Example 5.5.4 *Write generalized pigeonhole principle. Use any form of pigeonhole principle to solve the given problem.*

i) Assume that there are 3 mens and 5 womens in a party show that if these people are lined up in a row at least two women will be next to each other.

ii) Find the minimum number of students in the class to be sure that three of them are born in the same month. **SPPU : Dec.-11, Marks 4**

Solution : Please refer section 5.5 (II) for definition.

i) By using analogy of pigeon hole principle, we get

3 men = pigeonholes and 5 women = pigeon

Pigeons are more than pigeon holes.

∴ At least two pigeons share the same pigeon hole i.e. at least two women in a row will be next to each other.

ii) Let h = Number of pigeons = Number of students

n = Number of pigeon holes = Number of months = 12

Given that three students in the class are born in the same month.

$$\therefore \quad \left[\frac{n-1}{m}\right] + 1 = 3$$

$$\Rightarrow \quad \frac{n-1}{12} = 3 - 1 = 2$$

$$n = 2 \times 12 + 1 = 25$$

Therefore there are 25 minimum number of students in the class.

5.6 Discrete Numeric Functions

A function whose domain is a set of natural numbers including zero and whose range set is the set of real numbers, is called a discrete numeric function or numeric function. It is also known as a sequence. If $f : \mathbb{N} \cup \{0\} \rightarrow \Re$ is a discrete numeric function then f(0, f(1), f(2), f(3), … denote the value of function at 0, 1, 2, 3, …

The numeric function f is written as

$$f = \{f_0, f_1, f_2, \cdots\}$$

Hereafter, to denote numeric function, we use

$$< a_r > = \{a_0, a_1, a_2, a_3 \cdots a_r, \cdots\}$$

e.g. 1) $a_r = 2^k$ if $0 \leq k \leq 3$

$= k$ if $k > 3$

This discrete function can be written as

$$a = \{1, 2, 4, 8, 4, 5, 6, 7, 8, \ldots\}$$

2) $$a_k = \begin{cases} 100 + r, & 0 \le r \le 5 \\ r^2 & r \ge 6 \end{cases}$$

$\therefore$ This discrete function can be written as

$$a = \{100, 101, 102, 103, 104, 105, 6^2, 7^2, 8^2, 9^2, \cdots\}$$

5.6.1 Basic Operations on Numeric Functions

Let a and b be two numeric functions. Then

i) Addition : c = a + b is also a numeric function defined as

$$c_r = a_r + b_r$$

i.e. $$c_0 = a_0 + b_0, c_1 = a_1 + b_1, c_2 = a_2 + b_2 \text{ and so on.}$$

ii) Multiplication : d = ab is also a numeric function defined as $d_r = a_r b_r$,

$\forall \in \mathbb{N} \cup \{0\}$.

iii) Scaling : Let k be any real number then b = ak is a numeric function and k is called as scalling factor.

iv) Linearity : Let p and q be any real numbers, then x = pq + qb is a numeric function where

$$x_r = p\, a_r + q\, b_r$$

v) The convolution of two numeric functions : Let a and b be two numeric functions. The convolution of two functions a and b is denoted by c = a * b and defined as

$$c_r = a_0\, b_r + a_1\, b_{r-1} + a_2\, b_{r-2} + \ldots + a_r b_0 = \sum_{n=0}^{r} a_n\, b_{r-n}$$

Example 1 :

If $$a_r = \frac{1}{3^r}, \; r \ge 0 \text{ then } a = \left\{1, \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \ldots\right\}$$

and $$5a = \left\{5, \frac{5}{3}, \frac{5}{3^2}, \frac{5}{3^3}, \cdots\right\}$$

$\therefore$ $$5\, a_r = 5 \cdot \frac{1}{3^r}$$

Example 2 :

If $a_r = \begin{cases} 1 & \text{if } 0 \le r \le 2 \\ 3r & \text{if } r \ge 3 \end{cases}$ and $b_r = \begin{cases} 2^r + 1 & , 0 \le r \le 1 \\ r - 5 & , r \ge 2 \end{cases}$

then $a + b = c$

$$\therefore \quad c_r = a_r + b_r = \begin{cases} 2^{r+1} + 2 & , 0 \le r \le 1 \\ 1 + (-3) & , r = 2 \\ 3r + r - 5 & , r \ge 3 \end{cases}$$

$$= \begin{cases} 2^{r+1} + 2 & , 0 \le r \le 1 \\ -2 & , r = 2 \\ 4r - 5 & , r \ge 3 \end{cases}$$

and $d = ab$ and $d_r = a_r\, b_r = \begin{cases} 1\,(2^r + 1) & , 0 \le r \le 1 \\ 1\,(-3) & , r = 2 \\ 3r\,(r - 5) & , r \ge 3 \end{cases}$

$$d_r = \begin{cases} 2^{r+1} & , 0 \le r \le 1 \\ -3 & , r = 2 \\ 3r^2 - 15r & , r \ge 3 \end{cases}$$

Example 3 :

If $a_r = 3^r, r \ge 0$

$b_r = 5^r, r \ge 0$

then $c = a * b \Rightarrow c_r = \sum_{n=0}^{r} a_n\, b_{r-n} = a_0\, b_r + a_1\, b_{r-1} + a_2\, b_{r-2} + \cdots + a_r\, b_0$

5.6.2 Finite Differences of a Numeric Functions

I) Shift of a numeric function

The shift of a numeric function of a_r is denoted by $E[a_r]$ and defined as

$$E[a_r] = a_{r+1}$$

$$E^{-1}[a_r] = a_{r-1}$$

$$E^n[a_r] = a_{r+n}, \qquad E^{-n}[a_r] = a_{r-n}$$

Note : It is also denoted by $s^n[a_r] = E^n[a_r]$

Example 1 :

If $a_r = 5^r,\ r \geq 0$ then $E\,[a_r] = a_{r+1} = 5^{r+1}$

$$E^2[a_r] = a_{r+2} = 5^{r+2},\ E^{-n}[a_r] = a_{r-n} = 5^{r-n}$$

Example 2 :

If $$a_r = \begin{cases} r^3 - 2r + 5 & ; \quad 0 \leq r \leq 6 \\ 3 & ; \quad r \geq 7 \end{cases}$$

then $$E^3[a_r] = \begin{cases} 0 & , \quad 0 \leq r < 2 \\ (r+3)^3 - 2\,(r+3) + 5 & , \quad 3 \leq r \leq 9 \\ 3 & , \quad r \geq 10 \end{cases}$$

and $$E^{-2}[a_r = \begin{cases} (r-2)^3 - 2(r-2) + 5 & ; \quad 0 \leq r \leq 4 \\ 3 & ; \quad r \geq 5 \end{cases}$$

II) Forward difference of a numeric function

The forward difference of a numeric function a_r is denoted by $\Delta\, a_r$ and defined as

$$\Delta\, a_r = a_{r+1} - a_r\ ;\ r \geq 0$$

Example 1 :

If $a_r = 3^r\ ;\ r \geq 0$ then

$$\Delta\, a_r = a_{r+1} - a_r = 3^{r+1} - 3^r$$
$$= (3-1)\ 3^r = 2.3^r$$

Example 2 :

If $$a_r = \begin{cases} 3 & ; \quad 0 \leq r \leq 4 \\ r+2 & ; \quad r \geq 5 \end{cases}$$

then $\Delta\, a_r = a_{r+1} - a_r$

$$\therefore \quad a_{r+1} = \begin{cases} 3 & ; \quad 0 \leq r+1 \leq 4 \\ (r+1)+2 & ; \quad r+1 \geq 5 \end{cases}$$

$$a_{r+1} = \begin{cases} 3 & ; \quad 0 \leq r \leq 3 \\ r+3 & ; \quad r \geq 4 \end{cases}$$

$$\therefore \quad \Delta\, a_r = \begin{cases} 0 & ; \quad 0 \leq r \leq 3 \\ -3 + (4+3) & ; \quad r = 4 \\ (r+3) - (r+2) & ; \quad r \geq 5 \end{cases}$$

$$\Delta a_r = \begin{cases} 0 & ; \quad 0 \le r \le 3 \\ 4 & ; \quad r = 4 \\ 1 & ; \quad r \ge 5 \end{cases}$$

III) Backward difference of a numeric function

The backward difference of a numeric function a_r is denoted by ∇a_r and defined as

$$\nabla a_r = a_r - a_{r-1} \; ; \; r \ge 1$$

Examples

Example 1 :

Let $\quad a_r = 6^r \; ; \; r \ge 0$

$$a_{r-1} = 6^{r-1} \; ; \; r - 1 \ge 0 \text{ i.e. } r \ge 1$$

$$\therefore \quad \nabla a_r = a_r - a_{r-1} = 6^r - 6^{r-1} \; ; \; r \ge 1$$

$$= (6 - 1)\, 6^{r-1} \; ; \; r \ge 1$$

$$\nabla a_r = 5 \cdot 6^{r-1} \; ; \; r \ge 1$$

Example 2 :

If $\quad a_r = \begin{cases} 1 & ; \quad 0 \le r \le 2 \\ 3^r & ; \quad r \ge 3 \end{cases}$

$$a_{r-1} = \begin{cases} 1 & ; \quad 0 \le r - 1 \le 2 \\ 3^{r-1} & ; \quad r - 1 \ge 3 \end{cases}$$

$$= \begin{cases} 1 & ; \quad 1 \le r \le 3 \\ 3^{r-1} & ; \quad r \ge 4 \end{cases}$$

$$\nabla a_r = a_r - a_{r-1} = \begin{cases} 0 & ; \quad r = 0 \\ 1 & ; \quad r = 1 \\ 0 & ; \quad r = 2 \\ 26 & ; \quad r = 3 \\ 2 \times 3^{r-1} & ; \quad r \ge 4 \end{cases}$$

Example 5.6.1 *Determine a ∗ b for the following numeric functions*

$$a_r = \begin{cases} 1 & ; \quad 0 \le r \le 2 \\ 0 & ; \quad r \ge 3 \end{cases} \quad and \quad b_r = \begin{cases} r+1 & ; \quad 0 \le r \le 2 \\ 0 & ; \quad r \ge 3 \end{cases}$$

Solution : By the definition of a_r, the numeric function of a is given by

$$a = \{1, 1, 1, 0, 0, 0, \ldots\} = \{a_0, a_1, a_2, a_3, \ldots\}$$

$$b = \{1, 2, 3, 0, 0, 0 \ldots\} = \{b_0, b_1, b_2, b_3, \ldots\}$$

The convolution of a and b is a numeric function c such that

$$c = a * b \quad \text{where} \quad c_r = \sum_{n=0}^{r} a_n \, b_{r-n}$$

$$\therefore \quad c_r = a_0 \, b_r + a_1 \, b_{r-1} + a_2 \, b_{r-2} + \ldots + a_r \, b_0$$

$$\therefore \quad c_0 = a_0 \, b_0 = 1 \times 1 = 1$$

$$c_1 = a_0 \, b_1 + a_1 \, b_0 = 1.2 + 1.1 = 3$$

$$c_2 = a_0 \, b_2 + a_1 \, b_1 + a_2 \, b_0 = 1.3 + 1.2 + 1.1 = 6$$

$$c_3 = a_0 \, b_3 + a_1 \, b_2 + a_2 \, b_1 + a_3 \, b_0$$

$$= 1.0 + 1.3 + 1.2 + 1.0 = 5$$

$$c_4 = a_0 \, b_4 + a_1 \, b_3 + a_2 \, b_2 + a_3 \, b_1 + a_4 \, b_0$$

$$c_4 = 0 + 0 + 1.3 + 0 + 0 = 3$$

$$c_5 = 0, c_6 = 0, c_r = 0 \; ; \; r \geq 5$$

Thus the numeric sequence of c is given by

$$c = \{1, 3, 6, 5, 3, 0, 0, 0, \ldots\}$$

$$\therefore \quad c_r = \begin{cases} 1 & ; \quad r = 0 \\ 3 & ; \quad r = 1 \\ 6 & ; \quad r = 2 \\ 5 & ; \quad r = 3 \\ 3 & ; \quad r = 4 \\ 0 & ; \quad r \geq 5 \end{cases}$$

❑❑❑

Notes

SUBJECT CODE : 210241

As per Revised Syllabus of

SAVITRIBAI PHULE PUNE UNIVERSITY

Choice Based Credit System (CBCS)

S.E. (Computer) Semester - I

DISCRETE MATHEMATICS

(For END SEM Exam - 70 Marks)

Dr. H. R. Bhapkar

M.Sc. SET, Ph.D. (Mathematics)
MIT Art, Design and Technology University's
MIT School of Engineering, Lonikalbhor, Pune
Email : drhrbhapkar@gmail.com
Mobile : 9011227141

Dr. Rajesh Nandkumar Phursule

Ph.D. (Computer Science and Engineering)
Associate Professor,
Pimpri Chinchwad College of Engineering
Nigdi, Pune.

® TECHNICAL ®
PUBLICATIONS
SINCE 1993
An Up-Thrust for Knowledge

DISCRETE MATHEMATICS

(For END SEM Exam - 70 Marks)

Subject Code : 210241

S.E. (Computer) Semester - I

Published by :

TECHNICAL PUBLICATIONS
SINCE 1993 An Up-Thrust for Knowledge

Amit Residency, Office No.1, 412, Shaniwar Peth,
Pune - 411030, M.S. INDIA, Ph.: +91-020-24495496/97
Email : sales@technicalpublications.org Website : www.technicalpublications.org

Printer :
Yogiraj Printers & Binders
Sr.No. 10/1A,
Ghule Industrial Estate, Nanded Village Road,
Tal. - Haveli, Dist. - Pune - 411041.

ISBN 978-93-332-2155-9

9 789333 221559

SPPU 19

PREFACE

The importance of **Discrete Mathematics** is well known in various engineering fields. Overwhelming response to our books on various subjects inspired us to write this book. The book is structured to cover the key aspects of the subject **Discrete Mathematics.**

The book uses plain, lucid language to explain fundamentals of this subject. The book provides logical method of explaining various complicated concepts and stepwise methods to explain the important topics. Each chapter is well supported with necessary illustrations, practical examples and solved problems. All the chapters in the book are arranged in a proper sequence that permits each topic to build upon earlier studies. All care has been taken to make students comfortable in understanding the basic concepts of the subject.

Representative questions have been added at the end of each chapter to help the students in picking important points from that chapter.

The book not only covers the entire scope of the subject but explains the philosophy of the subject. This makes the understanding of this subject more clear and makes it more interesting. The book will be very useful not only to the students but also to the subject teachers. The students have to omit nothing and possibly have to cover nothing more.

We wish to express our profound thanks to all those who helped in making this book a reality. Much needed moral support and encouragement is provided on numerous occasions by our whole family. We wish to thank the **Publisher** and the entire team of **Technical Publications** who have taken immense pain to get this book in time with quality printing.

Any suggestion for the improvement of the book will be acknowledged and well appreciated.

Authors

Dr. H. R. Bhapkar

Dr. Rajesh N. Phursule

Dedicated to the Readers of the Book

SYLLABUS

Discrete Mathematics - (210241)

Credit Scheme	Examination Scheme and Marks
03	End_Semester (TH) : 70 Marks

Unit III Counting Principles

The Basics of Counting, rule of Sum and Product, Permutations and Combinations, Binomial Coefficients and Identities, Generalized Permutations and Combinations, Algorithms for generating Permutations and Combinations. **(Chapter-6)**

Unit IV Graph Theory

Graph Terminology and Special Types of Graphs, Representing Graphs and Graph Isomorphism, Connectivity, Euler and Hamilton Paths, the handshaking lemma, Single source shortest path- Dijkstra's Algorithm, Planar Graphs, Graph Colouring. **(Chapter-7)**

Unit V Trees

Introduction, properties of trees, Binary search tree, tree traversal, decisicon tree, prefix codes and Huffman coding, cut sets, Spanning Trees and Minimum Spanning Tree, Kruskal's and Prim's algorithms, The Max flow-Min Cut Theorem (Transport network). **(Chapter-8)**

Unit VI Algebraic Structures and Coding Theory

The structure of algebra, Algebraic Systems, Semi Groups, Monoids, Groups, Homomorphism and Normal Subgroups, and Congruence relations, Rings, Integral Domains and Fields, Coding theory, Polynomial Rings and polynomial Codes, Galois Theory -Field Theory and Group Theory. **(Chapter-9)**

Table of Contents

For Unit I and Unit II (Chapters 1 to 5)

Refer *Separate Book* for *In Sem Exam*

Unit - III

Unit - IV

Unit - V

Unit - VI

Notes

UNIT - III

6 Counting Principles

Syllabus

The Basics of Counting, rule of Sum and Product, Permutations and Combinations, Binomial Coefficients and Identities, Generalized Permutations and Combinations, Algorithms for generating Permutations and Combinations. Case Studies : Study Sudoku Solving algorithm for generation of new SUDOKU, Study Hand-shake puzzle and algorithm to solve it.

Contents

6.1 Introduction

Permutations and combinations deal with counting and enumeration of specified objects, patterns or designs. All techniques of the counting are very important in computer science and mathematics. In this chapter we will study basic of counting. We also develop basic ideas of permutations, combinations and discrete probability. Let us begin our study with two basic counting principles.

6.2 Basic Counting Principles

SPPU : Dec.-17

6.2.1 Sum Rule (Principle of Disjunctive Counting)

We know that, if $S = A \cup B$ and A and B are disjoint sets i.e. $A \cap B = \phi$ then

$$|S| = |A| + |B|$$

i.e. A and B are disjoint partitions of S.

Now we can extend this logic to state sum rule.

Sum rule : If one experiment E_1 has n_1 possible outcomes and another experiment E_2 has n_2 possible outcomes and E_1 and E_2 are disjoint (exclusive) then there are $n_1 + n_2$ possible outcomes when E_1 or E_2 take place.

Example 6.2.1 *How many ways can we select a girl or a boy representative from the class of 30 girls and 40 boys ?*

Solution :

$\Rightarrow$ Let E_1 = Set of girls in a class

E_2 = Set of boys in a class

$\therefore$ $|E_1| = 30$, $|E_2| = 40$, E_1 and E_2 are disjoint sets.

$\therefore$ $30 + 40 = 70$ ways.

$\Rightarrow$ A girl or a boy representative can be selected by 70 different ways.

Example 6.2.2 *How many ways can we get a sum of 7 or 11 when two different dice are rolled ?*

Solution : Two dice are different, therefore the ordered pairs (a, b) and (b, a) are different when $a \neq b$.

E_1 = Set of ordered pairs in which the sum is 7

$\therefore$ $E_1 = \{(1, 6), (2, 5), (3, 4)\ (4, 3), (5, 2), (1, 6)\}$

There are 6 ways to get sum 7

E_2 = Set of ordered pairs whose sum is 11.

$E_2 = \{(5, 6), (6, 5)\}$

$\therefore$ There are 2 ways to get sum 11.

E_1 and E_2 are mutually exclusive. i.e. $E_1 \cap E_2 = \phi$

Therefore we can get sum 7 or 11 in 6 + 2 = 8 different ways.

Note : We can extend this principle for finite number of experiments.

6.2.2 Product Rule (The Principle of Sequential Counting)

We have, if A and B are non empty sets and $|A| = n$, $|B| = n$. then the number of elements in the cartesian product of A and B is equal to $n \times m$

i.e. $|A \times B| = n \times m$

Now extend this analogy for the product rule.

Product Rule : If one experiment E_1 has n_1 possible outcomes and another experiment E_2 has n_2 possible outcomes then there are $n_1 \cdot n_2$ possible outcomes when the sequence of experiment E_1 first followed by E_2.

Example 6.2.3 *How many ways can we select a girl and a boy representatives from the class of 30 girls and 40 boys ?*

Solution : A girl representative can be selected by 30 ways.

A boy representative can be selected by 40 ways.

Therefore, a girl and a boy representative can be selected by $30 \times 40 = 1200$ different ways.

Example 6.2.4 *In how many different ways one can answer all questions of a true or false test consisting of 8 questions ?*

Solution : There are two ways of answering each question {T, F}.

Therefore by product rule, the number of ways in which all the 8 questions can be answered = $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8$ = **256** ways

6.3 Permutations

SPPU : Dec.-05, 14, May-19

An arrangement in a sequence of elements of a set is called a permutation of elements.

Depending upon the nature of arrangements, there are three types of permutations.

Type I) Permutations when all objects are distinct : A permutation of n objects taken r at a time is an arrangement of r objects out of n objects where $r \leq n$.

It is called r - permutations or r - arrangements and denoted by P (n, r) or ${}^nP_r$.

e.g. 1) Consider the three letters a, b, c. The arrangements of the letters a, b, c taken two at a time are ab, ba, ac, ca, bc, cb.

$\therefore$ The number of 2 - arrangements are 6. i.e. the number of permutations of 3 symbols taken two at a time = ${}^3P_2 = 6$.

Therefore as discussed above, the first place in the sequence can be filled up in n - ways, the second place in (n – 1) ways, the third place in (n – 2) ways and proceeding in this manner the r^{th} place can be filled up in n – (r – 1) = n – r + 1 ways.

$$\text{Hence } {}^nP_r = n\cdot(n-1)\cdot(n-2)\ldots(n-(r-1))$$

$$= \frac{n\cdot(n-1)(n-2)\ldots(n-r+1)\times(n-r)!}{(n-r)\ !}$$

$$\boxed{{}^nP_r = \frac{n!}{(n-r)!}\ ;\ 0 \le r \le n}$$

Properties : 1) $\quad {}^nP_n = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!} = n!$

2) $\quad {}^nP_1 = \dfrac{n!}{(n-1)!} = \dfrac{n(n-1)!}{(n-1)!} = n$

$$ {}^nP_2 = \frac{n!}{(n-2)!} = \frac{n(n-1)[(n-2)!]}{(n-2)!} = n(n-1)$$

$$ {}^nP_3 = n(n-1)(n-2) \text{ and so on.}$$

3) $\quad 0! = 1$

Example 6.3.1 *Given that A = {1, 2, 3, 4, 5, 6}, find the number of permutations of A taken i) 2 at a time ii) 3 at a time iii) 4 at a time iv) 5 at a time v) 6 at a time.*

Solution :

We have $\quad A = \{1, 2, 3, 4, 5, 6\}$

$\therefore \quad |A| = 6$

i) The permutation at 6 letters taken 2 at a time is ${}^6P_2 = \dfrac{6!}{(6-2)!} = \dfrac{6!}{4!} = \dfrac{6\times5\times4!}{4!} = 30$

ii) The permutation of 6 letters taken 3 at a time is ${}^6P_3 = \dfrac{6!}{(6-3)!} = \dfrac{6\times5\times4\times3!}{3!} = 120$

Similarly,

iii) $\quad {}^6P_4 = \dfrac{6!}{(6-2)!} = \dfrac{6!}{4!} = 30$

iv) $$^6P_5 = \frac{6\,!}{(6-5)\,!} = \frac{6\,!}{1\,!} = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

v) $$^6P_6 = \frac{6\,!}{(6-6)\,!} = \frac{6\,!}{0\,!} = 6\,! = 720$$

Example 6.3.2 *How many four digit numbers can be formed out of digits 1, 2, 3, ...9. if*

i) No repetition is permitted.

ii) How many of these will be greater than 3000.

Solution :

i) The number of ways of selecting 4 digits out of 9 digits is

$$^9P_4 = \frac{9\,!}{(9-4)\,!} = \frac{9\times 8\times 7\times 6\times 5\,!}{5\,!} = 3024$$

ii) There is a restriction that the 4 digit numbers so formed must be greater than 3000.

Therefore the thousandth position can be filled with numbers 3, 4, 5, 6, 7, 8, 9 i.e. the thousandth place can be selected in 7 different ways.

Now out of remaining 8 digits, hundredth position can be filled in 8 different ways, Tenth place can be filled in 7 different ways and unit place can be filled in 6 different ways.

Thus the total number of 4 digit numbers greater than 3000 can be formed in

$7 \times 8 \times 7 \times 6 = 2352$ ways.

Example 6.3.3 *i) Suppose repetitions are not permittable, how many four digit numbers can be formed from six digits 1, 2, 3, 5, 7, 8 ?*

ii) How many of such numbers are less than 4000 ?

iii) How many in (i) are even ?

iv) How many in (ii) are odd ?

v) How many in (i) contain both 3 and 5.

vi) How many in (i) are divisible by 10. **SPPU : Dec.-05**

Solution :

i) Out of 6 numbers, 4 digit numbers can be formed in $^6P_4$ ways.

$$\therefore \quad \text{Number of ways} = \frac{6\,!}{2\,!} = 360$$

ii) The four digit numbers which are less than 4000 are the numbers in which first digit is 1, 2 or 3 i.e. 1^{st} digit can be chosen in 3 ways, 2^{nd} digit can be any one of the remaining 5 digits. 3^{rd} digit can be any of the remaining 4 digit and the 4^{th} digit is any one of the remaining 3 digits.

Hence the total number of ways = $3 \times 5 \times 4 \times 3 = 180$.

iii) Those numbers ending in 2 or 8 are even numbers. Hence the last digit (4^{th} digit) can be chosen in 2 ways (the number 2 or 8). The first digit can be chosen in any one of the remaining 5 digits, 2^{nd} in any of the 4 digits and 3^{rd} in any of the 3 digits. Hence the total number of ways

iv) The numbers less than 4000 and are odd. The numbers ending with 1, 3, 5 or 7 are odd. The 4 digit numbers ending in 1 and less than 4000 should begin with eithor 2 or 3.

$$= (5 \times 4 \times 3) \times 2$$

TH	H	Ten	Unit place

= 120 ways

Then there are $2 \times 4 \times 3 \times 1 = 24$ such numbers. Similarly the number ending in 3 are 24. However the number ending with 5 or 7 are

TH UnitPlace

$\overset{1}{3} \times 4 \times 3 \times \overset{1}{2}$

= 72 ways.

Hence the total number which are odd and less than 4000 are 24 + 24 + 72 = 120

v) The digit 3 can occupy any of the 4 positions and the remaining 3 positions will be occupied by the digit 5. Hence the number of ways in which two positions are occupied by 3 and 5 will be 4×3 i.e. 12.

Now the remaining two positions will be filled by the remaining 4 numbers i.e. 1, 2, 7 and 8.

Hence out of remaining two positions one position can be occupied in 4 different ways and the remaining position will be occupied in 3 different ways.

Hence total number of 4 digit numbers in which both 3 and 5 are present = $12 \times 4 \times 3 = 144$.

vi) Not even a single number is divisible by 10 as there is no zero at unit's place.

Example 6.3.4 *A menu card in a restaurant displays four soups, five main courses, three desserts and 5 beverages. How many different menus can a customer select if,*

i) He selects one item from each group without omission.

ii) He chooses to omit the beverages, but selects one each from the other groups.

iii) He choose to omit the desserts but decides to take a beverage and one item each from the remaining groups.

Solution :

i) The customer can select the soup in 4 ways, the main course in 5 ways, the dessert in 3 ways and beverages in 5 ways.

Hence by product rule, the number of ways in which he can select one item each, without omission is $4 \times 5 \times 3 \times 5 = 300$.

ii) The number of ways in which he omit beverages $= 4 \times 5 \times 3 = 60$ ways.

iii) The number of ways in which he omit desserts but he takes all other items $= 4 \times 5 \times 5 = 100$ ways.

Example 6.3.5 *10 different M_1 books, 3 different M_2 books, 5 different M_3 books and 7 different D.S. books are to be arranged on a shelf. How many different arrangements are possible if*

i) The books in each subject must all be together

ii) Only M_3 books must be together.

Solution : i) M_1 books can be arranged among themselves in 10 ! ways, the M_2 books in 3 ! ways, M_3 books in 5 ! ways and D.S. books in 7 ! ways.

Hence the total number of arrangements $= 4! \cdot 10! \cdot 3! \cdot 5! \cdot 7!$

ii) Consider the 5 M_3 books as a single book. Then there are 21 books which can be arranged in 2 ! ways. In each of these arrangements the M_3 books can be arranged among themselves in 5! ways.

Hence the number of arrangements in $5! \cdot 2!$

Example 6.3.6 *2 mathematics papers and 5 other papers are to be arranged at an examination. Find the total number of ways if,*

i) Mathematics papers are consecutive. ii) Mathematics papers are not consecutive.

SPPU : May-19, Marks 3

Solution : i) Both mathematics papers (M_1 and M_2) are together, consider both M_1 and M_2 as single paper.

These two papers among themselves can be arranged in 2! ways.

Now 6 papers (as M_1 and M_2 is considered as single paper) can be arranged in 6! ways.

Hence total number of arrangements = $2! \cdot 6!$

ii) If M_1 or M_2 are not consecutive than they are to be arranged between the 4 gaps or at the 2 ends.

$$\begin{array}{ccccccccccc} M_1 & & M_1 & & M_1 & & M_1 & & M_1 & & M_1 \\ \text{or} & \square & \text{or} & \square & \text{or} & \square & \text{or} & \square & \text{or} & \square & \text{or} \\ M_2 & & M_2 & & M_2 & & M_2 & & M_2 & & M_2 \end{array}$$

Where $\square$ denotes other papers

Hence there are 6 places where mathematics papers can be arranged. Therefore, 2 mathematics papers can be arranged in 6 places in ${}^6P_2$ ways. Five other papers can be arranged among themselves in 5! ways.

Therefore total number of arrangements

$$= 5\,! \cdot {}^6P_2 \;=\; 5! \cdot 6 \cdot 5$$

$$= (120) \cdot (30) \;=\; 3600$$

Example 6.3.7 *How many permutations can be made out of the letter of word "COMPUTER" ? How many of these*

i) begin with C ii) end with R iii) begin with C and end with R iv) C and R occupy the end places

Solution : There are 8 letters in the word "COMPUTER" and all are distinct

$\therefore$ The total number of permutations of these letters is 8! = 40320.

i) Permutations begin with C :

The first position can be filled in only one way i.e. C and the remaining 7 letters can be arranged in 7! ways.

$\therefore$ The total no. of permutations begining with C = $1 \times 7! = 5040$

ii) Permutations end with R :

The Last position can be filled in only one way and the remaining 7 letters can arranged in 7! ways.

$\therefore$ The total no. of permutations ending with R be = $7! \times 1 = 5040$

iii) Permutation begin with C and end with R :

The first position can be filled in only one way i.e. C and the end position also can be filled in only one way i.e. R and the remaining 6 letters can be arranged in 6! ways.

$\therefore$ The required no. of permutations = $1 \times 6! \times 1 = 720$

iv) Permutation in which C and R occupy end places :

C and R occupy end positions in 2 ! ways i.e. CR or RC and the remaining 6 letters can be arranged in 6! ways.

$\therefore$ The total no. of required permutations = $2! \times 6! = 1440$

Permutations with restrictions :

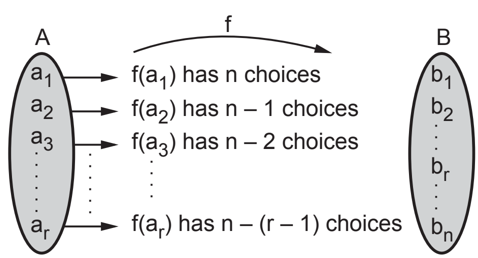
1) The number of permutations of n different objects taken r at a time in which p particular objects do not occur is ${}^{(n-p)}P_r$.

2) The number of permutations of n different objects taken r at a time in which p particular objects are present is ${}^{(n-p)}P_{r-p} \times {}^{r}P_p$.

Example 6.3.8 *Show that the number of injective functions from a set with r elements to a set with n elements is ${}^{n}P_r$; $r \leq n$.*

Solution : Let A and B be two sets with $|A| = r$ and $|B| = n$.

$$A = \{a_1, a_2, \ldots a_r\} \quad \text{and} \quad B = \{b_1, b_2, b_3 \ldots b_r, \ldots b_n\}$$



Let $f : A \rightarrow B$ be an injective function

Hence by product rule, the number of injective functions from A to B is $n(n-1)(n-2) \ldots (n-r+1) = {}^{n}P_r$

Type II) Permutations when all objects are not distinct

A) If r_1 objects are of one kind

r_2 objects are of second kind

r_3 objects are of third kind

$\vdots$

r_k objects are of k^{th} kind, where $r_1 + r_2 + \ldots r_k = n$

Then the number of permutations when all are taken at a time (i.e. $r_1, r_2, \ldots r_k$ are taken) is $\dfrac{n!}{r_1!\, r_2!\, r_3! \ldots r_k!}$

Example 6.3.9 *Find the number of permutations that can be made out of the letters i) MISSISSIPPI ii) ASSASSINATION* **SPPU : Dec.-14**

Solution : i) There are 11 letters in the word out of which S, I, P, M are distinct.

S appears 4 times

I appears 4 times

P appears 2 times

M appears 1 time

$\therefore$ The required no. of permutations $= \dfrac{11\ !}{4!\ 4!\ 2!\ 1!} = 34650$

ii) There are 13 letters of which A, I, N, S, T and O are different.

A appears 3 times

I appears 2 times

N appears 2 times

S appears 4 times

T appears 1 time

O appears 1 time

$\therefore$ The required no. of permutations $= \dfrac{13\ !}{3!\ 2!\ 2!\ 4!\ 1!\ 1!} = 10810800$

Example 6.3.10 *How many ways can the letters in the word "PIONEER" be arranged so that the two E's are always together.*

Solution : The word 'PIONEER' has two E's and remaining 5 letters are distinct. These distinct five letters can be arranged in 5 ways and for each such arrangement two E's can occupy any of the six remaining places. Hence the required no. of permutations are

$$6 \times 5! = 6! = 720.$$

Example 6.3.11 *How many seven digit numbers can be formed using digits 1, 7, 2, 7, 6, 7, 6 ?*

Solution : There are 7 digits out of which 7 is repeated 3 times, 6 is repeated twice, appears 1 time and 2 appear 1 time.

$\therefore$ The total no. of permutation $= \dfrac{7\ !}{3! \times 2! \times 1! \times 1!} = 420$

Type III) Permutations with repeated objects.

The number of different permutations of n distinct objects taken r at a time when every object is allowed to repeat any number of times is given by n^r.

Example 6.3.12 *How many 4 digits numbers can be formed by using the digits 2, 4, 6, 8 when repetition of digits are allowed ?*

Solution : We have 4 digits numbers.

No. of ways of filling unit's place = 4

No. of ways of filling ten's place = 4

No. of ways of filling hundred's place = 4

No. of ways of filling thousand's place = 4

Therefore the total number of 4 digits numbers = $4 \times 4 \times 4 \times 4 = 4^4 = 256$

Example 6.3.13 *How many 4 digits even numbers can be formed by using the digits 1, 3, 4, 6, 8 when repetition of digits are allowed ?*

Solution : We have 3 even numbers and 2 odd numbers. Therefore,

The no. of ways of filling unit's place = 3

The no. of ways of filling ten's place = 5

The no. of ways of filling hundred's place = 5

The no. of ways of filling thousand's place = 5

Thus, the total no. of required 4 digits numbers = $3 \times 5^3 = 3 \times 125 = 375$

Example 6.3.14 *In how many ways can 5 software projects be alloted to 6 final year students when all of five projects are not alloted to the same student.*

Solution : We have 5 projects and 6 students. Each project can be alloted in 6 ways.

Thus, the number of ways of alloting 5 projects = $6 \times 6 \times 6 \times 6 \times 6 = 6^5$

The number of ways in which all 5 projects are allotted to same student = 6.

Therefore, total number of ways to allocate 5 projects to 6 students = $6^5 - 6 = 7770$

Example 6.3.15 *A bit is either 0 or 1. A byte is a sequence of 8 bits. Find i) Number of bytes ii) Number of bytes that begin with 11 and end with 11.*

Solution : i) Total number of byte is $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8 = 256$.

ii) As the first two and last two bits are fixed i.e. 11 the remaining bits in the sequence are either 0 or 1.

$\therefore$ The required no. of total bytes = $2^4 = 16$.

Example 6.3.16 *Prove that the number of circular permutations of n different objects is $(n - 1)$!*

Solution : Let us consider that k be the number of required permutations.

For each such circular permutation of k, there are n corresponding linear permutations. We can start from every object of n objects in the circular permutation. Thus for k circular permutations, we have kn linear permutations.

Therefore $k \cdot n = n! \Rightarrow k = \frac{n!}{n} = (n-1)!$ Hence the proof.

Example 6.3.17 *How many ways can these letters A, B, C, D, E and F be arranged in a circle ?*

Solution : There are six letters. Hence the no. of ways to arrange these six letters in a circle is $(6-1)! = 5! = 120$.

Example 6.3.18 *In how many ways 10 programmers can sit on a round table to discuss the project so that project leader and a particular programmer always sit together ?*

Solution : There are 10 programmers. But project leader and particular programmer always sit together. So both become a single unit and hence there are

$(10 - 2 + 1) = 9$ remins

Thus these 9 units can be arreanged on round table in $(9-1)! = 8!$ ways.

The two programmer i.e. project leader and particular programmer can be arranged in 2! ways.

Therefore the required no. of ways $= 2! \times 8! = 80640$.

Example 6.3.19 *Determine the number of ways in which 5 software engineers and 6 electronics engineers can be sitted at a round table so that no two software engineers can sit together.* **SPPU : Dec.-16, 17, Marks 6**

Solution : There are 6 electronics engineers that can be arranged round a table in $(6-1)!$ ways. There are 5 software engineers and they are not to sit together, so there are six places for software engineers and can be placed in 6! ways as shown in Fig. 6.3.1

Therefore the required no. of ways $= (6-1)! \times 6! = 5! \times 6!$

$= 120 \times 720 = 86400$

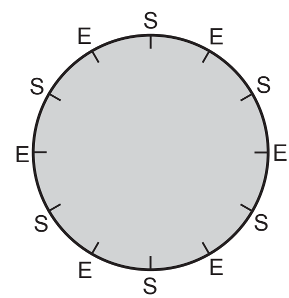


Fig. 6.3.1

6.4 Combination

SPPU : May-06, 14

A combination is a selection of some or all objects from a set of given objects where the order of the objects does not matter. In this context, we used mainly two words "Selection" and "arrangement". In selection, order of objects is immaterial i.e. selection is a set. But in arrangement, the order of objects is important it is not a set. Arrangement is a n-tupple. Arrangement is associated with permutation selection with combination.

I) Definition : The number of combinations of n different objects taken r at a time is given by ${}^{n}C_{r}$ and defined as

$$ {}^{n}C_{r} = \frac{n\,!}{r\,!\,(n-r)\,!} \quad ; \quad r \leq n $$

Properties :

1) $$ {}^{n}C_{r} = \frac{n\,!}{r!(n-r)\,!} = \frac{{}^{n}P_{r}}{r!} $$

2) $$ {}^{n}C_{n} = \frac{n\,!}{n!\,(n-n)\,!} = 1 $$

3) $$ {}^{n}C_{0} = \frac{n\,!}{0!\,(n-0)\,!} = \frac{n!}{n!} = 1 $$

4) $$ {}^{n}C_{1} = \frac{n!}{1!\,(n-1)\,!} = n \quad , \quad {}^{n}C_{2} = \frac{n\,!}{2!\,(n-2)\,!} = \frac{n\,(n-1)}{2} $$

$$ {}^{n}C_{2} = \frac{n\,(n-1)\,(n-2)}{3!} \quad , \quad {}^{n}C_{4} = \frac{n\,(n-1)\,(n-2)\,(n-3)}{4\,!} $$

Example 6.4.1 *Find the value of n if i)* ${}^{n}C_{n-2} = 10$ *and ii)* ${}^{25}C_{n+2} = {}^{25}C_{2n-1}$

Solution :

i) We have $${}^{n}C_{n-2} = 10 \Rightarrow \frac{n\,!}{(n-2)\,!\,(n-(n-2))\,!} = 10$$

$$\therefore \quad \frac{n\,(n-1)\,(n-2)\,!}{(n-2)\,!\,(2\,!)} = 10$$

$$n\,(n-1) = 10 \times 2 = 20$$

$$n^{2} - n - 20 = 0$$

$$\Rightarrow \quad (n-5)\,(n+4) = 0$$

$$n = 5 \text{ or } n = -4 \text{ as } n = -4 \text{ is not possible}$$

$$\boxed{n = 5}$$

ii) we know that ${}^{n}C_{r} = {}^{n}C_{n-2}$

Now ${}^{25}C_{n-2} = {}^{25}C_{n-2}$

$\Rightarrow$ Either $n + 2 = 2n - 1$ or $(n + 2) + (2n - 1) = 25$

$\therefore$ either $n = 3$ or $3n = 25 - 1 = 24$

$\therefore \quad n = 3$ or $n = 8$

$\therefore \qquad n = 3, 8.$

Example 6.4.2 *How many 16 bit strings are there containing exactly five 0's ?*

Solution : Each string of 16 bit has 16 digits. A 16-bit string having exactly five 0's is determined if we tell which bit are 0's. So, here order is immaterial.

$\therefore$ This can be done in ${}^{16}C_5$ ways.

$\therefore$ The total number of 16-bit strings is

$$^{16}C_5 = \frac{16!}{5! \times (16-5)!} = \frac{16\times 15\times 14\times 13\times 12\times 11!}{5\times 4\times 3\times 2\times 1\times 11!} = 4368$$

Example 6.4.3 *In how many ways can 30 late admitted students be assigned to three practical batches A, B, C if A can accomodate 10 students, B - 15 students and C - 5 students only ?*

Solution : The batch A can accomodate 10 students out of 30.

$\therefore$The batch A can be assigned 10 students in ${}^{30}C_{10}$ ways.

then batch B can be assigned 15 students in ${}^{20}C_{15}$ ways.

then batch C can be assigned 5 students in ${}^{5}C_{5}$ ways.

Therefore by the product rule, the total no of ways of assigned students is

$$^{30}c_{10} \times {}^{20}c_{15} \times {}^{5}c_{5} = \frac{30!}{10!\,(20!)} \times \frac{20!}{15! \times 5!} \times 1 = \frac{30!}{10!\; 15!\; 5!}$$

Example 6.4.4 *How many ways can we select a software development group of 1 project leader, 15 programmers and 6 data entry operators from a group of 5 project leaders 20 programmers and 25 data entry operators.*

Solution : One project leader can be selected from 5 project leaders in ${}^{5}C_{1} = 5$ ways.

15 programmers can be selected from 20 programmers in ${}^{20}C_{15}$ ways.

6 data entry operators can be selected from 25 data entry operators in ${}^{25}C_{6}$ ways.

Therefore the total number of ways to select the software development group is

${}^{5}C_{1} \times {}^{20}C_{15} \times {}^{25}C_{6} = 96101544000.$

Example 6.4.5 *From 10 programmers in how many ways can 5 be selected when i) A particular programmer is included everytime ii) A particular programmer is not included at all time.*

Solution : The number of ways to select 5 programmers from 10 is $^{10}C_5 = \frac{10!}{5!\,5!}$

$$= \frac{10\times 9\times 8\times 7\times 6}{5\times 4\times 3\times 2\times 1} = 252$$

i) When a particular programmer is included every time then the remaining 5 – 1 = 4 programmers can be selected from the remaining 10 – 1 = 9 programmers. This can be done in $^9C_4$

$$\therefore \qquad {}^9C_4 = \frac{9!}{4!\,5!} = \frac{9\times 8\times 7\times 5}{4\times 3\times 2} = 126$$

ii) When a particular programmer is not included at all then the 5 programmers can be selected from the remaining 10 – 1 = 9 programmers.

This can be done in $^9C_5$ ways.

$$\therefore \qquad {}^9C_5 = \frac{9!}{4!\,5!} = 126$$

Example 6.4.6 *A committee of 5 people is to be formed from a group of 4 men and 7 women. How many possible committees can be formed if at least 3 women are on the committee ?*

SPPU : May-14

Solution : If at least three women are on committee, it means committee with 3 women or 4 women or 5 women.

i) 3 women can be selected in $^7C_3$ ways.

2 men can be selected in $^4C_2$ ways.

The no. of ways this can be done is $^7C_3 \times {}^4C_2$ = 210 ways.

ii) 4 women and 1 man can be selected in $^7C_4$ and $^4C_1$ ways respectively.

$\therefore$ The number of ways to form a committee is $^7c_4 \times {}^4c_1$ = 140 ways.

iii) 5 women can be selected in $^7C_5$ ways.

$\therefore$ The no. of ways to form a committee is $^7C_5$ = 21 ways

Hence the total no. of ways a committee can be formed with at least 3 women is

210 + 140 + 21 = 371.

Example 6.4.7 *How many automobile license plates can be made if each plate contains two different letters followed by three different digits. Solve the problem if the first digit can not be zero.*

SPPU : May-06

Solution : The first position is a letter and can be selected from 26 letters in

$$^{26}C_1 = 26 \text{ ways.}$$

The second position is a letter and can be selected from 26 – 1 = 25 letters in

$$^{25}C_1 = 25 \text{ ways.}$$

For digits :

1) The first digit can be selected from 10 digits in $^{10}C_1 = 10$ ways.

2) The second digit can be selected from 9 digits in $^{9}C_1 = 9$ ways

3) The third digit can be selected from 8 digits in $^{8}C_1 = 8$ ways.

Therefore the total number of license plates

$$= 26 \times 25 \times 10 \times 9 \times 8 = 468000$$

Now, in license plate, the first digit can not be 0 then the first position can selected from 9 digits in $^{9}C_1$ ways.

The second digit can be zero, but one digit is already selected for the first position. Hence the second digit can be selected in $^{9}C_1$ ways. The third digit can be selected in $^{8}C_1$ ways.

Hence the total no. of required license plates are

$26 \times 25 \times 9 \times 9 \times 8 = 421200$.

Example 6.4.8 *In the discrete structure paper, there are 10 questions. In how many ways can an examiner select five questions in all the first question is compulsory.*

Solution : The first question is compulsory, so the examiner has to select 4 questions from the remaining 9 questions.

$\therefore$ The number of ways to select five questions is

$$1 \times {}^{9}C_4 = \frac{9!}{4!\,5!} = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2} = 126 \text{ ways.}$$

Example 6.4.9 *Determine the number of triangles that are formed by selecting points from a set of 12 points out of which 5 are colinear.*

Solution : By using 12 points, the number of triangles formed is $^{12}C_3$.

As five points are colinear i.e. lie on same line, they do not form any triangle. Thus $^{5}C_3$ triangles are lost.

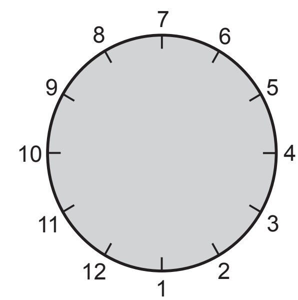
$\therefore$ The total number of triangles produced is

$$^{12}C_3 - {}^5C_3 = \frac{12!}{3! \times 9!} - \frac{5!}{3!\ 2!}$$

$$= \frac{12 \times 11 \times 10}{3 \times 2} - \frac{5 \times 4}{2} = 220 - 10 = 210$$

Example 6.4.10 *How many lines can be drawn through 12 points on a circle and line passes through exactly two points ?*

Solution : As all points on the circle are not colinear, thus no line will lost.



The total no. of lines drawn through 12 points is

$$^{12}C_2 = \frac{12\ !}{2!\ 10!} = \frac{12 \times 11}{2 \times 1} = 66$$

Example 6.4.11 *Determine the number of diagonals that can be drawn by joining the nodes of octgon.*

Solution : The number of lines that can be drawn by 2 points out of 8 points of octagon is $^8C_2$ = 28. Out of these 28 lines, 8 are the sides of the octagon.

$\therefore$ The number of diagonals = 28 – 8 = 20.

Example 6.4.12 *In a box, there are 40 floppy disks of which 4 are defective. Determine*

i) In how many ways we can select five floppy disks ?

ii) In how many ways we can select five non defective floppy disks ?

iii) In how many ways we can select five floppy disks containing exactly three defective disks ?

iv) In how many ways we can select five floppy disks containing at least 1 defective disk ?

