Quantum Field Theory 2nd Edition

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Preface to the Second Edition

The first edition of this book aimed to give an easily accessible introduction to QED and the unified theory of electromagnetic and weak interaction. In this edition, we have added five new chapters, giving an introduction to QCD and the methods used to understand it: in particular, path integrals and the renormalization group. At the same time, the treatment of electroweak interactions has been updated to take account of more recent experiments. When the first edition was published in 1984, only a handful of W and Z boson events had been observed and the experimental investigation of high energy electroweak interactions was in its infancy. Now it is a precise science, despite the fact that crucial questions about Higgs bosons and the nature of neutrinos remain unanswered.

The structure of the book is as follows. The first ten chapters deal with QED in the canonical formalism, and are little changed from the first edition. A brief introduction to gauge theories (Chapter 11) is then followed by two sections, which may be read independently of each other. They cover QCD and related topics (Chapters 12–15) and the unified electroweak theory (Chapters 16–19) respectively.

Sadly, my close friend and collaborator, Franz Mandl, passed away on 4th February 2009 after a long illness. He retained his passion for physics, and his active commitment to this project, almost to the end. It was a privilege to work with him, and he remains an inspiration to all who knew him.

November 2009 GRAHAM SHAW

Preface to the First Edition

Our aim in writing this book has been to produce a short introduction to quantum field theory, suitable for beginning research students in theoretical and experimental physics. The main objectives are: (i) to explain the basic physics and formalism of quantum field theory, (ii) to make the reader fully proficient in perturbation theory calculations using Feynman diagrams, and (iii) to introduce the reader to gauge theories which are playing such a central role in elementary particle physics.

The theory has been applied to two areas. The beginning parts of the book deal with quantum electrodynamics (QED) where quantum field theory had its early triumphs. The

last four chapters, on weak interactions, introduce non-Abelian gauge groups, spontaneous symmetry breaking and the Higgs mechanism, culminating in the Weinberg-Salam standard electro-weak theory. For reasons of space, we have limited ourselves to purely leptonic processes, but this theory is equally successful when extended to include hadrons. The recent observations of the $W^{\pm}$ and Z° bosons, with the predicted masses, lend further support to this theory, and there is every hope that it is the fundamental theory of electro-weak interactions.

The introductory nature of this book and the desire to keep it reasonably short have influenced both the level of treatment and the selection of material. We have formulated quantum field theory in terms of non-commuting operators, as this approach should be familiar to the reader from non-relativistic quantum mechanics and it brings out most clearly the physical meaning of the formalism in terms of particle creation and annihilation operators. We have developed the formalism only to the level we require in the applications. These concentrate primarily on calculations in lowest order of perturbation theory. The techniques for obtaining cross-sections, decay rates, and spin and polarization sums have been developed in detail and applied to a variety of processes, many of them of interest in current research on electro-weak interactions. After studying this material, the reader should be able to tackle confidently any process in lowest order.

Our treatment of renormalization and radiative corrections is much less complete. We have explained the general concepts of regularization and renormalization. For QED we have shown in some detail how to calculate the lowest-order radiative corrections, using dimensional regularization as well as the older cut-off techniques. The infra-red divergence and its connection with radiative corrections have similarly been discussed in lowest order only. The scope of this book precludes a serious study of higher order corrections in QED and of the renormalization of the electro-weak theory. For the latter the Feynman path integral formulation of quantum field theory seems almost essential. Regretfully, we were not able to provide a short and simple treatment of this topic.

This book arose out of lectures which both of us have given over many years. We have greatly benefited from discussions with students and colleagues, some of whom have read parts of the manuscript. We would like to thank all of them for their help, and particularly Sandy Donnachie who encouraged us to embark on this collaboration.

January 1984

FRANZ MANDL
GRAHAM SHAW

Notes

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In preparing this new edition, we have again benefited from discussions with many of our colleagues. We are grateful to them all, and especially Jeff Forshaw, who read all the new chapters and made many helpful suggestions. We are also grateful to Brian Martin: Sections 15.1 and 15.3.3 draw heavily on his previous work with one of us (G.S.) and he also generously helped prepare many of the new figures.

Illustrations

Some illustrations in the text have been adapted from diagrams that have been published elsewhere. In a few cases they have been reproduced exactly as previously published. We acknowledge, with thanks, permission from the relevant copyright holders to use such illustrations and this is confirmed in the captions.

Data

Except when otherwise stated, the data quoted in this book, and their sources, are given in "Review of Particle Physics", *Journal of Physics* **G33** (2006) 1 and on the Particle Data Group website http:pdg.lbl.gov, which is regularly updated.

Webpage

This book has its own website:www.hep.manchester.ac.uk/u/graham/qftbook.html. Any misprints or other necessary corrections brought to our attention will be listed on this page. We would also be grateful for any other comments about this book.

1

Photons and the Electromagnetic Field

1.1 Particles and Fields

The concept of photons as the quanta of the electromagnetic field dates back to the beginning of the twentieth century. In order to explain the spectrum of black-body radiation, Planck, in 1900, postulated that the process of emission and absorption of radiation by atoms occurs discontinuously in quanta. Einstein, by 1905, had arrived at a more drastic interpretation. From a statistical analysis of the Planck radiation law and from the energetics of the photoelectric effect, he concluded that it was not merely the atomic mechanism of emission and absorption of radiation which is quantized, but that electromagnetic radiation itself consists of photons. The Compton effect confirmed this interpretation.

The foundations of a systematic quantum theory of fields were laid by Dirac in 1927 in his famous paper on 'The Quantum Theory of the Emission and Absorption of Radiation'. From the quantization of the electromagnetic field one is naturally led to the quantization of any classical field, the quanta of the field being particles with well-defined properties. The interactions between these particles are brought about by other fields whose quanta are other particles. For example, we can think of the interaction between electrically charged particles, such as electrons and positrons, as being brought about by the electromagnetic field or as due to an exchange of photons. The electrons and positrons themselves can be thought of as the quanta of an electron–positron field. An important reason for quantizing such particle fields is to allow for the possibility that the number of particles changes as, for example, in the creation or annihilation of electron–positron pairs.

These and other processes of course only occur through the interactions of fields. The solution of the equations of the quantized interacting fields is extremely difficult. If the interaction is sufficiently weak, one can employ perturbation theory. This has been outstandingly successful in quantum electrodynamics, where complete agreement exists between theory and experiment to an incredibly high degree of accuracy. Perturbation

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theory has also very successfully been applied to weak interactions, and to strong interactions at short distances, where they become relatively weak.

The most important modern perturbation-theoretic technique employs Feynman diagrams, which are also extremely useful in many areas other than relativistic quantum field theory. We shall later develop the Feynman diagram technique and apply it to electromagnetic, weak and strong interactions. For this a Lorentz-covariant formulation will be essential.

In this introductory chapter we employ a simpler non-covariant approach, which suffices for many applications and brings out many of the ideas of field quantization. We shall consider the important case of electrodynamics for which a complete classical theory – Maxwell's – exists. As quantum electrodynamics will be re-derived later, we shall in this chapter, at times, rely on plausibility arguments rather than fully justify all steps.

1.2 The Electromagnetic Field in the Absence of Charges

1.2.1 The classical field

Classical electromagnetic theory is summed up in Maxwell's equations. In the presence of a charge density $\rho(\mathbf{x}, t)$ and a current density $\mathbf{j}(\mathbf{x}, t)$, the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ satisfy the equations

$$\nabla \cdot \mathbf{E} = \rho \tag{1.1a}$$

$$\nabla \wedge \mathbf{B} = \frac{1}{c}\,\mathbf{j} + \frac{1}{c}\,\frac{\partial \mathbf{E}}{\partial t} \tag{1.1b}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.1c}$$

$$\nabla \wedge \mathbf{E} = -\frac{1}{c}\,\frac{\partial \mathbf{B}}{\partial t} \tag{1.1d}$$

where, as throughout this book, rationalized Gaussian (c.g.s.) units are being used.[1]

From the second pair of Maxwell's equations [Eqs. (1.1c) and (1.1d)] follows the existence of scalar and vector potentials $\phi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$, defined by

$$\mathbf{B} = \nabla \wedge \mathbf{A}, \quad \mathbf{E} = -\nabla\phi - \frac{1}{c}\,\frac{\partial \mathbf{A}}{\partial t}. \tag{1.2}$$

Eqs. (1.2) do not determine the potentials uniquely, since for an arbitrary function $f(\mathbf{x}, t)$ the transformation

$$\phi \to \phi' = \phi + \frac{1}{c}\,\frac{\partial f}{\partial t}, \quad \mathbf{A} \to \mathbf{A}' = \mathbf{A} - \nabla f \tag{1.3}$$

leaves the fields $\mathbf{E}$ and $\mathbf{B}$ unaltered. The transformation (1.3) is known as a gauge transformation of the second kind. Since all observable quantities can be expressed in

[1] They are also called rationalized Lorentz–Heaviside units. In these units, the fine structure constant is given by $\alpha = e^2/(4\pi\hbar c) \approx 1/137$, whereas in unrationalized gaussian units $\alpha = e_{\text{unrat}}^2/\hbar c$, i.e. $e = e_{\text{unrat}}\sqrt{(4\pi)}$. Correspondingly for the fields $\mathbf{E} = \mathbf{E}_{\text{unrat}}/\sqrt{(4\pi)}$, etc.

terms of **E** and **B**, it is a fundamental requirement of any theory formulated in terms of potentials that it is gauge-invariant, i.e. that the predictions for observable quantities are invariant under such gauge transformations.

Expressed in terms of the potentials, the second pair of Maxwell's equations [Eqs. (1.1c) and (1.1d)] are satisfied automatically, while the first pair [Eqs. (1.1a) and (1.1b)] become

$$-\nabla^2 \phi - \frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = \Box\phi - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A}\right) = \rho \tag{1.4a}$$

$$\Box \mathbf{A} + \nabla\left(\frac{1}{c}\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A}\right) = \frac{1}{c}\mathbf{j} \tag{1.4b}$$

where

$$\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2. \tag{1.5}$$

We now go on to consider the case of the free field, i.e. the absence of charges and currents: $\rho = 0, \mathbf{j} = 0$. We can then choose a gauge for the potentials such that

$$\nabla \cdot \mathbf{A} = 0. \tag{1.6}$$

The condition (1.6) defines the Coulomb or radiation gauge. A vector field with vanishing divergence, i.e. satisfying Eq. (1.6), is called a transverse field, since for a wave

$$\mathbf{A}(\mathbf{x}, t) = \mathbf{A}_0 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$$

Eq. (1.6) gives

$$\mathbf{k} \cdot \mathbf{A} = 0, \tag{1.7}$$

i.e. **A** is perpendicular to the direction of propagation **k** of the wave. In the Coulomb gauge, the vector potential is a transverse vector. In this chapter we shall be employing the Coulomb gauge.

In the absence of charges, Eq. (1.4a) now becomes $\nabla^2 \phi = 0$ with the solution, which vanishes at infinity, $\phi \equiv 0$. Hence Eq. (1.4b) reduces to the wave equation

$$\Box \mathbf{A} = 0. \tag{1.8}$$

The corresponding electric and magnetic fields are, from Eqs. (1.2), given by

$$\mathbf{B} = \nabla \wedge \mathbf{A}, \quad \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \tag{1.9}$$

and, like **A**, are transverse fields. The solutions of Eq. (1.8) are the transverse electromagnetic waves in free space. These waves are often called the radiation field. Its energy is given by

$$H_{\text{rad}} = \frac{1}{2}\int (\mathbf{E}^2 + \mathbf{B}^2)\, d^3\mathbf{x}. \tag{1.10}$$

In order to quantize the theory, we shall want to introduce canonically conjugate coordinates (like x and p_x in non-relativistic quantum mechanics) for each degree of

freedom and subject these to commutation relations. At a given instant of time t, the vector potential $\mathbf{A}$ must be specified at every point $\mathbf{x}$ in space. Looked at from this viewpoint, the electromagnetic field possesses a continuous infinity of degrees of freedom. The problem can be simplified by considering the radiation inside a large cubic enclosure, of side L and volume $V = L^3$, and imposing periodic boundary conditions on the vector potential $\mathbf{A}$ at the surfaces of the cube. The vector potential can then be represented as a Fourier series, i.e. it is specified by the denumerable set of Fourier expansion coefficients, and we have obtained a description of the field in terms of an infinite, but denumerable, number of degrees of freedom. The Fourier analysis corresponds to finding the normal modes of the radiation field, each mode being described independently of the others by a harmonic oscillator equation. (All this is analogous to the Fourier analysis of a vibrating string.) This will enable us to quantize the radiation field by taking over the quantization of the harmonic oscillator from non-relativistic quantum mechanics.

With the periodic boundary conditions

$$\mathbf{A}(0, y, z, t) = \mathbf{A}(L, y, z, t), \text{etc.}, \tag{1.11}$$

the functions

$$\frac{1}{\sqrt{V}}\, \boldsymbol{\varepsilon}_r(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}}, \quad r = 1, 2, \tag{1.12}$$

form a complete set of transverse orthonormal vector fields. Here the wave vectors $\mathbf{k}$ must be of the form

$$\mathbf{k} = \frac{2\pi}{L}\,(n_1, n_2, n_3), \quad n_1, n_2, n_3 = 0, \pm 1, \ldots, \tag{1.13}$$

so that the fields (1.12) satisfy the periodicity conditions (1.11). $\boldsymbol{\varepsilon}_1(\mathbf{k})$ and $\boldsymbol{\varepsilon}_2(\mathbf{k})$ are two mutually perpendicular real unit vectors which are also orthogonal to $\mathbf{k}$:

$$\boldsymbol{\varepsilon}_r(\mathbf{k}) \cdot \boldsymbol{\varepsilon}_s(\mathbf{k}) = \delta_{rs}, \quad \boldsymbol{\varepsilon}_r(\mathbf{k}) \cdot \mathbf{k} = 0, \quad r, s = 1, 2. \tag{1.14}$$

The last of these conditions ensures that the fields (1.12) are transverse, satisfying the Coulomb gauge condition (1.6) and (1.7).[2]

We can now expand the vector potential $\mathbf{A}(\mathbf{x}, t)$ as a Fourier series

$$\mathbf{A}(\mathbf{x}, t) = \sum_{\mathbf{k}} \sum_{r} \left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}} \right)^{1/2} \boldsymbol{\varepsilon}_r(\mathbf{k}) \left[a_r(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}} + a_r^*(\mathbf{k}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} \right], \tag{1.15}$$

where $\omega_{\mathbf{k}} = c|\mathbf{k}|$. The summations with respect to r and $\mathbf{k}$ are over both polarization states $r = 1, 2$ (for each $\mathbf{k}$) and over all allowed momenta $\mathbf{k}$. The factor to the left of $\boldsymbol{\varepsilon}_r(\mathbf{k})$ has been introduced for later convenience only. The form of the series (1.15) ensures that the vector potential is real: $\mathbf{A} = \mathrm{A}^*$. Eq. (1.15) is an expansion of $\mathbf{A}(\mathbf{x}, t)$ at each instant of time t. The time-dependence of the Fourier expansion coefficients follows, since $\mathbf{A}$ must satisfy the

[2] With this choice of $\boldsymbol{\varepsilon}_r(\mathbf{k})$, Eqs. (1.12) represent linearly polarized fields. By taking appropriate complex linear combinations of $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ one obtains circular or, in general, elliptic polarization.

wave equation (1.8). Substituting Eq. (1.15) in (1.8) and projecting out individual amplitudes, one obtains

$$\frac{\partial^2}{\partial t^2}\, a_r(\mathbf{k},\, t) = -\, \omega_{\mathbf{k}}^2 a_r(\mathbf{k},\, t). \tag{1.16}$$

These are the harmonic oscillator equations of the normal modes of the radiation field. It will prove convenient to take their solutions in the form

$$a_r(\mathbf{k},\, t) = a_r(\mathbf{k})\, \exp\,(-\mathrm{i}\omega_{\mathbf{k}} t), \tag{1.17}$$

where the $a_r(\mathbf{k})$ are initial amplitudes at time $t = 0$.

Eq. (1.15) for the vector potential, with Eq. (1.17) and its complex conjugate substituted for the amplitudes a_r and a_r^*, represents our final result for the classical theory. We can express the energy of the radiation field, Eq. (1.10), in terms of the amplitudes by substituting Eqs. (1.9) and (1.15) in (1.10) and carrying out the integration over the volume V of the enclosure. In this way one obtains

$$H_{\mathrm{rad}} = \sum_{\mathbf{k}} \sum_{r} \hbar\omega_{\mathbf{k}} a_r^*(\mathbf{k}) a_r(\mathbf{k}). \tag{1.18}$$

Note that this is independent of time, as expected in the absence of charges and currents; we could equally have written the time-dependent amplitudes (1.17) instead, since the time dependence of a_r and of a_r^* cancels.

As already stated, we shall quantize the radiation field by quantizing the individual harmonic oscillator modes. As the interpretation of the quantized field theory in terms of photons is intimately connected with the quantum treatment of the harmonic oscillator, we shall summarize the latter.

1.2.2 Harmonic oscillator

The harmonic oscillator Hamiltonian is, in an obvious notation,

$$H_{\mathrm{osc}} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2,$$

with q and p satisfying the commutation relation $[q, p] = \mathrm{i}\hbar$. We introduce the operators

$$\left.\begin{matrix} a \\ a^\dagger \end{matrix}\right\} = \frac{1}{(2\hbar m\omega)^{1/2}}\, (m\omega q \pm \mathrm{i}p).$$

These satisfy the commutation relation

$$\left[a,\, a^\dagger\right] = 1, \tag{1.19}$$

and the Hamiltonian expressed in terms of a and $a^\dagger$ becomes:

$$H_{\mathrm{osc}} = \frac{1}{2}\hbar\omega\left(a^\dagger a + a a^\dagger\right) = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right). \tag{1.20}$$

This is essentially the operator

$$N \equiv a^\dagger a, \tag{1.21}$$

which is positive definite, i.e. for any state $|\Psi\rangle$

$$\langle\Psi|N|\Psi\rangle = \langle\Psi|a^\dagger a|\Psi\rangle = \langle a\Psi\,|\,a\Psi\rangle \geq 0.$$

Hence, N possesses a lowest non-negative eigenvalue

$$\alpha_0 \geq 0.$$

It follows from the eigenvalue equation

$$N|\alpha\rangle = \alpha|\alpha\rangle$$

and Eq. (1.19) that

$$Na|\alpha\rangle = (\alpha - 1)a|\alpha\rangle, \quad Na^\dagger|\alpha\rangle = (\alpha + 1)a^\dagger|\alpha\rangle, \tag{1.22}$$

i.e. $a|\alpha\rangle$ and $a^\dagger|\alpha\rangle$ are eigenfunctions of N belonging to the eigenvalues $(\alpha - 1)$ and $(\alpha + 1)$, respectively. Since α_0 is the lowest eigenvalue we must have

$$a|\alpha_0\rangle = 0, \tag{1.23}$$

and since

$$a^\dagger a|\alpha_0\rangle = \alpha_0|\alpha_0\rangle,$$

Eq. (1.23) implies $\alpha_0 = 0$. It follows from Eqs. (1.19) and (1.22) that the eigenvalues of N are the integers $n = 0, 1, 2, \ldots$, and that if $\langle n|n\rangle = 1$, then the states $|n \pm 1\rangle$, defined by

$$a|n\rangle = n^{1/2}|n - 1\rangle, \quad a^\dagger|n\rangle = (n + 1)^{1/2}|n + 1\rangle, \tag{1.24}$$

are also normed to unity. If $\langle 0|0\rangle = 1$, the normed eigenfunctions of N are

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle, \quad n = 0,\ 1,\ 2, \ldots. \tag{1.25}$$

These are also the eigenfunctions of the harmonic oscillator Hamiltonian (1.20) with the energy eigenvalues

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0,\ 1,\ 2,\ \ldots. \tag{1.26}$$

The operators a and $a^\dagger$ are called lowering and raising operators because of the properties (1.24). We shall see that in the quantized field theory $|n\rangle$ represents a state with n quanta. The operator a(changing $|n\rangle$ into $|n - 1\rangle$) will annihilate a quantum; similarly, $a^\dagger$, will create a quantum.

So far we have considered one instant of time, say $t = 0$. We now discuss the equations of motion in the Heisenberg picture.[3] In this picture, the operators are functions of time. In particular

$$i\hbar\frac{da(t)}{dt} = [a(t),\ H_{\text{osc}}] \tag{1.27}$$

[3] See the appendix to this chapter (Section 1.5) for a concise development of the Schrödinger, Heisenberg and interaction pictures.

with the initial condition $a(0) = a$, the lowering operator considered so far. Since H_{osc} is time-independent, and $a(t)$ and $a^\dagger(t)$ satisfy the same commutation relation (1.19) as a and $a^\dagger$, the Heisenberg equation of motion (1.27) reduces to

$$\frac{\mathrm{d}a(t)}{\mathrm{d}t} = -\mathrm{i}\omega a(t)$$

with the solution

$$a(t) = a\,\mathrm{e}^{-\mathrm{i}\omega t}. \tag{1.28}$$

1.2.3 The quantized radiation field

The harmonic oscillator results we have derived can at once be applied to the radiation field. Its Hamiltonian, Eq. (1.18), is a superposition of independent harmonic oscillator Hamiltonians (1.20), one for each mode of the radiation field. [The order of the factors in (1.18) is not significant and can be changed, since the a_r and a_r^* are classical amplitudes.] We therefore introduce commutation relations analogous to Eq. (1.19)

$$\left.\begin{aligned} \left[a_r(\mathbf{k}),\, a_s^\dagger(\mathbf{k}')\right] &= \delta_{rs}\,\delta_{\mathbf{k}\mathbf{k}'} \\ \left[a_r(\mathbf{k}),\, a_s(\mathbf{k}')\right] &= \left[a_r^\dagger(\mathbf{k}),\, a_s^\dagger(\mathbf{k}')\right] = 0 \end{aligned}\right\} \tag{1.29}$$

and write the Hamiltonian (1.18) as

$$H_{\text{rad}} = \sum_{\mathbf{k}} \sum_{r} \hbar\omega_{\mathbf{k}} \left(a_r^\dagger(\mathbf{k}) a_r(\mathbf{k}) + \frac{1}{2} \right). \tag{1.30}$$

The operators

$$N_r(\mathbf{k}) = a_r^\dagger(\mathbf{k}) a_r(\mathbf{k})$$

then have eigenvalues $n_r(\mathbf{k}) = 0, 1, 2, \ldots$, and eigenfunctions of the form (1.25)

$$|n_r(\mathbf{k})\rangle = \frac{\left[a_r^\dagger(\mathbf{k})\right]^{n_r(\mathbf{k})}}{\sqrt{n_r(\mathbf{k})!}}\,|0\rangle. \tag{1.31}$$

The eigenfunctions of the radiation Hamiltonian (1.30) are products of such states, i.e.

$$|\ldots\, n_r(\mathbf{k})\, \ldots\rangle = \prod_{\mathbf{k}_i} \prod_{r_i} |n_{r_i}(\mathbf{k}_i)\rangle, \tag{1.32}$$

with energy

$$\sum_{\mathbf{k}} \sum_{r} \hbar\omega_{\mathbf{k}} \left(n_r(\mathbf{k}) + \frac{1}{2} \right). \tag{1.33}$$

The interpretation of these equations is a straightforward generalization from one harmonic oscillator to a superposition of independent oscillators, one for each radiation mode $(\mathbf{k}, r)$. $a_r(\mathbf{k})$ operating on the state (1.32) will reduce the *occupation number* $n_r(\mathbf{k})$

of the mode $(\mathbf{k}, r)$ by unity, leaving all other occupation numbers unaltered, i.e. from Eq. (1.24):

$$a_r(\mathbf{k})|\ldots n_r(\mathbf{k})\ldots\rangle = [n_r(\mathbf{k})]^{1/2}\,|\ldots,\ n_r(\mathbf{k})-1,\ldots\rangle. \tag{1.34}$$

Correspondingly the energy (1.33) is reduced by $\hbar\omega_{\mathbf{k}} = \hbar c|\mathbf{k}|$. We interpret $a_r(\mathbf{k})$ as an annihilation (or destruction or absorption) operator, which annihilates one photon in the mode $(\mathbf{k}, r)$, i.e. with momentum $\hbar\mathbf{k}$, energy $\hbar\omega_{\mathbf{k}}$ and linear polarization vector $\boldsymbol{\varepsilon}_r(\mathbf{k})$. Similarly, $a_r^\dagger(\mathbf{k})$ is interpreted as a creation operator of such a photon. The assertion that $a_r(\mathbf{k})$ and $a_r^\dagger(\mathbf{k})$ are absorption and creation operators of photons with momentum $\hbar\mathbf{k}$ can be justified by calculating the momentum of the radiation field. We shall see later that the momentum operator of the field is given by

$$\mathbf{P} = \sum_{\mathbf{k}}\sum_{r}\hbar\mathbf{k}\left(N_r(\mathbf{k}) + \frac{1}{2}\right), \tag{1.35}$$

which leads to the above interpretation. We shall not consider the more intricate problem of the angular momentum of the photons, but only mention that circular polarization states obtained by forming linear combinations

$$-\frac{1}{\sqrt{2}}[\boldsymbol{\varepsilon}_1(\mathbf{k}) + i\boldsymbol{\varepsilon}_2(\mathbf{k})], \qquad \frac{1}{\sqrt{2}}[\boldsymbol{\varepsilon}_1(\mathbf{k}) - i\boldsymbol{\varepsilon}_2(\mathbf{k})], \tag{1.36}$$

are more appropriate for this. Remembering that $(\boldsymbol{\varepsilon}_1(\mathbf{k}), \boldsymbol{\varepsilon}_2(\mathbf{k}), \mathbf{k})$ form a right-handed Cartesian coordinate system, we see that these two combinations correspond to angular momentum $\pm\hbar$ in the direction $\mathbf{k}$ (analogous to the properties of the spherical harmonics $Y_1^{\pm 1}$), i.e. they represent right- and left-circular polarization: the photon behaves like a particle of spin 1. The third spin component is, of course, missing because of the transverse nature of the photon field.

The state of lowest energy of the radiation field is the vacuum state $|0\rangle$, in which all occupation numbers $n_r(\mathbf{k})$ are zero. According to Eqs. (1.30) or (1.33), this state has the energy $\frac{1}{2}\Sigma_{\mathbf{k}}\Sigma_r\hbar\omega_{\mathbf{k}}$. This is an infinite *constant*, which is of no physical significance: we can eliminate it altogether by shifting the zero of the energy scale to coincide with the vacuum state $|0\rangle$. This corresponds to replacing Eq. (1.30) by

$$H_{\text{rad}} = \sum_{\mathbf{k}}\sum_{r}\hbar\omega_{\mathbf{k}}a_r^\dagger(\mathbf{k})\,a_r(\mathbf{k}). \tag{1.37}$$

[The 'extra' term in Eq. (1.35) for the momentum will similarly be dropped. It actually vanishes in any case due to symmetry in the $\mathbf{k}$ summation.]

The representation (1.32) in which states are specified by the occupation numbers $n_r(\mathbf{k})$ is called the *number representation*. It is of great practical importance in calculating transitions (possibly via intermediate states) between initial and final states containing definite numbers of photons with well-defined properties. These ideas are, of course, not restricted to photons, but apply generally to the particles of quantized fields. We shall have to modify the formalism in one respect. We have seen that the photon occupation numbers $n_r(\mathbf{k})$ can assume all values $0,1,2,\ldots$. Thus, photons satisfy Bose–Einstein statistics. They are *bosons*. So a modification will be required to describe particles obeying Fermi–Dirac

statistics (*fermions*), such as electrons or muons, for which the occupation numbers are restricted to the values 0 and 1.

We have quantized the electromagnetic field by replacing the classical amplitudes a_r and a_r^* in the vector potential (1.15) by operators, so that the vector potential and the electric and magnetic fields become operators. In particular, the vector potential (1.15) becomes, in the Heisenberg picture [cf. Eqs. (1.28) and (1.17)], the time-dependent operator

$$\mathbf{A}(\mathbf{x}, t) = \mathbf{A}^+(\mathbf{x}, t) + \mathbf{A}^-(\mathbf{x}, t), \tag{1.38a}$$

with

$$\mathbf{A}^+(\mathbf{x}, t) = \sum_{\mathbf{k}} \sum_r \left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}}\right)^{1/2} \boldsymbol{\varepsilon}_r(\mathbf{k}) a_r(\mathbf{k})\, e^{i(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}} t)}, \tag{1.38b}$$

$$\mathbf{A}^-(\mathbf{x}, t) = \sum_{\mathbf{k}} \sum_r \left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}}\right)^{1/2} \boldsymbol{\varepsilon}_r(\mathbf{k}) a_r^\dagger(\mathbf{k})\, e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}} t)}. \tag{1.38c}$$

The operator $\mathbf{A}^+$ contains only absorption operators, $\mathbf{A}^-$ only creation operators. $\mathbf{A}^+$ and $\mathbf{A}^-$ are called the positive and negative frequency parts of $\mathbf{A}$.[4] The operators for $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ follow from Eqs. (1.9). There is an important difference between a quantized field theory and non-relativistic quantum mechanics. In the former it is the amplitudes (and hence the fields) which are operators, and the position and time coordinates $(\mathbf{x}, t)$ are ordinary numbers, whereas in the latter the position coordinates (but not the time) are operators.

Finally, we note that a state with a definite number ν of photons (i.e. an eigenstate of the total photon number operator $N = \Sigma_{\mathbf{k}} \Sigma_r N_r(\mathbf{k})$) cannot be a classical field, not even for $\nu \to \infty$. This is a consequence of the fact that $\mathbf{E}$, like $\mathbf{A}$, is linear in the creation and absorption operators. Hence the expectation value of $\mathbf{E}$ in such a state vanishes. It is possible to form so-called coherent states $|c\rangle$ for which $\langle c|\mathbf{E}|c\rangle$ represents a transverse wave and for which the relative fluctuation $\Delta\mathbf{E}/\langle c|\mathbf{E}|c\rangle$ tends to zero as the number of photons in the state, $\langle c|N|c\rangle$, tends to infinity, i.e. in this limit the state $|c\rangle$ goes over into a classical state of a well-defined field.[5]

1.3 The Electric Dipole Interaction

In the last section we quantized the radiation field. Since the occupation number operators $a_r^\dagger(\mathbf{k}) a_r(\mathbf{k})$ commute with the radiation Hamiltonian (1.37), the occupation numbers $n_r(\mathbf{k})$ are constants of the motion for the free field. For anything 'to happen' requires interactions with charges and currents so that photons can be absorbed, emitted or scattered.

The complete description of the interaction of a system of charges (for example, an atom or a nucleus) with an electromagnetic field is very complicated. In this section we shall consider the simpler and, in practice, important special case of the interaction occurring via

[4] This is like in non-relativistic quantum mechanics where a time-dependence $e^{-i\omega t}$ with $\omega = E/\hbar > 0$ corresponds to a positive energy, i.e. a positive frequency.

[5] For a discussion of coherent states see R. Loudon, *The Quantum Theory of Light*, Clarendon Press, Oxford, 1973, pp. 148–153. See also Problem 1.1.

the electric dipole moment of the system of charges. The more complete (but still non-covariant) treatment of Section 1.4 will justify some of the points asserted in this section.

We shall consider a system of N charges $e_1, e_2, \ldots, e_N$ which can be described non-relativistically, i.e. the position of e_i, $i = 1, \ldots, N$, at time t is classically given by $\mathbf{r}_i = \mathbf{r}_i(t)$. We consider transitions between definite initial and final states of the system (e.g. between two states of an atom). The transitions are brought about by the electric dipole interaction if two approximations are valid.

Firstly it is permissible to neglect the interactions with the magnetic field.

Secondly, one may neglect the spatial variation of the electric radiation field, causing the transitions, across the system of charges (e.g. across the atom). Under these conditions the electric field

$$\mathbf{E}_{\mathrm{T}}(\mathbf{r}, t) = -\frac{1}{c}\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t}, \tag{1.39}$$

resulting from the transverse vector potential (1.38) of the radiation field (we are again using the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$), can be calculated at *one* point somewhere inside the system of charges, instead of at the position of each charge.[6] Taking this point as the origin of coordinates $\mathbf{r} = 0$, we obtain for the interaction causing transitions, the electric dipole interaction H_{I} given by

$$H_{\mathrm{I}} = -\mathbf{D} \cdot \mathbf{E}_{\mathrm{T}}(0, t) \tag{1.40}$$

where the electric dipole moment is defined by

$$\mathbf{D} = \sum_i e_i \mathbf{r}_i. \tag{1.41}$$

Transitions brought about by the interaction (1.40) in first-order perturbation theory are called *electric dipole transitions*. Since $\mathbf{E}_{\mathrm{T}}$, like $\mathbf{A}$ [Eq. (1.38)], is linear in the photon absorption and creation operators, so is H_{I}. It follows that, in electric dipole transitions, one photon is emitted or absorbed. In the next section it will be shown that the electric dipole approximation is valid, provided the wavelength $\lambda = 2\pi/k$ of the radiation emitted or absorbed in the transition is very large compared to the linear dimensions R of the system of charges: $\lambda \gg R$. For example, for optical transitions in atoms, R is of the order of 1 Å and λ lies in the range 4000–7500 Å. Similarly, for gamma rays emitted by nuclei, R is of the order of a few fermis (1 f $= 10^{-15}$ m) and since $\lambda/2\pi = [197/(E \text{ in MeV})]$ f for a gamma ray of E MeV, the electric dipole approximation is valid up to quite high gamma-ray energies.

If there are selection rules forbidding a transition in the electric dipole approximation, it might still occur via the magnetic interactions or via parts of the electric interactions which are neglected in the dipole approximation. It may happen that a transition is strictly forbidden, i.e. cannot occur in first-order perturbation theory, even when the exact interaction is used as perturbation instead of H_{I} [Eq. (1.40)]. In such cases, the transition can still occur in higher orders of perturbation theory or, possibly, by some quite different mechanism.[7]

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[6] In Eq. (1.39) we have written $\mathbf{E}_{\mathrm{T}}$, since we now also have the Coulomb interaction between the charges, which makes a contribution $-\nabla\phi$ to the electric field. [See Eqs. (1.2) and (1.4a) and Section 1.4.]

[7] For selection rules for radiative transitions in atoms, see H. A. Bethe and R. W. Jackiw, *Intermediate Quantum Mechanics*, 2nd edn, Benjamin, New York, 1968, Chapter 11.

Let us now consider in some detail the emission and absorption of radiation in electric dipole transitions in atoms. The atom will make a transition from an initial state $|A\rangle$ to a final state $|B\rangle$ and the occupation number of one photon state will change from $n_r(\mathbf{k})$ to $n_r(\mathbf{k}) \pm 1$. The initial and final states of the system will be

$$\left.\begin{aligned} |A, n_r(\mathbf{k})\rangle &= |A\rangle|n_r(\mathbf{k})\rangle \\ |B, n_r(\mathbf{k})\pm 1\rangle &= |\mathrm{B}\rangle|n_r(\mathbf{k})\pm 1\rangle \end{aligned}\right\}, \tag{1.42}$$

where the occupation numbers of the photon states which are not changed in the transition are not shown. The dipole operator (1.41) now becomes:

$$\mathbf{D} = -e \sum_i \mathbf{r}_i \equiv -e\mathbf{x}, \tag{1.43}$$

where the summation is over the atomic electrons and we have introduced the abbreviation $\mathbf{x}$. The transverse electric field $\mathbf{E}_\mathrm{T}(0, t)$ which occurs in the interaction (1.40) is from Eqs. (1.38)

$$\begin{aligned} \mathrm{E}_\mathrm{T}(0, t) &= -\frac{1}{c}\frac{\partial \mathbf{A}(0, t)}{\partial t} \\ &= \mathrm{i} \sum_{\mathbf{k}} \sum_r \left(\frac{\hbar\omega_\mathbf{k}}{2V}\right)^{1/2} \boldsymbol{\varepsilon}_r(\mathbf{k}) \left[a_r(\mathbf{k})\, \mathrm{e}^{-\mathrm{i}\omega_\mathbf{k} t} - a_r^\dagger(\mathbf{k}) \mathrm{e}^{\mathrm{i}\omega_\mathbf{k} t}\right]. \end{aligned}$$

Let us consider radiative emission. The transition matrix element of the interaction (1.40) between the states (1.42) then is given by

$$\begin{aligned} &\langle B,\ n_r(\mathbf{k}) + 1|H_\mathrm{I}|A,\ n_r(\mathbf{k})\rangle \\ &\quad = \mathrm{i}\left(\frac{\hbar\omega_\mathbf{k}}{2V}\right)^{1/2} \langle n_r(\mathbf{k}) + 1|a_r^\dagger(\mathbf{k})|n_r(\mathbf{k})\rangle \langle B|\boldsymbol{\varepsilon}_r(\mathbf{k}) \cdot \mathbf{D}|A\rangle\ \mathrm{e}^{\mathrm{i}\omega_\mathbf{k} t} \\ &\quad = \mathrm{i}\left(\frac{\hbar\omega_\mathbf{k}}{2V}\right)^{1/2} [n_r(\mathbf{k}) + 1]^{1/2} \langle B|\boldsymbol{\varepsilon}_r(\mathbf{k}) \cdot \mathbf{D}|A\rangle\ \mathrm{e}^{\mathrm{i}\omega_\mathbf{k} t}, \end{aligned} \tag{1.44}$$

where the last line follows from Eq. (1.24).

The transition probability per unit time between initial and final states (1.42) is given by time-dependent perturbation theory as

$$w = \frac{2\pi}{\hbar} |\langle B,\ n_r(\mathbf{k}) + 1|H_\mathrm{I}|A,\ n_r(\mathbf{k})\rangle|^2\ \delta(E_A - E_B - \hbar\omega_\mathbf{k}) \tag{1.45}$$

where E_A and E_B are the energies of the initial and final atomic states $|A\rangle$ and $|B\rangle$.[8] The delta function ensures conservation of energy in the transition, i.e. the emitted photon's energy $\hbar\omega_\mathbf{k}$ must satisfy the Bohr frequency condition

$$\omega_\mathbf{k} = \omega \equiv (E_A - E_B)/\hbar. \tag{1.46}$$

[8] Time-dependent perturbation theory is, for example, developed in A. S. Davydov, *Quantum Mechanics*, 2nd edn, Pergamon, Oxford, 1976, see Section 93 [Eq. (93.7)]; E. Merzbacher, *Quantum Mechanics*, 2nd edn, John Wiley & Sons, Inc., New York, 1970, see Section 18.8; L. I. Schiff, Quantum Mechanics, 3rd edn, McGraw-Hill, New York, 1968, see Section 35.

The delta function is eliminated in the usual way from Eq. (1.45) by integrating over a narrow group of final photon states. The number of photon states in the interval $(\mathbf{k}, \mathbf{k} + \mathrm{d}\mathbf{k})$, all in the same polarization state ($\boldsymbol{\varepsilon}_1(\mathbf{k})$ or $\boldsymbol{\varepsilon}_2(\mathbf{k})$), is[9]

$$\frac{V\,\mathrm{d}^3\,\mathbf{k}}{(2\pi)^3} = \frac{Vk^2\,\mathrm{d}k\,\mathrm{d}\Omega}{(2\pi)^3}. \tag{1.47}$$

From Eqs. (1.44)–(1.47) we obtain the probability per unit time for an atomic transition $|A\rangle \rightarrow |B\rangle$ with emission of a photon of wave vector in the range $(\mathbf{k}, \mathbf{k} + \mathrm{d}\mathbf{k})$ and with polarization vector $\boldsymbol{\varepsilon}_r(\mathbf{k})$:

$$\begin{aligned} w_r\,\mathrm{d}\Omega = \int \frac{Vk^2\,\mathrm{d}k\,\mathrm{d}\Omega}{(2\pi)^3}\,\frac{2\pi}{\hbar}\,\delta(E_A - E_B - \hbar\omega_{\mathbf{k}}) \\ \times\left(\frac{\hbar\omega_{\mathbf{k}}}{2V}\right)[n_r(\mathbf{k}) + 1]\,|\langle B|\boldsymbol{\varepsilon}_r(\mathbf{k})\cdot\mathbf{D}|A\rangle|^2. \end{aligned} \tag{1.49}$$

If we perform the integration with respect to $k\ (= \omega_{\mathbf{k}}/c)$ and substitute (1.43) for $\mathbf{D}$, the last expression reduces to

$$w_r\,\mathrm{d}\Omega = \frac{e^2\omega^3\,\mathrm{d}\Omega}{8\pi^2\hbar c^3}\,[n_r(\mathbf{k}) + 1]\,|\boldsymbol{\varepsilon}_r(\mathbf{k})\cdot\mathbf{x}_{BA}|^2 \tag{1.50}$$

where $\mathbf{x}_{BA}$ stands for the matrix element

$$\mathbf{x}_{BA} \equiv \langle B|\mathbf{x}|A\rangle = \langle B|\sum_i \mathbf{r}_i|A\rangle. \tag{1.51}$$

The most interesting feature of Eq. (1.50) is the occurrence of the factor $[n_r(\mathbf{k}) + 1]$. $n_r(\mathbf{k})$ is the occupation number of photons in the $(\mathbf{k}, r)$ mode present initially, and thus the part of (1.50) proportional to $n_r(\mathbf{k})$ represents induced (or stimulated) emission, i.e. radiation which results from the radiation incident on the atom; classically, we can think of it as resulting from the forced oscillations of the electrons, and this term can be produced from a semiclassical theory of radiation.[10] However, even with no radiation present initially ($n_r(\mathbf{k}) = 0$), the transition probability (1.50) is different from zero. This corresponds to the spontaneous emission of radiation from an atom, and this cannot be derived from a semiclassical theory of radiation.

Eqs. (1.50) and (1.51) represent the basic result about emission of radiation in electric dipole transitions, and we only briefly indicate some consequences.

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[9] Since we are using a finite normalization volume V, we should be summing over a group of allowed wave vectors $\mathbf{k}$ [see Eq. (1.13)]. For large V (strictly $V \rightarrow \infty$)

$$\frac{1}{V}\sum_{\mathbf{k}} \rightarrow \frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathbf{k}. \tag{1.48}$$

The normalization volume V must of course drop out of all physically significant quantities such as transition rates etc.

[10] See, for example, L. I. Schiff, *Quantum Mechanics*, 3rd edn, McGraw-Hill, New York, 1968, Chapter 11, or Bethe and Jackiw, referred to earlier in this section, Chapter 10.

