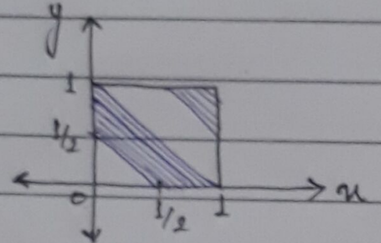
Rishu Toshniwal

2021201043

Assignment - 03

3.1 Continuous random variables

$$f_{X,Y}(x,y) = \begin{cases} c & \text{, shaded region of figure} \\ 0 & \text{, otherwise} \end{cases}$$



1.

$$c = \frac{1}{\text{area of shaded region}} \quad , \quad \text{since } \iint f_{x,y}(x,y) = 1$$

$$= \frac{1}{1/2} = 2 = c$$

2. Marginal PDF of x

$$f_x(x) = \int_{-\infty}^{\infty} f_{x,y}(x,y)\,dy$$

$$\left.\begin{aligned} &= \int_0^{1-x} 2\,dy + \int_{3/2-x}^{1} 2\,dy \\ &= [2y]_0^{1-x} + [2y]_{3/2-x}^{1} \\ &= 2(1-x) + 2\left[1 - \frac{3}{2} + x\right] \\ &= 2(1-x) + 2\left[x - \frac{1}{2}\right] \\ &= 1 \end{aligned}\right\} \text{ when } \frac{1}{2} < x < 1$$

Also $f_X(x) = \int_{1/2 - x}^{1-x} 2\,dy$ when $0 < x < 1/2$

$= \left[2y\right]_{1/2 - x}^{1-x}$

$= 2\left[1 - x - 1/2 + x\right]$

$= 1$

$\therefore\ f_X(x) = \begin{cases} 1 & 0 < x \le 1/2 \\ 1 & 1/2 < x \le 1 \end{cases}$

$\boxed{f_X(x) = 1}$ for $0 \le x \le 1$

Similarly,

<u>Marginal PDF of Y</u>

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,dx$$

Similar calculations will get us

$$f_Y(y) = \begin{cases} \int_{1/2-y}^{1-y} 2\,dx & 0 \le y \le 1/2 \\ \int_0^{1-y} 2\,dy + \int_{3/2-y}^{1} 2\,dy = 1 & 1/2 < y \le 1 \end{cases}$$

$\Rightarrow \boxed{f_Y(y) = 1}$ for $0 \le y \le 1$

3. $E(X \mid Y = \frac{1}{4}) = \int_{-\infty}^{\infty} x\, f_{X|Y}(x / y = 1/4)\,dx$

$= \int_{-\infty}^{\infty} x\, \dfrac{f_{XY}(x, y)}{f_Y(y = 1/4)}\,dx$

$$= \int_{-\infty}^{\infty} x \frac{f_{XY}(x,y)}{1} \, dx$$

Also, $f_{XY}(x,y) = \begin{cases} 2, & \text{area in shaded region OR } \frac{1}{4} \le x \le \frac{3}{4} \text{ [for } y = \frac{1}{4}] \\ 0, & \text{otherwise} \end{cases}$

$$\therefore \quad E\left(X \mid Y = \tfrac{1}{4}\right) = \int_{1/4}^{3/4} x \cdot 2 \, dx$$

$$= \left[x^2\right]_{1/4}^{3/4} = \boxed{\frac{1}{2}}$$

$$Var\left(X \mid Y = \tfrac{1}{4}\right) = \int_{-\infty}^{\infty} x^2 f_{X|Y}\left(x \mid y = \tfrac{1}{4}\right) dx$$

$$= \int_{1/4}^{3/4} x^2 \cdot 2 \, dx = \left[\frac{2x^3}{3}\right]_{1/4}^{3/4}$$

$$= \boxed{\frac{13}{48}}$$

4. $$f_{X|Y}\left(x \mid y = \tfrac{3}{4}\right) = \frac{f_{XY}(x,y)}{f_Y\left(y = \frac{3}{4}\right)}$$

$$f_{X|Y}\left(x \mid y = \tfrac{3}{4}\right) = f_{X,Y}(x,y)$$

If $y = \frac{3}{4}$ & $\quad \frac{1}{2} \le x + y \le 1$

$$1 \ge x + 1 \ge \frac{3}{2}$$

$$\boxed{\therefore \quad 1 \ge x \ge \frac{3}{4} \;\; \& \;\; \frac{1}{4} \ge x \ge 0}$$

Date
Page

$\therefore f_{X|Y}(x|y=\frac{3}{4})$

$= f_{X,Y}(x,y)$ where $\frac{1}{4} \geq x \geq 0$ & $1 \geq x \geq \frac{3}{4}$

$$f_{X|Y}(x|y=\tfrac{3}{4}) = \begin{cases} 2, & 0 \leq x \leq \frac{1}{4} \ \& \ \frac{3}{4} \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Q-5.2 Card Shuffling

1. T_i is a random card variably denoting number of shuffles to get i cards under card n.

The initial given state is 1, 2, 3, ... n with 1 at the top.

