1) Here 10 and 30 are upper bounds of 2 and 5, But 10 is the least upper bound of 2 and 5.

2) 5, 15, 3 are lower bounds of 30 and 45. But 15 is the greatest lower bound of 30 to 45.
3) This Poset has neither greatest element nor least element.
4) This poset has two maximal elements 45 and 30 as there is no element c such that $45 \leq C$ and $30 \leq C$.
5) This poset has three minimal elements 2, 3 and 5. because there is no element $x \in A$ such that $x \leq 2$, $x \leq 3$ and $x \leq 5$

4.10.4 Types of Lattices

A lattice is a poset in which every pair of elements has a least upper bound (lub) and a greatest lower (glb).

Let $(A, \leq)$ be a poset and $a, b \in A$ then lub of a and b is denoted by $a \vee b$. It is called the join of a and b.

i.e. $a \vee b = \text{lub}(a, b)$

The greatest lower bound of a and b is called the meet of a and b and it is denoted by $a \wedge b$

$$\therefore \quad a \wedge b = \text{glb}(a, b)$$

From the above discussion, it follows that a lattic is a mathematical structure with two binary operations $\vee$ (join) and $\wedge$ (meet). It is denoted by $\{L, \vee, \wedge\}$.

Examples :

Example 4.10.4 *Let A = {1, 2, 3} P (A) =* $\{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$ *show that P(A),* $\subseteq$*) is a lattice*

Solution : The Hasse diagram of the poset $(P(A), \subseteq)$ is given below :

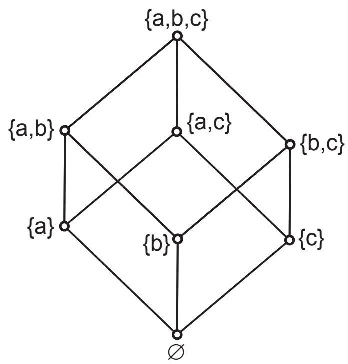
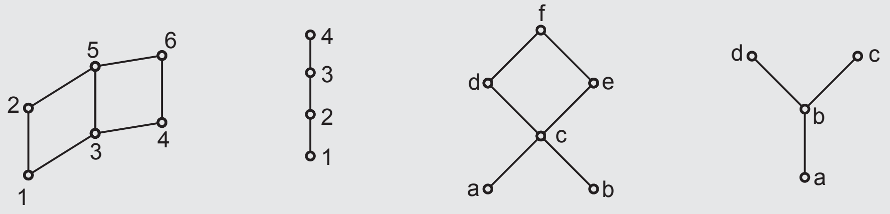


Fig. 4.10.4

Here every pair of elements of a poset has lub and glb. Hence $(P(A), \subseteq)$ is a lattice.

Example 4.10.5 *Determine which of the following posets are lattice.*



I II III IV

Fig. 4.10.5

Solution : I) In Fig. 4.10.5 (I), every pair of elements has lub and glb.

∴ It is a lattice.

II) In Fig. 4.10.5 (II), every pair of elements has lub and glb.

∴ It is a lattice.

II) In Fig. 4.10.5 (III), $a \wedge b$ does not exist.

∴ It is not a lattice.

IV) In Fig. 4.10.5 (IV), $c \vee d$ does not exist.

∴ It is not a lattice.

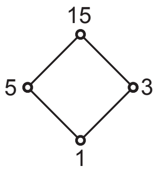
Example 4.10.6 *Let A be the set of positive factors of 15 and R be a relation on A s.t. R = {xRy | x divides y, x y ∈ A}. Draw Hasse diagram and give and* $\wedge$ and $\vee$ *for lattice.*

SPPU : Dec.-06

Solution : We have A = {1, 3, 5, 15}

R = {(1,1) (1,3) (1,5) (1, 15) (3,15) (5,15) (15,15)}

Hasse diagram of R is : **Table for $\wedge$ and $\vee$**



$\vee$	1	3	5	15
1	1	3	5	15
3	3	3	15	15
5	5	15	5	15
15	15	15	15	15

$\wedge$	1	3	5	15
1	1	1	1	1
3	1	3	1	3
5	1	1	5	5
15	1	3	5	15

Fig. 4.10.6

Every pair of elements has lub and glb. ∴ It is a lattice.

4.10.5 Properties of a Lattice

Let $(L, \wedge, \vee)$ be a lattice and $a, b, c \notin L$. Then L satisfies the following properties.

1) Commutative property

$$a \wedge b = b \wedge a \text{ and } a \vee b = b \vee a$$

2) Associative law

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

3) Absorption law

$$a \wedge (a \vee b) = a \text{ and } av\ (a \wedge b) = a$$

3) $a \wedge a = a$, $a \vee a = a$

4) $a \wedge b = a$ iff $a \vee b = b$

4.10.6 Types of Lattices

I) Bounded lattice : A lattice L is called a bounded lattice if it has a greatest element 1 and least element 0.

II) Sublattice : Let, $(L, \vee, \wedge)$ be a lattice. A non empty subset L_1 of L is called a sublattice of L if L_1 itself is a lattice w.r.t. the operations of L.

III) Distributive lattice : A lattice $(L, \vee, \wedge)$ is called a distributive lattice if for any elements $a,b,c \in L$, it satisfies the following properties,

i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

If the lattice does not satisfy the above properties then it is called a non distributive lattice.

Theorem : Let, $(L, \wedge, \vee)$ be a lattice with universal bounds 0 and 1 then for any $a \in L$, $a \vee 1 = 1, a \wedge 1 = a$, $0 \vee a = a$, $0 \wedge a = 0$.

III) Complement lattice : Let $(L, \wedge, \vee)$ be a lattice with universal bounds 0 and 1 for any $a \in L$, $b \in L$ is said to be complement of a if $a \vee b = 1$ and $a \wedge b = 0$.

A Lattice in which every element has a complement in that lattice, is called the complemented lattice.

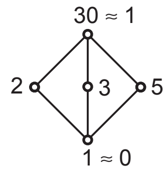


Fig. 4.10.7

e.g. 1) The Hasse diagram is here $0 \approx 1$ and $1 \approx 30$.

i) $2 \wedge 3 = 0$ and $2 \vee 3 = 1$, $2 \wedge 5 = 0$ and $2 \vee 5 = 1$

$\therefore$ 2 has two compliments 3 and 5

Hence the complement is not unique.

4.11 Principle of Duality

SPPU : Dec.-10, 11, 13, 14, 15, 16, May-14, 15, 19

Any statement about lattice involving $\wedge, \vee \leq, \geq$ remains true if '$\wedge$' is replaced by '$\vee$', '$\vee$ by $\wedge$', '$\leq$ by $\geq$', '$\geq$ by $\leq$', '0 by 1' and '1 by 0'.

e.g. 1) $a \vee (b \wedge c) = a \wedge (b \vee c)$

2) $a \wedge (b \vee 1) = a \vee (b \wedge 0)$

Examples

Example 4.11.1 *Let A = {1, 2, 3, 4, 6, 9, 12} Let a relation R on a set A is R = {(a,b)/ a divides b ∀ a, b ∈ A}. Give list of R. Prove that it is a partial ordering relation. Draw Hasse diagram of the same. Prove or disprove it is a lattice. Give two examples of chain and antichains.* **SPPU : Dec.-11, May-19, Marks 6**

Solution : We have A = {1, 2, 3, 4, 6, 9, 12}

and $R = \begin{Bmatrix} (1,1), (1,2), (1,3), (1,4)\ (1,6), (1,9), (1,12), (2,2), (2,4), (2,6), (2,12)\ (3,3), (3,6), (3,9) \\ (3,12), (4,4), (4,12), (6,6), (6,12), (9,9), (12,12) \end{Bmatrix}$

We know that for any $a \in A$, $a \mid a$ $\therefore$ aRa

$\therefore$ R is a reflexive relation.

