LandTrendR

Notation and definitions: For an $m \times n$ matrix A, an n-vector x, $I \subset \{1, \ldots, m\}$, $J \subset \{1, \ldots, n\}$, let A_{IJ} denote the submatrix of A formed from the rows indexed by I and the columns indexed by J, and x_J denote the subvector of x indexed by J. $A_{I\cdot}$ ($A_{\cdot J}$) are the rows (columns) of A indexed by I (J), respectively. An image is an $R \times C$ matrix D, where each D_{rc} (pixel) is an $S \times B$ matrix, whose (s, b) element $(D_{rc})_{sb}$ is the signal value at time index s and frequency band index b.

Algorithm LandTrendR.

for band $b = 1 : B$
  for row $r = 1 : R$
    for col $c = 1 : C$ **do**

Step 1: Despike

Let $u = (D^0_{rc})_{\cdot b}$ denote the raw time series data. For each time point t_i, $1 < i < S$, define $\Delta u_i = (D^0_{rc})_{(i+1)b} - (D^0_{rc})_{ib}$, $\nabla u_i = (D^0_{rc})_{ib} - (D^0_{rc})_{(i-1)b}$, $\mu\delta u_i = (D^0_{rc})_{(i+1)b} - (D^0_{rc})_{(i-1)b}$, $k_i = 1 - |\mu\delta u_i| / \max\{|\nabla u_i|, |\Delta u_i|\}$, and correction

$$\kappa_i = (\delta^2 u_i) k_i / 2 = \big((D^0_{rc})_{(i-1)b} - 2(D^0_{rc})_{ib} + (D^0_{rc})_{(i+1)}\big) k_i / 2.$$

For each i such that $k_i = \max\limits_{1<j<S} k_j$, update $(D_{rc})_{ib} := (D^0_{rc})_{ib} + \kappa_i$. Repeat iteratively until $\max\limits_{1<j<S} k_j < \upsilon$, some given despiking tolerance.

Step 2: Find potential breakpoints

Let $S^1 = (t_1, \ldots, t_S)$ be the original sequence of time points and $I^1 = (2, \ldots, S-1)$ denote the corresponding sequence of interior indices. Let

$$X = \begin{pmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_S \end{pmatrix}$$

be the Gram matrix for the time points $t_1, \ldots, t_S$, for ordinary least squares linear regression. The least squares fit to this data is given by

$$u(t) = \alpha_0 + \alpha_1 t$$

with coefficients

$$\alpha = (X^t X)^{-1} X^t u$$

and residuals

$$E^1(\alpha) = u - X\alpha.$$

Find the smallest index i_1 corresponding to the maximum absolute deviation, i.e.,

$$i_1 = \min\big\{i \mid i \in I^1 \text{ and } |E^1(\alpha)_i| = \max_{j \in I^1} |E^1(\alpha)_j|\big\}.$$

Split the sequence S^1 into two subsequences,

$$S_l^1 = (t_1, \ldots, t_{i_1}) \text{ and } S_r^1 = (t_{i_1}, \ldots, t_S).$$

Do linear regression on each of these and compute their respective mean squared errors, MSE_l and MSE_r. Suppose $|\text{MSE}_l| \leq |\text{MSE}_r|$. Then let $S^2 = S_r^1$ with interior index set $I^2 = (i_1 + 1, \ldots, S - 1)$ be the next candidate sequence for 'breakpoint search'.

Again, find the smallest index i_2 corresponding to the maximum absolute deviation

$$i_2 = \min\big\{i \mid i \in I^2 \text{ and } |E^2(\alpha^2)_i| = \max_{j \in I^2} |E^2(\alpha^2)_j|\big\}.$$

Again, split S^2 into two subsequences $S_l^2 = (t_{i_1}, \ldots, t_{i_2})$ and $S_r^2 = (t_{i_2}, \ldots, t_S)$, compute the least squares fit for each of these, choose the interval with higher MSE, and find the index i_3 corresponding to maximum absolute deviation. Recursively apply the algorithm until there are $\mu + \nu + 1$ breakpoints (including t_1 and t_S), where μ is the maximum number of segments allowed and ν is the maximum number of vertex overshoots (see Step 3 below) allowed. (In the rare circumstance that $\text{MSE}_l = \text{MSE}_r = 0$ at some iteration, there may be fewer than $\mu + \nu + 1$ breakpoints.)