$T_1 = \text{Geometric}(\frac{1}{n})$ ← keep shuffling until top card is placed beneath 'n'

$\therefore E(T_1) = E(\text{Geometric}(\frac{1}{n})) = n$

$\therefore$ Expected number of shuffles required to have 1st card under card 'n' is n.

Given 1 card is already below 'n', number of shuffles until second card is beneath 'n' is

$T_2 - T_1 = Y_2$ ← Number of shuffles to get 2nd card under 'n' given 1st card is under n

$Y_2 = \text{Geometric}(\frac{2}{n})$

$E(Y_2) = \frac{n}{2}$

Similarly, we can define Y_i as number of shuffles to get ith card under n given there are $i-1$ cards under n as

$$Y_i = T_i - T_{i-1} = \text{geometric}\left(\frac{i}{n}\right)$$

$$\boxed{\therefore \text{PDF}(T_i - T_{i-1}) = \text{geometric}\left(\frac{i}{n}\right)}$$

This is independent of T_{i-1} as number of shuffles done to reach a state has no bearing on number of shuffles required for next state, only nature of current state matters.

2. T_{n-1} shuffles means that card 'n' is now at the top, with $n-1$ cards below it. Another shuffle after it will move 'n' to one of n other positions, each of which is equally likely.

Even before this, we know that at each shuffle, the top card can be placed in any of the states below 'n' to get another distinct card below n. Since each state is equally likely, and the chances of a slot being chosen does not depend upon other words, we can say that the cards below 'n' are a uniformly random permutation

After T_{n-1} shuffles, we will have a uniformly random permutation of $(n-1)$ cards.

$$P(\text{any one arrangement below the top card}) = \frac{1}{(n-1)!}$$

After another shuffle, we can have n more distinct places for 'n' to go to, hence

$$P(\text{any one arrangement after } T_{n-1}+1 \text{ shuffles}) = \frac{1}{(n-1)!} \times \frac{1}{n} = \boxed{\frac{1}{n!}}$$

This is the probability of choosing any one arrangement out of $n!$ possible arrangements, which proves it as uniform

3. $T_{n-1} = T_{n-1} + T_{n-2} - T_{n-2} + T_{n-3} - T_{n-3} + T_{n-4} - T_{n-4}$

$\quad ------ + T_3 - T_3 + T_2 - T_2 + T_1 - T_1$

$= T_{n-1} - T_{n-2} + T_{n-2} - T_{n-3} - T_{n-4} + T_{n-4} ----$

$\quad --- T_3 - T_2 + T_2 - T_1 + T_1$

$= Y_{n-1} + Y_{n-2} + Y_{n-3} ---- Y_3 + Y_2 + T_1$

$T_{n-1} + 1 = Y_{n-1} + Y_{n-2} ---- Y_2 + T_1 + 1$

$E(T_{n-1}+1) = E[Y_{n-1} + Y_{n-2} ---- Y_2 + T_1 + 1]$

$= E[Y_{n-1}] + E[Y_{n-2}] ---- + E[Y_2] + E[T_1] + E[1]$

[By linearity of expectations]

$= \frac{n}{n-1} + \frac{n}{n-2} ---- \frac{n}{2} + n + 1$

$= n\left[\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} ---- \frac{1}{3} + \frac{1}{2} + \frac{1}{1}\right]$

$\boxed{E(T_{n-1}+1) \approx n \ln n}$ ← Harmonic number approximation.

4. The deck is truly shuffled at some $T = T_{n-1} + 1$

at this point, we have reached a stationary state, if we perceive this problem as a markov chain

Also,

$$E(T) = n \ln n$$

To have a truly shuffled deck with 99% confidence, let us assume we must shuffle 'k' times

Let $P'(T)$ be defined as probability that deck is truly random after T shuffles.

We must find k such that $P'(T \geq k) \geq \frac{99}{100}$

OR $P(T \geq k) \leq \frac{1}{100}$ — (1) i.e. after k shuffles probability of true shuffle not happen is less than 1%.

