MTL390 : Statistical Methods

Assignment-2

Name: Rishit Sanmukhani
Entry no: 2013CS10255

Data set: Uniform distribution $U(5,14)$

* Distribution = $U(a,b)$

Parameters = $\{a,b\}$

Now, $f_X(x;a,b) = \frac{1}{b-a}$, $x \in [a,b]$

- Considering $X_1, X_2, \ldots, X_n$ to be iid samples from $X \sim U(a,b)$

$$f_{\tilde{X}}(\tilde{x};a,b) = \prod_{i=1}^{n} f_X(x_i;a,b)$$

$$= \left(\frac{1}{b-a}\right)^n, \quad x_i \in [a,b], \forall i$$

- Now, above function is clearly a non-differentiable function. So can't maximize it via derivative.

$\therefore$ To maximize $\Rightarrow$ we have to minimize $(b-a)$

But $a \le x_i, \forall i$,
$b \ge x_i, \forall i$

$\therefore$ MLE estimates are:

$$\boxed{\begin{aligned} \hat{a} &= \min(x_i), \ i=1,2,\ldots,n \\ \hat{b} &= \max(x_i), \ i=1,2,\ldots,n \end{aligned}}$$

Now, I am using one of the estimate to convert $U(a,b)$ to $U(0,\theta)$

i.e $\hat{a} = \min(x_i) \approx 5$ (in my case)

$\therefore$ For my distribution, $\boxed{x_i' = x_i - 5}$

$\therefore$ I am changing my distribution from,

$$U(5,14) \longrightarrow U(0,9)$$
$$x_i \qquad\qquad x_i' = x_i - 5$$

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* New distribution : $U(0,\theta)$ | mean $\mu = \theta/2$
variance $\sigma = \theta^2/12$

Parameters $= \theta$

Q-1 i) method of moments estimate:

$$E(X) = \frac{\sum x_i}{n}$$

$$\therefore \frac{\theta}{2} = \frac{\sum x_i}{n}$$

$\therefore$ Estimator $\boxed{T(\tilde{x}) = \frac{2\sum x_i}{n}}$ $i=1,\dots,n$

ii) MLE estimate:

Likelihood $L(\tilde{x};\theta) = \frac{1}{\theta^n}$, $\forall x_i \in [0,\theta]$

To maximize, θ = minimize as possible

But $\theta \geq x_i, \forall i$

$\therefore \theta = \max(x_i)$

$\therefore \boxed{T(\tilde{x}) = \max(x_i), i = 1, \dots, n}$

Note: In case of $U(a,b)$ i) M.O. Moments $\hat{a} = \bar{x} - \sqrt{3}s$

$\hat{b} = \bar{x} + \sqrt{3}s$

(using, $\frac{a+b}{2} = \frac{\sum x_i}{n} = \bar{x}$

$\frac{(b-a)^2}{12} = \frac{\sum x_i^2}{n} = s^2$)

ii) MLE: $\hat{a} = \min(x_i)$

$\hat{b} = \max(x_i)$ $i = 1, \dots, n$

(proof shown earlier)

Q-2 (a) i) Methods of moments:

$T(\tilde{x}) = \frac{2\sum x_i}{n}, i = 1, \dots, n$

$\therefore E(T(\tilde{x})) = E\left(\frac{2\sum x_i}{n}\right)$

$= \frac{2}{n} \sum_{i=1}^{n} E(X_i)$

$= \frac{2}{n} * n E(X)$ ($\because X_i \to$ iid from X, $E(X_i) = E(X)$)

$= \frac{2}{n} * n * \frac{\theta}{2} = \theta$

$\therefore \boxed{E(T(\underset{\sim}{x})) = \theta} \implies$ $T(\underset{\sim}{x})$ unbiased estimator for θ.

$var(T(\underset{\sim}{x})) = \frac{4(n\, var(X))}{n^2}$ ($\because X_i \to$ iid from X)

$= \frac{4}{n^2} * n * \frac{\theta^2}{12}$

$$\boxed{var(T(\underset{\sim}{x})) = \frac{\theta^2}{3n}}$$

$\therefore var(T(\underset{\sim}{x})) \to 0$ as $n \to \infty$

$\therefore$ consistent estimator

ii) • MLE Estimate:

$X = T(\underset{\sim}{x}) = max(x_i) \quad i = 1, 2, \dots, n$

$\therefore P(X \leq x) = \prod_{i=1}^{n} P(X_i \leq x)$.

$= \prod_{i=1}^{n} \left(\frac{x}{\theta}\right) \quad , x \in [0, \theta]$

$= \left(\frac{x}{\theta}\right)^n$

$\therefore f_X(x;\theta) = \frac{n x^{n-1}}{\theta^n} \quad , x \in [0, \theta]$

$= 0$ otherwise.

