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Assignment-2
Linear Algebra

1. A transformation T is linear iff
(i) $T(x+y) = T(x) + T(y)$
(ii) $T(cx) = cT(x)$

$$T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$$

$$T[(x_1, x_2) + (y_1, y_2)] = T(x_1+y_1, x_2+y_2)$$
$$= (4(x_1+y_1) - 2(x_2+y_2), 3|x_2+y_2|)$$
$$= (4x_1 + 4y_1 - 2x_2 - 2y_2, 3|x_2+y_2|) \quad \text{- eqn (1)}$$

$$T(x_1, x_2) + T(y_1, y_2) = (4x_1 - 2x_2, 3|x_2|) + (4y_1 - 4y_2, 3|y_2|)$$
$$= (4x_1 + 4y_1 - 2x_2 - 2y_2, 3|x_2| + 3|y_2|) \quad \text{- eqn (2)}$$

Comparing 2nd term in eqn (1) and eqn (2), if they are equal -

Proof by counter example -
put $x_2 = 2$, $y_2 = -1$

$3|x_2+y_2| = 3|2+(-1)| = 3|1| = 3$
$3|x_2| + 3|y_2| = 3|2| + 3|1| = 6+3 = 9$
Clearly, they are not equal i.e. $3|x_2+y_2| \neq 3|x_2| + 3|y_2|$

$\therefore T(x+y) \neq T(x) + T(y)$
Hence T is not linear.

2. Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ projects each point (x_1, x_2, x_3) onto $x_1 x_2$ plane. We need to find matrix representation of linear transformation T.

<u>Solution</u> – T projects (x_1, x_2, x_3) onto $x_1 x_2$ plane which means –

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find the matrix such that:

$$A_{2\times3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Standard basis vectors of $\mathbb{R}^2 \to$

$$\left\{ f_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, f_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

Standard basis vectors of $\mathbb{R}^3 \to$

$$\left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T(e_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then the matrix $A_{2\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Hence, the linear transformation can be represented as:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Using $A_{2\times3}$, we can project each point (x_1, x_2, x_3) onto x_1-x_2 plane.

$$\boxed{\text{Matrix representation of linear transformation } T = A_{2\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{2\times3}}$$

3. (i) $\because$ $D(a+bx+cx^2+dx^3) = b+2cx+3dx^2$

$$\ker(D) = \{a+bx+cx^2+dx^3 : D(a+bx+cx^2+dx^3) = 0\}$$
$$= \{a+bx+cx^2+dx^3 : b+2cx+3dx^2 = 0\}$$

$b+2cx+3dx^2 = 0$ if and only if $b=c=d=0$

$$\therefore \ker(D) = \{a+bx+cx^2+dx^3 : b=c=d=0\}$$
$$\boxed{\ker(D) = \{a : a \text{ in } \mathbb{R}\}}$$

i.e. Kernel of D is the set of constant polynomials.

$\boxed{\text{The range of } D \text{ is all of } P_2}$, since every polynomial in P_2 is the image under D of some polynomial in P_3.

In other words, If $a+bx+cx^2$ is in P_2, then

$$a+bx+cx^2 = D\left(ax+\left(\frac{b}{2}\right)x^2+\left(\frac{c}{3}\right)x^3\right)$$

3.(ii). $S: P_1 \to \mathbb{R}$ be linear transformation defined by –

$$S(p(x)) = \int_0^1 p(x)\,dx$$

Let, $S(a+bx) = \int_0^1 (a+bx)\,dx$

$$= \left[ax + \frac{bx^2}{2}\right]_0^1$$

$$= \left(a+\frac{b}{2}\right) - 0 = a+\frac{b}{2}$$

$\therefore$ $\text{Ker}(S) = \{a+bx : S(a+bx) = 0\}$
$= \{a+bx : a+\frac{b}{2} = 0\}$

$= \left\{a+bx : a = -\frac{b}{2}\right\}$

$$\boxed{\ker(S) = \left\{-\frac{b}{2} + bx\right\}}$$

The range of S is $\mathbb{R}$, since every real number can be obtained as the image under S of some polynomial in P_1.