To sum over the two polarization states for a given $\mathbf{k}$, we note that $\boldsymbol{\varepsilon}_1(\mathbf{k})$, $\boldsymbol{\varepsilon}_2(\mathbf{k})$ and $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ form an orthonormal coordinate system. Hence,

$$\begin{aligned}\sum_{r=1}^{2} |\boldsymbol{\varepsilon}_r(\mathbf{k}) \cdot \mathbf{x}_{BA}|^2 &= \mathbf{x}_{BA} \cdot \mathbf{x}_{BA}^* - (\hat{\mathbf{k}} \cdot \mathbf{x}_{BA})(\hat{\mathbf{k}} \cdot \mathbf{x}_{BA}^*) \\ &= (\mathbf{x}_{BA} \cdot \mathbf{x}_{BA}^*)(1 - \cos^2\theta) \\ &= |\mathbf{x}_{BA}|^2 \sin^2\theta,\end{aligned}$$

where the last line but one defines the angle θ which the complex vector $\mathbf{x}_{BA}$ makes with $\hat{\mathbf{k}}$. Hence, from Eq. (1.50)

$$\sum_{r=1}^{2} w_r \, d\Omega = \frac{e^2 \omega^3}{8\pi^2 \hbar c^3} \, d\Omega [n_r(\mathbf{k}) + 1] |\mathbf{x}_{BA}|^2 \sin^2\theta. \tag{1.52}$$

For spontaneous emission, the total transition probability per unit time is obtained from the last equation, with $n_r(\mathbf{k}) = 0$, by integrating over all directions. Since

$$\int \sin^2\theta \, d\Omega = \frac{8\pi}{3},$$

we obtain

$$w_{\text{total}}(A \rightarrow B) = \frac{e^2 \omega^3}{3\pi\hbar c^3} |\mathbf{x}_{BA}|^2. \tag{1.53}$$

The lifetime τ of an excited atomic state $|A\rangle$ is defined as the reciprocal of the total transition probability per unit time to *all* possible final states $|B_1\rangle$, $|B_2\rangle$, ..., i.e.

$$\frac{1}{\tau} = \sum_n w_{\text{total}}(A \rightarrow B_n). \tag{1.54}$$

In particular, if the state $|A\rangle$ can decay to states with non-zero total angular momentum, Eq. (1.54) must contain a summation over the corresponding magnetic quantum numbers.

The selection rules for electric dipole transitions follow from the matrix element (1.51). For example, since $\mathbf{x}$ is a vector, the states $|A\rangle$ and $|B\rangle$ must have opposite parity, and the total angular momentum quantum number J of the atom and its z-component M must satisfy the selection rules

$$\Delta J = 0, \pm 1, \quad \text{not } J = 0 \rightarrow J = 0, \quad \Delta M = 0, \pm 1.$$

The second selection rule (not $J = 0 \rightarrow J = 0$) applies strictly to one-photon processes, not only in the electric dipole approximation. It is a consequence of the fact that there are no one-photon states with zero angular momentum. To form such a state from the spin 1 of the photon and a unit of orbital angular momentum requires all three components of the spin angular momentum, but because of the transversality of the radiation field, only two of the spin components are available [compare Eq. (1.36)].

Finally, we note that very similar results hold for the absorption of radiation in electric dipole transitions. The matrix element

$$\langle B, n_r(\mathbf{k}) - 1 | H_1 | A, n_r(\mathbf{k}) \rangle$$

corresponding to Eq. (1.44) now involves the factor $[n_\lambda(\mathbf{k})]^{1/2}$ instead of $[n_\lambda(\mathbf{k})+1]^{1/2}$. Our final result for emission, Eq. (1.50), also holds for absorption, with $[n_\lambda(\mathbf{k})+1]$ replaced by $[n_\lambda(\mathbf{k})]$, $d\Omega$ being the solid angle defining the incident radiation, and the matrix element $\mathbf{x}_{BA}$, Eq. (1.51), representing a transition from an atomic state $|A\rangle$ with energy E_A to a state $|B\rangle$ with energy $E_B > E_A$. Correspondingly, the frequency ω is defined by $\hbar\omega = E_B - E_A$ instead of Eq. (1.46).

1.4 The Electromagnetic Field in the Presence of Charges

After the special case of the electric dipole interaction, we now want to consider the general interaction of moving charges and an electromagnetic field. As this problem will later be treated in a relativistically covariant way, we shall not give a rigorous complete derivation, but rather stress the physical interpretation. As in the last section, the motion of the charges will again be described non-relativistically. In Section 1.4.1 we shall deal with the Hamiltonian formulation of the classical theory. This will enable us very easily to go over to the quantized theory in Section 1.4.2. In Sections 1.4.3 and 1.4.4 we shall illustrate the application of the theory for radiative transitions and Thomson scattering.

1.4.1 Classical electrodynamics

We would expect the Hamiltonian of a system of moving charges, such as an atom, in an electromagnetic field to consist of three parts: a part referring to matter (i.e. the charges), a part referring to the electromagnetic field and a part describing the interaction between matter and field.

For a system of point masses m_i, $i = 1, \ldots, N$, with charges e_i and position coordinates $\mathbf{r}_i$, the Hamiltonian is

$$H_{\mathrm{m}} = \sum_i \frac{\mathbf{p}_i^2}{2m_i} + H_{\mathrm{C}} \tag{1.55a}$$

where H_{C} is the Coulomb interaction

$$H_{\mathrm{C}} \equiv \frac{1}{2} \sum_{\substack{i,j \\ (i \neq j)}} \frac{e_i e_j}{4\pi |\mathbf{r}_i - \mathbf{r}_j|} \tag{1.55b}$$

and $\mathbf{p}_i = m_i \, d\mathbf{r}_i/dt$ is the kinetic momentum of the ith particle. This is the usual Hamiltonian of atomic physics, for example.

The electromagnetic field in interaction with charges is described by Maxwell's equations [Eqs. (1.1)]. We continue to use the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$, so that the electric field (1.2) decomposes into transverse and longitudinal fields

$$\mathbf{E} = \mathbf{E}_{\mathrm{T}} + \mathbf{E}_{\mathrm{L}},$$

where

$$\mathbf{E}_{\mathrm{T}} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{E}_{\mathrm{L}} = -\nabla\phi.$$

(A longitudinal field is defined by the condition $\nabla \wedge \mathbf{E}_L = 0$.) The magnetic field is given by $\mathbf{B} = \nabla \wedge \mathbf{A}$.

The total energy of the electromagnetic field

$$\frac{1}{2} \int (\mathbf{E}^2 + \mathbf{B}^2)\, d^3\mathbf{x}$$

can be written

$$\frac{1}{2} \int (\mathbf{E}_T^2 + \mathbf{B}^2)\, d^3\mathbf{x} + \frac{1}{2} \int \mathbf{E}_L^2\, d^3\mathbf{x}.$$

The last integral can be transformed, using Poisson's equation $\nabla^2 \phi = -\rho$, into

$$\frac{1}{2} \int \mathbf{E}_L^2\, d^3\mathbf{x} = \frac{1}{2} \int \frac{\rho(\mathbf{x}, t)\rho(\mathbf{x}', t)}{4\pi|\mathbf{x} - \mathbf{x}'|}\, d^3\mathbf{x}\, d^3\mathbf{x}'. \tag{1.56}$$

Thus the energy associated with the longitudinal field is the energy of the *instantaneous* electrostatic interaction between the charges. With

$$\rho(\mathbf{x}, t) = \sum_i e_i\, \delta(\mathbf{x} - \mathbf{r}_i(t))$$

Eq. (1.56) reduces to

$$\begin{aligned} \frac{1}{2} \int \mathbf{E}_L^2\, d^3\mathbf{x} &= \frac{1}{2} \sum_{i,j} \frac{e_i e_j}{4\pi|\mathbf{r}_i - \mathbf{r}_j|} \\ &= \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} \frac{e_i e_j}{4\pi|\mathbf{r}_i - \mathbf{r}_j|} \equiv H_C, \end{aligned} \tag{1.57}$$

where, in the last line, we have dropped the infinite self-energy which occurs for point charges. The term H_C has already been included in the Hamiltonian H_m, Eqs. (1.55), so we must take as additional energy of the electromagnetic field that of the transverse radiation field

$$H_{rad} = \frac{1}{2} \int (\mathbf{E}_T^2 + \mathbf{B}^2)\, d^3\mathbf{x}. \tag{1.58}$$

Eqs. (1.55) allow for the instantaneous Coulomb interaction of charges. To allow for the interaction of moving charges with an electromagnetic field, one must replace the matter-Hamiltonian (1.55a) by

$$H'_m = \sum_i \frac{1}{2m_i} \left(\mathbf{p}_i - \frac{e_i}{c}\mathbf{A}_i\right)^2 + H_C \tag{1.59}$$

where $\mathbf{A}_i = \mathbf{A}(\mathbf{r}_i, t)$ denotes the vector potential at the position $\mathbf{r}_i$ of the charge e_i at time t. In Eq. (1.59) $\mathbf{p}_i$ is the momentum coordinate canonically conjugate to the position coordinate $\mathbf{r}_i$, in the sense of Lagrangian mechanics, and it is related to the velocity $\mathbf{v}_i = d\mathbf{r}_i/dt$ of the ith particle by

$$\mathbf{p}_i = m_i \mathbf{v}_i + \frac{e_i}{c} \mathbf{A}_i.$$

It is only for $\mathbf{A} = 0$ that this conjugate momentum reduces to the kinetic momentum $m_i\mathbf{v}_i$. The justification for the form (1.59) for H'_{m} is that it gives the correct equations of motion for the charges (see Problem 1.2):

$$m_i \frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = e_i \left[\mathbf{E}_i + \frac{\mathbf{v}_i}{c} \wedge \mathbf{B}_i\right], \tag{1.60}$$

where $\mathbf{E}_i$ and $\mathbf{B}_i$ are the electric and magnetic fields at the instantaneous position of the ith charge.[11]

We can regroup the terms in Eq. (1.59) as

$$H'_{\mathrm{m}} = H_{\mathrm{m}} + H_{\mathrm{I}} \tag{1.61}$$

where H_{I}, the interaction Hamiltonian of matter and field, is given by

$$\begin{aligned} H_{\mathrm{I}} &= \sum_i \left\{ -\frac{e_i}{2m_i c} (\mathbf{p}_i \cdot \mathbf{A}_i + \mathbf{A}_i \cdot \mathbf{p}_i) + \frac{e_i^2}{2m_i c^2} \mathbf{A}_i^2 \right\} \\ &= \sum_i \left\{ -\frac{e_i}{m_i c} \mathbf{A}_i \cdot \mathbf{p}_i + \frac{e_i^2}{2m_i c^2} \mathbf{A}_i^2 \right\} . \end{aligned} \tag{1.62}$$

In the quantum theory $\mathbf{p}_i$, the momentum canonically conjugate to $\mathbf{r}_i$, will become the operator $-i\hbar\nabla_i$. Nevertheless, the replacement of $\mathbf{p}_i \cdot \mathbf{A}_i$ by $\mathbf{A}_i \cdot \mathbf{p}_i$ in the second line of Eq. (1.62) is justified by our gauge condition $\nabla_i \cdot \mathbf{A}_i = 0$. Eq. (1.62) represents the general interaction of moving charges in an electromagnetic field (apart from $\mathrm{H_C}$). It does not include the interaction of the magnetic moments, such as that due to the spin of the electron, with magnetic fields.

Combining the above results (1.55), (1.58), (1.59) and (1.62), we obtain for the complete Hamiltonian

$$H = H'_{\mathrm{m}} + H_{\mathrm{rad}} = H_{\mathrm{m}} + H_{\mathrm{rad}} + H_{\mathrm{I}}. \tag{1.63}$$

Just as this Hamiltonian leads to the correct equations of motion (1.60) for charges, so it also leads to the correct field equations (1.4), with $\nabla \cdot \mathbf{A} = 0$, for the potentials.[12]

1.4.2 Quantum electrodynamics

The quantization of the system described by the Hamiltonian (1.63) is carried out by subjecting the particles' coordinates $\mathbf{r}_i$ and canonically conjugate momenta $\mathbf{p}_i$ to the usual commutation relations (e.g. in the coordinate representation $\mathbf{p}_i \rightarrow i\hbar\nabla_i$), and quantizing the radiation field, as in Section 1.2.3. The longitudinal electric field $\mathbf{E}_{\mathrm{L}}$ does not provide any additional degrees of freedom, being completely determined via the first Maxwell equation $\nabla \cdot \mathbf{E}_{\mathrm{L}} = \rho$ by the charges.

The interaction H_{I} in Eq, (1.63) is usually treated as a perturbation which causes transitions between the states of the non-interacting Hamiltonian

$$H_0 = H_{\mathrm{m}} + H_{\mathrm{rad}}. \tag{1.64}$$

[11] For the Lagrangian and Hamiltonian formulations of mechanics which are here used see, for example, H. Goldstein, *Classical Mechanics*, 2nd edn, Addison-Wesley, Reading, Mass., 1980, in particular pp. 21–23 and 346.

[12] See W. Heitler, *The Quantum Theory of Radiation*, 3rd edn, Clarendon Press, Oxford, 1954, pp. 48–50.

The eigenstates of H_0 are again of the form

$$|A, \ldots n_r(\mathbf{k}) \ldots\rangle = |A\rangle \, |\ldots n_r(\mathbf{k}) \ldots\rangle,$$

with $|A\rangle$ and $|\ldots n_r(\mathbf{k}) \ldots\rangle$ eigenstates of H_m and H_{rad}.

Compared with the electric dipole interaction (1.40), the interaction (1.62) differs in that it contains a term quadratic in the vector potential. This results in two-photon processes in first-order perturbation theory (i.e. emission or absorption of two photons or scattering). In addition, the first term in (1.62) contains magnetic interactions and higher-order effects due to the spatial variation of $\mathbf{A}(\mathbf{x}, t)$, which are absent from the electric dipole interaction (1.40). These aspects are illustrated in the applications to radiative transitions and Thomson scattering which follow.

1.4.3 Radiative transitions in atoms

We consider transitions between two states of an atom with emission or absorption of one photon. This problem was treated in Section 1.3 in the electric dipole approximation, but now we shall use the interaction (1.62).

We shall consider the emission process between the initial and final states (1.42). Using the expansion (1.38) of the vector potential, we obtain the matrix element for this transition [which results from the term linear in $\mathbf{A}$ in Eq. (1.62)]

$$\begin{aligned}&\langle B, \, n_r(\mathbf{k}) + 1|H_I|A, \, n_r(\mathbf{k})\rangle \\ &\quad = -\frac{e}{m}\left(\frac{\hbar}{2V\omega_{\mathbf{k}}}\right)^{1/2} [n_r(\mathbf{k}) + 1]^{1/2} \, \langle B|\boldsymbol{\varepsilon}_r(\mathbf{k}) \cdot \sum_i \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}_i} \, \mathbf{p}_i|A\rangle \, \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}t}.\end{aligned} \tag{1.65}$$

Using this matrix element, one calculates the transition probability per unit time as in Section 1.3. Instead of Eqs. (1.50) and (1.51), one obtains:

$$w_r \, \mathrm{d}\Omega = \frac{e^2\omega \, \mathrm{d}\Omega}{8\pi^2 m^2 \hbar c^3} \, [n_r(\mathbf{k}) + 1] \left| \boldsymbol{\varepsilon}_r(\mathbf{k}) \cdot \langle B| \sum_i \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}_i} \mathbf{p}_i \, |A\rangle \right|^2. \tag{1.66}$$

These results go over into the electric dipole approximation if in the matrix elements in Eqs. (1.65) and (1.66) we can approximate the exponential functions by unity:

$$\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}_i} \approx 1. \tag{1.67}$$

This is justified provided the wavelength $\lambda = 2\pi/k$ of the radiation emitted in the transition is very large compared to the linear dimensions R of the system of charges (in our case, of the atom): $\lambda >> R$. The atomic wavefunctions $|A\rangle$ and $|B\rangle$ restrict the effective values of $\mathbf{r}_i$ to $r_i \lesssim R$, so that $\mathbf{k} \cdot \mathbf{r}_i \lesssim kR << 1$. We saw in Section 1.3 that this inequality is generously satisfied for optical atomic transitions. From the equation of motion $i\hbar\dot{\mathbf{r}}_i = [\mathbf{r}_i, \, H]$ and Eq. (1.46)

$$\langle B| \, \mathbf{p}_i|A\rangle = m\langle B| \, \dot{\mathbf{r}}_i|A\rangle = -\mathrm{i}m\omega\langle B| \, \mathbf{r}_i|A\rangle.$$

Hence, in the approximation (1.67), Eqs. (1.65) and (1.66) reduce to the electric dipole form, Eqs. (1.44) and (1.50).

If selection rules forbid the transition $|A\rangle$ to $|B\rangle$ via the electric dipole interaction, it may in general still occur via higher terms in the expansion of the exponentials

$$e^{-i\mathbf{k}\cdot\mathbf{r}_i} = 1 - i\mathbf{k}\cdot\mathbf{r}_i + \cdots.$$

With the second term, the expression within the modulus sign in Eq. (1.66) becomes

$$\boldsymbol{\varepsilon}_r(\mathbf{k})\cdot\langle B|\sum_i(-i\mathbf{k}\cdot\mathbf{r}_i)\mathbf{p}_i|A\rangle = -i\sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}\varepsilon_{r\alpha}(\mathbf{k})k_\beta\langle B|\sum_i r_{i\beta}p_{i\alpha}|A\rangle,$$

where α, β (= 1, 2, 3) label the Cartesian components of the vectors $\boldsymbol{\varepsilon}_r$, $\mathbf{k}$, $\mathbf{r}_i$ and $\mathbf{p}_i$. The matrix element can be written as the sum of an antisymmetric and a symmetric second-rank tensor

$$\langle B|\sum_i r_{i\beta}p_{i\alpha}|A\rangle = \frac{1}{2}\left\{\langle B|\sum_i(r_{i\beta}p_{i\alpha} - r_{i\alpha}p_{i\beta})|A\rangle + \langle B|\sum_i(r_{i\beta}p_{i\alpha} + r_{i\alpha}p_{i\beta})|A\rangle\right\}.$$

The first term contains the antisymmetric angular momentum operator and corresponds to the magnetic dipole interaction. (In practice this must be augmented by the spin part.) The symmetric term corresponds to the electric quadrupole interaction. The parity and angular momentum selection rules for the transitions brought about by these matrix elements are easily determined from their forms. We obtain in this way an expansion into electric and magnetic multipoles, i.e. photons of definite parity and angular momentum. As usual, a better procedure for such an expansion, except in the simplest cases, is to use spherical rather than Cartesian coordinates.[13]

The result (1.66) can again be adapted to the case of absorption of radiation by replacing the factor $[n_r(\mathbf{k}) + 1]$ by $n_r(\mathbf{k})$ and the appropriate re-interpretation of the matrix element, etc.

1.4.4 Thomson scattering

As a second illustration, we consider Thomson scattering, i.e. the scattering of photons of energy $\hbar\omega$ by atomic electrons, with $\hbar\omega$ large compared to the binding energies of the electrons, so that they can be considered as free electrons, but $\hbar\omega$ very small compared to the electron rest energy mc^2. In this case the energy $\hbar\omega'$ of the scattered photon is not changed: $\hbar\omega' = \hbar\omega$, since for small recoil momenta the recoil energy may be neglected.

The scattering from an initial state with one photon of momentum $\hbar\mathbf{k}$ and polarization $\boldsymbol{\varepsilon}_\alpha(\mathbf{k})$ (with $\alpha = 1$ or 2) to a final state with one photon of momentum $\hbar\mathbf{k}'$ and polarization $\boldsymbol{\varepsilon}_\beta(\mathbf{k}')$ (with $\beta = 1$ or 2) can occur in first-order perturbation theory via the term in $\mathbf{A}^2$ in the interaction (1.62). It can also occur in second-order perturbation theory via the term linear in $\mathbf{A}$ in Eq. (1.62), but one can show that

[13] See A. S. Davydov, *Quantum Mechanics*, 2nd edn, Pergamon, Oxford, 1976, Sections 81 and 95.

under our conditions the contribution of the second-order process is negligible.[14] The operator $\mathbf{A}^2(0, t)$ can, from Eq. (1.38), be written

$$\mathbf{A}^2(0, t) = \sum_{\mathbf{k}_1\mathbf{k}_2} \sum_{r, s} \frac{\hbar c^2}{2V(\omega_1 \omega_2)^{1/2}} (\boldsymbol{\varepsilon}_r(\mathbf{k}_1) \cdots \boldsymbol{\varepsilon}_s(\mathbf{k}_2)) \times \left[a_r(\mathbf{k}_1)e^{-i\omega_1 t} + a_r^\dagger(\mathbf{k}_1)\, e^{+i\omega_1 t}\right] \left[a_s(\mathbf{k}_2)e^{-i\omega_2 t} + a_s^\dagger(\mathbf{k}_2)\, e^{+i\omega_2 t}\right], \tag{1.68}$$

where $\omega_r \equiv c|\mathbf{k}_r|$, $r = 1, 2$. This operator can bring about the transition from the initial state $|\mathbf{k}, \alpha\rangle$ to the final state $|\mathbf{k}', \beta\rangle$ (we use a somewhat simplified, but unambiguous, notation) in two ways: either of the factors in square parentheses can act to absorb the initial photon, and the other factor then creates the final photon. One then obtains the matrix element for this transition from Eq. (1.62)

$$\langle \mathbf{k}', \beta| \frac{e^2}{2mc^2} \mathbf{A}^2(0, t) |\mathbf{k}, \alpha\rangle = \frac{e^2\hbar}{2mV(\omega\omega')^{1/2}} \boldsymbol{\varepsilon}_\alpha(\mathbf{k}) \cdot \boldsymbol{\varepsilon}_\beta(\mathbf{k}') e^{i(\omega' - \omega)t}$$

where $\omega = c|\mathbf{k}|$ and. $\omega' = c|\mathbf{k}'|$. The transition probability per unit time for a photon, initially in the state $|\mathbf{k}, \alpha\rangle$, to be scattered into an element of solid angle $d\boldsymbol{\Omega}$ in the direction $\mathbf{k}'$, and with polarization $\boldsymbol{\varepsilon}_\beta(\mathbf{k}')$, is given by

$$\begin{aligned} w_{\alpha \to \beta}(\mathbf{k}')\, d\Omega &= \frac{2\pi}{\hbar} \int \frac{Vk'^2\, dk'\, d\Omega}{(2\pi)^3} \delta(\hbar\omega' - \hbar\omega) \\ &\quad \times \left(\frac{e^2\hbar}{2mV}\right)^2 \left(\frac{1}{\omega\omega'}\right) \left[\boldsymbol{\varepsilon}_\alpha(\mathbf{k}) \cdot \boldsymbol{\varepsilon}_\beta(\mathbf{k}')\right]^2 \\ &= \frac{c}{V} \left(\frac{e^2}{4\pi mc^2}\right)^2 \left[\boldsymbol{\varepsilon}_\alpha(\mathbf{k}) \cdot \boldsymbol{\varepsilon}_\beta(\mathbf{k}')\right]^2 d\Omega \end{aligned}$$

where $|\mathbf{k}'| = |\mathbf{k}|$. Dividing this transition probability per unit time by the incident photon flux *(c/V)*, one obtains the corresponding differential cross-section

$$\sigma_{\alpha \to \beta}(\mathbf{k}')\, d\Omega = r_0^2 \left[\boldsymbol{\varepsilon}_\alpha(\mathbf{k}) \cdot \boldsymbol{\varepsilon}_\beta(\mathbf{k}')\right]^2 d\Omega, \tag{1.69}$$

where the classical electron radius has been introduced by

$$r_0 = \frac{e^2}{4\pi mc^2} = 2.818 \quad \text{fm}. \tag{1.70}$$

For an unpolarized incident photon beam, the unpolarized differential cross-section (i.e. the final polarization state is not observed) is obtained from Eq. (1.69) by summing over final and averaging over initial polarization states. We introduce the abbreviations $\boldsymbol{\varepsilon}_\alpha \equiv \boldsymbol{\varepsilon}_\alpha(\mathbf{k})$ and $\boldsymbol{\varepsilon}'_\beta \equiv \boldsymbol{\varepsilon}_\beta(\mathbf{k}')$. Since $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2$ and $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ form an orthonormal coordinate system,

$$\sum_{\alpha=1}^{2} \left(\boldsymbol{\varepsilon}_\alpha \cdot \boldsymbol{\varepsilon}'_\beta\right)^2 = 1 - \left(\hat{\mathbf{k}} \cdot \boldsymbol{\varepsilon}'_\beta\right)^2.$$

[14] See J. J. Sakurai, *Advanced Quantum Mechanics*, Addison-Wesley, Reading, Mass., 1967, p. 51.

Similarly

$$\sum_{\beta=1}^{2}\left(\hat{\mathbf{k}}\cdot\boldsymbol{\varepsilon}'_{\beta}\right)^2 = 1 - \left(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}'\right)^2 = \sin^2\theta$$

where θ is the angle between the directions $\mathbf{k}$ and $\mathbf{k}'$ of the incident and scattered photons, i.e. the angle of scattering. From the last two equations

$$\frac{1}{2}\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}\left(\boldsymbol{\varepsilon}_{\alpha}\cdot\boldsymbol{\varepsilon}'_{\beta}\right)^2 = \frac{1}{2}\left(2 - \sin^2\theta\right) = \frac{1}{2}\left(1 + \cos^2\theta\right) \tag{1.71}$$

and hence the unpolarized differential cross-section for scattering through an angle θ is from Eq. (1.69) given as

$$\sigma(\theta)\,\mathrm{d}\Omega = \frac{1}{2}r_0^2\left(1 + \cos^2\theta\right)\mathrm{d}\Omega. \tag{1.69a}$$

Integrating over angles, we obtain the total cross-section for Thomson scattering

$$\sigma_{\text{total}} = \frac{8\pi}{3}r_0^2 = 6.65\times10^{-25}\,\text{cm}^2. \tag{1.72}$$

1.5 Appendix: The Schrödinger, Heisenberg and Interaction Pictures

These three pictures (abbreviated S.P., H.P. and I.P.) are three different ways of describing the time development of a system. In this Appendix, we shall derive the relationships between the three pictures. Quantities in these three pictures will be distinguished by the labels S, H and I.

In the S.P., the time-dependence is carried by the states according to the Schrödinger equation

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|A,\ t\right\rangle_{\mathrm{S}} = H\left|A,\ t\right\rangle_{\mathrm{S}}, \tag{1.73}$$

where H is the Hamiltonian of the system in the S.P. This can formally be solved in terms of the state of the system at an arbitrary reference time t_0

$$\left|A,\ t\right\rangle_{\mathrm{S}} = U_{\mathrm{S}}(t)\left|A,\ t_0\right\rangle_{\mathrm{S}} \tag{1.74}$$

where $U_{\mathrm{S}}(t)$ is the unitary operator:

$$U_{\mathrm{S}}(t) = \mathrm{e}^{-\mathrm{i}H(t-t_0)/\hbar}. \tag{1.75}$$

By means of $U_{\mathrm{S}}(t)$ we can carry out a unitary transformation of states and operators (O) from the S.P. to the H.P., in which we define

$$\left|A\right\rangle_{\mathrm{H}} = U_{\mathrm{S}}^{\dagger}(t)\left|A,\ t\right\rangle_{\mathrm{S}} = \left|A,\ t_0\right\rangle_{\mathrm{S}} \tag{1.76}$$

and

$$O^{\mathrm{H}}(t) = U_{\mathrm{S}}^{\dagger}(t)\, O^{\mathrm{S}}\, U_{\mathrm{S}}(t). \tag{1.77}$$

At $t = t_0$, states and operators in the two pictures are the same. We see from Eq. (1.76) that in the H.P. state, vectors are constant in time; the time-dependence is carried by the Heisenberg operators. From Eq. (1.77)

$$H^{\mathrm{H}} = H^{\mathrm{S}} \equiv H. \tag{1.78}$$

Since the transformation from the S.P. to the H.P. is unitary, it ensures the invariance of matrix elements and commutation relations:

$${}_{\mathrm{S}}\langle B,\ t|O^{\mathrm{S}}|A,\ t\rangle_{\mathrm{S}} = {}_{\mathrm{H}}\langle B, t|O^{\mathrm{H}}(t)|A,\ t\rangle_{\mathrm{H}}, \tag{1.79}$$

and if O and P are two operators for which $[O^{\mathrm{S}}, P^{\mathrm{S}}] = \text{const.}$, then $[O^{\mathrm{H}}(t), P^{\mathrm{H}}(t)]$ equals the same constant.

Differentation of Eq. (1.77) gives the Heisenberg equation of motion

$$\mathrm{i}\hbar\, \frac{\mathrm{d}}{\mathrm{d}t}\, O^{\mathrm{H}}(t) = \left[O^{\mathrm{H}}(t),\ H\right]. \tag{1.80}$$

For an operator which is time dependent in the S.P. (corresponding to a quantity which classically has an explicit time dependence), Eq. (1.80) is augmented to

$$\mathrm{i}\hbar\, \frac{\mathrm{d}}{\mathrm{d}t}\, O^{\mathrm{H}}(t) = \mathrm{i}\hbar\, \frac{\partial}{\partial t}\, O^{\mathrm{H}}(t) + \left[O^{\mathrm{H}}(t),\ H\right]. \tag{1.81}$$

We shall not be considering such operators.

The I.P. arises if the Hamiltonian is split into two parts

$$H = H_0 + H_{\mathrm{I}}. \tag{1.82}$$

In quantum field theory, H_{I} will describe the interaction between two fields, themselves described by H_0. [Note that the suffix I on H_{I} stands for 'interaction'. It does not label a picture. Eq. (1.82) holds in any picture.] The I.P. is related to the S.P. by the unitary transformation

$$U_0(t) = \mathrm{e}^{-\mathrm{i}H_0(t\ -\ t_0)/\hbar} \tag{1.83}$$

i.e.

$$|A,\ t\rangle_{\mathrm{I}} = U_0^{\dagger}(t)\, |A,\ t\rangle_{\mathrm{S}} \tag{1.84}$$

and

$$O^{\mathrm{I}}(t) = U_0^{\dagger}(t)\, O^{\mathrm{S}}\, U_0(t). \tag{1.85}$$

Thus the relation between I.P. and S.P. is similar to that between H.P. and S.P., but with the unitary transformation U_0 involving the non-interacting Hamiltonian H_0, instead of U involving the total Hamiltonian H. From Eq. (1.85):

$$H_0^{\mathrm{I}} = H_0^{\mathrm{S}} \equiv H_0. \tag{1.86}$$

Differentiating Eq. (1.85) gives the differential equation of motion of operators in the I.P.:

$$i\hbar \frac{d}{dt} O^I(t) = [O^I(t), H_0]. \tag{1.87}$$

Substituting Eq. (1.84) into the Schrödinger equation (1.73), one obtains the equation of motion of state vectors in the I.P.

$$i\hbar \frac{d}{dt} |A, t\rangle_I = H_I^I(t) |A, t\rangle_I \tag{1.88}$$

where

$$H_I^I(t) = e^{iH_0(t-t_0)/\hbar} H_I^S e^{-iH_0(t-t_0)/\hbar}. \tag{1.89}$$

Finally, from the above relations, one easily shows that the I.P. and H.P. are related by

$$O^I(t) = U(t) O^H(t) U^\dagger(t) \tag{1.90}$$

$$|A, t\rangle_I = U(t) |A\rangle_H \tag{1.91}$$

where the unitary operator $U(t)$ is defined by

$$U(t) = e^{iH_0(t-t_0)/\hbar} \quad e^{-iH(t-t_0)/\hbar}. \tag{1.92}$$

The time development of the I.P. states follows from Eq. (1.91). From this equation

$$|A, t_1\rangle_I = U(t_1)|A\rangle_H = U(t_1)U^\dagger(t_2)|A, t_2\rangle_I.$$

Hence

$$|A, t_1\rangle_I = U(t_1, t_2)|A, t_2\rangle_I \tag{1.93}$$

where the unitary operator $U(t_1, t_2)$, defined by

$$U(t_1, t_2) = U(t_1)U^\dagger(t_2), \tag{1.94}$$

satisfies the relations

$$U^\dagger(t_1, t_2) = U(t_2, t_1) \tag{1.95a}$$

$$U(t_1, t_2)U(t_2, t_3) = U(t_1, t_3). \tag{1.95b}$$

Problems

1.1. The radiation field inside a cubic enclosure, which contains no charges, is specified by the state

$$|c\rangle = \exp\left(-\frac{1}{2}|c|^2\right) \sum_{n=0}^{\infty} \frac{c^n}{\sqrt{n!}} |n\rangle$$

where $c = |c| e^{i\delta}$ is any complex number and $|n\rangle$ is the state (1.31) in which there are n photons with wave vector $\mathbf{k}$ and polarization vector $\boldsymbol{\varepsilon}_r(\mathbf{k})$ present, and no others. Derive the following properties of the state $|c\rangle$.

(i) $|c\rangle$ is normalized: $\langle c|c\rangle = 1$.

(ii) $|c\rangle$ is an eigenstate of the destruction operator $a_r(\mathbf{k})$ with the complex eigenvalue c:

$$a_r(\mathbf{k})|c\rangle = c|c\rangle.$$

(iii) The mean number $\overline{N}$ of photons in the enclosure in the state $|c\rangle$ is given by

$$\overline{N} = \langle c|N|c\rangle = |c|^2 \tag{A}$$

where N is the total photon number operator.

(iv) The root-mean-square fluctuation ΔN in the number of photons in the enclosure in the state $|c\rangle$ is given by

$$(\Delta N)^2 = \langle c|N^2|c\rangle - \overline{N}^2 = |c|^2. \tag{B}$$

(v) The expectation value of the electric field $\mathbf{E}$ in the state $|c\rangle$ is given by

$$\langle c|\mathbf{E}|c\rangle = -\boldsymbol{\varepsilon}_r(\mathbf{k})2\left(\frac{\hbar\omega_{\mathbf{k}}}{2V}\right)^{1/2}|c|\sin(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}}t + \delta) \tag{C}$$

where V is the volume of the enclosure.

(vi) The root-mean-square fluctuation ΔE of the electric field in the state $|c\rangle$ is given by

$$(\Delta E)^2 = \langle c|\mathbf{E}^2|c\rangle - \langle c|\mathbf{E}|c\rangle^2 = \frac{\hbar\omega_{\mathbf{k}}}{2V}. \tag{D}$$

We noted in Section 1.2.3 that the expectation value of $\mathbf{E}$ in a state with a definite number of photons is zero, so that such a state cannot represent a classical field, even for very large photon numbers. In contrast, it follows from Eqs. (A)–(D) that the relative fluctuation in photon numbers

$$\frac{\Delta N}{\overline{N}} = \overline{N}^{-1/2}$$

tends to zero as $\overline{N} \to \infty$, and that the fluctuation ΔE becomes negligible for large field strengths, i.e. $|c\rangle$ goes over into a classical state in which the field is well defined as $\overline{N} \to \infty$. The state $|c\rangle$ is called a coherent state and represents the closest quantum-mechanical approach to a classical electromagnetic field. (For a full discussion, see the book by Loudon, quoted at the end of Section 1.2.)

1.2. The Lagrangian of a particle of mass m and charge q, moving in an electromagnetic field, is given by

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2}m\dot{\mathbf{x}}^2 + \frac{q}{c}\mathbf{A}\cdot\dot{\mathbf{x}} - q\phi$$

where $\mathbf{A} = \mathbf{A}(\mathbf{x}, t)$ and $\phi = \phi(\mathbf{x}, t)$ are the vector and scalar potentials of the electromagnetic field at the position $\mathbf{x}$ of the particle at time t.

(i) Show that the momentum conjugate to $\mathbf{x}$ is given by

$$\mathbf{p} = m\dot{\mathbf{x}} + \frac{q}{c}\mathbf{A} \tag{A}$$

(i.e. the conjugate momentum $\mathbf{p}$ is not the kinetic momentum $m\dot{\mathbf{x}}$, in general) and that Lagrange's equations reduce to the equations of motion of the particle [compare Eq. (1.60)]

$$m\frac{\mathrm{d}}{\mathrm{d}t}\dot{\mathbf{x}} = q\left[\mathbf{E} + \frac{1}{c}\dot{\mathbf{x}} \wedge \mathbf{B}\right], \tag{B}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields at the instantaneous position of the charge.

(ii) Derive the corresponding Hamiltonian [compare Eq. (1.59)]

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2 + q\phi,$$

and show that the resulting Hamilton equations again lead to Eqs. (A) and (B).

1.3. For Thomson scattering of an unpolarized beam of photons, obtain the differential cross-section for scattering through an angle θ, with the scattered radiation being linearly polarized in a given direction. By considering two mutually perpendicular such directions, use your result to re-derive Eq. (1.69a) for the unpolarized differential cross-section.

Show that for $\theta = 90°$, the scattered beam is 100% linearly polarized in the direction of the normal to the plane of scattering.

2

Lagrangian Field Theory

In the last chapter we quantized the electromagnetic field by Fourier analysing the classical field into normal modes and imposing harmonic oscillator commutation relations on the normal coordinates. We shall now take the fields at each point in space as the dynamical variables and quantize these directly. This approach generalizes the classical mechanics of a system of particles, and its quantization, to a continuous system, i.e. to fields.[1] One introduces a Lagrangian (actually, as we shall see, it is a Lagrangian density), from which the field equations follow by means of Hamilton's principle. One introduces momenta conjugate to the fields and imposes canonical commutation relations directly on the fields and the conjugate momenta. This formalism provides a systematic quantization procedure for any classical field theory derivable from a Lagrangian. Since this approach is equivalent to that of the last chapter, one can only obtain bosons in this way and a different formalism will be needed for fermions.

Another difference from Chapter 1 is that the theory will now be developed in a manifestly relativistically covariant form, and in Section 2.1 we shall define our relativistic notation. The classical Lagrangian field theory will be developed in Section 2.2, to be quantized in Section 2.3. An important feature of a Lagrangian field theory is that all its symmetry properties and the consequent conservation laws are contained in the Lagrangian density. We shall consider some of these aspects in Section 2.4.

[1] The relevant Lagrangian and Hamiltonian mechanics is, for example, developed in H. Goldstein, *Classical Mechanics*, 2nd edn, Addison-Wesley, Reading, Mass., 1980, Chapters 2 and 8, or in L. D. Landau and E. M. Lifshitz, *Mechanics*, Pergamon, Oxford, 1960, Sections 1–7 and Section 40.

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2.1 Relativistic Notation

We shall write x^μ ($\mu = 0, 1, 2, 3$) for the space–time four-vector with the time component $x^0 = ct$ and the space coordinates x^j ($j = 1, 2, 3$), i.e. $x^\mu = (ct, \mathbf{x})$. The components of four-vectors will be labelled by Greek indices, the components of spatial three-vectors by Latin indices.

By means of the metric tensor $g_{\mu\nu}$, with components

$$\left.\begin{aligned} &g_{00} = -g_{11} = -g_{22} = -g_{33} = +1 \\ &g_{\mu\nu} = 0 \text{ if } \mu \neq \nu \end{aligned}\right\}, \tag{2.1}$$

we define the covariant vector x_μ from the contravariant x^μ:

$$x_\mu = \sum_{\nu=0}^{3} g_{\mu\nu} x^\nu \equiv g_{\mu\nu} x^\nu. \tag{2.2}$$

In the last expression we have used the summation convention: repeated Greek indices, one contravariant and one covariant, are summed. From Eqs. (2.1) and (2.2) we have $x_\mu = (ct, -\mathbf{x})$.

We also define the contravariant metric tensor $g^{\lambda\mu}$ by

$$g^{\lambda\mu} g_{\mu\nu} = g^\lambda_\nu = \delta^\lambda_\nu \tag{2.3}$$

where δ^λ_ν is the usual Kronecker delta: $\delta^\lambda_\nu = 1$ if $\lambda = \nu$, and $\delta^\lambda_\nu = 0$ if $\lambda \neq \nu$. From Eqs. (2.1) and (2.3), $g^{\mu\nu} = g_{\mu\nu}$.

A Lorentz transformation

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_{\ \nu} x^\nu \tag{2.4}$$

leaves

$$x^\mu x_\mu = (x^0)^2 - \mathbf{x}^2 \tag{2.5}$$

invariant, i.e. $x'^\mu x'_\mu = x^\mu x_\mu$ is a scalar quantity. Hence

$$\Lambda^{\lambda\mu} \Lambda_{\lambda\nu} = \delta^\mu_\nu. \tag{2.6}$$

(In addition the matrix $\Lambda^{\lambda\mu}$ must be real to ensure the reality of the space–time coordinates.)

A four-component object s^μ (s_μ) transforming like x^μ (x_μ) under Lorentz transformations, and hence with $s^\mu s_\mu$ invariant, is a contravariant (covariant) four-vector. An example is the energy–momentum vector $p^\mu = (E/c, \mathbf{p})$. When no confusion can result, we shall often omit the tensor indices, e.g. we may write x for x^μ or x_μ.

The scalar product of two four-vectors a and b can be written in various ways:

$$ab = a^\mu b_\mu = a_\mu b^\mu = g_{\mu\nu} a^\mu b^\nu = \cdots = a^0 b^0 - \mathbf{a} \cdot \mathbf{b}. \tag{2.7}$$

Like $x^2 = x^\mu x_\mu$, so the scalar product ab is an invariant under Lorentz transformations.

The four-dimensional generalization of the gradient operator ∇ transforms like a four-vector. If $\phi(x)$ is a scalar function, so is

$$\delta\phi = \frac{\partial\phi}{\partial x^\mu}\delta x^\mu,$$

and hence

$$\frac{\partial\phi}{\partial x^\mu} \equiv \partial_\mu\phi \equiv \phi_{,\mu} \tag{2.8a}$$

is a covariant four-vector. Similarly

$$\frac{\partial\phi}{\partial x_\mu} \equiv \partial^\mu\phi \equiv \phi_{,}{}^{\mu} \tag{2.8b}$$

is a contravariant four-vector. Note that indices following a comma denote differentiation. Finally, we note that the operator $\Box$ is a scalar:

$$\partial^\mu\partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \equiv \Box. \tag{2.9}$$

2.2 Classical Lagrangian Field Theory

We consider a system which requires several fields $\phi_r(x), r=1,\ldots,N$, to specify it. The index r may label components of the same field [for example, the components of the vector potential $\mathbf{A}(x)$], or it may refer to different independent fields. We restrict ourselves to theories which can be derived by means of a variational principle from an action integral involving a Lagrangian density

$$\mathscr{L} = \mathscr{L}\left(\phi_r, \phi_{r,\alpha}\right) \tag{2.10}$$

where the derivative $\phi_{r,\alpha}$ is defined by Eq. (2.8a). The Lagrangian density (2.10), depending on the fields and their first derivatives only, is not the most general case possible, but it covers all theories discussed in this book and greatly simplifies the formalism.

We define the action integral $S(\Omega)$ for an arbitrary region Ω of the four-dimensional space-time continuum by

$$S(\Omega) = \int_\Omega \mathrm{d}^4x\,\mathscr{L}\left(\phi_r, \phi_{r,\alpha}\right), \tag{2.11}$$

where d^4x stands for the four-dimensional element $\mathrm{d}x^0\,\mathrm{d}^3\mathbf{x}$.

