MCS 1

Assignment-1

Probability Basics

1.1 Social Groups

Problem can be simply converted to counting number of labelled graph with 5 vertices that contains triangles.

Edges (x)	Total graph possible with x edges	Graphs containing triangle
0	${}^{10}C_0 = 1$	0
1	${}^{10}C_1 = 10$	0
2	${}^{10}C_2 = 45$	0
3	${}^{10}C_3 = 120$	10
4	${}^{10}C_4 = 210$	70
5	${}^{10}C_5 = 252$	180
6	${}^{10}C_6 = 210$	200
7	${}^{10}C_7 = 120$	120
8	${}^{10}C_8 = 45$	45
9	${}^{10}C_9 = 10$	10
10	${}^{10}C_{10} = 1$	1
Total	1024	636

Can never contain triangle

Always contain triangle

$\therefore$ Probability $= 636/1024$

Explanation →

3 edges containing a triangle $= {}^5C_3 = 10$

Selecting 3 vertices from 5 to form a triangle

4 edges containing a triangle →

$$^5C_3 * 7 = 10 * 7 = 70$$

$^5C_3$ → Selecting 3 vertices to form a triangle

7 → 3 edges forming a triangle and from remaining 7 edges we can choose anyone as 4^{th} edge

5 edges containing a triangle →

(i)

$$= {}^5C_3 * {}^3C_1 = 30$$

$^5C_3$ → Triangle

$^3C_1$ → One vertex from triangle connected to remaining 2 vertices

(ii)

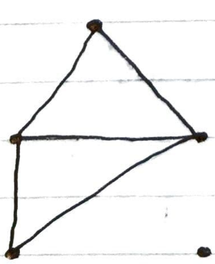
$$= {}^5C_3 * {}^3C_2 * 2 = 60$$

$^5C_3$ → Triangle

$^3C_2$ → Choose 2 vertices from triangle to connect with remaining vertices

2 → Selected 2 vertices can be connected in either ways

(iii)



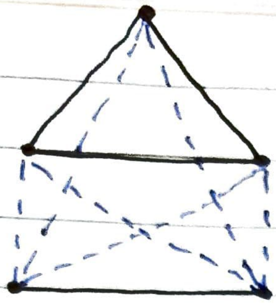
$$= {}^5C_4 * \frac{(4-1)!}{2} * 2 = 30$$

$^5C_4$ → Choosing 4 vertices

$\frac{(4-1)!}{2}$ → Circular permutations of 4 vertices

2 → Choose 1 diagonal from 2

(iv)



$$= {}^5C_3 \;*\; 1 \;*\; 6 \;=\; 60$$

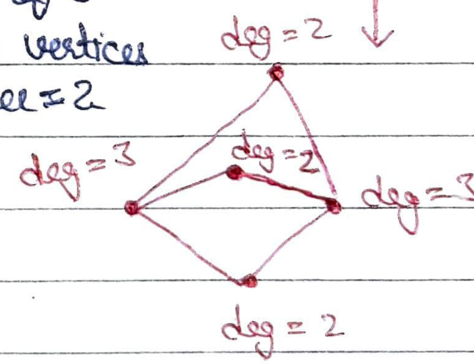
- ${}^5C_3$: Triangle
- 1: Edge between remaining 2 vertices is always selected as 4th edge
- 6: Choose any 1 from remaining 6 edges (dotted) as 5th edge

5 edges containing a triangle = 30 + 60 + 30 + 60 = 180

6 edges containing a triangle =
All graphs with 6 edges − Not containing a triangle

$${}^{10}C_6 \;-\; ({}^5C_2 * {}^3C_3)$$

- ${}^5C_2$: Choose 2 of 5 vertices whose degree = 3
- ${}^3C_3$: Choose 3 of 3 remaining vertices whose degree = 2



= 210 − 10

= 200

7 edges onwards always contain a triangle

1.2 Counting Cards →

(a) Total outcomes = 20! (arranging 20 cards)

No. of favourable outcomes = $^{15}C_{10} * 10! * 10!$
(All red cards are marked less than or equal to 15)

Choosing 10 from 15 numbers for red cards

Arranging red cards

Arranging blue cards

$$\text{Probability} = \frac{^{15}C_{10} * 10! * 10!}{20!}$$

$$= \frac{3003}{184756} = \frac{21}{1292}$$

(b) Total outcomes = 20! (arranging 20 cards)

Exactly 8 cards are assigned numbers less than or equal to 15 → $^{15}C_8 * {}^5C_2 * 10! * 10!$

Selecting 8 numbers from 15 numbers for red balls

Selecting 2 from 5 numbers for remaining 2 ~~balls~~ red balls

Arranging red balls

Arranging blue balls

$$\text{Probability} = \frac{^{15}C_8 * {}^5C_2 * 10! * 10!}{20!}$$

$$= \frac{64350}{184756} = \frac{225}{646}$$

(c) Total outcomes = 20! (arranging 20 cards)