3.(iii) $\text{Ker}(T) = \{A \text{ in } M_{22} : T(A) = O\}$
$= \{A \text{ in } M_{22} : A^T = O\}$

If $A^T = O$, then $A = (A^T)^T = O^T = O$
$\therefore$ $\ker(T) = \{O\}$ [$O \to$ Zero matrix of order 2×2]

For any matrix A in M_{22}, we have $A = (A^T)^T = T(A^T)$ (and A^T is also in M_{22}).

$$\therefore \text{Range}(T) = M_{22}$$

Given -

4. $[u_1, u_2, \ldots, u_n]$ are linearly independent in V.

To prove - $[[u_1]_B, [u_2]_B, \ldots, [u_n]_B]$ are linearly independent in $\mathbb{R}^n$

$\because u_1, u_2, \ldots, u_n$ are linearly independent. i.e.

$\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n = 0$ only when

$\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$ — (1)

We know that,

If $[v_B] = 0$ then $v = 0$ and $\boxed{\text{if } v = 0 \text{ then } [v]_B = 0}$

using above property,

if $\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n = 0$ then

$[\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n]_B = 0$

$\Rightarrow [\alpha_1 u_1]_B + [\alpha_2 u_2]_B + \ldots + [\alpha_n u_n]_B = 0$

$\Rightarrow \alpha_1 [u_1]_B + \alpha_2 [u_2]_B + \ldots + \alpha_n [u_n]_B = 0$

$\therefore [[u_1]_B, [u_2]_B \ldots [u_n]_B]$ is linearly independent in $\mathbb{R}^n$ bcz $[u_i]_B$ is $(n \times 1)$ vector

Given - $[[u_1]_B, [u_2]_B, \ldots [u_n]_B]$ is linearly independent in $\mathbb{R}^n$

To prove - $[u_1, u_2, \ldots u_n]$ are linearly independent in V.

$\because$ $[u_1]_B, [u_2]_B \ldots, [u_n]_B$ are linearly independent then $\alpha_1 [u_1]_B + \alpha_2 [u_2]_B + \ldots + \alpha_n [u_n]_B = 0$ only when $\alpha_1 = \alpha_2 \ldots = \alpha_n = 0$ — (1)

We know that,

$\boxed{\text{If } [v]_B = 0 \text{ then } v = 0}$ and if $v = 0$ then $[v]_B = 0$

eqn (1) can be rewritten as

$$[\alpha_1 u_1]_B + [\alpha_2 u_2]_B + \ldots + [\alpha_n u_n]_B = 0$$

$$[\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n]_B = 0$$

From the above property of coordinate vectors, if $[\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n]_B = 0$ then

$$\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n = 0$$

From (1), we know $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$

and we obtained $\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n = 0$

$\therefore$ $[u_1, u_2, \ldots u_n]$ is linearly independent in V

5. Any 2 bases for a vector space have the same number of vectors.

Hence, let $C = [a, b, c]$

and $P_{C \leftarrow B} = [\, [x]_C \;\; [1+x]_C \;\; [1-x+x^2]_C \,]$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix} \quad \text{(given)}$$

On equating

$x = 1*a + 0*b + (-1)*c$

$\Rightarrow \boxed{x = a - c}$ — eqn (1)

$1+x = 0*a + 1*b + 1*c$

$\boxed{1+x = 2b + c}$ — eqn (2)

$1-x+x^2 = 0*a + 1*b + 1*c$

$\boxed{1-x+x^2 = b+c}$ — eqn (3)

~~eqn (1) + eqn (3)~~

~~eqn (1) +~~

eqn (2) − eqn (3)

$\Rightarrow \boxed{2x - x^2 = b}$

put value of 'b' in eqn (3)