Solution : i) There are 40 floppy disks out of which we have to select 5 floppy disks in $^{40}C_5$ ways.

$$^{40}C_5 = \frac{40\ !}{5! \times (40-5)\ !} = \frac{40 \times 39 \times 38 \times 37 \times 36}{5 \times 4 \times 3 \times 2 \times 1} = 658008$$

ii) There are 40 – 4 = 36 nondefective floppy disks out of which we have to select 5. This can be done in $^{36}C_5$ ways.

$$\therefore \qquad ^{36}C_5 = \frac{36\ !}{5! \times (31)\ !} = \frac{36 \times 35 \times 34 \times 33 \times 32}{5 \times 4 \times 3 \times 2 \times 1} = 376992$$

iii) To select exactly three defective floppy disks out of 4 disks, we have $^4C_3$ ways and the remaining two floppy disks can be selected from 36 disks in $^{36}C_2$ ways.

Therefore, the required no. of ways = ${}^{4}C_3 \times {}^{36}C_2$

$$= \frac{4!}{3! \times 1!} \times \frac{36!}{2!\ 34!} = 4 \times \frac{36 \times 35}{2} = 2520$$

iv) There are 4 defective floppy disks out of which at least one must be selected. We know that the total number of ways to select 5 disks from 40 disks is ${}^{40}C_5$.

Also the number of ways to select 5 floppy disks with no defective is ${}^{36}C_5$ way.

Therefore the required no. of ways

$$= {}^{40}C_5 - {}^{36}C_5 = 658008 - 376992 = 281016$$

II) Combination with Repetitions : The number of ways to fill r slots from n categories with repetition allowed is c (n + r – 1, r) = c (n + r – 1, n – 1)

Therefore we have

$$^{(n+r-1)}C_r = {}^{(n+r-1)}C_{n-1} = C\,(n + r - 1, r)$$

and C (n + r – 1, r) = The number of non-negative integers solution of

$x_1 + x_2 + \ldots + x_n = r$

= The number of ways of placing r indistinguishable balls in n numbered boxes.

= The number of binary numbers with n–1 one's and r zero's.

= C (n + r – 1 ; n – 1)

= The number of r-combinations of $\{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_n\}$

Example 6.4.13 *How many 4 combinations of {1, 2, 3, 4, 5, 6} are there with unlimited repetition ?*

Solution : We have r = 4, n = 6

∴The number of 4 combinations of {1, 2, 3, 4, 5, 6} are

$$C\,(6 + 4 - 1, 4) = C\,(9, 4) = \frac{9!}{4!\ 5!} = \frac{9 \times 8 \times 7 \times 6 \times 5!}{4 \times 3 \times 2 \times 1 \times 5!} = 126.$$

Example 6.4.14 *Find the number of 3-combinations of $\{\infty \cdot a_1, \infty \cdot a_2, \infty \cdot a_3, \infty \cdot a_4\}$.*

Solution : We have n = 4, r = 3

∴The number of 3-combinations of the given set is

$$C\,(4 + 3 - 1, 3) = C\,(6, 3) = \frac{6!}{3!\ 3!} = \frac{6 \times 5 \times 4 \times 3!}{3 \times 2 \times 3!} = 20$$

Example 6.4.15 *The number of non negative integer solutions to $x_1 + x_2 + x_3 + x_4 = 20$*

Solution : we have r = 20, n = 4

$\therefore$ The number of non negative integer solutions

$$= C(4 + 20 - 1, 20) = C(23, 20) = \frac{23!}{20! \times 3!} = \frac{23 \times 22 \times 21}{3 \times 2} = 1771$$

Example 6.4.16 *The number of ways of placing 8 similar balls in 5 numbered boxes.*

Solution : The number of ways of placing r = 8 similar balls in n = 5 boxes is

$$C(5 + 8 - 1, 8) = c(12, 8) = 495.$$

Example 6.4.17 *Find the number of binary numbers with six 1's and 4 zero's.*

Solution : The number of binary numbers with 6 one's and 4

$$\text{zero's} = C(6 + 4, 4) = C(10, 4) = \frac{10 \times 9 \times 8 \times 7 \times 6!}{4 \times 3 \times 2 \times 1 \times 6!} = 210$$

Example 6.4.18 *How many integral solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 = 16$ where each $x_i \geq 2$?*

Solution : Let $x_i = y_i + 2$ where $y_i \geq 0$.

we have $x_1 + x_2 + x_3 + x_4 + x_5 = 16$

iff $y_i + 2 + y_2 + 2 + y_3 + 2 + y_4 + 2 + y_5 + 2 = 16$

Iff $y_1 + y_2 + y_3 + y_4 + y_5 = 16 - 10 = 6$

Thus the number of integral solutions of given equation is the same as the number of integral solutions of $y_1 + y_2 + y_3 + y_4 + y_5 = 6$

There are C (5 – 1 + 6, 6) = C (10, 6) such solutions

$$\therefore \quad C(10, 6) = \frac{10!}{6! \times 4!} = \frac{10 \times 9 \times 8 \times 7 \times 6!}{4 \times 3 \times 2 \times 1 \times 6!} = 210$$

Example 6.4.19 *How many integral solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 = 30$ where $x_1 \geq 2$, $x_2 \geq 3$, $x_3 \geq 4$, $x_4 \geq 2$, $x_5 \geq 0$.*

Solution : Let $x_1 = y_1 + 2$, $x_2 = y_2 + 3$, $x_3 = y_3 + 4$, $x_4 = y_4 + 2$, $x_5 = y_5 + 0$

$$\therefore \quad x_1 + x_2 + x_3 + x_4 + x_5 = 30$$

$$\Rightarrow \quad y_1 + 2 + y_2 + 3 + y_3 + 4 + y_4 + 2 + y_5 = 30$$

$$y_1 + y_2 + y_3 + y_4 + y_5 = 19$$

$\therefore$ The required number of integral solutions are

$$C(5 - 1 + 19, 19) = C(23, 19) = \frac{23 \times 22 \times 21 \times 20}{4 \times 3 \times 2 \times 1} = 8855$$

Theorem 1 : The number of integer solutions to $a_1 + a_2 + a_3 + \ldots + a_n = r$ when $a_1 \geq b_1$, $a_2 \geq b_2$, $a_3 \geq b_3$, … $a_n \geq b_n$ is C $(n + r - 1 - b_1 - b_2 - b_3 - \ldots b_n,$ $r - b_1 - b_2 - b_3 - \ldots - b_n)$

Theorem 2 : The number of ways to select r things from n categories with b total restrictions on the r things is C (n + r – 1 – b, r – b)

Theorem 3 : The number of ways to select r things from n categories with atleast 1 thing from each category is C (r – 1, r – n) ($\because$ b = n)

6.5 Generation of Permutation and Combination **SPPU : May-18**

I) Generation of permutations : Suppose we want to generate n! permutations of n distinct objects. For n = 1, 2, 3, it is simple but when n is large it is difficult to keep track of what we have written and make sure that we shall write down all permutations with no repetition or omissions.

An interesting problem is to find a systematic procedure for generating all n! permutations of a set with n distinct elements.

Suppose from the initial permutation 1, 2, 3, … n by using the next permutation procedure repeatedly we shall obtain all the permutations of 1, 2, 3, …, n. The last permutation is n, (n – 1), (n – 2), … 4, 3, 2, 1.

Procedure for next permutation :

Step 1 : Given a permutation $a_1, a_2, a_3, \ldots a_n$ of 1, 2, 3, …, n

Step 2 : Scan from right to left (L $\leftarrow$ R). Find the first m such that $a_m < a_{m+1}$

Step 3 : $\alpha = \min \{a_k \mid k = m + 1, m + 2, m + 3, \ldots n, a_k > a_m\}$

Step 4 : The next permutation is

$a_1 \; a_2 \; a_3 \; \ldots \; a_{m-1}, \alpha, x, x, x \ldots$

where x, x, x … are the remaining numbers arranged in the increasing order.

Algorithms :

I) Algorithm for generating the next Permutation in Lexicographic order :

Next permutation $[(a_1, a_2, \ldots, a_n)]$: Permutation of {1, 2, 3, …n} not equal to (n, (n – 1) (n – 2) … 3 2 1)]

$i := n - 1$

while $a_i > a_k$

k : = k – 1 {a_k is the smallest integer greater than a_i to the right of a_i} interchange a_i and a_k

r : = n

s : = j + 1

while r > s

begin

interchange a_r and a_s

r : = r – 1

s = s + 1

end

{This puts the tail end of permutation after the i^{th} position in increasing order}

Example 6.5.1 *Let n = 6 and given permutation is 125364 find the next permutation.*

Solution : Step 1 : Given permutation is 125364

Step 2 : Scan from right to left and find the first m such that. $a_m < a_{m+1}$

Now start with 5^{th} position ; As 6 > 4, m ≠ 5

As 3 < 6 and position of 3 is 4^{th} $\therefore$ m = 4.

Step 3 : $\alpha = \min \{a_k \mid a_k > a_m\} = \min \{6, 4\} = 4$

Step 4 : Replace element a_m by α i.e. 3 by 4, keep previous elements as its i.e. 125 and write all remaining elements in increasing order i.e. 36.

$\therefore$ The next permutation is 125436.

Example 6.5.2 *Find the next two permutations of 125436*

Solution : i) Given that $1254\underset{\uparrow}{3}6$

$\therefore$ m = 5 and α = min { 6 } = 6

$\therefore$ The next permutation is 125463.

ii) Find the next permutation of $125\underset{\uparrow}{4}63$

m = 4, α = min {6} = 6

$\therefore$ The next permutation is 125634.

Example 6.5.3 *Generate all permutations for n = 3 by next permutation method.*

SPPU : May-18, Marks 4

Solution : Consider the following table to generate all permutations for n = 3.

Sr. No.	Given permutation	Next permutation
1)	1 2 3 $m = 2, \alpha = 3$ ↑ (under 2)	1 3 2
2)	1 3 2 $m = 1, \alpha = 2$ ↑ (under 1)	2 1 3
3)	2 1 3 $m = 2, \alpha = 3$ ↑ (under 1)	2 3 1
4)	2 3 1 $m = 1, \alpha = 3$ ↑ (under 2)	3 1 2
5)	3 1 2 $m = 2, \alpha = 2$ ↑ (under 1)	3 2 1 Last permutation.

$\therefore$ The set of all permutations for n = 3 is

{(1 2 3) (1 3 2) (2 1 3) (2 3 1) (3 1 2) (3 2 1)}

II) Procedure to generate subsets of {1, 2, 3, … n}

Let $\{a_1, a_2, a_3, \ldots a_k\}$ be a subset of size k of {1, 2, 3, …n} with $a_1 < a_2 < a_3 < \ldots < a_k$.

The maximum possible value of a_k is n

The maximum possible value of $a_{k-1} = n - 1$

In general the maximum possible value of a_i is $n - k + i$. Consider the subset {1, 2, 3, … k – 1, k} If $k \neq n$, its maximum value then increase k by 1, so that the next subset {1, 2, 3, … k – 1, k + 1} is generated. We continue this procedure till we reached to {1, 2, 3 … (k – 1), n}. Now repeat the procedure for k – 1, if $k - 1 \neq n - 1$ then increase it by 1 and continue this process with k – 1 till we reached to {1, 2, 3, … (k – 2) (n – 1), n} Then move to (k – 2) and repeat the same process. In this manner, moving from right to left we finally reach to an element a_j such that a_j can be increased to a_{j+1} but no a_j with $i > j$ can be increased which means that at some stage a_i is equal to its maximum value $n - k + i$. This procedure terminates when a_1 reaches to its maximum value.

II) Algorithm for generating the next r-combination in lexicographic order.

Next r-combination $(\{a_1, a_2, a_3, \ldots a_r\})$: proper subset of {1, 2, 3, …n} not equal to {n – r + 1, n – r + 2, … n} : with $a_1 < a_2 < a_3 < \ldots a_r$)

i ; = r

while $a_i = n - r + i$

i : = i – 1

$a_i = a_2 + 1$

for j = i + 1 to r

$a_j = a_i + j - i$

Example 6.5.4 *Generate all subsets of size 4 of {1, 2, 3, 4, 5, 6}.*

Solution : Let us begin with {1, 2, 3, 4}. We know that for any subset $\{a_1, a_2, a_3, a_4\}$ with $a_1 < a_2 < a_3 < a_4$ the maximum possible value of a_4 is 6, a_3 is 5, a_2 is 4 and a_1 is 3.

i) For $a_4 = 4$:

Hence increasing 4 by 1 we obtain a subset {1, 2, 3, 5}. Since a_4 has not still reached to 6.

$\therefore$ Again increase 5 by 1, we get {1, 2, 3, 6}.

ii) For $a_3 = 3$:

The maximum value of a_3 is 5.

$\therefore$ Increase a_3 successively by 1 still, we reach to 5.

$\therefore \{1, 2, 4, \underset{\uparrow}{5}\}$, $\{1, 2, \underset{\uparrow}{4}\ 6\}$, $\{1, 2, 5, 6\}$

iii) For $a_2 = 2$: The maximum value of a_2 is 4

$\therefore$ We get $(1, \underset{\uparrow}{3}, 4, 5\}$, $\{1, 3, 4, \underset{\uparrow}{6}\}$, $\{1, 3, \underset{\uparrow}{5}, 6\}$, $\{1, 3, \underset{\uparrow}{4}, 6\}$

iv) For $a_1 = 1$: Maximum value of a_1 is 3.

$\therefore$ We get $\{\underset{\uparrow}{2}, 3, 4, 5\}$, $\{2, 3, 4, \underset{\uparrow}{6}\}$, $\{2, 3, \underset{\uparrow}{5}, 6\}$, {2, 4, 5, 6}, {3, 4, 5, 6}

Thus we obtain the following 15 subsets

{ {1, 2, 3, 4} {1, 2, 3, 5} {1, 2, 3, 6}

{1, 2, 4, 5} {1, 2, 4, 6} {1, 2, 5, 6}

{1, 3, 4, 5} {1, 3, 4, 6}, {1, 3, 5, 6} {1, 4, 5, 6}

{2, 3, 4, 5}, {2, 3, 4, 6}, {2, 3, 5, 6} {2, 4, 5, 6} {3, 4, 5, 6} }

6.6 Binomical Coefficients

In the previous sections we have studied permutations and combinations of set with n elements. By using the formula of combinations we can define binomial coefficients.

Generally, Bi means two and nomial means nominee's So binomial means an experiment which has only two outcomes. e.g. i) Tossing a coin ii) Result of any examination is eitherpass or fail.

In mathematics, binomial coefficients are numbers which are coefficients in the expansion of powers of binomial expressions i.e. $(x + y)^n$

6.6.1 Binomial Theorem

Let x, y be any variables and n be a non-negative integer. Then

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

$$= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

$$= \binom{n}{n}x^n y^0 + \binom{n}{n-1}x^{n-1}y + \binom{n}{n-2}x^{n-2}y^2 + \ldots + \binom{n}{1}xy^{n-1} + \binom{n}{0}x^0 y^n$$

$$= \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} \qquad (\because n_{c_r} = n_{c_{n-r}})$$

Examples :

1) $(x+y)^1 = x^1 + y^1 = \binom{1}{0}x + \binom{1}{1}y$

2) $(x+y)^2 = x^2 + 2xy + y^2 = \binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2$

3) $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$$= \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

$$= \binom{3}{3}x^3 + \binom{3}{2}x^2y + \binom{3}{1}xy^2 + \binom{3}{0}y^3$$

4) $(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}x^0y^4$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

Example 6.6.1 *What is the coefficient of $x^{13} y^7$ in the expansion of $(x+y)^{20}$?*

Solution : we know that the coefficient of $x^{n-r} y^r$ in the expansion of $(x+y)^n$ is

$$^n c_r = \binom{n}{r}$$

Here n = 20, r = 7, n − r = 13

∴ The required coefficient is

$$\binom{20}{7} = \frac{20\,!}{7\,! \times 13\,!} = \frac{20\times 19\times 18\times 17\times 16\times 15\times 14\times \cancel{13!}}{7\times 6\times 5\times 4\times 2\times \cancel{13!}} = 77520$$

Example 6.6.2 *What is the coefficient of $x^{12} y^{13}$ in the expansion of $(x-y)^{25}$?*

Solution : By the binomial theorem, the required coefficients

$$-\binom{25}{12} = \frac{-(25\,!)}{12!\ \ 13!} = -\ 5,200300$$

Example 6.6.3 *What is the coefficient of $x^8\, y^{12}$ in the expansion of $(2x-3y)^{20}$? Hence find the coefficient of $x^{12}\, y^8$*

Solution : we have

$$(2x-3y)^{20} = \sum_{r=0}^{20} \binom{20}{r} (2x)^{20-r}\ (-3y)^r$$

Here r = 12

$\therefore$ The coefficient of $x^8\ y^{12}$ is obtained by putting r = 12 in R.H.S. of above formula.

Hence the required coefficient $= \binom{20}{12} (2)^8\ (-3)^{12} = (125970)\ (2)^8\ (-3)^{12}$

Example 6.6.4 *What is the coefficient of x^{99} in the expansion of $(x-1)^{101}$?*

Solution : The required coefficient is $= \binom{101}{99} = 5050$

Example 6.6.5 *What is the coefficient of x^{98} in the expansion of $(x-1)^{101}$?*

Solution : We have

$$(x-1)^{101} = \sum_{r=0}^{101} \binom{101}{r} x^r\ (-1)^{n-r}$$

$\therefore$ Put r = 98 and x = 1

$\therefore$ The required coefficient $= \binom{101}{98} (-1)^3 = -\ 166650$

Example 6.6.6 *What is the coefficient of x^{99} in the expansion of $(2-x)^{19}$?*

Solution : We have

$$(2-x)^{19} = \sum_{r=0}^{19} \binom{19}{r} (2)^r\ (-x)^{19-r}$$

Put r = 10 and x = 1,

$\therefore$ The required coefficient is $= \binom{19}{10}\ 2^{10}\ (-1)^9 = -\ 92378 \times 2^{10}$

Example 6.6.7 *What is the coefficient of $x^{101}\, y^{99}$ in the expansion of $(2x-3y)^{200}$?*

Solution : We have

$$(2x-3y)^{200} = \sum_{r=0}^{200} \binom{200}{r} (2x)^r \ (-3y)^{n-r}$$

Put r = 101 at R.H.S. and x = 1 = y

$\therefore$ The required coefficient of $x^{101}\ y^{99}$ is

$$= \binom{200}{101} 2^{101} \ (-3)^{99}$$

$$= -\binom{200}{101} 2^{101} \ (3)^{99}$$

Example 6.6.8 *What is the coefficient of x^{10} in the expansion of $(x+\frac{1}{x})^{100}$?*

Solution : we have

$$(x+\frac{1}{x})^{100} = \sum_{r=0}^{100} \binom{100}{r} x^r \left(\frac{1}{x}\right)^{100-r}$$

$$= \sum_{r=10}^{100} \binom{100}{r} x^{2r-100}$$

put $2r - 100 = 10 \Rightarrow 2r = 110 \Rightarrow r = 55$

$\therefore$ The coefficient of x^{10} is $\binom{100}{55}$

Example 6.6.9 *What is the coefficient of x^k in the expansion of $(x^2-\frac{1}{x})^{100}$? Hence find the coefficient of x^{51}*

Solution : we have

$$(x^2-\frac{1}{x})^{100} = \sum_{r=0}^{100} \binom{100}{r} (x^2)^r \ (-\frac{1}{x})^{100-r}$$

$$= \sum_{r=0}^{100} \binom{100}{r} x^{2r} \left(-\frac{1}{x}\right)^{100-r}$$

$$= \sum_{r=0}^{100} \binom{100}{r} (-1)^{100-r} \ (x)^{3r-100}$$

Put $3r - 100 = k \Rightarrow r = \frac{1}{3}(k+100)$

$$\therefore \quad (x^2 - \frac{1}{x})^{100} = \sum_{k=-100}^{200} \binom{100}{\frac{1}{3}(k+100} (-1)^{100 - \frac{1}{3}(k+100)} (x)^k$$

Thus the coefficient of x^k in $(x^2 - \frac{1}{x})^{100}$ is

$$= \binom{100}{\frac{1}{3}(k+100} (-1)^{100 - \frac{1}{3}(k+100)}$$

Example 6.6.10 *What is the coefficient of x^{20} and x^{21} in $(x^2 - \frac{1}{x})^{100}$?*

Solution : i) coefficient of x^{20} :

Put k = 20 in example (13)

$\therefore$ The required coefficient is $= \binom{100}{40} (-1)^{100-40} = \binom{100}{40}$

ii) coefficient of x^{21} :

put k = 21 in example (13), we get

$$k = 3r - 100 \Rightarrow 3r = k - 100$$

$$3r = 21 - 100 = -79$$

$$r = -\frac{79}{3} \text{ which is not possible because } r \in N \text{ (natural number)}$$

$\therefore$ The required coefficient is zero.

Example 6.6.11 *Let n be a non negative integer, then S.T.*

$\sum_{r=0}^{n} \binom{n}{r} = 2^n$ *by i) Binomial theorem ii) Combinatorial method*

Solution : i) By Binomial theorem

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n} x^o y^n$$

Put x = y = 1, we get

$$(1+1)^n = 2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$$

$$2^n = \sum_{r=0}^{n} \binom{n}{r}$$

ii) Let A be a set with n elements. Then the power set of A has 2^n elements. Now find all subsets by different method.

There are subsets of A with zero elements 1, 2, 3…n elements {ϕ} is a subset of A with no element.

So there are $\binom{n}{o}$ subsets with no element (zero element) $\binom{n}{1}$ subsets with one element,

$\binom{n}{2}$ subsets with 2 elements… and $\binom{n}{n}$ subset

With n elements $\therefore$ By addition principle, the total number of subsets are given as follows :

$$2^n = \binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n}$$

$$2^n = \sum_{r=0}^{n}\binom{n}{r}$$

Example 6.6.12 *Let n be a positive integer then prove that* $\sum_{r=0}^{n}(-1)^r\binom{n}{r} = 0$

Solution : By Binomial theorem (put x = 1, y = – 1)

$$(0)^n = (1+(-1))^n = \sum_{r=0}^{n}\binom{n}{r}(-1)^r\ 1^{n-r} = \sum_{r=0}^{n}\binom{n}{r}(-1)^r$$

Example 6.6.13 *Prove that if n and r integers with* $1 \le r \le n$ *then* $r\binom{n}{r} = n\binom{n-1}{r-1}$ *combinatorially.*

Solution : A good mathematician can prove this identity by connecting it to some real life example of set. Suppose we have a set A with n elements, count the number of ways to select a subset with r elements from A and then element of this set by two different methods.

i) The number of ways to select subsets with r elements from set A with n element is $\binom{n}{r}$. The number of ways to select an element of this set is r.

Thus the total number of elements = $r\binom{n}{r}$

ii) The number of ways to select one element from set A is n.

The number of ways to select r – 1 elements from n – 1 elements is $\binom{n-1}{r-1}$

$\therefore$ The total no. of ways = $n\binom{n-1}{r-1}$

Hence $r\binom{n}{r} = n\binom{n-1}{r-1}$

Example 6.6.14 *Let n be a non negative integer then* $\sum_{r=0}^{n} 2^r \binom{n}{r} = 3^n$

Solution : By Binomial theorem, putting x = 1, y = 2, we get

$$(1+2)^n = \sum_{r=0}^{n} \binom{n}{r} 1^{n-r}\, 2^r = \sum_{r=0}^{n} 2^r \binom{n}{r}$$

Hence $\sum_{r=0}^{n} 2^r \binom{n}{r} = 3^n$

Example 6.6.15 *Let n and r be positive integers with* $r \le n$ *then prove that*

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

This is called as Pascal's Identity.

Solution : Let A be a set with n+1 elements. The number of ways to select subsets with r elements from A is $\binom{n+1}{r}$

Let $x \in A$ and $B = A - \{x\}$

There are two possibilities to select subset with r elements from set B with n elements.

i) The subset of A with r elements contain x with r – 1 elements of B

So there are $\binom{n}{r-1}$ subsets of A that contain x.

ii) The subset of A with r elements without x i.e. select r elements from the set $B = A - \{x\}$ with n elements.

This can be done in $\binom{n}{r}$ ways.

Thus $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$

Example 6.6.16 *Let m, n and r be non negative integers with* $0 \le r \le m$ *and* $0 \le r \le n$ *Then*

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

This is known as Vandermonde's Identity.

Solution : Let A and B two disjoint sets with $|A| = m$ and $|B| = n$

$\therefore \quad |A \cup B| = |A| + |B| = m+n$

The total number of ways to select r elements from $A \cup B$ is $\binom{m+n}{r}$

Alternately, to select r elements from $A \cup B$ is to select k elements from set B and r – k elements from set B is $\binom{n}{k}$. The number of ways to select r–k elements from A is $\binom{m}{r-k}$

Therefore by product rule, this can be done in $\binom{m}{r-k}\binom{n}{k}$ ways.

Hence the total number of ways to select r elements from $A \cup B$ is also equals to

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Example 6.6.17 *Let n be a non negative integer then P.T.* $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$

Solution : Put m = n = r in Vandermorde's identity, we get

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2 \qquad \left(\because \binom{n}{n-k} = \binom{n}{k}\right)$$

6.7 Case Studies

I) Sudoku : Sudoku puzzle is a number - placement puzzle. The objective is to fill 9×9 with digits so that each column, row and each of the nine 3×3 subgrids. Contain all the digits from 0 to 9. The puzzle developer provides a partially filled grid. Remaining grids have to fill by puzzle solver.

Here two types of algorithms are important, viz Sudoku Generator algorithm and Sudoku Solver algorithm.

1) Sudoku Generator Algorithm :

Step 1 : By using Sudoku rules and Sudoku solver algorithm, generate a full grid of numbers.

Step 2 : Now remove 1 value at a time from the grid.

Step 3 : After removing a value, apply Sudoku solver algorithm to check the grid can be solved and also count the number of solutions.

Step 4 : If resulting grid gives unique solution, goto step 2. Otherwise put the removed value again in the grid and remove other value.

Step 5 : Repeat the steps 2 to 4 several times to make the grid difficult to solve. Difficulty level of resulting grid is depends on step 2 to 4.

2) Sudoku Solver Algorithm :

Use the backtracking algorithm to find the solution -

1) The backtracking algorithm is a recursive algorithm.

2) It starts with adding a number in the grid. If the solution is not found, the algorithm backtracks and try another number.

3) Repeat step 2, until grid is full with all the rules of Sudoku puzzle.

Example :

5		4		7				
6			1	9	5			
	9	8					6	
8		9		6				3
			8		3			1
7				2				6
	6				7			
			4			6	3	5
					6		7	

Solution :

5	**3**	4	**6**	7	**8**	**9**	**1**	**2**
6	**7**	**2**	1	9	5	**3**	**4**	**8**
1	9	8	**3**	**4**	**2**	**5**	6	**7**
8	**5**	9	**7**	6	**1**	**4**	**2**	3
4	**2**	**6**	8	**5**	3	**7**	**9**	1
7	**1**	**3**	**9**	2	**4**	**8**	**5**	6
9	6	**1**	**5**	**3**	7	**2**	**8**	**4**
2	**8**	**7**	4	**1**	**9**	6	3	5
3	**4**	**5**	**2**	**8**	6	**1**	7	**9**

II) Handshake Puzzle/Problem :

Various handshaking problems are in circulation. The most common two are as follows

1) "In a room of n people, how many different hand shakes are possible."

The first person may shake hands with (n – 1) other people. The next person will shake hand (n – 2) other people, not counting the first person again. Counting like this gives total no. of handshakes are,

$$(n-1)+(n-2)+\ldots+2+1=\frac{n\cdot(n-1)}{2}$$

The no. of handshakes = ${}^{n}C_{2}$

$$= \frac{n!}{(n-2)!\ 2!}$$

$$= \frac{n(n-1)\cdot(n-2)!}{(n-2)!\cdot 2!} = \frac{n\cdot(n-1)}{2}$$

We will see one more common example as follows :

2) There are n married couples attending a party. Each person shakes hands with every person other than his spouse. Find the total number of handshakes.

$\Rightarrow$ There are n couples and each couple contains 2 persons.

$\therefore$ Total no. of persons = 2n

If each person is making handshake with other person.

Total no. of handshakes = ${}^{2n}C_2$

But each married couple is not making handshake among themselves.

$\therefore$ Total no. of handshakes = ${}^{2n}C_2 - n$

$$= \frac{2n!}{(2n-2)!2!} - n$$

$$= \frac{2n(2n-1)(2n-2)!}{(2n-2)!2!} - n$$

$$= n(2n-1) - n$$

$$= 2n^2 - n - n$$

$$= 2n^2 - 2n$$

$$= 2n(n-1)$$

Example :

1) In a birthday party, every person shakehand with every other person. If there was 36 handshakes in the party, how many persons were present in the party.

Solution : Total No. of handshakes $= \frac{n(n-1)}{2}$

$$\therefore \quad 36 = \frac{n(n-1)}{2}$$

$$\therefore \quad n(n-1) = 72$$

$$\therefore \quad \boxed{n = 9}$$

Total no. of persons in the party = 9.

6.8 University Questions with Answers

Dec. 2016

Q.1 *A bag contains 6 red and 8 green balls.*
i) If one ball is drawn at random, then what is the probability of the ball being green ?
ii) If two balls are drawn at random, then what is the probability that one is red and the other is green ? **[6]**

Ans. :

Total number of balls in the bag are 6 + 8 = 14.

i) The number of ways to draw one ball from the bag is $^{14}C_1 = 14$

The number of ways to draw one green ball from the bag is $^{8}C_1 = 8$

Hence, the required probability $= \frac{8}{14} = \frac{4}{7} = 0.57$

ii) The number of ways to draw two balls from the bag is $^{14}C_2 = \frac{14!}{2! \times 12!} = \frac{14 \times 13}{2}$

$= 91$

The number of ways to draw one red ball is $^{6}C_1 = 6$

The number of ways to draw one green ball is $^{8}C_1 = 8$

Hence, the required probability $= \frac{8 \times 6}{91} = \frac{48}{91} = 0.5274$

Q.2 *In how many ways can 6 men and 5 women be seated in a line so that no two women sit together ? In how many ways can 6 men and 5 women sit in a line so that women occupy the even places.* **(Refer example 6.3.19)** **[6]**

May 2017

Q.3 *How many words of three distinct letters can be formed from the letters of the word MAST ?* **[3]**

Ans. : Given word is "MAST"

$\therefore$ It involves 4 distinct letters.

To form words of three distinct letters,

The first letter can be selected in 4 ways.

The second letter can be selected in 3 ways

The third letter can be selected in 2 ways

$\therefore$ The total number of words are $4 \times 3 \times 2 = 24$

Q.4 *How many different seven-person committees can be formed each containing 3 women from an available set of 20 women and 4 men from an available set of 30 men.* **[3]**

Ans. : We have seven - person committee.

Each committee contains three women from 20 women and 4 men from 30 men.

$\therefore$ The required numbers of ways are

$$= {}^{20}C_3 \cdot {}^{30}C_4 = \frac{20!}{3!\,17!} \cdot \frac{30!}{4!\,26!}$$

$$= \frac{20\times19\times18}{3\times2} \cdot \frac{30\times29\times28\times27}{4\times3\times2\times1} = 104{,}13{,}900 \text{ ways}$$

Q.5 *How many distinguishable words that can be formed from the letters of MISSISSIPPI ?* **(Refer example 6.3.9)**. **[3]**

Q.6 *Compute the number of distinct five-card hands that can be dealt from a deck of 52 cards.* **[3]**

Ans. : The number of distinct five card hands that can be dealt from a deck of 52 cards $= \binom{52}{5} = \frac{52!}{5!\,47!} = 2598960$ ways.

Dec. 2017

Q.7 *Explain the rule of sum and products with exmples.* **(Refer sections 6.2.1 and 6.2.2)** **[4]**

Q.8 *Find out how many 5-digit number greater than 30,000 can be formed from the digits 1, 2, 3, 4, 5.* **[4]**

Ans. : There is restriction that the 5 digit numbers so formed must be greater than 3000. Therefore the ten thousandth place can be filled with numbers 3, 4 or 5 only. Hence the ten thousandth place can be selected in 3 different ways. Now out of remaining digits, fill remaining places.

$\therefore$ The Thousandth's place can be filled by 5 ways.

The Hundredth place can be filled by 5 ways.

The Tenth place can be filled by 5 ways.

The unit place can be filled by 5 ways.

$\therefore$ The total number of 5 digit numbers greater than 30,000 can be formed in

$3\times5\times5\times5\times5 = 3\times5^4$ ways.

Q.9 *Find the number of permutaions which can be made with the letters of the word ENGINEERING.* **[4]**

Ans. : There are 11 letters in the word "ENGINEERING" out of which E, N, G, I, R are distinct.

E appears 3 times

N appears 3 times

I appears 2 times

G appears 2 times

R appears 1 times

$\therefore$ The required number of permutations $= \dfrac{11!}{3!\,3!\,2!\,2!\,1!} = 277200.$

May 2018

Q.10 *Write an algorithm for generating permutation of {1, 2, ...n}. Apply it for n = 3 case.* **(Refer section 6.5 and example 6.5.3)** **[4]**

Q.11 *Solve the following :*

(i) How many different car number plates are possible with 2 letters followed by 3 digits ?

(ii) How many of these number plates begin with 'MH' ? **[4]**

Ans. : i) The first position is a letter and can be selected from 26 letters in ${}^{26}C_1 = 26$ ways. The second position is a letter and can be selected from 26 letters in ${}^{26}C_1 =$ 26 ways. The first digit can be selected in 10 ways. The second digit can be selected in 10 ways. The third digit can be selected in 10 ways.

Therefore the number of ways to select 3 digits is $10\times10\times10-1 = 999$

($\because$ 000 is not allowed).

Hence the required number of ways $= 26\times26\times999 = 675324$

ii) The total number of plates begin with 'MH' is $1\times1\times999 = 999$

Q.12 *In how many ways can a cricket team of eleven players be chosen out of a batch of 14 players. How many of them will :*

i) Include a particular player ii) Exclude a particular player **[4]**

Ans. : The number of ways to select 11 cricket team players from 14 is

$${}^{14}C_{11} = \frac{14\times13\times12\times11\,!}{11\,!\times3\,!} = 364.$$

i) When a particular player is included every time, then the remaining 11 – 1 = 10 players can be selected from remaining 14 – 1 = 13 players. This can be done in ${}^{13}C_{10} = \dfrac{13\times12\times11\times10!}{3!\times10!}$ = 286 ways

ii) When a particular player is excluded, then select 11 players from 14 – 1 = 13 players. This can be done in ${}^{13}C_{11} = \dfrac{13\times12\times11\,!}{2\,!\times11\,!}$ = 78 ways.

Dec. 2018

Q.13 *From a group of 7 men and 6 women, five persons are to be selected to form a committee so that at least 3 men are there on the committee. In how many ways can it be done ?* **[3]**

Ans. : Given that, there are at least 3 men on committee of 5 persons. It means committee will contains 3 men or 4 men or 5 men.

i) 3 men can be selected in ${}^{7}C_{3} = \dfrac{7\times6\times5}{3!}$ = 35 ways

2 women can be selected in ${}^{6}C_{2} = \dfrac{6\times5\times4!}{2\times4!}$ = 15 ways

$\therefore$ The number of ways this can de done is 33 × 15 = 525

ii) The number of ways 4 men and 1 woman can be selected is

${}^{7}C_{4}\times{}^{6}C_{1}$ = 35 × 6 = 210 days

iii) The number of ways 5 men and no woman can be selected is

${}^{7}C_{5}\times{}^{6}C_{0}$ = 21 × 1 = 21

Hence the total number of ways a committee can be formed with at least 3 men is

525 + 210 + 21 = 756 ways

Q.14 *How many 4-letter words with or without meaning, can be formed out of the letters of the word 'LOGARITHMS', if repetition of letters is not allowed ?* **[3]**

Ans. : There are 10 letters, and all letters are distinct.

$\therefore$ The number if 4 letters words from 10 distinct letters

$$= {}^{10}P_{4} = \frac{10\times9\times8\times7\times6!}{4\,!\times6!} = 210 \text{ ways}$$

Q.15 *If a committee has eight members :*

i) How many ways can the committee members be seated in a row ?

ii) How many ways can the committee select a president, vice-president and secretary ? **[6]**

Ans. : There are eight members in a committee

i) There are eight positions in a row

1	2	3	4	5	6	7	8

For the first position, there are 8 ways.

For the second position, there are 7 ways.

Similarly, the total number of ways the committee members can be seated in a row is 8!

$$= 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$$

ii) The number of ways the committee can select a president is ${}^{8}C_{1} = 8$ ways

The number of ways the committee can sets a vice president and secretary respectively is 7 and 6 ways.

Hence, the required number of ways = $8 \times 7 \times 6 = 336$.

May 2019

Q.16 *2 Mathematics papers and 5 other papers are to be arranged at an examination, find the total number of ways if, i) Mathematics papers are consecutive.*
(Refer example 6.3.6) **[3]**

Q.17 *In the expansion of $(1+x)^6$.What is the coefficient of x^3.* **[3]**

Ans. : The coefficient of x^3 in the expansion of $(1+x^6)$ is found as follows,

We have,

$$(1+x)^6 = \sum_{r=0}^{6} \binom{6}{r} (1)^r (x)^{6-r} = \sum_{r=0}^{6} \binom{6}{r} (x)^{6-r}$$

Put, $6-r = 3 \Rightarrow r = 3$

$\therefore$ The coefficient of x^3 is,

$$\binom{6}{3} = \frac{6}{(6-3)!\cdot 3!} = \frac{6\times5\times4\times3!}{3!\times3!} = \frac{6\times5\times4}{3\times2} = 20$$

Q.18 *If the letters of the word 'REGULATIONS' be arranged at random. What is the chance that there will be exactly 4 letters between R and E ?* **[3]**

Ans. :

Total number of letters = 11

R is at first place. So, E will be at 6^{th} place.

R is at second place. So, E will be at 7^{th} place.

And, so on.

Finally, R is at 6^{th} place and E will at 11^{th} place.

So total favorable cases = 6

If R and E are interchanged

Total favorable cases = 6×2

Remaining 9 letters can be arranged by 9! Ways

Total number of ways that R and E have 4 letters between them.

$$= 6 \times 2 \times 9!$$

$$= 4354560$$

Q.19 *Use Binomial theorem to expand $(x^4 + 2)^3$.* **[3]**

Ans. : Binomial theorem states that,

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

$$\therefore \sum_{r=0}^{3} \frac{3!}{(3-r)! \cdot r!} (x^4)^{3-r} (2)^r$$

Expand the summation,

$$= \frac{3!}{(3-0)! \cdot 0!} (x^4)^{3-0} \cdot 2^0 + \frac{3!}{(3-1)! \cdot 1!} (x^4)^{3-1} \cdot 2^1$$

$$+ \frac{3!}{(3-2)! \cdot 2!} (x^4)^{3-2} \cdot 2^2 + \frac{3!}{(3-3)! \cdot 3!} (x^4)^{3-3} \cdot 2^3$$

$$= 1 \cdot (x^4)^3 + 3 \cdot (x^4)^2 \cdot 2 + 3 \cdot (x^4) \cdot 2^2 + 1 \cdot (x^4)^0 \cdot 2^3$$

$$= x^{12} + 6x^8 + 12x^4 + 8$$

Dec. 2019

Q.20 *The company has 10 members on its board of directors. In how many ways can they elect a president, a vice president, secretary and a treasurer ?* **[3]**

Ans. :

$$^{10}P_4 = \frac{10!}{(10-4)!} = \frac{10!}{6!} = \frac{10\times9\times8\times7\times6!}{6!} = 10 \times 9 \times 8 \times 7$$

$$= 5040$$

Number of ways 10 members can elect president, vice precident a secretary and a treasures = 5040 ways.

Q.21 *Find 8^{th} term in the expansion of $(x+y)^{13}$.* **[3]**

Ans. : Binomial theorem states that,

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

In this question

x = x, y = y and n = 13

The 8^{th} term occures when r = 8 – 1 = 7

(since sum starting with r = 0)

$\therefore 8^{th}$ term is

$$\binom{13}{7} x^{13-7}\, y^7 = 1716\, x^6\, y^7$$

Q.22 *A box contains 6 white and 5 black balls. Find number of ways 4 balls can be drawn from the box, if : (i) Two must be white (ii) All of them must have same colour.* **[3]**

Ans. : Total no. of balls = 6 + 5 = 11

i) 4 balls needs to draw, out of that 2 must be white.

i.e. 2 white and 2 black

The numbers of ways to draw 2 white balls from the box is,

$$^6C_2 = \frac{6!}{2! \times 4!} = \frac{6\times5}{2} = 15 \text{ ways}$$

The number of ways to draw 2 black balls from the box is,

$$^5C_2 = \frac{5!}{2! \times 3!} = \frac{5\times4}{2} = 10$$

$\therefore$ The no. of ways to draw 4 balls from the box = 15 × 10 = 150

ii) All of them must have same colour.

i.e. Either 4 balls are white or black.

The number of ways to draw 4 white balls from the box

$$^6C_4 = \frac{6!}{4! \times 2!} = \frac{6 \times 5}{2} = 15 \text{ ways}$$

The number of ways to draw 4 black balls from box

$$^5C_4 = \frac{5!}{4!} = 5 \text{ ways}$$

$\therefore$ Total no. of ways to draw 4 balls from the box with all of them must have same colour = 15 + 5 = 20 ways

Q.23 *Expand $(3x-4)^4$ using binomial theorem.* **[3]**

Ans. : Binomial theorem states that,

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

$$\therefore \sum_{r=0}^{4} \frac{4!}{(4-r)!r!} (3x)^{4-r} (-4)^r$$

Expand the summation,

$$= \frac{4!}{(4-0)!0!}(3x)^{4-0} \cdot (-4)^0 + \frac{4!}{(4-1)!1!}(3x)^{4-1} \cdot (-4)$$

$$+ \frac{4!}{(4-2)!2!}(3x)^{4-2} \cdot (-4)^2 + \frac{4!}{(4-3)!3!}(3x)^{4-3} \cdot (-4)^3 + \frac{4!}{(4-4)!4!}(3x)^{4-4}(-4)^4$$

$$= 1 \cdot (3x)^4 (-4)^0 + 4 \cdot (3x)^3 \cdot (-4) + 6 \cdot (3x)^2 \cdot (-4)^2 + 4(3x)(-4)^3 + 3 \cdot (3x)^0 \cdot (-4)^4$$

$$= 81x^4 - 432x^3 + 864x^2 - 768x + 256$$

❑❑❑

UNIT - IV

7 Graph Theory

Syllabus

***Graph Terminology** and Special Types of Graphs, Representing Graphs and Graph Isomorphism, Connectivity, Euler and Hamilton Paths, the handshaking lemma, Single source shortest path-Dijkstra's Algorithm, Planar Graphs, Graph Colouring. Case Studies : Three Utility Problem, Web Graph, Google Map.*

Contents

7.1 Introduction

The development of graph theory is very much similar to the development of probability theory. The original work of graph theory was motivated by constant efforts to understand or solve real life problems. The graph theory is a very important area of applied mathematics. The origin of graph theory started with an important and practical problem of the Konigberg bridge in 1735.

A great mathematician Euler became the father of the graph theory, when in 1736 he solved a famous unsolved problem of his days called the konigberg bridge problem. This is today called as the first problem of graph theory. He created the first graph to replicate this problem.

Mathematicians, Mobius, Gaustar Kirchhoff, De-Morgan William Hamilton, Alfred Kempe, P. K. Tait, Narsingdeo and many more contributed several result of graph theory.

The graph theory because of it's applications to many disciplines has emerged as on of the useful branch of mathematics during the cast decade. There are several reasons for the acceleration of interest in the graph theory because its applications so some areas of computer engineering, communication science, electrical and civil engineering, cryptography operations research, physics, chemistry, genetics and many more.

7.2 Graphs

A graph is an ordered pair (V(G), E(G)) where

i) V(G) is non empty finite set of elements known as vertices or nodes. V(G) is called the vertex set.

ii) E(G) is a family of unordered pairs (not necessarily distinct) of elements of v, known as edges or arc or branches of G. E(G) is known as edge set.

Graphs are so named because they can be represented diagrammatically in the plane.

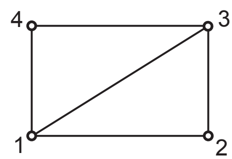
It is denoted by V(G, E).

a) Each vertex of G is represented by a point or small circle in the plane. In practical examples vertex set may be the set of states or cities or objects etc.

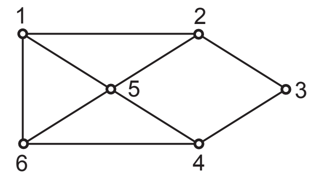
b) Every edge is represented by a continuous curve or straight line segment. Edges may be the route among states or cities or relation among objects etc. Diagrams of road maps, electrical circuits, chemical compounds, job scheduling family trees, all have two objects common namely vertices and edges.

Let us consider the following examples of graphs with V(G) and E(G).

1)



2)



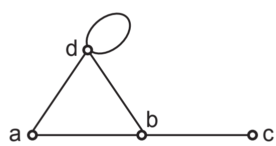
$V(G_1) = \{1, 2, 3, 4\}$

$E(G_1) = \{(1, 2), (1, 3)\ (1, 4)\ (2, 3), (3, 4)\}$

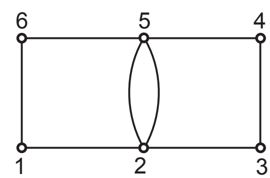
$V(G_2) = \{1, 2, 3, 4, 5, 6\}$

$E(G_2) = \{(1, 2), (1, 5), (1, 6), (2,5),$
$(2, 3), (3, 4), (4, 5), (4, 6), (5, 6)\}$

3)



4)



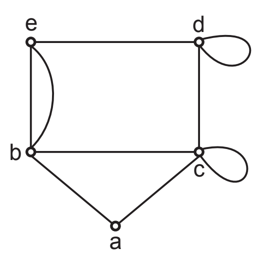
$V(G_3) = \{a, b, c, d\}$

$E(G_3) = \{(a, b), (a, d)\ (b, c)$
$(b, d), (d, d)\}$

$V(G_2) = \{1, 2, 3, 4, 5, 6\}$

$E(G_4) = \{(1, 2), (1, 6), (2, 3),(2,5), (2, 5),$
$(3, 4), (4, 5), (5, 6)\}$

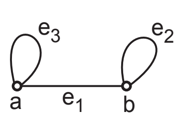
5)



$V(G_5) = \{a, b, c, d\}$

$E(G_5) = \{(a, b), (a, c), (b, e)\ (b, e), (b, c),$
$(c, c)\ (c, d)\ (d, d), (d, e)\}$

6)



$V(G_6) = \{a, b\}$

$E(G_6) = \{a, b)\ (a, a)\ (b, b) = \{e_1, e_2, e_3\}$

i) If x and y are two vertices of a graph G and unordered pair $\{x, y\} = (x, y) = e$ is an edge then we say that edge e joins x and y or e is incident to both vertices x and y.

In this case, vertices x and y are said to be incident one e.g. In example (1), $e = (2, 3)$ $\therefore$ e is incident at 2 and 3 and vertices 2, 3 are one incident on $e = (2, 3)$.

ii) Two vertices x and y are said to be adjacent to each other if the pair (x, y) is an edge of G.

If e = (x, y) is an edge of G then x and y are said to be end vertices of e and we can say that e is incident at x and y.

iii) Two edges e_1 and e_2 are said to be adjacent if they have a common vertex i.e. If e_1 and e_2 are adjacent then $e_1 = \{x, y\}$ and $e_2 = \{y, z\}$.

iv) An edge joining a vertex to itself is called a loop. E.g. In example (5) there are 2 loops (c, c) and (d, d).

v) A pair of vertices of a graph is joined by two or more edges, such edges are called as multiple or parallel edges.

