Assignment-1

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1.(a) Let, set of rational numbers be 'Q'
Assume field F to be the set of real numbers R

(i) Closure - If $a, b \in Q$
then $a+b \in Q$
$\therefore$ Closed under addition

(ii) Commutative - For any 2 rational numbers a & b
$a+b = b+a$
$\therefore$ Commutative under addition

(iii) Associative - For any 3 $a, b, c \in Q$
$(a+b)+c = a+(b+c)$
$\therefore$ Associative under addition

(iv) Identity element - ~~$a+0 = 0$~~
$a+0 = 0+a = a$
$\therefore$ '0' is identity element

(v) Inverse - $a+b = b+a = 0$
$b = -a$
$\therefore$ Inverse of 'a' is '$-a$' for any rational number

(vi) Scalar Multiplication -
Consider $5 \in Q$ and $\sqrt{3} \in R$
$$\underbrace{5}_{\text{Rational}} * \underbrace{\sqrt{3}}_{\text{Real}} = \underbrace{5\sqrt{3}}_{\text{Irrational}}$$

$\therefore$ Not closed under scalar multiplication

(vii) Let, $c \in F$ and $a, b \in Q$, then $c(a+b) = ca + cb$
$\therefore$ Scalar multiplication is distributive over addition

(viii) Let, $a, b \in F$ and let $c \in Q$ be a rational number
$\therefore (a+b)c = ac + bc$
$\therefore$ Scalar multiplication is distributive with respect to field addition.

(ix) For 2 scalars $a, b \in R$ and $c \in Q$,
$(ab)c = a(bc)$
$\therefore$ Associative multiplication is satisfied.

(x) For all $a \in Q$, $1 \cdot a = a$ where $1 \in R$
$\therefore$ Scalar identity exists.

$\therefore$ Set of rational numbers is not a vector space.
Because axiom 6 is violated.

1(b) Set of all $n \times n$ skew symmetric matrices with usual matrix addition and scalar multiplication.

(i) Closure - A and B are skew-symm. of order $n \times n$

$A^T = -A$ and $B^T = -B$

$$(A+B)^T = A^T + B^T .$$
$$= -A + (-B)$$
$$= -(A+B)$$

$(A+B)$ is also skew-symmetric

$\therefore$ Closed under matrix addition.

(ii) Commutative - Matrix addition is commutative in nature.

$\therefore$ Commutative under addition.

(iii) Associative - Matrix addition is associative in nature

$\therefore$ Associative under matrix addition.

(iv) Identity element - Let, A be skew-symm. matrix.

$$A + \underbrace{O}_{\text{zero matrix}} = O + A = A$$

Zero matrix is identity element.

(v) Inverse - ~~A~~ $A + B = O = B + A$ (Let B is inverse of A)

$B = -A$

$\therefore$ Inverse of matrix A is matrix $(-A)$

$\therefore$ Inverse exists

(vi) Closed under scalar multiplication -
Let A be a skew - symmetric matrix.

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & \\ -a_{12} & 0 & a_{23} & \cdots & \\ -a_{13} & -a_{23} & 0 & \cdots & \\ \vdots & \vdots & \vdots & \cdots & 0 \end{bmatrix}_{n \times n}$$

$$KA = \begin{bmatrix} 0 & Ka_{12} & Ka_{13} & \cdots \\ -Ka_{12} & 0 & +Ka_{23} & \\ -Ka_{13} & -Ka_{23} & 0 & \\ \vdots & \vdots & \vdots & \cdots 0 \end{bmatrix}_{n \times n}$$

(KA) is also skew-symmetric
$\therefore$ Closed under scalar multiplication.

(vii) Identity element - $\exists e = 1 \in R$ such that $e \cdot M = M$
for all skew-symmetric matrices 'M'
Identity element $= 1$

