MCS-1
Assignment-2

3.1 An expectation identity

We know that, for a discrete random variable $X \rightarrow$

$$P(X>i) = P(X=i+1) + P(X=i+2) + \ldots$$

Therefore,

$$P(X>0) = P(X=1) + P(X=2) + P(X=3) + P(X=4) + \ldots + \quad -(1)$$
$$P(X>1) = P(X=2) + P(X=3) + P(X=4) + \ldots \quad (2)$$
$$P(X>2) = P(X=3) + P(X=4) + \ldots \quad (3)$$

and so on

adding eqn (1), (2), (3) and so on

$$P(X>0) + P(X>1) + P(X>2) + \ldots = 1 \cdot P(X=1) + 2 \cdot P(X=2) + 3P(X=3) + \ldots$$

$$\sum_{x \geq 0} P[X>x] = \underbrace{0 \cdot P(X=0)} + 1 \cdot P(X=1) + 2 \cdot P(X=2) + 3 \cdot P(X=3) + \ldots$$

We can add this term bcz this term is zero.

↳ eqn (w)

We also know that, for a random variable $X \rightarrow$

$$E(X) = \sum_{i=0}^{\infty} iP_i = 0 \cdot P_0 + 1 \cdot P_1 + 2 \cdot P_2 + \ldots$$

↳ eqn (u)

From equation (w) and (u) $\longrightarrow$

$$\boxed{E(X) = \sum_{x \geq 0} P[X>x]}$$

3.2 Random Permutation

(a) Expected value of X

X : number of fixed points ($\pi(i) = i$)

Let, X_i be an indicator random variable, such that

$$X_i = \begin{cases} 1 & , \text{ if point 'i' is fixed} \\ 0 & , \text{ if point 'i' is not fixed} \end{cases}$$

Expected values of X_i →

$E[X_1] = 1/n$ [Among 'n' possibilities, there is only 1 position where a point is said to be fixed]

$E[X_2] = 1/n$

$E[X_3] = 1/n$

OR

$E[X_n] = 1/n$ $\left\{ E[X_i] = 1 \cdot \left(\frac{1}{n}\right) + 0 \cdot \left(1 - \frac{1}{n}\right) = \frac{1}{n} \right\}$

$$X = X_1 + X_2 + X_3 + \dots + X_n$$

$$E(X) = E[X_1 + X_2 + X_3 + \dots + X_n]$$

$$E(X) = E[X_1] + E[X_2] + E[X_3] + \dots + E[X_n]$$ {By using linearity of expectations}

$$E(X) = \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \dots \text{ upto } n \text{ times}$$

$$E(X) = n * \frac{1}{n}$$

$$\boxed{E(X) = 1}$$

3.2 (b) $|\Omega| = n * (n-1) * (n-2) * \cdots * 1 = n!$

Let, $P(X=i)$ denote the probability that 'i' points are fixed →

Choosing 'i' fixed points out of 'n' points

$$\therefore P(X=i) = \frac{{}^nC_i * D_{(n-i)}}{n!}$$

No. of possibilities that $(n-i)$ elements are not fixed

↳ Total possibilities (Sample space)

D_n represents derangement formula →

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$D_n = n! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right]$$

put value of D_n in $P(X=i)$

$$P(X=i) = \frac{{}^nC_i * D_{(n-i)}}{n!}$$

$$= \frac{\cancel{n!}}{\cancel{(n-i)!}\, i!} \times \frac{\cancel{(n-i)!}}{\cancel{n!}} \left[1 - \frac{1}{1!} + \frac{1}{2!} + \cdots + (-1)^{n-i} \frac{1}{(n-i)!}\right]$$

$$\boxed{P(X=i) = \frac{1}{i!} * \sum_{k=0}^{n-i} \left[(-1)^k \frac{1}{k!}\right]}$$

3.2(c) Let, X_i be an indicator random variable such that -

$$X_i = \begin{cases} 1, & \text{if point '}i\text{' is involved in swap} \\ 0, & \text{otherwise} \end{cases}$$