In example (4) (2, 5) (2, 5) are multiple edges.

7.3 Basic Definitions

SPPU : Dec.-17

1) **Multigraph :** A graph in which a pair of vertices is joined by two or more edges is called a multigraph or multiple graph.

i.e. A graph having multiple edges is called a multigraph. In examples (1), (2), (3), (6), graphs are not multigraphs and graphs in examples (4), (5) are multigraphs.

2) **Pseudograph :** A graph having loops but no multiple edges is called a Pseudograph.

Graphs in examples (3), (5) and (6) are pseudographs. A graph having only loops is called a Haary graph. For example :

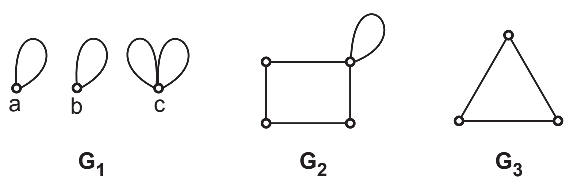
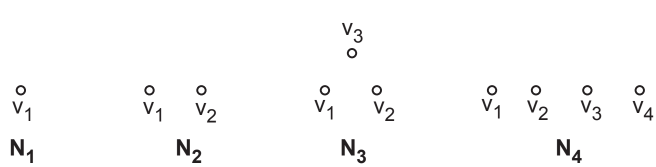


Fig. 7.3.1

Graph G is a pseudograph is well as Haary graph. Graph G_2 is Pseudograph but not Haary graph. Graph G_3 is neither Pseudo not Haary graphs.

3) **Simple graph :** A graph without loops and multiple edges is called a simple graph. Graphs in examples (1) and (2) are simple graphs. Graphs in examples (3), (4), (5), (6) are not simple graphs.

4) **Null graph :** A graph G (V, E) is said to be null graph if E is a empty set. Null graph on n vertices is denoted by N_n.



5) **Finite graph :** A graph G (V, E) in which V (x) and E (x) are finite sets is called a finite graph. Otherwise infinite graph.

6) **Directed graph :** A graph G (V, E) is said to be directed graph if the elements of E are an ordered pairs of vertices. E.g.

E = {(a, b), (b, c), (a, c)}

Here (a, c) $\neq$ (c, a), (c, a) $\notin$ E (G).

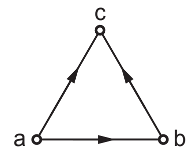


Fig. 7.3.2 Directed graph

A graph which is not directed is called Non-directed graph or graph.

7) **Weighted graph :** A graph G (V, E) in which some weight is assigned to every edge of G, is called weighted graph.

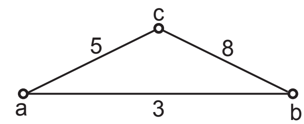


Fig. 7.3.3 Weighted graph

8) **Degree of a vertex :**

a) In a directed graph G the number of edges ending at vertex v is called the indegree of v. It is denoted by deg G^+ (v) or d^+ (v)

b) Outdegree : In a directed graph G, the number of edges beginning at vertex v is called the outdegree of v. It is denoted by deg G^- (v) or d^- (v).

c) The number of edges incident at a vertex v of a graph G with self loops counted twice is called the degree of the vertex v. It is denoted by d(v). A vertex of degree one is called pendent vertex. A vertex of degree zero is called isolated vertex. An edge incident at pendent vertex is called pendent edge.

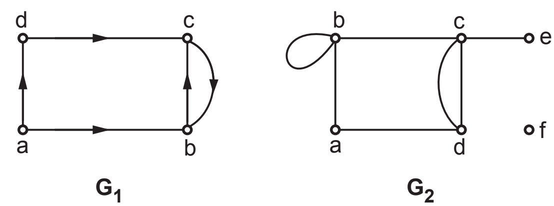


Fig. 7.3.4

e.g.

In graph G_1,

Vertices	Indegree	Outdegree
a	0	2
b	2	1
c	2	1
d	1	1

In graph G_2,

$$d(a) = 2 \quad , \quad d(b) = 2 + 2 = 4 \quad , \quad d(c) = 4 \quad , \quad d(e) = 1$$

$$d(d) = 3 \quad , \quad d(f) = 0$$

$\therefore$ f is an isolated vertex. e is a pendent vertex. An edge {c, e} is a pendent edge.

9) Order and size of graph : The number of vertices in a finite graph G is called the order of G. The number of edges in a finite graph G is called size of the graph. A graph of order n and size m is called (n, m) graph.

If G is a (p, q) graph then G has p vertices and q edges.

10) Degree sequence of a graph :

Let G be a graph with vertex set $V = \{v_1, v_2, v_3, \ldots v_n\}$ and $d_i = \deg(v_i)$ then the sequence $(d_1, d_2, d_3, \ldots d_n)$ in any order is called the degree sequence of G.

Note : 1) Vertices of G are ordered so that degree sequence is monotonically increasing.

2) Two graphs with same degree sequence are called to be degree equivalent. e.g.

$d(v_1) = 4$, $d(v_2) = 3$, $d(v_3) = 2$, $d(v_5) = 5$,

$d(v_6) = 3$), $d(v_4) = 1$

$\therefore$ Its degree seq. is (4, 3, 2, 1, 5, 3)

By relabelling vertices we may write degree sequence as (1, 2, 3, 3, 4, 5).

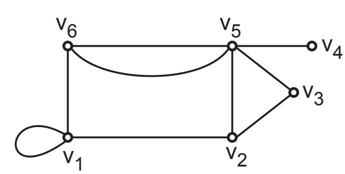


Fig. 7.3.5

Theorems :

Theorem 1 : Handshaking lemma : Let G (V, E) be any graph then $\sum_{v \in V} d(v) = 2q$ where q denotes the number of edges of G.

Proof : Let us argue by induction on q. Suppose q = 0 i.e. G has no edge i.e. E is an empty set. So d(v) = 0, $\forall$ v $\in$ V. $\therefore \sum d(v) = 2q = 0$.

Let G be a graph with q > 0 edges. Choose any edge e = {u, v} of G. Consider the graph G_1 obtained from G as follows :

i) The vertex set of G_1 is same as the vertex set of G i.e. $V(G_1) = V(G) = V = p$

ii) The edges of G_1 are all edges of G except e.

In other words, G_1 is obtained from G by deleting the edge e. $\therefore$ x is a (p, q – 1) graph.

$\therefore$ By induction principle, result is true for q–1 edges

i.e. $$\sum_{x \in V} d(x) = 2(q-1) \qquad \dots (7.3.1)$$

The degree of a vertex x other than u or v in G_1 is same as that at G. And the degree of u in G_1 is one less than the degree of u in G.

i.e. $$d_{G_1}(u) = d_G(u) - 1$$

Similarly $$d_{G_1}(v) = d_G(v) - 1$$

Hence equation (7.3.1) becomes

$$\sum_{\substack{x \in V \\ x \neq u, v}} d(x) + d_{G_1}(u) + d_{G_1}(v) = 2(q-1)$$

$$\sum_{\substack{x \in V \\ x \neq u, v}} d(x) + d_{G_1}(u) - 1 + d_G(v) - 1 = 2(q-2)$$

$\therefore$ $$\sum_{x \in V} d(x) = 2q \quad \text{Hence the proof.}$$

The result is so named because it implies that if several people shake hands, the total number of hands shaken must be even as two hands are involved in one handshake.

Note : If $\sum_{v \in V} d(v)$ = Odd number then there does not exist any graph with this degree sequence.

7.4 Matrix Representation of a Graph

SPPU : Dec.-09, 10, May-10, 18

We saw that graphs can be represented either set theoretically or diagrammatically. Graphs can also be represented by matrices. It is very much useful to store graphs in a computer.

7.4.1 Adjacency Matrix

Let G be a graph with n vertices and no parallel edges. The adjacency matrix of x is denoted by

$$A(G) = [a_{ij}]_{n \times n} \text{ and defined as}$$

$$a_{ij} = 1 \text{ if } v_i \text{ and } v_j \text{ are adjacent}$$

$$= 0 \text{ if } v_i \text{ and } v_j \text{ are not adjacent.}$$

Note : i) A (G) is asymmetric - binary matrix.

ii) The principal diagonal entries are all zeros if G has no loops.

iii) The i^{th} row sum = i^{th} column sum = $d(v_i)$

e.g. 1) The adjacent matrices of the following graphs are

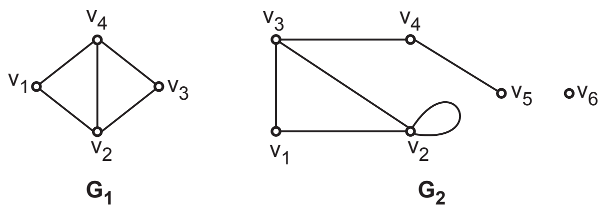


Fig. 7.4.1

$$A(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}_{4 \times 4} \end{array}, \quad A(G_2) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6 \times 6} \end{array}$$

The adjacency matrix for a multigraph G is a $n \times n$ Matrix $A(G) = \left[a_{ij}\right]_{n \times n}$ where

$$a_{ij} = \text{Number of edges joining } v_i \text{ and } v_j$$

The adjacency matrix of the following graph is

$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 & 1 \end{bmatrix} \end{array}$$

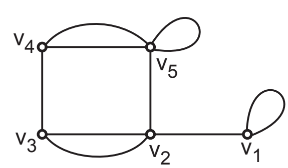


Fig. 7.4.2

7.4.2 Incidence Matrix

Let G be a graph with n vertices and m edges without self loops. The incidence matrix is denoted by X (G) or I (G) and defined as

$$X(G) = [x_{ij}]_{n \times m} \text{ where}$$

$$x_{ij} = 1 \text{ if } j^{th} \text{ edge is incident on } i^{th} \text{ vertex } v_i.$$

$$= 0 \text{ otherwise.}$$

X (G) is a $n \times m$ matrix whose n rows correspond to the n vertices and m columns correspond to m edges. The graph and it's incidence matrix are given below

$$X(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5 \times 7} \end{array}$$

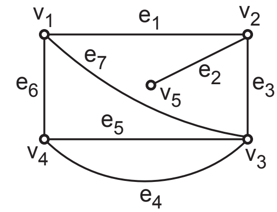


Fig. 7.4.3

Properties of Incidence Matrix

1) It contains only 0 and 1.
2) Each column in the incidence matrix has exactly two 1's appearing in that column.
3) The sum of elements in a row is equal to the degree of the corresponding vertex.
4) Two identical columns correspond to the parallel edges in graph.
5) A row with all 0's represents an isolated vertex.
6) A row with single 1 represents a pendent vertex.

The incidence matrix of a graph with loop is given as follows :

$$X(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{3 \times 3} \end{array}$$

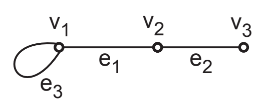


Fig. 7.4.4

7.4.3 Adjacency Matrix of a Diagraph (Directed Graph)

Let G be a directed graph with n vertices and without parallel edges. The adjacency matrix is denoted by

Where, $A(D) = [a_{ij}]_{n \times n}$

$a_{ij} = 1$ if there is an edge directed from v_i to v_j

$= 0$ otherwise

In network flow, adjacency matrix is also known as connection matrix or transition matrix. The adjancency matrix and diagraph are given below.

$$A(D) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6 \times 6} \end{array}$$

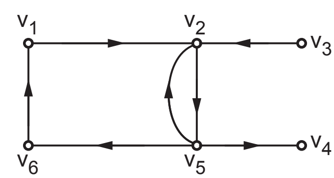


Fig. 7.4.5

7.4.4 Incidence Matrix of Diagraph

The incidence matrix of a diagraph with n vertices m edges and no self loops is a matrix.

$$X(G) = [x_{ij}]_{n \times m} \text{ where}$$

$$x_{ij} = 1 \text{ if } j^{th} \text{ edge } e_j \text{ is incident out of } i^{th} \text{ vertex } v_i.$$

$$= -1 \text{ if } j^{th} \text{ edge } e_j \text{ is incident into of vertex } v_i.$$

$$= 0 \text{ otherwise.}$$

A graph and its incidence matrix are given below :

$$X(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}_{6 \times 6} \end{array}$$

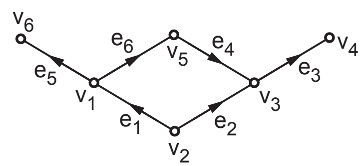


Fig. 7.4.6

Note : i) Sum of elements in each column of incidence matrix is zero.

Examples :

Example 7.4.1 *Show that in a graph the number of vertices of odd degree is even.*

Solution : Let G be a (p – q) graph.

By handshaking lemma $\sum_{v_i \in V} d(v_i) = 2q$

Now separate out vertices of even degree and odd degree

$$\therefore \quad \sum_{v_i \in V} d(v_i) = \sum_{\substack{x \in V \\ \text{even degree}}} d(x) + \sum_{\substack{y \in V \\ \text{odd degree}}} d(y) = 2q$$

$$\therefore \quad \sum_{\substack{v_i \in V \\ \text{odd degree}}} d(v_i) = 2q - \sum_{\substack{x \in V \\ \text{even degree}}} d(x) = \text{Even number}$$

$\therefore$ The sum of vertices of odd degree is even.

Hence the number of vertices of odd degree is even.

Example 7.4.2 *Show that the maximum number of edges in a simple graph with n vertices is* $\dfrac{n(n-1)}{2}$

Solution : Let G be a graph with n vertices m edges

$\therefore$ By handshaking lemma

$$\sum_{v \in V} d(v) = 2m \rightarrow$$

Let $x \in V$ $\therefore x$ must be adjacent to remaining $(x - 1)$ vertices $\therefore d(x) = n - 1$. $\forall x \in V$

$\therefore$ Equation (1) $\Rightarrow (n - 1) + (n - 1) + \ldots$ n times = 2 m

$$n(n - 1) = 2m$$

$$m = \frac{n(n-1)}{2}$$

Hence the maximum number of edges in any simple graph with n vertices is $\frac{n(n-1)}{2}$

Example 7.4.3 *How many nodes are necessary to construct a graph with exactly 8 edges in which each node is of degree 2.*

Solution : Let G be the required graph with n vertices. Each vertex is of degree 2.

By handshaking lemma

$$\sum_{i=1}^{n} d(v_i) = 2(8) = 16$$

$2 + 2 + 2 \ldots$ n times = 16

$$2n = 16$$

$$n = 8$$

Hence 8 vertices are required.

Example 7.4.4 *Determine the number of edges in a graph with 6 nodes, 2 of degree 4 and 4 of degree 2. Draw to such graphs.* **SPPU : Dec.-09**

Solution : Let G be the required graph with 6 nodes and m edges.

$\therefore$ By handshaking lemma

$$\sum_{v \in G} d(v) = 2m$$

$$d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) + l(v_6) = 2m$$

$$4 + 4 + 2 + 2 + 2 + 2 = 2m$$

$$2m = 16$$

$$m = 8$$

Hence 8 edges are required.

Two such graphs are given below.

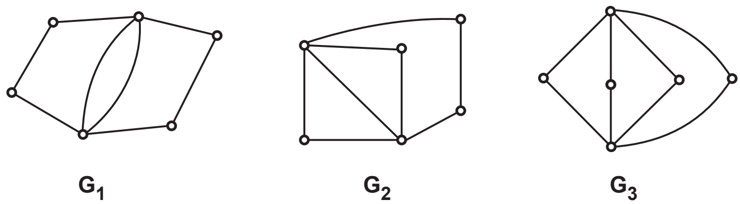


Fig. 7.4.7

Example 7.4.5 *Is it possible to construct a graph with 12 nodes such that 2 of the nodes have degree 3 and the remaining have degree 4.* **SPPU : Dec.-10**

Solution : Let G be the required graph with 12 vertices.

By handshaking lemma.

$$\sum_{v \in V(G)} d(v) = 2m$$

$$(3 + 3) + (4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4) = 2m$$

$$6 + 40 = 2m$$

$$\Rightarrow \quad m = 23$$

$\therefore$ It is possible to construct such graph.

Example 7.4.6 *Is graph exist for the degree sequence 4, 4, 3, 3, 2, 2, 1.*

Solution : Now apply handshaking lemma

$$\sum d(v) = 2m = \text{Even}$$

$$4 + 4 + 3 + 3 + 2 + 2 + 1 = \text{Even}$$

$$19 = \text{Even which is impossible}$$

$\therefore$ Such graph does not exist.

Example 7.4.7 *Let A = {1, 2, 3, 4, 5} and relation R is defined as R = {(1, 1) (2, 2), (3, 3) (4, 4) (5, 5)}. Draw graph G (A, R).*

Solution :

Fig. 7.4.8

This graph is known as Haary graph.

Example 7.4.8 *Find the adjacency matrices of the following graphs.*

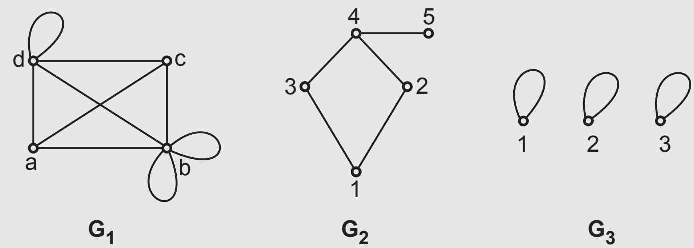


Fig. 7.4.9

Solution : The adjacency matrices for the graphs G_1, G_2, G_3 are given below :

$$A(G_1) = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}_{4\times 4} \end{array}$$

$$A(G_2) = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{5\times 5}$$

$$A(G_3) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \end{array}$$

Example 7.4.9 *How many simple labelled graphs with n vertices are there ?* **SPPU : May-10**

Solution : We know that a simple graph with n vertices has maximum possible number of edges $\frac{n(n-1)}{2} = m$ (say).

To construct a simple graph with e edges and n vertices, can be done in $\binom{m}{e}$ ways.

i.e. ${}^{m}C_{e}$ ways where $m = \frac{n(n-1)}{2}$ and $0 \le e \le m$

Hence the total number of ways to construct such graphs is given by

$$\binom{m}{0}+\binom{m}{1}+\binom{m}{2}+\ldots+\binom{m}{m}$$

$$= 2^m$$

$$= 2^{\left(\frac{n(n-1)}{2}\right)} \text{ ways.}$$

Example 7.4.10 *A man supposed to bring a tiger, a sheep a bag of cabbage across a river on a row boat. Boat is very small and he can carry one of these items on the boat at a time. Furthermore he can not leave the tiger with sheep nor the sheep with cabbage. Construct a graph to determine a possible way for the man to transport all items across the river.*

Solution : Let M = man, T = tiger, S = sheep, C = cabbage

Step 1 :

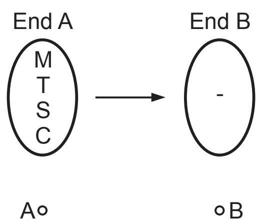


Fig. 7.4.10

Step 2 :

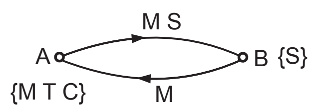


Fig. 7.4.10 (a)

Step 3 :

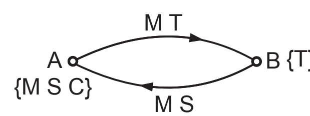


Fig. 7.4.10 (b)

Step 4 :

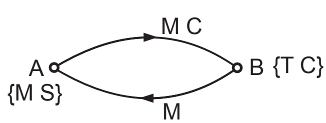


Fig. 7.4.10 (c)

Step 5 :

Fig. 7.4.10 (d)

Example 7.4.11 *Draw a non simple graph with degree sequence (1, 1, 3, 3, 3,, 4, 6, 7)*

Solution : G is non directed, therefore G permits self loops in it. The required graph is as follows :

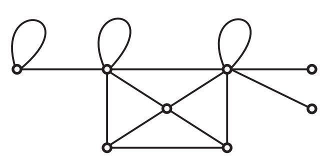


Fig. 7.4.11

Example 7.4.12 *Show that a simple graph of order 4 and size 7 does not exist.*

Solution : Let G be a simple graph with 4 vertices.

Then G has at most $\frac{n(n-1)}{2} = \frac{4 \times 3}{2} = 6$ edges.

But given that G has 7 edges which is contradiction.

$\therefore$ there can not be a simple graph with 4 vertices and 7 edges.

Example 7.4.13 *Show that every cubic graph has even number of vertices ?*

Solution : Let G be a cubic graph with n vertices.

$$\therefore \quad \sum_{v \in x} d(v) = 3P = \text{Even [by handshaking lemma]}$$

$\therefore$ 3P must be even but 3 is not even therefore P must be even.

Hence the proof.

7.5 Some Important Definitions

SPPU : Dec.-16, May-18

I) Regular Graph : A graph G is said to be r-regular graph if every vertex of G has degree r.

i) Regular graph of degree zero is called null graph.

ii) A regular graph of degree 3 is called cubic graph.

e.g.

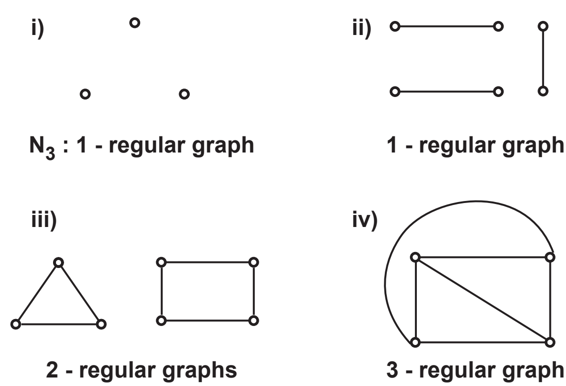


Fig. 7.5.1

II) Complete Graph : A simple graph G in which every pair of distinct vertices are adjacent is called a complete graph. If G is a complete graph on n vertices then it is denoted by K_n.

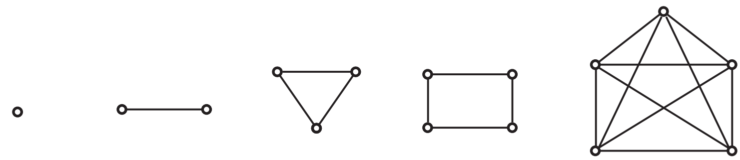
In a complete graph, there is an edge between every pair of distinct vertices.

In graph K_n, every vertex is adjacent to remaining n–1 vertices so degree of each vertex is n – 1.

Thus K_n is a (n – 1) - regular graph.

K_n has exactly $\frac{n(n-1)}{2}$ edges.

Consider the following examples :



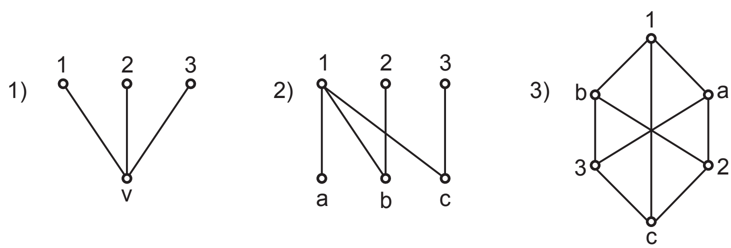
K_1 K_2 K_3 K_4 K_5

Fig. 7.5.2

III) Bipartite Graph : A graph G (v, E) is said to be bipartite graph if its vertex set can be partitioned into two disjoint subsets say v_1 and v_2 such that $v_1 \cup v_2 = v$ and $v_1 \cap v_2 = \phi$ and every edge of G joins a vertex of v_1 to a vertex of v_2.

In Bipartite graph, vertices of v_1 should not be adjacent. It is free from loops.

Following graphs are bipartite graphs



$v_1 = \{v\}$
$v_2 = \{1, 2, 3\}$

$v_1 = \{a, b, c\}$
$v_2 = \{1, 2, 3\}$

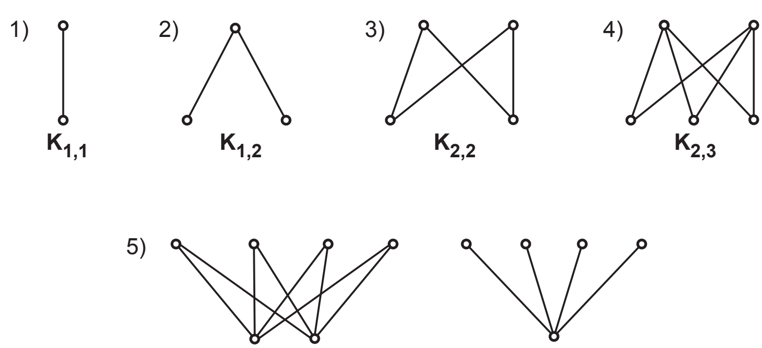
$v_1 = \{1, 2, 3\}$
$v_2 = \{a, b, c\}$

Fig. 7.5.3

IV) Complete Bipartite graph : A bipartite graph G (v, E), $v_1 \cup v_2 = v$ and $v_1 \cap v_2 = \phi$ is said to be complete Bipartite graph if each vertex of V_i is joined to every vertex of v_2 by a unique edge.

If $|v_1| = m$, $|v_2| = n$, then the complete bipartite graph $G(v_1 \cup v_2, E)$ is denoted by $K_{m, n}$

Examples :



$K_{1,1}$ $K_{1,2}$ $K_{2,2}$ $K_{2,3}$

$K_{4,2}$ $K_{4,1}$

Fig. 7.5.4

The graph $K_{1, n}$ is called as star graph.

7.6 Linked Representation of Graphs

In many practical problems which involve large graph (w.r.t. vertices or edges) are solved by using computer. The link representation also is called the adjacency structure which is used for storing the graphs in computer.

In this method the graph is represented by a linear array. After assigning the vertex, in any order say 1, 2, 3, ... n, we represent each vertex k by a linear array, whose first element is k and remaining elements are the vertices that are immediate successor of K. i.e. the vertices which have a directed path of length one from K.

For example consider the following graphs :

The vertices and their adjacent vertices are as follows :

1 : 2, 3

2 : 2

3 : 2, 4

4 : 1, 5

5 : 6

6 : 4

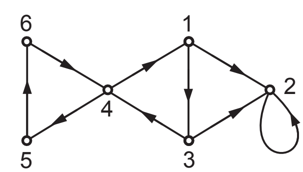


Fig. 7.6.1

This adjacency can be represented as

$$G = [1 : 2, 3 ; 2 : 2 ; 3 : 2, 4 ; 4 : 1, 5 ; 5 : 6 ; 6 : 4]$$

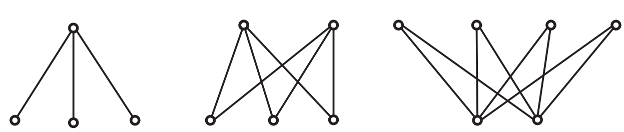
This is called the link presentation of the graph.

The link representation of a graph G stores the graph in the memory of computer by using its adjacency lists.

Examples :

Example 7.6.1 *Give two examples of the complete bipartite graph which are not regular.*

Solution :



$K_{1, 3}$ $K_{2, 3}$ $K_{4, 2}$

Fig. 7.6.2

$K_{1, 3}$ $K_{2, 3}$ $K_{4, 2}$

$K_{1, 3}$, $K_{2, 3}$, $K_{4, 2}$ all are complete bipartite graphs but not regular.

Example 7.6.2 *Is there exist any complete bipartite graph with 7 vertices and 14 edges ?*

Solution : First find all possible bipartitions of 7. They are 6 + 1, 5 + 2, 4 + 3.

We know that, if $G(v_1 \cup v_2, E)$ is a bipartite graph then the number edges in G is equal to $|v_1| \cdot |v_2|$

i.e. $|E| = |v_1| \cdot |v_2|$

Here $|E| = 14$ But 6.1 = 6, 5.2 = 10, 4.3 = 12

Therefore the complete bipartite graphs with 7 vertices has 6 or 10 or 12 edges only.

Therefore any complete bipartite graph with 7 vertices and 14 edges.

Example 7.6.3 *How many edges are there in i) K_{10} ii) $K_{5,3}$ iii) $K_{5,7}$*

Solution : i) The complete graph on 10 vertices K_{10} has $\frac{10 \times 9}{2} = 45$ edges

ii) $K_{5,3}$ has $5 \cdot 3 = 15$ edges

iii) $K_{5,7}$ graph has $5 \cdot 7 = 35$ edges

7.7 Isomorphism of Graphs

SPPU : Dec.-12, 18, May-18

In real life we come across so many similar objects or figures with respect to size, shape or orientation. Similarly there are a few concepts in graph theory which deal with the similarity of graphs w.r.t. number of vertices or number of edges, number of regions and so on. Among all such similarities the most important one is an isomorphism of graphs.

Definition : Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs. G_1 and G_2 are said to be isomorphic graphs if

i) There exists a bijective function $\phi : V_1 \to V_2$

ii) There exists a bijective function $\psi : E_1 \to E_2$ such that e = (x, y) is an edge in G_1 iff $(\phi(x), \phi(y))$ is an edge in G_2.

The pair of functions ϕ and ψ is called an isomorphism of G_1 and G_2. It is denoted by $G_1 \cong G_2$.

Suppose two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are isomorphic graphs. Then it is clear that

i) $|V_1| = |V_2|$ i.e. G_1 and G_2 must have same number of vertices.

ii) $|E_1| = |E_2|$ i.e. G_1 and G_2 must have same number of edges.

iii) G_1 and G_2 must have an equal number of vertices with the same degree.

iv) G_1 and G_2 must have an equal number of loops.

v) G_1 and G_2 must have same number of pendent.

vi) G_1 and G_2 must have same number of pendent edges.

vii) If u and v are adjacent in G_1 then the corresponding vertices in G_2 are also adjacent.

In general it is easier to prove two graphs are not isomorphic by proving that any one of the above property fails.

Consider the following example

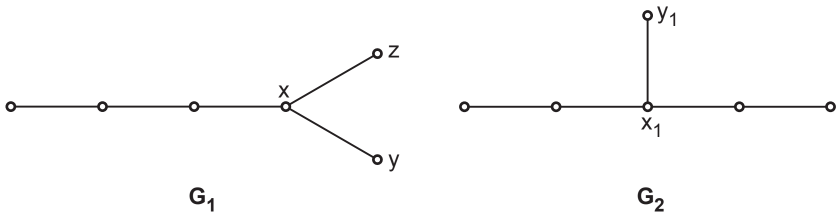


Fig. 7.7.1

These graphs have i) The same number of vertices.

ii) The same number of edges.

iii) An equal number of vertices of degree k.

These conditions are necessary for two graphs to be isomorphic but not sufficient.

In graph G_1, d (x) = 3, d (y) = 1, d (z) = 1 and x and y are adjacent to vertex x.

In graph G_2, d (x_1) = 3, d (y_1) = 1

there is only one pendent vertex adjacent to x_1. Hence adjacency is not preserved. Therefore G_1 is not isomorphic to G_2 i.e. $G_1 \not\cong G_2$.

Note : Isomorphism of graphs is an equivalence relation.

Examples :

Example 7.7.1 *Draw all isomorphic graphs on vertices 2 and 3.*

Solution : i) For 3 vertices.

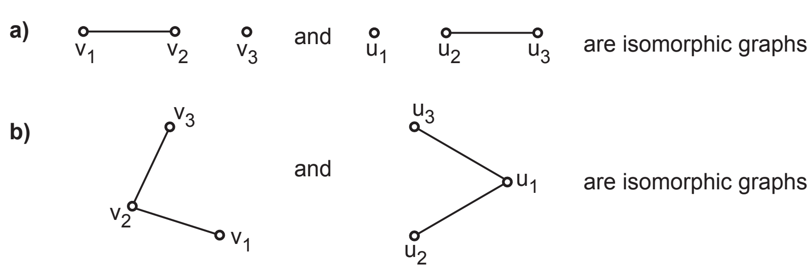


Fig. 7.7.2

ii) For two vertices.

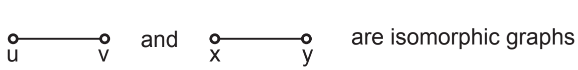


Fig. 7.7.2 (a)

Example 7.7.2 *Draw all non isomorphic graphs on 2 and 3 and 4 vertices.*

Solution : All non-isomorphic graphs on 2 vertices are

Fig. 7.7.3

All non isomorphic graphs on 3 vertices.

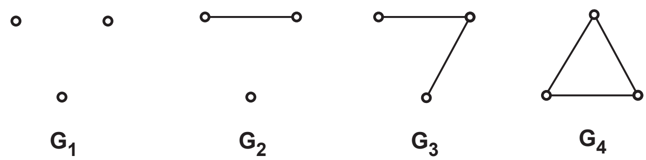


Fig. 7.7.3 (a)

All non isomorphic graphs on 4 vertices.

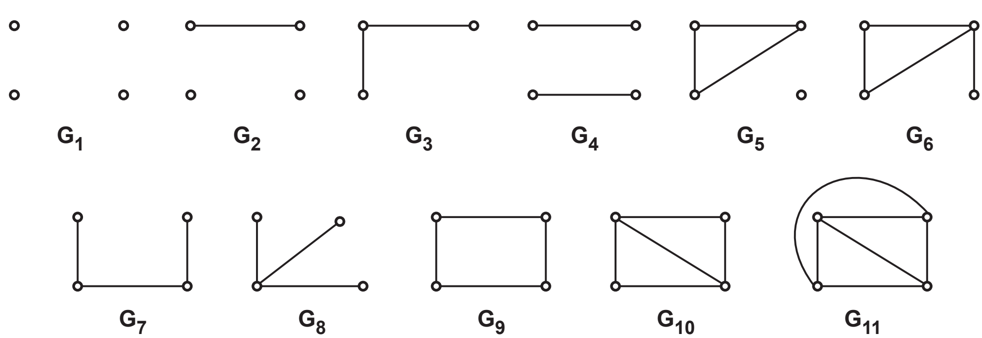


Fig. 7.7.3 (b)

Example 7.7.3 *Draw all non isomorphic graphs on 5 vertices and 5 edges.*

Solution : The following are non isomorphic graphs with 5 vertices and 5 edges.

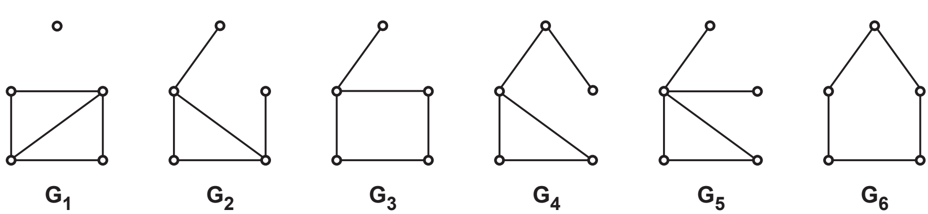


Fig. 7.7.4

Example 7.7.4 *Find whether the following pairs of graphs are isomorphic or not.*

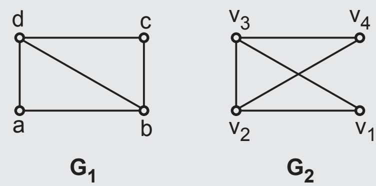


Fig. 7.7.5

Solution : i) Both the graphs have 4 vertices and 5 edges.

Both have 2 vertices of degree 3 and 2 vertices of degree 2.

$\therefore \phi : \{a, b, c, d\} \to \{v_1, v_2, v_3, v_4\}$ such that

$a \to v_1$

$c \to v_4$

$b \to v_2$

$d \to v_3$

ϕ preserves adjacency and non-adjacency of vertices.

$\psi \to E_1 \to E_2$ Is bijective.

$\therefore$ G_1 is isomorphic to Graph G_2.

ii)

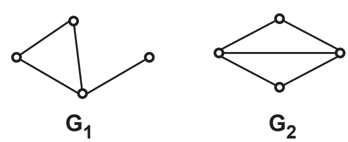


Fig. 7.7.5 (a)

As G_1 has 4 edges and G_2 has 5 edges, G_1 and G_2 are not isomorphic graphs.

iii) G_1 And G_2 are not isomorphic graphs because in G_1 vertices v_1 and v_3 of 4 degree are non adjacent while in G_2, the vertices x and y of degree 4 are adjacent.

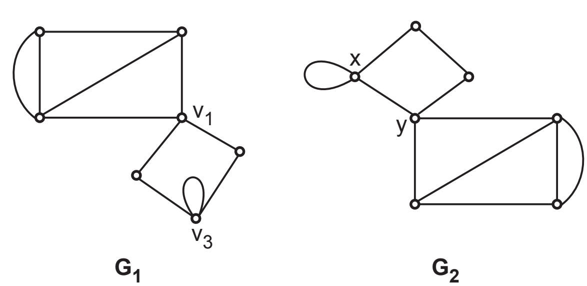


Fig. 7.7.5 (b)

Example 7.7.5 *Identify whether the given graphs are isomorphic or not* **SPPU : Dec.-12**

Solution :

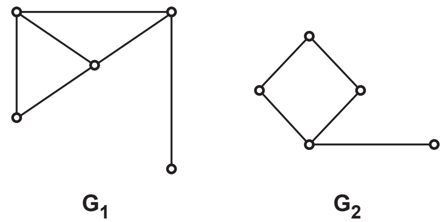


Fig. 7.7.6

In graph G_1, there are 2 vertices of degree 3. But in G_2, there is only one vertex of degree 3. So G_1 and G_2 are not isomorphic graphs.

ii)

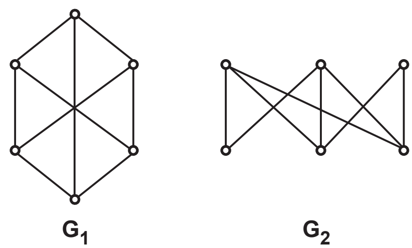


Fig. 7.7.6 (a)

Graph G_1 has 9 edges and G_2 has 8 edges.

Therefore G_1 and G_2 are not isomorphic graphs.

Example 7.7.6 *Determine whether graphs G and H are isomorphic or not. Justify your answer.* **SPPU : Dec.-18, Marks 6**

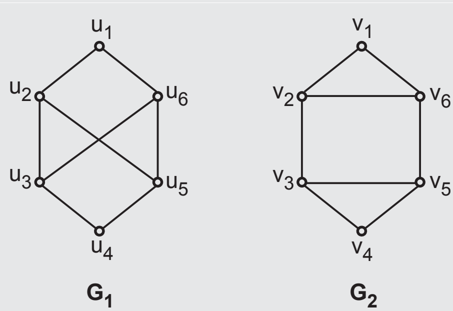


Fig. 7.7.7

Solution : G_1 And G_2 have 6 vertices, 8 edges.

Define $\phi : v\,(G_1) \rightarrow v\,(G_2)$ such that

$$u_1 \rightarrow v_1$$

$$u_2 \rightarrow v_2$$

$u_3 \rightarrow v_3$

$u_4 \rightarrow v_4$

$u_3 \rightarrow v_5$

$u_6 \rightarrow v_6$

ϕ is bijective mapping an edge $e = (u_2, u_5) \in E(G_1)$

but $\psi(e) = (\phi(u_2)), \phi(u_5)) = (v_2, v_5) \notin E(G_2)$

$\therefore G_1$ and G_2 are not isomorphic graphs.

Example 7.7.7 *Show that the following graphs are isomorphic*

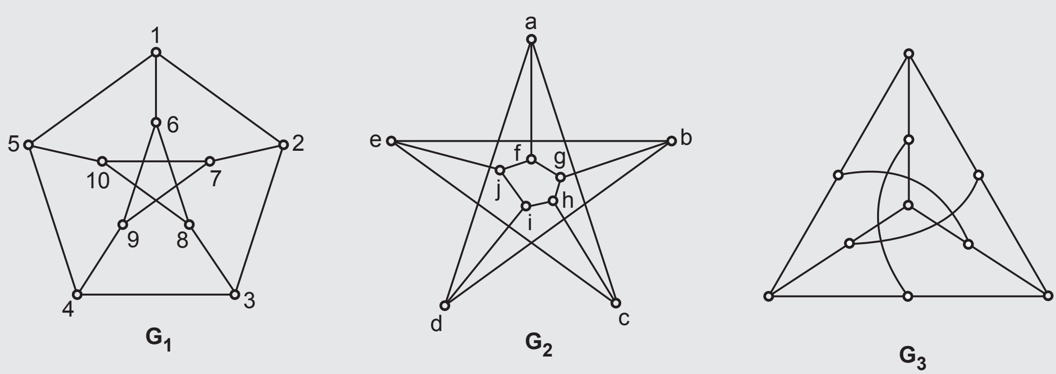


Fig. 7.7.8

Solution : All graphs G_1, G_2 and G_3 have 10 vertices and 15 edges.

All these graphs are 3-regular graphs. Also they preserve adjacency. Hence all graphs are isomorphic. Isomorphism is given by

$1 \rightarrow f$

$2 \rightarrow g$

$3 \rightarrow h$

$4 \rightarrow i$

$5 \rightarrow j$

$6 \rightarrow a$

$7 \rightarrow b$

$8 \rightarrow c$

$9 \rightarrow d$

$10 \rightarrow e$

In the similar way, we can show that G_1 and G_3 are isomorphic graphs.

Example 7.7.8 *Are the graphs isomorphic ? why ?*

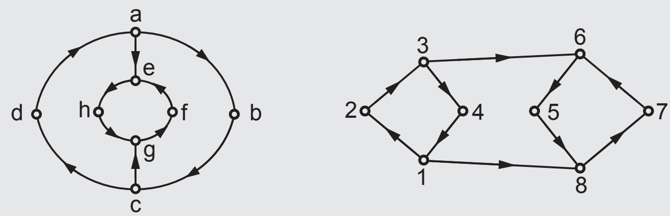


Fig. 7.7.9

Solution : Given graphs G_1 and G_2 have 8 vertices and 10 edges.

Both the graphs have 4 vertices of degree 2 and 4 vertices of degree 3. Also the adjacency is preserved.

$\phi : v(G_1) \rightarrow v(G_2)$ is defined as

$a \rightarrow 1$, $b \rightarrow 2$, $c \rightarrow 3$, $d \rightarrow 4$, $e \rightarrow 8$, $f \rightarrow 5$, $g \rightarrow 6$, $h \rightarrow 7$,

ϕ is bijective.

$\therefore$ G_1 and G_2 are isomorphic graphs.

7.8 New Graphs from Old Graphs

SPPU : May-07, 18, Dec.-09, 12

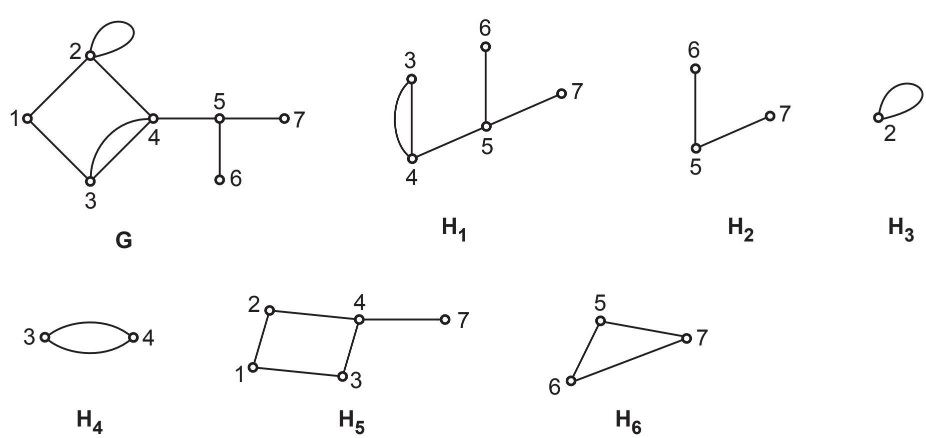
A good mathematical theory must contain sufficient number of models and examples. Moreover it must have methods to generate new objects from old ones.

In this section we derive new graphs from old graphs.

1) **Subgraphs :** Let G (V, E) be any graph. A graph H (V_1, E_1) is said to be subgraph of G if $V_1 \subseteq V$ and $E_1 \subseteq E$.

We also say that G is a supergraph of H.

e.g.

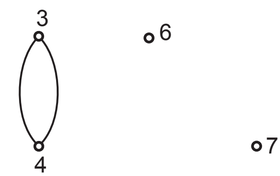


Graphs H_1, H_2, H_3 and H_4 are subgraphs of G. But graphs H_5 and H_6 are not subgraph as $(4, 7) \in E(H_5)$ but $(4, 7) \notin E(G)$ and $(6, 7) \in E(H_6)$ but $(6, 7) \notin E(G)$.

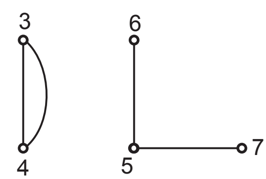
Properties :

1) Each graph is a subgraph of itself.
2) A subgraph of a subgraph of a graph G is a subgraph of G.
3) A graph G - {v} is a subgraph of G which is obtained from G by removing the vertex $v \in G$ and also the edges which are incident at v.
4) If $e \in (G)$ then G - e is a subgraph of G obtained from G by deleting the edge e.

In above example H_1 - {5} is



and H_1 - (4, 5) is given by



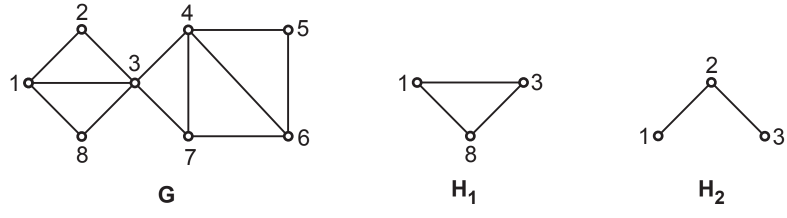
2) **Edge Disjoint Subgraphs :** Two subgraphs H_1 and H_2 of the graph G are said to be edge disjoint subgraphs of a graph G if there is no edge common between H_1 and H_2 i.e. $E(H_1) \cap E(H_2) = \phi$ It may have common vertex.

3) **Vertex Disjoint Subgraphs :** Two subgraphs H_1 and H_2 of the graph G are said to be vertex disjoint subgraphs if there is no vertex common between H_1 and H_2 i.e. $V(H_1) \cap V(H_2) = \phi$.

Note : 1) All vertex disjoint subgraphs are edge disjoint subgraphs.

4) **Spanning subgraph :** Let G (V, E) be any graph. A subgraph H of a graph G is said to be spanning subgraph if V (G) = V (H).

Example : Let G be the following graph :



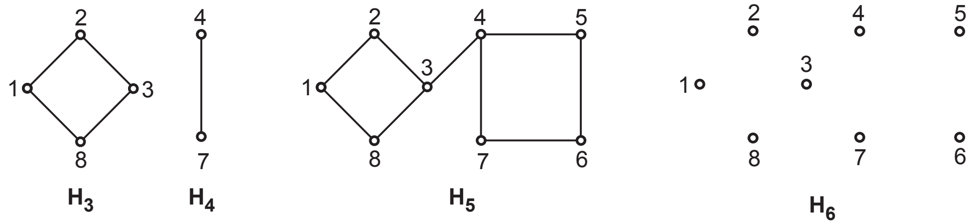


Fig. 7.8.1

Graphs H_1, H_2, ... H_6 are subgraphs of G.

H_1 and H_2 are edge disjoint subgraphs but not vertex disjoint subgraphs.

H_3 and H_4 are vertex disjoint subgraphs as well as edge disjoint subgraphs.

Subgraphs H_5 and H_6 are spanning subgraphs of G as $V(H_5) = V(H_6) = V(G)$.

5) **Factors of a Graph :** Let G be any graph. A k – factor of a graph G is defined to be a spanning subgraph of the graph with the degree of each of its vertex being K. i.e. K-factor is a K-regular graph.

When G has a 1-factor, say G_1, if the number of vertices are even and edges of G are point disjoint.

In particular, K_{2n+1} can not have a 1-factor but K_{2n} can have 1-factor of graph.