(viii) Associativity - For any skew-symmetric matrix A -
$(\alpha\beta) A = \alpha(\beta A)$

$$LHS = (\alpha\beta)A = (\alpha\beta) \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ -a_{12} & 0 & +a_{23} & & \vdots \\ -a_{13} & -a_{23} & 0 & & \vdots \\ \vdots & & & & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \alpha\beta a_{12} & \alpha\beta a_{13} & \cdots & \alpha\beta a_{1n} \\ -\alpha\beta a_{12} & 0 & \alpha\beta a_{23} & & \\ \vdots & \vdots & & & 0 \end{bmatrix}$$

$$RHS = \alpha(\beta A) = \alpha \begin{bmatrix} 0 & \beta a_{12} & - - - - & \beta a_{1n} \\ -\beta a_{12} & 0 & & \vdots \\ -\beta a_{13} & \vdots & & \vdots \\ \vdots & & & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \alpha\beta a_{12} & - - - - & \alpha\beta a_{1n} \\ -\alpha\beta a_{12} & 0 & & \vdots \\ -\alpha\beta a_{13} & \vdots & & \vdots \\ \vdots & & - - - - & 0 \end{bmatrix}$$

$LHS = RHS$

$\therefore$ Associativity satisfies

(ix) Distributive (Two ~~scalars~~ vectors, One ~~vector~~ scalar)

$\forall \alpha \in R$, $\forall A_1, A_2 \in M_{n \times n}$

$\alpha(A_1 + A_2) = \alpha A_1 + \alpha A_2$

$\therefore$ Distributive

(x) Distributive (Two scalars, One vector)

$\forall \alpha, \beta \in R$, $\forall A \in M_{n \times n}$

$(\alpha + \beta) A = \alpha A + \beta A$

$\therefore$ Distributive

$\therefore$ Set of all $n \times n$ skew-symmetric matrices $M_{n \times n}$ with matrix addition and scalar multiplication over the field R is a vector space.

1.(c) Set of all upper triangular matrices (2×2) with matrix addition and scalar multiplication.

(i) Closure - $A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$ $B = \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix}$

$$A+B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ 0 & a_{22}+b_{22} \end{bmatrix}$$

(A+B) is also upper triangular matrix.
Closed under matrix addition.

(ii) Commutative - Matrix addition is always commutative.
$\therefore$ $A+B = B+A$
$\therefore$ Commutative holds good.

(iii) Associative - Matrix addition is always associative.
$\therefore$ $(A+B)+C = A+(B+C)$
$\therefore$ Associativity holds good

(iv) Identity element - $A+Z = A = Z+A$
→ Zero matrix of order (2×2)
→ also upper triangular matrix

$$Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$$ $\therefore$ Identity element exists

(v) Inverse element - $A+A^{-1} = 0 = A^{-1}+A$

$$\boxed{A^{-1} = -A}$$

$A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$ $\Rightarrow A^{-1} = \begin{bmatrix} -a_{11} & -a_{12} \\ 0 & -a_{22} \end{bmatrix}$ $\therefore$ Inverse exists $\forall A \in M_{2\times 2}$

(vi) Closed under scalar multiplication –

$$A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}, \quad kA = \begin{bmatrix} ka_{11} & ka_{12} \\ 0 & ka_{22} \end{bmatrix}$$

(kA) matrix is also upper triangular matrix
$\therefore$ Closed under scalar multiplication.

(vii) Identity element –
$\exists e = 1 \in R$ such that $eA = A$; $\forall A \in M_{2\times2}$
$\therefore$ Identity element exists $(e = 1)$

(viii) Associativity –
$\forall \alpha, \beta \in R$ and $A \in M_{2\times2}$
such that $(\alpha\beta)A = \alpha(\beta A)$

$$LHS = (\alpha\beta)A = \alpha\beta \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} = \begin{bmatrix} \alpha\beta a_{11} & \alpha\beta a_{12} \\ 0 & \alpha\beta a_{22} \end{bmatrix}$$

$$RHS = \alpha(\beta A) = \alpha \begin{bmatrix} \beta a_{11} & \beta a_{12} \\ 0 & \beta a_{22} \end{bmatrix} = \begin{bmatrix} \alpha\beta a_{11} & \alpha\beta a_{12} \\ \alpha\beta a_{21} & \alpha\beta a_{22} \end{bmatrix}$$

$LHS = RHS$
Associativity under scalar multiplication.

(ix) Distributive (Two vectors, One scalar)
$\forall \alpha \in R$, $\forall A, B \in M_{2\times2}$ (upper triangular)

$$\alpha(A+B) = \alpha A + \alpha B$$

$\therefore$ Distributivity holds good.