$1 - x + x^2 = 2x - x^2 + c$

$\Rightarrow \boxed{c = 1 - 3x + 2x^2}$

put value of in eqn (1)

$x = a - (1 - 3x + 2x^2)$

$\Rightarrow a = 1 - 2x + 2x^2$

$\therefore$ Basis $C = [2x^2 - 2x + 1, \; 2x - x^2, \; 2x^2 - 3x + 1]$

6.(a) $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -4b \\ a+5b \end{bmatrix}$

Matrix representation of T with respect to basis -
$B = \{e_1, e_2\}$ where $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Then matrix $[T]_B = \left[[T(e_1)]_B \;\; [T(e_2)]_B \right]$

$$= \left[\begin{bmatrix} 0 \\ 1 \end{bmatrix}_B \;\; \begin{bmatrix} -4 \\ 5 \end{bmatrix}_B \right]$$

$$= \begin{bmatrix} 0 & -4 \\ 1 & 5 \end{bmatrix}$$

To find a basis C for $\mathbb{R}^2$ such that the matrix $[T_C]$ is a diagonal matrix

A matrix becomes diagonal when you work in an eigen basis i.e. basis made up of eigen vector
Hence, we need to find eigen vectors of $[T]_B$

Let, $A = [T_B]$

<u>To find eigen-values</u>: $|A - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & -4 \\ 1 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda(5-\lambda) + 4 = 0$$

$$(\lambda - 1)(\lambda - 4) = 0$$

$$\lambda = 1, 4$$

To find eigen-vectors :

(i) $\lambda = 1$;

$$(A - \lambda I)X = 0$$

$$(A - I)X = 0$$

$$\begin{bmatrix} -1 & -4 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x + 4y = 0$$

$$x = -4y$$

at $y = k$, $x = -4k$

$\therefore$ Eigen vector at $\lambda = 1$ is $k\begin{bmatrix} -4 \\ 1 \end{bmatrix}$

(ii) $\lambda = 4$;

$$(A - \lambda I)X = 0$$

$$\Rightarrow (A - 4I)X = 0$$

$$\begin{bmatrix} -4 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x + y = 0$$

$$x = -y$$

at $y = k$, $x = -k$

$\therefore$ Eigen vector at $\lambda = 4$ is $k\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\therefore$ The basis $C = \left\{ \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ and

$$[T]_C = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

6(b) $T: P_2 \to P_2$ defined by $T(p(x)) = P(3x+2)$

Matrix representation of T with respect to standard basis $B = \{1, x, x^2\}$

$T(1) = 1$; $T(x) = 3x+2$; $T(x^2) = (3x+2)^2$

$T(x^2) = 9x^2 + 12x + 4$

$[T]_B = [[T(1)]_B \; [T(x)]_B \; [T(x^2)]_B]$

$$= \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 12 \\ 0 & 0 & 9 \end{bmatrix} \to \text{Matrix representation of } T$$

To find a basis C such that the matrix $[T]_C$ is diagonal

<u>To find eigen-values:</u> $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 4 \\ 0 & 3-\lambda & 12 \\ 0 & 0 & 9-\lambda \end{vmatrix} = 0$$

$(1-\lambda)(3-\lambda)(9-\lambda) = 0$

$\lambda = 1, 3, 9$

<u>To find eigen vectors</u> $\to$

(i) $\lambda = 1$: $(A - \lambda I)X = 0$

$(A - I)X = 0$

$$\begin{bmatrix} 0 & 2 & 4 \\ 0 & 2 & 12 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$8z = 0 \Rightarrow z = 0$

$2y + 12z = 0 \Rightarrow y = 0$

$x = K$ (say)