Let $\bar{S} = (t_1, t_{i_1}, \ldots, t_{i_{\mu+\nu-1}}, t_S)$ be the final sequence of (sorted) breakpoints thus obtained, $\bar{I} = (1, i_1, \ldots, i_{\mu+\nu-1}, S)$ and $\bar{V} = (v_1, v_{i_1}, \ldots, v_S)$ be the corresponding index and 'vertex' sequences, where $v_i = (t_i, u_i)$.

Step 3: Cull by angle change

Define the sequence of angles

$$\alpha_j = \arccos\left(\frac{(v_{\bar{I}_j} - v_{\bar{I}_{j-1}}) \cdot (v_{\bar{I}_{j+1}} - v_{\bar{I}_j})}{\|(v_{\bar{I}_j} - v_{\bar{I}_{j-1}})\| \|(v_{\bar{I}_{j+1}} - v_{\bar{I}_j})\|}\right), \text{ for } j = 2, 3, \ldots, \mu + \nu.$$

Find $\bar{a} = \min\big\{i \mid \alpha_i = \min_j \alpha_j\big\}$, delete $v_{\bar{a}}$ from the sequence $\bar{V}$, and recalculate from this the angles with the new vertices. Repeat until reaching the sequence $V^* = (v_1, v_{l_1}, \ldots, v_{l_{\mu-1}}, v_S)$ with index sequence $L^* = (1, l_1, \ldots, l_{\mu-1}, S)$.

Step 4: Fit trajectories

Moving from $i = 1, \ldots, \mu$, consider consecutive vertices $v_{L_i^*}, v_{L_{i+1}^*} \in V^*$, one at a time, and an anchored regression fit

$$u_{AR}(t) = y_{L_i^*} + \alpha(t - t_{L_i^*}),$$

where $y_{L_i^*}$ is the 'fitted' value inferred from the fit in the preceding interval $\big(t_{L_{i-1}^*}, t_{L_i^*}\big)$, α is the solution to the least squares regression problem $u_{AR} \approx u$ at the points $t_{L_i^*+1}, \ldots, t_{L_{i+1}^*}$. For the special case $i = 1$, the coefficient $y_{L_1^*}$ is also estimated. The final result of this step will be a continuous piecewise linear function $P^*(t)$ covering the full domain. Further, let $Y^* = (y_1, y_{l_1}, \ldots, y_{l_{\mu-1}}, y_S)$ be the sequence of fitted values at the breakpoints with indices L^*. Call the tuple $\mathcal{M}^* = (P^*(t), L^*, V^*)$ a regression model. In addition, let $(y_1, y_2, \ldots, y_S)$ be the sequence of fitted values over all time points in S^1 as predicted by $\mathcal{M}^*$.

Step 5: Model statistics

The improvement in prediction from regression compared to the mean model is given by the random variable

$$X_1^2 = \sum_{i=1}^{S}(u_i - \bar{u})^2 - \sum_{i=1}^{S}(u_i - y_i)^2 = \sum_{i=1}^{S}(y_i - \bar{u})^2,$$

where $\bar{u}$ is the mean value of the observations. The squared distance of the observed values from the values predicted by the regression model is the random variable

$$X_2^2 = \sum_{i=1}^{S}(u_i - y_i)^2.$$

Assuming that $y_i - \bar{u}$ and $u_i - y_i$ are independent, normally distributed, and have variance one, X_1^2 and X_2^2 have a χ^2 distribution with degrees of freedom $d_1 = \mu$, $d_2 = S - \mu - 1$, respectively. Therefore, the ratio

$$F = \frac{X_1^2/d_1}{X_2^2/d_2}$$

has an F-distribution and F-statistics can be used for measuring the 'goodness' of fit of the regression model. Let f be the F-statistic for model $\mathcal{M}$. Calculate the p-value of this F-statistic:

$$Q(f|d_1, d_2) = 1 - I_{\frac{d_1 f}{d_1 f + d_2}}\left(\frac{d_1}{2}, \frac{d_2}{2}\right) = I_{\frac{d_2}{d_2 + d_1 f}}\left(\frac{d_2}{2}, \frac{d_1}{2}\right).$$

$Q(f|d_1, d_2)$ is the probability that $F > f$ and $I_x(a, b)$ denotes the regularized incomplete Beta function given by

$$I_x(a, b) = \frac{1}{B(a, b)} \int_0^x t^{a-1}(1-t)^{b-1} dt, \qquad a > 0, \quad b > 0,$$

where

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1} dt$$

is the (complete) Beta function, and

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \qquad \Re z > 0$$

is the Gamma function.

For model $\mathcal{M}^{(\mu)} = \mathcal{M}^*$, where the superscript corresponds to the number of segments in the model, let the p-value calculated in this step be $p^{(\mu)}$.