Let, X be a random variable representing total number of swaps.

$$X = \frac{1}{2}(X_1 + X_2 + \cdots + X_n)$$

(bcz each swap is counted exactly twice i.e. say $\pi(i) = j$ and $\pi(j) = i$ then $X_i = 1$ and $X_j = 1$ as both 'i' and 'j' are involved in a swap so it is counted twice but this is 1 swap only)

$$P[X_i = 1] = \frac{(n-1)}{n} * \frac{1}{(n-1)} = \frac{1}{n}$$

For point 'i' to be swapped it can be mapped to any point from (1 to n) except 'i' i.e. 'i' can be mapped to any of (n-1) choices out of n available choices

Say point 'i' is mapped to 'j' i.e. $\pi(i) = j$. Now, for a successful swap $\pi(j)$ must be equal to 'i' i.e. point 'j' can be mapped to only 1 choice (i.e. point 'i') out of remaining (n-1) choices.

$$E[X_i] = 0 \cdot P[X_i = 0] + 1 \cdot P[X_i = 1]$$

$$E[X_i] = P[X_i = 1]$$

$$E[X_i] = 1/n$$

Using linearity of expectations -

$$E[X] = \frac{1}{2} E[X_1 + X_2 + \cdots + X_n]$$

$$E[X] = \frac{1}{2}(E[X_1] + E[X_2] + \cdots + E[X_n])$$

$$E[X] = \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n} + \cdots \text{ upto 'n' times}\right)$$

$$E[X] = \frac{1}{2} * n * \frac{1}{n} \Rightarrow \boxed{E[X] = \frac{1}{2}}$$

$\therefore$ Expected number of swaps $= 1/2$

3.2(d) To prove $Pr[X>10] \leq 1/10$

Markov's inequality →

$$P(X \geq a) \leq \frac{E[X]}{a}$$

put $a = 10$

$$P(X \geq 10) \leq \frac{E[X]}{10} \quad [E(X)=1, \text{ proved earlier}]$$

$$\boxed{P(X \geq 10) \leq \frac{1}{10}} \quad \text{— eqn (1)}$$

Now,

$P(X \geq 10) = P(X=10) + P(X=11) + P(X=12) + \text{-----}$

$0 \leq P(X=10) \leq 1$ [Reason probability of an event]

$\therefore P(X \geq 10) \geq P(X=11) + P(X=12) + P(X=13) + \text{-----}$

(as one positive (or zero) term is removed from RHS)

$$\boxed{P(X \geq 10) \geq P(X>10)} \quad \text{— eqn (2)}$$

From eqn (1) and eqn (2) →

$$\boxed{P(X>10) \leq P(X \geq 10) \leq \frac{1}{10}}$$

$$\Rightarrow \boxed{P(X>10) \leq \frac{1}{10}} \quad \text{Hence proved}$$

Note - $(X \geq 10)$ is superset of $(X>10)$ and

$Pr(\text{superset}) \geq Pr(\text{subset})$

Using this logic we can also reach to equation (2)

4. Randomized Colouring

(i) Given, X_e is random variable such that -

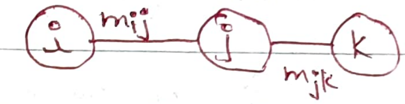
$$X_e = \begin{cases} 1 & , e \text{ is monochromatic} \\ 0 & , e \text{ is non-monochromatic} \end{cases}$$

Let, m_{ij} be the edge between vertex i and j

To prove - $\{X_e\}_{e \in E}$ are pairwise independent

<u>Case I</u> - Two edges share a common vertex

i.e. m_{ij} & m_{jk}.