As $a \mid b$ and $b \mid a \Rightarrow a = b$ $\therefore$ R is antisymmetric relation.

As $a \mid b$ and $b \mid c \Rightarrow a \mid c \Rightarrow$ R is a transitive relation.

$\therefore$ R is reflexive antisymmetric and transitive

$\therefore$ (A, R) is a poset and R is a partial ordering relation.

Hasse diagram is as follows :

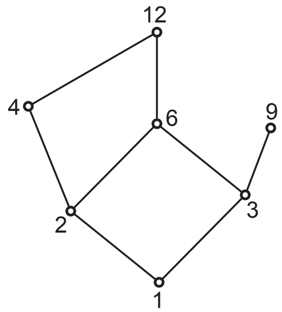


Fig. 4.11.1

In above diagram $6 \vee 9$ does not exist. $\therefore$ It is not a lattice.

i) Chains are

{1, 2, 4, 12}

{1, 2, 6, 12}

{1, 3, 6, 12}

ii) Antichains are

{6, 9} {9, 12}

{4, 9}

Example 4.11.2 *Determine whether the poset represented by each of the Hasse diagram are lattices. Justify your answer.*

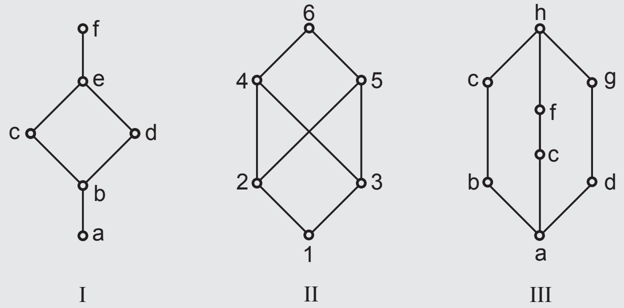


Fig. 4.11.2

SPPU : Dec.-10

Solution :

I) In Fig. 4.11.2 (I), every pair of element has glb and lub. ∴ It is a lattice.

II) In Fig. 4.11.2 (II), every pair of elements has lub and glb. ∴ It is a lattice.

III) In Fig. 4.11.2 (III), every pair of elements gas lub and glb. ∴ It is a lattice.

Example 4.11.3 *Show that the set of all divisors of 36 forms a lattice.* **SPPU : Dec.-14**

Solution : Let A = {1, 2, 3, 4, 6, 9,12, 18, 36 } and Let '≤' is a divisor of.

It's Hasse diagram is as follows.

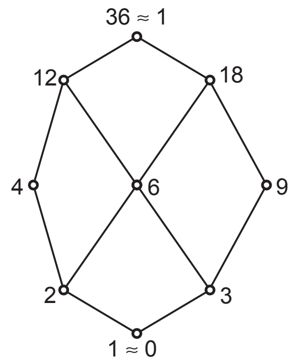


Fig. 4.11.3

The universal upper bound 1 is 36 and lower bound 0 is 1. Every pairs of elements of this poset has lub and glb.

∴ It is a lattice.

Example 4.11.4 *SLet n be a positive integer, S_n be the set of all divisors of n, Let D denote the relation of divisor. Draw the diagram of lattices for n = 24, 30, 6.* **SPPU : May-15**

Solution : Given that

i) We have $S_6 = \{1,2,3,6\}$, D is the relation of divisor.

ii) $S_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$

iii) $S_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

Diagrams of Lattices are as follows.

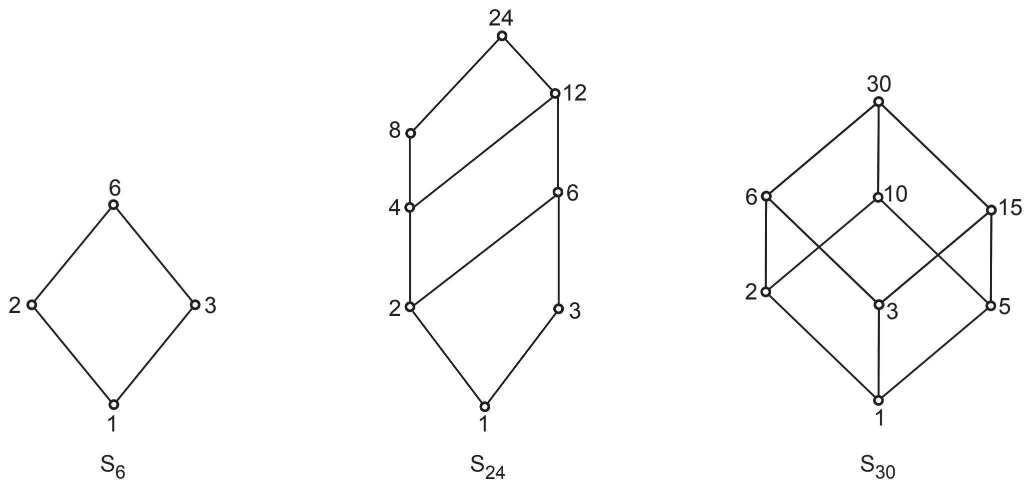


Fig. 4.11.4

Example 4.11.5 *Show that the set of all divisors of 70 forms a lattice.* **SPPU : Dec.-13, May-19, Marks 3**

Solution : Let A = {1, 2, 5, 7, 10, 14, 35, 70}

and Let '≤' is "a divisor of".

The universal upper bound 1 is 70 and the lower bound 0 is 1.

It's Hasse diagram is as follows :

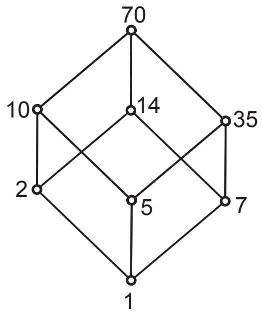


Fig. 4.11.5

Every pair of elements of A has $\wedge$ and $\vee$.

$\therefore$ It is a lattice [write table of $\wedge$ and $\vee$].

Example 4.11.6 *Let A = {1, 2, 3, 4 , 5, 6, 7, 8 ,9, 12, 18, 24} be ordered by the relation x divides y. Show that the relation is a partial ordering and draw Hasse diagram.*

SPPPU : Dec-15

Solution : We have A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 18, 24}

$$R = \{(x,y) \mid x \text{ divides } y, \text{ for } x, y \in A\}$$

$$R = \{(1, 2)\ (1, 3), (1, 4), (1, 5)\ (1, 6)\ (1, 7)\ (1, 8)\ (1, 9)\ (1, 12)\ (1, 18)\ (1, 24)$$
$$(2, 2)\ (2, 4)\ (2, 6)\ (2, 8)\ (2, 12)\ (2, 18)\ (2, 24)\ (5, 5)\ (6, 6)\ (6, 12)\ (6, 18)$$
$$(6, 24), (7, 7)\ (8, 8)\ (8, 24)\ (9, 9)\ (9, 18)\ (12, 12)\ (12, 24)\ (18, 18)\ (4, 24)\}$$

We have for any $x \in A$, $x|x \Rightarrow R$ is a reflexive for $x|y$ and $y|x \Rightarrow x = 0 \Rightarrow R$ is antisymmetric. If $x|y$ and $y|z \Rightarrow x|z$ $\therefore$ R is a transitive relation.