(x) Distributive (Two scalars, One vector)

$\forall\ \alpha, \beta \in R$, $\forall A \in M_{2*2}$ (upper triangular)

$$(\alpha+\beta) A = \alpha A + \beta A$$

$\therefore$ Distributivity holds good

$\therefore$ Set of all 2*2 upper triangular matrices M_{2*2} with matrix addition and scalar multiplication over the field R is a vector space.

2. Show that $R[0,1]$ forms a vector space over R
$R[0,1]$ {all functions $f: [0,1] \to R$ such that f is continuous}

(i) Closure under addition -
Suppose $f, g \in R[0,1]$.
$(f+g)(x) = f(x) + g(x) \quad \forall x \in [0,1]$

By algebra of continuous functions, $f+g$ is continuous on $[0,1]$. Since real numbers are closed under addition, $(f+g)(x)$ would also ~~be~~ mapped to a real number.

Hence, closed under addition.

(ii) Commutative property -
The above definition of f, g and x
$(g+f)(x) = g(x) + f(x) = \underset{\text{Real}}{f(x)} + \underset{\text{Real}}{g(x)}$
(commutativity of real numbers)

Hence, closed under commutativity

(iii) Additive Identity - $f, g, h \in R[0,1]$ and $x \in [0,1]$
$((f+g)+h)(x) = (f+g)(x) + h(x)$
$= f(x) + g(x) + h(x)$
$= f(x) + (g(x) + h(x))$
$= f(x) + (g+h)(x)$ $\quad$ ∴ Associativity holds good
$= (f + (g+h))(x)$

(iv) Additive identity – Let us define a function $f_0 \in R[0,1]$ that maps everything to 0. This is a constant function hence it is continuous.

Now, for any $f \in R[0,1]$ and $x \in [0,1]$

$$(f_0 + f)(x) = f_0(x) + f(x) = 0 + f(x)$$
$$= f(x)$$

$\therefore$ f_0 is the additive identity

(v) Additive inverse – Let us define a function 'g' that takes a number $x \in [0,1]$, calculates $f(x)$ for $f \in R[0,1]$ multiplies the answer by -1 and outputs the result.

$$f(x) + g(x) = g(x) + f(x) = f_0$$
$$f(x) + g(x) = g(x) + f(x) = 0$$
$$g(x) = (-1 * f(x))$$

Hence inverse exists.

(vi) Scalar multiplication –
Suppose $f \in R[0,1]$ and $\lambda \in R$.

$$(\lambda f)(x) = \lambda f(x) \qquad \text{(To prove)}$$

By algebra of continuous functions, λf is continuous on $[0,1]$

Hence, scalar multiplication is satisfied.

(vii) Distributive (Two vectors, One scalar)
Let, $f, g \in R[0,1]$ and $\lambda \in R$ be a scalar

$$\lambda(f+g)(x) = \lambda(f(x)+g(x))$$
$$= \lambda f(x) + \lambda g(x)$$

$$\Rightarrow \boxed{\lambda(f+g) = \lambda f + \lambda g}$$ $\therefore$ Distributivity holds good

(viii) Distributive (Two scalars, One vector)
Let, $f \in R[0,1]$ and $\lambda_1, \lambda_2 \in R$ be 2 scalars
$$(\lambda_1+\lambda_2) f(x) = \lambda_1 f(x) + \lambda_2 f(x)$$

$$\Rightarrow \boxed{(\lambda_1+\lambda_2) f = \lambda_1 f + \lambda_2 f}$$ Distributivity holds good

(ix) Associativity
Let, $f \in R[0,1]$ and scalars $c, d \in R$
$$c(df) = c(d f(x)) = cd f(x)$$
$$= (cd) f$$
$$\Rightarrow \boxed{c(df) = (cd)f}$$
$\therefore$ Associativity holds good

(x) Scalar Identity – Let, $f \in R[0,1]$
$1 * f = 1 * f(x) = f(x)$
$\because$ 1 is multiplicative identity for real numbers
Hence, scalar identity exists.

$\Rightarrow R[0,1]$ forms a vector space over R.

