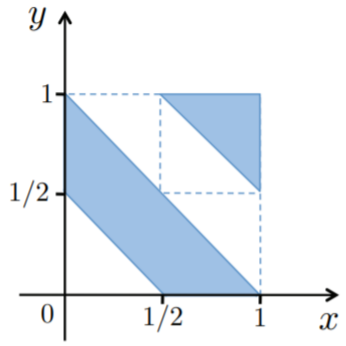
5 Assignment 3

5.1 Continuous Random Variables

A pair of jointly continuous random variables, X and Y, have a joint probability density function given by

$$f_{X,Y}(x,y) = \begin{cases} c, & \textit{in the shaded region of figure} \\ 0, & \textit{otherwise} \end{cases}$$



1. *Find c.* **(1)**

2. *Find marginal PDF of X and Y.* **(1.5)**

3. *Find* $\mathbb{E}(X|Y = 1/4)$ *and* $Var(X|Y = 1/4)$. **(1.5)**

4. *Find the conditional PDF for X given that* $Y = 3/4$. **(1)**

5.2 Card Shuffling

Consider a deck of cards numbered $1, \cdots, n$. *Suppose the cards are arranged as* $1, 2, \cdots, n$ *initially from top to bottom. Define a shuffle operation to be picking the top card and placing it at a random position bellow it (n positions are there including the top position). Suppose you do a sequence of shuffle operations to the deck of cards.*

1. *Now let* T_i *be the random variable which counts the number of shuffles that are performed until for the first time i cards lie below card numbered n (This is the card which was initially at the bottom). What is the PDF of the random variable* $T_i - T_{i-1}$? *(Need to also explain why it is so).*
   Hint: $T_i - T_{i-1}$ *is the random variable which is the number of shuffles that was done after* $i-1$ *cards went bellow n for the next card to go bellow n. Does it depend on* T_{i-1}?

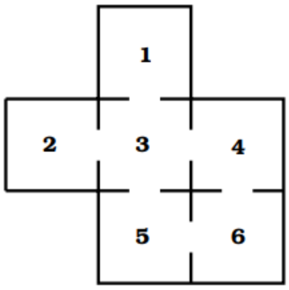
2. *Prove that after* $T_{n-1} + 1$ *shuffles, all possible* $n!$ *arrangements are equally likely.*

3. *Find the expected value of* $T_{n-1} + 1$. *Hint:* $T_{n-1} = T_1 + T_2 - T_1 + T_3 - T_2 + \cdots + T_{n-1} - T_{n-2}$

4. *Show that if you shuffle* $100 \cdot n \cdot \ln n$ *times, with* 99% *chance, the deck will be completely random (ie. all* $n!$ *possibilities are equally likely). Hint: The nth Harmonic number can be approximated by* $\ln n$ *(the natural logarithm of* n*).*

5.3

Suppose you are lost in a maze shown below. At each step you leave the room by choosing at random one of the doors out of the room.



1. *Give the transition matrix P for this Markov chain.*
2. *Show that it is irreducible but not aperiodic.*
3. *Find the stationary distribution*
4. *Now suppose there is a trap in Room 5 and you start in Room 1. Find the expected number of steps before reaching Room 5 for the first time, starting in Room 1.*
5. *Find the expected time to return to room 1.*