$\therefore$ R is a partial ordering relation, It's Hasse diagram is as follows.

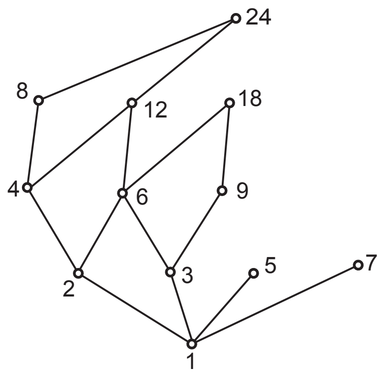


Fig. 4.11.6

Example 4.11.7 *Let x = {2, 3, 6, 12, 24, 36} and $x \leq y$ iff x divides y find*
i) Maximal element ii) Minimal element iii) Chain iv) Antichain v) Is Poset lattice

SPPU : May-14

Solution : We have x = {2, 3, 6, 12, 24, 36}

The relation 'R' = '≤'

∴ R = {(2, 2) (2, 6) (2, 12) (2, 24) (2, 36) (3, 3) (3, 6) (3, 12) (3, 24) (3, 36), (6, 6) (6, 12) (6, 24) (6, 36) (12, 12)(12, 24) (12, 36) (24, 24) (36, 36)}.

It's Hasse diagram is as follows .

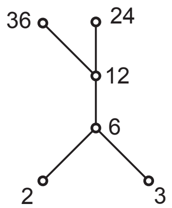


Fig. 4.11.7

i) Maximal elements are 24, 36

ii) Minimal elements are 2, 3

iii) Chain { 2, 6, 12, 24}, { 2, 6, 12, 36}, {3, 6, 12, 24}, {3, 6, 12, 36}

iv) Antichain : {2, 3} {24, 36}

v) The given poset is not a lattice as $2 \wedge 3$ does not exist.

Example 4.11.8 *The following is the Hasse diagram of the poset :*
{(a, b, c, d, e), <} Is it a lattice ? Justify

SPPU : Dec.-16, Marks 2

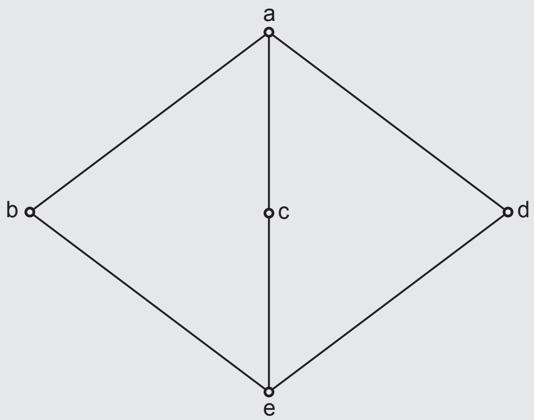


Fig. 4.11.8

Solution : Let

$$A = \{a, b, c, d, e\}$$

From the given Hasse diagram, every pair of element has glb and lub. Hence it is a lattice.

❑❑❑

Unit - II

5 Functions

Syllabus

Surjective, Injective and Bijective functions, Identity function, Partial function, Invertible function, Constant function, Inverse functions and Compositions of functions, The Pigeonhole Principle.

Contents

5.1 Introduction

The concept of relation was defined very generally in the preceeding chapter. We shall now discuss a particular class of relations called functions. Many concepts of computer science can be conveniently stated in the language of functions.

5.2 Function

SPPU : May-07, 08, 15, Dec.-07, 12,

Let A and B be non-empty sets. A function f from A to B is denoted by $f : A \rightarrow B$ and defined as a relation from A to B such that for every $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in f$.

If $(a, b) \in f$ then it can be written as f(a) = b. Functions are also called as Mappings or Transformations.

e.g.

1) f = {(a, b) (b, c) (c, a)} is a function on set A = {a, b, c}
2) f = {(a, b) (b, c) (b, a)} is not a function as (b, c) and $(b, a) \in f$.
3) f = {(a, b) (b, c)} is not a function on set A = {a, b, c} as for $C \in A \nexists$ any element x in A such that $(c, x) \in f$.

5.2.1 Important Definitions

I) Let $f : A \rightarrow B$ be a function such that f(a) = b then $b \in B$ is called the image of $a \in A$ and a is called the pre-image of b. The element a is called argument of f.

II) Let $f : A \rightarrow B$ be a function. The set A is known as domain set of f and the set B is known as co-domain set of f.

III) The range set of a function $f : A \rightarrow B$ is denoted by R(f) and defined as

$$R(f) = \{b / b \in B \text{ and } f(a) = b \text{ for some } a \in A\}$$

In other words, range of f is the set of all images of the elements of A under f.

Examples

Example 5.2.1 *If f is a function such that $f(x) = x^2 - 1$ where $x \in R$ find the values of f(–1), f(0), f(1), f(3) and range set of f.*

Solution : We have $f(x) = x^2 - 1$

$$\therefore \quad f(-1) = (-1)^2 - 1 = 1 - 1 = 0$$

$$f(0) = 0 - 1 = -1$$

$$f(1) = 1^2 - 1 = 0$$

$$f(3) = 3^2 - 1 = 8$$

For any $x \in \Re$, $x^2 - 1 \in \Re$

$\therefore$ The range of f is $R(f) = \{x / -1 \le x < \infty\}$

Example 5.2.2 *Let* $f : \Re \to \Re$ *where f is defined by*

a) $f(x) = \sqrt{x}$

b) $f(x) = x^2$

c) $f(x) = \sin x \; \forall x \in \Re$

Solution :

a) We have $f(x) = \sqrt{x}$ $\quad \therefore f(4) = \sqrt{4} = \pm 2$

$\therefore$ f is not a function. Therefore we can not find range of f.

b) We have $f(x) = x^2$, x^2 is positive real number

Hence range of f is $R(f) = \{x \mid 0 \le x < \infty\}$

c) $\forall x \in \Re$, sin x lies between –1 to +1

$$\therefore \quad R(f) = \{x \mid -1 \le x \le 1\}$$

5.2.2 Partial Functions

Let A and B be two non empty sets. A partial function f with domain set A and codomain set B is any function from A' to B where A' CA. For any $x \in A - A'$, f(x) is not defined.

To make the distinction more clear, the function which is not partial is called as a total function.

e.g.

1)

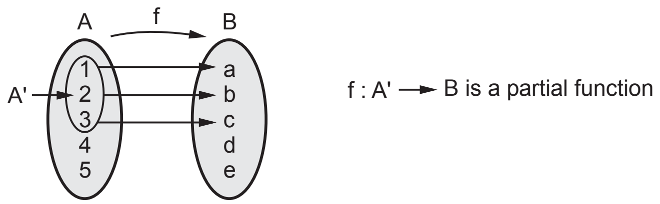


Fig. 5.2.1

2) The function $f : \Re \to \Re$ defined as $f(x) = \dfrac{1}{x}$ is a partial function. As it is not defined for x = 0.

3) The function $f : \Re \to \Re$ defined as $f(x) = \sqrt{x}$ is a partial function, as $\sqrt{x}$ is not defined for x < 0, in $\Re$.

5.2.3 Equality of Two Functions

Two functions $f : A \to B$ and $g : A \to B$ are said to be equal functions or identical functions iff.

$$f(x) = g(x) \; \forall \; x \in A$$

Note : Two functions $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \Re \to \Re$ defined as $f(x) = x, \forall \; x \in \mathbb{Z}$ and $g(x) = x, \forall \; x \in \Re$ are not identical or equal functions because their domains are not same.