3. $f(x) = x$, $g(x) = e^x$, $h(x) = e^{-x}$

To prove – f, g and h are linearly independent i.e.

$$af(x) + bg(x) + ch(x) = 0$$

has only 1 solution $\Rightarrow$ $\boxed{a = b = c = 0}$

$$af(x) + bg(x) + ch(x) = 0$$
$$ax + be^x + ce^{-x} = 0$$

at $x = 0$, $0 + b + c = 0 \Rightarrow \boxed{b + c = 0}$ – (1)

at $x = 1$, $a + be + \frac{c}{e} = 0$ – (2)

at $x = 1/2$, $\frac{a}{2} + b\sqrt{e} + \frac{c}{\sqrt{e}} = 0$ – (3)

using (1), $c = -b$ (put in (2) and (3))

$$a + be - \frac{b}{e} = 0 \Rightarrow \boxed{a + b\left(e - \frac{1}{e}\right) = 0}$$ – (4)

$$\boxed{\frac{a}{2} + b\left(\sqrt{e} - \frac{1}{\sqrt{e}}\right) = 0}$$ – (5)

eqn (4) $-$ 2 * eqn (5)

$$\left[a + b\left(e - \frac{1}{e}\right)\right] - \left[a + b\left(2\sqrt{e} - \frac{2}{\sqrt{e}}\right)\right] = 0$$

$$\underbrace{b\left(e - \frac{1}{e} - 2\sqrt{e} + \frac{2}{\sqrt{e}}\right)}_{\text{non-zero}} = 0$$

$\therefore \boxed{b = 0}$ put in (1) and (4)

$c = 0$, $a = 0$ $\Rightarrow$ $\boxed{a = b = c = 0}$

The only solution of the given equation is $a = b = c = 0$.
$\therefore$ f, g and h are linearly independent

4.(a) Set of all invertible matrices.

Addition - Let, A be an invertible matrix ($\det(A) \neq 0$)

$\therefore$ $-A$ be also invertible ($\det(-A) = -\det(A) \neq 0$)

$$A_{n \times n} + (-A)_{n \times n} = O_{n \times n}$$

$\hookrightarrow$ zero matrix of order $n \times n$

$O_{n \times n}$ is not invertible as $\det(O) = 0$

Hence not closed under addition.

$\therefore$ It is not a vector space.

(b) Set of all non-invertible matrices.

Proof by counter-example -

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

$\det(A) = \det(B) = 0$ [$\therefore$ Non-invertible]

$$A + B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$\det(A+B) = 2*2 - 0*0 = 4 \neq 0$ (Invertible)

Hence not closed under addition

$\therefore$ It is not a vector space.

4.(c) All A such that $AB = BA$ for a fixed matrix B in V.

Let, A_1 and A_2 matrices such that both commutes with matrix B

$$A_1B = BA_1 \quad \text{and} \quad A_2B = BA_2$$

<u>Addition</u> –
$$(A_1 + A_2)B = A_1B + A_2B$$
$$= BA_1 + BA_2$$
$$(A_1 + A_2)B = B(A_1 + A_2)$$

Clearly, $(A_1 + A_2)$ matrix and matrix B commutes
$\therefore$ Closed under addition.

<u>Scalar Multiplication</u> –

$\because$ Matrices are closed under scalar multiplication
i.e. $(KA_1) = K \cdot A_1$ because of this property
matrix (KA_1) also commutes with matrix B.

$$\therefore \quad (KA_1)B = B(KA_1)$$

Clearly, closed under scalar multiplication.
Hence, it is a vector space.

4.(d) Set of Idempotent matrix

Let, A be an idempotent matrix i.e. $A^2 = A$
and I is also an idempotent matrix i.e. $I^2 = I$
↳ (identity matrix)

Proof by contradiction -
Let us assume, set of idempotent matrices are closed under addition.

$$\therefore \quad (I+A) = (I+A)^2 \qquad \text{(this will be true)}$$
$$(I+A) = I^2 + A^2 + 2A$$
$$I + A = I + 3A$$
$$\Rightarrow \boxed{A = 0}$$

This indicates sum of 2 idempotent can be idempotent only if $A = 0$
Hence, our assumption is wrong.
∴ Set of idempotent matrices is not closed under addition.

∴ Not a subspace.

5.(a) V: all function $f: R \to R$

(a) all continuous functions

(i) F_0 is a function that maps everything to 0
Since it is a constant function, it is continuous
$\therefore$ 0 function is present in set.

(ii) closure under addition - Let f, g are continuous over R
$(f+g)x = f(x) + g(x) \quad \forall x \in R$
By algebra of continuous functions $(f+g)$ is continuous.
$\therefore$ Closed under addition.