Example 1)

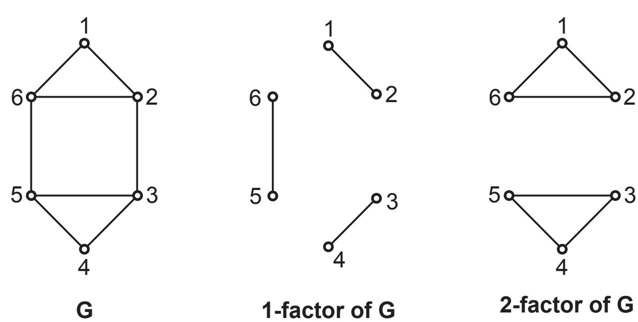


Fig. 7.8.2

Example 2)

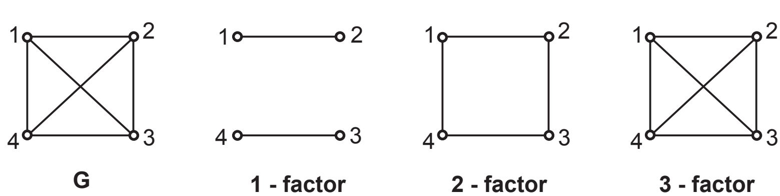


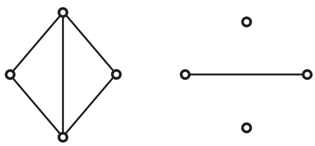
Fig. 7.8.3

6) **Complement of a Graph :** Let G be a simple graph. The complement of G is denoted by $\overline{G}$ is the graph whose vertex set is the same as the vertex set of G and in which two vertices are adjacent if and only if they are not adjacent in G.

A graph is said to be self complementary graph if it is isomorphic to its complement.

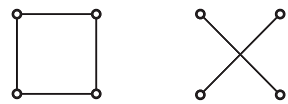
e.g.

i)



G $\quad$ $\overline{G}$

ii)



G $\quad$ $\overline{G}$

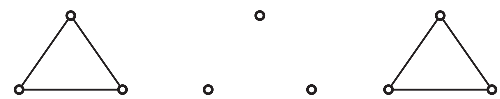
iii)

K_1 $\quad$ $\overline{K_1}$

iv)

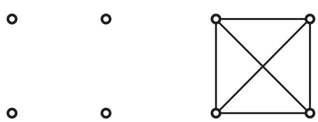
K_2 $\quad$ $\overline{K_2} = N_2$ $\quad$ $\overline{(\overline{K_2})}$

v)



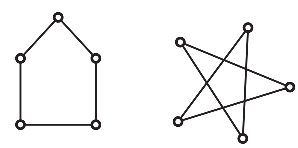
K_3 $\quad$ $\overline{K_3} = N_3$ $\quad$ $\overline{(\overline{K_3})} = \overline{N_3}$

vi)



N_4 $\quad$ $\overline{N_4} = K_4$

vii)



G $\quad$ $\overline{G} \cong G$

G is isomorphic to $\overline{G}$. $\therefore$ G is self complementary graph.

Note :

1) For any graph G, $\overline{(\overline{G})} = G$

2) The complement of the null graph on n vertices is the complete graph K_n on n vertices and vice versa.

3) K_1 is self complementary graph.

Examples :

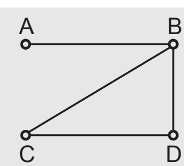
Example 7.8.1 *For the following graphs, determine whether H (V′, E′) is a subgraph of G where*

i) V′ = {A, B, C}, E′ = {(A, B), (A, F)}

ii) V′ = {B, C, D}, E′ = {(B, C), (B, D)}

iii) V′ = {A, B, C, D}, E′ = {(A, C)}

SPPU : May-07, Dec.09



G

Fig. 7.8.4

Solution : i) H is not a subgraph of G because F ∈ V (H)

but F ∉ V (G), so V (H) ⊄ V (G)

ii) Here $V' \subset V(G)$, $E' \subset E(G)$, so $H(V', E')$ is a subgraph of G.

iii) Here $V' \subset v(G)$, but $E' \not\subset E(G)$ Therefore $H(V', E')$ is not a subgraph of G.

Example 7.8.2 *Find the 1-factor, 2-factor graph of the following graphs :*

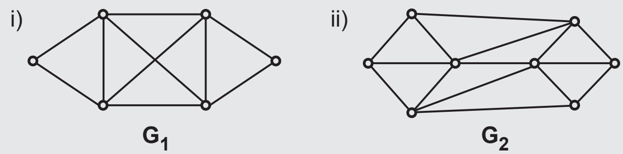


Fig. 7.8.5

Solution : i) A) 1-factor graphs :

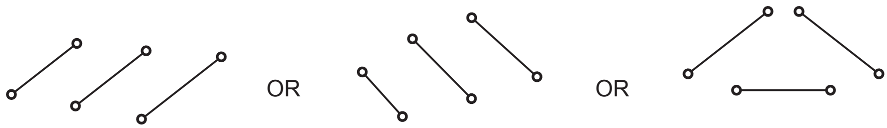


Fig. 7.8.5 (a)

B) 2-factor graphs :

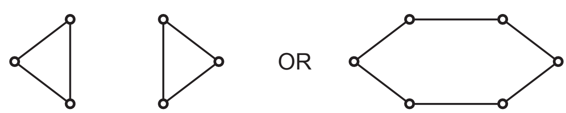


Fig. 7.8.5 (b)

ii) A) 1-factor graphs :

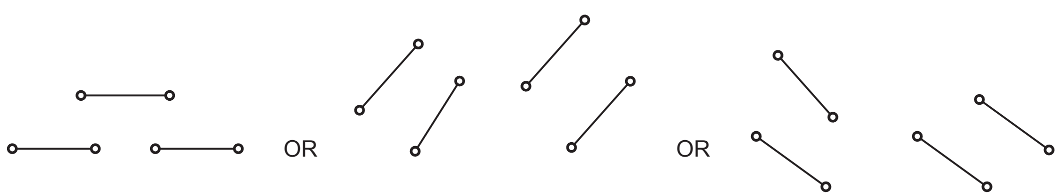


Fig. 7.8.5 (c)

B) 2-factor graphs :

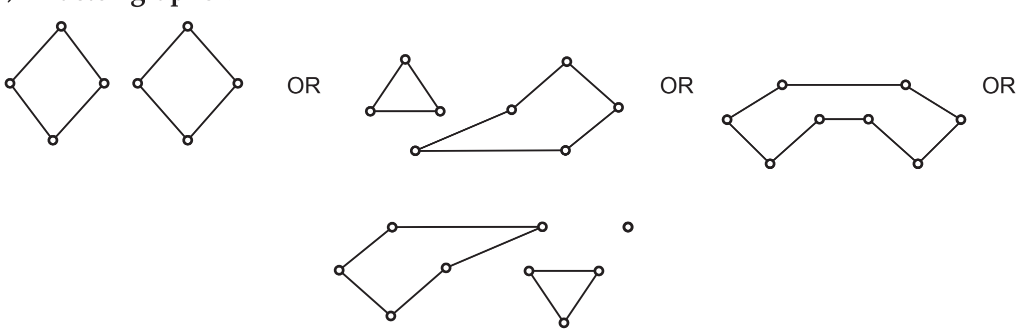


Fig. 7.8.5 (d)

Example 7.8.3 *Find the complement of the following graph. Is it self complementary.*

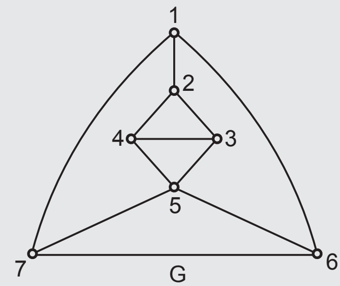


Fig. 7.8.6

Solution : The complement of G is as follows :

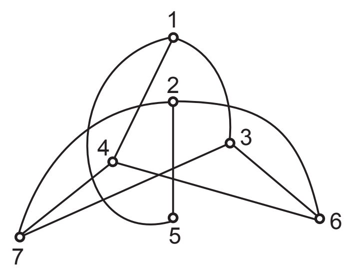


Fig. 7.8.6 (a)

It is not self complementary because G has a vertex of degree 4 but $\overline{G}$ do not have any vertex of degree 4.

Example 7.8.4 *Draw all self complementary graphs on 5 vertices.* **SPPU : Dec.-12**

Solution : The following graphs are self complementary graphs on 5 vertices.

Here $\overline{G_1} = G_2$ and $\overline{G_2} = G_1$

$\therefore G_1$ as well as G_2 are self complementary graphs.

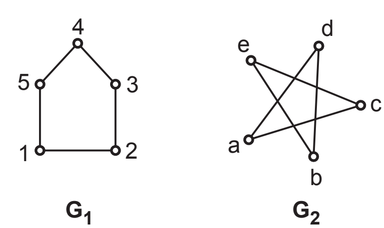


Fig. 7.8.7

Example 7.8.5 *Draw the complement of the following graphs.*

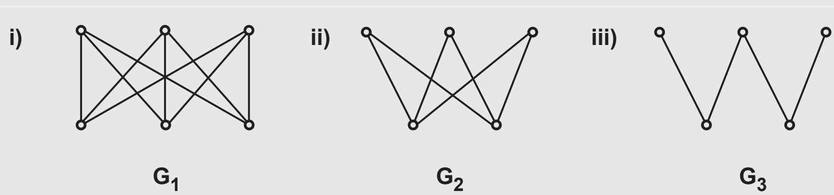
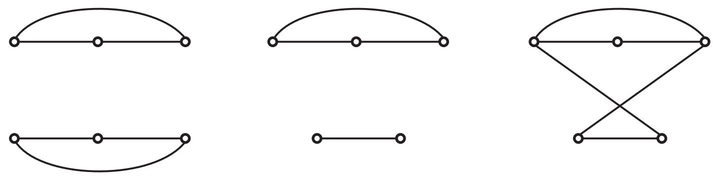


Fig. 7.8.8

Solution : The complements of given graphs are as follows :



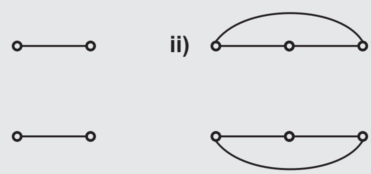
$\overline{G}_1$ $\overline{G}_2$ $\overline{G}_3$

Fig. 7.8.8 (a)

Note : The complement of the complete bipartite graph $K_{m, n}$ gives the complete graphs K_m and K_n and vice versa.

Example 7.8.6 *Find the complement of the following graphs :*

i) ii)

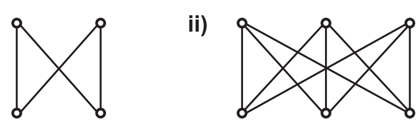


K_2 and K_2 K_3 and K_3

Fig. 7.8.9

Solution :

i) ii)



$K_{2,2}$ $K_{3,3}$

Fig. 7.8.9 (a)

7.9 Operations on Graphs

In this section, we define some standard operations of graphs like intersection, union, Ringsum etc.

A) Intersection of Two Graphs : The intersection of two graphs $G_1 (V_1, E_1)$ and $G_2 (V_2, E_2)$ is a graph G (V, E) whose vertex set is $V = V_1 \cap V_2$ and edge set is $E = E_1 \cap E_2$. The intersection of G_1 and G_2 is denoted by $G_1 \cap G_2$.

e.g.

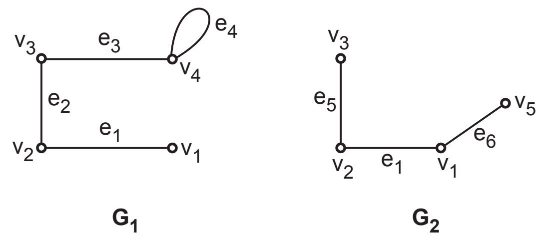
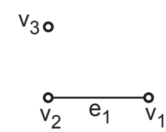


Fig. 7.9.1

$$V_1 = \{v_1, v_2, v_3, v_4\} \quad V_2 = \{v_1, v_2, v_3, v_5\}$$
$$E_1 = \{e_1, e_2, e_3, e_4\} \quad E_2 = \{e_1, e_5, e_6\}$$

Therefore $G = G_1 \cap G_2$ (v, E) where

$$V = V_1 \cap V_2 = \{v_1, v_2, v_3\}, E = E_1 \cap E_2 = \{e_1\}$$



$G_1 \cap G_2$

Fig. 7.9.2

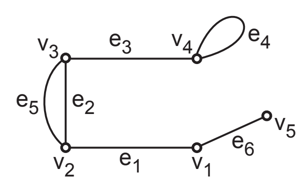
B) Union of Two Graphs : Let G_1 (V_1, E_1), G_2 (V_2, E_2) be two graphs. The union of G_1 and G_2 is denoted by $G_1 \cup G_2 = G$ (v, E) and it is a graph whose vertex set is $V = V_1 \cup V_2$ and Edge set is $E = E_1 \cup E_2$

Consider the graphs G_1 and G_2 as shown in above example :

The union of G_1 and G_2 is given by G (v, E)

where $V = V_1 \cup V_2 = \{v_1, v_2, v_3, v_4, v_5\}$

$V = V_1 \cup V_2 = \{e_1, e_2, e_3, e_4, e_5, e_6\}$



$G_1 \cup G_2$

Fig. 7.9.3

Note : Both graphs G_1 and G_2 are subgraphs of $G_1 \cup G_2$.

C) The Ring Sum of Two Graphs : The ring sum of two graphs G_1 (V_1, E_1) and G_2 (V_2, E_2) is denoted by $G = G_1 \oplus G_2$ (V, E) whose vertex set is $V = V_1 \cup V_2$ and the edge set consists of those edges which are either in E_1 or in E_2 but not in both i.e. $E = (E_1 \cup E_2) - (E_1 \cap E_2)$.

The ring sum of above graphs G_1 and G_2 is given by $G(V, E) = G_1 \oplus G_2$

$$V = \{v_1, v_2, v_3, v_4, v_5\} = V_1 \cup V_2$$
$$E = (E_1 \cup E_2) - (E_1 \cap E_2) = \{e_2, e_3, e_4, e_5, e_6\}$$

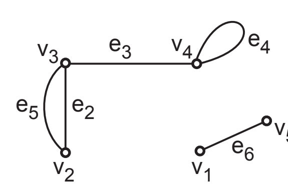


Fig. 7.9.4

D) Sum of Two Graphs : The sum of two vertex disjoint graphs G_1 (V_1, E_1) and G_2 (V_2, E_2) is denoted by $G_1 + G_2 = G$ (V, E) is defined as the graph whose vertex set is V $(G_1 \cup G_2)$ and consisting of edges which are in G_1 or G_2 together with the edges obtained by joining each vertex of G_1 to each vertex of G_2. Thus $G_1 + G_2$ is nothing but the graph $G_1 \cup G_2$ in which each vertex of G_1 is joined to each vertex of G_2 by an edge.

e.g. If

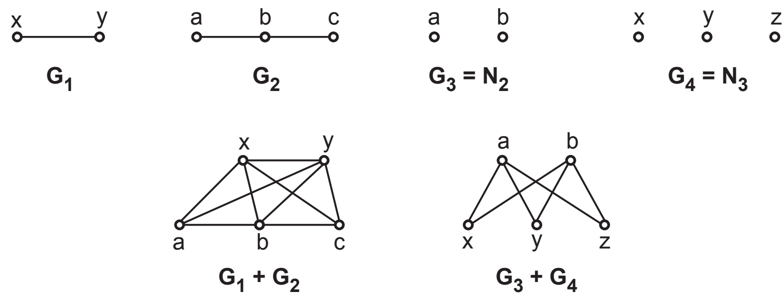


Fig. 7.9.5

Note : The sum $N_m + N_n$ of null graphs is nothing but the complete bipartite graph $K_{m, n}$.

E) Product of Two Graphs : Let $G_1 (V_1, E_1)$ and $G_2 (V_2, E_2)$ be two vertex disjoint graphs then the product of G_1 and G_2 is denoted by $G_1 \times G_2 = G (V, E)$ is a graph whose vertex set is $V = V_1 \times V_2$ and two edges (x_1, x_2) and (y_1, y_2) are adjacent if $x_1 = y_1$ and x_2 is adjacent to y_2 in G_2 or $x_2 = y_2$ and x_1 is adjacent to y_1 in G_1.

e.g. If

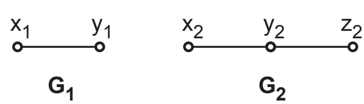


Fig. 7.9.6

Then $G_1 \times G_2$ is given below :

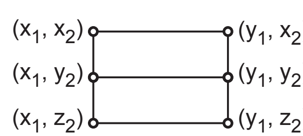
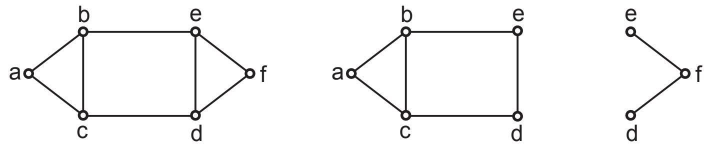


Fig. 7.9.7

F) Decomposition : A graph G is said to have been decomposed into two subgraphs H and K if $H \cup K = G$ and $H \cap K$ = Null graph i.e. each edge of G occurs either in H or in K but not in both. But vertices may occur in both. In this context isolated vertices are not considered.

e.g. The decomposition of G into H and K is given below :

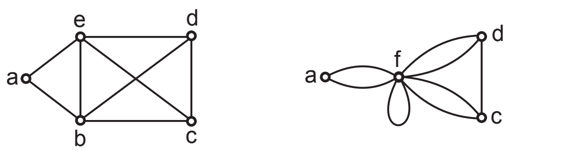


G H K

Fig. 7.9.8

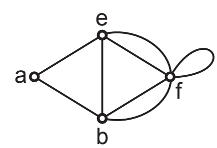
G) Fusion of vertices : A pair of vertices a and b in a graph G are said to be fused if a and b are replaced by a single new vertex say c such that every edge that was incident on either a or b or both is incident on the new vertex c. The fusion of two vertices do not change the number of edges but reduced number of vertices by 1.

e.g.



G **Graph after fusion of b and e**

Fig. 7.9.9



Graph after fusion of c and d

Fig. 7.9.10

7.10 Paths and Circuits

1) Path : An alternating sequence of vertices and edges $v_0 - e_1 - e_2 - e_3 - \ldots v_{n-1} - e_n - v_n$ beginning are ending with vertices in which each edge is incident with the two vertices immediately preceding if and following it is called a path.

The vertices v_o and v_n are called terminal vertices and $v_1, v_2, \ldots v_{n-1}$ are called its interior vertices.

e.g. Let G be the following graph.

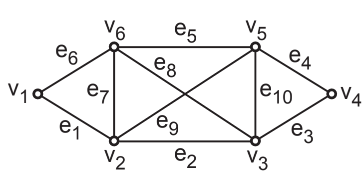


Fig. 7.10.1

Following are some examples of path

i) $v_1 - e_1 - v_2 - v_3$

ii) $v_2 - e_2 - v_3 - e_3 - v_4 - e_4 - v_5 - e_{10} - v_3 - e_2 - v_2$

iii) $v_6 - e_5 - v_5 - e_{10} - v_3 - e_8 - v_6$

iv) $v_1 - e_6 - v_6$

There are so many paths between every distinct pair of vertices of given graph G. Depending upon the nature of terminal vertices, there are two types at path.

A path in which terminal vertices are equal is called a **closed path**. A closed path is known as circuit. A path in which terminal vertices are distinct, is called an **open path**.

In above examples, paths in (i) and (iv) are open paths and (ii) and (iii) are closed paths.

7.10.1 Simple Path

A path in a graph G is said to be a simple path if the edges do not repeat in the path. Vertices may be repeated.

e.g. i) $v_1 - e_1 - v_2 - e_2 - v_3 - e_{10} - v_5$ is a simple path.

ii) $v_1 - e_1 - v_2 - e_2 - v_3 - e_{10} - v_5 - e_9 - v_2 - e_7 - v_6$ is a simple path in which v_3 is repeated.

iii) $v_3 - e_3 - v_4 - e_4 - v_5 - e_{10} - v_3 - e_8 - v_6$ is a simple path in which v_3 is repeated.

iv) $v_3 - e_3 - v_4 - e_4 - v_5 - e_{10} - v_3 - e_3 - v_4$ is not a simple path as an edge e_3 is repeated.

7.10.2 Elementary Path

A path in a graph G is said to be elementary path if vertices do not repeat in the path. Every elementary path is a simple path.

e.g. i) $v_1 - e_1 - v_2 - e_2 - v_3$ is an elementary path.

ii) $v_1 - e_1 - v_2 - e_2 - v_3 - e_8 - e_6 - e_7 - v_2$ is not an elementary path. But it is simple path.

7.10.3 Simple and Elementary Circuits

A closed path is known as circuit.

A simple path which is closed is called a simple circuit of graph.

In other words, A circuit in a graph G is said to be simple circuit if all edges of a circuit are distinct.

A circuit in a graph G is said to be elementary circuit if all vertices of a circuit are distinct except the terminal vertices i.e. the first and last vertices. The number of edges in any circuit (or path) is called the length of the circuit (or path).

In above graph G.

e.g. i) $v_1 - e_1 - v_2 - e_2 - v_3 - e_{10} - v_5 - e_9 - v_2 - e_1 - v_1$ is a circuit with e_1 repeated twice and v_2 is also repeated twice.

ii) $v_1 - e_1 - v_2 - e_2 - v_3 - e_{10} - v_5 - e_9 - v_2 - e_7 - v_6 - e_6 - v_1$ is a simple circuit but not elementary circuit as v_2 is repeated.

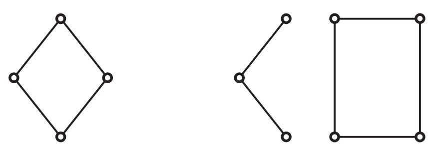
iii) $v_1 - e_1 - v_2 - e_7 - v_6 - e_6 - v_1$ is an elementary circuit.

7.11 Connected and Disconnected Graphs

A graph G is said to be connected graph if there exists a path between every pair of vertices. A graph which is not connected is called the disconnected graph.

A disconnected graph consists of two or more parts called components or blocks if each of which is connected and there is no path between two vertices if they belong to different components.

A connected graph has only one component, e.g.



G_1 G_2

Fig. 7.11.1

Graph G_1 is connected and G_2 is disconnected graph with two components.

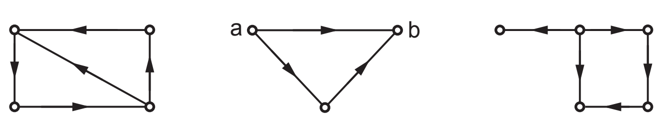
i) A directed graph is said to be connected if the undirected graph derived from it by ignoring the directions of the edges, is connected.

ii) A directed graph is said to be strongly connected if for every two vertice x and y in the graph there is a path from x to y as well as y to x.

iii) A directed graph (or diagraph) is said to be weakly connected if it is not strongly connected and its underlying graph is connected.

iv) A directed graph which is neither strongly connected nor weakly connected is called as disconnected digraph.

Consider the following examples :



G_1 G_2 G_3

Fig. 7.11.2

Graph G_1 is strongly connected. In graph G_2 there is no path from b to a. $\therefore G_2$ is not strongly connected. G_2 is weakly connected. G_3 is also weakly connected.

7.12 Connectivity of Graph

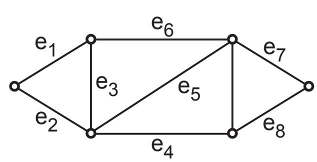
In this section, we study the measure, in some, sense, of the connectedness of a graph. For this, we must know how many edges or vertices are to be removed so that the number of components increases.

7.12.1 Edge Connectivity

A set of edges of a connected graph G whose removal disconnects G is called a disconnecting set of G. A cutset is defined as a minimal disconnecting set i.e. A minimal set of edges whose removal from G gives a disconnected graph is called a cutset.

If a cutset of a graph contains only one edge, then that edge is called as an isthmus or bridge. The number of edges in the smallest cutset of G is called the edge connectivity of G. It is denoted by λ (G).

e.g. Consider the following graph G.



G

Fig. 7.12.1

Cutsets of G are as follows :

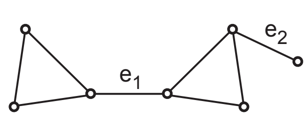
i) $\{e_4, e_5, e_6\}$,

ii) $\{e_1, e_3, e_6\}$,

iii) $\{e_1, e_2\}$,

A set $\{e_1, e_2, e_3\}$ is not a cutset because its subset $\{e_1, e_2\}$ is a cutset. The edge connectivity of graph G is 2. i.e. λ (G) = 2.

Consider the following graph G_1.



G_1

Fig. 7.12.2

Graph G_1 has edge connectivity 1 as $G_1 - \{e_1\}$ is a disconnected graph. $\therefore e_1$ is an isthmus or Bridge.

$G - e_2$ is also disconnected graph. $\therefore$ e_2 is also isthmus.

7.12.2 Vertex Connectivity

The vertex connectivity k(G) of a simple connected graph G is defined as the smallest number of vertices whose removal disconnects the graph.

In graph G, the sets $\{v_2, v_5, v_4\}$, $\{v_2, v_3, v_4, v_5\}$, $\{v_2, v_3\}$ disconnect graph G. The smallest set is $\{v_2, v_3\}$ $\therefore$ k (G) = 2.

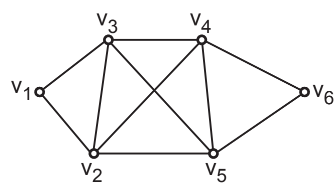


Fig. 7.12.3

1) A graph G is said to be k-connected if its vertex connectivity is k.
2) A graph G is said to be seperable graph if its vertex connectivity is one.
3) A vertex v of a connected graph G is said to be cut vertex if G – {v} is a disconnected graph.
4) $k(G) \leq \lambda(G) \leq \delta$

i.e. vertex connectivity ≤ edge connectivity ≤ minimum degree of a vertex in G and

$$\lambda(G) \leq \left[\frac{2e}{n}\right]$$

where e = number of edges in G.

n = no. of vertices in G.

7.13 Shortest Path Algorithm

SPPU : May-05, 07, 14, 15, 19, Dec.-06, 07, 12, 13, 14, 15, 16, 17, 18

Suppose there is associated to each edge e of a graph a real number w (e). w (e) is called the weight of e. A weighted graph is a graph in which each edge has a weight. The weight of graph G is the sum of weight of all edges of G. Weighted graph has many applications in communication networks. Given a railway network connecting several cities, determine a shortest route between two cities.

We consider the weighted graph where the vertices are the towns, rail roads are the edges and the weight represent the distance between directly linked cities. Therefore weights are non negative integers. The problem is to find a path of minimum weight connectivity two given cities. Of course, this is possible theoretically. One has to list all paths, find their weights and select minimum one. But for large networks (large number of vertices and edges) this may not be efficient. So we required different method to take such problems. The algorithm was found by Dijkstra (1959) and is known as Dijkstra's algorithm.

A) Dijkstra's algorithm to find the shortest path from the vertex a to vertex z of a graph G. Let G (v, E) be a simple graph and a, z $\in$ V.

Suppose L(x) is the label of the vertex which represents the length of the shortest path from the vertex a. W_{ij} = weight of an edge $e_{ij} = (v_i, v_j)$

Consider the following steps :

Step 1 : Let P be the set of those vertices which have permanent labels and T = set of all vertices of G.

$$\text{Set } L(a) = 0, L(x) = \infty \; ; \forall \; x \in T \text{ and } x \neq a$$

$$P = \phi \text{ and } T = v.$$

Step 2 : Select the vertex v in T which has the smallest label. This label is called the permanent label of v. Also set P as P $\cup$ {v} and T as T - {v}.

If v = z then L (z) is the length of the shortest path from the vertex a to z and stop the procedure.

Step 3 : If v $\neq$ z, then revise the labels of the vertices of T. i.e. The vertices which do not have permanent labels.

The new label of x in T is given by

$$L(x) = \min \{\text{old } L(x), L(v) + w(v, x)\}$$

where w (v, x) is the weight of the edge joining v and x. If there is no edge joining v and x then take w (v, x) = ∞.

Step 4 : Repeat the steps 2 and 3 until z gets the permanent label.

Examples :

Example 7.13.1 *Use Dijkstra's algorithm to find the shortest path between a and z.*

SPPU : May-05, 14, 19, Marks 8, Dec.-06, Marks 6

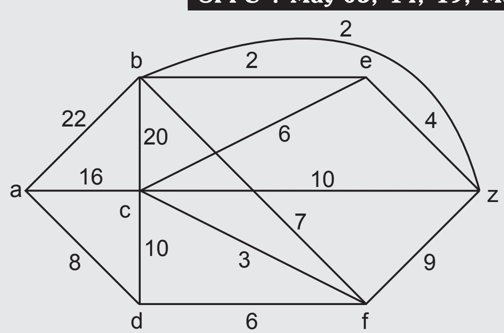


Fig. 7.13.1

Solution :

Step 1 : $P = \varnothing$, $T = \{a, b, c, d, e, f, z\}$

$$L\{a\} = 0$$

$$L\{x\} = \infty, \quad \forall\ x \in T, \quad x \neq a$$

Step 2 : $v = a$, the permanent label of a is 0

$$P = \{a\}, \quad T = \{b, c, d, e, f, z\}$$

$$L\{b\} = \min\{\text{old } L(b), L(a) + w(a, b)\}$$

$$= \min\{\infty, 0 + 22\} = 22$$

$$L\{c\} = \min\{\infty, 0 + 16\} = 16$$

$$L\{d\} = \min\{\infty, 0 + 8\} = 8$$

$$L\{e\} = \min\{\infty, 0 + \infty\} = \infty$$

$$L\{f\} = \min\{\infty, 0 + \infty\} = \infty$$

$$L\{z\} = \min\{\infty, 0 + \infty\} = \infty$$

$\therefore$ $L\{d\} = 8$ is the minimum label.

Step 3 : $v = d$, the permanent label of d is 8.

$$P = \{a, d\}, \quad T = \{b, c, e, f, z\}$$

$$L\{b\} = \min\{\text{old } L(b), L(d) + w(d, b)\}$$

$$= \min\{22, 8 + \infty\} = 22$$

$$L\{c\} = \min\{16, 8 + 10\} = 16$$

$$L\{e\} = \min\{\infty, 8 + \infty\} = \infty$$

$$L\{f\} = \min\{\infty, 8 + 6\} = 14$$

$$L\{z\} = \min\{\infty, 8 + \infty\} = \infty$$

$\therefore$ $L\{f\} = 14$ is the minimum label.

Step 4 : $v = f$, the permanent label of f is 14.

$$P = \{a, d, f\}, \quad T = \{b, c, e, z\}$$

$$L\{b\} = \min\{\text{old } L(b), L(f) + w(b, f)\}$$

$$= \min\{22, 14 + 7\} = 21$$

$$L\{c\} = \min\{16, 14 + 3\} = 16$$

$$L\{e\} = \min\{\infty, 14 + \infty\} = \infty$$

$$L\{z\} = \min\{\infty, 14 + 9\} = 23$$

$\therefore$ $L\{c\} = 16$ is the minimum label.

Step 5 : $v = c$, the permanent lable of c is 16.

$$P = \{a, d, f, c\}, \quad T = \{b, e, z\}$$

$$L\{b\} = \min\{\text{old } L(b), L(f) + w(f, b)\}$$

$$= \min\{21, 16 + 20\} = 21$$

$$L\{e\} = \min\{\infty, 16 + 6\} = 22$$

$$L\{z\} = \min\{23, 16 + 10\} = 23$$

$\therefore$ $L\{b\} = 2$ is the minimum label.

Step 6 : $v = b$, the permanent label of b is 21.

$$P = \{a, d, f, c, b\}, \quad T = \{e, z\}$$

$$L\{e\} = \min\{\text{old } L(e), \ L(b) + w(e, b)\}$$

$$= \min\{22, 21 + 2\} = 22$$

$$L\{z\} = \min\{23, 21 + 2\} = 23$$

$\therefore$ $L\{e\} = 22$ is the minimum label.

Step 7 : $v = e$, the permanent label of e is 22.

$$P = \{a, d, f, c, b, e\}, \quad T = \{z\}$$

$$L(z) = \min\{\text{old } L(z), L(e) + w(e, z)\}$$

$= \min\{23, 22 + 4\} = 23$ which is the minimum label

Step 8 : $v = z$, the permanent label of z is 23.

Hence the length of the shortest path from a to z is 23.

The shortest path is adfz or adfbz.

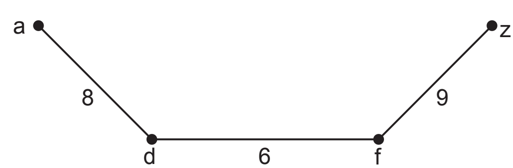


Fig. 7.13.2 (a)

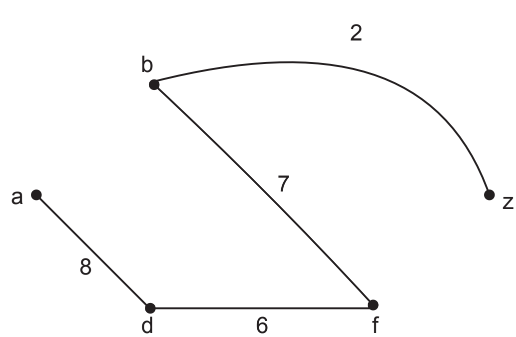


Fig. 7.13.2 (b)

Example 7.13.2 *Find the shortest path from a-z in the given graph using Dijkstra's algorithm.* **SPPU : May-07, Dec.-07, 15, 18, Marks 6**

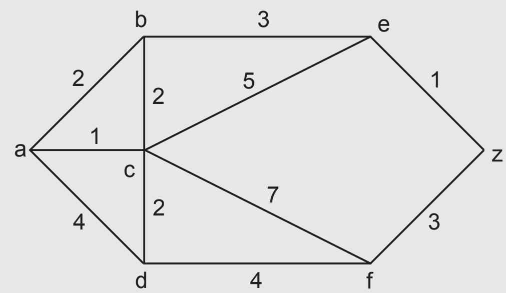


Fig. 7.13.3

Solution :

Step 1 : Set $P = \varnothing, \quad T = \{a, b, c, d, e, f, z\}$

$L\{a\} = 0$

$L\{x\} = \infty, \quad \forall x \in T, \quad x \neq a$

Step 2 : $v = a$, the permanent label of a is 0.

$P = \{a\}, \quad T = \{b, c, d, e, f, z\}$

$L\{b\} = \min\{\text{old } L(b), L(a) + w(a, b)\}$

$= \min\{\infty, 0 + 2\} = 2$

$L\{c\} = \min\{\infty, 0 + 1\} = 1$

$L\{d\} = \min\{\infty, 0 + 4\} = 4$

$L\{e\} = \min\{\infty, 0 + \infty\} = \infty$

$L\{f\} = \min\{\infty, 0 + \infty\} = \infty$

$L\{z\} = \min\{\infty, 0 + \infty\} = \infty$

$\therefore \quad L\{c\} = 1$ is the minimum label.

Step 3 : $v = c$, the permanent label of c is 1.

$P = \{a, c\}, \quad T = \{b, d, e, f, z\}$

$L\{b\} = \min\{2, 1 + 2\} = 2$

$L\{d\} = \min\{4, 1 + 2\} = 3$

$L\{e\} = \min\{\infty, 1 + 5\} = 6$

$L\{f\} = \min\{\infty, 1 + 7\} = 8$

$L\{z\} = \min\{\infty, 1+\infty\} = \infty$

$\therefore$ $L\{b\} = 2$ is the minimum label.

Step 4 : $v = b$, the permanent label of b is 2.

$P = \{a, c, b\}$, $T = \{d, e, f, z\}$

$L\{d\} = \min\{3, 2+\infty\} = 3$

$L\{e\} = \min\{6, 2+3\} = 5$

$L\{f\} = \min\{8, 2+\infty\} = 8$

$L\{z\} = \min\{\infty, 2+\infty\} = \infty$

$\therefore$ $L\{d\} = 3$ is the minimum label.

Step 5 : $v = d$, the permanent label of d is 3.

$P = \{a, c, b, d\}$, $T = \{e, f, z\}$

$L\{e\} = \min\{5, 3+\infty\} = 5$

$L\{f\} = \min\{8, 3+4\} = 7$

$L\{z\} = \min\{\infty, 3+\infty\} = \infty$

$\therefore$ $L\{e\} = 5$ is the minimum label.

Step 6 : $v = e$, the permanent label of e is 5.

$P = \{a, c, b, d, e\}$, $T = \{f, z\}$

$L\{f\} = \min\{7, 5+\infty\} = 7$

$L\{z\} = \min\{\infty, 5+1\} = 6$

$\therefore$ $L\{z\} = 6$ is the minimum label.

Step 7 : $v = z$, the permanent label of z is 6.

Hence the length of shortest path from a to z is 6.

The shortest path is a b e z.

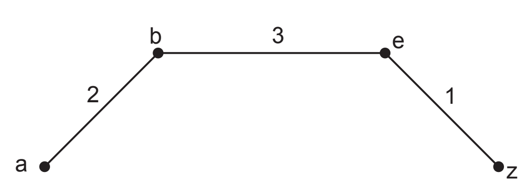


Fig. 7.13.3 (a)

Example 7.13.3 *Find the shortest path between a-z for the given graph : using Dijkstra's algorithm.* **SPPU : Dec.-12, 13, 14, 16, May-15, Marks 8**

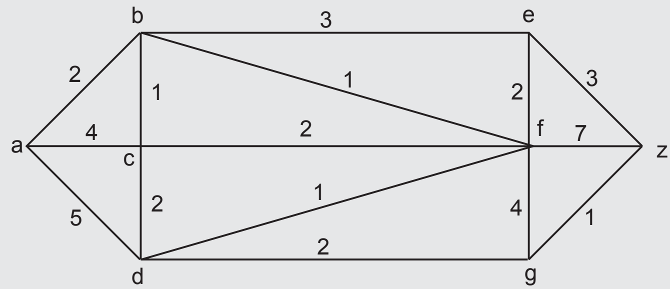


Fig. 7.13.4

Solution :

Step 1 : Set $P = \varnothing$, $T = \{a, b, c, d, e, f, g, z\}$

$L\{a\} = 0$

$L\{x\} = \infty$, $\forall\ x \in T$, $x \neq a$.

Step 2 : $v = a$, the permanent label of a is 0.

$P = \{a\}$, $T = \{b, c, d, e, f, g, z\}$

$L\{b\} = \min\{\text{old } L(b),\ L(a) + w(a, b)\}$

$= \min\{\infty, 0 + 2\} = 2$

$L\{c\} = \min\{\infty, 0 + 4\} = 4$

$L\{d\} = \min\{\infty, 0 + 5\} = 5$

$L\{e\} = L\{f\} = L\{g\} = L\{z\} = \infty$

$\therefore$ $L\{b\} = 2$ is the minimum label. The permanent label of b is 2.

Step 3 : $v = b$

$P = \{a, b\}$, $T = \{c, d, e, f, g, z\}$

$L\{c\} = \min\{L(c), L(b) + w\{b, c\}\}$

$= \min\{4, 2 + 1\} = 3$

$L\{d\} = \min\{5, 2 + \infty\} = 5$

$L\{e\} = \min\{\infty, 2 + 3\} = 5$

$L\{f\} = \min\{\infty, 2 + 1\} = 3$

$L\{g\} = L\{z\} = \infty$

$\therefore$ $L\{c\} = L\{f\} = 3$ are the minimum labels.

Step 4 : $v = c \text{ or } f$

Let $v = f$, permanent label of f is 3.

$P = \{a, b, f\}, \quad T = \{c, d, e, g, z\}$

$L\{c\} = \min\{3, 3 + 2\} = 3$

$L\{d\} = \min\{5, 3 + 1\} = 4$

$L\{e\} = \min\{5, 3 + 2\} = 5$

$L\{g\} = \min\{\infty, 3 + 4\} = 7$

$L\{z\} = \min\{\infty, 3 + 7\} = 10$

$\therefore$ $L\{c\} = 3$ is the minimum label.

Step 5 : $v = c$, permanent label of $c = 3$.

$P = \{a, b, f, c\}, \quad T = \{d, e, g, z\}$

$L\{d\} = \min\{4, 3 + 2\} = 4$

$L\{e\} = \min\{5, 3 + \infty\} = 5$

$L\{g\} = \min\{7, 3 + \infty\} = 7$

$L\{z\} = \min\{10, 3 + \infty\} = 10$

$\therefore$ $L\{d\} = 4$ is the minimum label.

Step 6 : $v = d$, permanent label of $d = 4$.

$P = \{a, b, f, c, d\}, \quad T = \{e, g, z\}$

$L\{e\} = \min\{5, 4 + \infty\} = 5$

$L\{g\} = \min\{7, 4 + 2\} = 6$

$L\{z\} = \min\{10, 4 + \infty\} = 10$

$\therefore$ $L\{e\} = 5$ is the minimum label.

Step 7 : $v = e$, permanent label of e is 5.

$P = \{a, b, f, c, d, e\}, \quad T = \{g, z\}$

$L\{g\} = \min\{6, 5 + \infty\} = 6$

$L\{z\} = \min\{10, 5 + 3\} = 8$

$\therefore$ $L\{g\} = 6$ is the minimum label.

Step 8 : $v = g$, permanent label of g is 6.

$P = \{a, b, f, c, d, e, g\}, \quad T = \{z\}$

L {z} = min {8, 6 + 1} = 7 which is the minimum label.

Step 9 : v = z, permanent label of z is 7.

Hence the length of shortest path from a to z is 7.

The shortest path is a b f d g z.

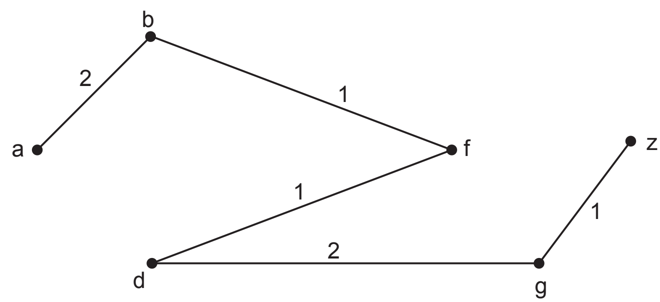


Fig. 7.13.4 (a)

7.14 Eulerian Graphs

SPPU : Dec.-09, May-19

7.14.1 Konigsberg's Seven Bridges

The city of Konigsberg, the capital of old East Prussia was founded by the tentonic knights in 1254. The river pregel flowed through the city which included an island called Kneiphof. There were seven bridges on this river connecting the island and various parts of the city as shown below :

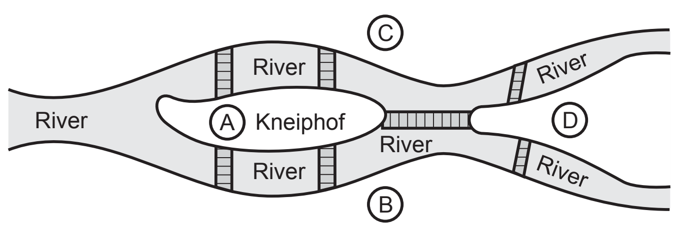


Fig. 7.14.1

The seven bridges problem of Konigsberg :

"Is it possible to cross the seven bridges in continuous walk without recrossing any of them ?"

On 26 August 1735, a swiss mathematician Leonhard Euler presented a paper which described the solution of this problem. Euler concluded that such a continuous walk over the bridges was impossible. This paper of Euler is considered as the origin of graph theory.

Euler represented the arrangement of the river and its bridges by means of a graph in which the land areas were the vertices and bridges were the edges. In this section we shall study the general problem of traversing the vertices and edges of a graph.

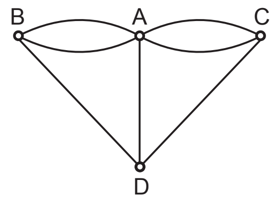


Fig. 7.14.2

In this section we shall study the general problem of traversing the vertices and edges of a graph.

7.14.2 Eulerian Path and Circuit

A path is called an Eulerian path if every edge of graph G appears exactly once in the path.

A circuit of a graph which contains every edge of graph exactly once is called the Eulerian circuit.

A graph which has an Eulerian circuit is called as Eulerian graph.

The problem of find Eulerian path is the same as the problem of drawing a network without lifting the pencil off the paper and without retracing an edge.

Consider the following graphs :

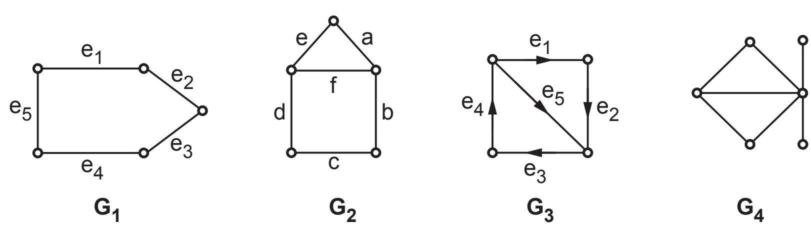


Fig. 7.14.3

In graph G_1, Eulerian circuit is e - e_2- e_3 - e_4 - e_5 - e_1

$\therefore G_1$ is an Eulerian graph.

In graph G_2, Eulerian circuit does not exist.

$\therefore G_2$ is not Eulerian graph.

In graph G_3, Eulerian path is $e_1 \rightarrow e_2 \rightarrow e_3 \rightarrow e_4 \rightarrow e_4 \rightarrow e_5$ but G_3 does not have any Eulerian circuit.

$\therefore G_3$ is not Eulerian graph.

G_4 is also not an Eulerian graph.

The existence of Eulerian paths and circuits in a graph depends upon the degree of vertices.

Theorem 1 : An undirected graph possesses an Eulerian path iff it is connected and has either zero or two vertices of odd degree.

Theorem 2 : An undirected graph possesses an Eulerian circuit iff it is connected and its vertices are all of even degree.

Theorem 3 : A directed graph possesses an Eulerian circuit iff it is connected and incoming degree of every vertex is equal to its outgoing degree.

Examples :

Example 7.14.1 *Draw a graph which has Eulerian path but not Eulerian circuit.*

Solution : This graph G has no Eulerian circuit because d (c) = 3 in G.

Eulerian path is

$c - e_3 - d - e_4 - e - e_5 - f - e_6 - e - e_2 - b - e_1 - a - e_7 - f$

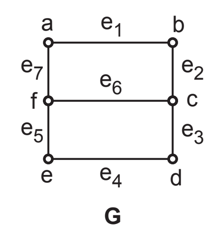


Fig. 7.14.4

Example 7.14.2 *For what values of n does K_n, the complete graph on n vertices have an Eulerian circuit ? an Eulerian path ?*

Solution : In the complete graph on n vertices (K_n), vertex is adjacent to remaining (n–1) vertices. So degree of each vertex is n–1.

If n is an odd, the degree of each vertex will be an even. Hence by theorem 1, Eulerian circuit exists and Graph is Eulerian.

For an Eulerian path the graph should have either zero or exactly two vertices of odd degree. It is possible only in K_2.

Fig. 7.14.5

Hence Eulerian path exists for K_2 only.

Example 7.14.3 *Find under what conditions $K_{m,n}$ the complete bipartite graph will have an Eulerian circuit* **SPPU : Dec.-09, May-19, Marks 3**

Solution : In $K_{m, n}$ consider the following cases.

Case 1 : m = n and both m and n are even :

In this case, degree of each vertex is even, Hence by theorem 1, $K_{m, n}$ will have an Eulerian circuit. For example $K_{1, 2}$ and $K_{4, 4}$

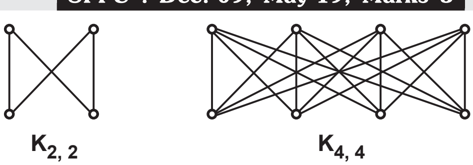
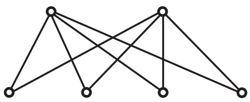


Fig. 7.14.6

Case 2 : If m = n and m, n are odd :

In this case degree of each vertex is odd. Hence Eulerian circuit will not exist.