5.2.4 Identity Function

Let A be any non empty set and function $f : A \to A$ is said to be the identity function if $f(x) = x, \forall \; x \in A$.

e.g.

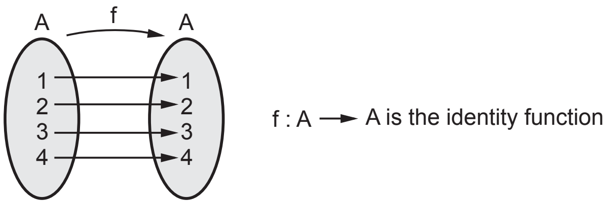


Fig. 5.2.2

5.2.5 Constant Function

A function $f : A \to B$ is said to be a constant function if $f(x) = \text{constant} = k \; ; \forall \; x \in A$.

The range set of a constant function consists of only one element.

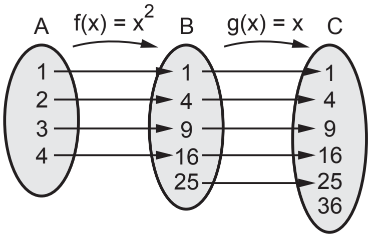
5.2.6 Composite Function

Let $f : A \to B$ and $g : B \to C$ be two functions.

The composite function of f and g is denoted by gof and defined as gof : $A \to C$ is a function such that $(gof)(a) = g[f(a)] \; \forall \, a \in A$.

Note : gof is defined only when the range of f is a subset of the domain of g.

e.g. 1)



Therefore gof : A → C

gof (1) = g[f(1)] = g(1) = 1

gof (2) = g[f(2)] = g(4) = 4

gof (3) = g[f(3)] = g(9) = 9

gof (4) = g[f(4)] = g(16) = 16

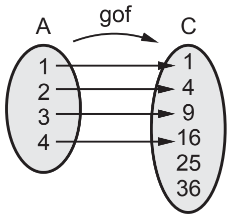


Fig. 5.2.3 **Fig. 5.2.4**

Example 5.2.3 *Let $f(x) = x+2$, $g(x) = x - 2$, $h(x) = 3x$, for $x \in \Re$ Where $\Re$ is the set of real numbers*

Find i) gof ii) fog iii) fof iv) hog v) gog vi) foh vii) hof viii) fohog ix) gofoh.

SPPU : May-08, 15, Dec.-12, Marks 4

Solution : Let $x \in \Re$ be any real number.

i) $gof(x) = g[f(x)] = g[x + 2] = x + 2 - 2 = x$

ii) $fog(x) = f[g(x)] = f[x - 2] = x - 2 + 2 = x$

iii) $fof(x) = f[f(x)] = f[x + 2] = x + 2 + 2 = x + 4$

iv) $hog(x) = h[g(x)] = h[x - 2] = 3(x - 2) = 3x - 6$

v) $gog(x) = g[g(x)] = g[x - 2] = x - 2 - 2 = x - 4$

vi) $foh(x) = f[h(x)] = f[3x] = 3x + 2$

vii) $hof(x) = h[f(x)] = h[x + 2] = 3(x + 2) = 3x + 6$

viii)
$$\begin{aligned} fohog(x) &= f[h(g(x))] = f[h(x - 2)] \\ &= f[3(x - 2)] = f(3x - 6) \\ &= 3x - 6 + 2 = 3x - 4 \end{aligned}$$

ix)
$$\begin{aligned} gofoh(x) &= g[f(h(x))] = g[f(3x)] \\ &= g[3x + 2] = 3x + 2 - 2 = 3x \end{aligned}$$

Example 5.2.4 *Let functions f and g be defined by $f(x) = 2x + 1$, $g(x) = x^2 - 2$*

Find a) gof(4) and fog (4)

b) gof(a + 2) and fog(a + 2)

c) fog(5)

d) gof(a + 3)

SPPU : May-07, Dec.-07, Marks 4

Solution :

a)
$$\begin{aligned} gof(4) &= g[f(4)] = g[2(4) + 1] = g[9] = 9^2 - 2 = 79 \\ fog(4) &= f[g(4)] = f(4^2 - 2) = f(14) = 2(14) + 1 = 29 \end{aligned}$$

b)
$$\begin{aligned} gof(a + 2) &= g[f(a + 2)] = g[2(a + 2) + 1] = g[2a + 5] \\ &= (2a + 5)^2 - 2 = 4a^2 + 20a + 23 \\ fog(a + 2) &= f[g(a + 2)] = f[(a + 2)^2 - 2] = f[a^2 + 4a + 2] \\ &= 2[(a^2 + 4a + 2] + 1 = 2a^2 + 8a + 5 \end{aligned}$$

c) $fog(5) = f[g(5)] = f[25 - 2] = f(23) = 2(23) + 1 = 47$

d) $gof(a + 3) = g[f(a + 3)] = g[2(a + 3) + 1] = g[2a + 7]$

$= [2a+7]^2 - 2 = 4a^2 + 28a + 47$

5.3 Special Types of Functions

SPPU : Dec.-10, 11, 16, 17, May-18

I) Let $f : A \to B$ be a function.

i) Function f is said to be one to one (or Injective) function if distinct elements of A are mapped into distinct elements of B.

i.e. f is one to one if

$$a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$$

OR $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

ii) Function f is said to be onto function if each element of B has at least one preimage in A.

OR

A function $f : A \to B$ is called onto (or surjective) function if the range set of f is equal to B.

iii) A function f is called a objective function if it is both one to one and onto.

iv) A function f is called into function if $\exists$ at least one element in B which has no preimage in A.

i.e. A function f is called into function if $R(f) \neq B$

2) Let a function $f : A \to B$ be a bijective function then $f^{-1} : B \to A$ is called the inverse mapping of f and defined as $f(b)^{-1} = a$ iff $f(a) = b$

It is also known as invertible mapping.

3) Characteristic function of a set :

Let U be a universal set and A be a subset of U.

Then the function $\psi_A : U \to \{0, 1\}$ defined by

$$\psi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

is called a characteristic function of the set A.

Examples

Example 5.3.1 *Give examples of functions of the following types by diagrams.*

a) Injective function but not surjective

b) Surjective but not injective

c) Neither injective nor surjective

d) Injective as well as surjective

e) Into function

f) Inverse function

Solution :

a) Injective but not surjective

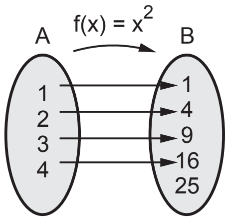


Fig. 5.3.1

b) Surjective but not injective

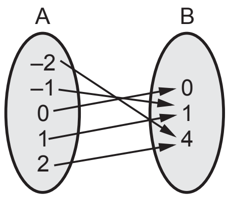


Fig. 5.3.2

c) Neither injective nor surjective

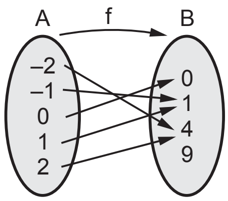


Fig. 5.3.3

d) Injective as well as surjective i.e. bijective

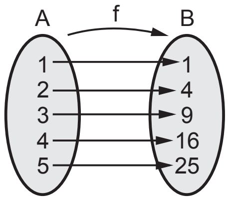


Fig. 5.3.4

e) Into function

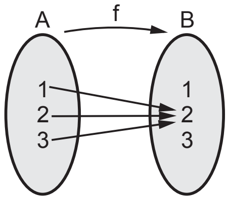


Fig. 5.3.5

f) Inverse function

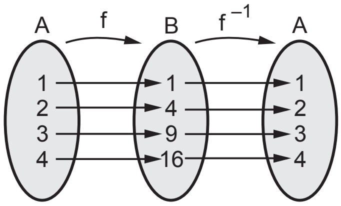


Fig. 5.3.6

Example 5.3.2 *Determine if each is a function. If yes is it injective, surjective, bijective ?*

a) Each person in the earth is assigned a number which corresponds to his age.

b) Each student is assigned a teacher.

c) Each country has assigned it's capital. **SPPU : Dec.-11, Marks 4**

Solution :

a) Every person has unique age

$\therefore$ It is a function. Two person's may have same age. $\therefore$ It is not injective. There is no person whose age is 300 years. $\therefore$ It is not surjective. $\therefore$ Function is not bijective.

b) It is a function. It is not injective. It is not surjective. $\therefore$ It is not bijective.

c) It is a function. It is injective as well as surjective. $\therefore$ It is bijective.