(iii) closure under scalar multiplication -
Let f be continuous over R and $\lambda \in R$
$\lambda f = (\lambda f(x)) \quad \forall x \in R$
$= \lambda f(x)$
By algebra of continuous function, λf is continuous over R.
$\therefore$ Closed under scalar multiplication.

Hence, it is a subspace.

(b) f such that $f(x^2) = f(x)^2$
(i) Zero vector of v should be present. Zero vector of V is just the function $f_0(x) = 0$, gives out 0 for any x $\therefore f(x^2) = 0 = [f(x)]^2$

$\therefore$ zero vector present

(ii) Closed under addition - Let $f(x)$ and $g(x)$ be arbitrary elements of set such that

$f(x^2) = (f(x))^2$ and $g(x^2) = [g(x)]^2$

$(f+g)(x) = f(x) + g(x)$

$\Rightarrow (f+g)(x^2) = f(x^2) + g(x^2) = [f(x)]^2 + [g(x)]^2$

$\neq [(f+g)(x)]^2$

$\therefore$ Not closed under addition.
Hence, it is not a subspace.

(c) $f(3) = 1 + f(-5)$

Take two arbitrary from V such that

$f(3) = 1 + f(-5)$

$g(3) = 1 + g(-5)$

$(f+g)(3) = 2 + f(-5) + g(-5)$

$= 2 + (f+g)(-5)$

$\neq 1 + (f+g)(-5)$

$\therefore$ Not closed under addition.

$\therefore$ Hence, it is not a subspace.

6. The first component of vector in Z inherits the vector space properties from V, while the second component of a vector in Z inherits vector space properties from W.

(a) Since V and W is a vector space.

$\forall (v_1, u_1), (v_2, u_2) \in V$, $(v_1, u_1) + (v_2, u_2)$
$= (v_1 + v_2, u_1 + u_2)$
$= (v_2 + v_1, u_2 + u_1)$

For first component $\Rightarrow$ $v_1 + v_2 = v_2 + v_1$

$\forall (x_1, w_1), (x_2, w_2) \in W$
$(x_1, w_1) + (x_2, w_2) = (x_1 + x_2, w_1 + w_2)$
$= (x_2 + x_1, w_2 + w_1)$

For second, $w_1 + w_2 = w_2 + w_1$

For both, additive commutativity holds good for Z

(b) Additive associativity -
$\forall (v_1, u_2), (v_2, u_2), (v_3, u_3) \in V$
$((v_1, u_1) + (v_2, u_2)) + (v_3, u_3) = (v_1, u_1) + ((v_2, u_2) + (v_3, u_3))$

$\Rightarrow (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$

$\forall (x_1, w_1), (x_2, w_2), (x_3, w_3) \in W$

$((x_1, w_1) + (x_2, w_2)) + (x_3, w_3) = (x_1, w_1) + ((x_2, w_2) + (x_3, w_3))$

$$\Rightarrow (w_1 + w_2) + w_3 = w_1 + (w_2 + w_3)$$

so by definition

$((v_1, w_1) + (v_2, w_2)) + (v_3, w_3) = (v_1, w_1) + ((v_2, w_2) + (v_3, w_3))$

where $(v_1, w_1), (v_2, w_2), (v_3, w_3) \in W$

so associativity holds good for Z

(c) Zero vectors

V, W are vector spaces $\Rightarrow \exists$ zero vector 0_V for V and zero vector 0_W for W

The zero vector 0_Z can be formed by taking component from $0_V, 0_W$

so $(0_V, 0_W)$ will be the zero vector

$(v, w) + (0_V, 0_W) = (v + 0_V, w + 0_W) = (v, w)$

(d) Additive Inverse

Since, $\forall (v, u) \in V$, $\exists (-v, -u)$ such that

$(v, u) + (-v, -u) = 0_V \Rightarrow \forall v, \exists -v$ such that $v + (-v) = 0$

Similarly for w,

So, $\forall (v, w) \in Z$, $\exists (-v, -w)$ such that $(v, w) + (-v, -w) = 0_Z$