$K_{2,\,4}$

Fig. 7.14.7

Case 3 : If m ≠ n but m and n are even :

In this case, degree of each vertex is even. So there exists an Eulerian circuit.

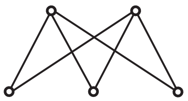


Fig. 7.14.8

Case 4 : If m ≠ n and either m is odd or n is odd or both are odd : then graph will have vertices of odd degree. Hence Eulerian circuit does not exist. e.g. K_2, 3.

Example 7.14.4 *Which of the following graphs possess Eulerian circuits or paths ?*

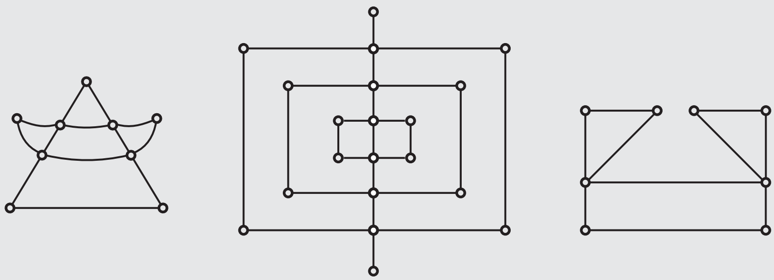


Fig. 7.14.9

Solution : In graph G_1, each vertex is of even degree, hence it possesses an Eulerian circuit.

In graph G_2 there are exactly 2 vertices of odd degree. Hence G_2 possesses an Eulerian path.

In graph G_3, each vertex is of even degree. Hence G_3 has an Eulerian circuit.

7.15 Hamiltonian Graphs

SPPU : Dec.-04, 09, 10, 12, 15, May-17

In this section, we introduce a class of graphs which possess a strinking similarity to Eulerian graphs.

We will now define Hamiltonian path and circuits of the connected graph.

A path in a connected graph G is called a Hamiltonian path if it contains every vertex of G exactly once.

A circuit in a connected graph G is called a Hamiltonian circuit if it contains every vertex of G exactly once.

A graph which has a Hamiltonian circuit is called a Hamiltonian Graph.

Consider the following graphs :

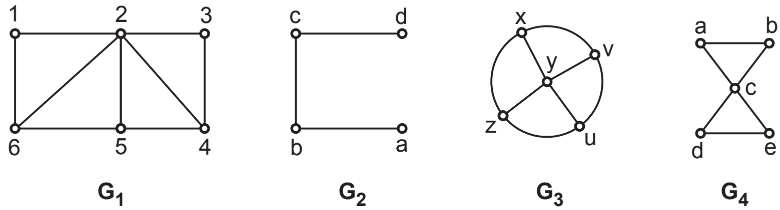


Fig. 7.15.1

In graph G_1, Hamiltonian circuit is 1-2-3-4-5-6-1

$\therefore G_1$ is a Hamiltonian graph.

In graph G_2, Hamiltonian path is a-b-c-d but Hamiltonian circuit does not exist.

$\therefore G_2$ is not Hamiltonian graph.

In graph G_3, Hamiltonian circuit is x-y-z-u-v-x

$\therefore G_3$ is Hamiltonian graph but it is not Eulerian.

In graph G_4, Hamiltonian path is a-b-c-d-e but Hamiltonian circuit does not exist.

$\therefore G_4$ is not Hamiltonian but Eulerian graph.

Remark : Hamiltonian graphs were named after W.R. Hamilton (in 1856) who described a mathematical game on the dodecahedron as given in Fig. 7.15.2

In this vertices are represented various cities and one is required to make a smallest spanning cycle.

Thus, dodecahedron graph is a Hamiltonian graph.

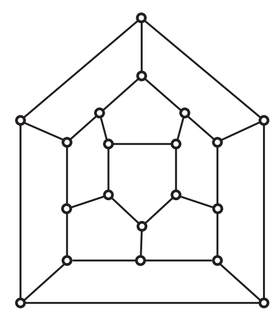


Fig. 7.15.2

Important theorems :

Theorem 1 : Let G be a simple connected graph on n vertices. If the sum of the degree of each pair of vertices in G is (n – 1) or large then there exists a Hamiltonian path in G.

Consider the following graphs :

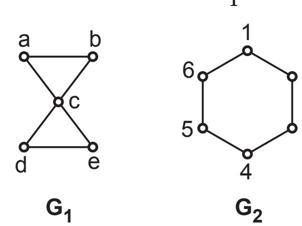


Fig. 7.15.3

In graph G_1, n = 5, degree sum of every pair of vertices is 4 or more. Hence ∃ a Hamiltonian path in G_1 which is a-b-c-d-e.

In graph G_2 n = 6, degree of every pair of vertices is 4 which is not equal to n – 1 = 5.

But ∃ a Hamiltonian path 1-2-3-4-5-6. Therefore the condition in theorem 1 is a sufficient but not a necessary condition for existence of Hamiltonian path.

Theorem 2 : If G (v, E) is a simple connected graph on n vertices and $d(v) = \frac{n}{2}$; $\forall\, v \in V$ then G will contain a Hamiltonian circuit.

This condition is sufficient condition but not necessary.

Theorem 3 : Let G (v, E) be a connected simple graph. If G has a Hamiltonian circuit then for every proper non empty subsets of v, the components in the graph G-S is less than or equal to the number of vertices in S.

Theorem 4 : A Hamiltonian graph contains no cut vertices and hence is 2-connected.

Example 7.15.1 *Show that the complete bipartite graph $K_{m,n}$ is Hamiltonian for $m = n$ and for $m \neq n$, $K_{m,n}$ is not Hamiltonian graph.*

Solution : In a complete bipartite graph $K_{m,n}$ for m = n i.e. $K_{n,n}$, degree of each vertex is n.

Therefore $d(v) \geq \frac{n}{2}$ for all $v \in v(K_{n,n})$

By theorem 2, G contains a Hamiltonian circuit.

Hence $K_{n,n}$ is a Hamiltonian graph.

If $m \neq n$, Let V_1 and V_2 be the partitions of the vertex set of $k_{m,n}$ where $|v_1| = m$ and $|v_2| = n$. Without loss of generality assume that m < n.

The graph $K_{m,n} - V_1$ is a null graph on n vertices.

Hence it is a disconnected graph with n components.

Therefore the number of components in $K_{m,n} - V_1 = n$ which is greater than the number of vertices in V_1

Hence by theorem 3, $K_{m,n}$ does not contain a Hamiltonian circuit when $m \neq n$.

Example 7.15.2 *Show that the complete graph K_n $(n \geq 3)$ is a Hamiltonian graph. What is the length of that circuit ? How many circuits exist in K_n ?* **SPPU : Dec.-09**

Solution : The complete graph K_n has n vertices, $n \geq 3$ and degree of each vertex is $n - 1$. As $n \geq 3$.

$$d(v) = n - 1 \geq \frac{n}{2} \; ; \forall\, v \in V(K_n).$$

Therefore by theorem 2, K_n has a Hamiltonian circuit. Hence K_n is a Hamiltonian graph.

Hamiltonian circuit contains all vertices of graph and length of circuit is the number of vertices present it. Hence in K_n, the length of the Hamiltonian circuit is n and there are $\frac{(n-1)!}{2}$ Hamiltonian circuits in K_n.

Example 7.15.3 *Find the Hamiltonian path and circuit in $K_{4,3}$?* **SPPU : Dec.-04**

Solution : The complete bipartite graph $K_{4,3}$ is given by

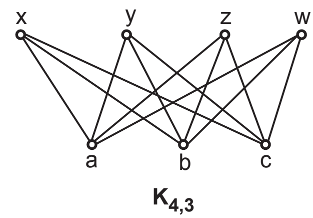


Fig. 7.15.4

In $K_{4,3}$, $4 \neq 3$ Hence it does not contain Hamiltonian circuit. Here degree of each vertex is either 3 or 4.

$\therefore$ For x, y any two vertices in $K_{4,3}$ $d(x) + d(y) = 7.1 = 6$

Hence by theorem 1, the graph $K_{4,3}$ has a Hamiltonian path. It is given by x a y b z c w.

Example 7.15.4 *Give an example of the following graphs*

a) Eulerian but not Hamiltonian.

b) Hamiltonian but not Eulerian.

c) Eulerian as well as Hamiltonian.

d) Neither Eulerian nor Hamiltonian.

Solution : a) Eulerian but not Hamiltonian graph.

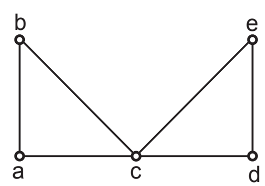


Fig. 7.15.5

Eulerian circuit : a b c d e c a

No Hamiltonian circuit.

b) Hamiltonian but not Eulerian

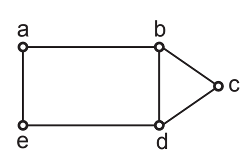


Fig. 7.15.5 (a)

Hamiltonian circuit : abcdea

No Eulerian circuit because d (b) = 3.

c) Eulerian and Hamiltonian graph.

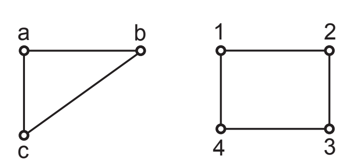


Fig. 7.15.5 (b)

Hamiltonian circuit : a-b-c-a, 1-2-3-4-1

Eulerian circuit : a-b-c-a, 1-2-3-4-1

d) Neither Eulerian nor Hamiltonian

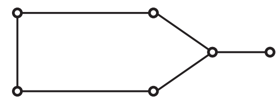


Fig. 7.15.5 (c)

No Hamiltonian circuit and no eulerian circuit.

Example 7.15.5 *Determine, if the following graphs are having the Hamiltonian circuit or path. Justify your answer.* **SPPU : Dec.-12, May-17, Marks 3**

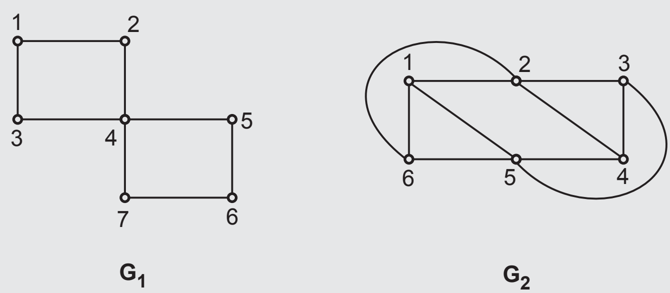


Fig. 7.15.6

Solution : In graph G_1, there are n = 7 vertices.

d (4) = 4 and all remaining vertices is 2.

So to draw Hamiltonian circuit, we have to visit vertex 4 twice. Which is not possible in Hamiltonian path. G_1 has no Hamiltonian circuit. Hamiltonian path is 1-2-3-4-5-6-7.

In graph G_2, there are 6 vertices and degree of each vertex is 3 or 4.

If we consider two vertices of lowest degree then also their sum is 6 which is equal to the number of vertices. So there exists a Hamiltonian path in G_2. $\therefore$ Path is 1-2-3-4-5-6.

In graph G_2, $d(x) = \frac{6}{2} = 3 \; ; \; x \in V(G_2)$

$\therefore$ By theorem 2, $\exists$ a Hamiltonian circuit

$\therefore$ Hamiltonian circuit is 1-2-3-4-5-6-1

Example 7.15.6 *Which of the following have a Euler circuit or path or Hamiltonian cycle ? Write the path or circuit* **SPPU : Dec.-10**

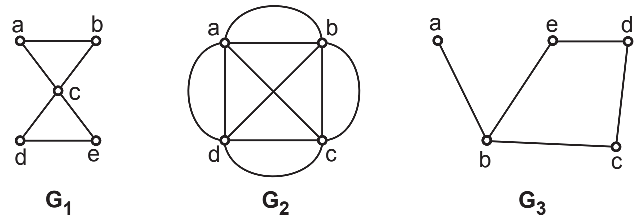


Fig. 7.15.7

Solution : In graph G_1, degree of each vertex is an even so $\exists$ an Eulerian circuit which is a-b-c-d-e-c-a.

In graph G, there are 5 vertices and degree sum of every pair of vertices is 4 or greater than 4. Hence there exists a Hamiltonian path in G_1 which is given by a-b-c-d-e. But there is no any Hamiltonian circuit as vertex c is a vertex. In graph G_2, degree of each vertex is 5 which is odd integer, so there is no Eulerian path in G_2, degree of each vertex is $5 > \frac{4}{2}$. Hence there exixts a Hamiltonian circuit which is given by a-b-c-d-a.

In G_3, Eulerian path is a-b-c-d-e-b No Eulerian circuit as d (a) = 1. Hamiltonian path is a-b-c-d-e.

No Hamiltonian cyle because b is a cut vertex.

7.15.1 The Travelling Salesman Problem (TSP)

A salesman is required to travel a number of cities during a trip. Given the distance among cities, in what order should he travel so that he travels as minimum distance as possible ? This is known as Travelling Salesman Problem (TSP).

In terms of graph theory, the TSP is to a Hamiltonian circuit with the smallest weight. In the case of K_n the problem can be solved theoretically by listing all the possible Hamiltonian circuits and select one which has least weight. But this method is highly impractical for the large graphs. In fact no efficient algorithm is there to solve TSP. It is therefore desirable to obtain a reasonably good but not an optimal solution.

One possible approach is to first find a Hamiltonian cycle and search for other Hamiltonian cycles of lesser weight. The simple method is as follows :

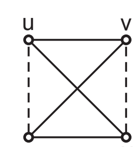


Fig. 7.15.8

Let C be the Hamiltonian circuit of a graph G.

Let further uv and xy be two no-adjacent edges of C such that the vertices u, v, x and y occur in that order in C.

If ux and vy are edges such that v (u x) + w (y v) < w (uv) + w (yx) then replace the edges uv and xy in (by ux and vy). The new cycle C' would still be Hamiltonian cycle and w (C') < w (C). This process can be continued until one gets a reasonably good Hamiltonian cycle.

7.15.2 Nearest Neighbour Method

In this method, we start with any arbitrary vertex and find the vertex which is nearest to it. Continuing this way and coming back to the starting vertex by travelling through all the vertices exactly once, we will get Hamiltonian cycle or circuit.

Consider the following steps to find Hamiltonian cycle by this method.

Step 1 : Start with any arbitrary vertex say v_1, choose the vertex closest to v_1 to form an initial path of one edge. Construct this path by selecting different vertices as described in step 2.

Step 2 : Let v_n be the latest vertex that was added to the path. Select the vertex v_{n+1} closest to v_n from all vertices that are not in the path and add this vertex to the path. Select those vertices which will not form a circuit in this stage.

Step 3 : Repeat step (2) till all the vertices of G are included in the path.

Step 4 : Lastly form a circuit by adding the edge connecting to v_1 and the last added vertex.

The circuit obtained using the nearest neighbour method will be the required Hamiltonian circuit.

Note : If we start with an arbitrary vertex in TSP then we may or may not minimum Hamiltonian circuit, But if we start with a vertex whose incident edge has the minimum weight in graph then we will get minimum Hamiltonian circuit as compared with arbitary starting vertex. For more details see example (2).

Examples :

Example 7.15.7 *Use nearest neighbour method to find the Hamiltonian circuit starting from a in the following graph. Find its weight.* **SPPU : Dec.-15**

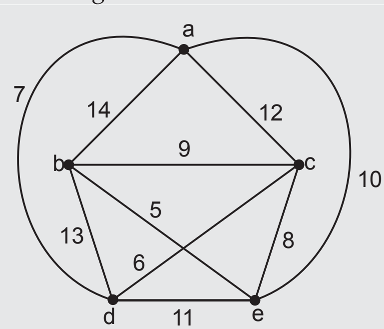


Fig. 7.15.9

Solution :

Step 1 : Let a be the starting vertex. Vertex a is adjacent to b, c, d, e. But minimum path is {a, d} which is the initial path.

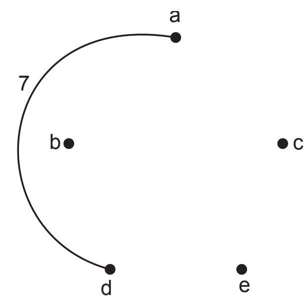


Fig. 7.15.9 (a)

Step 2 : There are three vertices adjacent to d. but closest one is C $\therefore$ The path is {a, d, c}.

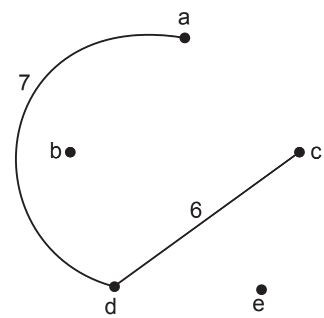


Fig. 7.15.9 (b)

Step 3 : There are 4 vertices adjacent to c. but closest is e $\therefore$ The path is {a, d, c, e}.

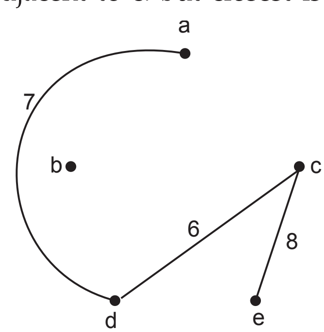


Fig. 7.15.9 (c)

Step 4 : There are 4 vertices adjacent to e. but closest is b $\therefore$ The path is {a, d, c, e, b}.

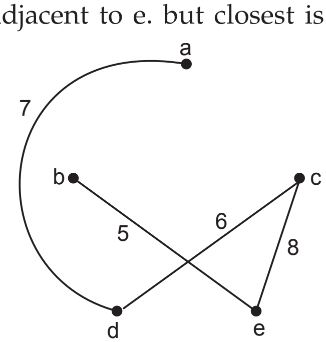


Fig. 7.15.9 (d)

Step 5 : Here are all vertices are corered so to complete Hamiltonian an circuit there should be a path from b to a.

$\therefore$ Hamiltonian circuit is {a, d, c, e, b, a}

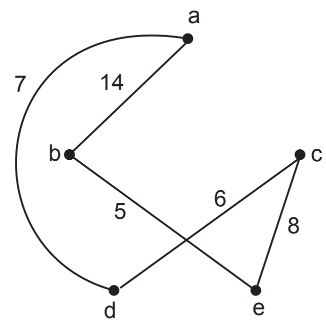


Fig. 7.15.9 (e)

Weight of the Hamiltonian circuit = 40.

Example 7.15.8 *For the following figure solve the following options :*

a) Use nearest neighbourhood method to find out Hamiltonian circuit for the graph in the following Fig. 7.15.10 starting at vertex a.

b) Repeat the part (a), starting at vertex 'd' instead.

c) Determine the minimum Hamiltonian circuit for the graph in the following Fig. 7.15.10

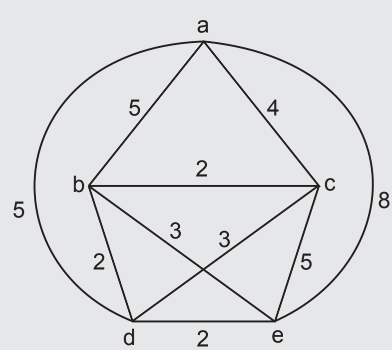


Fig. 7.15.10

Solution : a) Start with vertex a. There are 4 adjacent vertices b, c, d and e. The closest vertex is c (as length of the edge (a, c) = 4, which is minimum).

Step 1 : Initial path = {a, c}

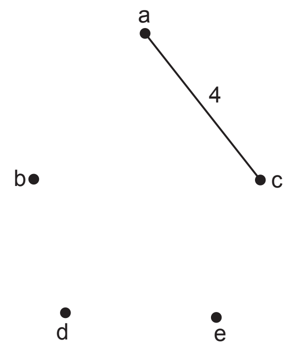


Fig. 7.15.10 (a)

Step 2 : There are three vertices adjacent to c, namely b, d, e (except a). The closest is b

Path = {a, c, b}

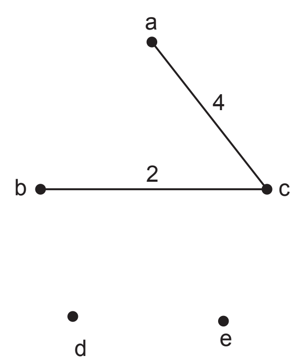


Fig. 7.15.10 (b)

Step 3 : There are two vertices adjacent to b, namely d and e. The closest is d

Path = {a, c, b, d}

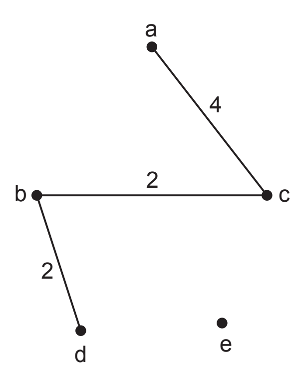


Fig. 7.15.10 (c)

Step 4 : path = {a, c, b, d, e}

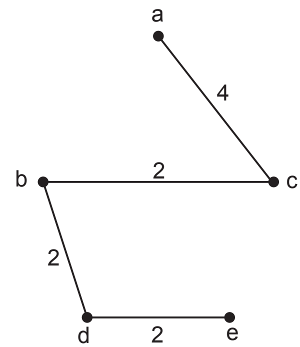


Fig. 7.15.10 (d)

Step 5 : Since all the vertices are traversed, to complete the Hamiltonian circuit, there should be a path from e to a.

Hamiltonian circuit = {a, c, b, d, e, a}

The weight of Hamiltonian circuit = 18

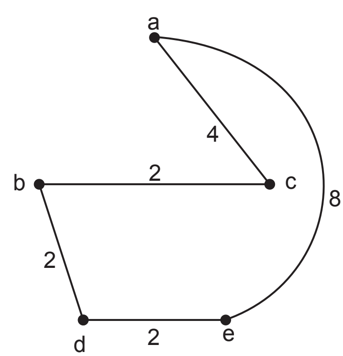


Fig. 7.15.10 (e)

(b) To find the Hamiltonian circuit starting from d, consider the vertex d. Since both the vertices b and e are nearest to d as both are at distance 2 from d, hence we can choose any vertex.

Let e be the vertex.

Step 1 : Initial path = {d, e}

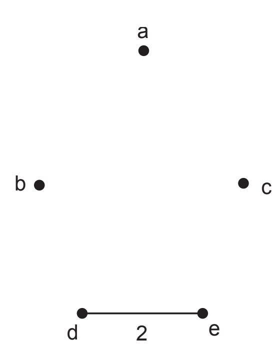


Fig. 7.15.10 (f)

Step 2 : a, b, c are adjacent to e but b is the nearest neighbour of e.

Path = {d, e, b}

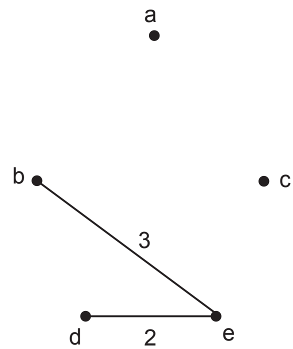


Fig. 7.15.10 (g)

Step 3 : Path = {d, e, b, c}

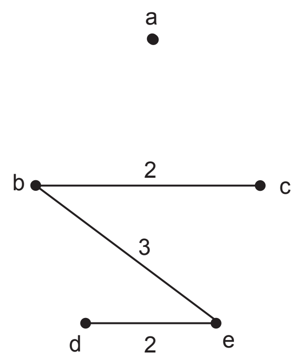


Fig. 7.15.10 (h)

Step 4 : Path = {d, e, b, c, a}

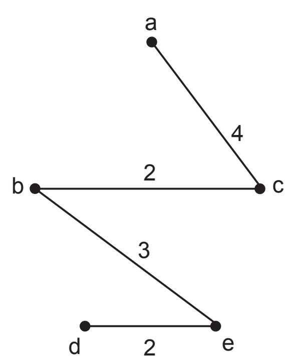


Fig. 7.15.10 (i)

Step 5 : Since all the vertices are traversed, to complete the Hamiltonian circuit, there should be a path from a to d.

Hamiltonian = {d, e, b, c, a, d}

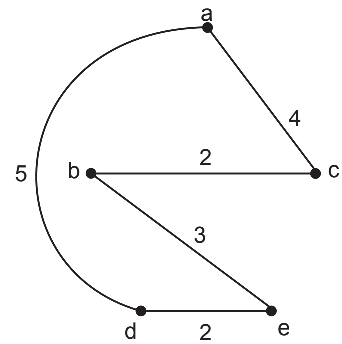


Fig. 7.15.10 (j)

The weight of the Hamiltonian circuit is 16.

C) The minimum weight of Hamilton circuit is 16.

7.16 Planar Graphs

SPPU : May-06, Dec.-08, 09, 10, 13

In this section we will study drawing of graphs without crossing edges on surfaces and deal with some important properties of such graphs.

Definition : A graph is said to be planar graph if it can be drawn on a plane such that no edges intersect or cross in a point other than their end vertices.

A graph G is said to be non-planar if it is not possible to draw graph be without crossing.

Examples :

1) Following graphs i.e. Planar graphs :

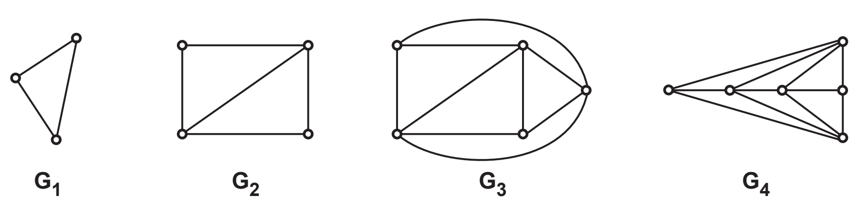


Fig. 7.16.1

2)

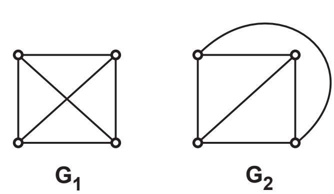


Fig. 7.16.2

Graph G_1 looks like non-planar graph out if we redraw graph we get graph G_2 which is planar.

7.16.1 Regions

A plane representation of a graph divides the plane into parts or regions. They are also known as faces or windows or meshes.

A region or face is characterised by the set edges forming it's boundary.

A region is said to be finite if it's area is finite. A region is said to be infinite or unbounded if it's area is infinite. Every planar graph has an infinite region.

Consider the graph given below :

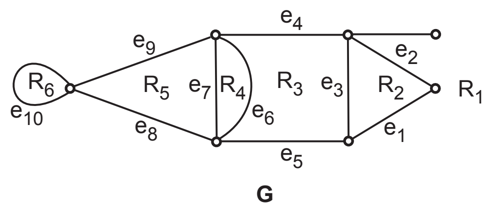


Fig. 7.16.3

The graph G has 6 regions, 7 vertices and 11 edges. Region R_1 is an infinite region known as exterior region. We have

$$R_2 = \{e_1, e_2, e_3\} = \text{Region bounded by } e_1, e_2, e_3$$

$$R_3 = \{e_3, e_4, e_5, e_6\}$$

$$R_4 = \{e_6, e_7\}, \quad R_6 = \{e_{10}\}.$$

It is observed that $n = 7$, $e = 11$, $r = 6$

$$\therefore \quad n + r - 2 = 7 + 6 - 2 = 11 = e$$

Now let us define Euler's formula.

7.16.2 Euler's Formula

Statement : For any connected planar graph G, with v number of vertices, e number of edges and r number of regions

$$\boxed{v - e + r = 2}$$

or $v + r - 2 = e$

Proof : Let G be a connected planar graph with v vertices, e edges and r regions. We shall prove the theorem by induction on e.

Step 1 : For $e = 0$, we get $v = r = 1$. Thus

$$v - e + r = 1 - 0 + 1 = 2$$

Hence result is true for $e = 0$

Step 2 : Let $e \geq 1$. Assume that the result is true for all connected planar graphs with less than e edges. Let G be a graph with v vertices, e edges and r regions

Step 3 : Case 1 : If G has a pendent vertex say x then $G - \{x\}$ is a connected graph with $v - 1$ vertices, $e - 1$ edges and r regions.

So by induction hypothesis

$$(v - 1) - (e - 1) + r = 2$$

$$v - e + r = 2$$

Case 2 : If G has no pendent vertex the G is a connected graph with circuit. Let e_1 be the edge of a circuit in G. Then $G - \{e_1\}$ is a connected graph with v vertices, $e - 1$ edges and $r - 1$ regions {If we remove edge from a circuit, then it reduces region by 1}.

By induction hypothesis

$$v - (e - 1) + (r - 1) = 2$$

$$\Rightarrow \quad v - e + r = 2$$

Thus by the principle of mathematical induction the result is true for all e.

Corollary 1 : If G (V, E) is a simple connected planar graph with v vertices and e edges then $e \leq 3v - 6$

Proof : Give that, G is a simple planar graph, so each region of G is bounded by three or more edges.

If G has r number of regions then the total number of edges in G is $e \geq 3r$.

Also each edge of G is included in exactly two regions of G. therefore $2e \geq 3r$

$$\Rightarrow \quad \frac{2e}{3} \geq r$$

Substitute these values in Euler's theorem, we get

$$v - e + r = 2$$

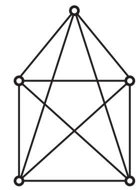
$$v - e + \frac{2e}{3} \geq 2$$

$$3v - e \geq 6$$

$$e \leq 3v - 6$$ Hence the proof.

Corollary 2 : Prove that, K_5 (the complete graph on 5 vertices) is not planar.

Proof : The complete graph on 5 vertices K_5 is given below :



K_5

Fig. 7.16.4

K_5 has 5 vertices and 10 edges. i.e. v = 5 and e = 10.

Now $3v - 6 = 15 - 6 = 9$

By corollary 1, $e \leq 3v - 6$

$10 \leq 9$ which is impossible.

Therefore K_5 is not planar graph.

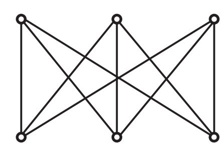
K_5 is the smallest planar graph with respect to number of vertices.

Consider the graph $K_{3, 3}$

Here $v = 6, e = 9,$

$3v - 6 = 18 - 6 = 12 > 9 = e$

i.e. $e \leq 3v - 6$



$K_{3, 3}$

Fig. 7.16.5

But $K_{3, 3}$ is not a planar graph.

$\therefore$ The graph $K_{3, 3}$ is the smallest non planar graph with respect to number of edges.

The graph K_5 is called the Kuratowski's first graph and $K_{3, 3}$ is called the Kuratowski's second graph.

In 1930, Kuratowski gave a necessary and sufficient condition for a graph to be planar.

Kuratowski's Theorem : A graph G is a planar if G does not contain any subgraph that is isomorphic to within vertices of degree two to either K_5 or $K_{3, 3}$

Two graphs are said to be isomorphic to within vertices of degree two if they are isomorphic or they can be reduced to isomorphic graphs by repeated insertion of vertices of degree 2 or by merging the edges which have exactly one common vertex of degree 2.

For example the following graph are isomorphic to within vertices of degree 2. (Homeomorphic)

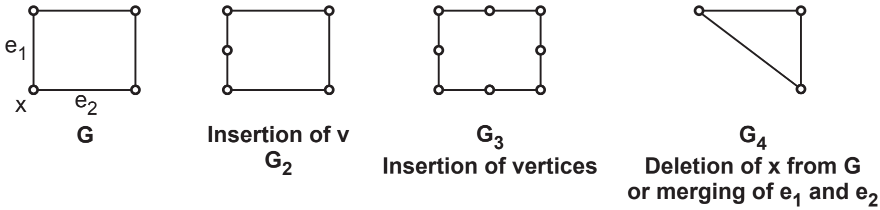


Fig. 7.16.6

Examples :

Example 7.16.1 *Every planar graph with at least 3 vertices contains a vertex of degree at the most 5.*

Solution : Without loss of generality assume that G is a plannar connected graph with v vertices and e edges.

If $v \leq 6$ there is nothing to prove.

Let $v > 6$ and there is no vertex of degree ≤ 5 then $2e = \sum_{x_i \in v(G)} d(x_i) \geq 6v$

i.e. $6v \leq 2e$ but $e \leq 3v - 6$

$\Rightarrow \quad 6v \leq 2e \leq 6v - 2$

$6v \leq 6v - 12$ which is impossible.

Hence there is a vertex of degree at the most 5.

Example 7.16.2 *Draw a planar representation of graphs given below if possible.*

SPPU : May-06, Dec.-09

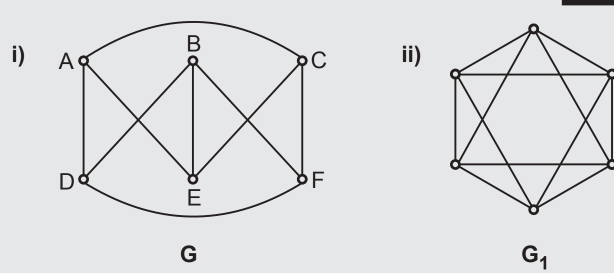


Fig. 7.16.7

Solution : The planar representation of G_1 and G_2 is as follows :

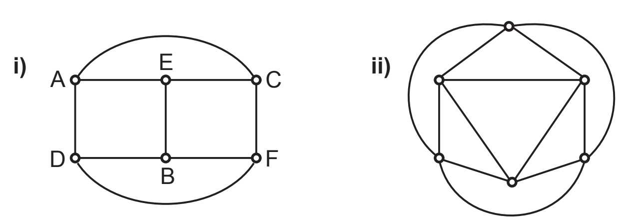
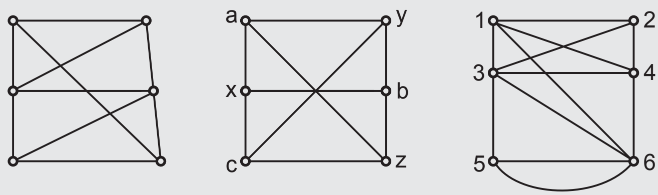


Fig. 7.16.7 (a)

Example 7.16.3 *Identify whether the graphs are planar or not Justify ?* **SPPU : Dec.-08**



(i) (ii) (iii)

Fig. 7.16.8

Solution : i)

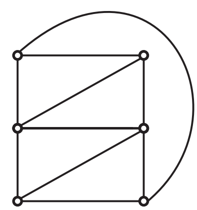


Fig. 7.16.8 (a)

Given graph is planar graph.

ii) Given graph is isomorphic to $K_{3,3}$

$\therefore$ Given graph is not planar.

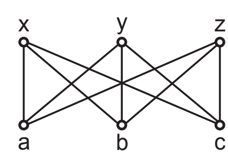


Fig. 7.16.8 (b)

iii)

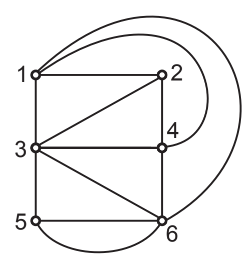


Fig. 7.16.8 (c)

$\therefore$ Given graph is planar.

Example 7.16.4 *Without drawing graph, prove that G is non planar graph.*

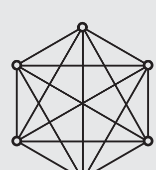


Fig. 7.16.9

Solution : Given graph is a complete graph on 6 vertices G has $\frac{n(n-1)}{2} = \frac{6 \times 5}{2} = 15$ edges.

$\therefore$ v = 6 and e = 15

$$3v - 6 = 3 \times 6 - 6 = 12$$

We know that for a planar graph $e \leq 3v - 6$

Here $15 \leq 3v - 6 = 12$ which is impossible.

Hence G is not planar graph.

Example 7.16.5 *Show that in a connected planar graph with 6 vertices, 12 edges each of region is bounded by 3 edges.* **SPPU : Dec.-10, 13**

Solution : According to Eulers theorem for planar graphs.

$$v - e + r = 2$$

Here $v = 6, e = 12$

$6 - 12 + r = 2 \Rightarrow r = 8$

We know that, each edge contributed twice in a regions we have 12 edges.

So $12 \times 2 = 24$ edges are distributed among 8 regions.

$\Rightarrow \frac{24}{8} = 3$ edges for each region.

So each region is bounded by 3 edges.

Example 7.16.6 *Prove that $K_{3,3}$ is not planar graph.*

Solution : $K_{3,3}$ has 6 vertices and 9 edges. Suppose $K_{3,3}$ is planar, then the boundary of each region has at least 4 edges because it is bipartite and contains no triangles. Each edge lies on boundary of two regions.

Therefore, $2e \geq \sum_{i=1}^{r}$ (the number of edges in the i^{th} region)

$$2e \geq 4r$$

$$2e \geq 4(2 + e - v)$$

$\Rightarrow \quad e \leq 2v - 4$

But $\quad e = 9$ and $v = 6$

$\therefore \quad 9 \leq 12 - 4 = 8$ which is impossible.

Hence $K_{3,3}$ is not planar graph.

7.17 Coloring of Graphs

The coloring of all vertices of a connected graph such that adjacent vertices have different colors is called a proper coloring or vertex coloring or simply a coloring of graphs.

A graph G is said to be properly colored graph if each vertex of G is colored according to a proper coloring.

e.g. 1) Consider the following graphs with proper coloring

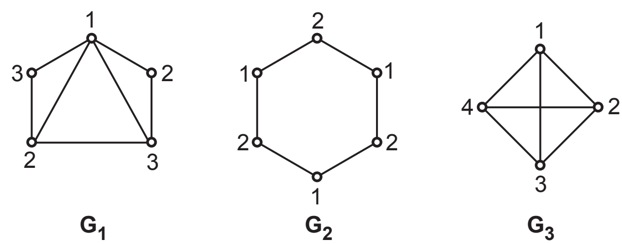


Fig. 7.17.1

7.17.1 Chromatic Number of Graph

The chromatic number of a graph G is denoted by X(G) and defined as the minimum number of colors required to color the vertices of G so that the adjacent vertices get different colors.

A graph G is said to be K-colorable if all vertices of G can be properly colored using at most K different colors. Obviously, a K-colorable graph is K+1 colorable.

If G is k-colorable then X (G) ≤ K.

e.g. In above example (1) X(G) = 3, X(G_2) = 2, G(G_3) = 4. If G is any graph with x(G) = K then the addition or deletion of loops or multiple edges do not change the chromatic number of that graph. Thus hereafter for a coloring of problem we consider only simple connected graphs.

In general, proper coloring of graph is very plain or simple but a proper coloring with minimum number of colors is not an easy task. We list some observation of coloring of graphs as follows :

i) Null graph is 1 - chromatic.

ii) The complete graph on n vertices is n-chromatic.

iii) Bipartite graph is 2-chromatic.

iv) Cycle graph with even edges is 2-chromatic and odd edges is 3-chromatic.

v) If graph is r-chromatic then it has r distinct partitions and that graph is called r-partite graph.

7.17.2 Chromatic Polynomial

We have studied the properly coloring of graph in many different ways using a sufficiently large number of colors.

The chromatic polynomial of a graph is denoted by P_n (λ) and defined as the number of ways of properly coloring of graph using λ or fewer colors

e.g. 1) The chromatic polynomial of the complete graph K_1 is P_1 $(\lambda) = \lambda$.

2) The chromatic polynomial of the complete graph K_n on n vertices is

$$P_n\ (\lambda) = \lambda\ (\lambda - 1)\ (\lambda - 2)\ \dots\ (\lambda - n + 1)$$

7.17.3 Coloring of Planar Graph

A map or atlas is a plane representation of a connected planar graph. Two regions of a planar graph G are said to be adjacent if they have an edge common.

The coloring of a planar graph or map means an assignment of a color to each region of a planar graph G such that adjacent regions have different colors.

A planar graph is - n colorable if minimum n different colors are required to color graph G.

Theorem 1 : (Four Color Theorem)

Every planar graph is 4 - colorable.

Initially it was a conjecture, but in 1979 Appel and Haken proved this. That's why this conjecture became theorem.

7.17.4 Open Problem of Coloring

A lot of research is done in the coloring of planar graphs, particularly coloring of vertices, or edges or regions of a planar graph.

The following open problem is stated by Dr. H. R. Bhapkar and proved partially first time.

Open Problem :

How many minimum colors will be required to color planar graph such that

i) Adjacent vertices have different colors

ii) Incident edges have different colors.

iii) Adjacent regions have different colors.

iv) A region, boundary edges and boundary vertices of that region have different colors.

This type of coloring is known as perfect coloring of G and denoted by PC (G).

We list some observations of perfect coloring of planar graph as follows :

i) If G is a null graph then PC (G) = 2

ii) If G is a chain graph when n vertices then,

$$pc\ (G) = \Delta\ (G) + 2$$

where $\Delta\ (G)$ = Highest degree of a vertex in G.

7.18 Web Graph

A directed graph whose nodes correspond to static pages on the web and whose arcs correspond to links among these pages, is called web graph.

There are severel reasons for the developement of this graph, Few of them are as follows :

i) Designing crawl strategies on the web.

ii) Understanding of the sociology of content creation on the web.

iii) Analyzing the behavior of web algorithms that make use of link information.

iv) Predicting the evolution of web structures such as bipartite cores and webrings and developing better algorithms for organizing and discovering them.

v) Predicting the emergence of very important new phenomena in the web graph.

We detail a number of experiments on a web crawl of approximately 200 million pages and 1.5 billion links. So the scale of this experiment is very large.

7.19 Graph Database

In computing, a graph database is database that uses graph structures for semantic queries with nodes, edges and properties to represent and store data. A key concept of the system is the graph (or edges or relationships) which directly relates data items in the store.

Alternately, Graph databases are based on graph theory graph databases employ nodes, edges and properties.

i) Nodes : Nodes representes entities such as people businesses, accounts or any other item you might want to keep track of. They are roughly the equivalent of the record, relations, the documents in a document database.

ii) Edges are known as relationships are the lines that connect nodes to other nodes, they represent the relationship among them. Edges are the key concepts in graph databse, representing an abstraction that is not directly implemented in other systems.

iii) Properties are pertinent information that relate to nodes.

Examples :

1) The set at all the users who's phone number contains the area code "141".

2) The set of all trees in the world with similar properties.

7.20 Google Map

- A google map is a big giant graph with nodes and edges to find out fastest and shortest route from source to destination.
- A graph theory can be applied to find fastest and shortest route in the following way.

1) A shortest path from source to destination - Dijkstra's Algorithm.

- We consider the weighted graph where the vertices are the towns, cities (sources) and edges are the roads. The weight between two nodes is the distance between two cities. Therefore weights are non-negative. The problem is find a path of minimum weight connecting two given cities. One has to list all paths, find their weights and select minimum one. But for a large network, this is not efficient. Dijkstra's algorithm is used to find out the shortest path from the vertex 'a' to the vertex 'z' of the graph. A detailed explanation of Dijkstra's algorithm is in section 7.13.

2) The Busiest Interconnection : The degree of vertex of directed graph is used to find out busiest interconnection. Each vertex in a directed graph and in degree and out degree. So such type of node (vertex) can be avoided during travel.

3) Find the nearest location :

Fig. 7.20.1

7.21 Case Studies

1) Three Utility Problem : It is a classical mathematical puzzle. It is also called as three cottages problem or water, gas and electricity problem. In 1917, Henry Dudeney states this problem is 'water, gas and electricity'. This problem is stated as follows :

- Suppose that there are three cottages H_1, H_2 and H_3 and each needs to be connected to each of the three utilities Water (W), Gas (G) and Electricity (E). Is there any way to make nine connections without any crossover of lines ?
- Consider the following graph :

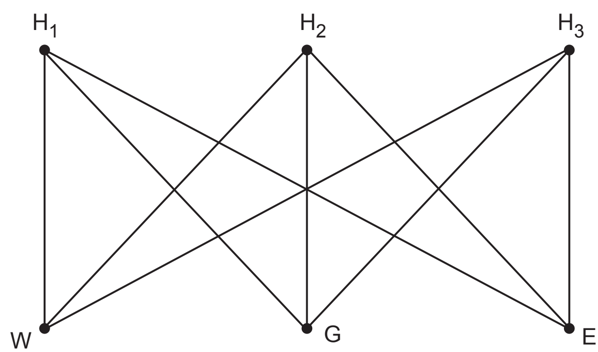


Fig. 7.21.1

- The above graph is complete bipartite graph $K_{3,3}$. Three, utility problem is a question whether this $K_{3,3}$ graph is planar. But $K_{3,3}$ is not a planar graph. In other words, there is no way to make all nine connections without line crossover. Some attempts are as follows :

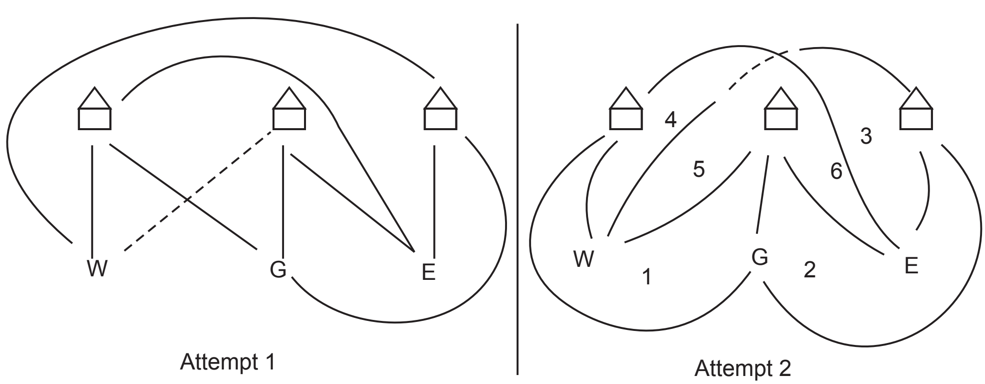


Fig. 7.21.2

- There is at least two lines crosses each other.

Proof : Eular's formula exists for planar graph as

$$F + V = E + 2 \text{ where}$$

$$V = \text{No. of vertices}$$

$$F = \text{No. of faces}$$

$$E = \text{No. of edges}$$

In above graph,

$$V = 6, E = 9, F = 4$$

$$\therefore \quad 4 + 6 \neq 9 + 2$$

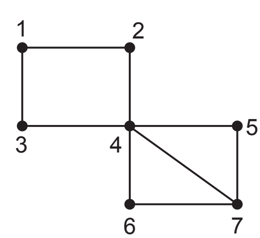
- Hence above graph is not planer. Hence there is no way to make all nine connection without line crossover.

2) Web graph : Refer section 7.18.

7.22 University Questions with Answers

Dec. 2016

Q.1 *Determine which of the graphs below represents Eulerian circuit, Eulerian path, Hamiltonian circuit and Hamiltonian path. Justify your answer.* **[4]**



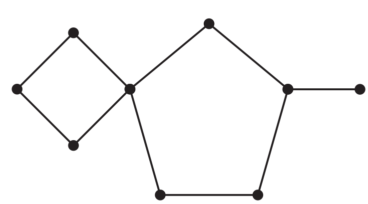


Fig. 7.22.1

Ans. : We know that,

i) An undirected graph possesses an Eulerian path iff it is connected and has either zero or two vertices of odd degree.

ii) An undirected graph possesses an Eulerian circuit iff it is connected and its vertices are all at even degree.

For graph (1)

Eulerian path : 4 - 3 - 1 - 2 - 4 - 5 - 7 - 4 - 6 - 7

Eulerian circuit does not exist.

Hamiltonian path and circuit do not exists.

For graph (2)

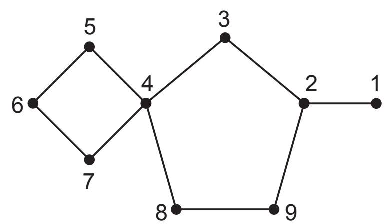


Fig. 7.22.2

Eulerian path : 1 - 2 - 3 - 4 - 5 - 6 - 7 - 4 - 8 - 9 - 2

Eulerian circuit does not exist.

Hamiltonian path and circuit do not exists.