Example 5.3.3 *Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$. Determine whether the relation R from A to B is a function. Justify. If it is function give the range :*

i) $R = [(a, 1), (b, 2), (c, 1), (d, 2)]$

ii) $R = [(a, 1), (b, 2), (a, 2), (c, 1), (d, 2)]$ **SPPU : Dec.-16, Marks 4**

Solution : Given that

$$A = \{a, b, c, d\}, B = \{1, 2, 3\}$$

i) $$R = \{(a, 1), (b, 2), (c, 1), (d, 2)\}$$

Every element of the domain set A is related to the unique element of set B.

$\therefore$ R is a funtion from $A \to B$.

Range of R is $\{1, 2\}$.

ii) $$R = \{(a, 1), (b, 2), (a, 2), (c, 1) (d, 2)\}$$

Here $a \to 1$ and $a \to 2$ hence a has two images 1 and 2 in set B. i.e. a has no unique image in set B. Thus given relation R is not a function.

Example 5.3.4 *What are relations and functions. Given a relation $R = \{(1, 4), (2, 2), (3, 10), (4, 8), (5, 6)\}$ and chek whether the following relations R_1, R_2, R_3 and R_4 is a function or not.*

$R_1 = \{(1, 4), (2, 4), (3, 4), (4, 4), (5, 4)\}$

$R_2 = \{(1, 2), (2, 4), (2, 10), (3, 8), (4, 6), (5, 4)\}$

$R_3 = \{(1, 6), (2, 2), (4, 4), (5, 10)\}$

$R_4 = \{(1, 6), (2, 2), (3, 2), (4, 4), (5, 10\}$ **SPPU : Dec.-17, Marks 6**

Solution : Refer sections 4.3 and 5.2

Domain set = $\{1, 2, 3, 4, 5\}$

Codomain = $\{2, 4, 6, 8, 10\}$.

Given that $R_1 = \{(1, 4), (2, 4), (3, 4), (4, 4), (5, 4)\}$

All elements are mapped to 4.

$\therefore$ R_1 is a constant function.

$R_2 = \{(1, 2), (2, 4), (2, 10), (3, 8), (4, 6), (5, 4)\}$

In R_2 $2 \to 4$ and $2 \to 10$ which is not possible in function.

$\therefore$ R_2 is not a function

$R_3 = \{(1, 6), (2, 2), (4, 4), (5, 10)\}$

As $3 \in D$ but 3 has no any image in R_3

$\therefore$ R_3 is not a function.

$R_4 = \{(1, 6), (2, 2), (3, 2), (4, 4), (5, 10)\}$

Every element of a domain set has unique image in codomain set so R_4 is a function.

Example 5.3.5 *Let A = B be the set of real numbers.*

$f : A \to B$ given by $f(x) = 2x^3 - 1$

$g : B \to A$ given by $g(y) = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}}$

Show that f is a bijective function and g is also bijective function :

SPPU : Dec.-10, Marks 4

Solution :

1) Suppose $f(x_1) = f(x_2)$

$\Rightarrow \quad 2x_1^3 - 1 = 2x_2^3 - 1$

$\Rightarrow \quad x_1^3 = x_2^3$

$\Rightarrow \quad x_1 = x_2$

$\therefore$ f is injective mapping

Let $y \in B$ and $f(x) = y \Rightarrow 2x^3 - 1 = y$

$\therefore \quad 2x^3 = 1 + y \Rightarrow x^3 = \frac{1+y}{2}$

$\Rightarrow \quad x = \sqrt[3]{\frac{1+y}{2}} = \sqrt[3]{\frac{y}{2} + \frac{1}{2}} \in A = \Re$ for any $y \in B$

$\therefore$ f is a surjective mapping.

Thus f is injective as well as surjective function.

Hence f is a bijective function.

We have $f(x) = y \Rightarrow x = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}} \Rightarrow f^{-1}(y) = x = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}} = g(y)$

Thus $f^{-1} = g$. We know that if f is bijective function then f^{-1} is also bijective. Hence g is bijective function.

Example 5.3.6 *Explain classification of functions with example.*

Solution : Depending upon the nature of function, there are mainly two functions.

1. **Algebraic function :** A function which consists of a finite number of terms involving powers and roots of the independent variable and four fundamental operations addition, subtraction, multiplication and division, is called an algebraic function.

There are three types of algebraic functions.

(a) Polynomial function : A function of the form

$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$, where n is positive integer, a_0, a_1, a_2, ... ,a_n are real numbers and $a_0 \neq 0$, is called polynomial function of x in degree n.

e.g. $f(x) = x^3 - 3x^2 + 2x - 5$

(b) Rational function : A function of the form $\frac{f(x)}{g(x)}$ where f(x) and g(x) are polynomial functions and $g(x) \neq 0$ is called rational function e.g. $\frac{x^2 - 3x + 5}{x^2 + 1}$

(c) Irrational function : A function involving radicals is called irrational function.
e.g. $f(x) = x^{2/3} + 5x^2 + 1, \sqrt[2]{x+1}$

2. **Transcendental function :** A function which is not algebraic is called transcendental function.

e.g. $f(x) = \sin x + x^3 + 5x$

(a) Trigonometric function : The six functions sin x, cos x, tan x, sec x, cosec x, cot x, where x is in radians are called trigonometric functions.

e.g. $f(x) = \sin x + \tan x$.

(b) Inverse trigonometric function : The six functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\text{cosec}^{-1} x$, $\sec^{-1} x$ and $\cot^{-1} x$ are called inverse trigonometric functions.

e.g. $f(x) = \cos^{-1} x + 5 \tan^{-1} x$

(c) Exponential function : A function of the form $f(x) = a^x$ $(a > 0)$ satisfying $a^x \cdot a^y = a^{x+y}$ and $a' = a$ is called exponential fucntion. e.g. $f(x) = 5^x$.

(d) Logarithmic function : The inverse function of the exponential function is called logarithmic function. e.g. $f(x) = \log x$.

Example 5.3.7 *If $f : A \rightarrow B$ is bijective function then f^{-1} is unique.*

Solution : Let f $A \rightarrow B$ is a bijective function.

Claim : Show that f^{-1} is unique.

Suppose f^{-1} is not unique, so there are two inverse functions say, g and h.