Favourable outcomes $\longrightarrow$

Arranging red and blue cards

$$\underbrace{({}^8C_5 * {}^3C_3)}_{} * ({}^6C_4 * {}^2C_2) * ({}^6C_3 * {}^3C_3) * \boxed{10! * 10!}$$

Choosing 5 red cards from 8 numbers (exactly 8 numbers are > 12) and remaining 3 ~~are~~ of 3 are chosen blue

Choosing 4 blue cards from 6 numbers (exactly 6 numbers are < 7) and remaining 2 of 2 are chosen red

From remaining 6 numbers (7 to 12) we need to choose 3 red and 3 blue balls remaining

$$= (56 * 1) * (15 * 1) * (20 * 1) * 10! * 10!$$

$$\text{Probability} = \frac{56 * 15 * 20 * 10! * 10!}{20!}$$

$$= \frac{16800}{184756 \ \cancel{184756}} = \frac{4200}{46189}$$

3.(a) Back to origin in 6 steps $\rightarrow$

There are 4 possible directions say left (L), right (R), Up (U), Down (D)

To return back to origin the following 2 conditions must hold true $\rightarrow$

(i) Number of Left = Number of Right

(ii) Number of Up = Number of Down

Case (a): 3 Down and 3 Up (DDDUUU)

$$\left({}^6C_3 * \frac{3!}{3!}\right) * \left({}^3C_3 * \frac{3!}{3!}\right) = 20$$

Choose 3 of 6 places for down

Arrangement of down

Choose 3 of remaining 3 places for Up

Arrangement of Up

Case (b): 3 Left and 3 Right (LLLRRR)

$$\left({}^6C_3 * \frac{3!}{3!}\right) * \left({}^3C_3 * \frac{3!}{3!}\right) = 20$$

Reason: Same as above

Case (c): 2 Down, 2 Up, 1 Left and 1 Right (DDUULR)

$$\left({}^6C_4 * \frac{4!}{2!2!}\right) * \left({}^2C_2 * 2!\right) = 180$$

Choose 4 places out of 6 for down-up

Arrangement of up and down

Choose 2 of remaining 2 places for Left and right

Arrangement for left and right

Case (d) : 2 Left , 2 Right , 1 Up , 1 Down (LLRRUD)

$$\left({}^6C_4 * \frac{4!}{2!\,2!}\right) * \left({}^2C_2 * 2!\right) = 180$$

Reason: same as previous

Total cases in which particle reach back to origin in 6 steps = 20 + 20 + 180 + 180
= 400

Total possible paths = 4^6 = 4096

Probability = $\frac{400}{4096}$ = $\frac{25}{256}$

(b) Back to origin in 8 steps by remaining in first quadrant →

To reach origin →
(i) Number of left = Number of right
(ii) Number of UP = Number of down

Always in first quadrant →
(i) $n(R) \geq n(L)$
(ii) $n(U) \geq n(D)$;
for every prefix of string of path

All 4 must conditions hold true.

This problem is very similar to the problem of counting balanced parenthesis.

To count balanced parenthesis, Catalan numbers are used.

K-th Catalan number is represented by →

$$\boxed{K_n = \frac{{}^{2n}C_n}{(n+1)}} \qquad n = 0, 1, 2, \ldots$$

Case (I): 4L and ~~4D~~ 4R

$$K_4 = \frac{{}^{8}C_4}{5} = 14$$

Case (II): 4D and 4U

$$K_4 = \frac{{}^{8}C_4}{5} = 14$$

Case (III): (3U - 3D) and (1L - 1R)

$$({}^{8}C_6 * K_3) * ({}^{2}C_2 * K_1)$$

Choose ~~8~~ 6 places out of 8 places for (3U-3D)

To make sure they are balanced

Choose 2 of remaining 2 places for (1L-1R)

To make sure they are balanced

$$= \left(28 * \frac{{}^{6}C_3}{4}\right) * \left(1 * \frac{{}^{2}C_1}{2}\right)$$

$$= (28 * 5) * (1 * 1) = 140$$

Case (IV): (3L-3R) and (1U - 1D)

$$({}^{8}C_6 * K_3) * ({}^{2}C_2 * K_1) = 140$$

Reason: same as above

Case (V): (2U - 2D) and (2L - 2R)

$$\left({}^{8}C_4 * K_2\right) * \left({}^{4}C_4 * K_2\right)$$

Choose 4 places out of 8 places for (2U-2D)

Balanced

Choose 4 of 4 remaining places for (2L-2R)

Balanced

$$\left(70 * \frac{{}^{4}C_2}{3}\right) * \left(1 * \frac{{}^{4}C_2}{3}\right)$$

$$= (70*2) * (1*2)$$

$$= 280$$

Total favourable outcomes $= 14+14 + 140 + 140 + 280$

$= 588$

Total possible paths $= 4^8 = 65536$

Probability $= \dfrac{588}{65536} = \dfrac{147}{16384}$