Q.2 *Define the graph K_n and K_{mn}* . **(Refer section 7.5 (ii) and (iv))** **[2]**

Q.3 *Use Dijkstra's algorithm to find the shortest path between a and z.* **(Refer example 7.13.3)** **[6]**

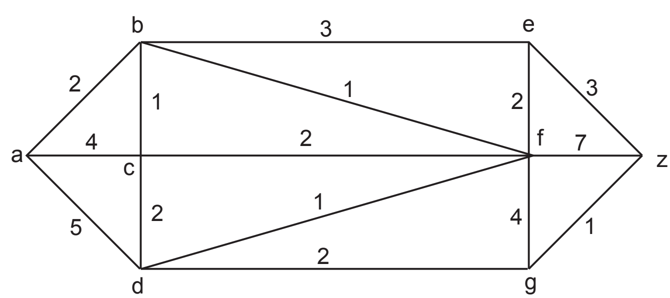
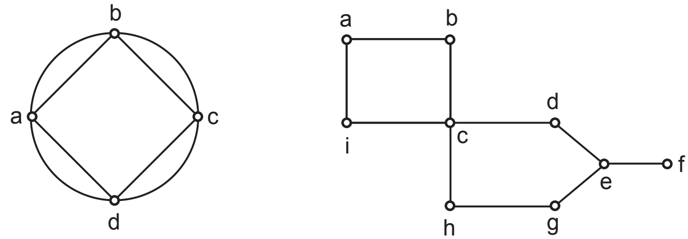


Fig. 7.22.3

May 2017

Q.4 *Check whether the graph has an Euler circuit, Euler path, Justify :* **[3]**



(Graph G_1) (Graph G_2)

Fig. 7.22.4

Ans. : For graph G_1 :

Enter circuit : a-b-c-d-a-b-c-d-a

Enter path : a-b-c-d-a-b-c-d-a

For graph G_2 :

Enter circuit does not exist

Enter path : f-e-g-h-c-i-a-b-c-d-e

Q.5 *How many colours required to colour $k_{m,n}$ why ?* **[3]**

Ans. : In $k_{m,n}$ graph, $V(k_{m,n}) = V_1 \cup V_2$

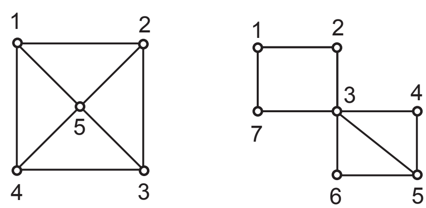
V_1 and V_2 are the partitions of $V(k_{m,n})$

$\therefore$ Vertices in V_1 are not adjacent to each other.

So assign color 1 to all vertices of V_1 and color 2 to all vertices of V_2

$\therefore$ Every $k_{m,n}$ is two colorable graph.

Q.6 *Determine whether the following graph has Hamiltonian circuit of Hamiltonian path.* **(Refer example 7.15.5)** **[3]**



(Graph 1) (Graph 2)

Fig. 7.22.5

Q.7 *Write 5 applications of graph theory in the field of data analytics.* **[3]**

Ans. : Applications of graph theory in the field of data analytics are given below.

1) Project Evaluation Review Technique (PERT).
2) Critical Path Method (CPM)
3) Game theory
4) Operations research
5) Combinatorial problems.

Dec 2017

Q.8 *Explain the directed and undireted grpah with suitable example.* **(Refer section 7.3(6))** **[4]**

Q.9 *Explain the Dijkstra's algorithm in details.* **(Refer section 7.13)** **[4]**

Q.10 *Define subgraph.*
Determine whether $H = H' = (V', E')$ is a subgraph of G(V, E) shown in Fig. 7.22.6 :
i) $V' = \{A, B, F\}$
$E' = \{(A, B), (A, F)\}$
ii) $V' = \{B, C, D\}$
$E' = \{(B, C), (B, D)\}$ **[4]**

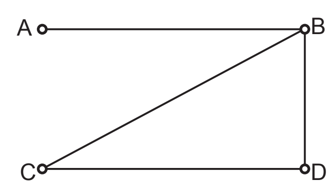


Fig. 7.22.6

Ans. : Refer section 7.8 (1).

Given that

$V = \{A, B, C, D\}$, $E = \{(A, B), (B, C), (B, D), (C, D)\}$

i) We have $V' = \{A, B, F\}$

As $F \in V'$ but $F \notin V$

$\therefore$ H is not a subgraph of G.

We have $V' = \{B, C, D\}$ $\therefore V' \subseteq V$

and $E' \{(B, C), (B, D)\}$ $\therefore E' \subset E$

$\therefore$ H is a subgraph of G.

May 2018

Q.11 *Consider a graph G(V, E) where $V = \{v_1, v_2, v_3\}$ and $\deg(v_2) = 4$:*
i) Does such simples graph exists ? If not, why ?
ii) Does such a multigraph exists ? If yes, give example. **[4]**

Ans. : Given that $V = \{v_1, v_2, v_3\}$, $\deg(v_2) = 4$

i) G_e has 3 vertices and degree of v_2 is 4.

V_2 has either a loop or multiple edges.

$\therefore$ G_e is not a simple graph.

ii) Multigraph with given conditions exists. Examples of such graphs are given below -

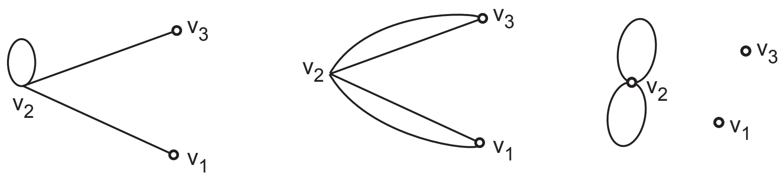


Fig. 7.22.7

Q.12 *Explain the following in brief :*

i) Subgraphs and spanning subgraph **(Refer section 7.8 (1) and (4))**

ii) Isomorphic graph **(Refer section 7.7)**

iii) Bipartite graph **(Refer section 7.5 (iii))**

iv) Adjacency matrix and incidence matrix of undirected graph.

(Refer sections 7.4.1 and 7.4.2) **[4]**

Q.13 *Apply Dijkastra's algorithm to find the shortest path from vertex v_1 to v_5 in the graph shown below in Fig. 7.22.8.* **[4]**

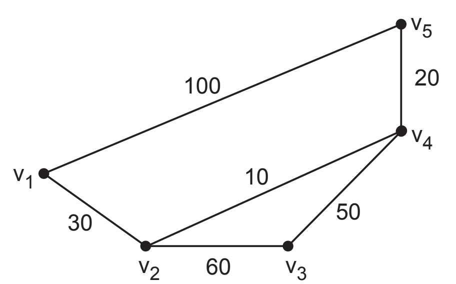


Fig. 7.22.8 Weighted graph

Ans. : Step 1 : P = ϕ and T = $\{v_1, v_2, v_3, v_4, v_5\}$

$L\{v_1\} = 0$, $L\{x\} = \infty$, $\forall$ $x \in T$, $x \neq v_1$

Step 2 : $v = v_1$, the permanent lable of v_1 is 0.

$P\{v_1\}$, $T = \{v_2, v_3, v_4, v_5\}$

$$L\{v_2\} = \min\{\text{old } L(v_2), L(v_1) + w(v_1, v_2)\}$$

$$= \min\{\infty, 0 + 30\} = 30$$

$$L\{v_3\} = \min\{\infty, 0 + \infty\} = \infty$$

$L\{v_4\} = \min\{\infty, 0 + \infty\} = \infty$

$L\{v_5\} = \min\{\infty, 0 + 100\} = 100$

$L\{v_2\} = 30$ is the minimum label.

Step 3 : $v = v_2$, the permanent label of v_2 is 30

$P = \{v_1, v_2\}$; $T = \{v_3, v_4, v_5\}$

$$L\{v_3\} = \min\{\text{old } L(v_3),\ L(v_2) + w(v_2, v_3)\}$$

$$= \min\{\infty, 30 + 60\} = 90$$

$L\{v_4\} = \min\{\infty, 30 + 10\} = 40$

$L\{v_5\} = \min\{\infty, 30 + \infty\} = \infty$

$L\{v_4\} = 40$ is the minimum label

Step 4 : $v = v_4$ the permanent label of v_4 is 40.

$P = \{v_1, v_2, v_4\}$; $T = \{v_3, v_5\}$

$L\{v_3\} = \min\{\infty, 40+50\} = 90$

$L\{v_5\} = \min\{\infty, 40+20\} = 60$

$L\{v_5\}$ is the minimum label.

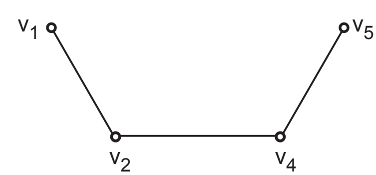


Fig. 7.22.9

Step 5 : $v = v_5$ the permanent label of v_5 is 60.

Hence the length of the shortest path from v_1 to v_5 is 60. The shortest path is $v_1 - v_2 - v_4 - v_5$.

Dec. 2018

Q.14 *Determine whether the two graphs are isomorphic or not. Explain.* **(Refer example 7.7.6)** **[6]**

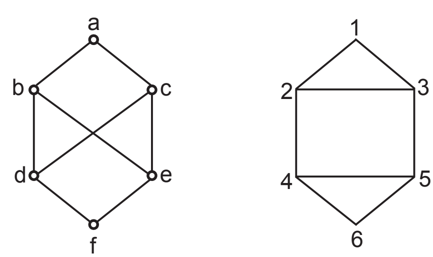


Fig. 7.22.10

Q.15 *Use Dijkstra's algorithm to find the shortest path between A and Z in figure :* **(Refer example 7.13.2)** **[6]**

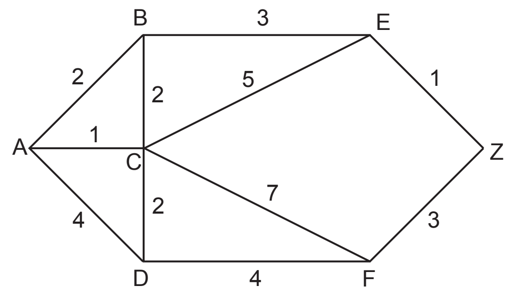


Fig. 7.22.11

May 2019

Q.16 *Use dijkstra's algorithm to find the shortest path between a and z.* **(Refer example 7.13.1)** **[6]**

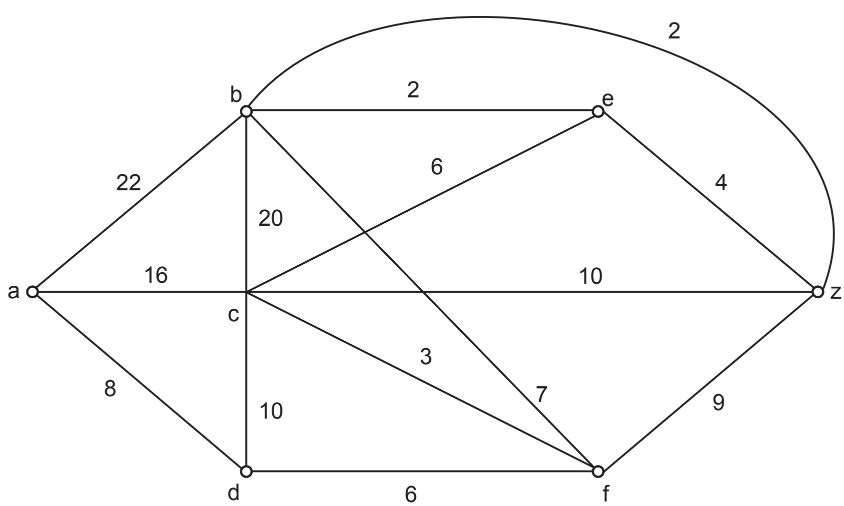


Fig. 7.22.12

Q.17 *Under what condition K_{mn} will have eulerian circuit ?* **(Refer example 7.14.3)** **[3]**

Q.18 *The graphs G and H with vertex sets V(G) and V(H), are drawn below. Determine whether or not G and H drawn below are isomorphic. If they are isomorphic, given a function g : V(G) –> V(H) that defines the isomorphism. If they are not explain why they are not.* **[3]**

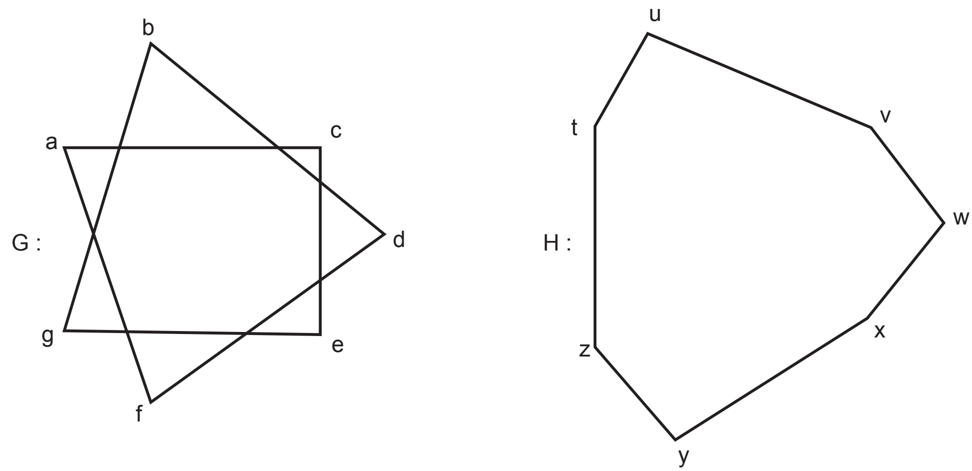


Fig. 7.22.13

Ans. : Number of vertices in both graphs = 7

Number of edges in both graphs = 7

Both graph have 7 vertices of degree 2

ϕ : {a, b, c, d, e, f, g} $\rightarrow$ {t, u, v, w, x, y, z}

Such that,

a $\rightarrow$ u, b $\rightarrow$ x, c $\rightarrow$ t, d $\rightarrow$ w, e $\rightarrow$ z, f $\rightarrow$ v, g $\rightarrow$ y

ϕ Preserves adjacency and non-adjacency of vertices

$\Psi \rightarrow E_1 \rightarrow E_2$ is bijective

Hence, both graphs G and H are isomorphic.

Dec. 2019

Q.19 *Can a simple graph exist with 15 vertices, each of degree five ?* **[3]**

Ans. : By the handshaking Lemma

$$2e = \sum_{u \in v} \deg(u) \text{ where}$$

$$e = \text{No. of edges and } v = \text{No. of vertices}$$

It suggests that sum of degrees of all the vertices is even.

But in given example

$$\sum_{u \in v} \deg(u) = 15 \times 5 = 75 \text{ which is odd}$$

Hence simple graph never exists with 15 vertices, each of degree 5.

Q.20 *For which values of n, m are the following graph regular :*
(i) K_n **[3]**

Ans. : A regular graph is a graph where each vertex has same degree

1) K_n - It is a complete graph

Complete graph is regular graph for any n.

Q.21 *Determine whether the following graphs are isomorphic to each other.* **[3]**

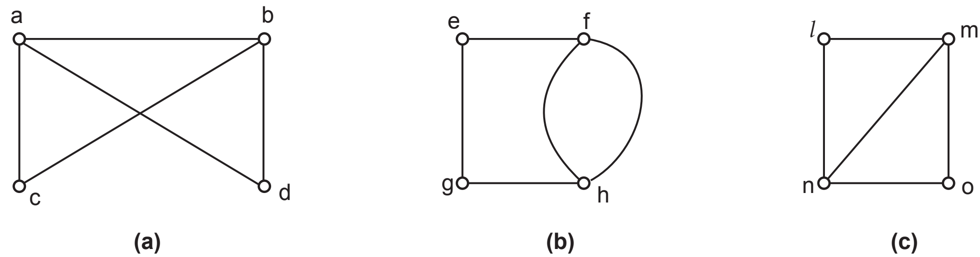


Fig. 7.22.14

Ans. :

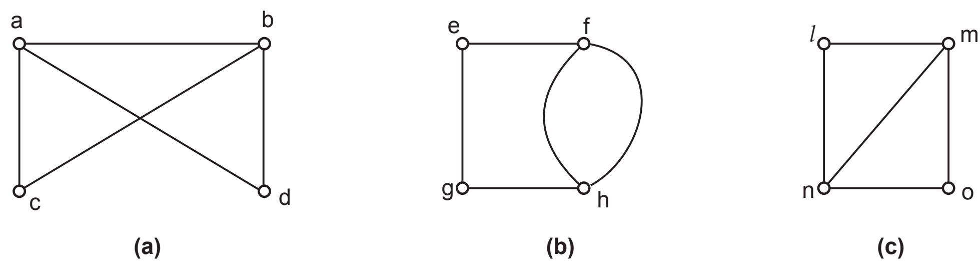


Fig. 7.22.15

All graphs have 4 vertices and 5 edges.

All graphs have 2 vertices of degree 3 and 2 vertices of degree 2.

i) $\therefore \phi : \{a, b, c, d\} \to \{m, n, l, o\}$ such that

$a \to m$

$b \to n$

$c \to l$

$d \to o$

ϕ preservers adjancy and non-adjancy of vertices for graph a and graph c.

$\psi \to E_1 \to E_2$ is bijective

ii) In graph a, vertex having degree 3 is connected to all vertices i.e. b with degree 3, C with with degree 2 and d with degree 2. But in graph B, f and h have degree 3 but not connected to all vertices.

Hence it is not preserving adjancy and non adjancy of vertices for graph a and graph b.

Hence graph a and graph C are isomophic graph. Graph B is not isomorphic to them hence A, B, C are not isomorphic.

Q.22 *How many regions would there be in a plane graph with 10 vertices each of degree 3.* **[3]**

Ans. : V = 10

According to handshake Lemma

$$2e = \sum_{u \in v} \deg(u)$$

$$2e = 10 \times 3$$

$$2e = 30$$

$$\therefore \quad e = 15$$

Total no. of edges = 15

According to Euler's formula

$$V - e + r = 2$$

$$\therefore \quad 10 - 15 + r = 2$$

$$\therefore \quad r = 7$$

$$\boxed{\text{Total no. of regions} = 07}$$

❑❑❑

Notes

UNIT - V

8 Trees

Syllabus

Introduction, Properties of trees, Binary search tree, Tree traversal, Decision tree, Prefix codes and Huffman coding, Cut sets, Spanning trees and minimum spanning tree, Kruskal's and prim's algorithms, The max flow-min cut theorem (Transport network). Case studies : Algebraic Expression Tree, Tic-Tac-Toe Geme Tree

Contents

8.1 Introduction

In this chapter we will study one of the simplest types of the connected graphs known as trees. This class of graphs has wide applications and has been the subject of study of many outstanding scientiests of different fields. Trees are discovered by Kircnnoff in 1847, while investigating the electrical networks. Sir Arthur Cayley used trees to study and enumerate isomers of saturated hydrocarbons.

The important applications of tree include searching, sorting, syntax checking and database managements. The tree is one of the most non-linear structures used for algorithm development in computer science.

8.2 Tree

SPPU : Dec.-16, May-17,18

A tree is a connected graph without any circuit i.e. tree is a connected acyclic graph. The collection or set of an acyclic graphs. (not necessarily connected) is called a forest.

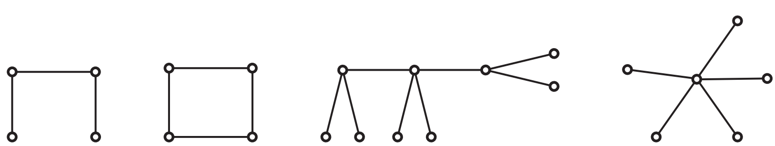
Examples :

Example 1

(i) (ii) (iii)

All these three graphs are trees. There are unique tree on one vertex, 2 vertices and 3 vertices.

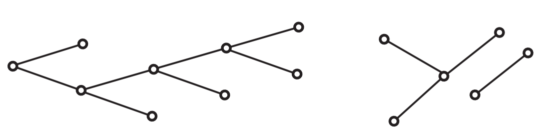
Example 2



(i) (ii) (iii) (iv)

Graphs (i), (iii) and (iv) are trees but (ii) is not a tree as it has a cycle.

Example 3 : Consider the following graph G.

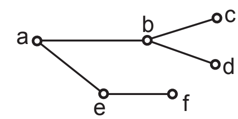


G

This graph G is not a tree because it is not connected graph. But G has a connected components i.e. G has 3 components which are trees. Therefore G is a forest.

A) A vertex of degree 1 in a tree is called a **leaf** or a **terminal node.** A vertex of a degree greater than one is called a **branch node** or **internal node.**

e.g. In a tree



c, d and e are leaves or terminal nodes and a, b, c are branch nodes.

B) Some properties of tree are obvious

i) Every edge of a tree is an isthmus. Conversely if every edge in a connected graph G is an isthmus then G is a tree.

ii) Every tree is a simple graph because loops and multiple edges field cycles.

iii) Every tree is a bipartite graph with $n \geq 2$ vertices.

Theorem 1 : A graph G is a tree iff there exists a unique path between every distinct pair of vertices of G.

Proof : Suppose G is a tree. So G is a connected and without circuits. We know that a circuit forms two or more paths. But G has no circuit so it has unique path between every pair of vertices.

Conversely assume that there is a unique path between every pair of vertices in G. This implies that G is a connected and G has no cycles. i.e. G is a connected acyclic graph.

Therefore G is a tree.

Theorem 2 : A graph G on n vertices is a tree iff G is connected and has exactly $n - 1$ edges.

Proof : Suppose graph G is a tree. Therefore is a connected and acyclic graph.

It is sufficient to prove that G has $n - 1$ edges only.

We can prove this by induction principle on n.

For $n = 1$, the result is obvious

Let $n > 1$, consider $G - e'$ for any $e' \in E(G)$

As G is a tree, e' is an isthmus of G.

$\therefore G - e'$ is a disconnected graph with two components say G_1 and G_2.

Now G_1 and G_2 are connected and a cyclic graphs as G_1 and G_2 are subgraph of G.

Let G_1 has n_1 vertices and m_1 edges and G_2 has n_2 vertices and m_2 edges

$\therefore$ By induction principle

$$m_1 = n_1 - 1 \qquad \text{and} \qquad m_2 = n_2 - 1$$

Therefore the number of edges in G is given by

$$e = (e_1 + e_2) + 1 = n_1 - 1 + n_2 - 1 + 1$$

$$= n_1 + n_2 - 1$$

$$e = n - 1 \qquad (\because \; n = n_1 + n_2)$$

Hence G has n – 1 edges only.

Conversely assume that G is connected graph and has exactly n – 1 edges.

Claim : Prove that G is a tree.

It is suffices to prove that G is noncyclic graph.

Suppose G contains a cycle C.

Let P denote the number of vertices in C

∴ The number of edges in C = P

As G is connected graph, the remaining n – P vertices must be connected to vertices in C. To connect each vertex of G which is not in C, we required n – P edges as each edge of G can connect only one vertex to the vertices in C.

Hence the total number of edges in G is given by

$$e = (n - P) + P = n \Rightarrow e = n \text{ which is contradiction}$$

∴ G has n – 1 edges only.

Thus G is a tree.

Theorem 3 : Let G be a graph with n vertices and m edges. If any of the following is true then all are true.

i) G is a tree.

ii) G is connected and m = n – 1

iii) G is acyclic graph and m = n – 1

iv) Every edge of G is an isthmus and G is connected.

v) There is exactly one path between every pair of vertices in G.

Theorem 4 : A non trivial tree contains at least two vertices of degree 1. (i.e. pendent vertices)

Proof : Let G be a tree with n vertices, so G is connected.

$$\therefore \quad d(x) \geq 1 \;\; ; \;\; \forall x \in V\;(G)$$

By handshaking lemma

$$\sum d(x) = 2 \times \text{Number of edges in G} = 2(n-1)$$

$$x \in V(G)$$

Suppose there is no vertex of degree 1.

Then $\quad 2n \le \sum_{x \in V(G)} d(x) = 2n - 2$

i.e. $\quad 2n \le 2n - 2 \Rightarrow n \le n - 1$ which is impossible.

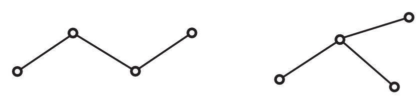
Thus there is at least one vertex of degree 1.

By a similar argument there is one more vertex of degree 1.

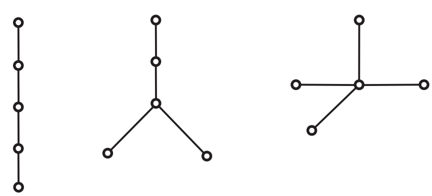
Hence every tree has at least two pendent vertices.

Example 8.2.1 *Draw all non isomorphic trees on 4 and 5 vertices.*

Solution : a) Non isomorphic trees on 4 vertices



b) Non isomorphic trees on 5 vertices



Example 8.2.2 *Is it possible to draw a tree with five vertices having degrees 1, 1, 2, 2, 2, 4 ?*

Solution : According to given degrees graph G has 6 vertices and total degree of graph is 1 + 1 + 2 + 2 + 2 + 4 = 12. So G has 6 edges as each edge contributes degree two.

Therefore G is a graph with 6 vertices and 6 edges which is contradiction to the fact that a tree on 6 vertices has 6 – 1 = 5 edges.

Hence the tree with given degree sequence of vertices does not exist.

Example 8.2.3 *Under what conditions trees are the complete bipartite graphs.*

Solution : Suppose T is a tree which is the complete bipartite graph.

Let $\quad T = K_{m,\, n}$

$\therefore$ T has m + n vertices and $(m + n - 1)$ edges.

But $K_{m,\, n}$ has $m \cdot n$ number of edges.

Therefore $\quad mn = m + n - 1$

$$mn - m - n + 1 = 0$$

$$(m - 1)(n - 1) = 0$$

$\Rightarrow$ $m = 1$ or $n = 1$

i.e. $m = 1$ and any n or $n = 1$ and any m

Hence $K_{1,n}$ and $K_{m,1}$ are the only complete bipartite graphs. These are known as star graphs.

Thus T is a star graph.

Example 8.2.4 *a) Is it possible to draw a tree with 10 vertices which has vertices either of degree 1 or 3 ?*

If possible draw tree. Is it possible to draw same type of tree with 13 vertices ?

b) For which values of n (number of vertices), such type of tree exist ?

Solution : Given that tree T has 10 vertices so it must have 9 edges.

Let x and y be the number of vertices of degree 1 and 3 in T respectively.

$$\therefore \quad x + y = 10 \qquad \text{... (1)}$$

By handshaking lemma

$$\sum_{V \in V(T)} d(v) = 2(\text{Number of edges in T})$$

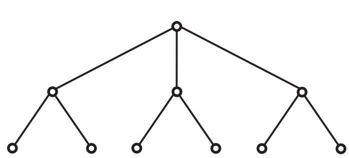
$$x + 3y = 2 \times 9 = 18$$

$$x + 3y = 18 \qquad \text{... (2)}$$

Solving equations (1) and (2) we get

$$y = 4 \quad \text{and} \quad x = 6$$

Therefore there are 6 vertices of degree 1 and 4 vertices of degree 3. Such type of graph is given below.



Now consider a tree with 13 vertices and 12 edges.

By using similar theory, we get

$$x + y = 13 \qquad \text{... (3)}$$

$$x + 3y = 24 \qquad \text{... (4)}$$

Solving (3) and (4), we get $1y = 11 \Rightarrow y = \frac{11}{2}$ and $x = \frac{15}{2}$ which is impossible.

Therefore it is not possible to draw such type of tree.

b) Let T be a tree with n vertices.

Let x and y be the number of vertices of degree 1 and 3 in T respectively. T has $n - 1$ edges.

Therefore $\quad x + y = n \quad$ ... (5)

By handshaking lemma $\quad x + 3y = 2(n - 1)$

$$x + 3y = 2n - 2 \quad \text{... (6)}$$

Equation (6) – Equation (5) $\Rightarrow 2y = 2n - 2 - n$

$$2y = n - 2$$

$$\Rightarrow \quad y = \frac{n-2}{2} = \frac{n}{2} - 1$$

$$\Rightarrow \quad x = n - y = n - \frac{n}{2} + 1 = \frac{n}{2} + 1$$

i.e. $x = \frac{n}{2}+1$ and $y = \frac{n}{2}-1$. These should be non negative integers.

Case 1 : If n is even then $x = \frac{n}{2}$ is odd

and $\quad \frac{n}{2}+1 = x$ is an even integer

$\frac{n}{2}-1 = y$ is an even integer

Thus the tree exists with required condition on even number of vertices.

$T_2, T_4, T_6, T_8 \ldots T_{2n}$ are such type of trees.

Case 2 : If n is odd then $\frac{n}{2}$ is not an integer there x and y are not an integer.

Hence the required tree does not exist on odd number of vertices.

8.3 Centre of a Tree

SPPU : Dec.-18

We know that the distance between two vertices in a connected graph i.e. is a length of the shortest path between that vertices. Before defining centre of a tree first find the distance between vertices of a tree which is defined as follows :

8.3.1 Eccentricity of a Vertex

Let G be a connected graph G and $v \in V(G)$. The eccentricity of a vertex v is denoted by E(v) or e(v) and defined as the distance from v to the vertex farthest from v in G.

i.e. $E(v)$ or $e(v) = \max\{d(v, v_i)/\forall v_i \in v(G)\}$

8.3.2 Centre of a Graph

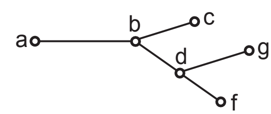
A vertex in a graph G with minimum eccentricity is called a centre of G and it's eccentricity is called as radius of G. It is denoted by r(G).

Consider the following examples

Example 1 :

$e(a) = 3,$ $e(b) = 2,$ $e(c) = 3$

$e(g) = 3,$ $e(d) = 2,$ $e(f) = 3$



G

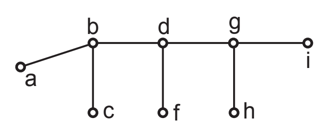
The smallest eccentricity is 2 for vertices b and d

$\therefore$ b and d are centre of G and r(G) = 2

Example 2

$e(a) = 4,$ $e(b) = 3,$ $e(c) = 4,$ $e(d) = 2,$

$e(f) = 3,$ $e(h) = 4,$ $e(i) = 4$



The minimum is 2 $\therefore$ Centre is at d and r(G) = 2

It is observed that the graph has more than one centre.

Theorem 1 : Prove that every tree has either one or two centres.

Proof : Let T be a tree. Then e(v) for a vertex v is at a vertex farthest from v. As T is a tree, it is attained at a pendent vertex.

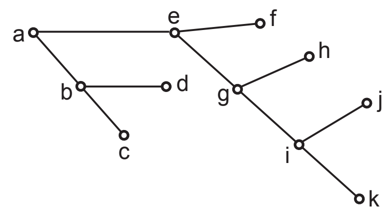
We know that every non trivial tree has at least two pendent vertices. Now, delete all pendent vertices from G, we get new graph G′ which is also a tree. Moreover as one edge at each pendent vertex is removed, the eccentricity of any vertex will be reduced by one. Therefore centres of G will still remain centres of G′. We continue the process with G′ until we arrive at K_2 or K_1. K_2 has two centres and K_1 has one centre. Hence the proof.

8.3.3 Cut Vertex of a Tree

We know that the vertex v whose removal from a connected graph G, disconnects the graph is called as cut vertex of G.

In any tree all vertices except pendent vertices are cut vertices.

e.g. Consider the following tree



G

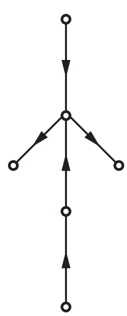
In graph G, a, b, c, g, i are cut vertices and c, d, f, h, j and k are not cut vertices.

8.4 Rooted Tree and Binary Tree

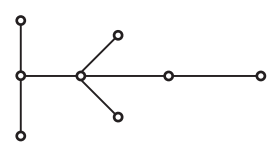
SPPU : Dec.-09, 16, 18, 19, May-18, 19

A connected acyclic, directed graph is called a **directed tree.** In other words, a directed graph is said to be a directed tree if it will become a tree when the directions of the edges are ignored.

e.g.



Directed tree



Non directed tree or tree

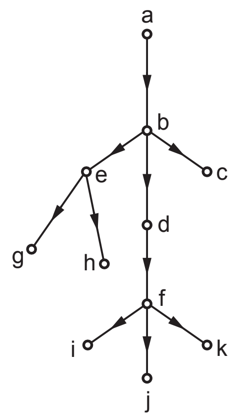
8.4.1 Rooted Tree

A directed tree is called a **rooted tree** if there is exactly one vertex whose incoming degree is zero and the incoming degrees of all other vertices are one.

The vertex with incoming degree 0 is called **the root** of the rooted tree. The vertex whose outgoing degree is zero is called **leaf** or **terminal node.**

A vertex whose outgoing and incoming degrees are non zero is called a **branch note** or an **internal node.** Consider the following example.

Given graph is a rooted tree a is the root of tree. Vertices b, e, d, f are called branch nodes. The vertices c, h, g, i, j and k are leaves.



Rooted tree

8.4.2 Level and Height of a Tree

A vertex v in a rooted tree is said to be at **level n** if there is a path of length n from the root to the vertex v.

The **height** of the tree is the maximum of the levels of its vertices.

Example

In the given graph G .... a is a root

Vertices b, c, d are at level 1

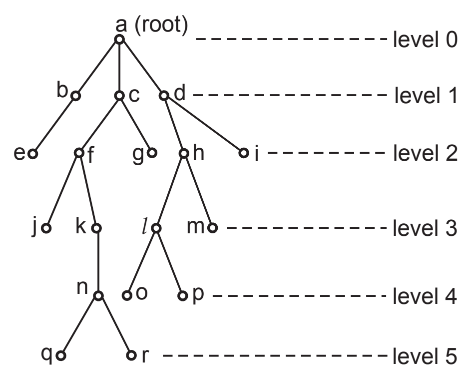
Vertices e, f, g, h and i are at level 2

Vertices j, k, l and m are at level 3

Vertices n, o and p are at level 4

Vertices q and r are at level 5

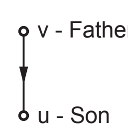
The maximum level is 5. Therefore the height of tree is 5.



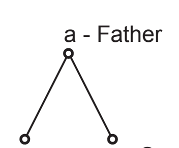
G

Rules for rooted tree : In a rooted tree

i) If the level of a vertex u is greater than the level of vertex v then u is below v.



ii) If vertex u is below vertex v and there is an edge from v to u then u is said to be son of v (or child of v) and v is said to be the father of u (or parent of u) i.e.

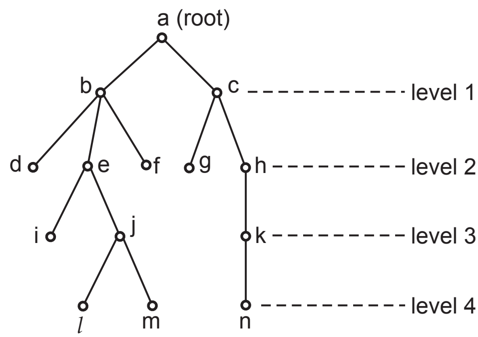


iii) Two vertices u and v are said to be brothers if they are the sons of the same vertex.

iv) A leaf is a vertex without children.

v) If $p = \{a, v_1, v_2, v_3, ... v_{n-1}, b\}$ is a path from a to b then b is called as descendent of a and a is called as ancestor of b.

Consider the following example



From above figure i) a is a root of the tree.

ii) b and c lie at level 1. $\therefore$ b and c are sons of root a i.e. a is father of b and c and b and c are brothers.

iii) b has three sons d, e, f

$\therefore$b is a father of d, e, f

c has 2 sons g and h

iv) e has two sons i and j

h has only one son k

v) i has no son j has two sons l and m. So k and l are brothers. k has one son n. n has no brother as n is a leaf.

l, m, n are known as descendent of a and a is ancestor of d, e, f, g, h, i, j, k, l, m, n and so on.

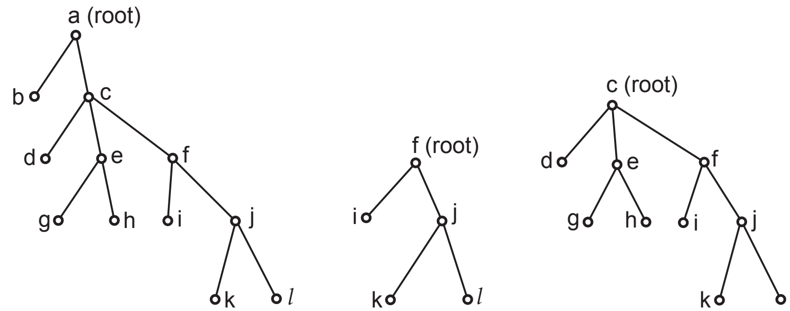
8.4.3 Subtrees

Let T be a rooted tree and $x \in V(T)$.

A vertex x together will all its descendants is called the **subtree** of T rooted at x.

The subtree corresponding to the root node is the **entire tree.** The subtree corresponding to any other node is called a **proper subtree.**

Example



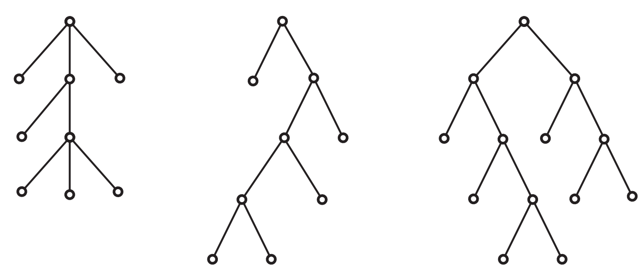
Tree **Subtree with root at f** **Subtree with root at c**

8.4.4 M-ary Trees

A rooted tree in which every interior node has at most m sons is called an **m-ary tree.**

A m-ary tree is said to be **regular m-ary tree** or full m-ary tree if every branch node has exactly m sons.

Consider the following examples.



3 - ary tree **2 - ary tree** **Full 2 - ary tree**

Theorem 1 : A regular m-ary tree with p interior nodes has $mp + 1$ nodes at all.

SPPU : Dec.-09

Proof : Let T be a regular m-ary tree with n vertices. Out of n vertices there are p interior vertices or branch nodes.

Therefore there are $t = n - p$ number of sons or leaves in T.

But given graph is regular and p interior nodes.

So the regular m-ary tree will have mp sons.

But root is not a son.

Therefore give tree has total (mp + 1) number of vertices

Hence $\boxed{n = mp + 1}$

8.5 Binary Tree

SPPU : Dec.-09, 16, May-10, 12, 17

So far we have discussed the tree and its properties. Now, we shall study about a special class of rooted tree known as binary trees. Binary trees play an important role in decision making. They are extensively used in the study of computer search methods, binary identification problems and coding theory. Binary tree is a particular case of m-ary tree.

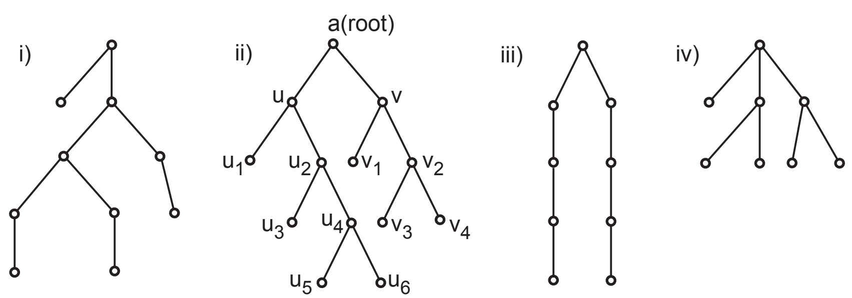
Definition :

An m-ary tree is known as **binary tree** if every branch node has at most 2 sons.

In other words, a tree in which there is exactly one vertex of degree 2 and each of the remaining vertices of degree or three, is called a binary tree.

A binary tree is called as **regular binary tree** or **full binary tree** if every branch node has exactly 2 sons or zero son.

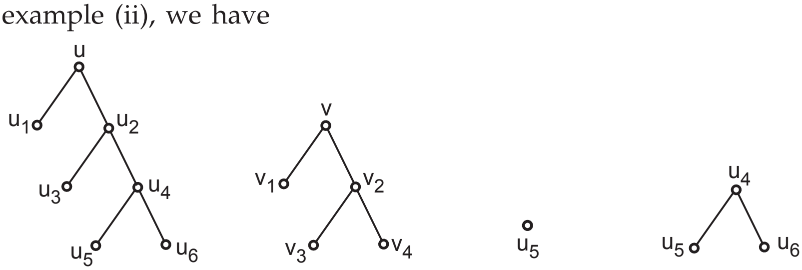
Consider the following examples



Binary tree **Full binary tree** **Binary tree** **Not binary tree**

In binary trees, instead of referring the first or second subtree of a branch node, we use to the left subtree or right subtree of the node.

In above example (ii), we have



Left subtree of a **Right subtree of a** **Left subtree of u_4** **Right subtree of u_2**

Note : The maximum number of vertices in a full binary tree with h levels is $1+2+2^2+ .... 2^h$.

$\therefore$ This is in geometric progression.

Let $$S_{h+1} = 1+2+2^2+ .... 2^h \quad ... (8.5.1)$$

$$2S_{h+1} = 2+2^2+2^3+ .... 2^h+2^{h+1} \quad ... (8.5.2)$$

$\therefore$ Equation (8.5.2) – Equation (8.5.1) $\Rightarrow$

$$\boxed{S_{h+1} = 2^{h+1}-1}$$

Theorem : The minimum height of a binary tree on n vertices is $[\text{Log}_2(n+1)-1]$ and maximum possible height is $\frac{n-1}{2}$. Assume that the root of T is at 0 level.

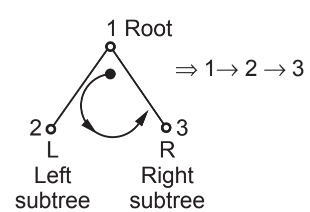
8.5.1 Binary Tree Traversal

Traversing means visiting or processing all the nodes of a tree. A binary tree traversal is the visiting of each node of a tree only once according to some sequence.

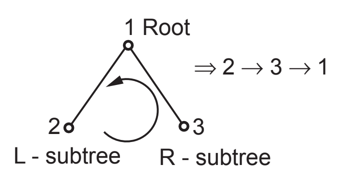
There are two types of traversing binary trees.

1) Depth-first traversal : In this method, the processing proceeds from the root or the most distant descendent of the first child. There are three types of Depth-First Traversal.

A) Pre order traversal : In this traversal, the root node is traversal first, followed by the left subtree and then the right subtree as shown below.

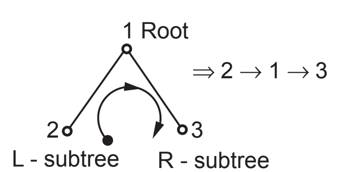


B) Post order traversal : If processes first the left subtree then the right subtree and then at the last root of the tree as shown below.

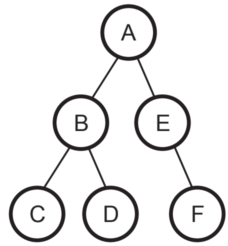


C) In order traversal : It processes the left subtree first then the root and at the last right subtree.

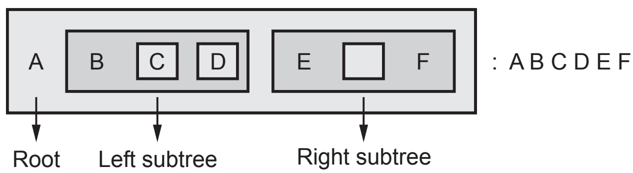
The prefix "in" means root is processed in between the subtrees. It is shown as below.



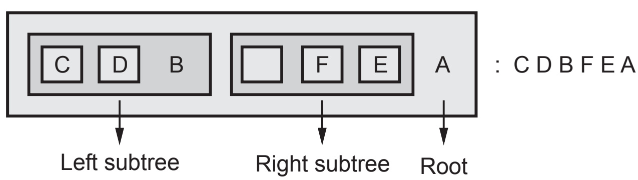
Consider the following binary tree.



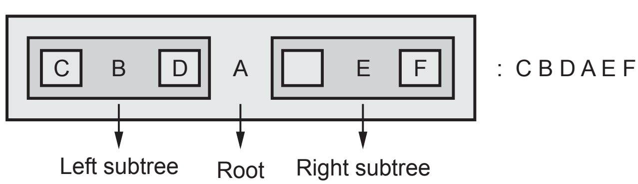
i) Pre order traversal



ii) Post order traversal

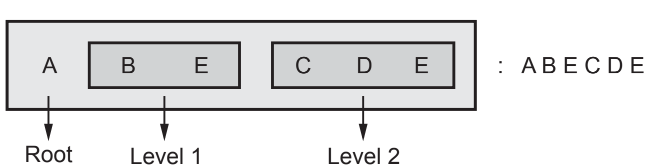


iii) In order traversal



2) Breadth first traversal : In this traversal, the processing proceeds horizontally from the root of all of its children, then to its childre's children and so on until all the nodes have been processed. That is first write node of zero level, then all nodes of level 1 then level 2 and so on.

The breadth first process for the above binary is shown below.



8.5.2 Binary Expression Tree

An algebraic expression can be conveniently expressed by its expression tree. An expression tree is a binary tree with the following properties.

i) Each leaf node is an operand

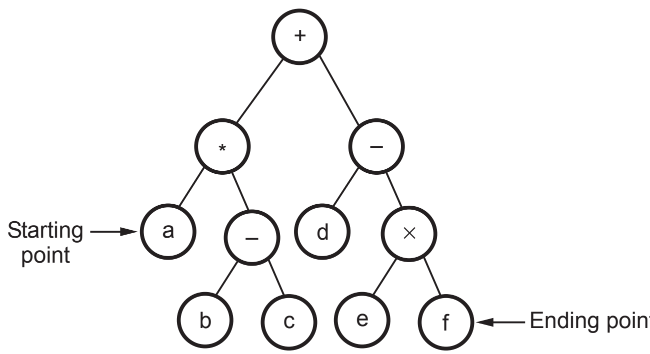
ii) The root and internal nodes are operators

iii)Subtrees are subexpressions, with the root being an operator.

In this context, we consider only standard arithmetic operations +, –, ×, /, .

For example

i)



Binary tree

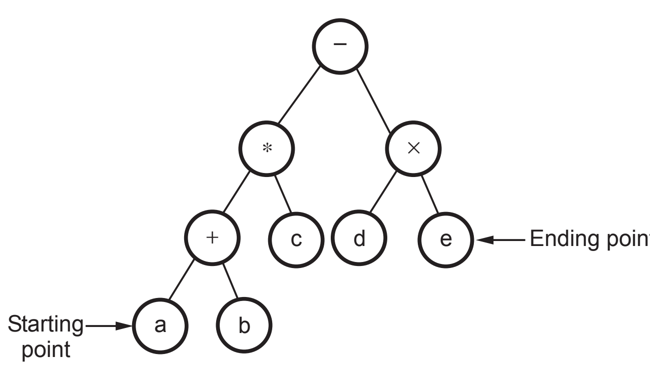
The expression of given binary tree is

$[a*(b-c)] + [d - (e \times f)]$

2) The binary tree of the expression

$[a+b)*c] - [d \times e]$

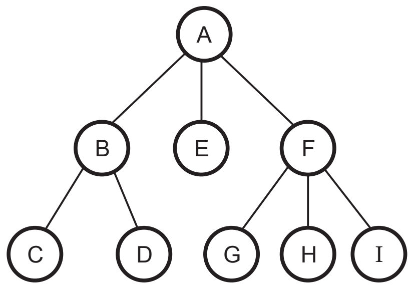
is given below



8.5.3 Conversion of General Tree to Binary Tree

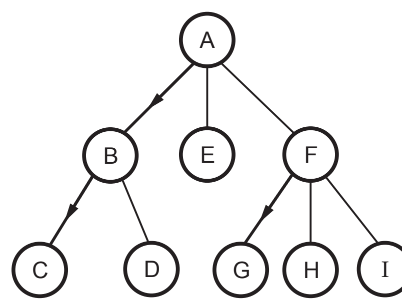
In general life we come across so many applications in which general trees are used. It is considerably easier to represent binary trees in programs than it is to represent general trees.