At $\lambda=1$, then eigen vector is $k\begin{bmatrix}1\\0\\0\end{bmatrix}$

(ii) $\underline{\lambda=3}$:

$$(A-\lambda I)X=0$$
$$\Rightarrow (A-3I)X=0$$
$$\begin{bmatrix}-2 & 2 & 4\\ 0 & 0 & 12\\ 0 & 0 & 6\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$6z=0 \quad \Rightarrow \quad z=0$

$-2x+2y+z=0$

$\Rightarrow \quad x=y$

say $(x=k)$

$\therefore y=k$

Eigen vector for $\lambda=3$ is $k\begin{bmatrix}1\\1\\0\end{bmatrix}$

(iii) $\underline{\lambda=9}$:

$$(A-\lambda I)X=0$$
$$(A-9I)X=0$$
$$\Rightarrow \begin{bmatrix}-8 & 2 & 4\\ 0 & -6 & 12\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$-6y+12z=0 \quad \Rightarrow \quad y=2z$

$-8x+2y+4z=0 \quad \Rightarrow \quad -8x+4z+4z=0$

$\Rightarrow x=z$

say $x=k$

$\therefore$ Eigen vector $\lambda=9$ is $k\begin{bmatrix}1\\2\\1\end{bmatrix}$

$\therefore$ The basis $C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$

$= \{1, 1+x, 1+2x+x^2\}$

and $[T]_C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

7.(a) To show that $B = \{x^2, x, 1\}$ is a basis for vector space of polynomials in x of degree < 3 over R -

We have to show that
(i) B is linearly independent
(ii) B spans V

Suppose c_0, c_1, c_2 are scalars such that
$c_0 x^2 + c_1 x + c_2 = 0$
put $x = 0$, we get $c_2 = 0$

$\therefore$ $c_0 x^2 + c_1 x = 0$
diff. w.r.to x, we get
$2c_0 x + c_1 = 0$
put $x = 0$, we get $c_1 = 0$

$\therefore$ $c_0 = c_1 = c_2 = 0$

Clearly, $c_0 x^2 + c_1 x + c_2 = 0$ and $c_0 = c_1 = c_2 = 0$
$\therefore$ $B = \{x^2, x, 1\}$ is linearly independent

B also spans P_2, since every polynomial in P_2 is a linear combination of 0^{th}, 1^{st} and 2^{nd} powers of x.

$\therefore$ B is a basis over polynomials in x of degree < 3 over R.

7(b) Show that T is linear transformation -

Given $T(x^2) = x+m$; $T: P^2 \to P^2$
$\qquad (V) \quad (W)$

$T(x) = (m-1)x$

$T(1) = x^2+m$

and Basis $B = \{x^2, x, 1\}$

Assuming B is basis for both V and W.

Using the following theorem, let us prove T is linear -

> Theorem - Let V, W be vector spaces over F
> Let $B = \{v_1, v_2, \ldots, v_n\}$ be a basis for V
> and $A = \{w_1, w_2, \ldots, w_n\}$ be any subset of W
>
> Then a transformation :
> $T(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n) = \alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_n w_n$
> is linear

Let $p(x) = a_1x^2 + b_1x + c_1 \in V$

$q(x) = a_2x^2 + b_2x + c_2 \in V$

then $T(p(x) + q(x))$

$= T(a_1x^2 + b_1x + c_1 + a_2x^2 + b_2x + c_2)$

$= T((a_1+a_2)x^2 + (b_1+b_2)x + (c_1+c_2))$

$= (a_1+a_2)(x+m) + (b_1+b_2)(m-1)x + (c_1+c_2)(x^2+m)$

(using theorem)

$= (c_1+c_2)x^2 + (a_1+a_2+b_1m+b_2m-b_1-b_2)x$
$+ (a_1m + a_2m + c_1m + c_2m)$

Now, $T(p(x)) = T(a_1x^2 + b_1x + c_1)$

$= a_1(x+m) + b_1(m-1)x + c_1(x^2+m)$

$= a_1x + a_1m + b_1(m-1)x + c_1x^2 + c_1m$

$= c_1x^2 + (a_1 + b_1m - b_1)x + (a_1m + c_1m)$

similarly,

$T(q(x)) = c_2x^2 + (a_2 + b_2m - b_2)x + (a_2m + c_2m)$

$T(p(x)) + T(q(x)) = (c_1+c_2)x^2 + (a_1+a_2+b_1m-b_1+b_2m-b_2)x$
$+ (a_1m + a_2m + c_1m + c_2m)$