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Now, $E(X) = \int_0^\theta x \cdot \frac{n x^{n-1}}{\theta^n}$

$= \frac{n}{\theta^n} \frac{\theta^{n+1}}{n+1} = \frac{n}{n+1}\theta \longrightarrow \theta$ as $n \longrightarrow \infty$

$E(X^2) = \int_0^\theta x^2 \frac{n x^{n-1}}{\theta^n}$

$= \frac{n}{\theta^n} \frac{\theta^{n+2}}{n+2} = \frac{n}{n+2}\theta^2$

Now, $E(X) = \frac{n\theta}{n+1} \neq \theta \Rightarrow$ $\boxed{\text{Biased estimator}}$

$Var(X) = E(X^2) - E(X)^2 = \frac{n\theta^2}{n+2} - \left(\frac{n}{n+1}\right)^2 \theta^2$

$= \left(\frac{n}{n+2} - \frac{n^2}{(n+1)^2}\right)\theta^2 = \frac{n\theta^2}{(n+2)(n+1)^2} \longrightarrow 0$

as $n \rightarrow \infty$

$\therefore$ $\boxed{\text{consistent estimator}}$

Since, estimator - i = unbiased
estimator - ii = biased,

efficiency can't be compared.

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2 (b) i) $T_0(\tilde{x}) = 2\frac{\sum x_i}{n}$

$$\boxed{\hat{\theta} = 9.0066}$$

ii) $T_0(\tilde{x}) = \max(x_i)$

$$\boxed{\hat{\theta} = 8.9987}$$

Q3 Using method of moments:

* $U(0,\theta)$

$E(x) = \int_0^{\theta} x f_x(x;\theta)\,dx$

$= \int_0^{\theta} x \cdot \frac{1}{\theta}\,dx$

$= \frac{\theta}{2}$

Now, $E(x) = \frac{\sum x_i}{n}$

$\therefore \frac{\theta}{2} = \frac{\sum x_i}{n} \Rightarrow \boxed{\hat{\theta} = 2\frac{\sum x_i}{n}}$

Note: $U(a,b)$

$E(x) = \frac{b+a}{2}$, $Var(x) = \frac{(b-a)^2}{12}$

$\therefore$ $\left.\begin{array}{l} \frac{b+a}{2} = \frac{\sum x_i}{n} = \bar{x} \\ \frac{(b-a)^2}{12} = \frac{1}{n}\sum(x_i-\bar{x})^2 = s^2 \end{array}\right\} \Rightarrow \begin{array}{l} \hat{a} = \bar{x} - \sqrt{3}s \\ \hat{b} = \bar{x} + \sqrt{3}s \end{array}$

where $\bar{x} = \sum x_i / n$

$s^2 = \sum(x_i-\bar{x})^2/n$

4 In our case, Cramer Rao lower bound can't be used to find UMVUE.

($\because$ condition - 3 is violated).

- Hence we will use Rao-Blackwell Theorem.

Step-1 $\tilde{g}(\tilde{X};\theta) = \frac{n+1}{n} \max(X_i)$

$\tilde{g}(\tilde{X})$ is an unbiased estimator for θ.

$\left(\because E(\max(X_i)) = \frac{n\theta}{n+1} \Rightarrow E(\tilde{g}(\tilde{X})) = \theta\right)$

Step-2 $T(\tilde{X}) = \max(X_i)$

clearly T is a sufficient statistics.

$\left(\because f_{\tilde{X}}(\tilde{x}) = \frac{I\{\theta > x\}}{\theta^n}\right.$, where $x = \max(x_i)$

$I\{\theta > x\} = 1$ if true
0 if false

Here $h(\tilde{x}) = 1$

$g(\tilde{x}) = \frac{I\{\theta > x\}}{\theta^n}$

Step-3 $T^*(t) = E\left(\tilde{g}(\tilde{X}) / T(\tilde{X}) = t\right)$

$= E\left(\frac{n+1}{n} \max(X_i) / \max(X_i) = t\right)$

$T^*(t) = \frac{n+1}{n} t$

$\therefore$ $\boxed{\dfrac{n+1}{n} \max(X_i) \text{ is UMVUE}}$

$$\text{Estimate} = \left(\frac{5000+1}{5000}\right)(8.9987)$$

$$= 9.0005$$

Q5 Here, we estimate μ = unknown.
But σ^2 = unknown

$\therefore$ we use T distribution.

But as $n \geq 30$, we use standard normal distribution.

$\therefore$ Interval for confidence is:

$$\left(\bar{x} - Z_{\alpha/2}\frac{s}{\sqrt{n}},\ \bar{x} + Z_{\alpha/2}\frac{s}{\sqrt{n}}\right)$$

$\alpha = 0.01 \Rightarrow (49.4083, 49.5982)$
$\alpha = 0.05 \Rightarrow (49.4309, 49.5757)$
$\alpha = 0.1 \Rightarrow (49.4423, 49.5642)$

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Now, we have

$$\mu = \frac{\theta}{2}$$

$$\therefore \theta = 2\mu$$

$\therefore$ confidence intervals for θ:

$\alpha = 0.01 \Rightarrow (8.8166, 9.1965)$
$\alpha = 0.05 \Rightarrow (8.8617, 9.1514)$
$\alpha = 0.1 \Rightarrow (8.8846, 9.1285)$