We now postulate that the equations of motion, i.e. the field equations, are obtained from the following variational principle, which is closely analogous to Hamilton's principle in mechanics. For an arbitrary region Ω, we consider variations of the fields,

$$\phi_r(x) \rightarrow \phi_r(x) + \delta\phi_r(x), \tag{2.12}$$

which vanish on the surface $\Gamma(\Omega)$ bounding the region Ω

$$\delta\phi_r(x) = 0 \text{ on } \Gamma(\Omega). \tag{2.13}$$

The fields ϕ_r may be real or complex. In the case of a complex field $\phi(x)$, the fields $\phi(x)$ and $\phi^*(x)$ are treated as two independent fields. Alternatively, a complex field $\phi(x)$ can be decomposed into a pair of real fields, which are then treated as independent fields. We now demand that for an arbitrary region Ω and the variation (2.12–2.13), the action (2.11) has a stationary value, i.e.

$$\delta S(\Omega) = 0. \tag{2.14}$$

Calculating $\delta S(\Omega)$ from Eq. (2.11), we obtain[2]

$$\begin{aligned}\delta S(\Omega) &= \int_\Omega \mathrm{d}^4x \left\{ \frac{\partial \mathscr{L}}{\partial \phi_r}\delta\phi_r + \frac{\partial \mathscr{L}}{\partial \phi_{r,\alpha}}\delta\phi_{r,\alpha} \right\} \\ &= \int_\Omega \mathrm{d}^4x \left\{ \frac{\partial \mathscr{L}}{\partial \phi_r} - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathscr{L}}{\partial \phi_{r,\alpha}} \right) \right\} \delta\phi_r + \int_\Omega \mathrm{d}^4x \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathscr{L}}{\partial \phi_{r,\alpha}}\delta\phi_r \right),\end{aligned} \tag{2.15}$$

where the last line is obtained by partial integration, since

$$\delta\phi_{r,\alpha} = \frac{\partial}{\partial x^\alpha}\delta\phi_r.$$

The last term in Eq. (2.15) can be converted to a surface integral over the surface $\Gamma(\Omega)$ using Gauss's divergence theorem in four dimensions. Since $\delta\phi_r = 0$ on Γ, this surface integral vanishes. If $\delta S(\Omega)$ is to vanish for arbitrary regions Ω and arbitrary variations $\delta\phi_r$, Eq. (2.15) leads to the Euler–Lagrange equations

$$\frac{\partial \mathscr{L}}{\partial \phi_r} - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathscr{L}}{\partial \phi_{r,\alpha}} \right) = 0, \quad r = 1, \ldots, N. \tag{2.16}$$

These are the equations of motion of the fields.

In order to quantize this classical theory by the canonical formalism of non-relativistic quantum mechanics, we must introduce conjugate variables. We are dealing with a system with a continuously infinite number of degrees of freedom, corresponding to the values of the fields ϕ_r, considered as functions of time, at each point of space $\mathbf{x}$. We shall again approximate the system by one having a countable number of degrees of freedom and ultimately go to the continuum limit.

Consider the system at a fixed instant of time t and decompose the three-dimensional space, i.e. the flat-space-like surface $t = \text{const.}$, into small cells of equal volume $\delta\mathbf{x}_i$, labelled by the index $i = 1, 2, \ldots$ We approximate the values of the fields within each cell by their values at, say, the centre of the cell $\mathbf{x} = \mathbf{x}_i$. The system is now described by the discrete set of generalized coordinates:

$$q_{ri}(t) \equiv \phi_r(i, t) \equiv \phi_r(\mathbf{x}_i, t), \quad r = 1, \ldots N, \quad i = 1, 2, \ldots \tag{2.17}$$

[2] In Eq. (2.15), and thereafter, summations over repeated indices r and α, occurring in products, is implied.

which are the values of the fields at the discrete lattice sites $\mathbf{x}_i$. If we also replace the spatial derivatives of the fields by their difference coefficients between neighbouring sites, we can write the Lagrangian of the discrete system as

$$L(t)=\sum_i \delta\mathbf{x}_i \mathscr{L}_i\left(\phi_r(i,t),\dot{\phi}_r(i,t),\phi_r(i',t)\right) \tag{2.18}$$

where the dot denotes differentiation with respect to time. The Lagrangian density in the ith cell, $\mathscr{L}_i$, depends on the fields at the neighbouring lattice sites i' on account of the approximation of the spatial derivatives. We define momenta conjugate to q_{ri} in the usual way as

$$p_{ri}(t)=\frac{\partial L}{\partial \dot{q}_{ri}} \equiv \frac{\partial L}{\partial \dot{\phi}_r(i,\,t)} \equiv \pi_r(i,t)\delta\mathbf{x}_i \tag{2.19}$$

where

$$\pi_r(i,t) \equiv \frac{\partial \mathscr{L}_i}{\partial \dot{\phi}_r(i,t)}. \tag{2.20}$$

The Hamiltonian of the discrete system is then given by

$$\begin{aligned} H &= \sum_i p_{ri}\dot{q}_{ri} - L \\ &= \sum_i \delta\mathbf{x}_i\left\{\pi_r(i,t)\dot{\phi}_r(i,t) - \mathscr{L}_i\right\}. \end{aligned} \tag{2.21}$$

With a view to going to the limit $\delta\mathbf{x}_i \to 0$, i.e. letting the cell size and the lattice spacing shrink to zero, we define the fields conjugate to $\phi_r(x)$ as

$$\pi_r(x) = \frac{\partial \mathscr{L}}{\partial \dot{\phi}_r}. \tag{2.22}$$

In the limit as $\delta\mathbf{x}_i \to 0, \pi_r(i,\ t)$ tends to $\pi_r(\mathbf{x}_i,\ t)$, and the discrete Lagrangian and Hamiltonian functions (2.18) and (2.21) become

$$L(t) = \int \mathrm{d}^3\mathbf{x}\,\mathscr{L}\left(\phi_r, \phi_{r,\alpha}\right) \tag{2.23}$$

and

$$H = \int \mathrm{d}^3\mathbf{x}\,\mathscr{H}(x), \tag{2.24}$$

where the Hamiltonian density $\mathscr{H}(x)$ is defined by

$$\mathscr{H}(x) = \pi_r(x)\dot{\phi}_r(x) - \mathscr{L}\left(\phi_r, \phi_{r,\alpha}\right), \tag{2.25}$$

and the integrations in Eqs. (2.23) and (2.24) are over all space, at time t. With our Lagrangian density, which does not depend explicitly on the time, the Hamiltonian H is, of course, constant in time. The conservation of energy will be proved in Section 2.4, where the expressions (2.24) and (2.25) for the Hamiltonian will also be re-derived.

As an example, consider the Lagrangian density

$$\mathscr{L} = \frac{1}{2}\left(\phi_{,\alpha}\phi^{,\alpha} - \mu^2\phi^2\right) \tag{2.26}$$

for a single real field $\phi(x)$, with μ a constant, which has the dimensions (length)$^{-1}$. In the next chapter we shall see that the quanta of this field are spinless neutral bosons with reduced Compton wavelength μ^{-1}, i.e. particles of mass $(\hbar\mu/c)$. The equation of motion (2.16) for this field is the Klein–Gordon equation

$$\left(\Box+\mu^2\right)\phi(x) = 0, \tag{2.27}$$

the conjugate field (2.22) is

$$\pi(x) = \frac{1}{c^2}\,\dot{\phi}(x) \tag{2.28}$$

and the Hamiltonian density (2.25) is

$$\mathscr{H}(x) = \frac{1}{2}\left[c^2\pi^2(x) + (\nabla\phi)^2 + \mu^2\phi^2\right]. \tag{2.29}$$

2.3 Quantized Lagrangian Field Theory

It is now easy to go from the classical to the quantum field theory by interpreting the conjugate coordinates and momenta of the discrete lattice approximation, Eqs. (2.17) and (2.19), as Heisenberg operators, and subjecting these to the usual canonical commutation relations:

$$\left.\begin{aligned}[\phi_r(j,\ t),\ \pi_s(j',t)] &= \mathrm{i}\hbar\frac{\delta_{rs}\delta_{jj'}}{\delta\mathbf{x}_j}\\ [\phi_r(j,\ t),\phi_s(j',\ t)] &= [\pi_r(j,\ t),\pi_s(j',\ t)] = 0\end{aligned}\right\}. \tag{2.30}$$

If we let the lattice spacing go to zero, Eqs. (2.30) go over into the commutation relations for the fields:

$$\left.\begin{aligned}[\phi_r(\mathbf{x},\ t),\pi_s(\mathbf{x}',\ t)] &= \mathrm{i}\hbar\ \delta_{rs}\ \delta(\mathbf{x}-\mathbf{x}')\\ [\phi_r(\mathbf{x},\ t),\phi_s(\mathbf{x}',\ t)] &= [\pi_r(\mathbf{x},\ t),\pi_s(\mathbf{x}',t)] = 0\end{aligned}\right\}, \tag{2.31}$$

since in the limit, as $\delta\mathbf{x}_j \to 0$, $\delta_{jj'}/\delta\mathbf{x}_j$ becomes the three-dimensional Dirac delta function $\delta(\mathbf{x}-\mathbf{x}')$, the points $\mathbf{x}$ and $\mathbf{x}'$ lying in the jth and j'th cell, respectively. Note that the canonical commutation relations (2.31) involve the fields at the same time; they are equal-time commutation relations. In the next chapter we shall obtain the commutators of the fields at different times.

For the Klein–Gordon field (2.26), Eqs. (2.31) reduce to the commutation relations:

$$\left.\begin{aligned}\left[\phi(\mathbf{x},\ t),\dot{\phi}(\mathbf{x}',\ t)\right] &= \mathrm{i}\hbar c^2\delta(\mathbf{x}-\mathbf{x}')\\ [\phi(\mathbf{x},\ t),\phi(\mathbf{x}',\ t)] &= \left[\dot{\phi}(\mathbf{x},\ t),\dot{\phi}(\mathbf{x}',\ t)\right] = 0\end{aligned}\right\}. \tag{2.32}$$

In the next chapter we shall study the Klein–Gordon field in detail.

2.4 Symmetries and Conservation Laws

It follows from the Heisenberg equation of motion of an operator $O(t)$

$$i\hbar\frac{dO(t)}{dt} = [O(t), H]$$

(we are not considering operators with explicit time dependence) that O is a constant of the motion provided

$$[O, H] = 0.$$

Constants of the motion generally stem from invariance properties of systems under groups of transformations, e.g. translational and rotational invariance lead to conservation of linear and angular momentum, respectively. Such transformations lead to equivalent descriptions of the system; for example, referred to two frames of reference related by a Lorentz transformation. Quantum-mechanically, two such descriptions must be related by a unitary transformation U under which states and operators transform according to

$$|\Psi\rangle \to |\Psi'\rangle = U|\Psi\rangle, \quad O \to O' = UOU^{\dagger}. \tag{2.33}$$

The unitarity of the transformation ensures two things. Firstly, operator equations are covariant, i.e. have the same form whether expressed in terms of the original or the transformed operators. In particular, this will be true of the commutation relations of the fields and of the equations of motion, e.g. Maxwell's equations will be covariant with respect to Lorentz transformations. Secondly, under a unitary transformation, amplitudes, and hence observable predictions, are invariant.

If one deals with continuous transformations, the unitary operator U can be written

$$U = e^{i\alpha T} \tag{2.34}$$

where $T = T^{\dagger}$ and α is a real continuously variable parameter. For $\alpha = 0$, U goes over into the unit operator. For an infinitesimal transformation

$$U \approx 1 + i\delta\alpha T$$

and Eq. (2.33) becomes

$$O' = O + \delta O = (1 + i\delta\alpha T)O(1 - i\delta\alpha T)$$

i.e.

$$\delta O = i\delta\alpha[T, O]. \tag{2.35}$$

If the theory is invariant under this transformation, the Hamiltonian H will be invariant, $\delta H = 0$, and taking $O = H$ in Eq. (2.35) we obtain $[T, H] = 0$, i.e. T is a constant of the motion.

For a field theory derived from a Lagrangian density $\mathscr{L}$, one can construct conserved quantities from the invariance of $\mathscr{L}$ under symmetry transformations. We shall show that for such a theory, the invariance of $\mathscr{L}$ leads to equations of the form

$$\frac{\delta f^{\alpha}}{\delta x^{\alpha}} = 0 \tag{2.36}$$

where the f^α are functions of the field operators and their derivatives. If we define

$$F^\alpha(t)=\int \mathrm{d}^3\mathbf{x} f^\alpha(\mathbf{x}, t), \tag{2.37}$$

where integration is over all space, then the continuity equation (2.36) gives

$$\frac{1}{c}\frac{\mathrm{d}F^0(t)}{\mathrm{d}t}=-\int \mathrm{d}^3\mathbf{x} \sum_{j=1}^{3} \frac{\partial}{\partial x^j} f^j(\mathbf{x}, t) = 0 \tag{2.36a}$$

where the last step follows by transforming the integral into a surface integral by means of Gauss's divergence theorem and assuming (as always) that the fields, and hence the f^j, tend to zero sufficiently fast at infinity.[3] Hence

$$F^0=\int \mathrm{d}^3\mathbf{x} f^0(\mathbf{x}, t) \tag{2.38}$$

is a conserved quantity. With $T=F^0$, the corresponding unitary operator is then given by Eq. (2.34).

The interpretation of the four-vector f^α follows from Eqs. (2.36)–(138). f^0/c and f^j are the three-dimensional volume and current densities of the conserved quantity F^0/c. Eq. (2.36a), applied to a finite three-dimensional volume V bounded by a surface S, then states that the rate of decrease of F^0/c within V equals the current of F^0/c flowing out through S. Correspondingly, the four-vector $f^\alpha(x)$, satisfying the conservation equation (2.36), is called a conserved current. (Strictly speaking, one should call it a four-current density.) The result, that the invariance of the Lagrangian density $\mathscr{L}$ under a continuous one-parameter set of transformations implies a conserved quantity, is known as Noether's theorem.

We apply these ideas to the transformation

$$\phi_r(x) \rightarrow \phi'_r(x) = \phi_r(x) + \delta\phi_r(x) \tag{2.39}$$

of the fields. The change induced in $\mathscr{L}$ is given by

$$\delta\mathscr{L} = \frac{\partial \mathscr{L}}{\partial \phi_r}\delta\phi_r + \frac{\partial \mathscr{L}}{\partial \phi_{r,\alpha}}\delta\phi_{r,\alpha} = \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathscr{L}}{\partial \phi_{r,\alpha}}\delta\phi_r\right),$$

where the last step follows since $\phi_r(x)$ satisfies the field equations (2.16), and summations over repeated indices r and α are implied as previously. If $\mathscr{L}$ is invariant under the transformation (2.39) so that $\delta\mathscr{L}=0$, the last equation reduces to the continuity equation (2.36) with

$$f^\alpha = \frac{\partial \mathscr{L}}{\partial \phi_{r,\alpha}}\delta\phi_r$$

[3] If one employs a finite normalization volume for the system, as we did in the last chapter, the surface integral vanishes on account of the periodic boundary conditions.

and the constant of the motion, from Eqs. (2.38) and (2.22), is

$$F^0 = c\int \mathrm{d}^3\mathbf{x}\pi_r(x)\delta\phi_r(x). \tag{2.40}$$

An important particular case of the above arises for complex fields ϕ_r, i.e. non-Hermitian operators in the quantized theory. ϕ_r and $\phi_r^\dagger$ are then treated as independent fields, as discussed earlier. Suppose $\mathscr{L}$ is invariant under the transformation

$$\left.\begin{aligned}\phi_r &\to \phi_r' = \mathrm{e}^{\mathrm{i}\varepsilon}\phi_r \approx (1+\mathrm{i}\varepsilon)\phi_r \\ \phi_r^\dagger &\to \phi_r^{\dagger\prime} = \mathrm{e}^{-\mathrm{i}\varepsilon}\phi_r^\dagger \approx (1-\mathrm{i}\varepsilon)\phi_r^\dagger\end{aligned}\right\} \tag{2.41}$$

where ε is a real parameter, and the right-hand expressions result for very small ε. From Eqs. (2.41)

$$\delta\phi_r = \mathrm{i}\varepsilon\phi_r, \quad \delta\phi_r^\dagger = -\mathrm{i}\varepsilon\phi_r^\dagger,$$

and Eq. (2.40) becomes

$$F^0 = \mathrm{i}\varepsilon c\int \mathrm{d}^3\mathbf{x}\left[\pi_r(x)\phi_r(x) - \pi_r^\dagger(x)\phi_r^\dagger(x)\right].$$

Since F^0 multiplied by any constant is also conserved, we shall, instead of F^0, consider

$$Q = -\frac{\mathrm{i}q}{\hbar}\int \mathrm{d}^3\mathbf{x}\left[\pi_r(x)\phi_r(x) - \pi_r^\dagger(x)\phi_r^\dagger(x)\right], \tag{2.42}$$

where q is a constant to be determined later. The reason for this change is that $\pm q$ will turn out to be the electric charges of the particles represented by the fields.

We evaluate the commutator $[Q, \phi_r(x)]$. Since ϕ_r and $\phi_r^\dagger$ are independent fields, ϕ_r commutes with all fields except π_r [see Eq. (2.31)]. Hence taking $(x')^0 = x^0 = ct$,

$$[Q, \phi_r(x)] = -\frac{\mathrm{i}q}{\hbar}\int \mathrm{d}^3\mathbf{x}'[\pi_s(x'), \phi_r(x)]\phi_s(x'),$$

and using the commutation relations (2.31) one obtains

$$[Q, \phi_r(x)] = -q\phi_r(x). \tag{2.43}$$

From this relation one easily verifies that if $|Q'\rangle$ is an eigenstate of Q with the eigenvalue Q', then $\phi_r(x)|Q'\rangle$ is also an eigenstate of Q belonging to the eigenvalue $(Q' - q)$, and correspondingly $\phi_r^\dagger(x)|(Q'\rangle$ belongs to $(Q' + q)$. In the next chapter we shall see that, consistent with these results, ϕ_r and $\phi_r^\dagger$ are linear in creation and absorption operators, with ϕ_r absorbing particles of charge $(+q)$ or creating particles of charge $(-q)$, while $\phi_r^\dagger$ absorbs particles of charge $(-q)$ or creates particles of charge $(+q)$. Hence, we interpret the operator Q, Eq. (2.42), as the charge operator. We have therefore shown that charge is conserved ($\mathrm{d}Q/\mathrm{d}t = 0$, $[Q, H] = 0$), provided the Lagrangian density $\mathscr{L}$ is invariant with respect to the transformation (2.41), which is known as a global phase transformation [corresponding to the fact that the phase ε in Eq. (2.41) is independent of x] or as a gauge transformation of the first kind. We see from Eq. (2.42) that we require complex, i.e. non-Hermitian, fields to represent particles with charge. Real, i.e. Hermitian, fields represent uncharged particles. Interpreting Eq. (2.42) as an operator involves the usual ambiguity

as to the order of factors. We shall have to choose these so that for the vacuum state $|(0)\rangle$, in which no particles are present, $Q|(0)\rangle=0$. We shall return to this point in the next chapter.

The unitary transformation corresponding to the phase transformation (2.41) can, from Eq. (2.34), be written

$$U = e^{i\alpha Q}. \tag{2.44}$$

Hence for infinitesimally small α, we obtain from Eq. (2.33)

$$\begin{aligned} \phi'_r &= e^{i\alpha Q}\phi_r e^{-i\alpha Q} \\ &= \phi_r + i\alpha[Q, \phi_r] = (1 - i\alpha q)\phi_r \end{aligned} \tag{2.45}$$

where the last line follows from Eq. (2.43). Comparing Eqs. (2.45) and (2.41), we see that they are consistent if we take $\varepsilon = -\alpha q$.

Although we have talked of electric charge, with which one is familiar, this analysis applies equally to other types of charge, such as hypercharge.

Conservation of energy and momentum and of angular momentum follows from the invariance of the Lagrangian density $\mathscr{L}$ under translations and rotations. Since these transformations form a continuous group we need only consider infinitesimal transformations. Any finite transformation can be built up through repeated infinitesimal transformations. In four dimensions these are given by

$$x_\alpha \to x'_\alpha \equiv x_\alpha + \delta x_\alpha = x_\alpha + \varepsilon_{\alpha\beta}x^\beta + \delta_\alpha, \tag{2.46}$$

where δ_α is an infinitesimal displacement and $\varepsilon_{\alpha\beta}$ is an infinitesimal antisymmetric tensor, $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$, to ensure the invariance of $x_\alpha x^\alpha$ under homogeneous Lorentz transformations ($\delta_\alpha = 0$).

The transformation (2.46) will induce a transformation in the fields which we assume to be

$$\phi_r(x) \to \phi'_r(x') = \phi_r(x) + \frac{1}{2}\varepsilon_{\alpha\beta}S^{\alpha\beta}_{rs}\phi_s(x). \tag{2.47}$$

In this section, summation over repeated indices r, s, labelling fields, as well as over Lorentz indices α, β, is implied. Here x and x' label the *same* point in space–time referred to the two frames of reference, and ϕ_r and ϕ'_r are the field components referred to these two coordinate systems. The coefficients $S^{\alpha\beta}_{rs}$ in Eq. (2.47) are antisymmetric in α and β, like $\varepsilon_{\alpha\beta}$, and are determined by the transformation properties of the fields. For example, for the vector potential $A_\alpha(x)$, Eq. (2.47) reduces to the transformation law of a vector.

Invariance under the transformations (2.46) and (2.47) means that the Lagrangian density expressed in terms of the new coordinates and fields has the same functional form as when expressed in the original coordinates and fields:

$$\mathscr{L}\big(\phi_r(x), \phi_{r,\alpha}(x)\big) = \mathscr{L}\big(\phi'_r(x'), \phi'_{r,\alpha}(x')\big). \tag{2.48}$$

(Here $\phi_{r,\alpha}(x') \equiv \partial\phi'_r(x')/\partial x'^\alpha$.) From Eq. (2.48) the covariance of the field equations, etc. follows; i.e. they will have the same form expressed in terms of either the original or the transformed coordinates and fields.

The conservation laws follow by expressing the right-hand side of Eq. (2.48) in terms of the original coordinates and fields by means of Eqs. (2.46) and (2.47). We shall first state and discuss these results, postponing their derivation to the end of this section.

For a translation (i.e. $\varepsilon_{\alpha\beta} = 0$), one obtains the four continuity equations

$$\frac{\partial \mathscr{T}^{\alpha\beta}}{\partial x^{\alpha}} = 0, \tag{2.49}$$

where

$$\mathscr{T}^{\alpha\beta} \equiv \frac{\partial \mathscr{L}}{\partial \phi_{r,\alpha}} \frac{\partial \phi_r}{\partial x_\beta} - \mathscr{L} g^{\alpha\beta}, \tag{2.50}$$

and the four conserved quantities are

$$cP^{\alpha} \equiv \int \mathrm{d}^3\mathbf{x}\, \mathscr{T}^{0\alpha} = \int \mathrm{d}^3\mathbf{x} \left\{ c\pi_r(x) \frac{\partial \phi_r(x)}{\partial x_\alpha} - \mathscr{L} g^{0\alpha} \right\}. \tag{2.51}$$

P^{α} is the energy-momentum four-vector, with

$$\left.\begin{aligned} cP^0 &= \int \mathrm{d}^3\mathbf{x} \left\{ \pi_r(x)\dot{\phi}_r(x) - \mathscr{L}\left(\phi_r, \phi_{r,\alpha}\right) \right\} \\ &= \int \mathrm{d}^3\mathbf{x}\, \mathscr{H} = H \end{aligned}\right\} \tag{2.51a}$$

being the Hamiltonian, Eqs. (2.24) and (2.25), and

$$P^j = \int \mathrm{d}^3\mathbf{x} \pi_r(x) \frac{\partial \phi_r(x)}{\partial x_j} \tag{2.51b}$$

being the momentum components of the fields. This interpretation will be confirmed when we come to express these operators in the number representation. Correspondingly $\mathscr{T}^{\alpha\beta}$ is called the energy-momentum tensor.

For a rotation (i.e. $\delta_\alpha = 0$) Eqs. (2.46)–(2.48) give the continuity equations

$$\frac{\partial \mathscr{M}^{\alpha\beta\gamma}}{\partial x^{\alpha}} = 0, \tag{2.52}$$

where

$$\mathscr{M}^{\alpha\beta\gamma} \equiv \frac{\partial \mathscr{L}}{\partial \phi_{r,\alpha}} S^{\beta\gamma}_{rs} \phi_s(x) + \left[x^{\beta} \mathscr{T}^{\alpha\gamma} - x^{\gamma} \mathscr{T}^{\alpha\beta} \right], \tag{2.53}$$

and the six conserved quantities (note that $\mathscr{M}^{\alpha\beta\gamma} = -\mathscr{M}^{\alpha\gamma\beta}$) are

$$\begin{aligned} cM^{\alpha\beta} &= \int \mathrm{d}^3\mathbf{x}\, \mathscr{M}^{0\alpha\beta} \\ &= \int \mathrm{d}^3\mathbf{x} \left\{ \left[x^{\alpha} \mathscr{T}^{0\beta} - x^{\beta} \mathscr{T}^{0\alpha} \right] + c\pi_r(x) S^{\alpha\beta}_{rs} \phi_s(x) \right\}. \end{aligned} \tag{2.54}$$

For two space-like indices ($i, j = 1, 2, 3$), M^{ij} is the angular momentum operator of the field (M^{12} being the z-component, etc.). Remembering that $\mathscr{T}^{oi}/c$ is the momentum density of the field [see Eqs. (2.51)], we interpret the term in square brackets in Eq. (2.54)

as the orbital angular momentum and the last term as the intrinsic spin angular momentum.

We return to the derivation of the continuity equations (2.49) and (2.52).[4] The variation of a function $\phi_r(x)$ with the argument unchanged was defined in Eq. (2.39) as

$$\delta\phi_r(x) \equiv \phi'_r(x) - \phi_r(x). \tag{2.55a}$$

In addition, we now define the variation

$$\delta_T\phi(x) \equiv \phi'_r(x\prime) - \phi_r(x). \tag{2.55b}$$

which results from changes of both the form and the argument of the function. We can then write

$$\begin{aligned}\delta_T\phi_r(x) &= \left[\phi'_r(x') - \phi_r(x')\right] + \left[\phi_r(x') - \phi_r(x)\right] \\ &= \delta\phi_r(x') + \frac{\partial\phi_r}{\partial x_\beta}\delta x_\beta,\end{aligned} \tag{2.56}$$

where δx_β is given by Eq. (2.46). To first order in small quantities this can be written

$$\delta_T\phi_r(x) = \delta\phi_r(x) + \frac{\partial\phi_r}{\partial x_\beta}\delta x_\beta. \tag{2.57}$$

We can similarly write Eq. (2.48) as

$$\begin{aligned}0 &= \mathscr{L}\left(\phi'_r(x'), \phi_{r,\alpha}(x')\right) - \mathscr{L}\left(\phi_r(x), \phi_{r,\alpha}(x)\right) \\ &= \delta\mathscr{L} + \frac{\partial\mathscr{L}}{\partial x^\alpha}\delta x^\alpha.\end{aligned} \tag{2.58}$$

For $\delta\mathscr{L}$ we obtain, since $\phi_r(x)$ satisfies the field equations (2.16),

$$\begin{aligned}\delta\mathscr{L} &= \frac{\partial\mathscr{L}}{\partial\phi_r}\delta\phi_r + \frac{\partial L}{\partial\phi_{r,\alpha}}\delta\phi_{r,\alpha} \\ &= \frac{\partial}{\partial x^\alpha}\left\{\frac{\partial\mathscr{L}}{\partial\phi_{r,\alpha}}\delta\phi_r\right\} = \frac{\partial}{\partial x^\alpha}\left\{\frac{\partial\mathscr{L}}{\partial\phi_{r,\alpha}}\left[\delta_T\phi_r - \frac{\partial\phi_r}{\partial x_\beta}\delta x_\beta\right]\right\}.\end{aligned} \tag{2.59}$$

We combine Eqs. (2.58) and (2.59) to obtain the continuity equations

$$\frac{\partial f^\alpha}{\partial x^\alpha} = 0, \tag{2.60}$$

where

$$f^\alpha \equiv \frac{\partial\mathscr{L}}{\partial\phi_{r,\alpha}}\delta_T\phi_r - \mathscr{T}^{\alpha\beta}\delta x_\beta \tag{2.61}$$

with $\mathscr{T}^{\alpha\beta}$ given by Eq. (2.50).

We first consider translations, i.e. $\varepsilon_{\alpha\beta} = 0$, so that from Eqs. (2.46) and (2.47) $\delta x_\beta = \delta_\beta$ and $\delta_T\phi_r = 0$. Eq. (2.61) reduces to $f^\alpha = -\mathscr{T}^{\alpha\beta}\,\delta x_\beta$, and, since the four displacements δ_β

[4] The reader may omit the rest of this section as the details of this derivation will not be required later on.

are independent of each other, Eq. (2.60) reduces to the four continuity equations (2.49) for energy and momentum conservation.

Finally, we consider rotations, i.e. $\delta_\alpha = 0$. From Eqs. (2.46) and (2.47) for δx_β and $\delta_T \phi_r$, and the antisymmetry of $\varepsilon_{\alpha\beta}$, Eq. (2.61) becomes

$$f^\alpha = \frac{1}{2}\varepsilon_{\beta\gamma}\, \mathscr{M}^{\alpha\beta\gamma} \tag{2.62}$$

where $\mathscr{M}^{\alpha\beta\gamma}$ is the tensor defined in Eq. (2.53). Since the rotations $\varepsilon_{\beta\gamma}$ are independent of each other, Eq. (2.60) reduces to the continuity equations (2.52).

Problems

2.1. Show that replacing the Lagrangian density $\mathscr{L} = \mathscr{L}(\phi_r, \phi_{r,\alpha})$ by

$$\mathscr{L}' = \mathscr{L} + \partial_\alpha \Lambda^\alpha(x),$$

where $\Lambda^\alpha(x)$, $\alpha = 0, \ldots, 3$, are arbitrary functions of the fields $\phi_r(x)$, does not alter the equations of motion.

2.2. The real Klein–Gordon field is described by the Hamiltonian density (2.29). Use the commutation relations (2.31) to show that

$$[H, \phi(x)] = -i\hbar c^2 \pi(x), \quad [H, \pi(x)] = i\hbar(\mu^2 - \nabla^2)\phi(x),$$

where H is the Hamiltonian of the field.

From this result and the Heisenberg equations of motion for the operators $\phi(x)$ and $\pi(x)$, show that

$$\dot{\phi}(x) = c^2 \pi(x), \quad (\Box + \mu^2)\phi(x) = 0.$$

2.3. Show that the Lagrangian density

$$\mathscr{L} = -\frac{1}{2}[\partial_\alpha \phi_\beta(x)][\partial^\alpha \phi^\beta(x)] + \frac{1}{2}[\partial_\alpha \phi^\alpha(x)][\partial_\beta \phi^\beta(x)] + \frac{\mu^2}{2}\phi_\alpha(x)\phi^\alpha(x)$$

for the real vector field $\phi^\alpha(x)$ leads to the field equations

$$[g_{\alpha\beta}(\Box + \mu^2) - \partial_\alpha \partial_\beta]\phi^\beta(x) = 0,$$

and that the field $\phi^\alpha(x)$ satisfies the Lorentz condition

$$\partial_\alpha \phi^\alpha(x) = 0.$$

2.4. Use the commutation relations (2.31) to show that the momentum operator of the fields

$$P^j = \int d^3\mathbf{x}\, \pi_r(x) \frac{\partial \phi_r(x)}{\partial x_j} \tag{2.51b}$$

satisfies the equations

$$[P^j, \phi_r(x)] = -i\hbar \frac{\partial \phi_r(x)}{\partial x_j}, \quad [P^j, \pi_r(x)] = -i\hbar \frac{\partial \pi_r(x)}{\partial x_j}.$$

Hence show that any operator $F(x) = F(\phi_r(x), \pi_r(x))$, which can be expanded in a power series in the field operators $\phi_r(x)$ and $\pi_r(x)$, satisfies

$$[P^j, F(x)] = -i\hbar \frac{\partial F(x)}{\partial x_j}.$$

Note that we can combine these equations with the Heisenberg equation of motion for the operator $F(x)$

$$[H, F(x)] = -i\hbar c \frac{\partial F(x)}{\partial x_0}$$

to obtain the covariant equations of motion

$$[P^\alpha, F(x)] = -i\hbar \frac{\partial F(x)}{\partial x_\alpha},$$

where $P^0 = H/c$.

2.5. Under a translation of coordinates

$$x_\alpha \rightarrow x'_\alpha = x_\alpha + \delta_\alpha \quad (\delta_\alpha = \text{a constant four-vector})$$

a scalar field $\phi(x_\alpha)$ remains invariant:

$$\phi'(x'_\alpha) = \phi(x_\alpha), \quad \text{i.e.} \phi'(x_\alpha) = \phi(x_\alpha - \delta_\alpha).$$

Show that the corresponding unitary transformation

$$\phi(x) \rightarrow \phi'(x) = U\phi(x)U^\dagger$$

is given by $U = \exp[-i\delta_\alpha P^\alpha/\hbar]$, where P^α is the energy-momentum four-vector of the field, Eqs. (2.51). (You may find the results of the previous problem helpful.)

3

The Klein–Gordon Field

In Chapter 1 we quantized the electromagnetic field by Fourier analysing it and imposing harmonic oscillator commutation relations on the Fourier expansion coefficients. This approach naturally led to photons. In the last chapter, a different procedure, the canonical quantization formalism, led directly to quantized field operators. We shall now Fourier analyse these field operators and we shall see that the Fourier coefficients, which are now also operators, satisfy the same commutation relations as the absorption and creation operators of the number representation. In this way the interpretation in terms of field quanta is regained.

In this chapter we shall consider relativistic material particles of spin 0. Photons, which are much more complicated on account of their transverse polarization will be treated in Chapter 5.

3.1 The Real Klein–Gordon Field

For particles of rest mass m, energy and momentum are related by

$$E^2 = m^2c^4 + c^2\mathbf{p}^2. \tag{3.1}$$

If the particles can be described by a single scalar wavefunction $\phi(x)$, the prescription of non-relativistic quantum mechanics

$$\mathbf{p} \to -\mathrm{i}\hbar\boldsymbol{\nabla}, \quad E \to \mathrm{i}\hbar\,\partial/\partial t \tag{3.2}$$

leads to the Klein–Gordon equation (2.27):

$$\left(\Box + \mu^2\right)\phi(x) = 0, \tag{3.3}$$

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where ($\mu \equiv mc/\hbar$). The interpretation of Eq. (3.3) as a single-particle equation leads to difficulties. These are related to defining a positive-definite particle density and to the two signs of the energy E which result from Eq. (3.1). We shall not discuss these difficulties, but only mention that they are typical of relativistic single-particle equations. We shall see that such difficulties do not occur in the many-particle theories which result when the fields, such as the Klein–Gordon field $\phi(x)$, are quantized.[1]

We know from Eq. (2.54), for the angular momentum of the field, that a single scalar field possesses orbital, but no spin, angular momentum, i.e. it represents particles of spin 0. Hence, the Klein–Gordon equation affords the appropriate description of π-mesons (pions) and K-mesons, both of which have spin 0.

We shall now consider a real scalar field $\phi(x)$, satisfying the Klein–Gordon equation (3.3). Such a field corresponds to electrically neutral particles. Charged particles, described by a complex field, will be dealt with in the next section.

We know from Section 2.2 that the Klein–Gordon equation (3.3) can be derived from the Lagrangian density

$$\mathscr{L} = \frac{1}{2}\left(\phi_{,\alpha}\phi^{,\alpha} - \mu^2\phi^2\right), \tag{3.4}$$

and that the field conjugate to ϕ is

$$\pi(x) = \frac{\partial \mathscr{L}}{\partial \dot{\phi}} = \frac{1}{c^2}\dot{\phi}(x). \tag{3.5}$$

On quantization, the real field ϕ becomes a Hermitian operator, $\phi^\dagger = \phi$, satisfying the equal-time commutation relations (2.32):

$$\left.\begin{aligned}\left[\phi(\mathbf{x}, t), \dot{\phi}(\mathbf{x}', t)\right] &= i\hbar c^2 \delta(\mathbf{x} - \mathbf{x}') \\ \left[\phi(\mathbf{x}, t), \phi(\mathbf{x}', t)\right] &= \left[\dot{\phi}(\mathbf{x}, t), \dot{\phi}(\mathbf{x}', t)\right] = 0\end{aligned}\right\}. \tag{3.6}$$

To establish contact with particles, we expand $\phi(x)$ in a complete set of solutions of the Klein–Gordon equation:

$$\phi(x) = \phi^+(x) + \phi^-(x) \tag{3.7a}$$

where

$$\phi^+(x) = \sum_{\mathbf{k}} \left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}}\right)^{1/2} a(\mathbf{k}) e^{-ikx} \tag{3.7b}$$

and

$$\phi^-(x) = \sum_{\mathbf{k}} \left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}}\right)^{1/2} a^\dagger(\mathbf{k}) e^{ikx}. \tag{3.7c}$$

[1] This is often referred to as second quantization, in contrast to the derivation of the single-particle wave equations by means of the substitution (3.2).

These equations are analogous to Eqs. (1.38) for the photon field. The summations are again over the wave vectors $\mathbf{k}$ allowed by the periodic boundary conditions, but k^0 and $\omega_{\mathbf{k}}$ are now given by

$$k^0 = \frac{1}{c}\omega_{\mathbf{k}} = +\left(\mu^2 + \mathbf{k}^2\right)^{1/2}, \tag{3.8a}$$

i.e. k is the wave four-vector of a particle of mass $m = \mu\hbar/c$, momentum $\hbar\mathbf{k}$ and energy

$$E = \hbar\omega_{\mathbf{k}} = +\left[m^2c^4 + c^2(\hbar\mathbf{k})^2\right]^{1/2}. \tag{3.8b}$$

The fact that each operator $a(\mathbf{k})$ occurs paired with its adjoint $a^\dagger(\mathbf{k})$ in Eqs. (3.7) ensures that ϕ is Hermitian.

From Eqs. (3.7) and the commutation relations (3.6), one easily obtains the commutation relations for the operators $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$. We shall leave the details for the reader (see Problem 3.1) and only quote the important result:

$$\left.\begin{aligned}\left[a(\mathbf{k}), a^\dagger(\mathbf{k}')\right] &= \delta_{\mathbf{k}\mathbf{k}'} \\ \left[a(\mathbf{k}), a(\mathbf{k}')\right] &= \left[a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')\right] = 0\end{aligned}\right\}. \tag{3.9}$$

These are precisely the harmonic oscillator commutation relations [Eqs. (1.19) and (1.29)] and all the results can at once be taken over from Section 1.2. In particular, the operators

$$N(\mathbf{k}) = a^\dagger(\mathbf{k})a(\mathbf{k}) \tag{3.10}$$

have as their eigenvalues the occupation numbers

$$n(\mathbf{k}) = 0, 1, 2, \ldots, \tag{3.11}$$

and, correspondingly, $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ are the annihilation and creation operators of particles with momentum $\hbar\mathbf{k}$ and energy $\hbar\omega_{\mathbf{k}}$, given by Eq. (3.8).

The Hamiltonian and momentum operators of the Klein–Gordon field are, from Eqs. (2.51), (3.4) and (3.5), given by

$$H = \int \mathrm{d}^3\mathbf{x}\,\frac{1}{2}\left[\frac{1}{c^2}\dot{\phi}^2 + (\nabla\phi)^2 + \mu^2\phi^2\right] \tag{3.12}$$

and

$$\mathbf{P} = -\int \mathrm{d}^3\mathbf{x}\,\frac{1}{c^2}\dot{\phi}\,\nabla\phi. \tag{3.13}$$

Substituting the expansion (3.7) in the last two equations one obtains

$$H = \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}}\left(a^\dagger(\mathbf{k})a(\mathbf{k}) + \frac{1}{2}\right), \tag{3.14}$$

$$\mathbf{P} = \sum_{\mathbf{k}} \hbar\mathbf{k}\left(a^\dagger(\mathbf{k})a(\mathbf{k}) + \frac{1}{2}\right), \tag{3.15}$$

confirming our interpretation of $[a^\dagger(\mathbf{k})a(\mathbf{k})]$ as the number operator for particles with wave vector $\mathbf{k}$. From the last two equations, one also sees directly that the momentum $\mathbf{P}$ is a constant of the motion for the free Klein–Gordon field. (Nevertheless we prefer the discussion of Section 2.4, because it reveals the fundamental and general connection between symmetries and conservation laws.)

From Eq. (3.14) we see that the state of lowest energy, the ground state, of the Klein–Gordon field is the vacuum state $|0\rangle$, in which no particles are present (all $n(\mathbf{k}) = 0$). We can also characterize this state by

$$a(\mathbf{k})|0\rangle = 0, \quad \text{all } \mathbf{k}, \tag{3.16a}$$

or, expressed in terms of the field operators (3.7), by

$$\phi^+(x)|0\rangle = 0, \quad \text{all } x. \tag{3.16b}$$

The vacuum has the infinite energy $\frac{1}{2}\Sigma_{\mathbf{k}}\hbar\omega_{\mathbf{k}}$. As discussed for the radiation field, only energy differences are observable. Hence, this infinite constant is harmless and easily removed by measuring all energies relative to the vacuum state.

One can avoid the explicit occurrence of such infinite constants by normal ordering of operators. In a *normal product*, all absorption operators stand to the right of all creation operators in each product of operators. Denoting the normal product by N(. . .) we have, for example,

$$\mathrm{N}\big(a(\mathbf{k}_1)a(\mathbf{k}_2)a^\dagger(\mathbf{k}_3)\big) = a^\dagger(\mathbf{k}_3)a(\mathbf{k}_1)a(\mathbf{k}_2), \tag{3.17}$$

and

$$\begin{aligned}\mathrm{N}[\phi(x)\phi(y)] &= \mathrm{N}[(\phi^+(x) + \phi^-(x))(\phi^+(y) + \phi^-(y))] \\ &= \mathrm{N}[\phi^+(x)\phi^+(y)] + \mathrm{N}[\phi^+(x)\phi^-(y)] \\ &\quad + \mathrm{N}[\phi^-(x)\phi^+(y)] + \mathrm{N}[\phi^-(x)\phi^-(y)] \\ &= \phi^+(x)\phi^+(y) + \phi^-(y)\phi^+(x) \\ &\quad + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y),\end{aligned} \tag{3.18}$$

where the order of the factors has been interchanged in the second term, i.e. all positive frequency parts ϕ^+ (which contain only absorption operators) stand to the right of all negative frequency parts ϕ^- (which contain only creation operators).[2] Normal ordering does not fix the order of absorption or creation operators, each amongst themselves, but since each of these commute amongst themselves, such different ways of writing a normal product are equal; for example, expression (3.17) also equals $a^\dagger(\mathbf{k}_3)a(\mathbf{k}_2)a(\mathbf{k}_1)$. Hence in arranging a product of operators in normal order, one simply treats them as though all commutators vanish.

Clearly, the vacuum expectation value of any normal product vanishes. We redefine the Lagrangian density $\mathscr{L}$ and all observables, such as the energy–momentum or angular momentum of the field, or their densities, as normal products. We are free to do this, as it merely corresponds to a particular order of factors before quantization. With observables

[2] This definition of the normal product will be modified when fermions are introduced. Another notation commonly used for the normal product $\mathrm{N}(AB\ldots L)$ is $:AB\ldots L:$.

defined as normal products, their vacuum expectation values vanish. In particular, Eqs. (3.14) and (3.15) become

$$P^{\alpha}=(H/c,\mathbf{P})=\sum_{\mathbf{k}}\hbar k^{\alpha}a^{\dagger}(\mathbf{k})a(\mathbf{k}). \tag{3.19}$$

From the vacuum state $|0\rangle$, one constructs particle states in the same way as was done for photons in Section 1.2. For example, one-particle states are linear superpositions of

$$a^{\dagger}(\mathbf{k})|0\rangle, \qquad \text{all } \mathbf{k}; \tag{3.20a}$$

two-particle states are linear superpositions of

$$a^{\dagger}(\mathbf{k})a^{\dagger}(\mathbf{k}')|0\rangle, \qquad \text{all } \mathbf{k} \text{ and } \mathbf{k}' \neq \mathbf{k}, \tag{3.20b}$$

and

$$\frac{1}{\sqrt{2}}\left[a^{\dagger}(\mathbf{k})\right]^2|0\rangle, \qquad \text{all } \mathbf{k}, \tag{3.20c}$$

and so on. With the vacuum state normalized, i.e. $\langle 0|0\rangle = 1$, the states (3.20) are also normalized. That is the purpose of the factor $1/\sqrt{2}$ in Eq. (3.20c). Similar factors occur for more than two particles.