Step 6: Generate more (simpler) models

Begin with the model $\mathcal{M}^{(\mu)} = (P^{(\mu)}, L^{(\mu)}, V^{\mu)}) = (P^*(t), L^*, V^*)$.

(i) Assume that a disturbance always corresponds to a positive slope while a negative slope indicates recovery.

First look for negative slopes at interior vertices. Let $V^{(\mu)}_{i^{(\mu)}} \neq V^{(\mu)}_1$ be the left vertex (leftmost in case of a tie) of the segment with steepest negative slope. Delete $L^{(\mu)}_{i^{(\mu)}}$ from the index sequence $L^{(\mu)}$, giving the shorter sequence $L^{(\mu-1)}$.

However, if no negative slopes are found or, if the steepest negative segment happens to be the leftmost segment of the current model, then for each interior vertex $v^{(\mu)}_{L_i}$ in the model, consider the point to point connect using the vertices immediately to the left and right of $v^{(\mu)}_{L_i}$, i.e., $v^{(\mu)}_{L_{i-1}}$ and $v_{L^{(\mu)}_{i+1}}$:

$$u^{(\mu)}_{PP}(t) = \frac{y_{L^{(\mu)}_{i+1}} - y_{L^{(\mu)}_{i-1}}}{t_{L^{(\mu)}_{i+1}} - t_{L^{(\mu)}_{i}}}(t - t_{L^{(\mu)}_{i-1}}) + y_{L^{(\mu)}_{i-1}}, \quad i \in L^{(\mu)} \setminus \{1, S\},$$

and calculate the $\mathrm{MSE}_{L^{(\mu)}_i}$ for vertex $v_{L^{(\mu)}_i}$ as

$$\mathrm{MSE}_{L^{(\mu)}_i} = \frac{1}{t_{L^{(\mu)}_{i+1}} - t_{L^{(\mu)}_{i-1}}} \sum_{L^{(\mu)}_{i-1} \le k \le L^{(\mu)}_{i+1}} \Big(u^{(\mu)}_{PP}(t_k) - u_k\Big)^2, \quad i \in L^{(\mu)} \setminus \{1, S\}.$$

Then define $i^{(\mu)} = \min\big\{i \mid \mathrm{MSE}_{L^*_i} = \min_j \mathrm{MSE}_{L^*_j}\big\}$, the index of the vertex dropping which leads to least MSE. Delete $L^{(\mu)}_{i^{(\mu)}}$, resulting in the shorter sequence $L^{(\mu-1)}$.

(ii) Remove the corresponding vertex from $V^{(\mu)}$ giving $V^{(\mu-1)}$, and as in Step 4 generate the new piecewise linear fit $P^{(\mu-1)}(t)$ and model $\mathcal{M}^{(\mu-1)} = \big(P^{(\mu-1)}(t), L^{(\mu-1)}, V^{(\mu-1)}\big)$.

(iii) Calculate the p-value $p^{(\mu-1)}$ for this model.

Proceeding in this way, generate a total of μ models $\mathcal{M}^{(i)}$, $i = \mu, \ldots, 1$.

Step 7: Pick best model $\mathcal{M}_{i^*}$.

Let i^* be the smallest index i corresponding to the models $\mathcal{M}^{(i)}$ whose p-value is less than a user defined recovery threshold τ, i.e., $i^* = \min\{i \mid p^{(i)} \le \tau,\, i = 1, \ldots, \mu\}$.

Remark. Check the linear segment slopes. If, for any model $\mathcal{M}^{(i)}$ under consideration, the recovery (to a global baseline) happens quicker than the quickest disturbance (from a global baseline), that model is discarded.

Step 8: Alternate approach.

If no models are found using Steps 1–7, repeat Steps 4–7 with following modifications:

Step $4'$**.** Instead of computing the continuous piecewise linear approximation $P^*(t)$ one segment at a time, going from left to right, compute $P^*(t)$ using all the data at once. This is done by expressing $P^*(t)$ as a linear combination of B-splines of order 2 with knot sequence $(t_1, t_1, t_{l_1}, t_{l_2}, \ldots, t_{l_{\mu-1}}, t_S, t_S)$, and then solving a linear least squares problem for the $\mu + 1$ coefficients of these B-spline basis functions. (Note that there is no need to use the Levenberg-Marquardt algorithm as proposed in [R. E. Kennedy, Z. Yang, and W. B. Cohen, 2010]).

Step $6'$**.** Skip directly to the point to point connect approach, without looking for negative slopes at all.

end

end

end