(e) Identity element -

$\forall (v,u) \in V$, $\quad 1\cdot(v,u) = (v,u) \Rightarrow 1 * v = v$

$\forall (x,w) \in W$, $\quad 1\cdot(x,w) = (x,w) \Rightarrow 1 * w = w$

From definition,

$$1(v,w) = (1*v, 1*w) \quad \text{where } c = 1$$

but

$$(1*v, 1*w) = (v,w) \Rightarrow 1\cdot(v,w) = (v,w)$$

(f) $\forall (a,b) \in F$, $\forall (v,u) \in V$

$a(b(v,u)) = (ab)(v,u)$ &

$\forall (x,w) \in W$, $a(b(x,w)) = (ab)(x,w)$

$\Rightarrow a(bv) = (ab)v$ & $a(bw) = (ab)w$

$\Rightarrow \forall (v,w) \in Z$, $a(b(v,w)) \cong a(bv, bw)$

where $c = b$

$(a(bv), a(bw))$ where $c = a$

$((ab)v, (ab)w)$ by $(*)$

$\underset{=}{\text{def}}$ $(ab)(v,w)$ where $c = ab$

Distributivity holds good

(g) $\forall c \in F$, $\forall (v_1,u_1), (v_2,u_2) \in V$

$c((v_1,u_1) + (v_2,u_2)) = c(v_1,u_1) + c(v_2,u_2)$

$\Rightarrow c(v_1+v_2) = cv_1 + cv_2$

$\forall (x_1, w_1), (x_2, w_2) \in W, \quad c((x_1, w_1) + (x_2, w_2))$
$= c(x_1, w_1) + c(x_2, w_2)$
$\Rightarrow c(w_1 + w_2) = cw_1 + cw_2$

$c((v_1, w_1) + (v_2, w_2)) = (c(v_1 + v_2), c(w_1 + w_2))$
$= (cv_1 + cv_2, cw_1 + cw_2)$
$= c(v_1, w_1) + c(v_2, w_2)$

(h) $\forall a, b \in F, \quad \forall (v, u) \in V$
$(a+b)(v, u) = a(v, u) + b(v, u)$.
$= (av + bv, au + bu)$
$\Rightarrow (a+b)v = av + bv$

$\forall (x, w) \in W, \quad (a+b)(x, w) = a(x, w) + b(x, w)$
$= (ax + bx, aw + bw)$

$\Rightarrow (a+b)w = aw + bw$

$\forall (v, w) \in Z, \quad (a+b)(v, w) = (av + bv, aw + bw)$
$= a(v, w) + b(v, w)$

So, Z is a vector space over a field F with the given operation.

7. Let $v_1, v_2, ----, v_n \in S_1$ be the pairwise distinct and $a_1, a_2, -----, a_n \in F$ are such that

$$a_1v_1 + a_2v_2 + ---- + a_nv_n = 0 \quad -(1)$$

such that $a_1, a_2, ----, a_n$ solves the vector equation

Since, we are given $S_1 \subseteq S_2$ so the vectors.

$v_1, v_2, ----, v_n$ are also in S_2 which is assumed to be linearly independent consequently

$$a_1 = a_2 = ---- a_n = 0$$

Thus, we have shown for any finite set of vectors $v_1, v_2, ---v_n$ in S_1.
The vector equation (1) has only the trivial solution.

Hence, S_1 is linearly independent.

8. To prove - Subset W of a vector space V is a subspace of V ~~an~~ if and only if $span(W) = W$

* For $span(W) = W$, we need to prove that $span(W) \subseteq W$ and $W \subseteq span(W)$

For $W \subseteq span(W)$, since W is a subspace.
Let, $z \in W$, by its definition we can write the linear combination of z.

$$z = u_1x_1 + u_2x_2 + \cdots + u_nx_n \in W$$

For $u_i \in F$ and $x_i \in W$

so, $W \subseteq span(W)$ — (1)

For any $u \in span(W)$
$u = a_1v_1 + a_2v_2 + \cdots + a_nv_n$

For some $v_1, v_2, \ldots v_n \in W$ and some scalars $a_1, a_2, \ldots a_n$
Since W is a subspace of V and $v_1, v_2, \ldots v_n \in W$

$$u = a_1v_1 + a_2v_2 + \cdots + a_nv_n \in W$$

so, $\boxed{span(W) \subseteq W}$ — (2)

From (1) and (2)
W is a subspace of V, then $span(W) = W$