The particles of the Klein–Gordon field are bosons; the occupation numbers can take on any value $n(\mathbf{k}) = 0, 1, 2, \ldots$ Eq. (3.20b) illustrates another aspect of boson states: they are symmetric under interchange of particle labels. Since all creation operators commute with each other, we have

$$a^{\dagger}(\mathbf{k})a^{\dagger}(\mathbf{k}')|0\rangle = a^{\dagger}(\mathbf{k}')a^{\dagger}(\mathbf{k})|0\rangle. \tag{3.21}$$

3.2 The Complex Klein–Gordon Field

We shall now extend the treatment of the last section to the complex Klein–Gordon field. The new feature this introduces is, as we know from Section 2.4, that we can associate a conserved charge with the field. For the real field this was not possible. We shall concentrate on this aspect of a conserved charge. In other respects, the real and complex fields are very similar, and we shall only quote the main results leaving verification to the reader.

For the complex Klein–Gordon field, the Lagrangian density (3.4) is replaced by

$$\mathcal{L} = \mathrm{N}\left(\phi^{\dagger}{}_{,\alpha}\phi^{\alpha} - \mu^2\phi^{\dagger}\phi\right), \tag{3.22}$$

where we have at once written the quantized operator as a normal product. The field ϕ and its adjoint $\phi^{\dagger}$, treated as independent fields, lead to the Klein–Gordon equations

$$(\Box+\mu^2)\phi(x) = 0, \qquad (\Box+\mu^2)\phi^{\dagger}(x) = 0. \tag{3.23}$$

The fields conjugate to ϕ and $\phi^{\dagger}$ are

$$\pi(x) = \frac{1}{c^2}\dot{\phi}^{\dagger}(x), \qquad \pi^{\dagger}(x) = \frac{1}{c^2}\dot{\phi}(x), \tag{3.24}$$

and the equal-time commutation relations (2.31) become

$$\left.\begin{array}{c}\left[\phi(\mathbf{x},\ t),\dot{\phi}^{\dagger}(\mathbf{x}',\ t)\right]=\mathrm{i}\hbar c^2\delta(\mathbf{x}-\mathbf{x}') \\ \left[\phi(\mathbf{x},\ t),\ \phi(\mathbf{x}',\ t)\right]=\left[\phi(\mathbf{x},\ t),\ \phi^{\dagger}(\mathbf{x}',\ t)\right]=\left[\dot{\phi}(\mathbf{x},\ t),\ \dot{\phi}(\mathbf{x}',\ t)\right] \\ =\left[\dot{\phi}(\mathbf{x},\ t),\dot{\phi}^{\dagger}(\mathbf{x}',\ t)\right]=\left[\phi(\mathbf{x},\ t),\ \dot{\phi}(\mathbf{x}',\ t)\right]=0\end{array}\right\}. \tag{3.25}$$

Analogously to Eqs. (3.7), we write the Fourier expansions of the fields as

$$\phi(x)=\phi^{+}(x)+\phi^{-}(x)=\sum_{\mathbf{k}}\left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}}\right)^{1/2}\left[a(\mathbf{k})\mathrm{e}^{-\mathrm{i}kx}+b^{\dagger}(\mathbf{k})\mathrm{e}^{\mathrm{i}kx}\right] \tag{3.26a}$$

and

$$\phi^{\dagger}(x)=\phi^{\dagger+}(x)+\phi^{\dagger-}(x)=\sum_{\mathbf{k}}\left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}}\right)^{1/2}\left[b(\mathbf{k})\mathrm{e}^{-\mathrm{i}kx}+a^{\dagger}(\mathbf{k})\mathrm{e}^{\mathrm{i}kx}\right]. \tag{3.26b}$$

($\phi^{\dagger+}$ and $\phi^{\dagger-}$ are the positive and negative frequency parts of $\phi^{\dagger}$.)

From Eqs. (3.25) and (3.26a), one obtains the commutation relations

$$\left[a(\mathbf{k}),a^{\dagger}(\mathbf{k}')\right]=\left[b(\mathbf{k}),b^{\dagger}(\mathbf{k}')\right]=\delta_{\mathbf{k}\mathbf{k}'}, \tag{3.27a}$$

and the commutator of any other pair of operators vanishes, i.e.

$$\left[a(\mathbf{k}),a(\mathbf{k}')\right]=\left[b(\mathbf{k}),b(\mathbf{k}')\right]=\left[a(\mathbf{k}),b(\mathbf{k}')\right]=\left[a^{\dagger}(\mathbf{k}),b(\mathbf{k}')\right]=0. \tag{3.27b}$$

From the commutation relations (3.27), it follows that we can interpret $a(\mathbf{k})$ and $a^{\dagger}(\mathbf{k})$, and $b(\mathbf{k})$ and $b^{\dagger}(\mathbf{k})$, as absorption and creation operators of two types of particles – we shall call them a-particles and b-particles – and

$$\mathrm{N}_a(\mathbf{k})=a^{\dagger}(\mathbf{k})a(\mathbf{k}),\qquad \mathrm{N}_b(\mathbf{k})=b^{\dagger}(\mathbf{k})b(\mathbf{k}), \tag{3.28}$$

as the corresponding number operators with eigenvalues 0, 1, 2, ... Hence a number representation can be set up as before, with states containing a- and b-particles generated by means of the creation operators $a^{\dagger}$ and $b^{\dagger}$ from the vacuum state $|0\rangle$, which is now defined by

$$a(\mathbf{k})|0\rangle=b(\mathbf{k})|0\rangle=0,\qquad \text{all } \mathbf{k}, \tag{3.29a}$$

or equivalently by

$$\phi^{+}(x)|0\rangle=\phi^{\dagger+}(x)|0\rangle=0,\qquad \text{all } x. \tag{3.29b}$$

Expressed in terms of the absorption and creation operators, the energy-momentum operator (2.51) of the complex Klein–Gordon field assumes the form we expect

$$P^{\alpha}=(H/c,\mathbf{P})=\sum_{\mathbf{k}}\hbar k^{\alpha}(N_a(\mathbf{k})+N_b(\mathbf{k})). \tag{3.30}$$

We now turn to the charge. From the invariance of the Lagrangian density (3.22) under the phase transformation (2.41) follows the conservation of charge Q, Eq. (2.42), which now takes the form

$$Q = \frac{-iq}{\hbar c^2} \int d^3\mathbf{x} N\left[\phi^\dagger(x)\phi(x) - \phi(x)\phi^\dagger(x)\right]. \tag{3.31}$$

The corresponding charge–current density is given by

$$s^\alpha(x) = (c\rho(x), \mathbf{j}(x)) = \frac{-iq}{\hbar} N\left[\frac{\partial\phi^\dagger}{\partial x_\alpha}\phi - \frac{\partial\phi}{\partial x_\alpha}\phi^\dagger\right], \tag{3.32}$$

which obviously satisfies the continuity equation

$$\frac{\partial s^\alpha(x)}{\partial x^\alpha} = 0. \tag{3.33}$$

Expressed in terms of creation and absorption operators, Eq. (3.31) becomes

$$Q = q \sum_{\mathbf{k}} [N_a(\mathbf{k}) - N_b(\mathbf{k})] \tag{3.34}$$

which clearly commutes with the Hamiltonian H, Eq. (3.30).

It follows from Eq. (3.34) that one must associate charges $+q$ and $-q$ with a- and b-particles, respectively. Apart from the sign of the charge, a- and b-particles have identical properties. Furthermore, the theory is completely symmetric between them, as one sees from Eqs. (3.26)–(3.34). Interchanging a and b merely changes the sign of Q. This result is not restricted to spin 0 bosons, but holds generally. The occurrence of antiparticles in association with all particles of non-zero charge is a fundamental feature of relativistic quantum field theory, which is fully vindicated by experiment.

An example of a particle–antiparticle pair is the pair of charged pi-mesons. Taking $q = e(>0)$, one can identify the π^+- and π^- -mesons with the a- and b-particles of the complex Klein–Gordon field. On the other hand, for a real field, the charge operator Q, Eqs. (3.31) or (3.34), is identically zero, and such a field corresponds to a neutral meson, such as the π^0.

The above considerations are not restricted to electric charge. The invariance of the Lagrangian density $\mathscr{L}$ under phase transformations would allow conservation of other additive quantities, which by analogy one would call some kind of charge other than electric. The above argument would lead to the occurrence of pairs of particles and antiparticles differing from each other in the sign of this new kind of charge. Because of this, even electrically neutral particles may possess antiparticles. This situation does occur in nature. The electrically neutral pseudo-scalar K^0-meson possesses an antiparticle, the $\overline{K}^0$-meson, which is also electrically neutral. K^0 and $\overline{K}^0$ possess opposite hypercharge, $Y = \pm 1$, and are represented by a complex Klein–Gordon field ϕ. Hypercharge is very nearly conserved (unlike electric charge, which is always exactly conserved), which is why it is a useful concept. To be specific, hypercharge is conserved in the strong interactions which are responsible for nuclear forces and associated production of strange particles, but it is not conserved in the weak interactions (about 10^{12} times weaker than the strong interactions) responsible for the decay of strange particles.

Instead of treating the complex Klein–Gordon field directly in terms of ϕ and $\phi^\dagger$ as independent fields, as we have done, one can define two real Klein–Gordon fields, ϕ_1 and ϕ_2, by

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \qquad \phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2), \tag{3.35}$$

and use these as independent fields. We shall not give the development in terms of the real fields as the two approaches are closely related and very similar. Since the fields ϕ_1 and ϕ_2 are real, the creation and annihilation operators associated with them cannot describe charged particles, and it is only linear combinations of them, corresponding to the complex fields (3.35), which describe charged particles. Consequently, when dealing with conserved charges, it is, in general, more natural to work directly with the complex fields.

3.3 Covariant Commutation Relations

While the equations of motion obtained using the Lagrangian formalism are manifestly covariant, this is not so obvious for the field commutation relations derived by the canonical formalism, since these single out equal times. Taking the real Klein–Gordon field as a typical example, we shall illustrate the covariance of the commutation relations by calculating the commutator $[\phi(x),\ \phi(y)]$ for two arbitrary space–time points x and y. Since this commutator is a scalar, it must equal an invariant function.

Writing $\phi = \phi^+ + \phi^-$, we note that

$$[\phi^+(x), \phi^+(y)] = [\phi^-(x), \phi^-(y)] = 0, \tag{3.36}$$

since $\phi^+(\phi^-)$ contains only absorption (creation) operators. Hence

$$[\phi(x),\ \phi(y)] = [\phi^+(x),\ \phi^-(y)] + [\phi^-(x),\ \phi^+(y)], \tag{3.37}$$

and we need only evaluate the first commutator on the right-hand side of this equation. From Eqs. (3.7) one obtains

$$\begin{aligned}[\phi^+(x), \phi^-(y)] &= \frac{\hbar c^2}{2V} \sum_{\mathbf{k}\mathbf{k}'} \frac{1}{(\omega_{\mathbf{k}}\omega_{\mathbf{k}'})^{1/2}} \left[a(\mathbf{k}), a^\dagger(\mathbf{k}')\right] e^{-ikx + ik'y} \\ &= \frac{\hbar c^2}{2(2\pi)^3} \int \frac{d^3\mathbf{k}}{\omega_{\mathbf{k}}} e^{-ik(x-y)}\end{aligned} \tag{3.38}$$

where we have taken the limit $V \to \infty$ [see Eq. (1.48)], and in the last integral $k_0 = \omega_{\mathbf{k}}/c$. We introduce the definition

$$\Delta^+(x) \equiv \frac{-ic}{2(2\pi)^3} \int \frac{d^3\mathbf{k}}{\omega_{\mathbf{k}}} e^{-ikx}, \qquad k_0 = \omega_{\mathbf{k}}/c, \tag{3.39}$$

since this and related functions will occur repeatedly.[3] Eq. (3.38) can then be written

$$[\phi^+(x), \phi^-(y)] = i\hbar c \Delta^+(x - y), \tag{3.40}$$

[3] There is no generally accepted definitions of these Δ-functions, with different definitions differing by constant factors, so care is required in using the literature.

and

$$[\phi^-(x), \phi^+(y)] = -i\hbar c\Delta^+(y-x) \equiv i\hbar c\Delta^-(x-y), \tag{3.41}$$

defining the function $\Delta^-(x)$. From Eqs. (3.40), (3.41) and (3.37) we obtain the commutation relation

$$[\phi(x), \phi(y)] = i\hbar c\Delta(x-y) \tag{3.42}$$

with $\Delta(x)$ defined by

$$\Delta(x) \equiv \Delta^+(x) + \Delta^-(x) = \frac{-c}{(2\pi)^3}\int \frac{d^3\mathbf{k}}{\omega_\mathbf{k}} \sin kx. \tag{3.43}$$

We see that $\Delta(x)$ is a real odd function, as required by the commutation relation (3.42), which (like $\Delta^\pm$) satisfies the Klein–Gordon equation

$$(\Box_x + \mu^2)\Delta(x-y) = 0. \tag{3.44}$$

The Δ-function (3.43) can be written

$$\Delta(x) = \frac{-i}{(2\pi)^3}\int d^4k\,\delta(k^2-\mu^2)\varepsilon(k_0)e^{-ikx} \tag{3.45}$$

where $d^4k = d^3\mathbf{k}\,dk_0$, the k_0-integration is over $-\infty < k_0 < \infty$ and $\varepsilon(k_0)$ is defined by

$$\varepsilon(k_0) = \frac{k_0}{|k_0|} = \begin{cases} +1, & \text{if } k_0 > 0 \\ -1, & \text{if } k_0 < 0. \end{cases} \tag{3.46}$$

The equivalence of the definitions (3.43) and (3.45) is easily established; for example, by writing the δ-function in Eq. (3.45) as

$$\delta(k^2-\mu^2) = \delta[k_0^2 - (\omega_\mathbf{k}/c)^2] = \frac{c}{2\omega_\mathbf{k}}\left[\delta\left(k_0 + \frac{\omega_\mathbf{k}}{c}\right) + \delta\left(k_0 - \frac{\omega_\mathbf{k}}{c}\right)\right] \tag{3.47}$$

and performing the k_0-integration.

The invariance of $\Delta(x)$ under proper Lorentz transformations (i.e. involving neither space nor time reflections) is obvious from Eq. (3.45), since each factor in the integrand is Lorentz-invariant [$\varepsilon(k_0)$ is invariant since proper Lorentz transformations do not interchange past and future].

The Lorentz-invariance of $\Delta(x)$ enables one to give a new interpretation to the equal-time commutation relation

$$[\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = i\hbar c\Delta(\mathbf{x}-\mathbf{y}, 0) = 0 \tag{3.48}$$

which we had earlier [Eq. (3.6)].[4] The invariance of $\Delta(x-y)$ implies that

$$[\phi(x), \phi(y)] = i\hbar c\Delta(x-y) = 0, \quad \text{for} (x-y)^2 < 0, \tag{3.49}$$

i.e. the fields at any two points x and y, with space-like separation, commute. Hence if the field is a physical observable, measurements of the fields at two points with space-like

[4] This also follows directly from Eq. (3.43), since for $x^0 = 0$ the integrand is an odd function of k.

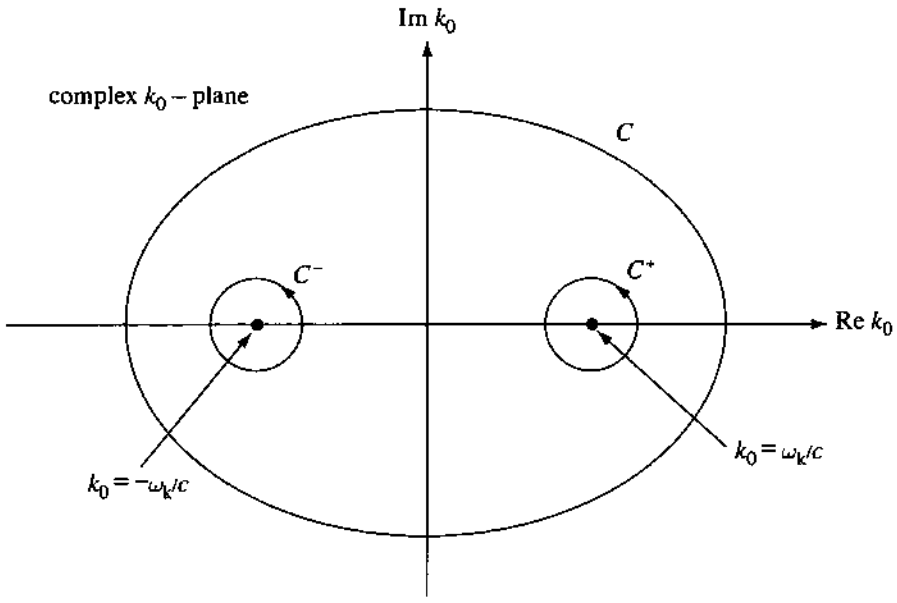

Figure 3.1 *Contours for the integral representation (3.50) of the functions $\Delta^{\pm}(x)$ and $\Delta(x)$*

separation must not interfere with each other. This is known as microcausality, since for a space-like separation, however small, a signal would have to travel with a velocity greater than the speed of light in order to cause interference, contrary to the special theory of relativity. When discussing the connnection between spin and statistics, at the end of Section 4.3, we shall see that the microcausality condition (3.49) is of fundamental importance, even if the field itself is not a physical observable.

A particularly useful way of representing Δ-functions is as contour integrals in the complex k_0-plane. The functions $\Delta^{\pm}(x)$ are given by

$$\Delta^{\pm}(x) = \frac{-1}{(2\pi)^4} \int_{C^{\pm}} \frac{\mathrm{d}^4 k \mathrm{e}^{-ikx}}{k^2 - \mu^2} \tag{3.50}$$

with the contours C^+ and C^-, for Δ^+ and Δ^- respectively, shown in Fig. 3.1. Performing the contour integrations, one picks up the residues from one or other of the poles at $k_0 = \pm\omega_k/c$, and Eq. (3.50) reduces to the definitions (3.39) and (3.41) of $\Delta^{\pm}(x)$. The function $\Delta(x)$, Eq. (3.43), is represented by the same integral (3.50) with the contour C shown in Fig. 3.1. Other Δ-functions are obtained by a different choice of contour.

3.4 The Meson Propagator

We shall now derive and discuss a Δ-function which is of great importance in quantum field theory. Its power and utility, particularly for the development of a systematic covariant perturbation theory, was first fully realized by Feynman. We shall again consider the real Klein–Gordon field.

To start with, we note that the Δ^+-function can be written as the vacuum expectation value of a product of two field operators. We have from Eq. (3.40) that

$$\begin{aligned} i\hbar c\Delta^+(x - x') &= \langle 0|[\phi^+(x), \phi^-(x')]|0\rangle = \langle 0|\phi^+(x)\phi^-(x')|0\rangle \\ &= \langle 0|\phi(x)\phi(x')|0\rangle. \end{aligned} \tag{3.51}$$

We define the *time-ordered* or T-product by

$$\mathrm{T}\{\phi(x)\phi(x')\} = \begin{cases} \phi(x)\phi(x'), & \text{if } t > t' \\ \phi(x')\phi(x), & \text{if } t' > t \end{cases} \tag{3.52}$$

($t \equiv x^0/c$, etc.), i.e. the operators are written in chronological order with time running from right to left: 'earlier' operators operate 'first'.[5] Using the step function

$$\theta(t) = \begin{cases} 1, & \text{if } t > 0 \\ 0, & \text{if } t < 0, \end{cases} \tag{3.53}$$

the T-product can be written

$$\mathrm{T}\{\phi(x)\phi(x')\} = \theta(t - t')\phi(x)\phi(x') + \theta(t' - t)\phi(x')\phi(x). \tag{3.54}$$

The Feynman Δ-function Δ_F is defined by the vacuum expectation value of this T-product:

$$i\hbar c\, \Delta_F(x - x') \equiv \langle 0 | \mathrm{T}\{\phi(x)\phi(x')\} | 0 \rangle. \tag{3.55}$$

From Eqs. (3.51) and (3.41), this leads to the explicit definition

$$\Delta_F(x) = \theta(t)\Delta^+(x) - \theta(-t)\Delta^-(x). \tag{3.56a}$$

Thus

$$\Delta_F(x) = \pm\Delta^{\pm}(x). \qquad \text{if } t \gtrless 0. \tag{3.56b}$$

We would like to be able to visualize the meaning of Δ_F, Eq. (3.55). For $t > t'$, this vacuum expectation value becomes $\langle 0|\phi(x)\phi(x')|0\rangle$. We can think of this expression as representing a meson being created at x', travelling to x, and being annihilated at x. The corresponding expression for $t' > t$, $\langle 0|\phi(x')\phi(x)|0\rangle$, admits a similar interpretation as a meson created at x, propagating to x', where it is absorbed. These two situations are illustrated schematically in Fig. 3.2. The dashed lines represent the propagation of the meson in the direction of the arrow, from x' to x or vice versa. Hence Δ_F, or the vacuum expectation value (3.55), is referred to as the *Feynman propagator* for the mesons of the Klein–Gordon field. We shall, briefly, call it the *meson propagator*, to distinguish it from the fermion and photon propagators to be introduced later.

To illustrate how these propagators arise, we shall consider qualitatively nucleon–nucleon scattering. In this process there will be two nucleons, but no mesons, present in the initial and final states (i.e. before and after the scattering). The scattering, i.e. the interaction, corresponds to the exchange of virtual mesons between the nucleons. The

[5] The definition (3.52) will be modified for fermions.

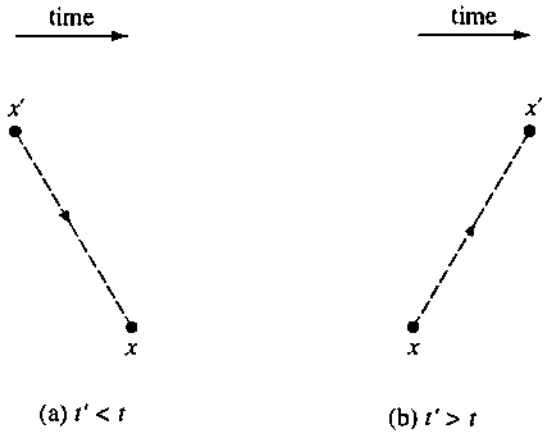

Figure 3.2 *The meson propagator (3.55)*

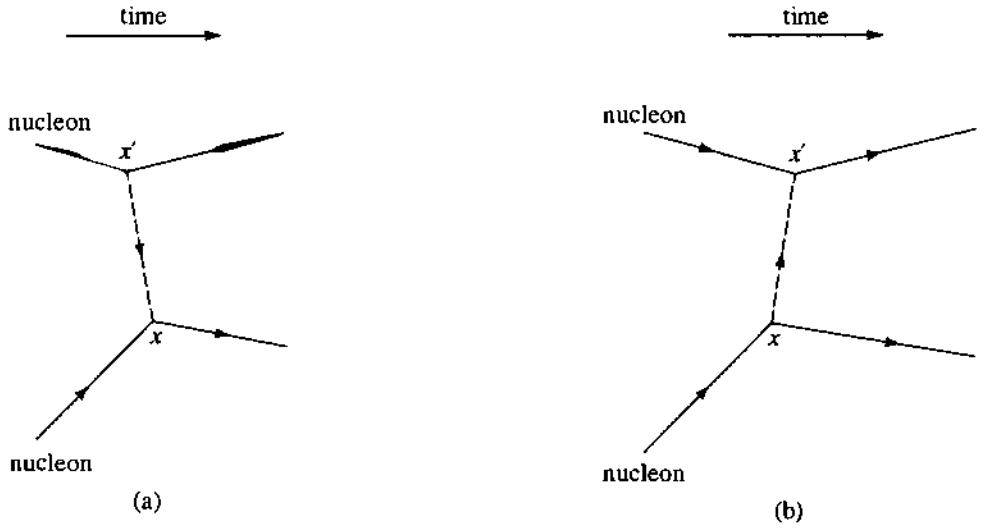

Figure 3.3 *Contribution from one-meson exchange to nucleon–nucleon scattering,* (a) t′ < t; (b) t′ > t

simplest such process is the one-meson exchange, schematically illustrated in Fig. 3.3. The continuous lines represent the nucleons, the dashed lines the mesons. As before, two situations arise according to whether $t > t'$ or $t' > t$. In the actual calculation, all values of x and x' are integrated over, corresponding to emission and absorption of the meson occurring at any two space–time points.

It is interesting to note that the division into the two types of process (a) and (b) of Fig. 3.3, depending on whether $t > t'$ or $t' > t$, is not Lorentz-invariant for $(x - x')$, a space-like separation. In this case what constitutes 'later' and what 'earlier' depends on the frame of reference. On the other hand, considering both cases *together* leads to the covariant Feynman propagator (3.55), which we represent by the single diagram in Fig. 3.4. No time-ordering is implied in this diagram and correspondingly there is no arrow on the meson line.

We have here introduced the ideas of *Feynman graphs* or diagrams. We shall deal with these fully later and shall see that they are a most useful way of picturing the mathematics.

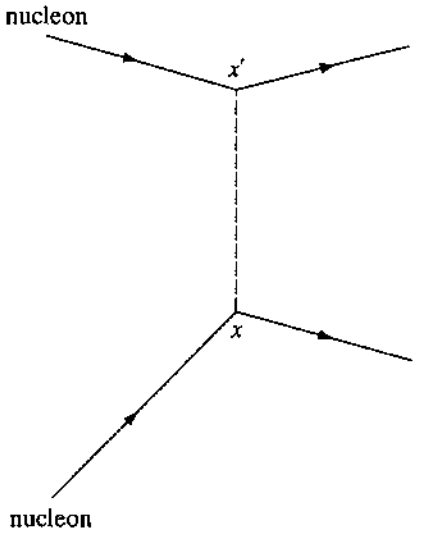

Figure 3.4 *Feynman graph for the one-meson contribution to nucleon–nucleon scattering*

But the reader must be warned not to take this pictorial description of the mathematics as a literal description of a process in space and time. For example, our naive interpretation of the meson propagator would imply that, for $(x - x')$, a space-like separation, the meson travels between the two points with a speed greater than the velocity of light. It is, however, possible to substantiate the above description if, instead of considering propagation between two points x and x', one calculates the probability for emission and absorption in two appropriately chosen four-dimensional regions.[6]

In the following we shall frequently require a representation of the meson propagator, not in coordinate, but in momentum space. This is given by the following integral representation, similar to Eq. (3.50) for $\Delta^{\pm}(x)$:

$$\Delta_F(x) = \frac{1}{(2\pi)^4} \int_{C_F} \frac{d^4k e^{-ikx}}{k^2 - \mu^2}, \tag{3.57}$$

where the contour C_F is shown in Fig. 3.5. To verify Eq. (3.57), we evaluate the contour integral. For $x^0 > 0$, we must complete the contour C_F in the lower half k_0-plane [since $\exp(-ik_0x^0) \to 0$ for $k_0 \to -i\infty$], and, comparing Eqs. (3.57) and (3.50), we obtain $\Delta_F(x) = \Delta^+(x)$, in agreement with Eq. (3.56b). For $x^0 < 0$, completion of the contour in the upper half k_0-plane similarly leads to agreement with Eq. (3.56b).

Instead of deforming the contour as in Fig. 3.5, we can move the poles an infinitesimal distance η off the real axis, as shown in Fig. 3.6, and perform the k_0-integration along the whole real axis, i.e. we replace Eq. (3.57) by

[6] See the article by G. Källén in *Encyclopedia of Physics*, vol. V, part 1, Springer, Berlin, 1958, Section 23. An English translation of this article, entitled *Quantum Electrodynamics*, has been published by Springer, New York, 1972, and by Allen & Unwin, London, 1972.

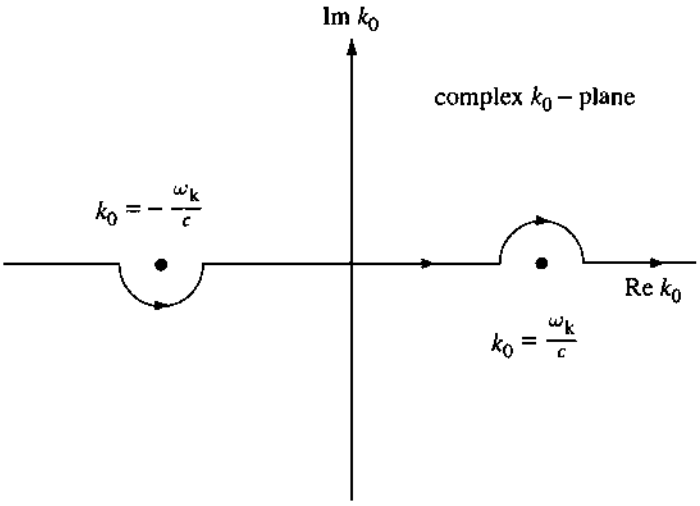

Figure 3.5 *The contour C_F for the meson propagator Δ_F, Eq. (3.57)*

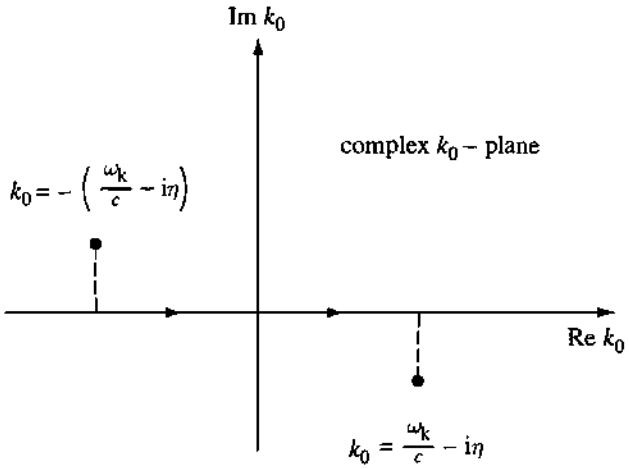

Figure 3.6 *Contour and displaced poles for the meson propagator Δ_F, Eq. (3.58)*

$$\begin{aligned}\Delta_F(x) &= \frac{1}{(2\pi)^4}\int \frac{d^4k e^{-ikx}}{k_0^2 - \left(\frac{\omega_k}{c} - i\eta\right)^2} \\ &= \frac{1}{(2\pi)^4}\int \frac{d^4k e^{-ikx}}{k^2 - \mu^2 + i\varepsilon},\end{aligned} \tag{3.58}$$

where $\varepsilon = 2\eta\omega_k/c$ is a small positive number which we let tend to zero after integration. In Eq. (3.58), integration with respect to each of the four variables $k_0, \ldots, k_3$ is along the whole real axis $(-\infty,\infty)$.

The arguments of this section at once generalize to the case of the complex scalar field, discussed in Section 3.2. The charged meson propagator is now given by

$$\left\langle 0\left|\, \mathrm{T}\left\{\phi(x)\phi^{\dagger}(x')\right\}\right|0\right\rangle = \mathrm{i}\hbar c\Delta_{\mathrm{F}}(x-x'), \tag{3.59}$$

where $\Delta_{\mathrm{F}}(x)$ is the same function [Eqs. (3.56)–(3.58)] as for the real field. The interpretation of the vacuum expectation value (3.59) in terms of the emission, propagation and reabsorption of particles or antiparticles, depending on whether $t' < t$ or $t' > t$, is left to the reader.

Problems

3.1. From the expansion (3.7) for the real Klein–Gordon field $\phi(\mathrm{x})$, derive the following expression for the absorption operator $a(\mathbf{k})$:

$$a(\mathrm{k}) = \frac{1}{(2\hbar c^2 V\omega_{\mathbf{k}})^{1/2}} \int \mathrm{d}^3\mathbf{x}\mathrm{e}^{\mathrm{i}kx}\left(\mathrm{i}\dot{\phi}(x) + \omega_{\mathbf{k}}\phi(x)\right).$$

Hence derive the commutation relations (3.9) for the creation and annihilation operators from the commutation relations (3.6) for the fields.

3.2. With the complex Klein–Gordon fields $\phi(\mathrm{x})$ and $\phi^{\dagger}(\mathrm{x})$ expressed in terms of two independent real Klein–Gordon fields $\phi_1(\mathrm{x})$and $\phi_2(\mathrm{x})$ by Eqs. (3.35), and with $\phi_r(\mathrm{x})$ expanded in the form

$$\phi_r(x) = \sum_{\mathbf{k}}\left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}}\right)^{1/2}\left[a_r(\mathbf{k})\mathrm{e}^{-\mathrm{i}kx} + a_r^{\dagger}(\mathbf{k})\mathrm{e}^{\mathrm{i}kx}\right], \qquad r = 1,2,$$

show that

$$a(\mathbf{k}) = \frac{1}{\sqrt{2}}\left[a_1(\mathbf{k}) + \mathrm{i}a_2(\mathbf{k})\right], \qquad b(\mathbf{k}) = \frac{1}{\sqrt{2}}\left[a_1(\mathbf{k}) - \mathrm{i}a_2(\mathbf{k})\right].$$

Hence derive the commutation relations (3.25) from those for the real fields, and the commutation relations (3.27) from those for $a_r(\mathbf{k})$ and $a_r^{\dagger}(\mathbf{k}), \mathrm{r} = 1,2$.

3.3. From Eq. (3.58), or otherwise, show that the Feynman Δ-function satisfies the inhomogeneous Klein–Gordon equation

$$\left(\Box + \mu^2\right)\Delta_{\mathrm{F}}(x) = -\delta^{(4)}(x).$$

3.4. Derive Eq. (3.59) for the charged meson propagator, and interpret it in terms of emission and re-absorption of particles and antiparticles.

3.5. Charge conjugation for the complex Klein–Gordon field $\phi(x)$ is defined by

$$\phi(x) \to \mathscr{C}\phi(x)\mathscr{C}^{-1} = \eta_{\mathrm{c}}\phi^{\dagger}(x) \tag{A}$$

where $\mathscr{C}$ is a unitary operator which leaves the vacuum invariant, $\mathscr{C}|0\rangle = |0\rangle$, and η_{c} is a phase factor.

Show that under the transformation (A) the Lagrangian density (3.22) is invariant and the charge-current density (3.32) changes sign.

Derive

$$\mathscr{C}\, a(\mathbf{k})\, \mathscr{C}^{-1} = \eta_c b(\mathbf{k}), \quad \mathscr{C}\, b(\mathbf{k})\, \mathscr{C}^{-1} = \eta_c^* a(\mathbf{k})$$

for the absorption operators, and hence show that

$$\mathscr{C}|a, \mathbf{k}\rangle = \eta_c^* |b, \mathbf{k}\rangle, \quad \mathscr{C}|b, \mathbf{k}\rangle = \eta_c |a, \mathbf{k}\rangle$$

where $|a, \mathbf{k}\rangle$ denotes the state with a single a-particle of momentum $\mathbf{k}$ present, etc. ($\mathscr{C}$ is called the charge conjugation operator. It interchanges particles and anti-particles: $a \leftrightarrow b$. The phase η_c is arbitrary and is usually set equal to unity, $\eta_c = 1$.)

3.6. The parity transformation (i.e. space inversion) of the Hermitian Klein–Gordon field $\phi(x)$ is defined by

$$\phi(\mathbf{x}, t) \rightarrow \mathscr{P}\, \phi(\mathbf{x}, t)\, \mathscr{P}^{-1} = \eta_p \phi(-\mathbf{x}, t) \tag{B}$$

where the parity operator $\mathscr{P}$ is a unitary operator which leaves the vacuum invariant, $\mathscr{P}|0\rangle = |0\rangle$, and $\eta_P = \pm 1$ is called the intrinsic parity of the field. Show that the parity transformation leaves the Lagrangian density (3.4) invariant.

Show that

$$\mathscr{P}|\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_n\rangle = \eta_P^n |-\mathbf{k}_1, -\mathbf{k}_2, \ldots, -\mathbf{k}_n\rangle$$

for an arbitrary n-particle state.

For any operators A and B

$$e^{i\alpha A} B e^{-i\alpha A} = \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} B_n$$

where

$$B_0 = B, \quad B_n = [A, B_{n-1}], \quad n = 1, 2, \ldots,$$

holds identically. Hence prove that

$$\mathscr{P}_1\, a(\mathbf{k})\, \mathscr{P}_1^{-1} = i a(\mathbf{k}), \quad \mathscr{P}_2\, a(\mathbf{k})\, \mathscr{P}_2^{-1} = -i\eta_P a(-\mathbf{k}),$$

where the $a(\mathbf{k})$ are the annihilation operators of the field, and $\mathscr{P}_1$ and $\mathscr{P}_2$ are given by

$$\mathscr{P}_1 = \exp\left[-i\frac{\pi}{2} \sum_{\mathbf{k}} a^\dagger(\mathbf{k}) a(\mathbf{k})\right], \quad \mathscr{P}_2 = \exp\left[i\frac{\pi}{2}\eta_P \sum_{\mathbf{k}} a^\dagger(\mathbf{k}) a(-\mathbf{k})\right].$$

Hence, show that the operator $\mathscr{P} = \mathscr{P}_1\, \mathscr{P}_2$ is unitary and satisfies Eq. (B), i.e. it gives an explicit expression for the parity operator $\mathscr{P}$.

4

The Dirac Field

We now wish to consider systems of particles which satisfy the Pauli exclusion principle, i.e. which obey Fermi–Dirac statistics, so-called *fermions*. We saw in Chapter 2 that the canonical quantization formalism necessarily leads to bosons. On the other hand, the harmonic oscillator quantization, used in Chapter 1, allows an *ad hoc* modification which leads to Fermi–Dirac statistics. This modification was first introduced in 1928 by Jordan and Wigner and consists in replacing the commutation relations between absorption and creation operators by anticommutation relations. We shall develop this general formalism in Section 4.1.

In the remainder of this chapter, this formalism will be applied to the Dirac equation, i.e. to relativistic material particles of spin $\frac{1}{2}$. One of the distinctions between bosons and fermions is that the former always have integral spin (0, 1, . . .) whereas the latter must have half-integral spin ($\frac{1}{2}$, $\frac{3}{2}$, . . .). We shall see that this connection between spin and statistics is an essential feature of relativistic quantum field theory.

4.1 The Number Representation for Fermions

In Sections 1.2.2 and 1.2.3 we derived a number representation for bosons from the quantization of a system of independent harmonic oscillators. We shall now modify this formalism so as to obtain a number representation for fermions.

The essence of our earlier treatment can be stated as follows. We had operators a_r, $a_r^\dagger$, $r = 1, 2, \ldots$, satisfying the commutation relations

$$\left[a_r, a_s^\dagger\right] = \delta_{rs}. \quad [a_r, a_s] = \left[a_r^\dagger, a_s^\dagger\right] = 0, \tag{4.1}$$

and defined operators

$$N_r = a_r^\dagger a_r. \tag{4.2}$$

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It then follows from the operator identity

$$[AB,\ C] = A[B,\ C] + [A,\ C]B \tag{4.3}$$

that

$$[N_r,\ a_s] = -\delta_{rs}a_s, \quad [N_r,\ a_s^\dagger] = \delta_{rs}a_s^\dagger. \tag{4.4}$$

The interpretation of a_r, $a_r^\dagger$ and N_r as absorption, creation and number operators, follows from Eqs. (4.2) and (4.4). In particular, N_r has the eigenvalues $n_r = 0, 1, 2\ldots$ The vacuum state $|0\rangle$ is defined by

$$a_r\,|0\rangle = 0, \text{ all } r, \tag{4.5}$$

and other states are built up from the vacuum state as linear superpositions of states of the form

$$\left(a_{r_1}^\dagger\right)^{n_1}\left(a_{r_2}^\dagger\right)^{n_2}\ldots\,|0\rangle. \tag{4.6}$$

It is a remarkable fact that there is an alternative way of deriving the relations (4.4). Define the *anticommutator* of two operators A and B by

$$[A,\ B]_+ \equiv AB\ + BA. \tag{4.7}$$

We then have a second operator identity, analogous to (4.3),

$$[AB,\ C] = A[B, C]_+ - [A,\ C]_+B. \tag{4.8}$$

Suppose now that the operators a_r, $a_r^\dagger$, $r = 1, 2, \ldots$, instead of satisfying the commutation relations (4.1) satisfy the anticommutation relations

$$\left[a_r,\ a_s^\dagger\right]_+ = \delta_{rs}, \quad [a_r,\ a_s]_+ = \left[a_r^\dagger,\ a_s^\dagger\right]_+ = 0; \tag{4.9}$$

in particular

$$(a_r)^2 = \left(a_r^\dagger\right)^2 = 0. \tag{4.9a}$$

One verifies from Eqs. (4.2), (4.8) and (4.9) that for the anticommuting operators [i.e. satisfying Eqs. (4.9)] the *same* commutation relations (4.4) hold which were previously derived for the commuting operators [i.e. satisfying Eqs. (4.1)]. This again leads to the interpretation of a_r, $a_r^\dagger$ and N_r as absorption, creation and number operators, but, from Eqs. (4.9), we now have

$$N_r^2 = a_r^\dagger a_r a_r^\dagger a_r = a_r^\dagger\left(1 - a_r^\dagger a_r\right)a_r = N_r, \tag{4.10}$$

whence

$$N_r(N_r - 1) = 0; \tag{4.10a}$$

i.e., for the anticommuting creation and absorption operators, the number operator N_r has the eigenvalues $n_r = 0$ and $n_r = 1$ only, i.e. we are dealing with Fermi–Dirac statistics.

The vacuum state $|0\rangle$ is again defined by Eq. (4.5). The state in which one article is in the state r is

$$\left|1_r\right\rangle = a_r^\dagger\,|0\rangle. \tag{4.11}$$

For the two-particle states we have from the anticommutation relations (4.9) that for $r \neq s$

$$|1_r 1_s\rangle = a_r^\dagger a_s^\dagger |0\rangle = -a_s^\dagger a_r^\dagger |0\rangle = -|1_s 1_r\rangle \tag{4.12}$$

i.e. the state is antisymmetric under interchange of particle labels, as required for fermions. For $r = s$, we have

$$|2_r\rangle = (a_r^\dagger)^2 |0\rangle = 0, \tag{4.13}$$

thus regaining the earlier result that two particles cannot be in the same single-particle state.

In conclusion, we would like to note the fundamental difference, in spite of their superficial similarity, in the derivation of the boson and fermion results of this section. The boson commutation relations (4.1) are a direct consequence of the canonical commutation relations of non-relativistic quantum mechanics [compare the derivation of Eq. (1.19)]. We have no such foundation for the fermion anticommutation relations (4.9).

4.2 The Dirac Equation

We shall now consider the classical field theory of the Dirac equation, in preparation for going over to the quantized field theory in the next section.[1] The Dirac equation describes material particles of spin $\frac{1}{2}$. We shall see that in the quantum field theory, antiparticles again necessarily occur, e.g. for electrons these are the positrons. Because of our later applications to quantum electrodynamics, we shall for definiteness speak of electrons and positrons in this chapter, but the theory is equally applicable to other spin $\frac{1}{2}$ material particles such as muons and quarks.