Sample space for an edge = {RR, RG, RB, BR, BG, BB, GR, GG, GB}

(colour of 2 vertices)

Edge is monochromatic = {RR, GG, BB}

$P(m_{ij}) = 3/9 = 1/3$

Similarly, $P(m_{jk}) = 3/9 = 1/3$

Size of Sample space for 3 vertices = $3 * 3 * 3 = 27$

(so that 2 edges can share a vertex)

3 possible color of vertex i | 3 possible colors of vertex j | 3 possible colours of vertex j

[m_{ij} is monochromatic $\cap$ m_{jk} is monochromatic]

(vertex i and j are of same color) (vertex j and k are of same color)

= {RRR, GGG, BBB}

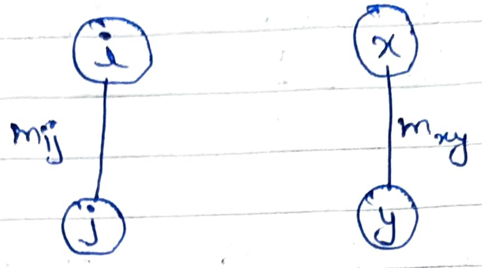
$$P(m_{ij} \cap m_{jk}) = \frac{3}{27} = \frac{1}{9}$$

$$P(m_{ij}) \cdot P(m_{jk}) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

Clearly, $\boxed{P(m_{ij} \text{ and } m_{jk}) = P(m_{ij}) \cdot P(m_{jk})}$

↳ Independent events

<u>Case II</u> - m_{ij} and m_{xy} i.e. 2 edges do not share any vertex.



$P(m_{ij}) = 1/3$
$P(m_{xy}) = 1/3$ } Probability of an edge to be monochromatic (shown in previous part)

Size of same space for 4 vertices $= 3 * 3 * 3 * 3 = 81$
(Colors of i, j, x and y respectively)

[m_{ij} and m_{xy} are monochromatic] $= 3*1 * 3*1 = 9$

Vertex 'i' can take any color but vertex 'j' has only 1 choice i.e. color of vertex 'i' only then m_{ij} will be monochromatic

Same logic for 'y' and 'z'

$$P(m_{ij} \cap m_{xy}) = \frac{9}{81} = \frac{1}{9}$$

$$P(m_{ij}) \cdot P(m_{xy}) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

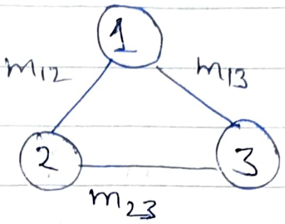
Clearly, $\boxed{P(m_{ij} \cap m_{xy}) = P(m_{ij}) \cdot P(m_{xy})}$

↳ Independent

Hence, in both cases, pair of 2 edges are monochromatic are independent.
So, we can say that $\{X_e\}_{e \in E}$ are pairwise independent

To prove - All monochromatic edges are not independent.

Proof by counter-example -
Consider the graph below -



$P(m_{12}) = 1/3$
$P(m_{23}) = 1/3$
$P(m_{13}) = 1/3$

Size of sample space for 3 vertices $= 3 * 3 * 3 = 27$
(Possible colors of vertices 1, 2 and 3)

(m_{12}, m_{23}, m_{13} are monochromatic) =
(colour of vertex 1 and 2 is same,
colour of vertex 2 and 3 is same,
colour of vertex 1 and 3 is same) = {RRR, GGG, BBB}

$P(m_{12} \cap m_{23} \cap m_{13}) = \frac{3}{27} = \frac{1}{9}$

$P(m_{12}) \cdot P(m_{23}) \cdot P(m_{13}) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{27}$

Clearly, $\boxed{P(m_{12} \cap m_{23} \cap m_{13}) \neq P(m_{12}) \cdot P(m_{23}) \cdot P(m_{13})}$

↳ Not independent

2. 'Y' be the random variable representing no. of non-chromatic edges.