Now, let us explain the method to convert general tree to binary tree with the help of the following example.

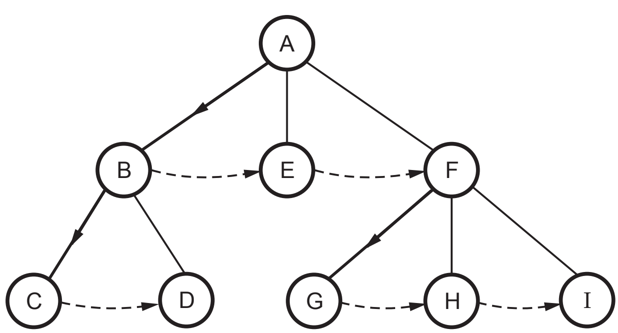


Consider the following steps

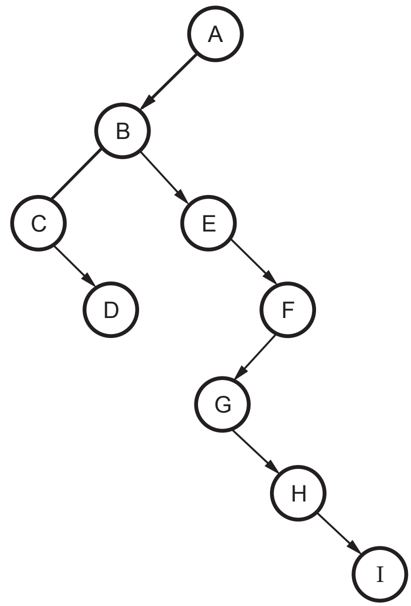
Step 1) To convert it into binary tree, we first identify the branch from the parent to its first or left most child. These branches from each parent become left pointers in the binary tree.



Step 2) Connect sibling, starting with the left most or first child, using a branch for each sibling to its right sibling. They are the right pointers in the binary tree.

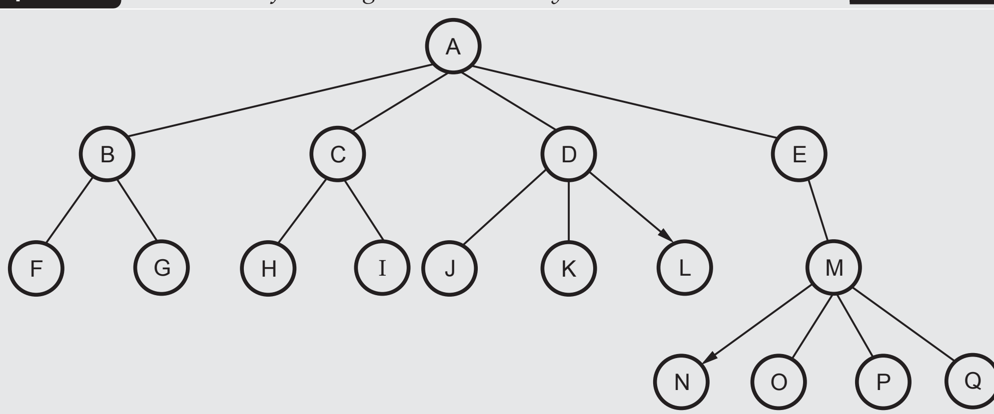


Step 3) Now remove all unneeded branches from the parent to its children. Therefore remove $A \rightarrow E$, $B \rightarrow D$, $A \rightarrow F$, $F \rightarrow H$, $F \rightarrow I$ we get the following required binary tree.

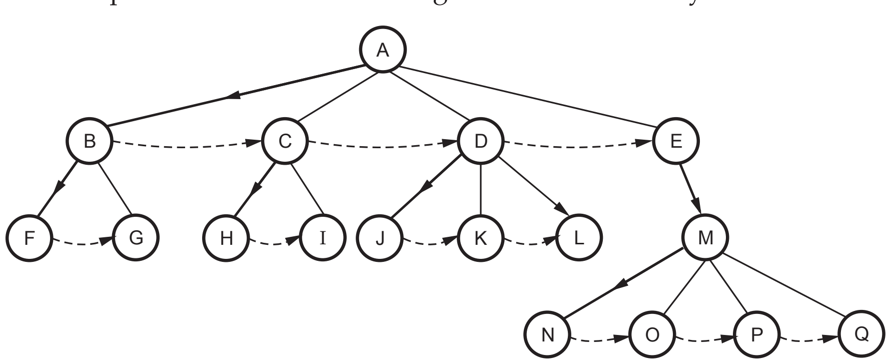


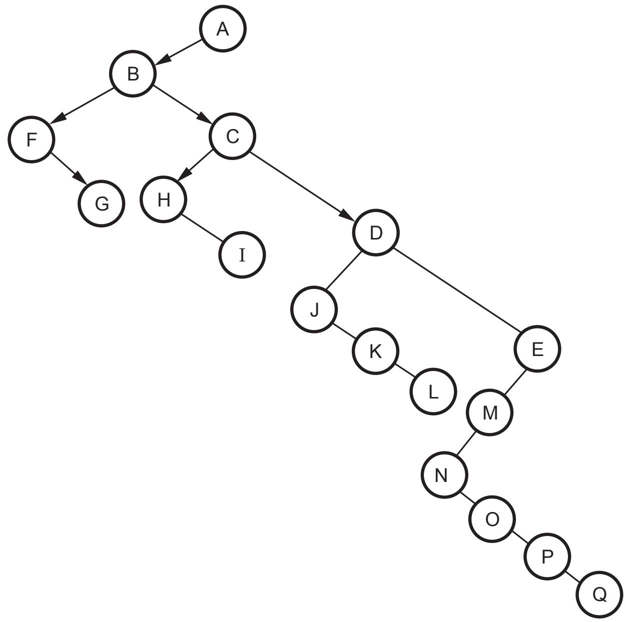
Resulting binary tree

Example 8.5.1 *Convert the following tree into binary tree.* **SPPU : Dec.-09**



Solution : The steps involve to convert the given tree into binary tree are as follows

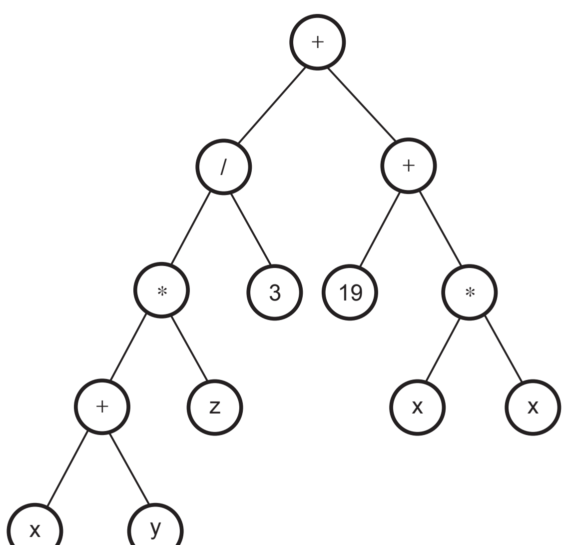




Binary tree

Example 8.5.2 *Construct the labeled tree of the following algebraic expression* $(((x+y)*z)/3)+(19+(x*x))$ **SPPU : May-10**

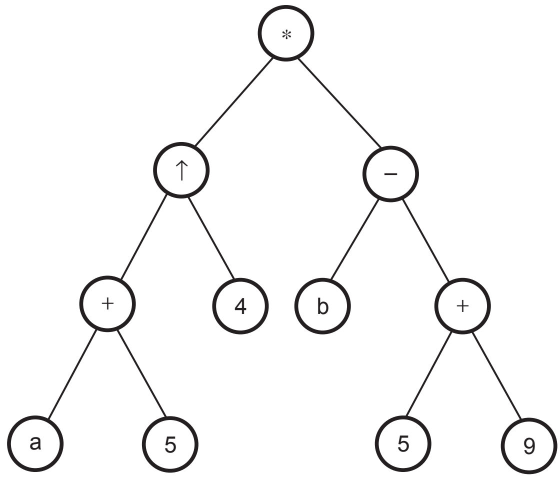
Solution : The expression tree for the given algebraic expression is given below :



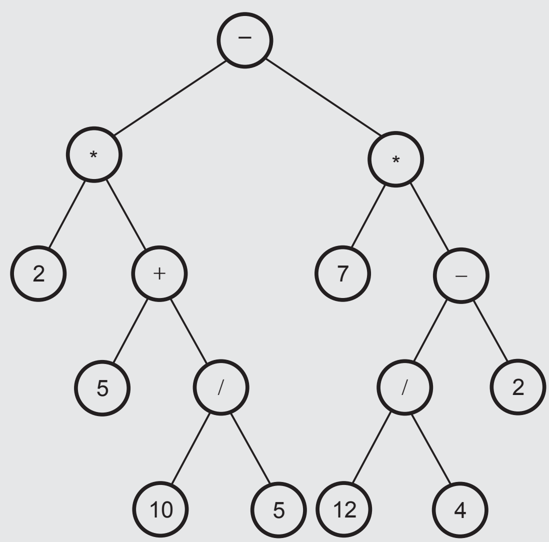
Example 8.5.3 *Represent the expression* $((a+5)\uparrow 4)*(b-(5+9))$ *using a binary tree.*

SPPU : May-12

Solution : The binary tree for the given expression is as follows :



Example 8.5.4 *Write and evaluate the expression tree shown below.* **SPPU : Dec.-09**

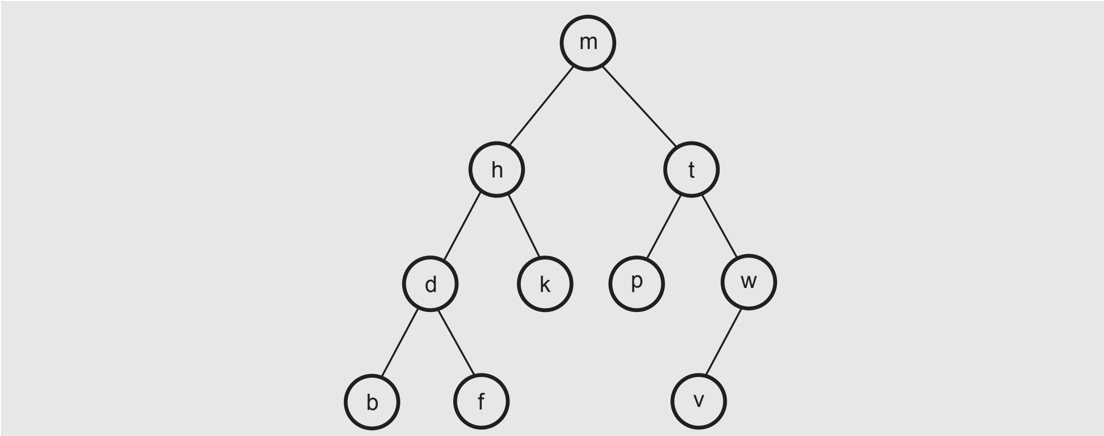


Solution : The algebraic expression of the given binary tree is

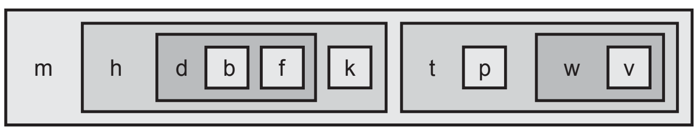
$(2*((5)+(10/5))) - ((7)*(((12/4)-(2))))$

It's value is $2*(5+2)-(7*(3-1)) = 14 - 7 = \mathbf{14}$

Example 8.5.5 *Find the preorder, postorder and inorder traversal of the following tree.* **SPPU : May-10**

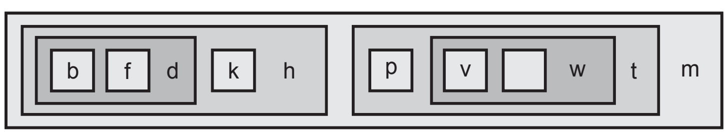


Solution : i) Preorder traversal



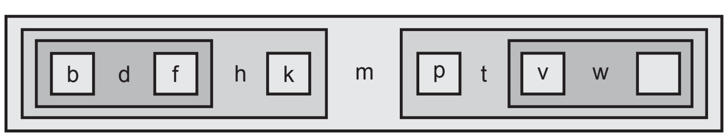
i.e. m b d b f k t p w v

ii) Postorder traversal



i.e. b f d k h p v w t m

iii) Inorder traversal



i.e. b d f h k m p t v w

Example 8.5.6 *How many leaves does a full binary tree with h levels have ?*

Solution : A full binary tree with h levels has 2^h leaves.

Example 8.5.7 *How many internal vertices does a full binary tree with h levels have ?*

Solution : The total number of vertices in full binary tree with h levels

$$= 2^{h+1} - 1$$

Out of which 2^h are leaves.

$$\text{Hence internal vertices} = (2^{h+1}-1)-2^h$$
$$= 2^h(2-1)-1$$
$$= 2^h-1$$

Example 8.5.8 *How many vertices will a full ternary tree with h levels have ?*

Solution : The number of vertices in a full ternary tree with h levels is

$$S = 1+3+3^2+3^3+ \ldots + 3^h$$
$$S = \frac{3^{h+1}-1}{3-1} = \frac{3^{h+1}-1}{2}$$

Hence the total number of vertices $= \dfrac{3^{h+1}-1}{2}$

Example 8.5.9 *What is the total number of nodes in a full binary tree with 20 leaves ?*

Solution : Let n be the number of nodes in a full binary tree.

Then $n = mP + 1$

$m = 2$

$\Rightarrow$ $n = 2P + 1$ Where P is the number of branch nodes.

The number of nodes in a tree is the sum of branch nodes and leaves.

Hence $n = P + 20 \Rightarrow P = n - 20$

$\Rightarrow$ $n = 2n - 40 + 1 = 2n - 39$

$2n - n = 39$

$n = 39$

Hence the total number of nodes in a full binary tree with 20 leaves are 39.

Example 8.5.10 *Can be have a ternary tree with exactly 20 nodes ?*

Solution :

We have $n = mP + 1$

$20 = 3P + 1$

$3P = 20 - 1 = 19$

$P = \dfrac{19}{3}$ Which is impossible

Therefore such tree does not exist.

Example 8.5.11 *Show that the number of vertices in a binary tree is odd.*

Solution : Let T be a binary tree with n vertices. T contains exactly one vertex of degree 2 and the remaining vertices of T are of degree one or three. Therefore the number of vertices of odd degree are n – 1. But for any graph the number of vertices of odd degree must be even. So n – 1 must be even.

Hence n is odd.

Example 8.5.12 *If T is a binary tree on n vertices then the number of pendent vertices in T is* $\frac{n+1}{2}$.

Solution : Let T be a binary tree with n vertices. Out of these vertices P vertices are pendent vertices and one vertex is of degree 2.

Therefore remaining all vertices are of degree 3.

Thus there are n – P – 1 vertices are of degree 3.

The number of edges in T is n – 1.

$\therefore$ Total degree of T is $2(n-1)$

Thus $$2 + P \times 1 + 3(n - P - 1) = 2(n - 1) = \text{Total degree}$$

$$P + 3n - 3P - 1 = 2n - 2$$

$$n + 1 = 2P$$

$$\boxed{P = \frac{n+1}{2}}$$

Hence the number of pendent vertices are $\frac{n+1}{2}$.

Example 8.5.13 *Find the maximum of possible height of a binary tree with 13 vertices and draw graph.*

Solution :

We have $n = 13$

The maximum possible height of the binary tree is

$$\frac{n-1}{2} = \frac{13-1}{2} = 6$$

The required graph as shown in Fig. 8.5.1

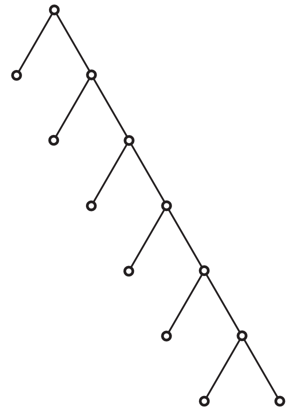


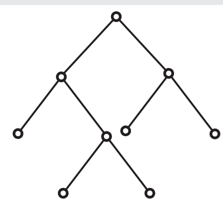
Fig. 8.5.1

Example 8.5.14 *Find the minimum height of the binary tree with 9 vertices.*

Solution : The minimum height of a binary tree

$$= [\log_2 (n+1) - 1] = \log_2 (g+1) - 1 = 3$$

It's graph is given below



8.6 Prefix Code and Binary Search Trees

SPPU : Dec.-12, 14, 15, 16, 18, May-08, 15, 17, 19

A **code** is simply an organized way of representing all information using a optimal set of symbols.

We know that modern computers rely on a binary code system consisting only of 0 and 1. Text messages, images, computer programs all such types of information is stored in computer by using coded form as a sequence of 1's and 0's.

The codes of the word may or may not be same length. So it is very difficult to convert codes into words. Therefore the codes should have some special characteristics so that easily we can convert that codes into unique meaningful words. Let us define prefix code.

Definition

A set of sequences is said to be a **prefix code** if no sequences in the set is a prefix of another sequence in the set.

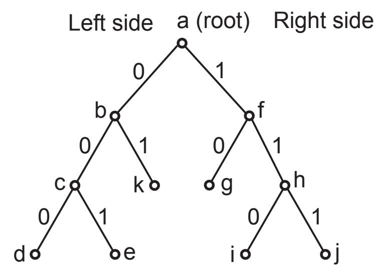
For example the set {000, 001, 01, 10, 11} is a prefix code as no sequence of symbols is present at the beginning of another sequences. All these sequence are distinct.

The set {1, 00, **00**0, **00**01} is not a prefix code because the sequence 00 is the prefix of the sequences 000 and 0001.

A question comes in everyone's mind "How to construct prefix code to a full binary trees?"

We can solve this question by adding some flavours and binary codes to a full binary trees.

For a given full binary tree, we label the two branches incident from each internal node with 0 and 1. For the left branch we assign 0 and for the right branch we assign 1 of every rooted tree or subtree. Consider the following example of full binary tree.



In given figure root a has 2 sons. Left son is ab and right is af, so assign 0 to ab and 1 to af. Now b has two sons. Assign 0 to left son be and 1 to right son bk. Similarly assign 0 or 1 to every edge of a tree.

Now assign to each leaf, a sequence of 0's and 1's which is the sequence of labels of the edges in the path from the root to the leaf.

For example, d is a leaf and the path a to d is a – b – c – d and their respective labels are 0 – 0 – 0 so the prefix code of d is 000.

For leaf e, path is a – b – c – e and labels 0 – 0 – 1

∴ The prefix code of e is 001

Thus the prefix code of above tree is {000, 001, 01, 10, 110, 111}.

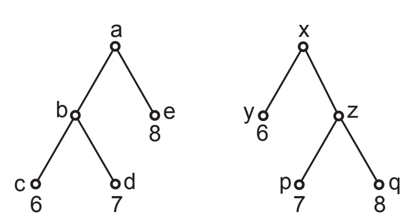
8.6.1 Optimal Tree

Let T be any full binary tree and $W_1, W_2, W_3, \ldots W_t$ be the weights of the leaves (terminal vertices) then the weight W of the full binary tree is given by

$$W(T) = \sum_{i=0}^{t} W_i l_i$$

Where $l_i = l(i)$ is the length of the path of the leaf i from the root of the tree. The full binary tree is called an optimal tree if its weight is minimum.

For example, suppose 6, 7, 8 are the weights of the leaves in a full binary tree as given below.



T_1 T_2

In T_1, $l(c) = 2$, $l(d) = 2$, $l(e) = 1$

∴ The weight of T is given by

$$W(T_1) = 6 \times l(c) + 7 \times l(d) + 8 \times l(e)$$

$$= 6 \times 2 + 7 \times 2 + 8 \times 1$$

$$= 12 + 14 + 8 = 34$$

In T_2, $l(y) = 1$, $l(p) = 2$, $l(q) = 2$

$$\therefore \quad W(T_2) = 6 \times 1 + 7 \times 2 + 8 \times 2$$

$$= 6 + 14 + 16 = 36$$

Hence $W(T_1) < W(T_2)$

Thus T_1 is the optimal tree for the weights 6, 7, 8.

8.6.2 Huffman Algorithm to Find an Optimal Tree

Let $W_1, W_2, W_3, \ldots W_t$ be the weights of the leaves and it is required to construct an optimal binary tree.

The following steps of an algorithm gives the required optimal binary tree.

Step 1 : Arrange the weights in increasing order

Step 2 : Consider two leaves with the minimum weights W_1 and W_2. Replace these two leaves and their father by a leaf. Assign weight $W_1 + W_2$ to this new leaf.

Step 3 : Repeat the step 2 for the weights $W_1, W_2, W_3, \ldots W_t$ until no weight remains.

Step 4 : The tree obtained in this way is an optimal tree for given weights and stop.

8.6.3 Optimal Prefix Code

A binary prefix code obtained from a optimal tree is called an optimal prefix code.

8.6.4 Binary Search Trees

In computer science, a binary search tree (BST) is a binary tree which has the following properties.

i) Each node has a value (weight)

ii) A total order is defined on these values.

iii) The left subtree of a node contains only values less than the nodes values.

iv) The right subtree of a node contains only values greater than or equal to the nodes values.

BST is used to construct more abstract data structures such as sets, multisets and associated arrays.

If a BST allows repeated values then it represents a multiset.

For multisets, we used everything in the right subtree is either > or = to the value of node.

For non repeated sets (All elements are distinct)

We used > for right subtree and < for left subtree

Let us formulate precisely the problem of searching for an item an ordered list.

i) Let $k_1, k_2, \ldots. k_n$ be the n items in an ordered list which are known as keys.

ii) Assume that $k_1 < k_2 < k_3 < \ldots. < k_n$

iii) Given an item x, our problem is to search the keys and determine whether x is equal to one of the keys or x falls between keys k_1 and k_{i+1} for some i.

iv) A search procedure consists of a sequence of comparisons between x and the keys k_i where each comparison gives.

i) $x = k_i$ or ii) $x > k_i$ iii) $x < k_i$

We explain now how we can describe a search procedure using a binary tree representation.

We define a search tree for the keys $k_1, k_2, \ldots. k_n$ to be a binary tree with n branch nodes and n + 1 leaves. The branch nodes are labeled as $k_1, k_2, \ldots. k_n$ and the leaves are labeled $k_0, k_1, k_2, \ldots. k_n$ such that for the branch node with the label k_i, its left subtree contains only vertices with labels k_j for j > i and its right subtree contains only vertices with labels k_j for $j \geq i$

Starting with the root of a search tree, we compare a given item x with label of root k_i.

If $x = k_i$ then search is completed.

If $x < k_i$, then we compare x with left son of the root.

If $x > k_i$ then we compare x with right son of the root.

Such comparison continues till, we find a match for x or till a leaf is reached. If a leaf labeled k_j is reached, it means that x is larger than the key k_j but less than the key k_{j+1}. If leaf labeled k_0 is reached it means that $x < k_1$.

If leaf labeled k_n is reached it means that $x > k_n$.

Example 8.6.1 *For the following set of weights construct optimal binary prefix code.*

$\alpha - 5$

$\beta - 6$

$\gamma - 6$

$\delta - 11$

$\in - 20$

SPPU : Dec.-14

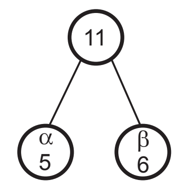
Solution : Here we use Huffman's coding method.

Step 1 : Sort the weights or frequencies of the given letters in increasing order in a queue.

$$\begin{array}{ccccc} \alpha & \beta & \gamma & \delta & \in \\ 5 & 6 & 6 & 11 & 20 \\ \uparrow & \uparrow & & & \end{array}$$

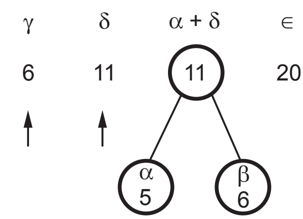
{$\uparrow$ indicates the first 2 smallest weights}

Consider the two symbols with lowest weights say α, β. The root of the first subtree has a weight 5 + 6 = 11.



As 6 < 11, it can not be placed at the beginning. It has to be placed an appropriate position means after δ. The first subtree is given below

Step 2 : Again rewrite sequence of weights in increasing order by replacing and β by new subtree weight as

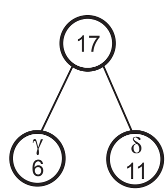


The first two smallest weights are 6 and 11.

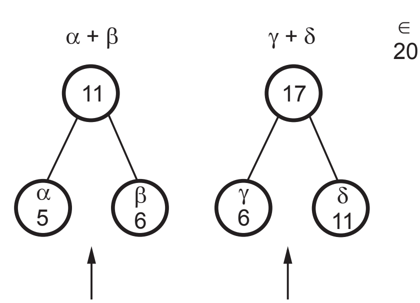
$\therefore$ The root of new subtree has a weight

$$6 + 11 = 17$$

The new subtree is as follows



Step 3 : Rewrite sequence of weight in increasing order as

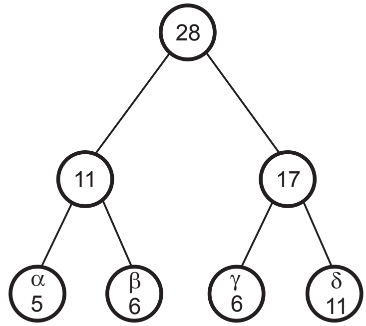


The first two smallest weights are 11 and 17.

$\therefore$ The root of new subtree has a weight

$$11 + 17 = 28$$

The new subtree is as follows

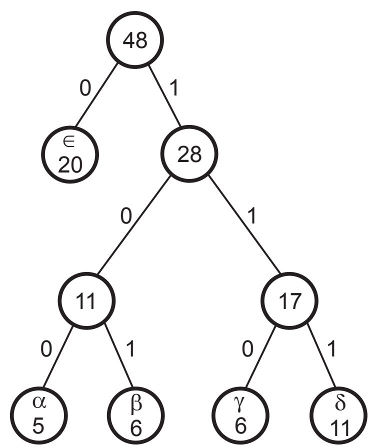


Step 4 : Rewrite the sequence of weights in increasing order

$$\begin{matrix} \in & \\ 20 & 28 \\ \uparrow & \uparrow \end{matrix}$$

The root of new subtree is 20 + 28 = 48

∴ The new subtree is as follows



This is the optimal binary tree.

Symbols leaves	α	β	γ	δ	$\in$
Binary prefix code	100	101	110	111	0

The weight of the optimal tree is

$$W = 20 \times 1 + 5 \times 3 + 6 \times 3 + 6 \times 3 + 11 \times 3$$

$$= 20 + 5 + 18 + 18 + 33 = \mathbf{104}$$

Note : All readers are requested to understand example 8.6.1 properly and then see next examples. Hereafter we have given solutions in shortforms.

Example 8.6.2 *Suppose data items A, B, C, D, E, F, G occur in the following frequencies.*

Data Items	A	B	C	D	E	F	G
Weight	10	30	5	15	20	15	05

Construct a Huffman code for the data.

What is the minimum weighted path length. **SPPU : May-08, 19, Dec.-14**

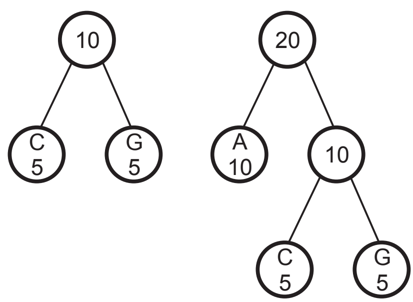
Solution : Consider the following steps

Step 1 : Sequence in increasing order is as follows

C D A D F E B

5 5 10 15 15 20 30

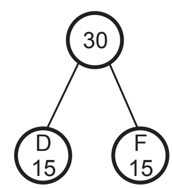
↑ ↑



Step 2 : Sequence : D F F

15, 15, 20, 20, 30

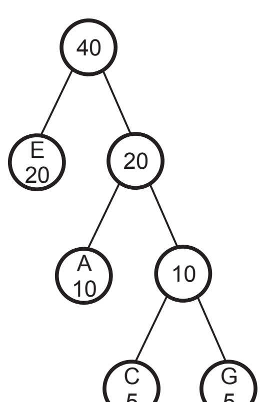
↑ ↑



Step 3 : Sequence : E B

20, 20, 30, 30

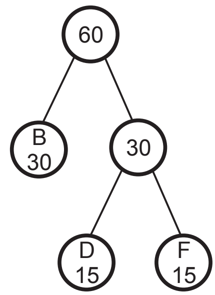
↑ ↑



Step 4 : Sequence : B

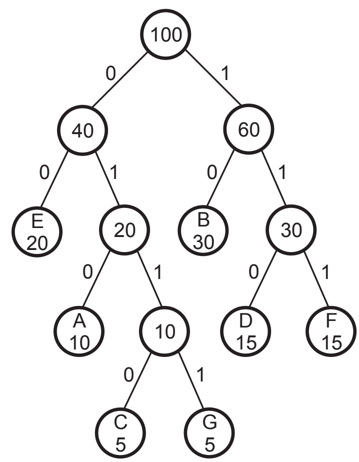
30, 30, 40

↑ ↑



Step 5 : Sequence : 40, 60

↑ ↑



Items	Binary prefix code
5	0111 or 0110
10	010
15	110 or 111
20	00
30	11

The minimum weight path length for the vertices as follows

$A \to 3, \; B \to 2, \; C \to 4, \; D \to 3, \; E \to 2, \; F \to 3, \; G \to 4$

$\therefore$ The minimum weight of tree is

$$W = 10 \times 3 + 30 \times 2 + 5 \times 4 + 15 \times 3$$
$$+ 20 \times 2 + 15 \times 3 + 5 \times 4 = \mathbf{260}$$

Example 8.6.3 *For the following set of weights, construct optimal binary prefix code. For each weight in the set, give the corresponding code words.*
8, 9, 12, 14, 16, 19.

Solution :

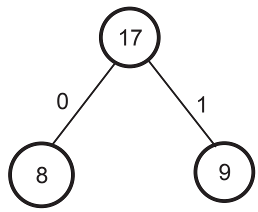
Step 1 : Increasing order of weights :

8, 9, 12, 14, 16, 19

↑ ↑

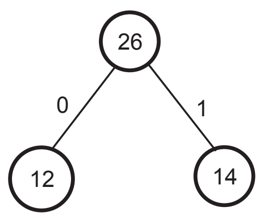
(↑ : First two smallest weights)

The first sub tree is as follows

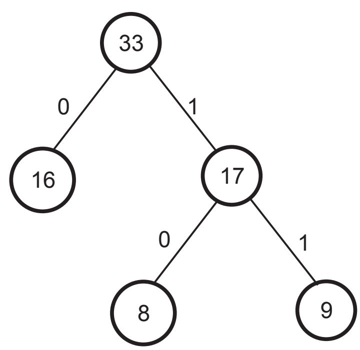


Step 2 : Rewrite sequence of weights in increasing order

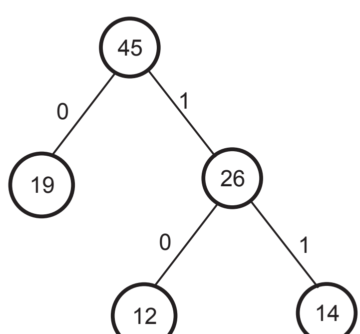
12, 14, 16, 17, 19
↑ ↑



Step 3 : Sequence : 16, 17, 19, 26,
↑ ↑

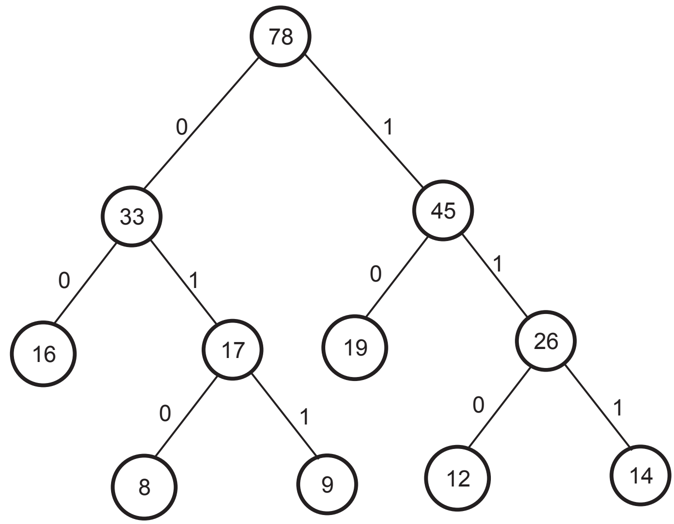


Step 4 : Sequence : 19, 26, 33,
↑ ↑



Step 5 : Sequence : 33, 45
↑ ↑

Symbols	Binary Prifix Codes
16	00
8	010
9	011
19	10
12	110
14	111



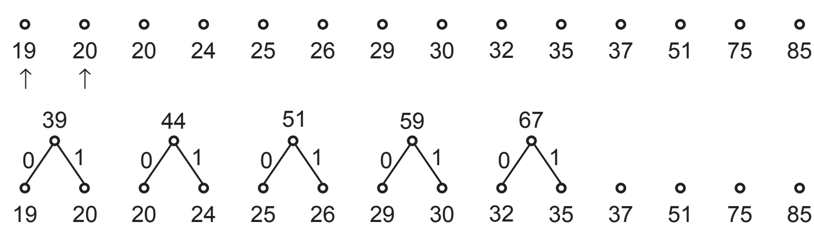
Example 8.6.4 *A secondary storage media contains information in files with different formats. The frequency of different types of files is as follows.*

Exe (20), bin (75), bat (20), jpeg (85), dat (51), doc (32), sys (26), c (19), cpp (25), bmp (30), avi (24), prj (29), 1st (35), zip (37).

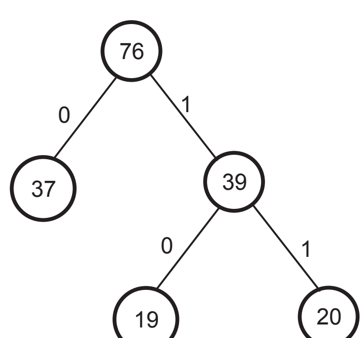
Construct the Huffman code for this. **SPPU : May-15, Dec.-15**

Solution : Consider the following steps

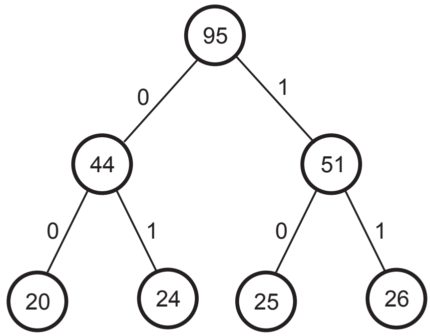
Step 1 : Sequence :



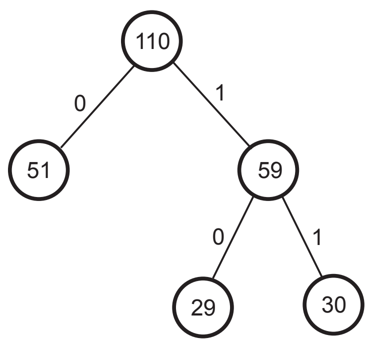
Step 2 : Sequence : 37, 39, 44, 51, 51, 59, 67, 75, 85
↑ ↑



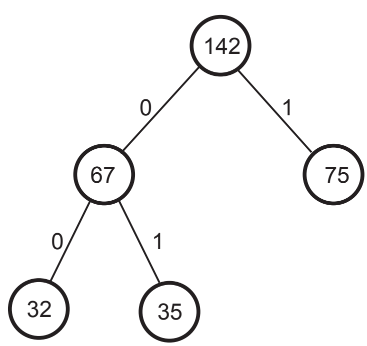
Step 3 : Sequence : 44, 51, 51, 59, 67, 75, 76, 85



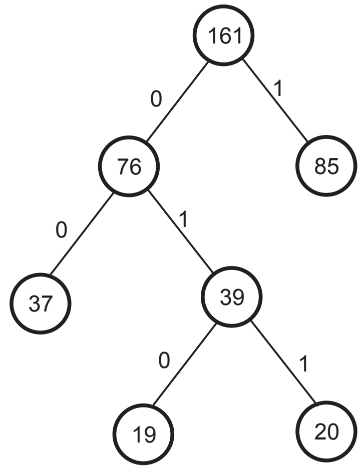
Step 4 : Sequence : 51, 59, 67, 75, 76, 85, 95



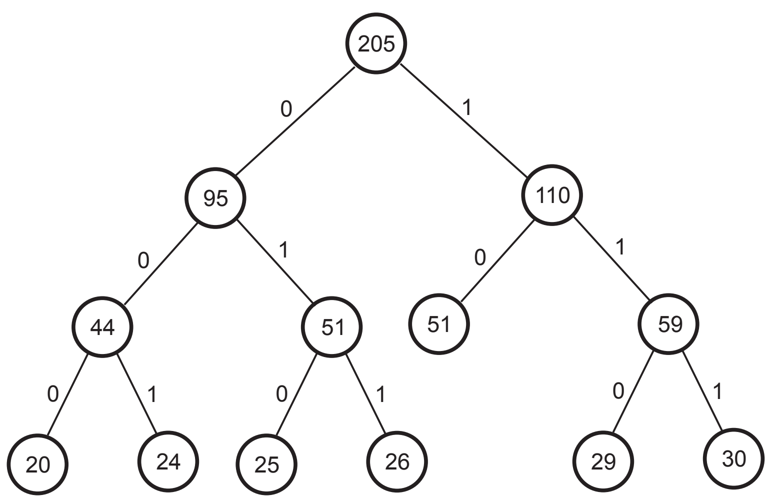
Step 5 : Sequence : 67, 75, 76, 85, 95, 100



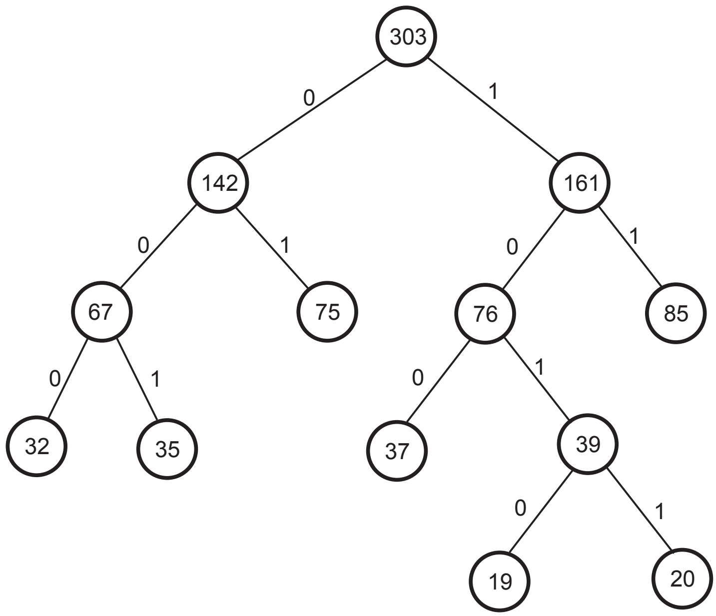
Step 6 : Sequence : 76, 85, 95, 110, 142,



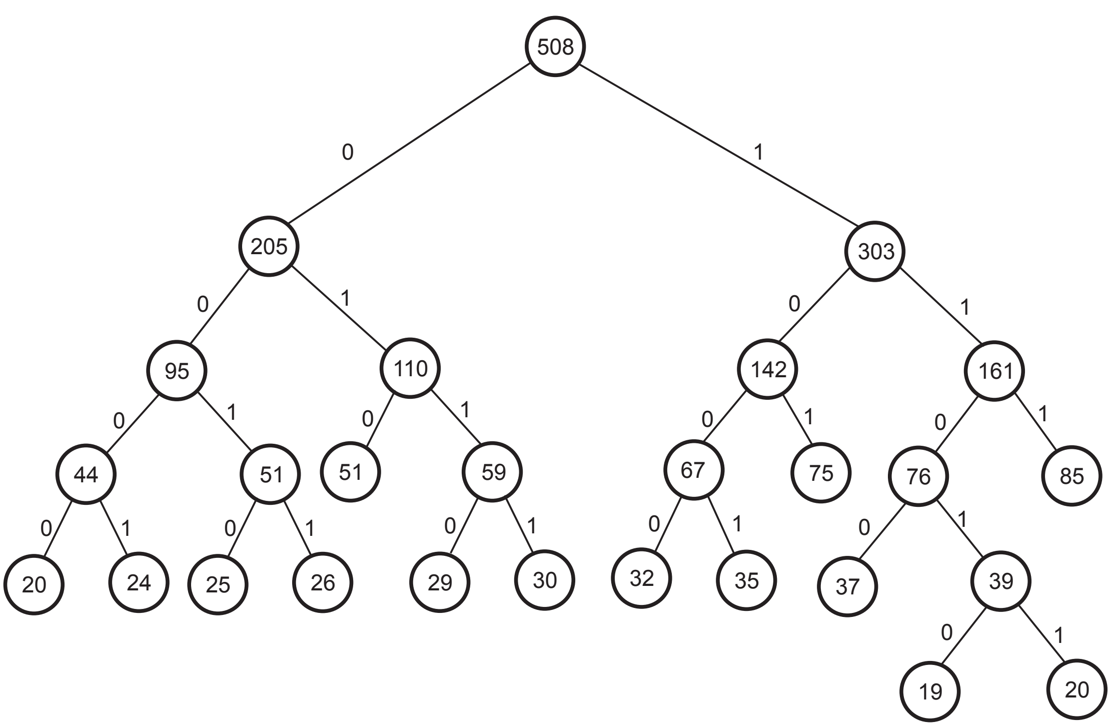
Step 7 : Sequence : 95, 110, 142, 161,
↑ ↑



Step 8 : Sequence : 142, 161, 205
↑ ↑



Step 9 : Sequence : 205, 303,
↑ ↑



Numbers	Binary prifix codes
20	0000
24	0001
25	0010
26	0011
51	010
29	0110
30	0111
32	1000
35	1001
75	101
37	1100
19	11010
20	11011
85	111

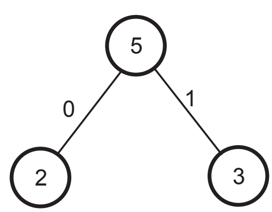
Example 8.6.5 *For the following sets of weights, construct an optimal binary prefix code. For each weight in the set, give the corresponding code word*

i) 2, 3, 5, 7, 9, 13

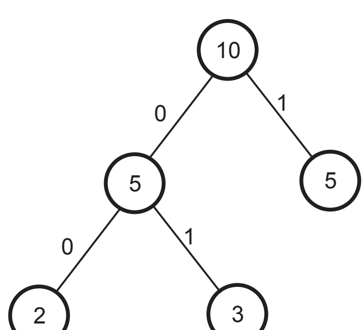
ii) 8, 9, 10, 11, 13, 15, 22 **SPPU : Dec.-12**

Solution : Consider the following steps.

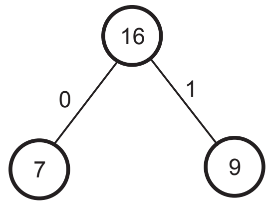
i) Step 1 : Sequence : 2, 3, 5, 7, 9, 13
↑ ↑



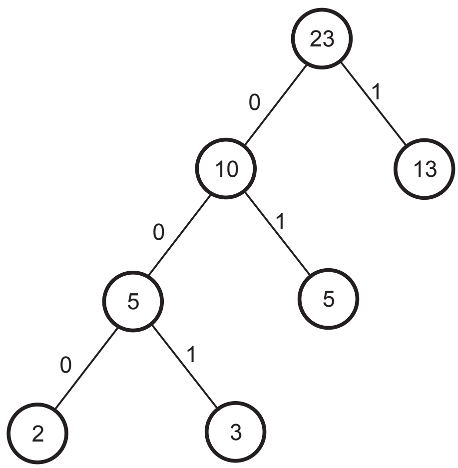
Step 2 : Sequence : 5, 5, 7, 9, 13,
↑ ↑



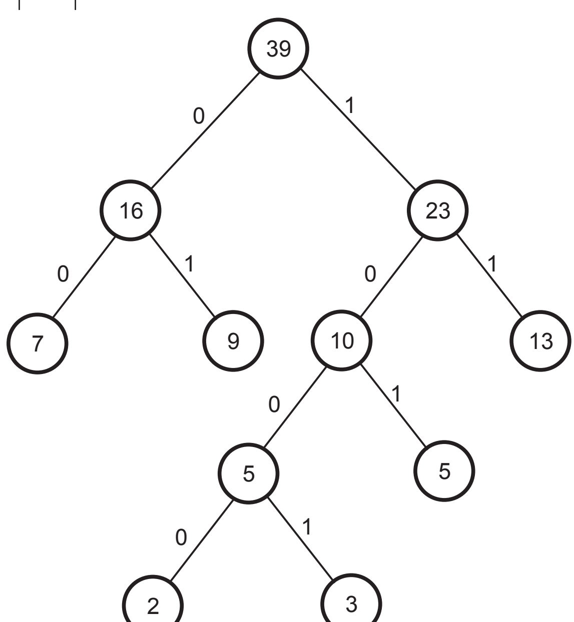
Step 3 : Sequence : 7, 9, 10, 13
↑ ↑



Step 4 : Sequence : 10, 13, 16,
↑ ↑



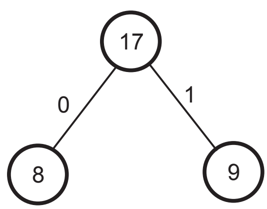
Step 5 : Sequence : 16, 23
↑ ↑



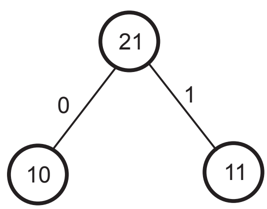
Symbols	Binary prefix code
7	00
9	01
2	1000
3	1001
5	101
13	11

ii) Consider the following steps

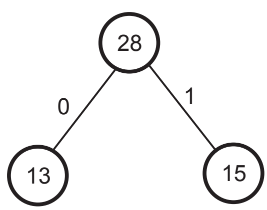
Step 1 : Sequence : 8, 9, 10, 11, 13, 15, 22
↑ ↑



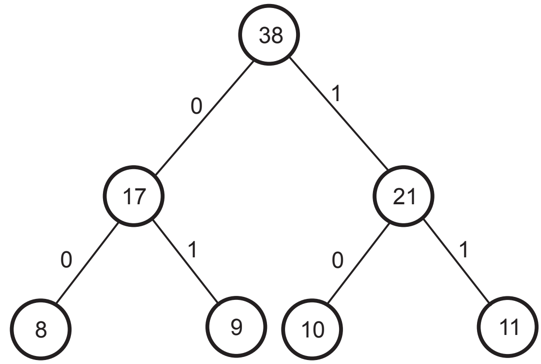
Step 2 : Sequence : 10, 11, 13, 15, 17, 22
↑ ↑



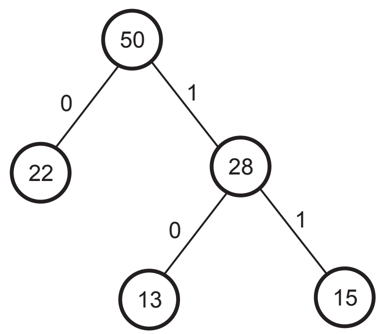
Step 3 : Sequence : 13, 15, 17, 21, 22,
↑ ↑



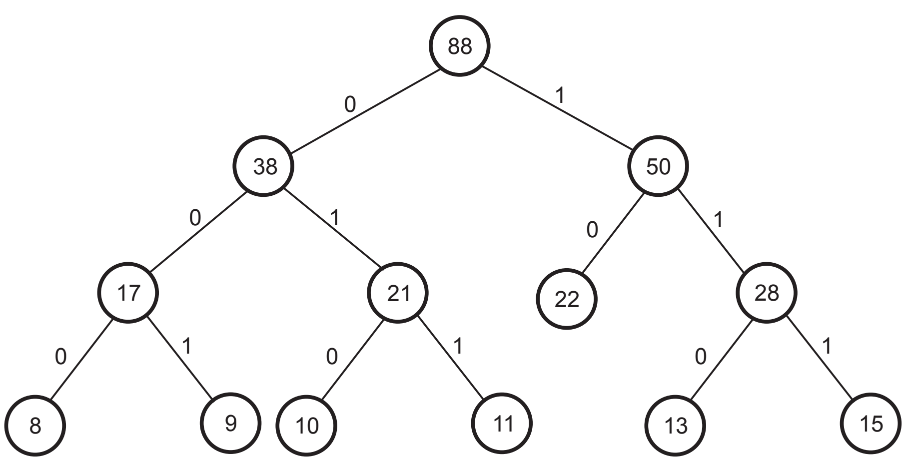
Step 4 : Sequence : 17, 21, 22, 28,
↑ ↑



Step 5 : Sequence : 22, 28, 38
↑ ↑



Step 6 : Sequence : 38, 50
↑ ↑



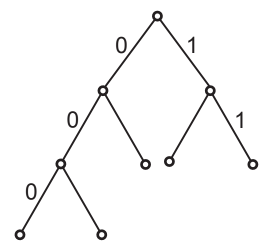
Symbols	Binary prefix code
8	000
9	001
10	010
11	011
22	10
13	110
15	111

Example 8.6.6 *State whether the given set is a prefix code. Justify {000, 001, 01, 10, 11}*

SPPU : May-08, Dec.-16

Solution : The given set which contains 5 elements will be a prefix code if we can construct a full binary tree with five leaves.