The Dirac equation for particles of rest mass m

$$i\hbar \frac{\partial \psi(x)}{\partial t} = \left[c\boldsymbol{\alpha} \cdot (-i\hbar \boldsymbol{\nabla}) + \beta mc^2\right] \psi(x)$$

can be written

$$i\hbar \gamma^\mu \frac{\partial \psi(x)}{\partial x^\mu} - mc\psi(x) = 0 \tag{4.14}$$

where

$$\gamma^0 = \beta, \quad \gamma^i = \beta \alpha_i, \quad i = 1, 2, 3,$$

are Dirac 4×4 matrices which satisfy the anticommutation relations

$$[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu} \tag{4.15}$$

[1] The reader is assumed to be familiar with the elementary theory of the Dirac equation, as discussed in, for example, L. I. Schiff, *Quantum Mechanics*, 3rd edn, McGraw-Hill, New York, 1968, pp. 472–488. Further results of the Dirac theory, which will be needed later, are derived and summarized in Appendix A.

and the Hermiticity conditions $\gamma^{0\dagger} = \gamma^0$ and $\gamma^{j\dagger} = -\gamma^j$ for $j = 1, 2, 3$, which can be combined into[2]

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0. \tag{4.16}$$

Correspondingly, $\psi(x)$ is a spinor wavefunction with four components $\psi_\alpha(x)$, $\alpha = 1, \ldots, 4$. The indices labelling spinor components and matrix elements will usually be suppressed.[3] Although it is at times convenient to use a particular matrix representation, this is generally not necessary. We shall formulate the theory in a representation-free way and only assume that the γ-matrices satisfy the anticommutation and Hermiticity relations (4.15) and (4.16). This will facilitate use of the most convenient representation in a given situation.

The adjoint field $\overline{\psi}(x)$ is defined by

$$\overline{\psi}(x) = \psi^\dagger(x)\gamma^0 \tag{4.18}$$

and satisfies the adjoint Dirac equation

$$i\hbar \frac{\partial\overline{\psi}(x)}{\partial x^\mu}\gamma^\mu + mc\overline{\psi}(x) = 0. \tag{4.19}$$

The Dirac equations (4.14) and (4.19) can be derived from the Lagrangian density

$$\mathscr{L} = c\overline{\psi}(x)\left[i\hbar\gamma^\mu \frac{\partial}{\partial x^\mu} - mc\right]\psi(x) \tag{4.20}$$

by varying the action integral (2.11) independently with respect to the fields ψ_α and $\overline{\psi}_\alpha$. From Eq. (4.20) one obtains for the conjugate fields of ψ_α and $\overline{\psi}_\alpha$

$$\pi_\alpha(x) = \frac{\partial\mathscr{L}}{\partial\dot{\psi}_\alpha} = i\hbar\psi^\dagger_\alpha, \quad \overline{\pi}_\alpha(x) = \frac{\partial\mathscr{L}}{\partial\dot{\overline{\psi}}_\alpha} \equiv 0. \tag{4.21}$$

The Hamiltonian and the momentum of the Dirac field are, from Eqs. (2.51), (4.20) and (4.21), given by

$$H = \int d^3\mathbf{x}\overline{\psi}(x)\left[-i\hbar c\gamma^j\frac{\partial}{\partial x^j} + mc^2\right]\psi(x) \tag{4.22}$$

and

$$\mathbf{P} = -i\hbar\int d^3\mathbf{x}\psi^\dagger(x)\nabla\psi(x). \tag{4.23}$$

---

[2] These conditions ensure the Hermiticity of the Dirac Hamiltonian

$$H = c\gamma^0\gamma^j(-i\hbar\,\partial/\partial x^j) + mc^2\gamma^0. \tag{4.17}$$

[3] In case of doubt, the reader should write the indices out explicitly; e.g. Eq. (4.14) becomes

$$\sum_{\beta=1}^{4} i\hbar\gamma^\mu_{\alpha\beta}\frac{\partial\psi_\beta(x)}{\partial x^\mu} - mc\psi_\alpha(x) = 0, \quad \alpha = 1, \ldots, 4. \tag{4.14a}$$

Eq. (4.22), of course, also follows from the usual definition of the Hamiltonian density (2.25) applied to the present case.

The angular momentum of the Dirac field follows similarly from Eq. (2.54). The transformation of the field under an infinitesimal Lorentz transformation, i.e. Eq. (2.47), is, in the case of the Dirac field, given by

$$\psi_\alpha(x) \to \psi'_\alpha(x') = \psi_\alpha(x) - \frac{\mathrm{i}}{4}\varepsilon_{\mu\nu}\sigma^{\mu\nu}_{\alpha\beta}\psi_\beta(x), \tag{4.24}$$

where summation over $\mu, \nu = 0, \ldots, 3$ and $\beta = 1, \ldots, 4$ is implied, and where $\sigma^{\mu\nu}_{\alpha\beta}$ is the (α, β) matrix element of the 4×4 matrix

$$\sigma^{\mu\nu} \equiv \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]. \tag{4.25a}$$

Eq. (4.24) is derived in Appendix A, Eq. (A.60). Eq. (2.54) now gives for the angular momentum of the Dirac field

$$\mathbf{M} = \int \mathrm{d}^3\mathbf{x}\psi^\dagger(x)[\mathbf{x} \wedge (-\mathrm{i}\hbar\boldsymbol{\nabla})]\psi(x) + \int \mathrm{d}^3\mathbf{x}\psi^\dagger(x)\left(\frac{\hbar}{2}\boldsymbol{\sigma}\right)\psi(x), \tag{4.26}$$

where the 4×4 matrices

$$\boldsymbol{\sigma} = (\sigma^{23}, \sigma^{31}, \sigma^{12}) \tag{4.25b}$$

are the generalizations for the Dirac theory of the 2×2 Pauli spin matrices. We see that the two terms in Eq. (4.26) represent the orbital and spin angular momenta of particles of spin $\frac{1}{2}$.

The Lagrangian density (4.20) is invariant under the phase transformation (2.41). Hence Eq. (2.42) leads to the conserved charge

$$Q = q\int \mathrm{d}^3\mathbf{x}\psi^\dagger(x)\psi(x), \tag{4.27}$$

and the charge–current density

$$s^\alpha(x) = (c\rho(x), \mathbf{j}(x)) = cq\bar{\psi}(x)\gamma^\alpha\psi(x) \tag{4.28}$$

satisfies the continuity equation (i.e. conservation equation)

$$\frac{\partial s^\alpha}{\partial x^\alpha} = 0. \tag{4.29}$$

The continuity equation also follows directly from the Dirac equations (4.14) and (4.19).

In order to quantize the Dirac field in the next section, we shall expand it in a complete set of solutions of the Dirac equation and then impose appropriate anticommutation relations on the expansion coefficients. To conclude this section, we shall therefore specify a complete orthonormal set of solutions of the Dirac equation (4.14).

We shall again consider a cubic enclosure, of volume V, with periodic boundary conditions. A complete set of plane wave states can then be defined as follows. For each momentum $\mathbf{p}$, allowed by the periodic boundary conditions, and positive energy

$$cp_0 = E_\mathbf{p} = +(m^2c^4 + c^2\mathbf{p}^2)^{1/2}, \tag{4.30}$$

the Dirac equation (4.14) possesses four independent solutions. These will be written

$$u_r(\mathbf{p})\frac{e^{-ipx/\hbar}}{\sqrt{V}}, \quad v_r(\mathbf{p})\frac{e^{ipx/\hbar}}{\sqrt{V}}, \quad r = 1, 2 \tag{4.31}$$

i.e. $u_r(\mathbf{p})$ and $v_r(\mathbf{p})$ are constant spinors satisfying the equations

$$(\not{p} - mc)u_r(\mathbf{p}) = 0, \quad (\not{p} + mc)v_r(\mathbf{p}) = 0, \quad r = 1, 2. \tag{4.32}$$

Here we introduce the very convenient notation $\not{A}$ (called A slash), which is defined for any four-vector A_μ by

$$\not{A} \equiv \gamma^\mu A_\mu. \tag{4.33}$$

Because of their time dependence, the solutions (4.31) involving u_r and v_r are referred to as positive and negative energy solutions respectively. We shall use these terms merely as a way of labelling the u- and v-solutions. We shall not follow up their interpretation in the single-particle theory, the resulting difficulties and the reinterpretation in terms of the hole theory. We shall see that the second quantization of the theory (i.e. when ψ and $\psi^\dagger$ become operators) leads directly to the interpretation in terms of particles and antiparticles without the intellectual contortions of the hole theory.[4]

The two-fold degeneracies of the two positive and the two negative energy solutions for a given momentum p result from the possible spin orientations. For the Dirac equation, only the longitudinal spin components (i.e. parallel to $\pm\mathbf{p}$) are constants of the motion, and we shall choose these spin eigenstates for the solutions (4.31). With

$$\sigma_\mathbf{p} = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{|\mathbf{p}|}, \tag{4.34}$$

where $\boldsymbol{\sigma}$ is defined in Eqs. (4.25a) and (4.25b), we then choose the spinors in Eqs. (4.31) so that

$$\sigma_\mathbf{p}u_r(\mathbf{p}) = (-1)^{r+1}u_r(\mathbf{p}), \quad \sigma_\mathbf{p}v_r(\mathbf{p}) = (-1)^r v_r(\mathbf{p}), \quad r = 1, 2. \tag{4.35}$$

The asymmetry in labelling u- and v-spinors will be convenient for labelling the spin properties of particles and antiparticles.

We normalize the spinors u_r and v_r so that

$$u_r^\dagger(\mathbf{p})u_r(\mathbf{p}) = v_r^\dagger(\mathbf{p})v_r(\mathbf{p}) = \frac{E_\mathbf{p}}{mc^2}. \tag{4.36}$$

They then satisfy the orthonormality relations

$$\left.\begin{aligned} u_r^\dagger(\mathbf{p})u_s(\mathbf{p}) = v_r^\dagger(\mathbf{p})v_s(\mathbf{p}) = \frac{E_\mathbf{p}}{mc^2}\delta_{rs} \\ u_r^\dagger(\mathbf{p})v_s(-\mathbf{p}) = 0 \end{aligned}\right\}, \tag{4.37}$$

and the states (4.31) form a complete orthonormal set of solutions of the free-particle Dirac equation, normalized to $E_\mathbf{p}/mc^2$ in a volume V. These and other properties of the plane wave solutions (4.31) are discussed further in Appendix A.

[4] This remark is in no way meant to denigrate Dirac's tremendous intellectual achievement of inventing the hole theory originally.

4.3 Second Quantization

In order to quantize the Dirac field, we expand it in terms of the complete set of plane wave states (4.31):

$$\begin{aligned}\psi(x) &= \psi^+(x) + \psi^-(x) \\ &= \sum_{r\mathbf{p}} \left(\frac{mc^2}{VE_{\mathbf{p}}}\right)^{1/2} [c_r(\mathbf{p})u_r(\mathbf{p})\, e^{-ipx/\hbar} + d_r^\dagger(\mathbf{p})v_r(\mathbf{p})\, e^{ipx/\hbar}]\end{aligned} \tag{4.38a}$$

and hence the conjugate field $\bar{\psi} = \psi^\dagger\gamma^0$ has the expansion

$$\begin{aligned}\bar{\psi}(x) &= \bar{\psi}^+(x) + \bar{\psi}^-(x) \\ &= \sum_{r\mathbf{p}} \left(\frac{mc^2}{VE_{\mathbf{p}}}\right)^{1/2} [d_r(\mathbf{p})\bar{v}_r(\mathbf{p})\, e^{-ipx/\hbar} + c_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})\, e^{ipx/\hbar}]\end{aligned} \tag{4.38b}$$

where $\bar{u}_r = u_r^\dagger\gamma^0$, etc. The summations in Eqs. (4.38) are over the allowed momenta $\mathbf{p}$ and the spin states, labelled by $r = 1, 2$.[5] The factors $(mc^2/VE_{\mathbf{p}})^{1/2}$ will be convenient for the subsequent interpretation of the expansion coefficients. We have written $c_r^\dagger$ and $d_r^\dagger$ for two of these, anticipating that they will become operators on second quantization.

Eqs. (4.38) are closely analogous to the expansions of the complex Klein–Gordon field, Eqs. (3.26). However, the Dirac equation describes spin $\frac{1}{2}$ particles, such as electrons, which obey the Pauli principle and Fermi–Dirac statistics. Following the treatment in Section 4.1, we shall therefore impose the following *anti*commutation relations on the expansion coefficients:

$$[c_r(\mathbf{p}), c_s^\dagger(\mathbf{p}')]_+ = [d_r(\mathbf{p}), d_s^\dagger(\mathbf{p}')]_+ = \delta_{rs}\delta_{\mathbf{pp}'}, \tag{4.39a}$$

and all other anticommutators vanish, i.e. with $c_r \equiv c_r(\mathbf{p})$, $c_s \equiv c_s(\mathbf{p}')$, etc.:

$$\left.\begin{aligned}[c_r, c_s]_+ = [c_r^\dagger, c_s^\dagger]_+ = [d_r, d_s]_+ = [d_r^\dagger, d_s^\dagger]_+ = 0 \\ [c_r, d_s]_+ = [c_r, d_s^\dagger]_+ = [c_r^\dagger, d_s]_+ = [c_r^\dagger, d_s^\dagger]_+ = 0\end{aligned}\right\}. \tag{4.39b}$$

If we define the operators

$$N_r(\mathbf{p}) = c_r^\dagger(\mathbf{p})c_r(\mathbf{p}), \quad \bar{N}_r(\mathbf{p}) = d_r^\dagger(\mathbf{p})d_r(\mathbf{p}), \tag{4.40}$$

the interpretation of $c_r, c_r^\dagger, N_r$ and $d_r, d_r^\dagger, \bar{N}_r$ as absorption, creation and number operators of two kinds of particles, both fermions, follows from the anticommutation relations (4.39), analogously to the development in Section 4.1.

The vacuum state $|0\rangle$ is defined by

$$c_r(\mathbf{p})|0\rangle = d_r(\mathbf{p})|0\rangle = 0, \quad \text{all } \mathbf{p}, \quad \text{and } r = 1, 2, \tag{4.41}$$

or, equivalently, by

$$\psi^+(x)|0\rangle = \bar{\psi}^+(x)|0\rangle = 0, \quad \text{all } x. \tag{4.42}$$

[5] We have chosen particular spin states u_r and v_r, but it should be clear to the reader that one may equally well use any other orthonormal spin states. The following arguments remain valid; only the interpretation of the spin properties of the states has to be modified.

States containing particles are generated from the vacuum state by means of the creation operators. As in Section 4.1, one sees that these states have all the properties characteristic of fermions [i.e. equations analogous to Eqs. (4.12) and (4.13) hold].

To obtain the physical properties of the particles associated with the c- and d-operators, we express the constants of the motion in terms of them. (The reader should be able to have a good guess at most of these properties.) In Section 4.2, we derived expressions for the energy, momentum, angular momentum and charge of the Dirac field [see Eqs. (4.22), (4.23), (4.26) and (4.27)]. However, these operators do not necessarily have the value zero for the vacuum state. We found a similar situation for the Klein–Gordon field [see Eqs. (3.15) and (3.16)]. As in the latter case, we automatically measure quantities relative to the vacuum state if we redefine the expressions for the constants of the motion with the operators ordered as normal products (i.e. in any product, absorption operators occur to the right of creation operators) so that vacuum values necessarily vanish.

For fermions we must modify our earlier definition of the normal product. In arranging a product of boson operators in normal order, one treats them as though all commutators vanish [see Eqs. (3.18) and (3.19)]. For fermion operators, one treats them as though all *anti*commutators vanish, e.g. with $\psi_\alpha \equiv \psi_\alpha(x)$ and $\psi_\beta \equiv \psi_\beta(x')$, etc., one has

$$\begin{aligned} N(\psi_\alpha \psi_\beta) &= \mathrm{N}[(\psi_\alpha^+ + \psi_\alpha^-)(\psi_\beta^+ + \psi_\beta^-)] \\ &= \psi_\alpha^+ \psi_\beta^+ - \psi_\beta^- \psi_\alpha^+ + \psi_\alpha^- \psi_\beta^+ + \psi_\alpha^- \psi_\beta^- \end{aligned} \tag{4.43}$$

which should be compared with Eq. (3.19) for bosons. Similar results hold if in Eq. (4.43) one or both operators are replaced by their adjoint operators, $\overline{\psi}$, or for products of more than two fields.[6]

With the expressions for the constants of the motion, i.e. Eqs. (4.22), (4.23) and (4.26)–(4.28), modified to be normal products, e.g.

$$H = \int \mathrm{d}^3\mathbf{x}\, \mathrm{N}\left\{\overline{\psi}(x)\left[-\mathrm{i}\hbar c\gamma^j \frac{\partial}{\partial x^j} + mc^2\right]\psi(x)\right\}, \tag{4.22a}$$

etc., we substitute the expansions (4.38) for ψ and $\overline{\psi}$. Using the orthonormality properties of the single-particle states (4.31) and, in the calculation of H, that they are solutions of the Dirac equation, we obtain for the energy, momentum and charge operators

$$H = \sum_{r\mathbf{p}} E_{\mathbf{p}}[N_r(\mathbf{p}) + \overline{N}_r(\mathbf{p})] \tag{4.44}$$

$$\mathbf{P} = \sum_{r\mathbf{p}} \mathbf{p}[N_r(\mathbf{p}) + \overline{N}_r(\mathbf{p})] \tag{4.45}$$

[6] The fermion operators ψ and $\overline{\psi}$ are non-commuting quantities, not only through their dependence on the absorption and creation operators, but also as four-component spinors. Hence care is required in changing the order of operators; e.g. if O is a 4 × 4 matrix (such as a product of γ-matrices) then

$$\mathrm{N}(\overline{\psi} O \psi) = \overline{\psi}_\alpha^+ O_{\alpha\beta} \psi_\beta^+ - \psi_\beta^- O_{\alpha\beta} \overline{\psi}_\alpha^+ + \overline{\psi}_\alpha^- O_{\alpha\beta} \psi_\beta^+ + \overline{\psi}_\alpha^- O_{\alpha\beta} \psi_\beta^-, \tag{4.43a}$$

i.e. suppressing the spinor indices the second term on the right-hand side is $-\psi^- O^{\mathrm{T}} \overline{\psi}^+$, where O^{T} is the transposed matrix: $O^{\mathrm{T}}_{\beta\alpha} = O_{\alpha\beta}$ In case of doubt, the reader should write the spinor indices out explicitly.

$$Q = -e \sum_{r\mathbf{p}} [N_r(\mathbf{p}) - \overline{N}_r(\mathbf{p})]. \tag{4.46}$$

In the last equation we have taken the parameter q to be the charge of the electron: $q = -e < 0$. Hence identifying the mass m in the Dirac equation with the mass of the electron, we can interpret the particles associated with the c- and d-operators as electrons and positrons, respectively.

To identify the spin properties, we calculate the spin angular momentum in the states $c_r^\dagger(\mathbf{p})|0\rangle$ and $d_r^\dagger(\mathbf{p})|0\rangle$, containing one electron or one positron of momentum $\mathbf{p}$. From Eqs. (4.26) and (4.34) we define the longitudinal spin operator, i.e. in the direction of motion $\mathbf{p}$, by

$$S_\mathbf{p} = \frac{\hbar}{2} \int \mathrm{d}^3\mathbf{x} \mathrm{N}[\psi^\dagger(x)\sigma_\mathbf{p}\psi(x)]. \tag{4.47}$$

It is left to the reader to verify that

$$\begin{aligned} S_\mathbf{p} c_r^\dagger(\mathbf{p})|0\rangle &= (-1)^{r+1} \frac{\hbar}{2} c_r^\dagger(\mathbf{p})|0\rangle, \\ S_\mathbf{p} d_r^\dagger(\mathbf{p})|0\rangle &= (-1)^{r+1} \frac{\hbar}{2} d_r^\dagger(\mathbf{p})|0\rangle, \quad r = 1, 2. \end{aligned} \tag{4.48}$$

We see from Eqs. (4.48) that in both the electron state $c_r^\dagger(\mathbf{p})|0\rangle$ and the positron state $d_r^\dagger(\mathbf{p})|0\rangle$, the spin component in the direction of motion has the value $+\hbar/2$ for $r = 1$, and the value $-\hbar/2$ for $r = 2$. We refer to these two spin states, i.e. spin parallel and antiparallel to the direction of motion, as having positive (right-handed) and negative (left-handed) *helicity*, respectively. (Right- and left-handed here specifies the screw sense of the spin in the direction of motion.) We shall call $S_\mathbf{p}$ the helicity operator of a spin $\frac{1}{2}$ particle (whether electron or positron) with momentum $\mathbf{p}$.

It follows from Eqs. (4.44)–(4.46) and (4.48) that, as for the complex Klein–Gordon field, the theory is completely symmetric between particles (electrons) and antiparticles (positrons). These have the same properties, except for the reversal of the sign of the electric charge. (As a result, other electromagnetic properties, such as the magnetic moments, have opposite signs.)

The symmetry of the theory between particles and antiparticles is not obvious from the expansions of the field operators ψ and $\overline{\psi}$, Eqs. (4.38). This is due to the fact that we have not chosen a specific spinor representation and in most representations the positive and negative energy spinors will look very different. The expansions (4.38) only manifest the particle–antiparticle symmetry for representations of a particular kind, known as Majorana representations. Labelling the γ-matrices in a Majorana representation with the subscript M, the defining property of a Majorana representation is that

$$\gamma_\mathrm{M}^{\mu *} = -\gamma_\mathrm{M}^\mu, \quad \mu = 0, \ldots, 3, \tag{4.49}$$

where the asterisk denotes complex conjugation, i.e. all four γ-matrices are pure imaginary. A particular Majorana representation is given in Appendix A, Eqs. (A.79). Here we only require the defining property (4.49).

We see from Eq. (4.49) that in a Majorana representation, the operator

$$\left(\mathrm{i}\hbar\gamma_{\mathrm{M}}^{\mu}\frac{\partial}{\partial x^{\mu}} - mc\right) \tag{4.49a}$$

is real. Hence if ψ_{M} is a solution of the Dirac equation in a Majorana representation, so is its complex conjugate ψ_{M}^{*}. It follows that if we denote the positive energy solutions (4.31) by

$$u_{\mathrm{M}r}(\mathbf{p})\frac{\mathrm{e}^{-\mathrm{i}px/\hbar}}{\sqrt{V}}, \quad r = 1, 2, \tag{4.50a}$$

in a Majorana representation, then the corresponding negative energy solutions are

$$u_{\mathrm{M}r}^{*}(\mathbf{p})\frac{\mathrm{e}^{\mathrm{i}px/\hbar}}{\sqrt{V}}, \quad r = 1, 2. \tag{4.50b}$$

Hence the expansions (4.38) become, in a Majorana representation,

$$\left.\begin{aligned}\psi_{\mathrm{M}}(x) &= \sum_{r\mathbf{p}}\left(\frac{mc^2}{VE_{\mathrm{p}}}\right)^{1/2}\left[c_r(\mathbf{p})u_{\mathrm{M}r}(\mathbf{p})\ \mathrm{e}^{-\mathrm{i}px/\hbar} + d_r^{\dagger}(\mathbf{p})u_{\mathrm{M}r}^{*}(\mathbf{p})\ \mathrm{e}^{\mathrm{i}px/\hbar}\right]\\ \psi_{\mathrm{M}}^{\dagger T}(x) &= \sum_{r\mathbf{p}}\left(\frac{mc^2}{VE_{\mathrm{p}}}\right)^{1/2}\left[d_r(\mathbf{p})u_{\mathrm{M}r}(\mathbf{p})\ \mathrm{e}^{-\mathrm{i}px/\hbar} + c_r^{\dagger}(\mathbf{p})u_{\mathrm{M}r}^{*}(\mathbf{p})\ \mathrm{e}^{\mathrm{i}px/\hbar}\right]\end{aligned}\right\}. \tag{4.51}$$

In the last equation we gave the expansion for $\psi^{\dagger\mathrm{T}}$ rather than $\overline{\psi}$ to bring out the complete symmetry between particles and antiparticles. The absorption operators $c_r(\mathbf{p})$ and $d_r(\mathbf{p})$ are multiplied by the same single-particle wavefunctions and thus are absorption operators of particles and antiparticles in the same single-particle state, i.e. with the same momentum, energy and helicity. The same is true of the creation operators. Having used a Majorana representation to manifest the particle–antiparticle symmetry of the field operators, we shall now revert to the representation-free formulation of Eqs. (4.31) and (4.38), in which this symmetry is masked.

The anticommutation relations (4.39) for the creation and absorption operators imply anticommutation relations for the Dirac field operators ψ and $\overline{\psi}$. From Eqs. (4.39) and the expansions (4.38) of the fields, one obtains

$$[\psi_{\alpha}(x), \psi_{\beta}(y)]_{+} = [\overline{\psi}_{\alpha}(x), \overline{\psi}_{\beta}(y)]_{+} = 0, \tag{4.52a}$$

$$[\psi_{\alpha}^{\pm}(x), \overline{\psi}_{\beta}^{\mp}(y)]_{+} = \mathrm{i}\left(\mathrm{i}\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} + \frac{mc}{\hbar}\right)_{\alpha\beta}\Delta^{\pm}(x-y) \tag{4.52b}$$

where $\Delta^{\pm}(x)$ are the invariant Δ-functions introduced for the Klein–Gordon equation, Eqs. (3.39) and (3.41). Eqs. (4.52a) are obvious. The derivation of Eqs. (4.52b) is left as an exercise for the reader.

Omitting suffixes, i.e. considered as a 4×4 matrix equation, we can write Eqs. (4.52b) as

$$[\psi^{\pm}(x), \overline{\psi}^{\mp}(y)]_{+} = \mathrm{i}S^{\pm}(x-y) \tag{4.53a}$$

where the 4×4 matrix functions $S^{\pm}(x)$ are defined by

$$S^{\pm}(x) = \left(\mathrm{i}\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} + \frac{mc}{\hbar}\right)\Delta^{\pm}(x). \tag{4.54a}$$

From the last two equations

$$[\psi(x), \overline{\psi}(y)]_{+} = \mathrm{i}S(x-y) \tag{4.53b}$$

where, analogously to $\Delta(x) = \Delta^{+}(x) + \Delta^{-}(x)$, we defined

$$S^{\pm}(x) = S^{+}(x) + S^{-}(x) = \left(\mathrm{i}\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} + \frac{mc}{\hbar}\right)\Delta(x). \tag{4.54b}$$

From the representations of the various Δ-functions, obtained in Chapter 3, Eqs. (4.54) provide representations of the corresponding S-functions. For example, from the integral representation (3.50) for $\Delta^{\pm}(x)$, we can write Eqs. (4.54a) as

$$S^{\pm}(x) = \frac{-\hbar}{(2\pi\hbar)^4}\int_{C^{\pm}} \mathrm{d}^4p\ \mathrm{e}^{-\mathrm{i}px/\hbar}\frac{\not{p} + mc}{p^2 - m^2c^2}, \tag{4.55a}$$

where the contours $C^{\pm}$ in the complex p_0-plane are anticlockwise closed paths enclosing the poles at $p_0 = \pm(E_{\mathbf{p}}/c)$, corresponding to Fig. 3.1 for the complex $k_0 (= p_0/c)$-plane. Since

$$(\not{p} \pm mc)(\not{p} \mp mc) = p^2 - m^2c^2,$$

the last equation is often abbreviated into the symbolic form

$$S^{\pm}(x) = \frac{-\hbar}{(2\pi\hbar)^4}\int_{C^{\pm}} \mathrm{d}^4p\ \frac{\mathrm{e}^{-\mathrm{i}px/\hbar}}{\not{p} - mc}. \tag{4.55b}$$

4.3.1 The spin-statistics theorem

We conclude Section 4.3 with a brief discussion on the connection between spin and statistics of particles. So far, we have quantized the Dirac equation according to the anticommutation relations (4.39) in order to obtain Fermi–Dirac statistics for electrons. It is interesting to ask what the consequences would be if we quantize the Dirac equation according to Bose–Einstein statistics, i.e. by replacing all the anticommutators in Eqs. (4.39) by commutators. With this change, the energy of the field, again calculated from Eq. (4.22a), is not given by Eq. (4.44), but by

$$H = \sum_{r\mathbf{p}} E_{\mathbf{p}}[N_r(\mathbf{p}) - \overline{N}_r(\mathbf{p})]. \tag{4.56}$$

We are now dealing with Bose–Einstein statistics, and the occupation number operators $N_r(\mathbf{p})$ and $\overline{N}_r(\mathbf{p})$ can take on all values 0, 1, 2, ... Hence, the Hamiltonian (4.56) does not possess a lower bound. If we demand the existence of a state of lowest energy (i.e. a stable ground state), we must quantize the Dirac equation according to Fermi–Dirac statistics.

One may similarly ask what the consequences are of quantizing the Klein–Gordon field according to Fermi–Dirac statistics. In Section 3.3 we referred to microcausality, i.e. the

requirement that two observables $A(x)$ and $B(y)$ must be compatible if $(x-y)$ is a space-like interval, i.e.

$$[A(x), B(y)] = 0, \quad \text{for } (x-y)^2 < 0. \tag{4.57}$$

We have seen that the observables of the fields, such as the energy–momentum densities or the charge–current densities, are bilinear in the field operators [see, for example, Eqs. (3.12), (3.13), (3.32), (4.22), (4.23) and (4.28)]. Using the operator identities (4.3) and (4.8), one can show that for Eq. (4.57) to hold for such bilinear observables, the fields themselves must either commute or anticommute for $(x-y)$ a space-like interval. For the real Klein–Gordon field we must have either

$$[\phi(x), \phi(y)] = 0, \quad \text{for } (x-y)^2 < 0,$$

or

$$[\phi(x), \phi(y)]_+ = 0, \quad \text{for } (x-y)^2 < 0.$$

We know that the first of these relations holds if the Klein–Gordon field is quantized according to Bose-Einstein statistics [compare Eq. (3.50)]. It is easy to show that neither relation holds if we quantize according to Fermi–Dirac statistics, i.e. replace the commutators by anticommutators in the commutation relations (3.9). Hence, the requirement of microcausality forces us to quantize the Klein–Gordon field according to Bose–Einstein statistics.

These conclusions generalize to interacting particles and other spin values. Particles with integral spin must be quantized according to Bose–Einstein statistics, particles with half-integral spin according to Fermi–Dirac statistics. The 'wrong' spin-statistics connections lead to the two types of difficulties we found above. This spin-statistics theorem, to which no exception is known in nature, represents an impressive success for relativistic quantum field theory.

4.4 The Fermion Propagator

In Section 3.4 we introduced the meson propagator. Corresponding to Eq. (3.55), we now define the Feynman fermion propagator as

$$\left\langle 0 | \mathbf{T}\left\{ \psi(x)\overline{\psi}(x') \right\} | 0 \right\rangle \tag{4.58}$$

where spinor indices have again been suppressed. For fermion fields, the time-ordered product is defined by

$$\left.\begin{aligned} T\{\psi(x)\overline{\psi}(x')\} &= \theta(t-t')\psi(x)\overline{\psi}(x') - \theta(t'-t)\overline{\psi}(x')\psi(x) \\ &= \begin{cases} \psi(x)\overline{\psi}(x'), & \text{if } t > t' \\ -\overline{\psi}(x')\psi(x), & \text{if } t' > t \end{cases} \end{aligned}\right\} \tag{4.59}$$

(where $t = x^0/c$, etc.). This definition differs by a factor (–1) in the $t' > t$ term from the corresponding boson definition, Eqs. (3.52) and (3.54). This change in sign reflects the anticommutation property of fermion fields. (A similar difference occurred in the definition of the normal products of boson and fermion fields.)

In order to calculate the fermion propagator (4.58), using Eq. (4.59), we note that

$$\begin{aligned}\langle 0|\psi(x)\bar{\psi}(x')|0\rangle &= \langle 0|\psi^+(x)\bar{\psi}^-(x')|0\rangle \\ &= \langle 0|[\psi^+(x)\,,\bar{\psi}^-(x')]_+|0\rangle = iS^+(x-x'),\end{aligned} \tag{4.60a}$$

where we used Eq. (4.53a); similarly

$$\langle 0|\bar{\psi}(x')\psi(x)|0\rangle = iS^-(x-x'). \tag{4.60b}$$

Combining Eqs. (4.58)–(4.60), we obtain the fermion propagator:

$$\langle 0|\mathrm{T}\left\{\psi(x)\bar{\psi}(x')\right\}|0\rangle = iS_F(x-x') \tag{4.61}$$

where $S_F(x)$ is defined, analogously to Eqs. (3.56) and (4.54), by

$$S_F(x) = \theta(t)S^+(x) - \theta(-t)S^-(x) = \left(i\gamma^\mu\frac{\partial}{\partial x^\mu} + \frac{mc}{\hbar}\right)\Delta_F(x). \tag{4.62}$$

Corresponding to the integral representation (3.58) for $\Delta_F(x)$, $S_F(x)$ can be written

$$S_F(x) = \frac{\hbar}{(2\pi\hbar)^4}\int \mathrm{d}^4p\ \mathrm{e}^{-ipx/\hbar}\frac{\not{p} + mc}{p^2 - m^2c^2 + i\varepsilon}, \tag{4.63}$$

where the integration in the complex p_0-plane is along the whole real axis: $-\infty < p_0 < \infty$. (Compare Fig. 3.6.)

As for the meson propagator, it is useful to visualize the fermion propagator in terms of Feynman diagrams. (As mentioned before, one must not take this interpretation too literally.)

For $t' < t$, the contribution to the fermion propagator (4.61) stems from the term (4.60a) and thus leads to the interpretation of (4.61) as creation of an electron at x', its propagation to x and its annihilation at x. On the other hand, for $t < t'$, the contribution to (4.61) comes from (4.60b) and is pictured as the emission of a positron at x, and its propagation to x' where it is annihilated. The two cases are illustrated in Fig. 4.1. Note that in both diagrams the arrow on the fermion line points *from* the vertex associated with the $\bar{\psi}$-field (x') *to* the vertex associated with the ψ-field (x), i.e. the arrow runs in the same direction as time for electrons, in the opposite direction for positrons.

These ideas are illustrated in Fig. 4.2, which shows two of the leading contributions, in lowest order of perturbation theory, to Compton scattering by electrons. Fig. 4.2(a) represents an electron propagating in the direction of the arrow on the fermion line, emitting the final photon at the vertex x' and absorbing the initial photon at x. Fig. 4.2(b) represents the corresponding process for $t < t'$. The initial photon is annihilated at x, creating an electron–positron pair, i.e. the final electron and a positron which propagates to x, where it annihilates with the initial electron to produce the final-state photon. Note that the arrow is always in the same sense along a fermion line, and in *both* diagrams is from the x'-vertex (associated with the $\bar{\psi}$ operator) to the x-vertex (associated

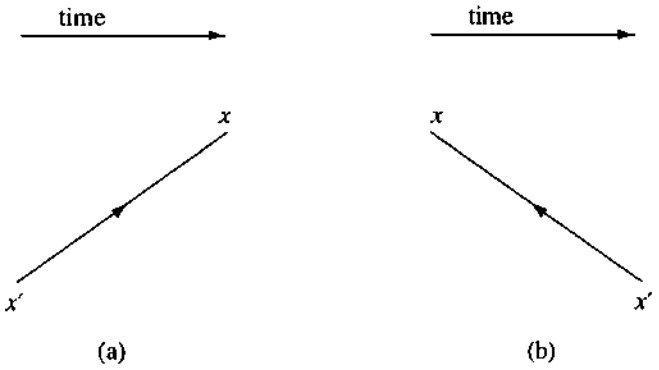

Figure 4.1 (a) $t' < t$: *electron propagated from* x' *to* x;
(b) $t' > t$: *positron propagated from* x *to* x'

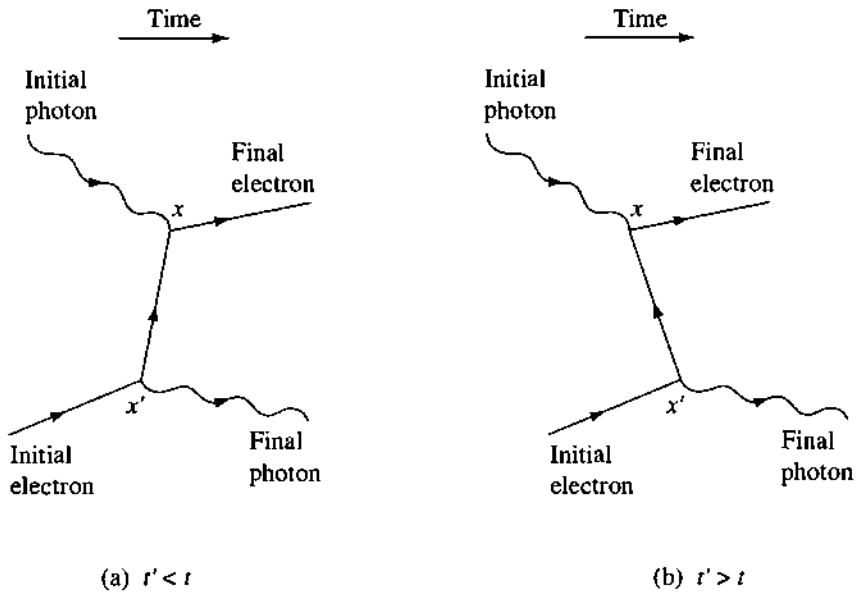

Figure 4.2 *Contributions to Compton scattering: time-ordered graphs*

with ψ) on the internal fermion line. Thus the two diagrams are topologically equivalent, i.e. they can be continuously deformed into each other.

The fermion propagator (4.61) includes both the contributions from Figs. 4.2(a) and (b), and we represent it by the single Feynman diagram in Fig. 4.3, in which no time ordering of the vertices x and x' is implied, and consequently no time direction is attached to the internal fermion line joining x' and x. The orientation of the line $x'x$ is of no significance in this diagram. However, we shall continue to interpret external lines (i.e. lines entering or leaving a diagram from outside) in the same way for Feynman graphs as for time-ordered graphs. A line entering a diagram from the left-hand side will be interpreted as a particle present initially, one leaving a diagram on the right-hand side as one present finally. With these conventions, Figs. 4.4(a) and (b) represent Compton scattering by electrons and positrons, without further labelling being necessary. The

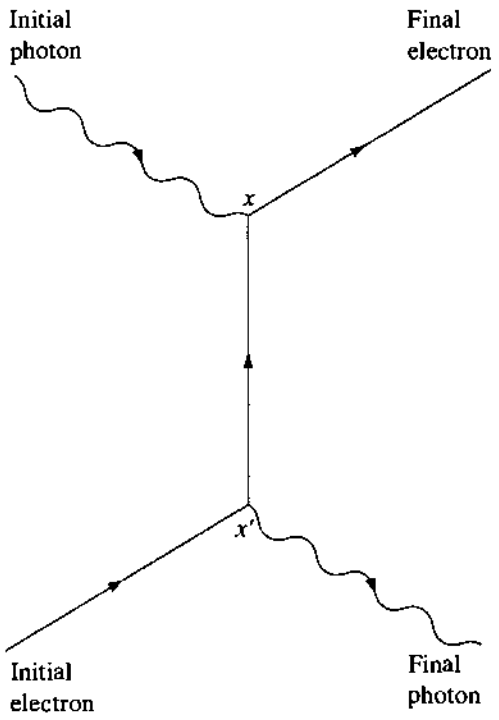

Figure 4.3 *Contribution to Compton scattering: Feynman graph corresponding to the time-ordered graphs of Figure 4.2*

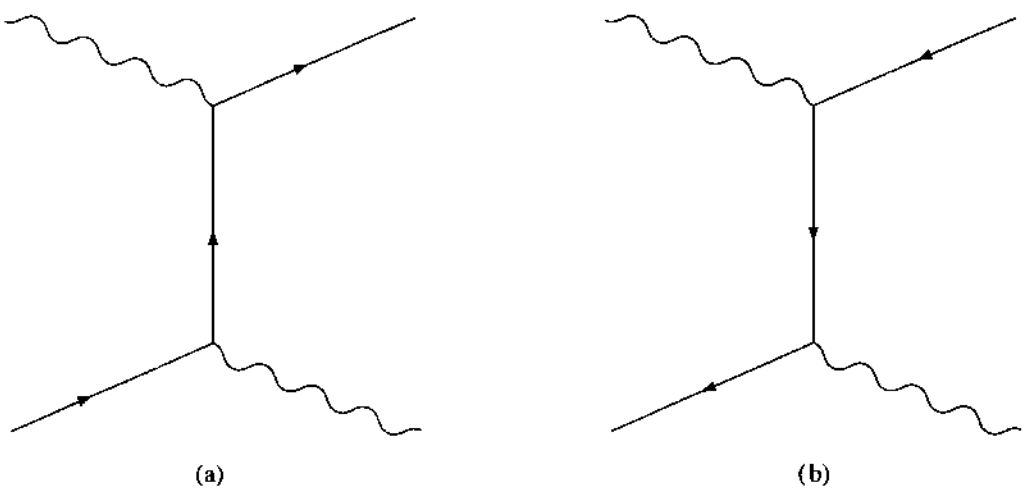

Figure 4.4 *Compton scattering:* (a) *by electrons;* (b) *by positrons*

arrows on the fermion lines are required to distinguish electrons (arrows on external lines from left to right) from positrons (arrows on external lines from right to left). Although no arrows are required on the photon lines, we shall at times use arrows on external photon lines to emphasize initially and finally present quanta. In the following we shall frequently use such Feynman diagrams.

4.5 The Electromagnetic Interaction and Gauge Invariance

We shall now consider the interaction of relativistic electrons with an electromagnetic field, specified by the scalar and vector potentials $\phi(x)$ and $\mathrm{A}(x)$. For this purpose we shall take over the procedure which is successful in non-relativistic quantum mechanics. In the latter case, making the substitution

$$\mathrm{i}\hbar\frac{\partial}{\partial t} \rightarrow \mathrm{i}\hbar\frac{\partial}{\partial t} - q\phi(x), \qquad -\mathrm{i}\hbar\boldsymbol{\nabla} \rightarrow -\mathrm{i}\hbar\boldsymbol{\nabla} - \frac{q}{c}\mathbf{A}(x), \tag{4.64a}$$

in the free-particle Schrödinger equation leads to the correct wave equation for a particle of charge q in this field. [The corresponding classical result is contained in Eq. (1.59).]