By Markov inequality,

$$P(T \geq k) \leq \frac{E(T)}{k}$$

$$\Rightarrow P(T \geq k) \leq \frac{n \ln n}{k} \quad - (2)$$

from (1) and (2)

$$\frac{n \ln n}{k} < \frac{1}{100} \Rightarrow \boxed{k \geq 100 n \ln n}$$

∴ We must shuffle at least $100 n \ln n$ times to achieve desired probability of randomness.

5.3 Maze

1. Transition matrix will be given as

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

2. It is irreducable because each room is reachable from any other room in some arbitrary number of steps.

The given matrix is not aperiodic. Given just state 2, we can depart and return to it in any even number of steps, but can never do so in odd number of steps. Hence, 2 is periodic with a period of 2. This alone proves the given markov chain is not aperiodic

3. Let stationary distribution for each state 1-6 be marked as

$$\pi = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ [\pi_1 \; \pi_2 \; \pi_3 \; \pi_4 \; \pi_5 \; \pi_6] \end{array}$$

For stationary distribution,

$$\pi = \pi P$$

$$[\pi_1 \; \pi_2 \; \pi_3 \; \pi_4 \; \pi_5 \; \pi_6] = [\pi_1 \; \pi_2 \; \pi_3 \; \pi_4 \; \pi_5 \; \pi_6] \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

$\therefore$ We get

$$n_1 = \frac{n_3}{4} \quad -① \qquad , \qquad n_2 = \frac{n_3}{4} \quad -②$$

$$n_3 = n_1 + n_2 + \frac{n_4}{2} + \frac{n_5}{2} \quad -③$$

$$n_4 = \frac{n_3}{4} + \frac{n_6}{2} \quad -④$$

$$n_5 = \frac{n_3}{4} + \frac{n_6}{2} \quad -⑤$$

$$n_6 = \frac{n_4}{2} + \frac{n_5}{2} \quad -⑥$$

$$n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = 1 \quad -⑦$$

Solving the above, we get

$$n_1 = \frac{1}{12}, \quad n_2 = \frac{1}{12}, \quad n_3 = \frac{1}{3}, \quad n_4 = \frac{1}{6}, \quad n_5 = \frac{1}{6}, \quad n_6 = \frac{1}{6}$$

$$\Rightarrow \quad n = \left[\frac{1}{12} \;\; \frac{1}{12} \;\; \frac{1}{3} \;\; \frac{1}{6} \;\; \frac{1}{6} \;\; \frac{1}{6} \right]$$

4. Let s_i = expected number of steps to reach state s given starting point as state i

$$s_1 = 1 + s_3 \quad -①$$

$$s_2 = 1 + s_3 \quad -②$$

$$s_3 = 1 + \frac{s_4}{4} + \frac{s_5}{4} + \frac{s_2}{4} + \frac{s_1}{4} \quad -③$$

$$s_4 = 1 + \frac{s_6}{2} + \frac{s_3}{2} \quad -④$$

$$s_5 = 0 \quad -⑤$$

$$S_6 = 1 + \frac{S_4}{2} + \frac{S_5}{2} \quad - \quad (6)$$

From (5) & (6)

$$S_6 = 1 + \frac{S_4}{2} \quad - \quad (7)$$

From (7) & (4)

$$S_4 = 1 + \frac{1}{2} + \frac{S_4}{4} + \frac{S_3}{2}$$

$$S_4 = \frac{4}{3}\left(\frac{3}{2} + \frac{S_3}{2}\right)$$

$$S_4 = \frac{2}{3}\left(S_3 + 3\right)$$

$$S_4 = 2 + \frac{2S_3}{3} \quad - \quad (8)$$

By putting (1), (5) & (8) in (3)

$$S_3 = 1 + \frac{1}{2} + \frac{S_3}{6} + \frac{S_5}{4} + \frac{S_3+1}{4} + \frac{S_3+1}{4}$$

$$\Rightarrow S_3 = 1 + \frac{1}{2} + \frac{S_3}{6} + \frac{S_3+1}{2} \qquad [S_5 = 0]$$

$$S_3 = \frac{3}{2} + \frac{S_3 + 3S_3 + 3}{6}$$

$$6S_3 = 9 + S_3 + 3S_3 + 3$$

$$2S_3 = 12$$

$$S_3 = 6$$

$$\therefore \boxed{S_1 = 7}$$

5. From part 3, $\pi_1 = \frac{1}{12}$

$\therefore$ Mean return time to state 1 is given as $\frac{1}{\pi_1} = 12$.