Let n be the number of vertices and P be the number of interior vertices in a full binary tree with five leaves.



Then $\quad n = P + 5 \Rightarrow P = n - 5 \quad \dots (1)$

and $\quad n = 2P + 1 = 2(n-5) + 1$

$\Rightarrow 2n - 2P = -9 \quad \dots (2)$

Solving (1) and (2) we get n = 9 and P = 4.

Therefore we have a full binary tree with 9 vertices and five leaves. Such binary tree is shown in the figure.

8.7 Spanning Trees

SPPU : Dec.-11,13,14,15,16,17,18, May-05,14,15,18,19

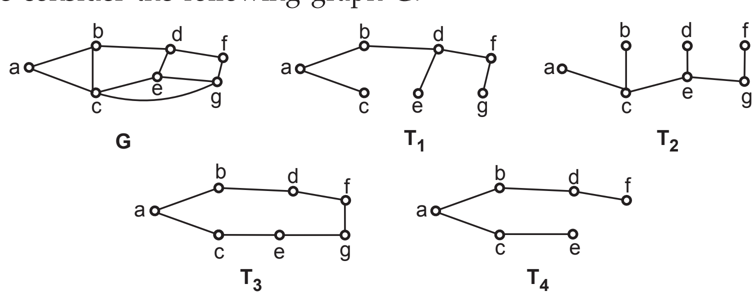
We know that a subgraph H of a graph G is a **Spanning subgraph** of G if V(G) = V(H).

Now define spanning subgraph by using tree.

Definition :

A spanning subgraph T of a connected graph G is said to be **spanning tree** of G if T is a tree. In other words, A subgraph T of a graph G is said to be spanning tree if V(T) = V(G).

For example consider the following graph G.



Trees T_1 and T_2 are spanning trees of graph G.

T_3 is not a tree so it is not spanning tree of G.

T_4 is a tree but $g \notin V(T_4)$, so T_4 is not spanning tree.

8.7.1 Minimum Spanning Tree

Let G be a weighted connected graph. A spanning tree of G is called a **Minimum Spanning tree** if it's weight is minimum.

Minimum spanning trees are using in constructing or designing a network of roads, railway lines, TV cable line, Telephone lines, Electric lines etc. for connecting number of cities or objects.

The connector problem is to design a network that connect all required stations such that the total cost of construction is least.

There are several spanning trees.

The minimum spanning tree may or may not be unique. There are any realistic situations in which we have two paths with exactly same weights.

There are two algorithms to find minimum spanning trees.

8.7.2 Prim's Algorithm

Let G(V, E) be a connected weighted graph.

To construct minimum spanning tree of G consider the following steps.

Step 1 : Select any vertex v_o in graph G.

Set $T = \{v_o, \phi\}$

Step 2 : Find edge $e_i = (v_o, v_i)$ in E such that its one end vertex is $v_o \in T$ and its weight is minimum.

$\therefore$ New set $T = \{\{v_0, v_1\}, \{e_i\}\}$

Step 3 : Choose next edge $e_k = (v_k, v_j)$ in such a way that it's one end vertex $v_k \in T$ and other vertex $v_j \notin T$ and weight of e_k is as small as possible. Again join vertex v_j and edge e_k to T.

Step 4 : Repeat the step 3 until T contains all the vertices of G. The set T will give the minimum spanning tree of the graph G.

8.7.3 Kruskal Algorithm

In 1956, Kruskal developed an algorithm to find a minimum spanning tree of a given weighted connected graph.

Let G(V, E) be a weighted connect graph.

Consider the following steps.

Step 1 : Pick up an edge e_i of G such that its weight $W(e_i)$ is minimum.

(If there are more edges of the minimum weight then select all those edges which do not form a circuit).

Step 2 : If edges $e_1, e_2, \ldots e_n$ have been chosen then pick an edge e_{n+1} such that

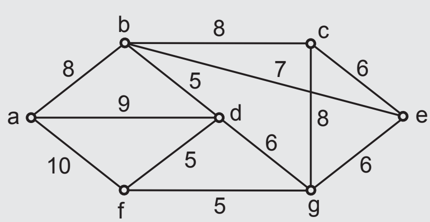
i) $e_{n+1} \neq e_i$ for $i = 1, 2, \ldots n$

ii) The edges $e_1, e_2, \ldots e_n, e_{n+1}$ do not form a circuit.

iii) $W(e_{n+1})$ is as small as possible subject to condition (ii).

Step 3 : Stop when step two cannot be implemented.

Example 8.7.1 *Find the minimum spanning tree for the graph given in the following figure using Prim's algorithm.* **SPPU : Dec.-14**



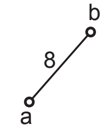
Solution : Consider the following steps for the construction of the minimum spanning tree.

Step 1 : Select $a \in G$ as a starting vertex

$\therefore \qquad T = \{\{a\}, \phi\}$

Step 2 : Vertex a is adjacent to vertices b, d and f. Among these edges minimum weight is {a, b} = 8

$\therefore \qquad T = \{\{a, b\}, \{e_1\}\}$

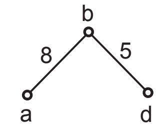


Step 3 : Vertex a is adjacent to f, d.

Vertex b is adjacent to c, d, e

The minimum weight is of an edge {b, d} = 5

$\therefore \qquad T = \{\{a, b, d\}, \{e_1, e_2\}\}$



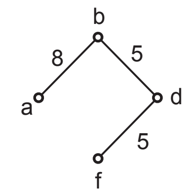
Step 4 : Vertex a is adjacent to f and af = 10

Vertex b is adjacent to c, e and bc = 8, be = 7

Vertex d is adjacent to f and g and df = 5, dg = 6

Among all these weights minimum is 5 = df

$\therefore$ $T = \{\{a, b, d, f\}\{e_1, e_2, e_3\}\}$



Step 5 : Vertex is adjacent to b, d, f but all are in T.

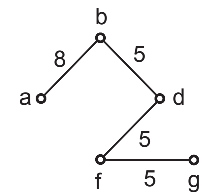
Vertex b is adjacent to c and e

Vertex d is adjacent to g.

Vertex f is adjacent to g

Among all these edges minimum weight is of fg.

$\therefore$ $T = \{\{a, b, d, f, g\}\{e_1, e_2, e_3, e_4\}\}$

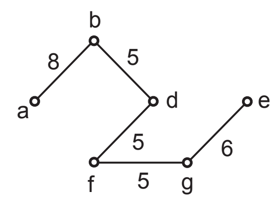


Step 6 : Vertex g is adjacent to c and e

Vertex b is adjacent to c and e

Among these edges the minimum weight is ge = 6

$\therefore$ $T = \{\{a, b, d, f, g, e\}\{e_1, e_2, e_3, e_4, e_5\}\}$

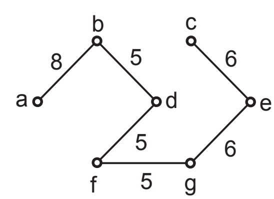


Step 7 : Now only one vertex is remaining.

The vertex C is adjacent to b, e, g.

The minimum weight is ec = 6

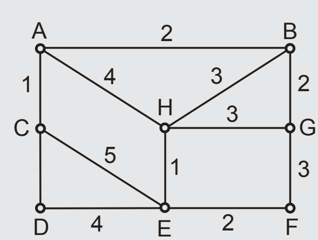
$\therefore$ $T = \{\{a, b, d, f, g, e, c\}\{e_1, e_2, e_3, e_4, e_5, e_6\}\}$



The graph obtained is the minimum spanning tree of weight

$= 8 + 5 + 5 + 5 + 6 + 6 = 35$

Example 8.7.2 *Use Prim's algorithm to find minimum spanning tree. Take A as starting vertex.* **SPPU : Dec.-11, 16**



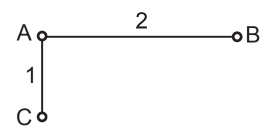
Solution : Consider the following steps for the construction of the minimum spanning tree.

Step 1 : Vertex A is starting vertex

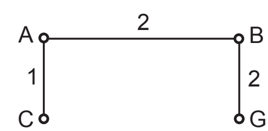
$$T = \{\{A\}, \phi\}$$

Step 2 : $T = \{\{A, C\}, e_1\}$

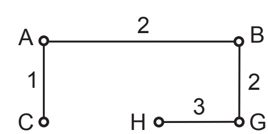
Step 3 : $T = \{\{A, C, B\}, \{e_1, e_2\}\}$



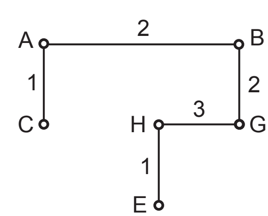
Step 4 : $T = \{\{A, C, B, G\}, e_1, e_2, e_3\}$



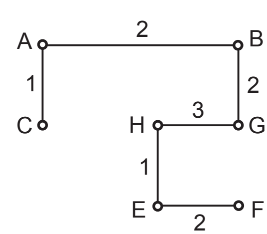
Step 5 : $T = \{\{A, C, B, G, H\}, e_1, e_2, e_3, e_4\}$



Step 6 : $T = \{\{A, C, B, G, H, E\}, \{e_1, e_2, e_3, e_4, e_5\}\}$



Step 7 : $T = \{\{A, C, B, G, H, E, F\}, \{e_1, e_2, e_3, e_4, e_5, e_6\}\}$

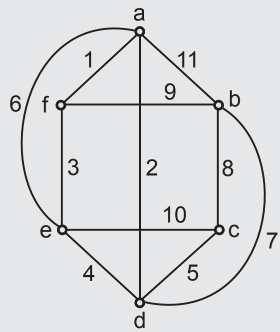


The graph obtained is the minimum spanning tree of weight

$$= 1 + 2 + 2 + 3 + 1 + 2 = 11$$

Example 8.7.3 *Determine minimum spanning tree for the given graph by Prism's algorithm.*

SPPU : May-14, 18

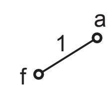


Solution : Consider the following steps for the construction of the minimum spanning tree.

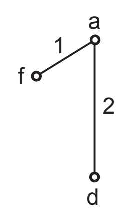
Step 1 : Starting with vertex a

$$T = \{\{a\}, \phi\}$$

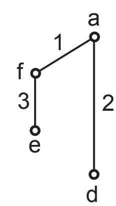
Step 2 : $T = \{\{a, f\}, e_1\}$



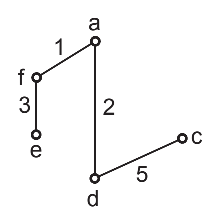
Step 3 : $T = \{\{a, f, d\}, \{e_1, e_2\}\}$



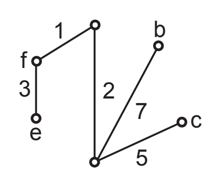
Step 4 : $T = \{\{a, f, d, e\}, \{e_1, e_2, e_3\}\}$



Step 5 : $T = \{\{a, f, d, e, c\}, \{e_1, e_2, e_3, e_4\}\}$



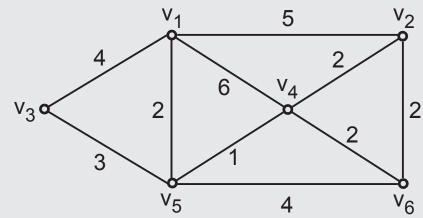
Step 6 : $T = \{\{a, f, d, e, c, b\}, \{e_1, e_2, e_3, e_4, e_5\}\}$



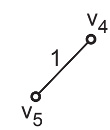
The graph obtained is the minimum spanning tree of weight

$$= 1 + 2 + 1 + 5 + 7 = 18$$

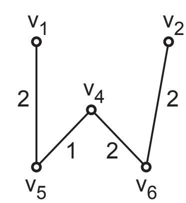
Example 8.7.4 *Find the minimum spanning tree for graph given below by Kruskal's algorithm.* **SPPU : Dec.-13**



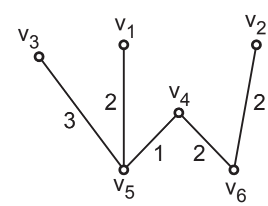
Solution : Consider the following steps for the construction of the minimum spanning tree.



Step 1 : The minimum weight in given graph is 1, so select an edge $\{v_4, v_5\}$



Step 2 : The minimum weight is 2 in remaining graph. There are four edges of weight 2. These edges form a circuit so select edges which do not form a circuit with selected edge. We select three edges $\{v_5, v_1\}$, $\{v_4, v_6\}$, $\{v_6, v_2\}$

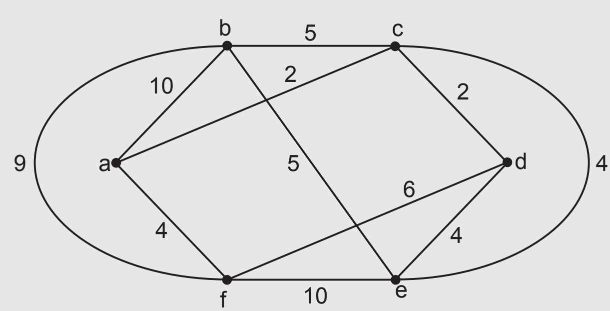


Step 3 : The minimum weight is 3 in the remaining graph so select $\{v_5, v_3\}$

The graph obtained is the minimum spanning tree of weight

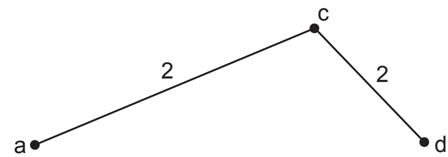
$$= 3 + 2 + 1 + 2 + 2 = \mathbf{10}$$

Example 8.7.5 *Use the Kruskal's algorithm to find the minimum spanning tree for the graph shown.* **SPPU : May-05**

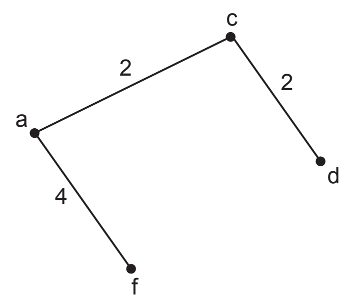


Solution : Consider the following steps for the construction of the minimum spanning tree.

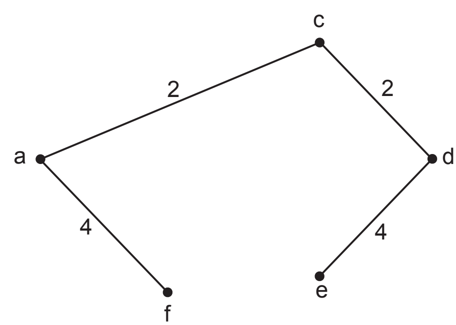
Step 1 : The minimum weight is 2 associated with two edges. These edges do not form a circuit. So select these to edges.



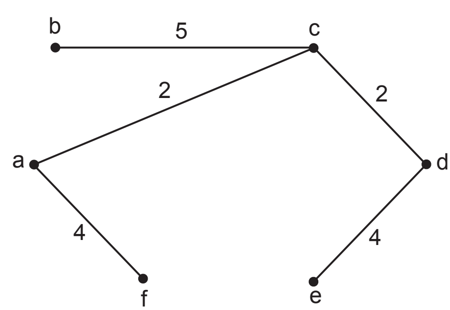
Step 2 : Minimum weight = 4



Step 3 : Maximum weight = 4

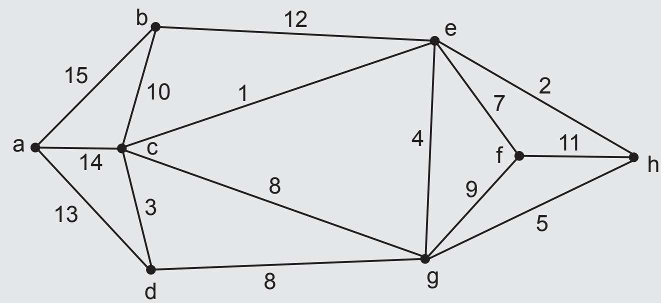


Step 4 : Minimum weight = 4 associated with {c, e} which forms a circuit so select next minimum weight = 5



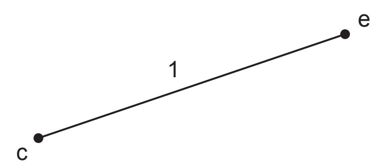
The graph is the minimum spanning tree and its total weight is 17.

Example 8.7.6 *Obtain the minimum spanning tree for the following graph. Obtain the total cost of minimum spanning tree.* **SPPU : May-15, Dec.-15**

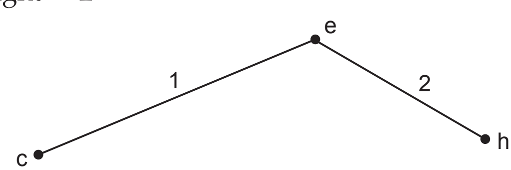


Solution : Using Kruskal algorithm, the minimum spanning tree is obtained as follows :

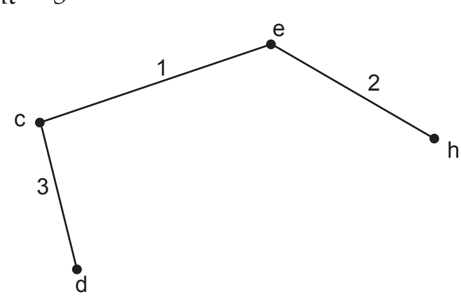
Step 1 : Minimum weight in a given graph is 1 associated with edge {a, e}



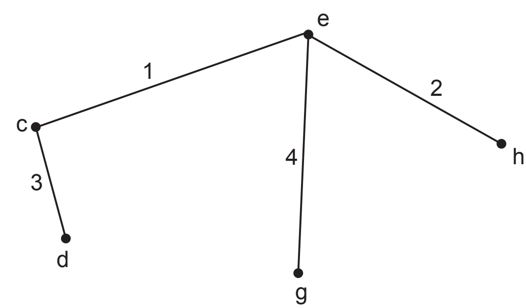
Step 2 : Minimum weight = 2



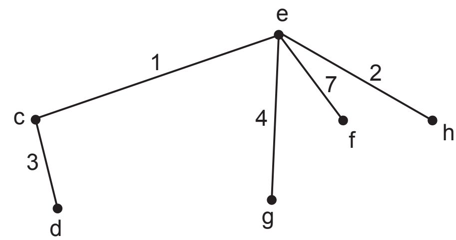
Step 3 : Minimum weight = 3



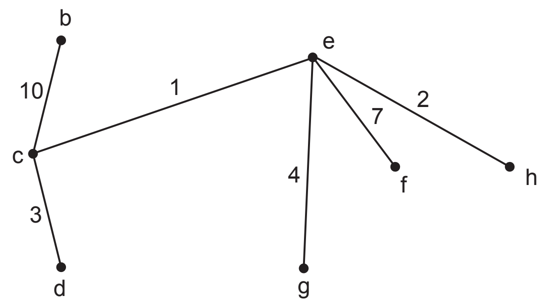
Step 4 : Minimum weight = 4



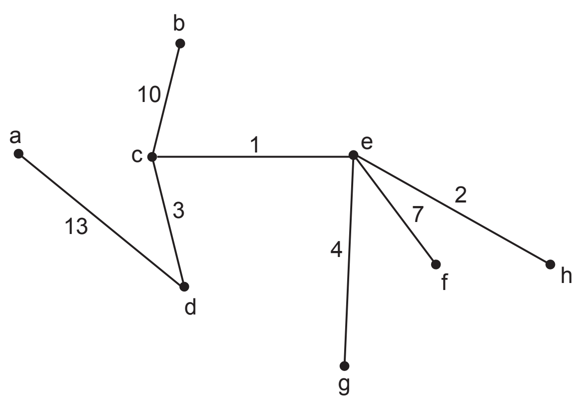
Step 5 : Minimum weight = 7



Step 6 : Minimum weight = 10



Step 7 : Minimum weight = 13



The obtained graph is the minimum spanning tree of the given graph. Its total cost is 40.

8.8 Fundamental Circuits and Cutsets

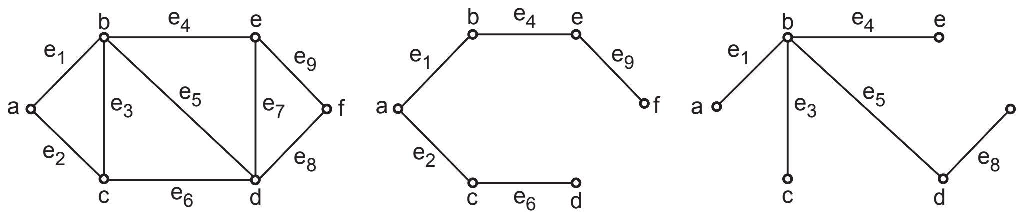
SPPU : Dec.-07, 12, 14, 15, 16, May-07, 15

Let G be a connected graph and T be a spanning tree of G. An edge of a tree is called a **branch.** An edge of G which is not in T is called chord of T. T + e contains a unique circuit called fundamental circuit of G with respect to T.

A fundamental circuit of a connected graph G is always with respect to a spanning tree of G.

Therefore the different spanning trees will have different fundamental circuits.

Consider the following graph G with two spanning trees T_1 and T_2.



G **T_1** **T_2**

In graph G, vertex set V(G) = {a, b, c, d, e, f}

Edge set $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$

A) For Spanning Tree T_1 :

Branches of T_1 are e_1, e_2, e_4, e_6, e_9

Chords of T_1 are e_3, e_5, e_4, e_8

Consider the following table of chords and corresponding fundamental circuits.

Chords	Corresponding Fundamental Circuits
e_3	$\{e_1, e_2, e_3\}$
e_5	$\{e_1, e_2, e_6, e_5\}$
e_7	$\{e_1, e_2, e_4, e_6, e_7\}$
e_8	$\{e_1, e_2, e_4, e_6, e_9, e_8\}$

B) For Spanning Tree T_2

Branches of T_2 are e_1, e_3, e_5, e_4, e_8

Chords of T_2 are e_2, e_4, e_6, e_7, e_9

Consider the following table of chords and corresponding fundamental circuits.

Chords	Corresponding Fundamental Circuits
e_2	$\{e_1, e_3, e_2\}$
e_6	$\{e_3, e_5, e_6\}$
e_7	$\{e_4, e_5, e_7\}$
e_9	$\{e_4, e_5, e_8, e_9\}$

Fundamental Cutsets

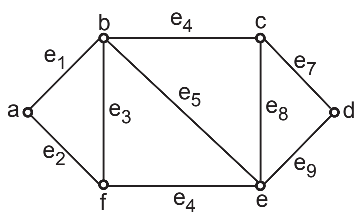
Let T be k spanning tree of a connected graph G. Since every edge e of a tree is an isthmus or bridge, T-e splits into two components say T_1 and T_2. But
$V(T) = V(T_1) \cup V(T_2) = V(G)$

The set E of edges of G which join a vertex in $V(T_1)$ to a vertex in $V(T_2)$ is a cutset of G.

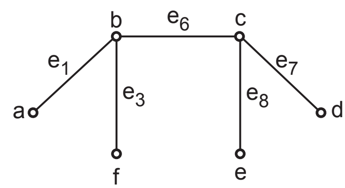
A cutset of G obtained in this manner is called a fundamental cutset of G with respect to T. To each edge of edge there is a fundamental cutset and every fundamental cutset is obtained in this way. Thus the number of fundamental cutsets of G w.r.t. T is the number of branches of T.

Theorem : Let G be a connected graph with n vertices then its spanning tree has $n-1$ edges and there are $n-1$ fundamental cutsets only.

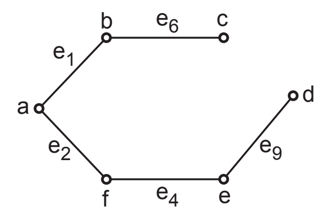
e.g. Consider the following and its spanning tree.



G



T_1



T_2

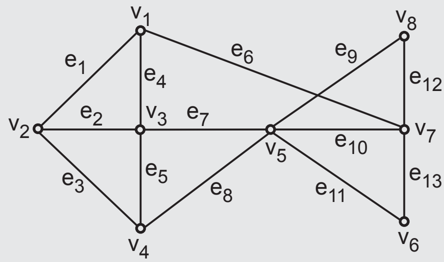
A) For Spanning Tree T_1

Branches of T_1 are e_1, e_3, e_6, e_7, e_8

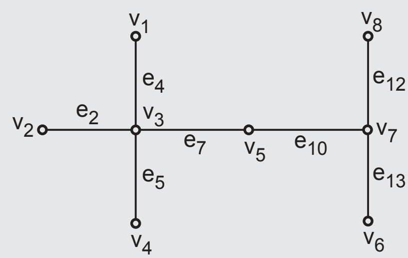
Consider the following table

Branches T_1	Corresponding Fundamental Cutset
e_1	$\{e_1, e_2\}$
e_3	$\{e_3, e_2, e_4\}$
e_6	$\{e_6, e_5, e_4\}$
e_7	$\{e_7, e_9\}$
e_8	$\{e_8, e_4, e_9, e_5\}$

Example 8.8.1 *Find the fundamental system of cutset for the graph G shown below w.r.t. the spanning tree T.* **SPPU : May-15, Dec.-12, 15**



G

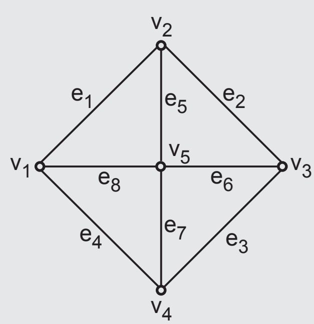


T

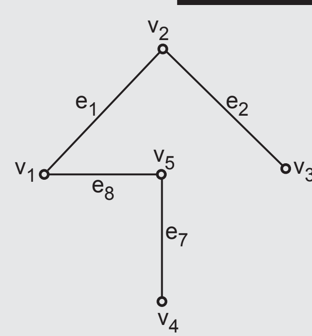
Solution : The spanning tree T has 7 branches $\{e_2, e_4, e_5, e_7, e_{10}, e_{12}, e_{13}\}$. Therefore are seven fundamental cutsets of G w.r.t. T which are given below :

Branch	Fundamental Cutset
e_2	$\{e_2, e_1, e_3\}$
e_4	$\{e_4, e_1, e_6\}$
e_5	$\{e_5, e_3, e_8\}$
e_7	$\{e_7, e_6, e_8\}$
e_{10}	$\{e_{10}, e_6, e_9, e_{11}\}$
e_{12}	$\{e_{12}, e_9\}$
e_{13}	$\{e_{13}, e_{11}\}$

Example 8.8.2 *Find the fundamental cutsets and fundamental circuits of the following graph w.r.t. given spanning tree.* **SPPU : May-07, Dec.-07, 14**



G



T

Solution : Here the spanning tree has 4 branches e_1, e_2, e_7, e_8. Therefore there are 4 fundamental cutsets corresponding to each branch of T which are given below.

Branch	Corresponding Fundamental Cutset
e_1	$\{e_1, e_5, e_6, e_3\}$
e_2	$\{e_2, e_6, e_3\}$
e_7	$\{e_7, e_3, e_4\}$
e_8	$\{e_8, e_4, e_5, e_6, e_3\}$

The chords of T are e_3, e_4, e_5, e_6

Therefore there are 4 fundamental circuits corresponding to each chord of T which are given below :

Chord	Corresponding Fundamental Cutset
e_3	$\{e_1, e_2, e_7, e_8, e_3\}$
e_4	$\{e_4, e_7, e_8\}$
e_5	$\{e_5, e_1, e_8\}$
e_6	$\{e_6, e_1, e_2, e_8\}$

8.9 Network Flows

SPPU : Dec.-12, 13, 14, 16, 17, 18, May-07, 14, 17, April-11

In a network of telephone lines, highways, railroads, pipelines of oils (gas or water) and so on, it is essential to know the maximum rate of flow that is possible from one station to another in the network. This type of network is represented by a weighted directed graph in which the vertices are stations and edges are lines through which the given commodity (water, gas, oils, etc.) flows

Transport Network

A weighted directed connected graph is said to be a **transport network** if the following conditions are satisfied.

i) It is without loops.

ii) There is one and only one vertex in the graph that has no incoming edge. It is known as **source.**

iii) There is one and only one vertex in the graph that has no outgoing edge. It is known as **sink.**

iv) The weight of each edge is a non negative real number. It is known as capacity of that edge. The capacity of the (i, j) edge is denoted by W(i, j).

A flow ϕ, in a transport network is an assignment of a non negative number $\phi(i, j)$ to each edge (i, j) such that

i) $$\phi(i, j) \leq W(i, j) \quad \text{for each } (i, j)$$

i.e. the amount of material to be transported through a route (edge) cannot greater than the capacity of that route.

ii) $$\sum_{\text{all } i} \phi(i, j) = \sum_{\text{all } k} \phi(i, j) \quad \text{for each vertex j except the source a and sink z.}$$

i.e. (Incoming material flow = Outgoing material flow)

The value of flow is given by

$$\phi_v = \sum_{\text{all } i} \phi(a, i) - \sum_{\text{all } k} \phi(k, z)$$

For a given flow an edge (i, j) is said to be **saturated** if $\phi(i, j) = W(i, j)$

A edge (i, j) is said to be unsaturated if ϕ (i, j) < W(i, j)

8.9.1 Maximum Flow

The maximum flow in a transport network is a flow that achieves the largest possible value.

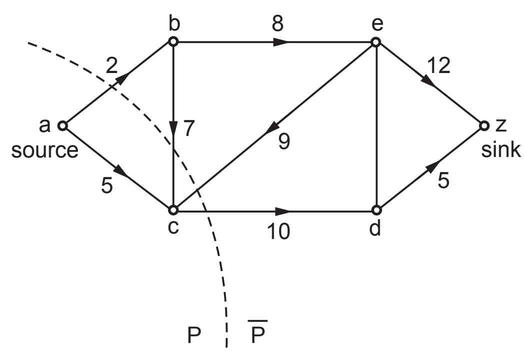
Cut : A cut in a transport network is a cutset of the undirected graph, obtained from the transport network by ignoring the direction of the edges that separate the source from sink.

The notation $(P, \overline{P})$ is used to denote a cut that divides the vertices into two subsets P and $\overline{P}$ where the subset P contains the source and the subset $\overline{P}$ contains the sink.

The capacity of a cut is denoted by $w(P, \overline{P})$ and defined as the sum of the capacities of those edges incident from the vertices in P to the vertices in $\overline{P}$.

i.e. $$w(P, \overline{P}) = \sum_{i \in p \text{ and } j \in \overline{P}} w(i, j)$$

Consider the following example.



The dashed line in the above figure identifies a cutset that separates subset of vertices {a, c} = P from the subset of vertices $\overline{P}$ = {b, d, e, z}

The capacity of this cut is

$$w(P, \overline{P}) = w(a, b) + w(c, d)$$

$$= 2 + 10 = 12$$

Theorem 1 : The value of any flow in a transport network is less than or equal to the capacity of any cut in the network.

Theorem 2 : For any cut $(P, \overline{P})$, the value of a flow equal to the sum of flows in the edges from the vertices in P to the vertices in $\overline{P}$ minus the sum of flows in the edges from the vertices in $\overline{P}$ to the vertices in P.

Theorem 3 : In a given transport network G, the maximum value of a flow from source a to sink z is equal to minimum value of the capacities of all the cuts in G that separate the source a from the sink z.

e.g. : Consider transport network in the given Fig. 8.9.1 below

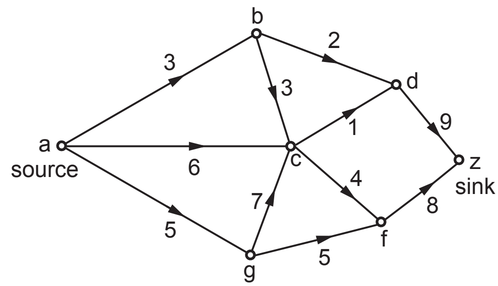


Fig. 8.9.1

Vertex a is a source and z is a sink.

The vertex set of some such cut are listed in the following table.

Vertex set P	Capacity w(p, $\overline{p}$)
{a}	14(3 + 6 + 5)
{a, b}	16(5 + 6 + 3 + 2)
{a, c}	13(3 + 5 + 1 + 4)
{a, d}	23(3 + 6 + 5 + 9)
{a, f}	22(3 + 6 + 5 + 8)
{a, b, c}	15(5 + 2 + 1 + 4 + 3)
{a, b, d}	23(6 + 5 + 3 + 9)
{a, b, c, g}	12(2 + 1 + 4 + 5)

8.9.2 Constructing a Maximum Flow in a Transport Network

For any cut (P, $\overline{P}$) the values of a flow in a transport network equal the sum of flows in the edges from the vertices in P to the vertices in $\overline{P}$ minus the sum of flows in the edges from the vertices in $\overline{P}$ to the vertices in P.

Whenever we can construct a flow ϕ the value of which is equal to the capacity of some cut, we can be certain that ϕ is a maximum flow.

8.9.3 Labelling Procedure for Finding Maximum Flow in the Network

1) The source a is labelled (–, ∞). It means that (out from nowhere) the source can supply can infinite amount of material to the other vertices.

2) A vertex b that is adjacent from a is labelled $(a^+, \Delta b)$, where Δb is equal to $w(a, b) - \phi(a, b)$,

if $w(a, b) > \phi(a, b)$;

i.e. $$\Delta b = w(a, b) - \phi(a, b) \qquad [\text{if } w(a, b) > \phi(a, b)]$$

The vertex is not labelled if $w(a, b) = \phi(a, b)$

3) Scan and label all the remaining vertices adjacent to a. Also scan and label all the vertices adjacent to labelled vertices.

Suppose vertex q is adjacent to labelled vertex b, then q is labelled as $(b^+, \Delta q)$

where $$\Delta q = \min\{\Delta b, [w(b, q) - \phi(b, q)]\}$$

if $w(b, q) > \phi(b, q)$

Vertex q is not labelled if $w(b, q) = \phi(b, q)$

We can also label vertex q as $(\bar{b}, \Delta q)$ where

$$\Delta q = \min[\Delta b, \phi(q, b)] \quad \text{if} \quad \phi(q, b) > 0$$

4) Repeat step 3 till we reach to sink z.

5) If we repeat this labelling procedure, two cases shall arise while labelling the sink z.

Case I :

i) Sink z is labelled, say with a label $(y^+, \Delta z)$ [z is never labelled $(y^-, \Delta z)$]

ii) Vertex y can be labeled as $(q^+, \Delta y)$ or $(q^-, \Delta y)$ for some adjacent vertex q.

iii) If y is labelled $(q^+, \Delta y)$ (q^+ means increase in flow) then we increase the flow in edge (q, y) to $\phi(q, y)$ to Δz. Similarly for the label $(q^-, \Delta y)$ we decrease the flow in edge (q, y) to $\phi(q, y) - \Delta y$.

iv) This process is continued back to source a till the value of flow is increased by amount Δz.

v) Again start the labelling procedure to further increase value of flow in the network.

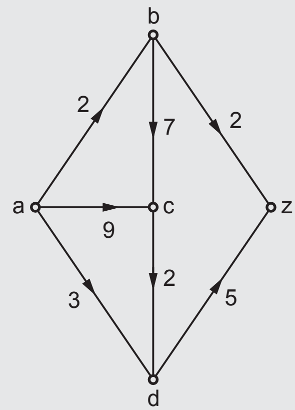
Case II :

i) If the sink z is not labelled, then denote all labelled vertices as P and all unlabelled vertices as $\bar{P}$.

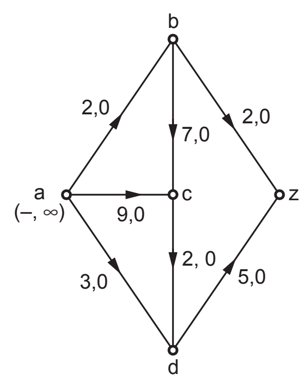
ii) The fact that sink z is not labeled means flow each edge directed from vertices of P to vertices of $\bar{P}$. is equal to capacity of cut $(P, \bar{P})$ is thus maximum flow.

Example 8.9.1 *Determine the maximal flow in the following transport network.*

SPPU : Dec.-12, May-14



Solution : Step 1 : Assign the flow zero to each edge and the label (–, ∞) to the source a.



Step 2 : The vertices b, c, d are adjacent to the source a.

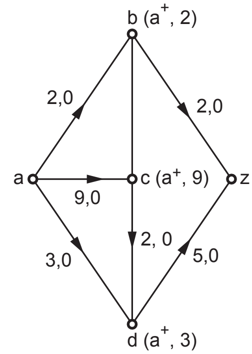
Therefore, we label the vertices b, c, d.

For the vertex b

$$w(a, b) - \phi(a, b) = 2$$

i.e. $\Delta b = 2$

Similarly $\Delta c = 9$ and $\Delta d = 3$



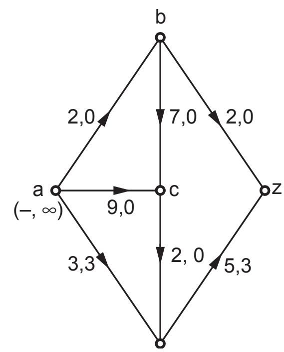
Now sink z is adjacent to both vertices b and d, so we arbitrarily choose vertex d and label.

sin z as $(d^+, 5)$ as

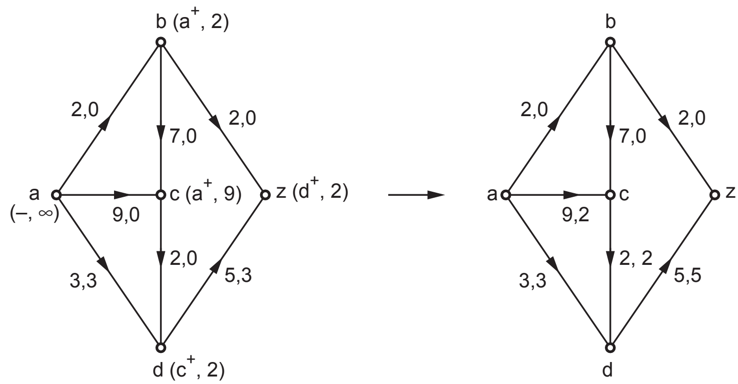
$$w(d,\ z) - \phi\,(d,\ z) \;=\; 5 - 0 = 5$$

and $$\Delta z \;=\; \min\{\Delta d,\ \{w(d,\ z) - \phi\,(d,\ z)\}$$
$$=\; \min\{3,\ 5\} = 3$$

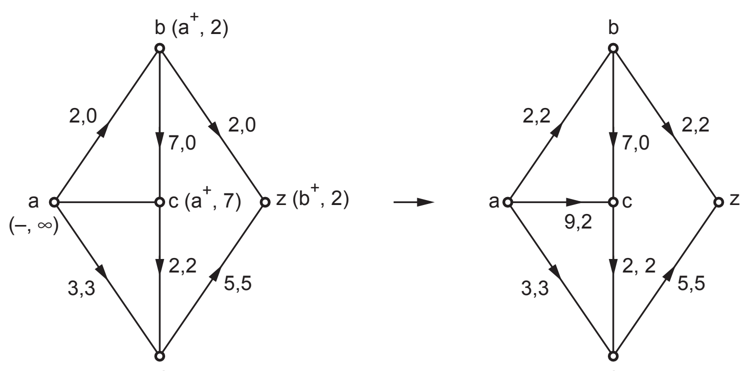
According to label of sink, we now adjust flow of edge (d, z) and edge (a, d)



Step 3 :

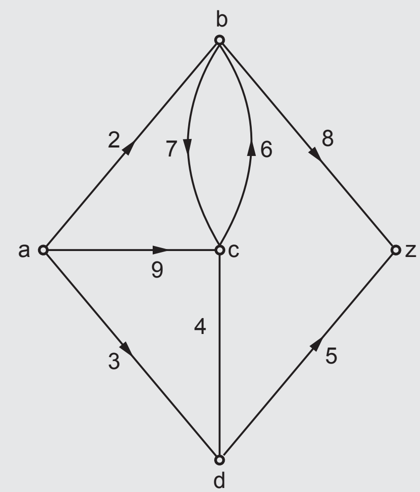


Step 4 :

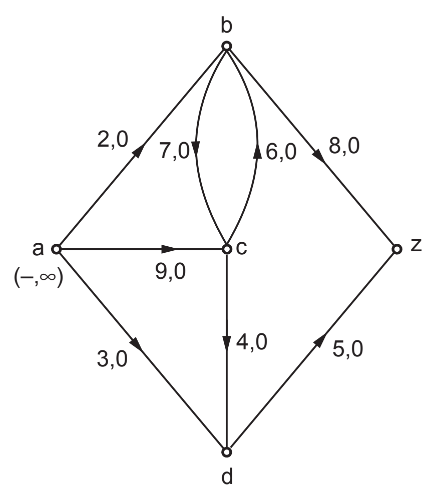


From the step 4, it is clear that the maximum flow is 7.

Example 8.9.2 *Determine the maximal flow in the flowing transport network.* **SPPU : Dec.-13**



Solution : Step 1 : Assign the flow zero to each edge and the label (–, ∞) to the source a.

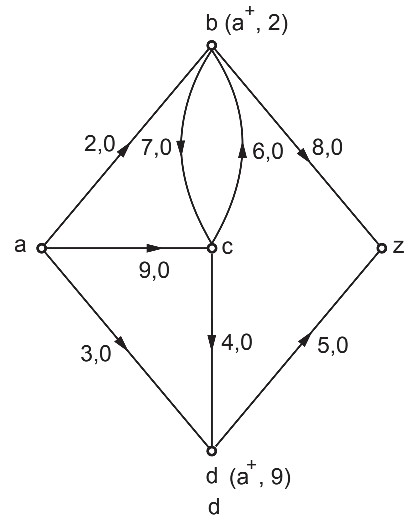


Step 2 : The vertices b, c, d are adjacent to the source a. Therefore, we label the vertices b, c, d for the vertex b.

$w(a, b) - \phi(a, b) = 2$

i.e. $\Delta b = 2$,

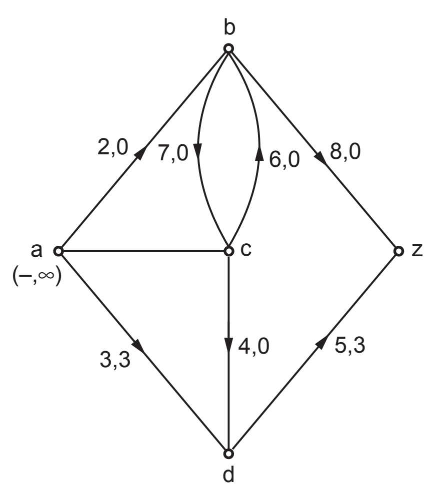
Similarly $\Delta c = 9$, $\Delta d = 3$



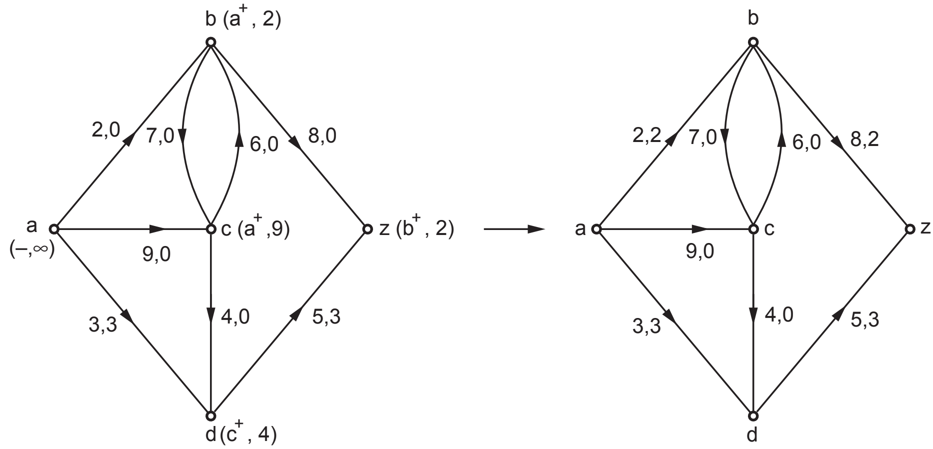
Now sink z is adjacent to both vertices b and d, so we can choose any vertex b or d for labelling of sink z. Let us choose the vertex d. The label of sink z is $(d^+, 5)$ as $w(d, z) - \phi\,(d, z) = 5 - 0 = 5$

and $\Delta z = \min\{\Delta d, w(d, z) - \phi\,(d, z)\} = \min\{3, 5\} = 3$

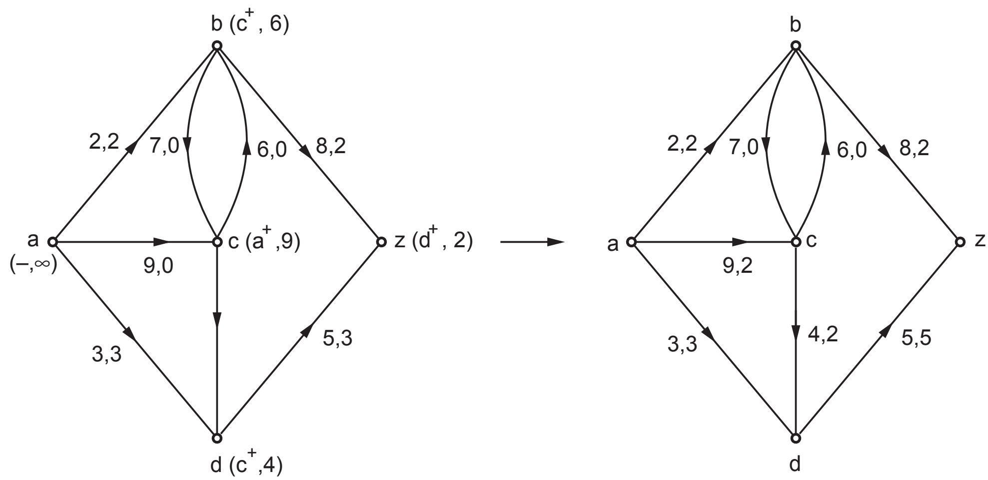
According to label of sink, we now adjust flow of edge (d, z) and edge (a, d)



Step 3 :



Step 4 :



Step 5 :