The substitution (4.64a) is usually referred to as the 'minimal substitution'. In terms of the four-vector potential $A^{\mu}(x) = (\phi, \mathrm{A})$, the minimal substitution takes the explicitly covariant form

$$\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}} \rightarrow D_{\mu} = \left[\partial_{\mu} + \frac{\mathrm{i}q}{\hbar c}A_{\mu}(x)\right]. \tag{4.64b}$$

We shall assume that this substitution also correctly introduces the electromagnetic interaction into the Dirac equation. With the replacement (4.64b), and $q = -e$ for electrons, the Dirac equation (4.14) and the Lagrangian density (4.20) become

$$(\mathrm{i}\hbar\gamma^{\mu}\partial_{\mu} - mc)\psi(x) = -\frac{e}{c}\gamma^{\mu}A_{\mu}(x)\psi(x), \tag{4.65}$$

and

$$\left.\begin{aligned}\mathscr{L} &= c\bar{\psi}(x)(\mathrm{i}\hbar\gamma^{\mu}D_{\mu} - mc)\psi(x)\\ &= \mathscr{L}_0 + \mathscr{L}_{\mathrm{I}}\end{aligned}\right\} \tag{4.66}$$

where $\mathscr{L}_0$ is the Lagrangian density of the free Dirac field, i.e.

$$\mathscr{L}_0 = c\bar{\psi}(x)(\mathrm{i}\hbar\gamma^{\mu}\partial_{\mu} - mc)\psi(x), \tag{4.67}$$

and $\mathscr{L}_{\mathrm{I}}$ is the interaction Lagrangian density

$$\mathscr{L}_{\mathrm{I}} = e\bar{\psi}(x)\gamma^{\mu}\psi(x)A_{\mu}(x), \tag{4.68}$$

which couples the conserved current $s^{\mu}(x) = c(-e)\bar{\psi}\gamma^{\mu}\psi$, Eq. (4.28), to the electromagnetic field.

To obtain the complete Lagrangian density for electrodynamics, we must add to Eq. (4.66) the Lagrangian density $\mathscr{L}_{\mathrm{rad}}$ of the radiation field, i.e. of the electromagnetic field in the absence of charges. This division is analogous to that of the Hamiltonian in Chapter 1, Eqs. (1.61)–(1.63). $\mathscr{L}_{\mathrm{rad}}$ depends on the potential $A_{\mu}(x)$ only, and we shall study it in the next chapter.

We know that it is only the electromagnetic fields E and **B** which have physical significance, not the potential A_{μ} itself, i.e. the theory must be invariant under the gauge transformations of the potentials, Eqs. (1.3). The latter can be written in the covariant form

$$A_{\mu}(x) \rightarrow A'_{\mu}(x) = A_{\mu}(x) + \partial_{\mu}f(x) \tag{4.69a}$$

where $f(x)$ is an arbitrary function. The invariance of the theory under gauge transformations follows from that of the Lagrangian density. $\mathscr{L}_{\text{rad}}$ has this invariance property, as we shall see in the next chapter. However, applying (4.69a) to Eq. (4.66), we obtain

$$\mathscr{L} \to \mathscr{L}' = \mathscr{L} + e\overline{\psi}(x)\gamma^{\mu}\psi(x)\partial_{\mu}f(x) \tag{4.70}$$

i.e. $\mathscr{L}$ is not gauge invariant with respect to (4.69a). We can restore gauge invariance by demanding that coupled with the gauge transformation (4.69a) of the electromagnetic potentials, the Dirac fields transform according to

$$\left.\begin{aligned} \psi(x) \to \psi'(x) &= \psi(x)\, \mathrm{e}^{\mathrm{i}ef(x)/\hbar c} \\ \overline{\psi}(x) \to \overline{\psi}'(x) &= \overline{\psi}(x)\, \mathrm{e}^{-\mathrm{i}ef(x)/\hbar c} \end{aligned}\right\}. \tag{4.69b}$$

Under the coupled transformations (4.69a) and (4.69b), the Lagrangian densities (4.67) and (4.68) transform according to

$$\mathscr{L}_0 \to \mathscr{L}'_0 = \mathscr{L}_0 - e\overline{\psi}(x)\gamma^{\mu}\psi(x)\partial_{\mu}f(x) \tag{4.71a}$$

$$\mathscr{L}_{\mathrm{I}} \to \mathscr{L}'_{\mathrm{I}} = \mathscr{L}_{\mathrm{I}} + e\overline{\psi}(x)\gamma^{\mu}\psi(x)\partial_{\mu}f(x). \tag{4.71b}$$

Consequently, $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{\mathrm{I}}$ remains invariant under the coupled transformations.

Eqs. (4.69b) are called a *local* phase transformation, since the phase factors depend on x. In the special case that $f(x) = \text{const.}$, Eqs. (4.69b) reduce to a *global* phase transformation, considered in Section 2.4, where we saw that invariance under a global phase transformation leads to a conserved charge. We have now seen that gauge invariance of the theory requires invariance when simultaneously transforming the electromagnetic potentials according to the gauge transformation (4.69a) and the Dirac fields according to the local phase transformation (4.69b), and we shall in future refer to these coupled transformations as gauge transformations. It can be shown that the application of Noether's theorem to the invariance with respect to these (coupled) gauge transformations does not lead to a new conservation law, but only reproduces the conservation of charge.

In what follows we shall assume that Eq. (4.68) gives the correct interaction of quantum electrodynamics. One could try and add other gauge-invariant and Lorentz-invariant local interaction terms, but these can be excluded if we impose another restriction – renomalizability of the theory – as shown in Section 11.3.2. The ultimate justification for taking Eq. (4.68) as the correct interaction lies, of course, in the complete agreement between some of physics' most precise experiments and theoretical predictions based on this interaction.

Problems

4.1. From Eq. (4.53b), or otherwise, derive the equal-time anticommutation relation

$$[\psi(x), \overline{\psi}(y)]_+|_{x_0 = y_0} = \gamma^0 \delta(\mathbf{x} - \mathbf{y}).$$

4.2. Show that the functions $S(x)$ and $S_{\mathrm{F}}(x)$ are solutions of the homogeneous Dirac equation and of an inhomogeneous Dirac equation respectively.

4.3. Show that the charge–current density operator

$$s^{\mu}(x) = -ec\overline{\psi}(x)\gamma^{\mu}\psi(x)$$

of the Dirac equation satisfies the relation

$$[s^{\mu}(x), s^{\nu}(y)] = 0, \quad \text{for } (x-y)^2 < 0.$$

This relation shows that the charge–current densities, which are observable quantities, at two different space–time points x and y, are compatible, provided the interval $(x-y)$ is space-like, as required by microcausality.

4.4. Show that if in the expansion (3.7) of the real Klein–Gordon field $\phi(x)$ we impose the anticommutation relations

$$[a(\mathbf{k}),\ a^{\dagger}(\mathbf{k}')]_{+} = \delta_{\mathbf{k}\mathbf{k}'}, \quad [a(\mathbf{k}),\ a(\mathbf{k}')]_{+} = [a^{\dagger}(\mathbf{k}),\ a^{\dagger}(\mathbf{k}')]_{+} = 0,$$

then for $(x-y)$, a space-like interval:

$$[\phi(x), \phi(y)] \neq 0 \ \text{ and } \ [\phi(x)\,, \phi(y)]_{+} \neq 0.$$

[We know from the discussion at the end of Section 4.3 that either the commutator or the anticommutator of $\phi(x)$ and $\phi(y)$ must vanish for $(x-y)$ space-like, if the bilinear observables constructed from ϕ are to satisfy the microcausality condition (4.57).]

4.5. For a Dirac field, the transformations

$$\psi(x) \to \psi'(x) = \exp(\mathrm{i}\alpha\gamma_5)\psi(x), \quad \psi^{\dagger}(x) \to \psi^{\dagger'}(x) = \psi^{\dagger}(x)\exp(-\mathrm{i}\alpha\gamma_5),$$

where α is an arbitrary real parameter, are called chiral phase transformations.

Show that the Lagrangian density (4.20) is invariant under chiral phase transformations in the zero-mass limit $m = 0$ only, and that the corresponding conserved current in this limit is the axial vector current $J^{\alpha}_{\mathrm{A}}(x) \equiv \overline{\psi}(x)\gamma^{\alpha}\gamma_5\psi(x)$.

Deduce the equations of motion for the fields

$$\psi_{\mathrm{L}}(x) \equiv \frac{1}{2}(1-\gamma_5)\psi(x), \qquad \psi_{\mathrm{R}}(x) \equiv \frac{1}{2}(1+\gamma_5)\psi(x)$$

for non-vanishing mass, and show that they decouple in the limit $m = 0$. Hence show that the Lagrangian density

$$\mathscr{L}(x) = \mathrm{i}\hbar c\overline{\psi}_{\mathrm{L}}(x)\gamma^{\mu}\partial_{\mu}\psi_{\mathrm{L}}(x)$$

describes zero-mass fermions with negative helicity only, and zero-mass antifermions with positive helicity only. (This field is called the Weyl field and can be used to describe the neutrinos in weak interactions in the approximation of zero mass.)

5

Photons: Covariant Theory

In our discussion of the electromagnetic field in Chapter 1, we saw that only the transverse radiation field corresponds to independent dynamical degrees of freedom, and we only quantized this transverse field. On the other hand, the instantaneous Coulomb interaction between charges is fully determined by the charge distribution and, in the formulation of Chapter 1, is treated as a classical potential. This formulation of quantum electrodynamics is closely related to the classical theory and so facilitates interpretation in familiar terms. However, the decomposition of the fields into transverse and longitudinal components is clearly frame dependent and so hides the Lorentz invariance of the theory.

An explicitly Lorentz-covariant formulation of the theory is essential for a complete development of quantum electrodynamics. This is required to establish the renormalizability of the theory, i.e. the possibility of carrying out calculations to all orders of perturbation theory with finite self-consistent results, and it is very helpful in practice in calculating such higher-order radiative corrections.

We shall therefore in this chapter develop a covariant theory starting, in Section 5.1, from an explicitly covariant formulation of classical electrodynamics in which all four components of the four-vector potential $A^\mu(x) = (\phi, \mathbf{A})$ are treated on an equal footing. This corresponds to introducing more dynamical degrees of freedom than the system possesses and these will later have to be removed by imposing suitable constraints.

The quantized theory, derived in Section 5.2 by quantizing all four components of the four-vector potential $A^\mu(x)$, looks on the face of it very different from the theory of Chapter 1. However, the two formulations are equivalent, as we shall illustrate when discussing the photon propagator in Section 5.3.

5.1 The Classical Fields

To express Maxwell's equations in covariant form, we introduce the antisymmetric field tensor

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$$F^{\mu\nu}(x) = \begin{array}{c} \begin{matrix} \nu \to 0 & 1 & 2 & 3 \end{matrix} \\ \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \end{array} \begin{matrix} \mu \\ \downarrow \\ 0 \\ 1 \\ 2 \\ 3 \end{matrix} \tag{5.1}$$

In terms of $F^{\mu\nu}$ and the charge-current density $s^\mu(x) = (c\rho(x), \mathbf{j}(x))$, Maxwell's equations (1.1) become

$$\partial_\nu F^{\mu\nu}(x) = \frac{1}{c} s^\mu(x) \tag{5.2}$$

$$\partial^\lambda F^{\mu\nu}(x) + \partial^\mu F^{\nu\lambda}(x) + \partial^\nu F^{\lambda\mu}(x) = 0. \tag{5.3}$$

Since $F^{\mu\nu}$ is antisymmetric, Eq. (5.2) at once gives

$$\partial_\mu s^\mu(x) = 0, \tag{5.4}$$

i.e. consistency requires conservation of the current to which the electromagnetic field is coupled.

The field $F^{\mu\nu}$ can be expressed in terms of the four-vector potential $A^\mu(x) = (\phi, \mathbf{A})$ by

$$F^{\mu\nu}(x) = \partial^\nu A^\mu(x) - \partial^\mu A^\nu(x) \tag{5.5}$$

which is identical with Eqs. (1.2). In terms of the potentials, Eqs. (5.3) are satisfied identically, and Eqs. (5.2) become

$$\Box A^\mu(x) - \partial^\mu(\partial_\nu A^\nu(x)) = \frac{1}{c} s^\mu(x). \tag{5.6}$$

These equations are Lorentz covariant, and they are also invariant under the gauge transformation

$$A^\mu(x) \to A'^\mu(x) = A^\mu(x) + \partial^\mu f(x). \tag{5.7}$$

The field equations (5.6) can be derived from the Lagrangian density

$$\mathscr{L}' = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) - \frac{1}{c} s_\mu(x) A^\mu(x) \tag{5.8}$$

by treating the four components $A^\mu(x)$ as the independent fields in the variational principle (2.11)–(2.14). The form of this Lagrangian density ensures the correct behaviour of the field equations (5.6) under Lorentz and gauge transformations.[1]

---

[1] $\mathscr{L}'$ is clearly Lorentz invariant. Under the gauge transformation (5.7)

$$\mathscr{L}' \to \mathscr{L}' - \frac{1}{c} s_\mu(x) \partial^\mu f(x) = \mathscr{L}' - \frac{1}{c} \partial^\mu [s_\mu(x) f(x)], \tag{5.9}$$

on account of current conservation. Although $\mathscr{L}'$ is not invariant, it follows from Eqs. (2.11)–(2.14) that adding a four-divergence to the Lagrangian density does not alter the field equations, i.e. their gauge invariance is ensured by the form of $\mathscr{L}'$. (See Problem 2.1.)

Unfortunately, the Lagrangian density (5.8) is not suitable for carrying out the canonical quantization. Eq. (5.8) leads to the conjugate fields

$$\pi^{\mu}(x) = \frac{\partial \mathscr{L}}{\partial \dot{A}_{\mu}} = -\frac{1}{c} F^{\mu 0}(x).$$

The antisymmetry of $F^{\mu\nu}$ then implies $\pi^{0}(x) \equiv 0$, and this is plainly incompatible with the canonical commutation relations (2.31) which we wish to impose.

A Lagrangian density which is suitable for quantization, first proposed by Fermi, is

$$\mathscr{L} = -\frac{1}{2} (\partial_{\nu} A_{\mu}(x))(\partial^{\nu} A^{\mu}(x)) - \frac{1}{c} s_{\mu}(x) A^{\mu}(x). \tag{5.10}$$

From Eq. (5.10) one obtains the conjugate fields

$$\pi^{\mu}(x) = \frac{\partial \mathscr{L}}{\partial \dot{A}_{\mu}} = -\frac{1}{c^2} \dot{A}^{\mu}(x) \tag{5.11}$$

which are now all non-vanishing so that the canonical quantization formalism can be applied.

The Lagrangian density (5.10) leads to the field equations

$$\Box A^{\mu}(x) = \frac{1}{c} s^{\mu}(x). \tag{5.12}$$

Comparison with Eqs. (5.6) shows that Eqs. (5.12) are only equivalent to Maxwell's equations if the potential $A^{\mu}(x)$ satisfies the constraint

$$\partial_{\mu} A^{\mu}(x) = 0. \tag{5.13}$$

Hence, to carry out the quantization, but end up with Maxwell's equations, we must in the first place quantize the theory for the general Lagrangian density (5.10), ignoring the constraint (5.13), and after quantization impose Eq. (5.13) or an equivalent constraint as a subsidiary condition. We shall consider this point in detail in the next section.

In the classical theory, starting from potentials $A^{\mu}(x)$ in an arbitrary gauge, we can always perform a gauge transformation (5.7) so that the transformed potentials $A'^{\mu}(x)$ satisfy the subsidiary condition (5.13). We achieve this by choosing the function $f(x)$ in Eq. (5.7) as a solution of

$$\partial_{\mu} A^{\mu}(x) + \Box f(x) = 0. \tag{5.14}$$

The subsidiary condition (5.13) does not specify the potentials uniquely. If the potentials $A^{\mu}(x)$ satisfy Eq. (5.13), so will any potentials $A'^{\mu}(x)$ obtained by the gauge transformation (5.7), provided the gauge function $f(x)$ satisfies

$$\Box f(x) = 0. \tag{5.15}$$

The subsidiary condition (5.13) is called the Lorentz condition. Its imposition represents a restriction on the choice of gauge. Any gauge in which Eq. (5.13) holds is called a Lorentz guage.

Using a Lorentz gauge has some important advantages. Firstly, the Lorentz condition (5.13) is a Lorentz-covariant constraint. This is in contrast to the condition for the Coulomb gauge, Eq. (1.6),

$$\nabla \cdot \mathbf{A} = 0,$$

which decomposes fields into transverse and longitudinal components and so is manifestly frame dependent. Secondly, the field equations (5.12) in a Lorentz gauge are much simpler than the corresponding Eqs. (5.6) in a general gauge. In particular, in the free field case ($s^\mu(x) = 0$) Eqs. (5.12) reduce to

$$\Box A^\mu(x) = 0. \tag{5.16}$$

Eq. (5.16) is the limit of the Klein–Gordon equation (3.3) for particles with mass zero. This will enable us to adapt many of our earlier results when considering the covariant quantization of the electromagnetic field.

Eq. (5.16) enables us to expand the free electromagnetic field $A^\mu(x)$ in a complete set of solutions of the wave equation, in close analogy to the expansion (3.7) for the Klein–Gordon field:

$$A^\mu(x) = A^{\mu+}(x) + A^{\mu-}(x) \tag{5.16a}$$

where

$$A^{\mu+}(x) = \sum_{r\mathbf{k}} \left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}} \right)^{1/2} \varepsilon_r^\mu(\mathbf{k}) a_r(\mathbf{k})\, \mathrm{e}^{-ikx} \tag{5.16b}$$

and

$$A^{\mu-}(x) = \sum_{r\mathbf{k}} \left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}} \right)^{1/2} \varepsilon_r^\mu(\mathbf{k}) a_r^\dagger(\mathbf{k})\, \mathrm{e}^{ikx}. \tag{5.16c}$$

The summations in these equations are over wave vectors $\mathbf{k}$ allowed by the periodic boundary conditions, and

$$k^0 = \frac{1}{c}\omega_{\mathbf{k}} = |\mathbf{k}|. \tag{5.17}$$

The summation over r, from $r = 0$ to $r = 3$, corresponds to the fact that for the four-vector field $A^\mu(x)$ there exist, for each $\mathbf{k}$, four linearly independent polarization states. These are described by the polarization vectors $\varepsilon_r^\mu(\mathbf{k})$, $r = 0, \ldots, 3$, which we choose to be real, and which satisfy the orthonormality and completeness relations

$$\varepsilon_r(\mathbf{k})\varepsilon_s(\mathbf{k}) = \varepsilon_{r\mu}(\mathbf{k})\varepsilon_s^\mu(\mathbf{k}) = -\zeta_r\, \delta_{rs}, \quad r, s = 0, \ldots, 3, \tag{5.18}$$

$$\sum_r \zeta_r \varepsilon_r^\mu(\mathbf{k})\varepsilon_r^\nu(\mathbf{k}) = -g^{\mu\nu}, \tag{5.19}$$

where

$$\zeta_0 = -1, \quad \zeta_1 = \zeta_2 = \zeta_3 = 1. \tag{5.20}$$

The classical potentials $A^\mu(x)$, $\mu = 0, \ldots 3$, are of course real quantities. Anticipating their interpretation in the quantized theory as operators, we have denoted the expansion coefficients in Eqs. (5.16) by a_r and $a_r^\dagger$.

Eqs. (5.16) should be compared with Eqs. (1.38). The latter expand the radiation field in terms of two transverse polarization states for each value of $\mathbf{k}$ and, in addition, we had the instantaneous Coulomb interaction between charges. Eqs. (5.16) give an expansion of the *total* field $A^\mu(x)$ in terms of four polarization states for each value of $\mathbf{k}$. We shall see in Section 5.3 that the two extra polarization states provide a covariant description of the instantaneous Coulomb interaction.

For many purposes one only requires the properties (5.18) and (5.19) of the polarization vectors. However a specific choice of polarization vectors in one given frame of reference often facilitates the interpretation. We shall choose these vectors as

$$\varepsilon_0^\mu(\mathbf{k}) = n^\mu \equiv (1,\ 0,\ 0,\ 0), \tag{5.21a}$$

$$\varepsilon_r^\mu(\mathbf{k}) = (0,\ \boldsymbol{\varepsilon}_r(\mathbf{k})), \quad r = 1,\ 2,\ 3, \tag{5.21b}$$

where $\boldsymbol{\varepsilon}_1(\mathbf{k})$ and $\boldsymbol{\varepsilon}_2(\mathbf{k})$ are mutually orthogonal unit vectors, which are also orthogonal to $\mathbf{k}$, and

$$\boldsymbol{\varepsilon}_3(\mathbf{k}) = \mathbf{k}/|\mathbf{k}|, \tag{5.22a}$$

i.e.

$$\mathbf{k} \cdot \boldsymbol{\varepsilon}_r(\mathbf{k}) = 0, \quad r = 1,\ 2; \quad \boldsymbol{\varepsilon}_r(\mathbf{k}) \cdot \boldsymbol{\varepsilon}_s(\mathbf{k}) = \delta_{rs}, \quad r, s = 1,\ 2,\ 3. \tag{5.22b}$$

ε_1^μ and ε_2^μ are called transverse, ε_3^μ longitudinal polarizations, and ε_0^μ scalar or time-like polarization.

For later use we note that $\varepsilon_3^\mu(\mathbf{k})$ can be written in the covariant form

$$\varepsilon_3^\mu(\mathbf{k}) = \frac{k^\mu - (kn)n^\mu}{[(kn)^2 - k^2]^{1/2}}. \tag{5.22c}$$

This expression comes about since $(kn)n^\mu$ subtracts off the time-like component of k^μ, and the denominator makes ε_3^μ a space-like unit vector. We have not set $k^2 = 0$ in Eq. (5.22c), as it would be for a real photon, since we shall later require the more general case $k^2 \neq 0$.

Real polarization vectors correspond to linear polarization. To describe circular or elliptic polarization would require complex polarization vectors and corresponding modifications of Eqs. (5.18) and (5.19).

5.2 Covariant Quantization

We now apply the canonical formalism of Chapter 2 to quantize the free electromagnetic field, using the Lagrangian density (5.10) with $s_\mu(x) = 0$ and, in the first place, ignoring the Lorentz condition (5.13). With the fields $\pi^\mu(x)$ conjugate to $A_\mu(x)$ given by Eqs. (5.11), the equal-time commutation relations (2.31) become

$$\left.\begin{aligned} \left[A^\mu(\mathbf{x},\ t),\ A^\nu(\mathbf{x}',\ t)\right] = 0, \quad \left[\dot{A}^\mu(\mathbf{x},\ t),\ \dot{A}^\nu(\mathbf{x}',\ t)\right] = 0, \\ \left[A^\mu(\mathbf{x},\ t),\ \dot{A}^\nu(\mathbf{x}',\ t)\right] = -\mathrm{i}\hbar c^2 g^{\mu\nu}\,\delta(\mathbf{x} - \mathbf{x}') \end{aligned}\right\} \tag{5.23}$$

Apart from the factor $(-g^{\mu\nu})$, these equations are identical with the commutation relations (3.6) of four independent Klein–Gordon fields, and each component $A^{\mu}(x)$ satisfies the wave equation (5.16) which is the limit of the Klein–Gordon equation (3.3) for particles of mass zero. [Both these points can be appreciated by comparing the Lagrangian densities (5.10) and (3.4).] This similarity enables us to take over earlier mathematical results, although their physical interpretation will have to be re-examined, taking into account the factor $(-g^{\mu\nu})$.

In Section 3.3, we derived the covariant commutation relations (3.42) for the Klein–Gordon field. From these we can at once write down the covariant commutation relations for the $A^{\mu}(x)$:

$$\left[A^{\mu}(x),\, A^{\nu}(x')\right] = \mathrm{i}\hbar c D^{\mu\nu}(x - x'), \tag{5.24}$$

where

$$D^{\mu\nu}(x) = \lim_{m\to 0}\ [-g^{\mu\nu}\Delta(x)], \tag{5.25}$$

and $\Delta(x)$ is the invariant Δ-function (3.43).

The Feynman photon propagator is similarly given by

$$\langle 0|\mathbf{T}\left\{A^{\mu}(x)A^{\nu}(x')\right\}|0\rangle = \mathrm{i}\hbar c D_{\mathrm{F}}^{\mu\nu}(x - x'), \tag{5.26}$$

where

$$D_{\mathrm{F}}^{\mu\nu}(x) = \lim_{m\to 0}\,[-g^{\mu\nu}\,\Delta_{\mathrm{F}}(x)] = \frac{-g^{\mu\nu}}{(2\pi)^4}\int \frac{\mathrm{d}^4k\,\mathrm{e}^{-\mathrm{i}kx}}{k^2 + i\varepsilon}, \tag{5.27}$$

as is seen from Eqs. (3.55) and (3.58). The photon propagator will be discussed fully in the next section.

To gain the photon interpretation of the quantized fields, we substitute the field expansions (5.16) in the commutation relations (5.23), with the result

$$\left.\begin{aligned}\left[a_r(\mathbf{k}),\, a_s^{\dagger}(\mathbf{k}')\right] &= \zeta_r\,\delta_{rs}\,\delta_{\mathbf{k}\mathbf{k}'}\\ \left[a_r(\mathbf{k}),\, a_s(\mathbf{k}')\right] &= \left[a_r^{\dagger}(\mathbf{k}),\, a_s^{\dagger}(\mathbf{k}')\right] = 0\end{aligned}\right\}. \tag{5.28}$$

From Eq. (5.20) $\zeta_r = 1$ for $r = 1, 2, 3$ so that for these values of r, Eqs. (5.28) are the standard boson commutation relations (3.9) leading to the usual number representation for transverse photons ($r = 1, 2$) and longitudinal photons ($r = 3$). For $r = 0$ (scalar photons) $\zeta_0 = -1$, and it consequently looks as though the usual roles of absorption and creation operators must be interchanged for $a_0(\mathbf{k})$ and $a_0^{\dagger}(\mathbf{k})$. However, effecting only this change results in other difficulties, and the standard formalism must be modified more radically. Of the several procedures available, we shall follow that due to Gupta and to Bleuler.

In the Gupta–Bleuler theory, the operators $a_r(\mathbf{k}), r = 1, 2, 3$ *and* 0, are interpreted as absorption operators, $a_r^{\dagger}(\mathbf{k}), r = 1, 2, 3$ *and* 0, as creation operators for transverse, longitudinal *and* scalar photons. The vacuum state $|0\rangle$ is defined as the state in which there are no photons of any kind present, i.e.

$$a_r(\mathbf{k})|0\rangle = 0, \quad \text{all } \mathbf{k}, \quad r = 0\ \ldots, 3, \tag{5.29a}$$

or, equivalently,

$$A^{\mu+}(x)|0\rangle = 0, \quad \text{all } x, \quad \mu = 0, \ldots, 3. \tag{5.29b}$$

The operators $a_r^\dagger(\mathbf{k})$ operating on the vacuum state $|0\rangle$ create the one-photon states

$$|1_{\mathbf{k}r}\rangle = a_r^\dagger(\mathbf{k})|0\rangle \tag{5.30}$$

in which one transverse ($r = 1, 2$), longitudinal ($r = 3$) or scalar ($r = 0$) photon of momentum $\mathbf{k}$ is present.

To justify this interpretation of the operators a_r and $a_r^\dagger$, we consider the Hamiltonian operator of the field. From Eq. (2.51a) this is given by

$$H = \int \mathrm{d}^3\mathbf{x} \mathrm{N}\left[\pi^\mu(x)\dot{A}_\mu(x) - \mathscr{L}(x)\right], \tag{5.31}$$

which, as usual, is to be taken as a normal product. On substituting the free Lagrangian density corresponding to Eq. (5.10), Eq, (5.11) for $\pi^\mu(x)$ and the expansions (5.16) for the fields, Eq. (5.31) becomes

$$H = \sum_{r\mathbf{k}} \hbar\omega_{\mathbf{k}}\zeta_r a_r^\dagger(\mathbf{k})a_r(\mathbf{k}). \tag{5.32}$$

Despite the minus sign ($\zeta_0 = -1$) associated with the scalar photons in Eq. (5.32), this energy is positive definite. For example, for the one-photon states (5.30) one easily obtains, using the commutation relations (5.28),

$$\begin{aligned} H|1_{\mathbf{k}r}\rangle &= \sum_{\mathbf{q}s} \hbar\omega_{\mathbf{q}}\zeta_s a_s^\dagger(\mathbf{q})a_s(\mathbf{q})a_r^\dagger(\mathbf{k})|0\rangle \\ &= \hbar\omega_{\mathbf{k}} a_r^\dagger(\mathbf{k})|0\rangle, \quad r = 0, \ldots, 3, \end{aligned}$$

i.e. the energy has the positive value $\hbar\omega_{\mathbf{k}}$ for transverse, longitudinal and scalar photons. Correspondingly, we must define the number operators by

$$N_r(\mathbf{k}) = \zeta_r a_r^\dagger(\mathbf{k})a_r(\mathbf{k}), \tag{5.33}$$

and these definitions, together with the commutation relations (5.28), lead to consistent number representations for all types of photons.

Although the formalism, as far as we have developed it, seems satisfactory, there are some difficulties which show up if we calculate the normalization of photon states. For example, the norm of the state (5.30) is

$$\langle 1_{\mathbf{k}r}|1_{\mathbf{k}r}\rangle = \langle 0|a_r(\mathbf{k})a_r^\dagger(\mathbf{k})|0\rangle = \zeta_r\langle 0|0\rangle = \zeta_r$$

(if we normalize $|0\rangle$ to $\langle 0|0\rangle = 1$), and for a scalar photon this norm is negative. More generally, one can show that for any state containing an odd number of scalar photons the norm is negative. At first sight this looks like a serious difficulty, since the probability interpretation of quantum mechanics depends on states having positive norms. However, no scalar or longitudinal photons have ever been observed. Both these points are related to the fact that so far we have ignored the Lorentz condition (5.13), so that our theory is not yet equivalent to Maxwell's equations. We must now try and impose the Lorentz condition.

Unfortunately, we cannot simply take the Lorentz condition (5.13) as an operator identity. Eq. (5.13) is incompatible with the commutation relations (5.24), since

$$\left[\partial_\mu A^\mu(x), A^\nu(x')\right] = i\hbar c \partial_\mu D^{\mu\nu}(x - x')$$

and this is not identically zero.

This problem was resolved by Gupta and Bleuler by replacing the Lorentz condition (5.13) by the weaker condition

$$\partial_\mu A^{\mu +}(x)|\Psi\rangle = 0, \tag{5.34}$$

involving absorption operators only. Eq. (5.34) is a restriction on the states which are allowed by the theory. From Eq. (5.34) and its adjoint

$$\langle\Psi|\partial_\mu A^{\mu -}(x) = 0$$

it follows that the Lorentz condition holds for expectation values:

$$\langle\Psi|\partial_\mu A^\mu(x)|\Psi\rangle = \langle\Psi|\partial_\mu A^{\mu +}(x) + \partial_\mu A^{\mu -}(x)|\Psi\rangle = 0. \tag{5.35}$$

This ensures that the Lorentz condition and hence Maxwell's equations hold as the classical limit of this theory.

In order to understand the meaning of the subsidiary condition (5.34), we express it in momentum space. On substituting Eqs. (5.16b) and (5.21), (5.22) for $A_\mu^+(x)$ and $\varepsilon_r^\mu(\mathbf{k})$, we obtain the conditions

$$[a_3(\mathbf{k}) - a_0(\mathbf{k})]|\Psi\rangle = 0, \quad \text{all } \mathbf{k}. \tag{5.36}$$

This is a constraint on the linear combinations of longitudinal and scalar photons, for each value of $\mathbf{k}$, that may be present in a state. It places no restriction on the transverse photons that may be present.

The effect of the subsidiary condition (5.36) becomes apparent if we calculate the expectation value of the energy of an allowed state $|\Psi\rangle$. Since from Eq. (5.36) and its adjoint we have

$$\langle\Psi|a_3^\dagger(\mathbf{k})a_3(\mathbf{k}) - a_0^\dagger(\mathbf{k})a_0(\mathbf{k})|\Psi\rangle = \langle\Psi|a_3^\dagger(\mathbf{k})[a_3(\mathbf{k}) - a_0(\mathbf{k})]|\Psi\rangle = 0,$$

it follows from Eq. (5.32) that

$$\langle\Psi|H|\Psi\rangle = \langle\Psi|\sum_{\mathbf{k}}\sum_{r=1}^{2}\hbar\omega_k a_r^\dagger(\mathbf{k})a_r(\mathbf{k})|\Psi\rangle, \tag{5.37}$$

i.e. only the transverse photons contribute to the expectation value of the energy as a consequence of the subsidiary condition. The same is true for all other observables.

Thus, as a result of the subsidiary condition, in free space, observable quantities will involve transverse photons only. This explains our earlier assertion that longitudinal and scalar photons are not observed as free particles. Only transverse photons are so observed, corresponding to the two degrees of freedom (for each $\mathbf{k}$) of the radiation field, which we found in the non-covariant formalism of Chapter 1 where we worked in the Coulomb gauge. In the covariant treatment, although they don't show up as free particles, the presence of longitudinal and scalar photons is not ruled out

altogether. Of the resulting additional two degrees of freedom (for each **k**), one is removed by the subsidiary condition (5.36). The other can be shown to correspond to the arbitrariness in choice of Lorentz gauge. More specifically, one can show that altering the allowed admixtures of longitudinal and scalar photons is equivalent to a gauge transformation between two potentials, both of which are in Lorentz gauges. (See Problems 5.2 and 5.3.)

For free fields (i.e. no charges present), it is then simplest to work in a gauge such that the vacuum is represented by the state $|0\rangle$ in which no photons of any kind are present [see Eq. (5.29a)]. But the vacuum could also be described by any state containing no transverse and only allowed admixtures of scalar and longitudinal photons. This description would merely correspond to a different choice of Lorentz gauge. The situation is entirely analogous for states containing transverse photons.

For the electromagnetic field in the presence of charges, the situation is more complicated. We can no longer ignore the longitudinal and scalar photons. When discussing the photon propagator in the next section, we shall see that longitudinal and scalar photons play an important role as virtual particles in intermediate states and provide a covariant description of the instantaneous Coulomb interaction of Chapter 1. However, in this case too, one need consider only transverse photons in initial and final states of scattering processes. This corresponds to a particular choice of gauge and the fact that one can consider particles initially and finally, when they are far apart, as free. In Section 6.2 we shall return to this idea of switching the interaction between colliding particles on and off adiabatically as they approach and as they move apart.

We have developed the Gupta–Bleuler formalism only to the limited extent to which it is needed in applications. It is possible to develop a more complete systematic formalism in which states with negative norm do not appear as a blemish in Hilbert space, but occur in a self-consistent manner in a function space with an indefinite metric. For most purposes, this complete formalism is not required.[2]

5.3 The Photon Propagator

In Section 3.4 we interpreted the Klein–Gordon propagator (3.56) as the exchange of a virtual meson in an intermediate state. We now expect a similar interpretation for the photon propagator (5.26) but, corresponding to the four-vector nature of the field $A^{\mu}(x)$ and the resulting four independent polarization states, we expect the exchange of four kinds of photons, two corresponding to transverse polarization and one each to longitudinal and scalar polarization. This description differs markedly from that of Chapter 1, where only transverse radiation occurred, but no longitudinal or scalar radiation. Instead we had the instantaneous Coulomb interaction between charges. We shall see that these two descriptions are indeed equivalent.

[2] The interested reader is referred to S. N Gupta, *Quantum Electrodynamics*, Gordon and Breach, New York, 1977; G. Källén, *Quantum Electrodynamics*, Springer, New York, 1972, and Allen & Unwin, London, 1972; or J.M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons*, 2nd edn, Springer, New York, 1976, Section 6.3.

To establish this interpretation in terms of photon exchange we consider the momentum space propagator $D_F^{\mu\nu}(k)$, related to the configuration space propagator $D_F^{\mu\nu}(x)$ Eq. (5.27), by

$$D_F^{\mu\nu}(x) = \frac{1}{(2\pi)^4} \int d^4k D_F^{\mu\nu}(k)\, e^{-ikx}. \tag{5.38}$$

From Eqs. (5.27) and (5.19) we obtain

$$D_F^{\mu\nu}(k) = \frac{-g^{\mu\nu}}{k^2 + i\varepsilon} = \frac{1}{k^2 + i\varepsilon} \sum_r \zeta_r \varepsilon_r^\mu(\mathbf{k}) \varepsilon_r^\nu(\mathbf{k}). \tag{5.39}$$

In order to interpret this expression, we use the special frame of reference in which the polarization vectors $\varepsilon_r^\mu(\mathbf{k})$ are given by Eqs. (5.21) and (5.22). The last equation then becomes:

$$D_F^{\mu\nu}(k) = \frac{1}{k^2 + i\varepsilon} \left\{ \sum_{r=1}^{2} \varepsilon_r^\mu(\mathbf{k}) \varepsilon_r^\nu(\mathbf{k}) + \frac{[k^\mu - (kn)n^\mu][k^\nu - (kn)n^\nu]}{(kn)^2 - k^2} + (-1) n^\mu n^\nu \right\}. \tag{5.40}$$

This equation exhibits the contributions to the photon propagator from transverse, longitudinal and scalar photons.

By analogy with the meson case, we interpret the first term in Eq. (5.40),

$$_T D_F^{\mu\nu}(k) \equiv \frac{1}{k^2 + i\varepsilon} \sum_{r=1}^{2} \varepsilon_r^\mu(\mathbf{k}) \varepsilon_r^\nu(\mathbf{k}), \tag{5.41a}$$

as the exchange of transverse photons. In the language of Chapter 1, it corresponds to the interaction of charges via the transverse radiation field.

The interpretation of the remaining two terms in Eq. (5.40) follows not from considering longitudinal and scalar photons separately, but from combining them into a term proportional to $n^\mu n^\nu$ plus the remainder. Eq. (5.40) then becomes

$$D_F^{\mu\nu}(k) = {}_T D_F^{\mu\nu}(k) + {}_C D_F^{\mu\nu}(k) + {}_R D_F^{\mu\nu}(k), \tag{5.42}$$

where

$$_C D_F^{\mu\nu}(k) \equiv \frac{n^\mu n^\nu}{(kn)^2 - k^2}, \tag{5.41b}$$

$$_R D_F^{\mu\nu}(k) \equiv \frac{1}{k^2 + i\varepsilon} \left[\frac{k^\mu k^\nu - (kn)(k^\mu n^\nu + k^\nu n^\mu)}{(kn)^2 - k^2} \right], \tag{5.41c}$$

and it is these linear combinations, both of which involve longitudinal and scalar photons, which allow a simple interpretation.

We first consider Eq. (5.41b) in configuration space. From Eqs. (5.38) and (5.21a) we obtain

$$
\begin{aligned}
{}_{\mathrm{C}}D_{\mathrm{F}}^{\mu\nu}(x) &= \frac{g^{\mu 0} g^{\nu 0}}{(2\pi)^4} \int \frac{\mathrm{d}^3\mathbf{k}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}}{|\mathbf{k}|^2} \int \mathrm{d}k^0\, \mathrm{e}^{-\mathrm{i}k^0 x^0} \\
&= g^{\mu 0}\, g^{\nu 0} \frac{1}{4\pi|\mathbf{x}|}\, \delta(x^0).
\end{aligned}
\tag{5.43}
$$

This expression has the time dependence $[\delta(x^0)]$ and the space dependence $[1/|\mathbf{x}|]$ characteristic of an instantaneous Coulomb potential. Thus we see that the exchange of longitudinal and scalar photons, represented by the term (5.43), corresponds to the instantaneous Coulomb interaction between charges. In Chapter 1, we quantized the transverse radiation field only and treated the instantaneous Coulomb interaction as a classical potential, corresponding to the fact that the instantaneous Coulomb field does not represent independent dynamical degrees of freedom, but is fully determined by the charges. In the present treatment, the longitudinal and scalar field components are also quantized and the instantaneous Coulomb interaction emerges as an exchange of longitudinal and scalar photons.

Finally, we must discuss the remainder term (5.41c). In Chapter 1, the complete electromagnetic interaction between charges was represented in terms of the interactions via the transverse radiation field and the instantaneous Coulomb fields. Both these have been accounted for in the present treatment, and for the two treatments to be equivalent, the contribution of the remainder term (5.41c) to all observable quantities must vanish. This is indeed the case, the basic reason being that the electromagnetic field only interacts with the conserved charge–current density $s^\mu(x)$, Eqs. (5.2) and (5.4). We shall illustrate this for a simple example.

We shall see in Section 7.1, Eq. (7.14), that the scattering of charges by each other is, in lowest order of perturbation theory, given by the matrix element of the operator

$$
\int \mathrm{d}^4x \int \mathrm{d}^4y s_1^\mu(x) D_{\mathrm{F}\mu\nu}(x-y) s_2^\nu(y). \tag{5.44}
$$

Here $s_1^\mu(x)$ and $s_2^\mu(y)$ are the two interacting charge–current densities. It is clear from Eq. (5.43) that the contribution of ${}_{\mathrm{C}}D_{\mathrm{F}\mu\nu}(x-y)$ to (5.44) corresponds to the instantaneous Coulomb interaction between the charge densities $\rho_1(\mathbf{x}, x^0) = s_1^0(\mathbf{x}, x^0)/c$ and $\rho_2(\mathbf{y}, x^0) = s_2^0(\mathbf{y}, x^0)/c$. Similarly, the transverse propagator ${}_{\mathrm{T}}D_{\mathrm{F}\mu\nu}(x-y)$ accounts for the electromagnetic interaction between the current densities $\mathbf{j}_1(x) = \mathbf{s}_1(x)$ and $\mathbf{j}_2(y) = \mathbf{s}_2(y)$.

The contribution to (5.44) of the remainder term ${}_{\mathrm{R}}D_{\mathrm{F}\mu\nu}(x-y)$ is easily shown to vanish, on account of current conservation. Transforming this contribution to expression (5.44) into momentum space, one obtains

$$
\frac{1}{(2\pi)^4} \int \mathrm{d}^4k s_1^\mu(-k)\, {}_{\mathrm{R}}D_{\mathrm{F}\mu\nu}(k) s_2^\nu(k), \tag{5.45}
$$

where the momentum transforms $s_r^\mu(k)$, $r = 1, 2$, are defined, analogously to Eq. (5.38), by

$$
s_r^\mu(x) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4k s_r^\mu(k)\, \mathrm{e}^{-\mathrm{i}kx}, \quad r = 1, 2. \tag{5.46}
$$

The current conservation equations, $\partial_\mu s_r^\mu(x) = 0$, translated into momentum space, become

$$k_\mu s_r^\mu(k) = 0, \quad r = 1, 2. \tag{5.47}$$

We see from the explicit form (5.41c) that each term in ${}_{\mathrm{R}}D_{\mathrm{F}\mu\nu}(k)$ is proportional to either k_μ or k_ν or both. Hence it follows from Eq. (5.47) that the expression (5.45) vanishes.

This completes our discussion of the equivalence of the two formulations of quantum electrodynamics. In doing this, we employed a special frame of reference leading to a division of the fields into transverse, longitudinal and scalar parts. In general, such a division is not required, and we shall work with manifestly covariant expressions involving summations over all four polarization states. In particular, the photon propagator, Eqs. (5.38) and (5.39), which will be very important in the development of quantum electrodynamics, has this property.

Problems

5.1. Show that the Lagrangian density obtained from

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x)$$

by adding the term $-\frac{1}{2}(\partial_\mu A^\mu(x))(\partial_\nu A^\nu(x))$, i.e.