Let $x_1, x_2 \in A$, $\exists$ y in B such that

$$f^{-1}(y) = x_1 \Rightarrow g(y) = x_1 \Rightarrow f(x_1) = y$$

and $$f^{-1}(y) = x_1 \Rightarrow h(y) = x_2 \Rightarrow f(x_2) = y$$

This implies $f(x_1) = f(x_2)$, but f is 1 – 1 function.

$$\therefore \quad x_1 = x_2$$

$$\Rightarrow \quad g(y) = h(y)\ \forall\ y$$

$$\Rightarrow \quad g \equiv h \quad \text{i.e. g and h are equal function.}$$

Hence universe of f is unique.

Example 5.3.8 *If $f : A \rightarrow B$ and $g : B \rightarrow C$ are bijective functions the gof is also bijective.*

Solution : Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two bijective functions then gof $A \rightarrow C$ is a function.

(i) Let $x_1, x_2 \in A$ and suppose $gof(x_1) = gof(x_2)$

$$g[f(x_1)] = g[f(x_2)]$$

$$\Rightarrow \quad f(x_1) = f(x_2) \qquad \text{... (}\because\text{ g is 1 – 1 function)}$$

$$\Rightarrow \quad x_1 = x_2 \qquad \text{... (}\because\text{ f is 1 – 1 function)}$$

$\Rightarrow$ gof is 1 – 1 function.

(ii) Let $z \in C$.

$\therefore$ $\exists$ y in B such that g (y) = z and $\exists$ x in A such that f(x) = y.

$\therefore$ $g(y) = z \quad \Rightarrow \quad z = g(f(x)) = gof(x)$ where $x \in A$

$\Rightarrow$ gof is onto function.

Hence gof is bijective function.

Example 5.3.9 *Determine whether the function is bijective.*

$$f : I \rightarrow I \text{ such that } f(i) = \frac{i}{2} \quad \text{if } i \text{ is even}$$

$$= \frac{(i-1)}{2} \quad \text{if } i \text{ is odd}$$

Solution : Let a and b be any integers such that $a \neq b$.

There are three possibilities

Case 1 : If a and b are odd

$$\therefore \quad f(a) = \frac{a-1}{2}, \quad f(b) = \frac{b-1}{2}$$

$$f(a) = f(b) \Rightarrow \frac{a-1}{2} = \frac{b-1}{2}$$

$$\Rightarrow a = b$$

Case 2 : If a and b are even

$$\text{then} \quad f(a) = \frac{a}{2}, \quad f(b) = \frac{a}{2}$$

$$f(a) = f(b) \Rightarrow \frac{a}{2} = \frac{b}{2} \Rightarrow a = b$$

Case 3 : If a is odd and b is even

then $\quad f(a) = \dfrac{a-1}{2} \quad$ and $\quad f(b) = \dfrac{b}{2}$

Now, $\quad f(a) = f(b) \Rightarrow \dfrac{a-1}{2} = \dfrac{b}{2} \Rightarrow \dfrac{a}{2} - \dfrac{b}{2} = \dfrac{1}{2}$

$$\Rightarrow \quad a - b = 1 \quad \therefore \quad a \neq b$$

In particular a = 7, b = 6

$$f(7) = \frac{7-1}{2} = 3 \qquad f(6) = \frac{6}{2} = 3$$

$$f(7) = f(6) \text{ but } 6 \neq 7$$

Thus f is not one one function. Hence f is not bijective function.

Example 5.3.10 *Let $f(x) = ax^2 + b$ and $g(x) = cx^2 + d$, where a, b, c, d are constants. Determine for which values of constants $gof(x) = fog(x)$.*

Solution : Given that, $f(x) = ax^2 + b$ and $g(x) = cx^2 + d$

Suppose $\quad fog(x) = gof(x)$

$$f[g(x)] = g[f(x)]$$

$$f[cx^2 + d] = g[ax^2 + b]$$

$$\Rightarrow \quad a[cx^2 + d]^2 + b = c[ax^2 + b]^2 + d$$

$$\Rightarrow \quad a[c^2x^4 + 2cdx^2 + d^2] + b = c(a^2x^4 + 2abx^2 + b^2) + d$$

$$\Rightarrow \quad ac^2x^4 + 2acdx^2 + ad^2 + b = ca^2x^4 + 2cabx^2 + cb^2 + d$$

$\Rightarrow \quad$ Coefficient of x^4 = coefficient of x^4

and $\quad$ Coefficient of x^2 = coefficient of x^2

$$\Rightarrow \quad ac^2 = ca^2 \Rightarrow a = c$$

and $\quad 2acd = 2acb \Rightarrow b = d$

Thus $\quad a = c$ and $b = d$

Example 5.3.11 *Show that the functions $f(x) = x^3 + 1$ and $g(x) = (x-1)^{1/3}$ are converse to each other.*

Solution : We have, $g(x) = (x-1)^{1/3}$

Let $\quad y = g(x) = (x - 1)^{1/3}$

$$\Rightarrow \quad y^3 = x - 1$$

$$\Rightarrow \quad x = y^3 + 1 \qquad \text{... (Which is f (x))}$$

Hence f (x) and g (x) are converse to each other.

Example 5.3.12 *Let $f : X \to Y$ and X and Y are set of real numbers. Find f^{-1} if (i) $f(x) = x^2$; (ii) $f(x) = \dfrac{2x-1}{5}$*

Solution : Let $f : X \to Y$ and $X, Y \subseteq \Re$

(i) $f(x) = x^2$

Let $f(x) = y$

$$x^2 = y$$

$$\Rightarrow \quad y = x^2$$

$$\Rightarrow \quad y = \pm\sqrt{x}$$

Therefore, given function is not one to one. Hence f^{-1} does not exist.

(ii) Let $f(x) = y$

$$\frac{2x-1}{5} = y$$

$$\Rightarrow \quad 2x = 5y + 1$$

$$x = \frac{5y+1}{2} \quad \forall \ y \in \Re$$

Function f is 1 – 1 and onto.

$\therefore$ The inverse function of f is given as,

$$f^{-1}(x) = \frac{1+5y}{2}$$

Example 5.3.13 *Let X = {a, b, c}. Define $f : X \to X$ such that f = {(a, b) (b, a) (c, c)}. Find (i) f^{-1} ; (ii) f^2 ; (iii) f^3 ; (iv) f^4*

Solution : Given fucntion f is bijective function.

(i) $f^{-1} = \{(b, a)\ (a, b)\ (c, c)\}$

(ii) $f^2 = fof = \{(a, a)\ (b, b)\ (c, c)\}$

$$fof(a) = f[f(a)] = f(b) = a$$

$$fof(b) = f[f(b)] = f(a) = b$$

$$fof(c) = f[f(c)] = f(c) = c$$

(iii) $f^3(a) = f[f^2(a)] = f(a) = b$

$f^3(b) = f[f^2(b)] = f(b) = a$

$f^3(c) = f[f^2(c)] = f(c) = c$

$f^2 = \{(a, b), (b, a), (c, c)\}$

(iv) $f^4 = f^2 \circ f^2$

$f^4(a) = f^2[f^2(a)] = f^2(a) = a$

$f^4(b) = f^2[f^2(b)] = f^2(b) = b$

$f^4(c) = f^2[f^2(c)] = f^2(c) = c$

$\therefore \quad f^4 = \{(a, a)\ (b, b)\ (c, c)\}$

Example 5.3.14 *Let $A = \{1, 2, 3\}$, $B = \{1, 2, 3\}$ and f_1 and f_2 are functions from A to B.*

$f_1 = \{(1, 2), (2, 3), (3, 1)\}$

$f_2 = \{(1, 2), (2, 1), (3, 3)\}$

Compute $f_1 \circ f_2$ and $f_2 \circ f_1$ **SPPU : May-18, Marks 4**

Solution : We have $f_1 \circ f_2(x) = f_1[f_2(x)]$

$\therefore \quad f_1[f_2(1)] = f_1[2] = 3$

$f_1[f_2(2)] = f_1[1] = 2$

$f_1[f_2(3)] = f_1[3] = 1$

Now, $f_2 \circ f_1(x) = f_2[f_1(x)]$

$f_2[f_1(1)] = f_2[2] = 1$

$f_2[f_1(2)] = f_2[3] = 3$

$f_2[f_1(3)] = f_2[1] = 2$

Example 5.3.15 *Let $X = \{1, 2, 3\}$, $Y = \{p, q\}$ and $z = \{a, b\}$*

Let $f : X \to Y$ be $\{(1, p), (2, p), (3, q)\}$ and $g : Y \to Z$ be

$g = \{(p, b)\ (q, b)\}$. Find gof and show it pictorically.