<u>To find</u> - $E[Y]$

Let, Y_i be an indicator random variable such that -

$$Y_i = \begin{cases} 1 & , \text{ edge 'i' is not monochromatic} \\ 0 & , \text{ otherwise (monochromatic)} \end{cases}$$

$$E[Y_i] = 1 \cdot P(\text{not monochromatic}) + 0 \cdot P(\overset{\text{mono}}{\text{chromatic}})$$

$$E[Y_i] = 1 \cdot \frac{2}{3} + 0 \cdot \frac{1}{3}$$

$$E[Y_i] = \frac{2}{3}$$

$$Y = Y_1 + Y_2 + Y_3 + \text{---} \quad \underbrace{|E|}_{\text{no. of edges}} \text{ times}$$

By linearity of expectations,

$$E[Y] = E[Y_1] + E[Y_2] + E[Y_3] \text{ ---} \quad |E| \text{ times}$$

$$= \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \text{ ----} \quad |E| \text{ times}$$

$$\boxed{E[Y] = \frac{2}{3}|E|}$$

3. To prove – For any graph G, there exist a colour assignment such that number of non-monochromatic edges $\geq \frac{2}{3}|E|$

Let, for any arbitrary graph G, number of colour assignments possible $= n$
and 'a_i' represent number of non-monochromatic edges in i-th colour assignment.

Proof by contradiction – Let us assume, there exist no such colour assignment, for a graph G, in which $a_i \geq \frac{2}{3}|E|$

i.e. $a_i < \frac{2}{3}|E| \quad \forall i \in \{1, 2, \ldots, n\}$

Sum of all non-monochromatic edges in all the colour assignments $= a_1 + a_2 + a_3 + \cdots + a_n$

$< \frac{2}{3}|E| \quad < \frac{2}{3}|E| \quad \frac{2}{3}|E| \quad \frac{2}{3}|E|$

$$\text{Sum} < n * \frac{2}{3}|E| \qquad -(1)$$

Expectation of non-chromatic edges $=$ $\dfrac{\text{Sum of all non-chromatic edges in all color assignment}}{\text{Number of assignment}}$

$$E[Y] < \frac{\cancel{n} * \frac{2}{3}|E|}{\cancel{n}} \qquad \text{(using (1))}$$

$$E[Y] < \frac{2}{3}|E|$$

but $E[Y] = \frac{2}{3}|E|$ (proved in previous part)

Hence, our initial assumption is wrong.
Therefore, for any graph G, there exists a colour assignment such that number of non-chromatic edges $\geq \frac{2}{3}|E|$

4. To prove - $P(Y \geq |E|/2) \geq \frac{1}{3}$.

Let,
X be a random variable representing no. of monochromatic edges
& Y " " " " " " " non-monochromatic edges

$E[Y] = \frac{2}{3}|E|$ (using 4.2 part)

$E[X] = 1 - E[Y]$ ($\because$ X and Y are complement of each other)
$E[X] = |E|/3$

By Markov's Inequality →
$$P(X \geq a) \leq \frac{E[X]}{a}$$

$$P(X \geq |E|/2) \leq \frac{|E|/3}{|E|/2}$$

$$P(X \geq |E|/2) \leq \frac{2}{3}$$

subtracting both sides from '1' →
$$1 - P(X \geq |E|/2) \geq 1 - \frac{2}{3}$$

$(1 - P(X \geq |E|/2)) = P(Y \geq |E|/2)$
↳ because X and Y are complimentary events

$$\boxed{P(Y \geq |E|/2) \geq \frac{1}{3}}$$

4.5 Algorithm to find a colour assignment in which no. of non-chromatic edges $\geq |E|/2$ with probability ≥ 0.99

(i) Set $S \leftarrow \{V \rightarrow \{R, G, B\}\}$ // Set of all colour assignments
(ii) count $\leftarrow 0$ // to store no. of color assignments in which non-chromatic edges $\geq |E|/2$
(iii) total $\leftarrow 0$ // all color assignments we observed

(iv) while (true)
{
(v) Select a random color assignment $a \in S$
(vi) total $\leftarrow$ total + 1
(vii) if (no. of non-chromatic edges in 'a' $\geq |E|/2$)
(viii) count $\leftarrow$ count + 1

(ix) if (count / total ≥ 0.99)
(x) break;
}

(xi) return total ; // indicates that we need to pick 'total' number of assignments to get probability of non-chromatic edges $\geq |E|/2$ is at least 0.99