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) - \frac{1}{2}(\partial_\mu A^\mu(x))(\partial_\nu A^\nu(x)),$$

is equivalent to the Lagrangian density, proposed by Fermi:

$$\mathscr{L} = -\frac{1}{2}(\partial_\nu A_\mu(x))(\partial^\nu A^\mu(x)).$$

5.2. From the commutation relations (5.28) show that

$$\left[a_3(\mathbf{k}) - a_0(\mathbf{k}),\ a_3^\dagger(\mathbf{k}) - a_0^\dagger(\mathbf{k})\right] = 0.$$

Show that the most general state representing the physical vacuum, i.e. the state in which there are no transverse photons present, but which contains the most general allowed admixture of scalar and longitudinal photons, is given by

$$|\Psi_{\mathrm{SL}}\rangle = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \ldots c(n_1, n_2, \ldots) \prod_{i=1}^{\infty} (\alpha_i^\dagger)^{n_i} |0\rangle$$

where

$$\alpha_i^\dagger \equiv a_3^\dagger(\mathbf{k}_i) - a_0^\dagger(\mathbf{k}_i),$$

$\mathbf{k}_i$ are the allowed wave vectors [see Eq. (1.13)], and $|0\rangle$ is the vacuum state in which there are no photons of any kind present. Show that the norm of this state is given by

$$\langle \Psi_{\mathrm{SL}} | \Psi_{\mathrm{SL}} \rangle = |c(0, 0, \ldots)|^2.$$

What is the most general state in which there are a definite number of transverse photons, with definite momenta and polarization vectors, present?

5.3. $|\Psi_T\rangle$ is a state which contains transverse photons only. Let

$$|\Psi'_{\mathrm{T}}\rangle = \left\{1 + a\left[a_3^\dagger(\mathbf{k}) - a_0^\dagger(\mathbf{k})\right]\right\}|\Psi_{\mathrm{T}}\rangle,$$

where a is a constant. Show that replacing $|\Psi_{\mathrm{T}}\rangle$ by $|\Psi'_{\mathrm{T}}\rangle$ corresponds to a gauge transformation, i.e.

$$\langle\Psi'_{\mathrm{T}}|A^\mu(x)|\Psi'_{\mathrm{T}}\rangle = \langle\Psi_{\mathrm{T}}|A^\mu(x) + \partial^\mu\Lambda(x)|\Psi_{\mathrm{T}}\rangle,$$

where

$$\Lambda(x) = \left(\frac{2\hbar c^2}{V\omega_{\mathbf{k}}^3}\right)^{1/2} \mathrm{Re}\left(ia\,\mathrm{e}^{-ikx}\right).$$

5.4. By making the minimal substitution

$$\partial_\alpha\phi(x) \to D_\alpha\phi(x) = \left[\partial_\alpha + \frac{ie}{\hbar c}A_\alpha(x)\right]\phi(x)$$

$$\partial_\alpha\phi^\dagger(x) \to [D_\alpha\phi(x)]^\dagger = \left[\partial_\alpha - \frac{ie}{\hbar c}A_\alpha(x)\right]\phi^\dagger(x)$$

in the Lagrangian density (3.22) of the complex Klein–Gordon field $\phi(x)$, derive the Lagrangian density $\mathscr{L}_{\mathrm{I}}(x)$ for the interaction of the charged bosons, described by the field $\phi(x)$, with the electromagnetic field $A^\alpha(x)$.

Assuming that this interaction is invariant under the charge conjugation transformation $\mathscr{C}$, show that

$$\mathscr{C}A^\alpha(x)\mathscr{C}^{-1} = -A^\alpha(x).$$

[The transformation properties of $\phi(x)$ and of $s^\alpha(x)$, Eq. (3.32), under charge conjugation were discussed in Problem 3.5.]

Hence show that a single-photon state $|\mathbf{k}, r\rangle$ is an eigenstate of $\mathscr{C}$ with eigenvalue -1.

6

The *S*-Matrix Expansion

We shall now progress from the discussion of the free fields to the realistic and much more interesting case of fields in interaction, in which particles can be scattered, created and destroyed. In essence, this requires solving the coupled non-linear field equations for given conditions. In quantum electrodynamics, for example, one must solve the inhomogeneous wave equation (5.12) with the Dirac current density (4.28) as source term. This is an extremely difficult problem, which has only been solved in perturbation theory, i.e. the Hamiltonian of the system is divided into that of the free fields plus an interaction term. The latter is treated as a perturbation, which is justifiable if the interaction is sufficiently weak. For quantum electrodynamics, where the coupling of photons and electrons is measured by the small dimensionless fine structure constant $\alpha \approx 1/137$, this approach is outstandingly successful, not only in calculating processes in lowest order of perturbation theory, but also in calculating higher-order corrections.

In the Heisenberg picture, which we have so far been using, this programme is still very complex, and it was decisive for the successful development of the theory to work instead in the interaction picture. In Section 6.2 we shall study the equations of motion of the interacting fields in the interaction picture and we shall obtain a perturbation series solution suitable for collision processes. This solution, known as the *S*-matrix expansion, is due to Dyson. The Dyson expansion of the *S*-matrix is of great importance, since it contains the complete information about all collision processes in a form suitable for extracting the transition amplitude for a specific process to any order of perturbation theory. A systematic procedure for doing this will be developed in Section 6.3.

Before proceeding with these topics, we shall, in Section 6.1, introduce natural units, which considerably simplify details of the following calculations.

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6.1 Natural Dimensions and Units

We have so far used c.g.s. units, in which the fundamental dimensions, in terms of which quantities are expressed, are mass (M), length (L) and time (T). In relativistic quantum field theory, expressions and calculations are much simplified if one uses natural units (n.u.). In natural units one takes mass, action (A) and velocity (V) as fundamental dimensions and chooses $\hbar$ as unit of action and the velocity of light c as unit of velocity. Hence $\hbar = c = 1$ in natural units, and c.g.s. expressions are transformed into natural units by putting $\hbar = c = 1$. In such n.u. expressions, all quantities have the dimensions of a power of M. Since

$$L = \frac{A}{MV} \text{ and } T = \frac{A}{MV^2} \tag{6.1}$$

one has the general result that a quantity which has the c.g.s. dimensions

$$M^p L^q T^r = M^{p-q-r} A^{q+r} V^{-q-2r}, \tag{6.2}$$

has the n.u. dimensions M^{p-q-r}. In natural units, many quantities have the same dimension. For example, the momentum–energy relation for a particle of mass m becomes, in natural units,

$$E^2 = m^2 + \mathbf{p}^2 = m^2 + \mathbf{k}^2, \tag{6.3}$$

so that mass, momentum, energy and wave number all have the same natural dimension M. The c.g.s. expression for the dimensionless fine structure constant

$$\alpha = \frac{e^2}{4\pi\hbar c} = \frac{1}{137.04} \text{ (c.g.s.)} \tag{6.4a}$$

becomes

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137.04} \quad \text{(n.u.)} \tag{6.4b}$$

so that, in natural units, electric charge is dimensionless (M^0).

From the general relation (6.2), or by using particular equations, one easily derives the n.u. dimensions of all quantities, and some of the more important ones are listed in Table 6.1.

Working in natural units it is very easy to obtain numerical results in any system of units. A quantity in natural units will have a dimension M^n. To convert this quantity to whatever c.g.s. units are convenient, one merely multiplies it by such powers of $\hbar$ and c, expressed in the appropriate units, as to give it the correct c.g.s. dimensions. One frequently interprets M as an energy and measures it in MeV. The conversion factors

$$\hbar = 6.58 \times 10^{-22} \text{ MeV} \cdot \text{s} \tag{6.5a}$$

$$\hbar c = 1.973 \times 10^{-11} \text{ MeV} \cdot \text{cm} \tag{6.5b}$$

then enable one easily to express quantities in terms of MeV, centimetres and seconds. Two examples will illustrate this.

Table 6.1 *The c.g.s dimension $M^pL^qT^r$ and the n.u. dimensions $M^n = M^{p-q-r}$ of some quantities*

Quantity	c.g.s.			n.u.
	p	q	r	n
Action	1	2	−1	0
Velocity	0	1	−1	0
Mass	1	0	0	1
Length	0	1	0	−1
Time	0	0	1	−1
Lagrangian or Hamiltonian densities	1	−1	−2	4
Fine structure constant α	0	0	0	0
Electric charge	$\frac{1}{2}$	$\frac{3}{2}$	−1	0
Klein–Gordon field $\phi(x)^*$	$\frac{1}{2}$	$\frac{1}{2}$	−1	1
Electromagnetic field $A^\mu(x)^*$	$\frac{1}{2}$	$\frac{1}{2}$	−1	1
Dirac fields $\psi(x)$ and $\overline{\psi}(x)^*$	0	$-\frac{3}{2}$	0	$\frac{3}{2}$

* The dimensions of the fields can, for example, be obtained from the Lagrangian densities, Eqs. (3.4), (5.10) and (4.20)

The Thomson cross-section (1.72) becomes, in natural units,

$$\sigma = \frac{8\pi}{3}\frac{\alpha^2}{m^2}. \tag{6.6}$$

With $m = 0.511$ MeV, we convert the right-hand side of this equation to cm^2 by multiplying by $(\hbar c \text{ in MeV} \cdot \text{cm})^2$ which, from Eq. (6.5b), gives

$$\sigma = \frac{8\pi}{3}\alpha^2 \frac{(1.973 \times 10^{-11}\,\text{MeV}\cdot\text{cm})^2}{(0.511\,\text{MeV})^2} = 6.65 \times 10^{-25}\,\text{cm}^2.$$

Secondly, we quote[1] the n.u. expression for the lifetime τ of the positronium ground state $1\,^1S_0$. It is given by

$$\tau = \frac{2}{\alpha^5}\frac{1}{m}, \tag{6.7}$$

where m is the mass of the electron. With m in MeV, we must multiply Eq. (6.7) by ($\hbar$ in MeV·s), Eq. (6.5a), to obtain

$$\tau = \frac{2}{\alpha^5}\frac{(6.58 \times 10^{-22}\,\text{MeV}\cdot\text{s})}{(0.511\,\text{MeV})} = 1.24 \times 10^{-10}\,\text{s}.$$

This conversion factor is of course the same for converting any lifetime τ from natural units to seconds, the essential points being that in natural units τ has the dimension M^{-1} and must be expressed in $(\text{MeV})^{-1}$.

These examples illustrate how very easy it is to obtain numerical results in any c.g.s. units from equations expressed in natural units. No advantage is gained by tediously retaining

[1] See J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons*, 2nd edn, Springer, New York, 1976, p. 286, Eq. (12-108).

factors of $\hbar$ and c throughout a calculation or by converting a n.u. equation into c.g.s. form by inserting the appropriate factors of $\hbar$ and c prior to substituting numerical values.

Although rarely required, the c.g.s. form of an equation is easily obtained from its n.u. form. In a sum of terms, one must multiply each term by appropriate powers of $\hbar$ and c to make all the terms have the same c.g.s. dimensions. [E.g. a factor $(E + k)$, with E interpreted as an energy and k a wave number, could be turned into $(E + c\hbar k)$ or into $(E/c + \hbar k)$, etc.] To obtain the correct c.g.s. dimensions for the whole expression, it must be multiplied by a factor $\hbar^a c^b$ with the exponents a and b determined from dimensional arguments. Usually they are easily guessed.

From now on we shall in general work in natural units.

6.2 The *S*-Matrix Expansion

So far we have mainly considered the free, i.e. non-interacting, fields, using the Heisenberg picture (H.P.), in which state vectors are constant in time and the operators carry the full time dependence.

We now turn to the study of the interacting fields. For example, in quantum electrodynamics (QED), the interacting electron–positron and electromagnetic fields are described by the Lagrangian density

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1 \tag{6.8}$$

with the free-field Lagrangian density

$$\mathscr{L}_0 = \mathrm{N}[\bar{\psi}(x)(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi(x) - \frac{1}{2}(\partial_\nu A_\mu(x))(\partial^\nu A^\mu(x))] \tag{6.9}$$

and the interaction Lagrangian density

$$\mathscr{L}_1 = \mathrm{N}[-s^\mu(x)A_\mu(x)] = \mathrm{N}[e\bar{\psi}(x)\not{A}(x)\psi(x)] \tag{6.10}$$

[see Eqs. (4.66)–(4.68) and (5.10)]. In Eqs. (6.9) and (6.10) we have written the free-field and the interaction Lagrangian densities as normal products. This ensures, as for the free-field cases considered earlier, that the vacuum expectation values of all observables, e.g. energy or charge, vanish. Corresponding to the division (6.8), the complete Hamiltonian H of the system is split into the free-field Hamiltonian H_0 and the interaction Hamiltonian H_1:

$$H = H_0 + H_1. \tag{6.11}$$

As discussed at the beginning of this chapter, we shall employ the interaction picture (I.P.) which leads to two essential simplifications.[2]

Firstly, in the I.P., the operators satisfy the Heisenberg-like equations of motion (1.87), but involving the free Hamiltonian H_0 only, not the complete Hamiltonian H.

Secondly, if the interaction Lagrangian density $\mathscr{L}_1$ does not involve derivatives (and we shall restrict ourselves to this case until Chapter 19), the fields canonically conjugate to the

[2] The interaction picture, and its relation to the Heisenberg and Schrödinger pictures, is discussed in the appendix to Chapter 1 (Section 1.5). The reader who is not intimately familiar with this material is advised to study this appendix in depth at this stage.

interacting fields and to the free fields are identical. (For example in QED $\partial\mathscr{L}/\partial\dot{\psi}_\alpha = \partial\mathscr{L}_0/\partial\dot{\psi}_\alpha$, etc.) Since the I.P. and the H.P. are related by a unitary transformation, it follows that in the I.P., the interacting fields satisfy the same commutation relations as the free fields.

Thus in the I.P., the interacting fields satisfy the same equations of motion and the same commutation relations as the free-field operators. Consequently, we can take over the many results derived for free fields (in Chapters 3–5), as also true for the interacting fields in the I.P. In particular, the complete sets of plane wave states which we obtained continue to be solutions of the equations of motion, resulting in the same plane wave expansions of the field operators as before, the same number representations and the same explicit forms for the Feynman propagators.

In the I.P., the system is described by a time-dependent state vector $|\Phi(t)\rangle$. According to Eqs. (1.88) and (1.89), $|\Phi(t)\rangle$ satisfies the equation of motion

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\Phi(t)\rangle = H_{\mathrm{I}}(t)|\Phi(t)\rangle, \tag{6.12}$$

where

$$H_{\mathrm{I}}(t) = \mathrm{e}^{\mathrm{i}H_0(t-t_0)} H_{\mathrm{I}}^{\mathrm{S}} \mathrm{e}^{-\mathrm{i}H_0(t-t_0)} \tag{6.13}$$

is the interaction Hamiltonian in the I.P., with $H_{\mathrm{I}}^{\mathrm{S}}$ and $H_0 = H_0^{\mathrm{S}}$ being the interaction and free-field Hamiltonians in the Schrödinger picture (S.P.). $H_{\mathrm{I}}(t)$ is obtained by replacing, in $H_{\mathrm{I}}^{\mathrm{S}}$, the S.P. field operators by the time-dependent free-field operators. In Eqs. (6.12) and (6.13) we have omitted the labels I, used in Eqs. (1.88) and (1.89) to distinguish the I.P., as we shall be working exclusively in the I.P. in what follows.

Eq. (6.12) is a Schrödinger-like equation with the time-dependent Hamiltonian $H_{\mathrm{I}}(t)$. With the interaction 'switched off' (i.e. we put $H_{\mathrm{I}} \equiv 0$), the state vector is constant in time. The interaction leads to the state $|\Phi(t)\rangle$ changing with time. Given that the system is in a state $|i\rangle$ at an initial time $t = t_i$, i.e.

$$|\Phi(t_i)\rangle = |i\rangle, \tag{6.14}$$

the solution of Eq. (6.12) with this initial condition gives the state $|\Phi(t)\rangle$ of the system at any other time t. It follows from the Hermiticity of the operator $H_{\mathrm{I}}(t)$ that the time development of the state $|\Phi(t)\rangle$ according to Eq. (6.12) is a unitary transformation. Accordingly, it preserves the normalization of states,

$$\langle\Phi(t)|\Phi(t)\rangle = \text{const.}, \tag{6.15}$$

and, more generally, the scalar product.

Clearly the formalism which we are here developing is not appropriate for the description of bound states, but it is particularly suitable for scattering processes. In a collision process the state vector $|i\rangle$ will define an initial state, long before the scattering occurs ($t_i = -\infty$), by specifying a definite number of particles, with definite properties and far apart from each other so that they do not interact. (For example, in QED, $|i\rangle$ would specify a definite number of electrons, positrons and photons with given momenta, spins and polarizations.) In the scattering process, the particles will come close together, collide (i.e. interact) and fly apart again. Eq. (6.12) determines the state $|\Phi(\infty)\rangle$ into which the initial state

$$|\Phi(-\infty)\rangle = |i\rangle, \tag{6.14a}$$

evolves at $t = \infty$, long after the scattering is over and all particles are far apart again. The S-matrix relates $|\Phi(\infty)\rangle$ to $|\Phi(-\infty)\rangle$ and is defined by

$$|\Phi(\infty)\rangle = S|\Phi(-\infty)\rangle = S|i\rangle. \tag{6.16}$$

A collision can lead to many different final states $|f\rangle$, and all these possibilities are contained within $|\Phi(\infty)\rangle$. (For example, an electron–positron collision may result in elastic scattering, bremsstrahlung (i.e. emission of photons), pair annihilation, etc.) Each of these final states $|f\rangle$ is specified in a way analogous to $|i\rangle$.

The transition probability that after the collision (i.e. at $t = \infty$) the system is in the state $|f\rangle$ is given by

$$|\langle f|\Phi(\infty)\rangle|^2. \tag{6.17}$$

($|\Phi(\infty)\rangle$ and $|i\rangle$ are assumed normed to unity.) The corresponding probability amplitude is

$$\langle f|\Phi(\infty)\rangle = \langle f|S|i\rangle \equiv S_{fi}. \tag{6.18}$$

With the state $|\Phi(\infty)\rangle$ expanded in terms of a complete orthonormal set of states,

$$|\Phi(\infty)\rangle = \sum_f |f\rangle\langle f|\Phi(\infty)\rangle = \sum_f |f\rangle S_{fi}, \tag{6.19}$$

the unitarity of the S-matrix can be written

$$\sum_f |S_{fi}|^2 = 1. \tag{6.20}$$

Eq. (6.20) expresses the conservation of probability. It is more general than the corresponding conservation of particles in non-relativistic quantum mechanics, since now particles can be created or destroyed.

In order to calculate the S-matrix we must solve Eq. (6.12) for the initial condition (6.14a). These equations can be combined into the integral equation

$$|\Phi(t)\rangle = |i\rangle + (-\mathrm{i}) \int_{-\infty}^{t} \mathrm{d}t_1 H_{\mathrm{I}}(t_1)|\Phi(t_1)\rangle. \tag{6.21}$$

This equation can only be solved iteratively. The resulting perturbation solution, as a series in powers of H_{I}, will only be useful if the interaction energy H_{I} is small. This is the case for QED, where the dimensionless coupling constant characterizing the photon–electron interaction is the fine structure constant $\alpha \approx 1/137$.

Solving Eq. (6.21) by iteration

$$|\Phi(t)\rangle = |i\rangle + (-\mathrm{i}) \int_{-\infty}^{t} \mathrm{d}t_1 H_{\mathrm{I}}(t_1)|i\rangle + (-\mathrm{i})^2 \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 H_{\mathrm{I}}(t_1)H_{\mathrm{I}}(t_2)|\Phi(t_2)\rangle,$$

and so on, we obtain, in the limit $t \to \infty$, the S-matrix

$$S = \sum_{n=0}^{\infty} (-\mathrm{i})^n \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \ldots \int_{-\infty}^{t_{n-1}} \mathrm{d}t_n H_{\mathrm{I}}(t_1) H_{\mathrm{I}}(t_2) \ldots H_{\mathrm{I}}(t_n) \tag{6.22a}$$

$$= \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2 \ldots \int_{-\infty}^{\infty} \mathrm{d}t_n \mathrm{T}\{H_{\mathrm{I}}(t_1) H_{\mathrm{I}}(t_2) \ldots H_{\mathrm{I}}(t_n)\}. \tag{6.22b}$$

Here, the time-ordered product $\mathrm{T}\{\ldots\}$ of n factors is the natural generalization of the definitions (3.52) and (4.59) for two factors, i.e. the factors are ordered so that later times stand to the left of earlier times, and all boson (fermion) fields are treated as though their commutators (anticommutators) vanish. The equivalence of the two forms (6.22a) and (6.22b) only holds if H_{I} contains an even number of fermion factors (as in QED), so that the reordering process introduces no extra factors (-1). The equivalence of the two forms holds separately for each term of the series. Its verification is left as an exercise for the reader. Finally we rewrite Eq. (6.22b) in terms of the interaction Hamiltonian density $\mathscr{H}_{\mathrm{I}}(x)$ to obtain the explicitly covariant result

$$S = \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int \ldots \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 \ldots \mathrm{d}^4x_n \mathrm{T}\{\mathscr{H}_{\mathrm{I}}(x_1) \mathscr{H}_{\mathrm{I}}(x_2) \ldots \mathscr{H}_{\mathrm{I}}(x_n)\}, \tag{6.23}$$

the integrations being over all space–time. This equation is the Dyson expansion of the S-matrix. It forms the starting point for the approach to perturbation theory used in this book.

We have seen that the amplitude for a particular transition $|i\rangle \to |f\rangle$ is given by $\langle f|S|i\rangle$. To pick out from the expansion (6.23) the parts which contribute to this matrix element is a complex problem to which we shall return in the next section, but we must first discuss the specification of the initial and final states $|i\rangle$ and $|f\rangle$.

In the above perturbation formalism the states $|i\rangle$ and $|f\rangle$ are, as usual, eigenstates of the unperturbed free-field Hamiltonian H_0, i.e. with the interaction switched off ($H_{\mathrm{I}} = 0$). This description appears wrong, since the particles we are dealing with are real physical particles, even when far apart. An electron, even when far away from other electrons, is surrounded by its photon cloud; it is a real electron, not a bare electron without its own electromagnetic field. Hence, the use of bare particle states $|i\rangle$ and $|f\rangle$ requires justification. One possible procedure is to appeal to the adiabatic hypothesis in which the interaction $H_{\mathrm{I}}(t)$ is replaced by $H_{\mathrm{I}}(t) f(t)$. The function $f(t)$ is chosen so that $f(t) = 1$ for a sufficiently long interval $-T \leq t \leq T$, and $f(t) \to 0$ monotonically as $t \to \pm\infty$. [In QED, for example, we could replace the elementary charge e by the time-dependent coupling constant $ef(t)$.] In this way the initial and final states are described by bare particles. During the interval $-\infty < t \leq -T$, the equation of motion (6.12), with $H_{\mathrm{I}}(t)$ replaced by $H_{\mathrm{I}}(t)f(t)$, generates the real physical particles from the bare particles, and during the interval $|t| \leq T$ we are dealing with the physical particles and the full interaction $H_{\mathrm{I}}(t)$. In particular, the full interaction is effective during the interval $-\tau \leq t \leq \tau$, while the particles are sufficiently close together to interact (i.e. we must choose $T \gg \tau$). The essence of the adiabatic hypothesis is that the scattering, which occurs during the interval

$|t| \leq \tau$, cannot depend on our description of the system a long time before the scattering ($t << -\tau$) or a long time after the scattering ($t >> \tau$). Only at the end of a calculation do we take the limit $T \to \infty$. Of course, if we calculate a process in lowest-order perturbation theory [i.e. we use only the term of lowest order n in Eq. (6.23) which gives a non-vanishing result] then the interaction is exclusively used to cause the transition and not also to convert bare into real particles. We may then take the limit $T \to \infty$ from the start of the calculation and work with the full interaction $H_I(t)$.

6.3 Wick's Theorem

We must now see how to obtain from the S-matrix expansion (6.23) the transition amplitude $\langle f|S|i\rangle$ for a particular transition $|i\rangle \to |f\rangle$ in a given order of perturbation theory. The Hamiltonian density $\mathscr{H}_I(x)$ in Eq. (6.23) involves the interacting fields, each linear in creation and absorption operators. Hence the expansion (6.23) will describe a large number of different processes. However, only certain terms of the S-matrix will contribute to a given transition $|i\rangle \to |f\rangle$. For these terms must contain just the right absorption operators to destroy the particles present in $|i\rangle$, and they must contain the right creation operators to emit the particles present in $|f\rangle$. They may also contain additional creation and absorption operators, which create particles which are subsequently re-absorbed. These particles are only present in intermediate states and are called virtual particles.

Calculations can be greatly simplified by avoiding the explicit introduction of virtual intermediate particles. This can be achieved by writing the S-matrix expansion as a sum of normal products, since, in a normal product, *all* absorption operators stand to the *right* of *all* creation operators. Such an operator first absorbs a certain number of particles and then emits some particles. It does not cause emission and re-absorption of intermediate particles. Each of these normal products will effect a particular transition $|i\rangle \to |f\rangle$, which can be represented by a Feynman graph, similar to those introduced in Chapters 3 and 4.

Consider, for example, Compton scattering ($e^- + \gamma \to e^- + \gamma$). The QED interaction Hamiltonian density is, from Eq. (6.10),

$$\mathscr{H}_I(x) = -\mathscr{L}_I(x) = -e\mathrm{N}[\overline{\psi}(x)\not{A}(x)\psi(x)]. \tag{6.24}$$

Since the negative (positive) frequency parts A^-, $\overline{\psi}^-$, ψ^- (A^+, ψ^+ $\overline{\psi}^+$) are linear in creation (absorption) operators for photons, electrons and positrons respectively, the only normal product which contributes to Compton scattering is

$$\overline{\psi}^- A^- \psi^+ A^+.$$

The method for expanding the S-matrix as a sum of normal products, which we shall now describe, is due to Dyson and Wick.

We first of all summarize the general definition of a normal product. Let $Q, R, \ldots, W$ be operators of the type $\psi^\pm, A^\pm$, etc., i.e. each is linear in either creation or absorption operators, then

$$\mathrm{N}(QR \ldots W) = (-1)^P (Q'R' \ldots W'). \tag{6.25a}$$

Here $Q', R', \ldots, W'$ are the operators $Q, R, \ldots, W$ reordered, so that all absorption operators (i.e. positive frequency parts) stand to the right of all creation operators (i.e. negative frequency parts). The exponent P is the number of interchanges of neighbouring fermion operators required to change the order $(QR \ldots W)$ into $Q'R' \ldots W')$. We generalize the definition (6.25a) by requiring the normal product to obey the distributive law

$$\mathrm{N}(RS \ldots + VW \ldots) = \mathrm{N}(RS \ldots) + \mathrm{N}(VW \ldots). \tag{6.25b}$$

The QED interaction (6.24) is a normal product of field operators. We shall find that in other cases too, the interaction Hamiltonian density can be written as a normal product, i.e.

$$\mathscr{H}_I(x) = \mathrm{N}\{A(x)B(x) \ldots\}, \tag{6.26}$$

where each of the fields $A(x)$, $B(x)$, $\ldots$, is linear in creation and absorption operators. Hence, we must consider the expansion into a sum of normal product of 'mixed' T-product (i.e. a T-product whose factors are normal products), such as occurs in the S-matrix expansion (6.23).

From the definition of the normal product, we have, for two field operators $A \equiv A(x_1)$ and $B \equiv B(x_2)$, that

$$AB - \mathrm{N}(AB) = \left\{ \begin{array}{ll} [A^+, B^-]_+, & \text{for two fermion fields} \\ [A^+, B^-], & \text{otherwise} \end{array} \right\}. \tag{6.27}$$

For two fermion fields, the anticommutators, and in all other cases, the commutators, are c-numbers, i.e. they do not involve creation or annihilation operators. [We had examples in Eqs. (3.40) and (4.53a).] Hence, the right-hand side of Eq. (6.27) is always a c-number. It is given by $\langle 0|AB|0\rangle$, as follows by taking the vacuum expectation value of Eq. (6.27). Hence Eq. (6.27) becomes:

$$AB = \mathrm{N}(AB) + \langle 0|AB|0\rangle. \tag{6.28}$$

Since

$$\mathrm{N}(AB) = \pm \mathrm{N}(BA), \tag{6.29}$$

the minus sign applying in the case of two fermion fields, the plus sign in all other cases, it follows from Eq. (6.28) that for $x_1^0 \neq x_2^0$

$$\mathrm{T}\{A(x_1)B(x_2)\} = \mathrm{N}\{A(x_1)B(x_2)\} + \langle 0|\mathrm{T}\{A(x_1)B(x_2)\}|0\rangle. \tag{6.30}$$

The case of equal times, $x_1^0 = x_2^0$, will be considered below.

The special notation

$$\overbrace{A(x_1)B(x_2)} \equiv \langle 0|\mathrm{T}\{A(x_1)B(x_2)\}|0\rangle \tag{6.31}$$

will be convenient for this vacuum expectation value, which will be called the *contraction* of $A(x_1)$ and $B(x_2)$. Being a vacuum expectation value, it will vanish unless one of the field operators A and B creates particles which the other absorbs. The non-vanishing

contractions are, of course, just the Feynman propagators, e.g. Eqs. (3.59), (3.60), (4.61) and (5.26):

$$\phi(x_1)\phi(x_2) = \mathrm{i}\Delta_{\mathrm{F}}(x_1 - x_2) \tag{6.32a}$$

$$\phi(x_1)\phi^{\dagger}(x_2) = \phi^{\dagger}(x_2)\phi(x_1) = \mathrm{i}\Delta_{\mathrm{F}}(x_1 - x_2) \tag{6.32b}$$

$$\psi_{\alpha}(x_1)\bar{\psi}_{\beta}(x_2) = -\bar{\psi}_{\beta}(x_2)\psi_{\alpha}(x_1) = \mathrm{i}S_{\mathrm{F}\alpha\beta}(x_1 - x_2) \tag{6.32c}$$

$$A^{\mu}(x_1)A^{\nu}(x_2) = \mathrm{i}D_{\mathrm{F}}^{\mu\nu}(x_1 - x_2). \tag{6.32d}$$

To generalize Eq. (6.30) to several operators $A \equiv A(x_1), \ldots, M \equiv M(x_m), \ldots$, the generalized normal product is defined by

$$\begin{aligned} &N(A\,BC\,DEF\ldots\,JKLM\ldots \\ &\qquad = (-1)^{P}\,AK\,BC\,EL\,\ldots\,\mathrm{N}(DF\ldots JM\ldots) \end{aligned} \tag{6.33}$$

where P is the number of interchanges of neighbouring fermion operators required to change the order $(ABC\ldots)$ to $(AKB\ldots)$; for example

$$\begin{aligned} &\mathrm{N}(\psi_{\alpha}(x_1)\,\psi_{\beta}(x_2)A^{\mu}(x_3)\bar{\psi}_{\gamma}(x_4)\bar{\psi}_{\delta}(x_5)) \\ &\qquad = (-1)\,\psi_{\beta}(x_2)\bar{\psi}_{\delta}(x_5)\mathrm{N}(\psi_{\alpha}(x_1)A^{\mu}(x_3)\bar{\psi}_{\gamma}(x_4)). \end{aligned} \tag{6.34}$$

For the case of unequal times (i.e. $x_i^0 \neq x_j^0$, for $i \neq j$), Wick has proved the following generalization of Eq. (6.30):

$$\begin{aligned} \mathrm{T}(ABCD\ldots WXYZ) &= \mathrm{N}(ABCD\ldots WXYZ) \\ &+ \mathrm{N}(AB\,C\ldots YZ) + \mathrm{N}(ABC\ldots YZ) + \cdots + \mathrm{N}(ABC\ldots YZ) \\ &+ \mathrm{N}(AB\,CD\ldots YZ) + \cdots + \mathrm{N}(AB\ldots WX\,YZ) + \cdots. \end{aligned} \tag{6.35}$$

On the right-hand side of this equation appears the sum of all possible generalized normal products that can be formed from $(ABCD\ldots WXYZ)$, the first, second and third lines representing all terms with no, one and two contractions, and so on. Each term on the right-hand side of this equation contains all the factors in the same order in which they occur in the T-product on the left-hand side.

Eq. (6.35) states Wick's theorem. We shall not reproduce its proof, which is by induction, and so not very illuminating.[3]

[3] G. C. Wick, *Phys. Rev.* **80** (1950) 268.

With the interaction (6.26), the S-matrix expansion (6.23) contains the mixed T-products

$$\mathrm{T}\{\mathscr{H}_{\mathrm{I}}(x_1)\ldots\mathscr{H}_{\mathrm{I}}(x_n)\}=\mathrm{T}\{\mathrm{N}(AB\ldots)_{x_1}\ldots\mathrm{N}(AB\ldots)_{x_n}\}. \tag{6.36}$$

Wick extended the theorem (6.35) to include such mixed T-products. In each factor $\mathrm{N}(AB\ldots)_{x_r}$ we replace $x_r=(x_r^0,\mathbf{x}_r)$ by $\xi_r=(x_r^0\pm\varepsilon,\mathbf{x}_r)$, $(\varepsilon<0)$, depending on whether the substitution is made in the creation or absorption part of the field. Hence

$$\mathrm{T}\{\mathrm{N}(AB\ldots)_{x_1}\ldots\mathrm{N}(AB\ldots)_{x_n}\}=\lim_{\varepsilon\to 0}\mathrm{T}\{(AB\ldots)_{\xi_1}\ldots(AB\ldots)_{\xi_n}\}, \tag{6.37}$$

the normal and chronological orderings within each group $(AB\ldots)_{\xi_r}$ being the same on account of the $\pm\varepsilon$ in ξ_r^0. On expanding the right-hand side of Eq. (6.37) by Wick's theorem *before* going to the limit $\varepsilon\to 0$, contractions within one group $(AB\ldots)_{\xi_r}$ (i.e. over equal-times operators when $\varepsilon\to 0$) vanish as the group is already in normal order. We thus have the desired result: the mixed T-product (6.36) can be expanded according to Eq. (6.35), provided contractions over equal times are omitted:

$$\mathrm{T}\{\mathrm{N}(AB\ldots)_{x_1}\ldots\mathrm{N}(AB\ldots)_{x_n}\}=\mathrm{T}\{(AB\ldots)_{x_1}\ldots(AB\ldots)_{x_n}\}_{\text{no e.t.c.}} \tag{6.38}$$

where 'no e.t.c.' stands for 'no equal-times contractions'.

Eqs. (6.35) and (6.38) represent the desired result, enabling us to expand each term in the S-matrix expansion (6.23) into a sum of generalized normal products. Each of these normal products corresponds to a definite process, characterized by the operators not contracted, which absorb and create the particles present in the initial and final states respectively. The non-vanishing contractions which occur in these generalized normal products are the Feynman propagators (6.32), corresponding to virtual particles being emitted and re-absorbed in intermediate states. In the next chapter we shall see how to evaluate these individual contributions to $\langle f|S|i\rangle$, which result from the application of Wick's theorem.

7

Feynman Diagrams and Rules in QED

In the last chapter we obtained the S-matrix expansion (6.23) and Wick's theorem for writing the terms in this expansion as a sum of normal products. In this chapter we shall show how to calculate the matrix element $\langle f|S|i\rangle$ for a transition from an initial state $|i\rangle$ to a final state $|f\rangle$ in a given order of perturbation theory. For definiteness, we shall give this development for the important case of QED. Once this case is understood, the corresponding formalism for others is easily derived.

In Section 7.1 we shall show how to pick out from the S-matrix expansion the terms which contribute to $\langle f|S|i\rangle$ in a given order of perturbation theory. These terms are easily identified. They are those normal products which contain the appropriate destruction and creation operators to destroy the particles present in the initial state $|i\rangle$ and create those present in the final state $|f\rangle$.

In Section 7.2 we shall evaluate the transition amplitude $\langle f|S|i\rangle$ in momentum space. This leads to Feynman diagrams as a way of interpreting the terms in the Wick expansion. There exists a one-to-one correspondence between the diagrams and the terms, which can be summarized in simple rules. These enable one to write down transition amplitudes directly from the Feynman graphs, rather than proceed *ab initio* from Wick's theorem. In Section 7.3 we shall state these rules, known as Feynman rules, for QED. We shall have obtained these rules from the Dyson–Wick formalism, but historically they were first derived by Feynman using a strongly intuitive approach.

In the first three sections of this chapter we shall consider QED as the interaction of the electron–positron field with the electromagnetic field. In the last section (Section 7.4) we shall extend QED to include, in addition to the electron–positron field, other leptons, namely the muon and tauon.

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7.1 Feynman Diagrams in Configuration Space

The processes to which the individual terms in the S-matrix expansion (6.23), i.e. in

$$S = \sum_{n=0}^{\infty} S^{(n)} \equiv \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int \ldots \int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n \mathrm{T}\{\mathscr{H}_{\mathrm{I}}(x_1) \ldots \mathscr{H}_{\mathrm{I}}(x_n)\}, \tag{7.1}$$

contribute are of course determined by the nature of the interaction $\mathscr{H}_{\mathrm{I}}(x)$. For QED this is given by Eq. (6.24):

$$\left.\begin{aligned} \mathscr{H}_{\mathrm{I}}(x) &= -e\mathrm{N}\{\overline{\psi}(x)\not{A}(x)\psi(x)\} \\ &= -e\mathrm{N}\left\{\left(\overline{\psi}^{+} + \overline{\psi}^{-}\right)\left(\not{A}^{+} + \not{A}^{-}\right)\left(\psi^{+} + \psi^{-}\right)\right\}_x \end{aligned}\right\} \tag{7.2}$$

With $\psi^{+}(\overline{\psi}^{-})$, $\overline{\psi}^{+}(\psi^{-})$ and $A^{+}(A^{-})$ being linear in absorption (creation) operators of electrons, positrons and photons, respectively, the interaction (7.2) gives rise to eight basic processes, e.g. the term $-e\mathrm{N}(\overline{\psi}^{+}A^{-}\psi^{+})_x$ corresponds to the annihilation of an electron–positron pair with the creation of a photon.

Using the conventions for Feynman diagrams explained at the end of Section 4.4, we can represent these eight processes by the Feynman graphs of Fig. 7.1, which have been grouped into pairs. The graphs in each pair correspond to absorption or emission of a photon, together with: (a) the scattering of an electron, (b) the scattering of a positron, (c) pair annihilation or (d) pair creation. These diagrams illustrate the basic processes to which the QED interaction gives rise and will be referred to as the basic vertex part. All other QED Feynman diagrams are built up by combining such basic vertex parts.

The diagrams of Fig. 7.1 also represent the processes arising from the first-order term $S^{(1)}$ in Eq. (7.1). However, these are not real physical processes, i.e. for none of them can

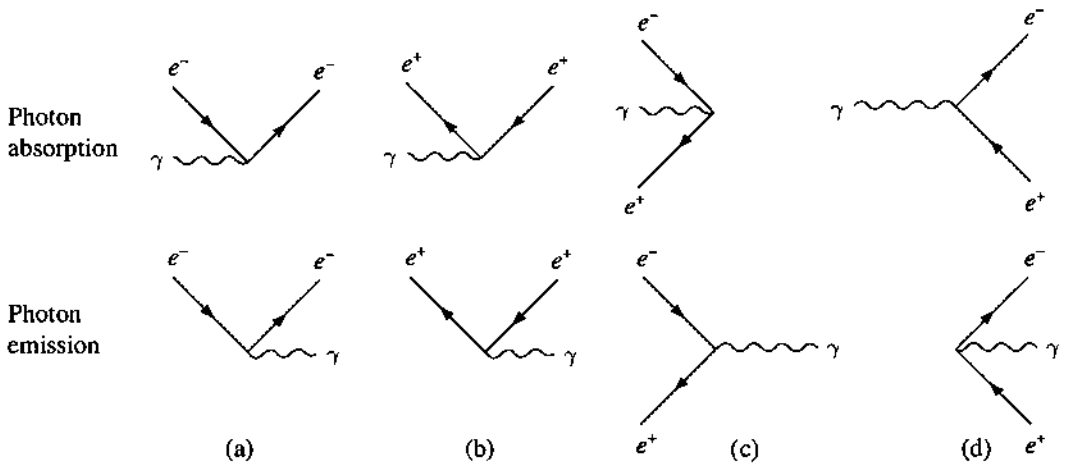


Figure 7.1 *The Feynman diagrams of the eight basic processes of the QED interaction* $\mathscr{H}_{\mathrm{I}}(x) = -e\mathrm{N}(\overline{\psi}\not{A}\psi)_x$. *(a)* e^- *scattering; (b)* e^+ *scattering; (c)* e^+e^- *annihilation; (d)* e^+e^- *creation*

energy and momentum be conserved for real physical particles for which we must have $k^2 = 0$ for photons, and $p^2 = m^2$ for fermions. Consequently

$$\left\langle f \left| S^{(1)} \right| i \right\rangle = 0 \tag{7.3a}$$

for these transitions, as will be shown explicitly in the next section. More generally

$$\left\langle f \left| S^{(n)} \right| i \right\rangle = 0 \tag{7.3b}$$

for any unphysical process, i.e. for a transition between real physical states which violates a conservation law of the theory. This follows, since S generates a solution of the equations of motion, so that

$$\langle f | S | i \rangle = 0 \tag{7.3c}$$

for an unphysical process, and since Eq. (7.1) is a power series in the coupling constant e.

To obtain real processes, we must go at least to the second-order term $S^{(2)}$ in Eq. (7.1). This term contains two factors $\mathscr{H}_{\mathrm{I}}$. Its expansion by Wick's theorem into a sum of normal products corresponds to all meaningful ways of joining two basic vertex parts into a Feynman diagram, as we shall now see.

Application of Wick's theorem, Eqs. (6.35) and (6.38), to $S^{(2)}$ leads to

$$S^{(2)} = \sum_{\mathrm{i=A}}^{F} S_i^{(2)} \tag{7.4}$$

where

$$S_{\mathrm{A}}^{(2)} = -\frac{e^2}{2!} \int \mathrm{d}^4 x_1 \mathrm{d}^4 x_2 \mathrm{N}[(\bar{\psi} A\!\!\!/ \psi)_{x_1} (\bar{\psi} A\!\!\!/ \psi)_{x_2}] \tag{7.5a}$$

$$S_{\mathrm{B}}^{(2)} = -\frac{e^2}{2!} \int \mathrm{d}^4 x_1 \mathrm{d}^4 x_2 \{\mathrm{N}[(\bar{\psi} A\!\!\!/ \psi)_{x_1} (\bar{\psi} A\!\!\!/ \psi)_{x_2}] + \mathrm{N}[(\bar{\psi} A\!\!\!/ \psi)_{x_1} (\bar{\psi} A\!\!\!/ \psi)_{x_2}]\} \tag{7.5b}$$

$$S_{\mathrm{C}}^{(2)} = -\frac{e^2}{2!} \int \mathrm{d}^4 x_1 \mathrm{d}^4 x_2 \mathrm{N}[(\bar{\psi} \gamma^\alpha A_\alpha \psi)_{x_1} (\bar{\psi} \gamma^\beta A_\beta \psi)_{x_2}] \tag{7.5c}$$

$$S_{\mathrm{D}}^{(2)} = -\frac{e^2}{2!} \int \mathrm{d}^4 x_1 \mathrm{d}^4 x_2 \{\mathrm{N}[(\bar{\psi} \gamma^\alpha A_\alpha \psi)_{x_1} (\bar{\psi} \gamma^\beta A_\beta \psi)_{x_2}] + \mathrm{N}[(\bar{\psi} \gamma^\alpha A_\alpha \psi)_{x_1} (\bar{\psi} \gamma^\beta A_\beta \psi)_{x_2}]\} \tag{7.5d}$$

$$S_{\mathrm{E}}^{(2)} = -\frac{e^2}{2!}\int \mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{N}[\overline{\psi}\not{A}\ \psi)_{x_1}(\overline{\psi}\ \not{A}\psi)_{x_2}] \tag{7.5e}$$

$$S_{\mathrm{F}}^{(2)} = -\frac{e^2}{2!}\int \mathrm{d}^4x_1\mathrm{d}^4x_2(\overline{\psi}\,\gamma^\alpha A_\alpha\,\psi)_{x_1}(\overline{\psi}\gamma^\beta A_\beta\psi)_{x_2}. \tag{7.5f}$$

The first of these terms, $S_A^{(2)}$, Eq. (7.5a), is not very interesting. It corresponds to two processes of the kind illustrated in Fig. 7.1 going on independently of each other. Like $S^{(1)}$, this term does not lead to any real transitions.