Solution : Given that, $f = \{(1, p), (2, p), (3, q)\}$

$g = \{(p, b)\ (q, b)\}$

and $gof(x) = g[f(x)]$ and $gof : X \to Z$

$gof(1) = g[f(1)] = g(p) = b$

$gof(2) = g[f(2)] = g(p) = b$

$gof(3) = g[f(3)] = g(q) = b$

$gof = \{(1, b)\ (2, b)\ (3, b)\}$

It's pictorial representation is

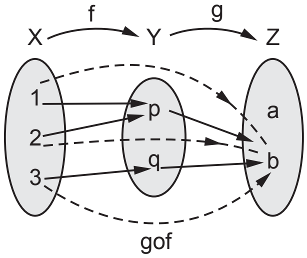


Fig. 5.3.7

Example 5.3.16 *Let R be the set of real numbers and f, g, h : $\Re \to \Re$ such that*

$f(x) = x + 2$, $g(x) = \dfrac{1}{x^2+1}$ $h(x) = 3$. Compute

(i) $f^{-1} g(x)$; (ii) $hf\,(gf^{-1})$ $[h\,f\,(x)]$

Solution : (i) Given that $f(x) = x + 2$

Let
$$f(x) = y = x + 2 \Rightarrow x = y - 2$$
$$f^{-1}(y) = y - 2$$
$$f^{-1}[g(x)] = f^{-1}\left[\frac{1}{x^2+1}\right] = \frac{1}{x^2+1} - 2$$
$$= \frac{1-2x^2-2}{(x^2+1)} = \frac{-(2x^2+1)}{x^2+1}$$

(ii) We have $f(x) = x + 2$

$$h\,f(x) = h[f(x)] = h[x+2] = 3$$
$$g\,f^{-1}(x) = g[f^{-1}(x)] = g[x-2]$$
$$= \frac{1}{(x-2)^2+1} = \frac{1}{x^2-2x+5}$$
$$(g\,f^{-1})[h\,f(x)] = g\,f^{-1}(3) = \frac{1}{3^2-2(3)+5}$$
$$= \frac{1}{9-6+6} = \frac{1}{8}$$
$$\therefore \quad h\,f\,(g\,f^{-1})\,(h\,f(x)) = h\,f\left[\frac{1}{8}\right] = h\left[f\left(\frac{1}{8}\right)\right]$$
$$= h\left[\frac{1}{8}+2\right] = h\left[\frac{17}{8}\right] = 3$$

i.e. $h\,f\,(g\,f^{-1})\,(h\,f(x)) = 3$

Example 5.3.17 *Show that f, g : $N \times N \to N$ as f (x, y) = x + y. g(x, y) = xy are onto but not one-one*

Solution : (i) Suppose $f(x_1, y_1) = f(x_2, y_2)$

$\Rightarrow \quad x_1 + y_1 = x_2 + y_2$

$\Rightarrow \; x_1 \neq x_2$ and $y_1 \neq y_2$

e.g. $x_1 = 5$, $x_2 = 2$, $y_1 = 4$, $y_2 = 7$.

$\therefore$ f is not one to one.

Now, $g(x_1, y_1) = g(x_2, y_2)$

$\Rightarrow \quad x_1 y_1 = x_2 y_2$

$\Rightarrow$ x_1 may or may not be equal to x_2

and y_1 may or may not be equal to y_2

$\therefore$ g is not one to one.

(ii) $f(x, y) = x + y$

$\therefore$ Every element of N can be written as the sum of two elements of N. Hence f is onto.

Now, $g(x, y) = xy$

Every element of can be written as the product of two elements of N.

$\therefore$ g is onto.

5.4 Infinite Sets and Countability

SPPU : May-18, 19

5.4.1 Infinite Set

A set A is said to be an infinite set if there exists an injective mapping (function) $f : A \to A$ such that f(A) is a proper subset of A.

If no such injective function exists, then set is finite.

Examples

1) Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that $f(n) = 2n$, $\forall\, n \in N$ = Natural number set.

There range set = $f(\mathbb{N})$ = {Set of positive even natural numbers} C $\mathbb{N}$

$\therefore$ $\mathbb{N}$ is an infinite set.

2) Define $f : \Re \to \Re$ such that $f(x) = \begin{cases} x+2 & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}$

f is an injective mapping and for $x \geq 0$, $f(x) = x + 2$

i.e. $f(x) \geq 2$ and for $x < 0$, $f(x) = x < 0$

$\therefore$ Range set = $f(\Re) = \{y \in \Re / f(x) \geq 2 \text{ or } f(x) < 0\}$

$\therefore 1 \notin f(\Re)$

Thus $f(\Re) \overset{\subset}{\neq} \Re$

Hence $\Re$ is an infinite set.

I) Properties of an infinite sets

1) If A is an infinite set the $A \times A$, P (A) are infinite sets.
2) If A and B are non empt sets and either A or B is an infinite set then $A \times B$ is an infinite set.
3) If either A or B is an infinite set then $A \cup B$ is an infinite set.
4) If $A \subseteq B$ and A is an infinite set then B is also an infinite set.

i.e. the superset of an infinite set is an infinite.

5.4.2 Countable Sets

We know that the codinality of a set is the number of elements of that set. If set is finite then, we can list elements as 1, 2, 3, …. Therefore every finite set is countable. As $|\phi| = 0$, the null set is also countable. A question remains same for an infinite sets. Let us define countable infinite sets.

Definition :

An infinite set A is said to be countable if there exists a bijection $f : \mathbb{N} \to A$

$$\therefore \quad A = \{ f(1), f(2), f(3), \ldots\}$$

A countably infinite set is also known as a denumarable set.

i.e. If A is a denumerable set then we can least elements of A as $a_1, a_2, a_3, \ldots a_n \ldots$ or f (1), f (x) … f (n) … .

e.g. 1) ϕ is countable

2) $A = \{1, 2, 3, 4, \ldots 1000\}$ is countable as $|A| = 1000$.
3) As $f : \mathbb{N} \to \mathbb{N}$ defined as $f(x) = n$, $\therefore$ $\mathbb{N}$ is countable
4) The set of integers is countable

As $f : N \to \mathbb{Z}$ such that $f(n) = \dfrac{n+1}{2}$ if n = 1, 3, 5, …

$$= \frac{-n}{2} \text{ if } n = 0, 2, 4, 6, \ldots$$

is bijective mapping.

5) The set of rational numbers is countable.

6) The set of real numbers is not countable

7) The set of complex numbers is not countable.