$\therefore T(p(x) + q(x)) = T(p(x)) + T(q(x))$

Now, let $\alpha \in R$ then,

~~T(αp)~~ $T(\alpha p(x)) = T(a_1\alpha x^2 + b_1\alpha x + c_1\alpha)$

$= a_1\alpha(x+m) + b_1\alpha(m-1)x +$
$c_1\alpha(x^2+m)$

and $\alpha T(p(x)) = \alpha[c_1x^2 + (a_1+b_1m-b_1)x + (a_1m+b_1m)]$

$= c_1\alpha x^2 + (a_1\alpha + b_1\alpha m - b_1\alpha x) + (a_1\alpha m + c_1\alpha m)$

$\therefore T(\alpha p(x)) = \alpha T(p(x))$

Hence, T is a linear transformation.

7.(c) Matrix representation of T relative to given basis

$T(x^2) = x+m$ (given)

$\Rightarrow 0 \cdot x^2 + 1 \cdot x + 1 \cdot m$

$\Rightarrow \begin{bmatrix} x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ m \end{bmatrix}$

Similarly, $T(x) = (m-1)x$

$\Rightarrow 0 \cdot x^2 + (m-1)x + 0$

Column representation for $T(x) = (m-1)x$ is

$= \begin{bmatrix} 0 \\ m-1 \\ 0 \end{bmatrix}$

Similarly $T(1) = x^2 + m$

$= 0 \cdot x^2 + 0 \cdot x + 1 \cdot m$

Column representation $= \begin{bmatrix} 1 \\ 0 \\ m \end{bmatrix}$

$\therefore$ Matrix representation of T relative to given basis $= \begin{bmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{bmatrix}$

7.(d) Find kernel (T)

$TV = 0$ (V = vector, T = matrix from ques 7(c))

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} z \\ x + y(m-1) \\ xm + zm \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$z = 0$ — (1)

$x + ym - y = 0$ — (2)

$xm + zm = 0$ — (3)

<u>Case I</u> - assume $m = 0$

then $z = 0$, $x = y$

$\therefore$ kernel (T) $= \begin{bmatrix} x \\ -x \\ 0 \end{bmatrix}$

<u>Case II</u> assume $m = 1$

$z = 0$, $y =$ any value, $x = 0$

$\therefore$ kernel (T) $= \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}$

Case III $m \neq 0$ and $m \neq 1$

we get $x = y = z = 0$

$$\text{kernel}(T) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

7.(e) Image of T for all values of m –

Case I $m = 0$

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Number of nullity $= 1$

$\therefore$ Rank $= 2$

$\therefore$ Range of T is $\{(0,1,0), (1,0,0)\}$

Case II $m = 1$

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Nullity $= 1$

$\therefore$ Rank $= 2$

Range of T is $\{(0,1,1), (1,0,1)\}$

Case III $m \neq 1$ and $m \neq 0$

Here all column vectors are Independent

$\therefore$ Range of T is $\{(0,1,m), (0,m-1,0), (1,0,m)\}$

8. We know that,

$E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$; $E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$; $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Coordinate vector of $A = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ which satisfies

$aE_{22} + bE_{21} + cE_{12} + dE_{11} = A$

$\Rightarrow a\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} d & c \\ b & a \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Here, $a = 4$, $b = 3$, $c = 2$, $d = 1$

$\therefore$ Coordinate vector of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ with respect to basis $B = [E_{22}, E_{21}, E_{12}, E_{11}]$ is $\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$