8) The set of real numbers in [a, b], a < b is not countable.

9) The countable union of countable sets is countable.

Properties of countable sets :

1) A subset of a countable set is countable.

2) Let A and B be countable sets then $A \cup B$ is countable.

$\Rightarrow$ Define $f : A \cup B \to \mathbb{N}$ as $f(a_i) = 2i - 1$

and $f(b_i) = 2i$

f is bijective $\therefore A \cup B$ is countable

3) Prove that the set of rational numbers is countably infinite.

SPPU : May-18, 19, Marks 4

Proof : We know that the countable union of countable sets is countable. Therefore it is sufficient to prove that the set of rational numbers in [0, 1] is countable.

We have to prove that $\exists$ atleast one function f.

$f : [0, 1] \to N$ such that f is injective.

We arrange the rational numbers of the interval according to increasing denominators as

$0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots$

then the one to one correspondence is as follows

$0 \leftrightarrow 1$

$1 \leftrightarrow 2$

$\frac{1}{2} \leftrightarrow 3$

$\frac{1}{3} \leftrightarrow 4$

$\frac{2}{3} \leftrightarrow 5$

$\frac{1}{4} \leftrightarrow 6$ and so on

Hence set of rationals in [0, 1] is countable. Thus the set of rational numbers is countable and as the set is infinite, it is countably infinite.

4) The set of irrational numbers is uncountable.

Proof : We know that $\Re = Q \cup \overline{Q}$ where Q = set of rational numbers, $\overline{Q}$ = set of irrational numbers.

Suppose $\overline{Q}$ is countable $\Rightarrow Q \cup \overline{Q} = \Re$ is countable which is contradiction.

$\therefore$ $\overline{Q}$ is not countable i.e. uncountable.

5) The set of real numbers in (0, 1) is not countable. Assume that the set is countable. $\therefore A = \{x_1, x_2, x_3, \cdots x_n, \cdots\}$.

Proof : Any real number in (0, 1) can be written in a unique decimal without an infinite string of 9's at the end. i.e. 0.3459999 will be represented as 0.345000. Let the infinite sequence be given by,

$$1 \to x_1 = 0 \cdot a_{11}\ a_{12}\ a_{13}\ \ldots$$

$$2 \to x_2 = 0 \cdot a_{21}\ a_{22}\ a_{23}\ \ldots$$

$$3 \to x_3 = 0 \cdot a_{31}\ a_{32}\ a_{33}\ \ldots$$

$$\vdots \qquad \vdots$$

$$n \to x_n = 0.a_{n1}\ a_{n2}\ a_{n3}\ \ldots$$

$$\vdots \qquad \vdots$$

Construct a new number $y = 0 \cdot b_1\, b_2\, b_3\ \ldots$

Where $b_i = 0$ if $a_{ii} \neq 0$

$b_i = 1$ if $a_{ii} = 0$

Hence $b_1 \neq a_{11}$, $b_2 \neq a_{22}$, $b_3 \neq a_{33}\ \ldots\ b_n \neq a_{nn}$

$\therefore b_i \neq a_{ii}, \forall i$

$\therefore y \neq x_1, y \neq x_i\ \forall i$

Hence y is not in the list of numbers $\{x_1, x_2, \cdots x_n \cdots\}$ Thus $y \in (0, 1)$ and it is different from elements in $\{x_1, x_2, \cdots x_n \cdots\}$ which is contradiction that A is countable.

Hence A is not countable.

Thus the set of real numbers is not countable.

5.5 Pigeon Hole Principle

SPPU : Dec.-09, 11, 12

I) This principle states that if there are n + 1 pigeons and only n pigeon holes then two pigeons will share the same whole.

This principle is stated by using the analogy of the bijective mapping i.e. If A and B are any two sets such that |A| > |B| then there does not exist bijective mapping from A to B.

Examples

Example 5.5.1 *If 11 shoes are selected from 10 pairs of shoes then there must be a pair and matched shoes among the selection.*

Solution : In the pigeonhole principle, 11 shoes are pigeons and the 10 pairs are the pigeon holes.

Example 5.5.2 *Show that if seven numbers from 1 to 12 are chosen then two of them will add upto 13.*

Solution : We have A = {1, 2, 3, 4, 5, … 12}

We form the six different sets each containing 2 numbers that add uptot 13.

$A_1 = \{1, 12\}, A_2 = \{2, 11\}, A_3 = \{3, 10\}, A_4 = \{4, 9\}, A_5 = \{5, 8\}, A_6 = \{6, 7\}$

Each of the seven numbers chosen must belong to one of these sets. As there are only six sets, by pigeonhole principle two of the chosen numbers must belong to the same set and their sum is 13.

II) The extended pigeon hole principle

If n pigeons are assigned to m pigeon holes, then one of the pigeon holes must be occupied by at least $\left[\frac{n-1}{m}\right] + 1$ pigeons. It is also known as generalized pigeon hole principle. Here $\left[\frac{n-1}{m}\right]$ is the integer division of n – 1 by m. e.g. $\left[\frac{9}{2}\right] = 4$, $\left[\frac{16}{5}\right] = 3$, $\left[\frac{8}{3}\right] = 2$.

Examples

Example 5.5.3 *Show that 7 colours are used to paint 50 bicycles, then at least 8 bicycles will be of same colour.* **SPPU : Dec.-09, 12, Marks 4**

Solution : By the extended pigeonhole principle, at least $\left[\frac{n-1}{m}\right] + 1$ pigeons will occupy one piegeonhole.

Here n = 50, m = 7 and m < n then

$$\left[\frac{50-1}{7}\right] + 1 = 7 + 1 = 8$$

Thus 8 bicycles will be of the same colour.

Example 5.5.4 *Write generalized pigeonhole principle. Use any form of pigeonhole principle to solve the given problem.*

i) Assume that there are 3 mens and 5 womens in a party show that if these people are lined up in a row at least two women will be next to each other.

ii) Find the minimum number of students in the class to be sure that three of them are born in the same month. **SPPU : Dec.-11, Marks 4**

Solution : Please refer section 5.5 (II) for definition.

i) By using analogy of pigeon hole principle, we get

3 men = pigeonholes and 5 women = pigeon

Pigeons are more than pigeon holes.

∴ At least two pigeons share the same pigeon hole i.e. at least two women in a row will be next to each other.

ii) Let h = Number of pigeons = Number of students

n = Number of pigeon holes = Number of months = 12

Given that three students in the class are born in the same month.

$$\therefore \quad \left\lfloor \frac{n-1}{m} \right\rfloor + 1 = 3$$

$$\Rightarrow \quad \frac{n-1}{12} = 3 - 1 = 2$$

$$n = 2 \times 12 + 1 = 25$$

Therefore there are 25 minimum number of students in the class.

5.6 Discrete Numeric Functions

A function whose domain is a set of natural numbers including zero and whose range set is the set of real numbers, is called a discrete numeric function or numeric function. It is also known as a sequence. If $f : \mathbb{N} \cup \{0\} \rightarrow \Re$ is a discrete numeric function then f(0, f(1), f(2), f(3), … denote the value of function at 0, 1, 2, 3, …

The numeric function f is written as

$$f = \{f_0, f_1, f_2, \cdots\}$$

Hereafter, to denote numeric function, we use

$$< a_r > = \{a_0, a_1, a_2, a_3 \cdots a_r, \cdots\}$$

e.g. 1) $a_r = 2^k$ if $0 \le k \le 3$

$= k$ if $k > 3$