The two terms in $S_B^{(2)}$, Eq. (7.5b), are identically equal to each other, as is seen by permuting the operators. This requires care, since the fermion fields are anticommuting operators and four-component spinors. Permuting the two groups $(\overline{\psi}\not{A}\psi)$ involves an even permutation of fermion operators and the spinor indices of each group are self-contained.[1] Hence

$$\mathrm{N}[(\overline{\psi}\not{A}\psi)_{x_1}(\overline{\psi}\not{A}\psi)_{x_2}] = \mathrm{N}[(\overline{\psi}\not{A}\psi)_{x_2}(\overline{\psi}\not{A}\ \psi)_{x_1}]. \tag{7.6}$$

Using this result and interchanging the integration variables $x_1 \leftrightarrow x_2$ in the second term of Eq. (7.5b), one obtains

$$S_{\mathrm{B}}^{(2)} = -e^2\int \mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{N}[(\overline{\psi}\not{A}\,\psi)_{x_1}(\overline{\psi}\not{A}\ \psi)_{x_2}]. \tag{7.7}$$

This expression contains one fermion contraction. This is given by the fermion propagator (6.32c), which is a c-number and corresponds to a virtual intermediate fermion. For $t_2 < t_1$, we can think of it as a virtual electron propagating from x_2 to x_1, for $t_1 < t_2$ as a virtual positron propagating from x_1 to x_2. As explained in Section 4.4, no time-ordering is implied in the present formalism – indeed, all space–time points x_1 and x_2 are summed over – and these two cases are combined and jointly referred to as a virtual fermion propagating from x_2 (associated with $\overline{\psi}$) to x_1 (associated with ψ). In addition to this propagator, expression (7.7) contains two uncontracted fermion and two uncontracted photon operators. These absorb or create particles present initially or finally, so-called *external particles*. The operator $S_{\mathrm{B}}^{(2)}$ contributes to many real processes. (To conserve energy and momentum, the initial and final states must each contain two particles.) Since the operators in $S_{\mathrm{B}}^{(2)}$ are in normal order, it is easy to pick out the terms which contribute to a given process.

One of these processes is Compton scattering

$$\gamma + e^- \rightarrow \gamma + e^-, \tag{7.8}$$

[1] The reader can always resolve any cases of doubt by writing out explicitly the spinor indices.

already mentioned in Section 4.4. This process corresponds to selecting the positive frequency part $\psi^+(x_2)$ of $\psi(x_2)$ to absorb the initial electron, and the negative frequency part $\overline{\psi}^-(x_1)$ of $\overline{\psi}(x_1)$ to create the final electron. But either $\not{A}^+(x_1)$ or $\not{A}^+(x_2)$ can absorb the initial photon and correspondingly $\not{A}^-(x_2)$ or $\not{A}^-(x_1)$ must emit the final photon. Thus the part of Eq. (7.7) which causes Compton scattering is

$$S^{(2)}(\gamma e^- \to \gamma e^-) = S_a + S_b \tag{7.9}$$

where

$$S_a = -e^2 \int d^4x_1 d^4x_2 \overline{\psi}^-(x_1)\gamma^\alpha i S_F(x_1 - x_2)\gamma^\beta A_\alpha^-(x_1)A_\beta^+(x_2)\psi^+(x_2) \tag{7.10a}$$

$$S_b = -e^2 \int d^4x_1 d^4x_2 \overline{\psi}^-(x_1)\gamma^\alpha i S_F(x_1 - x_2)\gamma^\beta A_\beta^-(x_2)A_\alpha^+(x_1)\psi^+(x_2). \tag{7.10b}$$

In Eqs. (7.10), the operators have been put in a normal order and we have substituted Eq. (6.32c) for the fermion contraction.

The contributions S_a and S_b to Compton scattering are represented by the Feynman graphs in Figs. 7.2(a) and (b). The latter is the same as Fig. 4.3. (Remember that, except for the conventions about initial and final lines, there is no time ordering in Feynman graphs, so that the same graph can be drawn in many different ways.) In Fig. 7.2 we have attached the appropriate Lorentz indices (α, β) to vertices, and particle labels (γ, e^-) to external lines. We shall often omit these as redundant.

The other real processes described by Eq. (7.7) are Compton scattering by positrons, and the two-photon pair annihilation and creation processes, i.e.

$$\text{(i)}\ \gamma + e^+ \to \gamma + e^+, \quad \text{(ii)}\ e^+ + e^- \to \gamma + \gamma, \quad \text{(iii)}\ \gamma + \gamma \to e^+ + e^-. \tag{7.11}$$

The corresponding Feynman diagrams are shown in Figs. 7.3–7.5. We leave it to the reader to write down the operators for the first two of these processes. For the pair creation process one obtains

$$S^{(2)}(2\gamma \to e^+e^-) = -e^2 \int d^4x_1 d^4x_2 \overline{\psi}^-(x_1)\gamma^\alpha i S_F(x_1 - x_2)\gamma^\beta \psi^-(x_2)A_\alpha^+(x_1)A_\beta^+(x_2). \tag{7.12}$$

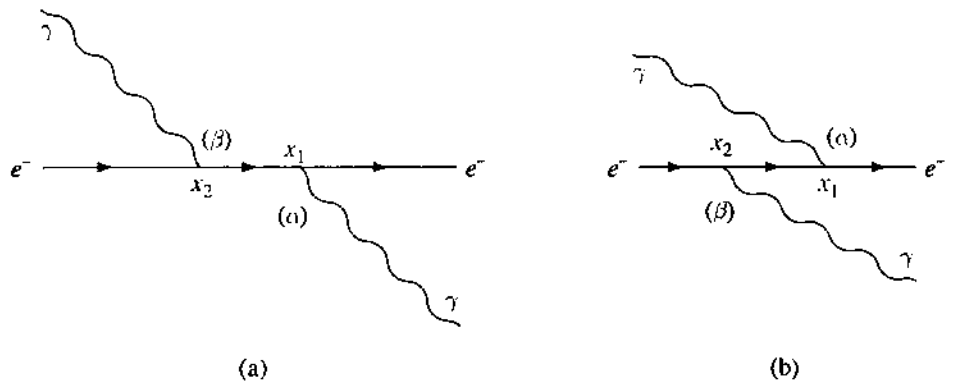


Figure 7.2 *The contributions S_a, S_b, Eqs. (7.10), to Compton scattering*

Although we have only shown one diagram in Fig. 7.5, Eq. (7.12) actually gives two contributions, since the operator $A_\beta^+(x_2)$ can absorb either of the initially present photons, with $A_\alpha^+(x_1)$ absorbing the other. A similar situation exists for the pair-annihilation process.

We next consider Eq. (7.5c). This term contains four uncontracted fermion operators. Accordingly, the real processes to which this term gives rise are fermion–fermion scattering: $e^- - e^-, e^+ - e^+$ or $e^- - e^+$ scattering, according to which positive and negative frequency parts are selected from the external fermion fields. The photon–photon contraction in Eq. (7.5c) describes the interaction between the charges as the exchange of transverse, longitudinal and scalar photons. This photon propagator occurs associated with two conserved current operators $S^\mu(x) = (\overline{\psi}\gamma^\mu\psi)_x$. As discussed in Section 5.3 [particularly the discussion of Eq. (5.44)], this covariant formulation is equivalent to the usual description of the interaction in terms of the instantaneous Coulomb interaction together with the exchange of transverse photons.

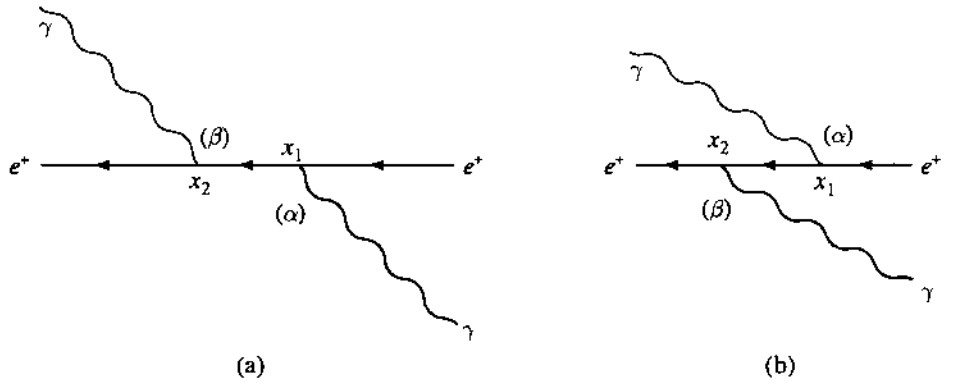


Figure 7.3 *The Feynman diagrams for Compton scattering by positrons*

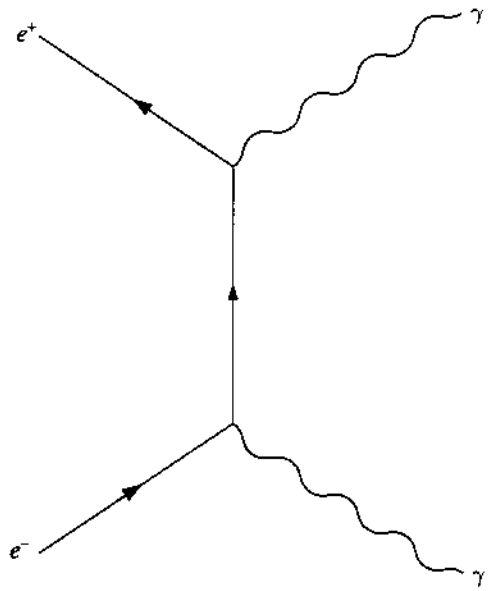


Figure 7.4 *The Feynman diagram for* $e^+ + e^- \rightarrow \gamma + \gamma$

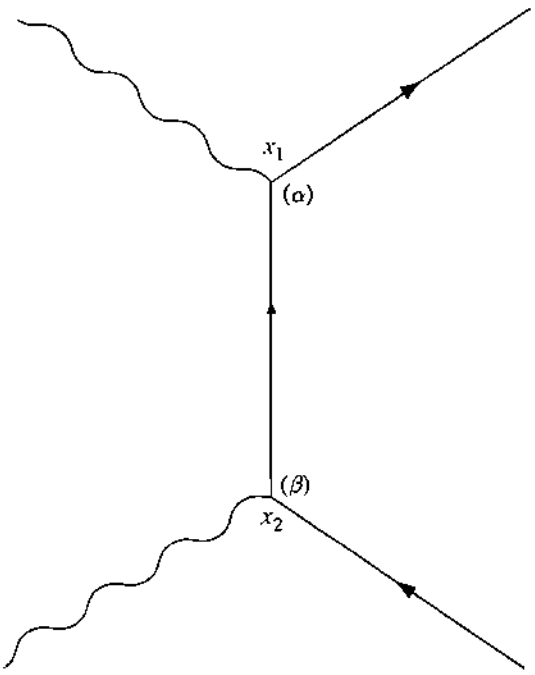


Figure 7.5 *The Feynman diagram for* $\gamma + \gamma \rightarrow e^+ + e^-$, *Eq. (7.12)*

We next consider electron–electron scattering,

$$e^- + e^- \rightarrow e^- + e^-, \tag{7.13}$$

known as Møller scattering, in more detail. The part of the operator (7.5c) describing this process is

$$S^{(2)}(2e^- \rightarrow 2e^-) = \frac{-e^2}{2!}\int \mathrm{d}^4x_1\mathrm{d}^4x_2 \mathrm{N}\left[(\overline{\psi}^-\gamma^\alpha\psi^+)_{x_1}(\overline{\psi}^-\gamma^\beta\psi^+)_{x_2}\right] \mathrm{i}D_{\mathrm{F}\alpha\beta}(x_1 - x_2), \tag{7.14}$$

where we substituted Eq. (6.32d) for the photon contraction.

Let us label the initial and final electron states 1, 2 and 1′, 2′ respectively, i.e. with an obvious notation we are considering the transition

$$|i\rangle = c^\dagger(2)c^\dagger(1)|0\rangle \rightarrow |f\rangle = c^\dagger(2')c^\dagger(1')|0\rangle. \tag{7.15}$$

Eq. (7.14) gives four contributions to the transition (7.15), since either initial electron can be absorbed by either ψ^+ operator, and either final electron can be emitted by either $\overline{\psi}^-$ operator. These four terms comprise two pairs, which differ only by the interchange of the integration variables $x_1 \leftrightarrow x_2$ in Eq. (7.14). We need consider only one of these pairs and multiply the result by a factor 2. The remaining two terms are represented by the Feynman graphs in Fig. 7.6.

We had another case of two identical contributions to a process, related to the interchange $x_1 \leftrightarrow x_2$, in connection with Eq. (7.5b). This represents a general result. The nth order term $S^{(n)}$ in the S-matrix expansion (7.1) contains a factor $1/n!$ and n integration variables $x_1, x_2, \ldots, x_n$. These are only summation variables and can be attached to the n

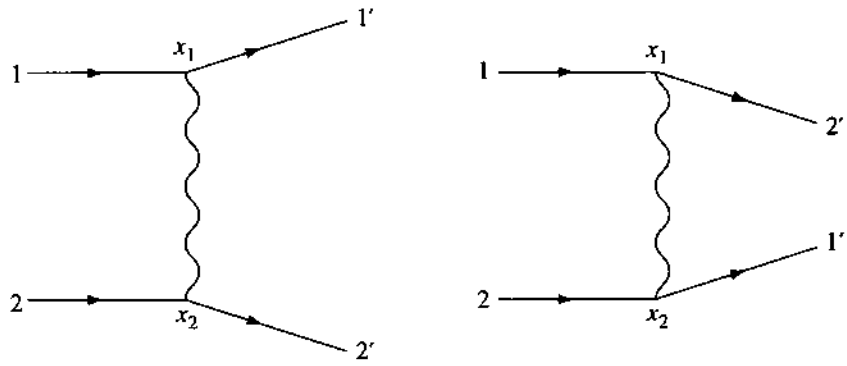


Figure 7.6 *The two diagrams for electron-electron scattering (Møller scattering)*

vertices of a given Feynman graph in $n!$ ways. We can omit the factor $1/n!$ if we consider only topologically different Feynman diagrams, i.e. diagrams which differ only in the labelling of vertices are considered the same. Some care is required in interpreting this statement. For example, the two diagrams of Fig. 7.6 are topologically different from each other because the two final electrons have different properties. (These were labelled $1'$ and $2'$. In practice they are the momenta and spins.) Permuting x_1 and x_2 does not interchange the two graphs of Fig. 7.6. As we shall see, their contributions occur with a relative minus sign and correspond to the 'direct minus exchange scattering', which the reader should recognize, from non-relativistic quantum mechanics, as characteristic of two identical fermions.

In order to obtain explicit expressions for these two contributions, let

$$\psi_j^+(x) = c(j) f_j(x), \quad \overline{\psi}_j^-(x) = c^\dagger(j) g_j(x) \tag{7.16}$$

be the parts of the operators $\psi^+(x)$ and $\overline{\psi}^-(x)$ proportional to $c(j)$ and $c^\dagger(j)$ respectively. $j = 1, 2, 1', 2'$ labels the electron states involved in the process. The part of the S-matrix operator (7.14) which effects the transition $|i\rangle \to |f\rangle$, Eq. (7.15), is then given by

$$S^{(2)}(e^-(1) + e^-(2) \to e^-(1') + e^-(2')) = S_a + S_b \tag{7.17a}$$

where S_a and S_b correspond to Figs. 7.6, and are given by

$$S_a = -e^2 \int d^4x_1 d^4x_2 N\left[(\overline{\psi}_{1'}^- \gamma^\alpha \psi_1^+)_{x_1} (\overline{\psi}_{2'}^- \gamma^\beta \psi_2^+)_{x_2}\right] iD_{F\alpha\beta}(x_1 - x_2) \tag{7.17b}$$

$$S_b = -e^2 \int d^4x_1 d^4x_2 N\left[(\overline{\psi}_{2'}^- \gamma^\alpha \psi_1^+)_{x_1} (\overline{\psi}_{1'}^- \gamma^\beta \psi_2^+)_{x_2}\right] iD_{F\alpha\beta}(x_1 - x_2). \tag{7.17c}$$

The relative minus sign of the two contributions is implied by the normal products in these equations. To arrange the creation and annihilation operators in both cases in the same order, e.g. as $c^\dagger(1')c^\dagger(2')c(1)c(2)$, requires the normal products in Eqs. (7.17b) and (7.17c) to be reordered equal to $-\overline{\psi}_{1'}^-(x_1)\overline{\psi}_{2'}^-(x_2)\psi_1^+(x_1)\psi_2^+(x_2)$ and $+\overline{\psi}_{1'}^-(x_2)\overline{\psi}_{2'}^-(x_1)\psi_1^+(x_1)\psi_2^+(x_2)$, respectively. Using Eqs. (7.16), we obtain from Eqs. (7.17) the transition amplitude

$$\begin{aligned}\langle f|S^{(2)}(2e^- \to 2e^-)|i\rangle &= \{-e^2 \int \mathrm{d}^4x_1\ \mathrm{d}^4x_2 g_{1'}(x_1)\gamma^\alpha f_1(x_1) g_{2'}(x_2)\gamma^\beta f_2(x_2) \mathrm{i}D_{\mathrm{F}\alpha\beta}(x_1 - x_2)\} \\ &\quad - \{1' \leftrightarrow 2'\}\end{aligned} \tag{7.18}$$

where the term $\{1' \leftrightarrow 2'\}$ is just the first expression in braces with the labels $1'$ and $2'$ of the two final electron states interchanged. Our final result (7.18) has the desired form of a 'direct' amplitude minus an 'exchange' amplitude, the two amplitudes being transformed into each other by exchanging the single-particle states of the two electrons in the final state. In non-relativistic quantum mechanics, this result follows through the use of anti-symmetric wavefunctions according to Pauli's principle. In the above field-theoretic derivation, the anticommutativity of the fermion field operators is the crucial element.

These arguments generalize. Whenever the initial or final state contains several identical fermions, one obtains a completely antisymmetric transition amplitude $\langle f|S|i\rangle$. For example, if the initial state $|i\rangle$ contains s positrons in states $1, 2, \ldots, s$, the corresponding S-matrix operator will contain s uncontracted operators $\mathrm{N}(\overline{\psi}(x_1)\overline{\psi}(x_2)\ldots\overline{\psi}(x_s))$. Any one of these operators $\overline{\psi}(x_1), \ldots, \overline{\psi}(x_s)$ can absorb the positron in state 1, and so on, giving $s!$ terms, whose sum is completely antisymmetric in the labels $1, 2, \ldots, s$, since the operators $\overline{\psi}(x_1), \ldots, \overline{\psi}(x_s)$ anticommute. An analogous argument holds for several identical final state fermions.

More curiously, the fact that the operator $\psi(x)$ can absorb an electron or create a positron implies that transition amplitudes are antisymmetric with respect to initial electron and final positron states. (A similar argument applies of course to $\overline{\psi}(x)$ and initial positrons and final electrons.) We have an example of this in electron–positron scattering,

$$e^+ + e^- \to e^+ + e^-,$$

known as Bhabha scattering. The part of the operator (7.5c) describing this process must contain the uncontracted operators $\overline{\psi}^+$, ψ^+, ψ^- and $\overline{\psi}^-$ to absorb and create the particles present initially and finally. As in the case of Møller scattering, four terms contribute, which again reduce to two by the general argument given above. It is left as an exercise to the reader to derive from Eq. (7.5c) the following expression for the S-matrix operator for Bhabha scattering:

$$S^{(2)}(e^+e^- \to e^+e^-) = S_\mathrm{a} + S_\mathrm{b} \tag{7.19a}$$

where

$$S_\mathrm{a} = -e^2 \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 \mathrm{N}\left[(\overline{\psi}^-\gamma^\alpha\psi^+)_{x_1}(\overline{\psi}^+\gamma^\beta\psi^-)_{x_2}\right] \mathrm{i}D_{\mathrm{F}\alpha\beta}(x_1 - x_2), \tag{7.19b}$$

$$S_\mathrm{b} = -e^2 \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 \mathrm{N}\left[(\overline{\psi}^-\gamma^\alpha\psi^-)_{x_1}(\overline{\psi}^+\gamma^\beta\psi^+)_{x_2}\right] \mathrm{i}D_{\mathrm{F}\alpha\beta}(x_1 - x_2). \tag{7.19c}$$

The Feynman graph for S_a is shown in Fig. 7.7(a). It represents the scattering by photon exchange, as occurred for electron–electron scattering (Fig. 7.6). However, in the term S_b both initial particles are annihilated at x_2 and the final electron–positron pair is created at x_1. It corresponds to the *annihilation diagram* of Fig. 7.7(b). As in electron–electron scattering, there is a relative sign factor (–1) between the two contributions implicit in

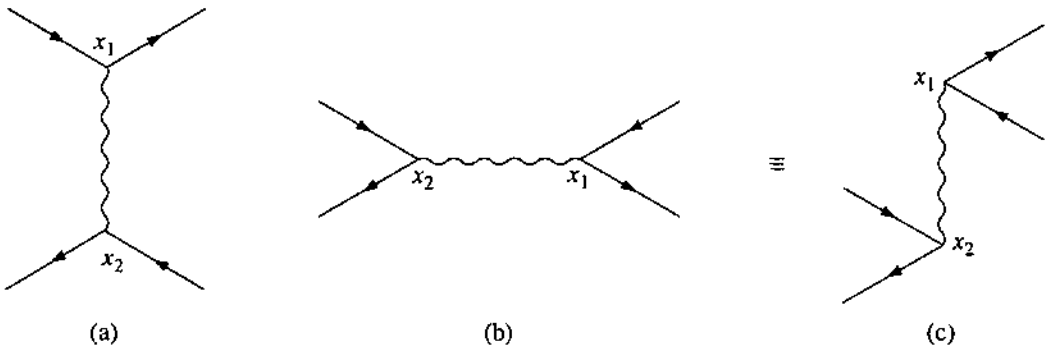


Figure 7.7 *The contributions S_a and S_b to electron–positron scattering (Bhabha scattering): (a) represents photon exchange; (b) and (c) are equivalent ways of representing the pair-annihilation process*

the normal products, which becomes explicit if the creation and annihilation operators are brought into the same normal order in both cases. That the diagrams of Figs. 7.7(a) and (b) are related by the interchange of an initial electron state and a final positron state is brought out by 'deforming' diagram 7.7(b) into diagram 7.7(c). Comparing diagrams 7.7(a) and (c) one sees that the latter is obtained from the former by interchanging the initial electron line at x_1 and the final positron line at x_2.

We shall now discuss briefly the remaining second-order terms $S_D^{(2)}$ to $S_F^{(2)}$, Eqs. (7.5d)–(7.5f).

Eq. (7.5d) contains two uncontracted fermion fields and gives rise to two processes according to whether the fermion present initially and finally is an electron or a positron. The two terms in Eq. (7.5d) are again equal to each other. For the electron case, this equation reduces to

$$\begin{aligned} &S^{(2)}(e^- \to e^-) \\ &\quad = -e^2 \int \mathrm{d}^4x_1\ \mathrm{d}^4x_2 \overline{\psi}^-(x_1)\gamma^\alpha \mathrm{i}S_F(x_1 - x_2)\gamma^\beta \psi^+(x_2)\mathrm{i}D_{F\alpha\beta}(x_1 - x_2), \end{aligned} \tag{7.20}$$

which corresponds to the diagram of Fig. 7.8. It represents a modification of the properties of a *bare* electron due to its interaction with the radiation field. It is one of the processes – in fact the simplest – which converts a bare electron into a *physical* electron, i.e. one surrounded by its photon cloud. This interaction changes the energy of the system, that is, the mass of the physical electron as compared with that of the bare electron. This is known as the *self-energy* of the electron, and Fig. 7.8 is called a self-energy diagram. Its evaluation leads to a divergent integral. These divergent self-energy effects can be eliminated by incorporating them in the properties of the physical electron. This is the process of *renormalization*, which will be studied in Chapter 9.

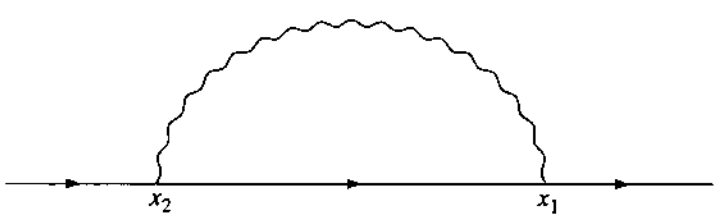


Figure 7.8 *The electron self-energy $S^{(2)}(e^- \to e^-)$, Eq. (7.20)*

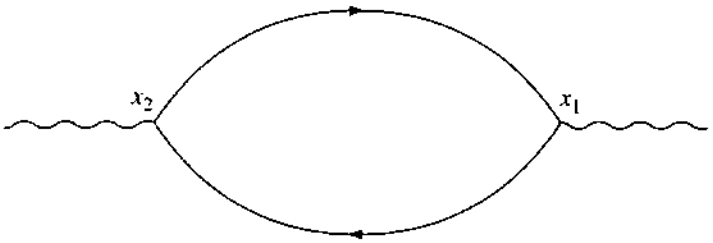


Figure 7.9 *The photon self-energy (vacuum polarization) $S^{(2)}(\gamma \to \gamma)$, Eq. (7.21)*

Fig. 7.9 similarly describes a *photon self-energy* arising from the term $S_E^{(2)}$, Eq. (7.5e). The interaction between the electromagnetic and the electron–positron fields enables the photon to create a virtual electron–positron pair, which subsequently annihilates again. An external electromagnetic field (for example the field of a heavy nucleus) will modify the distribution of these virtual electron–positron pairs, i.e. it will 'polarize the vacuum' in much the same way in which it would polarize a dielectric. For this reason, such photon self-energy graphs are called *vacuum polarization* diagrams. Like the electron self-energy, they lead to infinities, which are again eliminated by re-normalization (see Chapter 9).

Eq. (7.5e) for the photon self-energy can be written

$$S^{(2)}(\gamma \to \gamma) = -e^2 \int \mathrm{d}^4x_1\, \mathrm{d}^4x_2 \mathrm{N}[(\overline{\psi} \not{A}^- \psi)_{x_1} (\overline{\psi}\, \not{A}^+ \psi)_{x_2}]. \tag{7.21}$$

Writing the spinor indices out explicitly, we can re-express the normal product in Eq. (7.21) as

$$\begin{aligned} &\mathrm{N}[(\overline{\psi}_\lambda \not{A}^-_{\lambda\mu} \psi_\mu)_{x_1} (\overline{\psi}_\sigma \not{A}^+_{\sigma\tau} \psi_\tau)_{x_2}] \\ &= (-1)\, \psi_\tau(x_2) \overline{\psi}_\lambda(x_1) \not{A}^-_{\lambda\mu}(x_1)\, \psi_\mu(x_1) \overline{\psi}_\sigma(x_2) \not{A}^+_{\sigma\tau}(x_2) \\ &= (-1)\mathrm{Tr}[\mathrm{i}S_\mathrm{F}(x_2 - x_1) \not{A}^-(x_1) \mathrm{i}S_\mathrm{F}(x_1 - x_2) \not{A}^+(x_2)]. \end{aligned} \tag{7.22}$$

(Here $\not{A}^-_{\lambda\mu}(x) \equiv \gamma^\alpha_{\lambda\mu} A^-_\alpha(x)$, etc.)

The minus sign in the last equation is characteristic of *closed fermion loops* (i.e. closed loops consisting of fermion lines only), which always involve the transposition of a single fermion operator from one end of a product of such factors to the other, i.e. an odd number of interchanges. The trace in Eq. (7.22) is equally characteristic. It corresponds to summing over all spin states of the virtual electron–positron pair. (The connection between spin sums and traces will be discussed in Section 8.2.)

Finally, Fig. 7.10 shows the graph representing Eq. (7.5f). This diagram has no external lines and consequently does not cause any transitions. One can show that such *vacuum diagrams* (i.e. diagrams without external lines) may be omitted altogether, at any rate in elementary applications.

This completes our initial analysis of the various terms which occur on decomposing $S^{(2)}$ into normal products. We have seen that the terms obtained correspond to specific

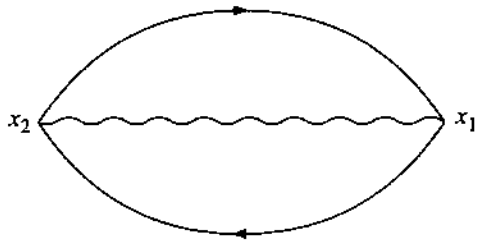


Figure 7.10 *The simplest vacuum diagram, Eq. (7.5f)*

processes and how Feynman diagrams greatly aid their interpretation. No new features occur for the higher-order terms $S^{(3)}, \ldots$ The methods developed are sufficient to deal with such higher- order processes, although the complexity of the mathematics increases rapidly with order.

7.2 Feynman Diagrams in Momentum Space

In the last section we developed a technique for deriving the S-matrix operator which generates a particular transition $|i\rangle \to |f\rangle$ in a given order. In practice, one is usually interested in the corresponding matrix element $\langle f|S^{(n)}|i\rangle$. The states $|i\rangle$ and $|f\rangle$ are usually specified by the particles of known momenta, and spin and polarization properties present initially and finally. Explicit calculations of the matrix elements lead to a re-interpretation of the Feynman graphs as diagrams in momentum space. By studying some specific cases we shall see that these diagrams are closely related to the mathematical expressions they represent. It is possible to formulate a set of rules which enable one to write down the matrix elements directly from the Feynman diagrams without detailed calculations. These Feynman rules, which will be given in the next section, are the linchpin of practical calculations in perturbation theory.

Calculation of the matrix elements, with $|i\rangle$ and $|f\rangle$ specified as momentum eigenstates of the particles present, essentially corresponds to Fourier transforming the fields into momentum space in order to pick out the appropriate absorption and creation operators. For the propagators, these Fourier transforms are given, from Eqs. (6.32c), (6.32d), (4.63) and (5.27), by

$$\underbrace{\psi(x_1)\overline{\psi}(x_2)} = \mathrm{i}S_{\mathrm{F}}(x_1 - x_2) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4 p \mathrm{i}S_{\mathrm{F}}(p)\mathrm{e}^{-\mathrm{i}p(x_1-x_2)} \tag{7.23a}$$

$$\underbrace{A^{\alpha}(x_1)A^{\beta}(x_2)} = \mathrm{i}D_{\mathrm{F}}^{\alpha\beta}(x_1 - x_2) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4 k \mathrm{i}D_{\mathrm{F}}^{\alpha\beta}(k)\mathrm{e}^{-\mathrm{i}k(x_1-x_2)} \tag{7.23b}$$

where

$$S_{\mathrm{F}}(p) = \frac{\not{p} + m}{p^2 - m^2 + \mathrm{i}\varepsilon} \equiv \frac{1}{\not{p} - m + \mathrm{i}\varepsilon} \tag{7.24a}$$

$$D_F^{\alpha\beta}(k) = \frac{-g^{\alpha\beta}}{k^2 + i\varepsilon}. \tag{7.24b}$$

The Fourier expansions of the uncontracted fields ψ, $\bar{\psi}$ and A_α are given by Eqs. (4.38) and (5.16). The effect of the uncontracted operators ψ^+, $\bar{\psi}^+$ and A_α^+, which occur in a term of the S-matrix expansion, acting on $|i\rangle$, is to give the vacuum state $|0\rangle$. For example, it follows from Eqs. (4.38) and (5.16) that

$$\psi^+(x)|e^-\mathbf{p}\rangle = |0\rangle\left(\frac{m}{VE_\mathbf{p}}\right)^{1/2} u(\mathbf{p})e^{-ipx} \tag{7.25a}$$

$$\bar{\psi}^+(x)|e^+\mathbf{p}\rangle = |0\rangle\left(\frac{m}{VE_\mathbf{p}}\right)^{1/2} \bar{v}(\mathbf{p})e^{-ipx} \tag{7.25b}$$

$$A_\alpha^+(x)|\gamma\mathbf{k}\rangle = |0\rangle\left(\frac{1}{2V\omega_\mathbf{k}}\right)^{1/2} \varepsilon_\alpha(\mathbf{k})e^{-ikx} \tag{7.25c}$$

Here we have suppressed the spin and polarization labels. For example, $|e^-\mathbf{p}\rangle$ and $|\gamma\mathbf{k}\rangle$ stand for the one-electron and one-photon states

$$|e^-\mathbf{p}\rangle \equiv |e^-\mathbf{p}r\rangle = c_r^\dagger(\mathbf{p})|0\rangle, \quad |\gamma\mathbf{k}\rangle \equiv |\gamma\mathbf{k}r\rangle = a_r^\dagger(\mathbf{k})|0\rangle, \quad r = 1, 2,$$

and $u(\mathbf{p})$ and $\varepsilon_\alpha(\mathbf{k})$ are short for $u_r(\mathbf{p})$ and $\varepsilon_{r\alpha}(\mathbf{k})$. In the following, we shall frequently simplify the notation in this way, writing $c(\mathbf{p})$ for $c_r(\mathbf{p})$, etc.

The effect of the uncontracted operators $\bar{\psi}^-$, ψ^- and A_α^-, which occur in a term of the S-matrix expansion, acting on $|0\rangle$, is to produce the final state $|f\rangle$. In particular, we find from Eqs. (4.38) and (5.16) that

$$\bar{\psi}^-(x)|0\rangle = \sum |e^-\mathbf{p}\rangle\left(\frac{m}{VE_\mathbf{p}}\right)^{1/2} \bar{u}(\mathbf{p})e^{ipx} \tag{7.26a}$$

$$\psi^-(x)|0\rangle = \sum |e^+\mathbf{p}\rangle\left(\frac{m}{VE_\mathbf{p}}\right)^{1/2} v(\mathbf{p})e^{ipx} \tag{7.26b}$$

$$A_\alpha^-(x)|0\rangle = \sum |\gamma\mathbf{k}\rangle\left(\frac{1}{2V\omega_\mathbf{k}}\right)^{1/2} \varepsilon_\alpha(\mathbf{k})e^{ikx}, \tag{7.26c}$$

where the summations are over spin and polarization states, as well as momenta. It is straightforward to generalize the results (7.25) and (7.26) to states involving several particles.

Using Eqs. (7.23)–(7.26) it is easy to calculate S-matrix elements, as the following examples will show.

7.2.1 The first-order terms $S^{(1)}$

The Feynman graphs resulting from the first-order term

$$S^{(1)} = ie \int d^4x N(\bar{\psi} A\!\!\!/ \psi)_x \tag{7.27}$$

are just the basic vertex diagrams of Fig. 7.1. Let us calculate the matrix element $\langle f|S^{(1)}|i\rangle$ for one of these processes, namely for electron scattering with emission of a photon, illustrated in Fig. 7.11. In this figure, we state the energy–momentum four-vectors of the particles involved, but their spin and polarization labels have been suppressed, as discussed above. Fig. 7.11 represents the transition

$$|i\rangle = |e^-\mathbf{p}\rangle = c^\dagger(\mathbf{p})|0\rangle \rightarrow |f\rangle = |e^-\mathbf{p}'; \gamma\mathbf{k}'\rangle = c^\dagger(\mathbf{p}')a^\dagger(\mathbf{k}')|0\rangle, \tag{7.28}$$

i.e. $|i\rangle$ consists of an electron of momentum $\mathbf{p}$ (and spin state $s = 1, 2$), and $|f\rangle$ of an electron of momentum $\mathbf{p}'$ (and spin state $s' = 1, 2$) plus a photon of momentum $\mathbf{k}'$ (and polarization state $r' = 1, 2$). From Eqs. (7.25)–(7.28) we obtain

$$\begin{aligned}\langle f|S^{(1)}|i\rangle &= \langle e^-\mathbf{p}';\gamma\mathbf{k}'|ie\int d^4x \bar{\psi}^-(x)\gamma^\alpha A_\alpha^-(x)\psi^+(x)|e^-\mathbf{p}\rangle \\ &= ie\int d^4x\left[\left(\frac{m}{VE_{\mathbf{p}'}}\right)^{1/2}\bar{u}(\mathbf{p}')e^{ip'x}\right]\gamma^\alpha\left[\left(\frac{1}{2V\omega_{\mathbf{k}'}}\right)^{1/2}\varepsilon_\alpha(\mathbf{k}')e^{ik'x}\right] \\ &\quad\times\left[\left(\frac{m}{VE_{\mathbf{p}}}\right)^{1/2}u(\mathbf{p})e^{-ipx}\right].\end{aligned} \tag{7.29}$$

The x-dependent terms in this expression give

$$\int d^4x \exp\left[ix(p' + k' - p)\right] = (2\pi)^4\delta^{(4)}(p' + k' - p), \tag{7.30}$$

where we have anticipated going to the limits of an infinite volume, $V \rightarrow \infty$, and an infinite time interval during which the transition may occur. From Eqs. (7.29) and (7.30) we obtain

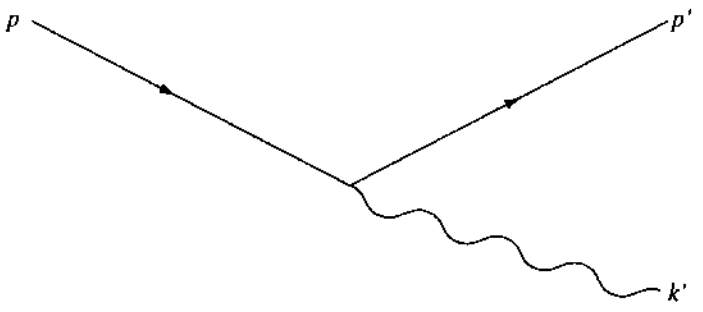


Figure 7.11 *The process $e^- \rightarrow e^- + \gamma$. The four-momenta of the particles are shown. The spin and polarization labels (s, s' and r) have been suppressed, as explained in the text*

$$\langle f|S^{(1)}|i\rangle = \left[(2\pi)^4\delta^4\left(p' + k' - p\right)\left(\frac{m}{VE_{\mathbf{p}}}\right)^{1/2}\left(\frac{m}{VE_{\mathbf{p}'}}\right)^{1/2}\left(\frac{1}{2V\omega_{\mathbf{k}'}}\right)^{1/2}\right]\mathcal{M} \quad (7.31)$$

where

$$\mathcal{M} = ie\bar{u}(\mathbf{p}')\not{\epsilon}(\mathbf{k}' = \mathbf{p} - \mathbf{p}')u(\mathbf{p}). \quad (7.32)$$

Eqs. (7.31) and (7.32) are our final result. $\mathcal{M}$ is called the *Feynman amplitude* for the process represented by the Feynman graph in Fig. 7.11. Since this diagram is labelled by the momenta (and the implied spin and polarization labels) of the particles involved, it is called a Feynman diagram in momentum space, in contrast to the configuration-space diagrams of the last section, e.g. Fig. 7.1(a).

The δ-function in Eq. (7.31) arose from the x-integration in Eq. (7.29) over the three exponential functions associated with the two fermion lines and the photon line, which meet at the vertex x. This δ-function ensures conservation of energy and momentum for this process: $p = p' + k'$. [Correspondingly, the argument of the polarization vector $\epsilon_\alpha(\mathbf{k}')$ in Eq. (7.32) was written $\mathbf{k}' = \mathbf{p} - \mathbf{p}'$.] We shall see that for more complicated Feynman diagrams, such a δ-function is obtained in this way for each vertex, ensuring energy–momentum conservation at each vertex and consequently for the process as a whole.

For the process $e^- \rightarrow e^- + \gamma$ and the other first-order processes, energy–momentum conservation is incompatible with the conditions for real particles ($p^2 = p'^2 = m^2$, $k'^2 = 0$, in our case), so these are not real processes, as stated earlier.

7.2.2 Compton scattering

As a second example, we calculate the matrix element for Compton scattering, for which the S-matrix operator and Feynman graphs were given in Eqs. (7.9) and (7.10) and Fig. 7.2. Their counterparts in momentum space are shown in Fig. 7.12, corresponding to the transition

$$|i\rangle = c^\dagger(\mathbf{p})a^\dagger(\mathbf{k})|0\rangle \rightarrow |f\rangle = c^\dagger(\mathbf{p}')a^\dagger(\mathbf{k}')|0\rangle. \quad (7.33)$$

The S-matrix element for this transition is derived from Eqs. (7.9) and (7.10). Using Eqs. (7.23a), (7.25) and (7.26), one obtains

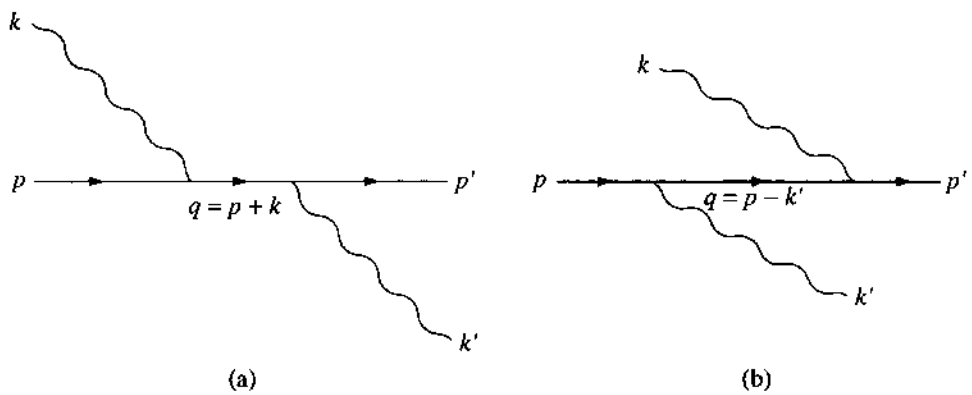


Figure 7.12 Compton scattering by electrons

$$\langle f|S_a|i\rangle = -e^2 \int d^4x_1\, d^4x_2 \left[\left(\frac{m}{VE_{\mathbf{p}'}}\right)^{1/2} \bar{u}(\mathbf{p}')e^{ip'x_1}\right]\left[\left(\frac{1}{2V\omega_{\mathbf{k}'}}\right)^{1/2} \not{\epsilon}(\mathbf{k}')e^{ik'x_1}\right]$$
$$\times \frac{1}{(2\pi)^4}\int d^4q\, iS_F(q)e^{-iq(x_1-x_2)}$$
$$\times \left[\left(\frac{1}{2V\omega_{\mathbf{k}}}\right)^{1/2} \not{\epsilon}(\mathbf{k})e^{-ikx_2}\right]\left[\left(\frac{m}{VE_{\mathbf{p}}}\right)^{1/2} u(\mathbf{p})e^{-ipx_2}\right]. \tag{7.34}$$

Note that u and $\bar{u}$ are four-component spinors and that S_F and the factors $\not{\epsilon}$ are 4×4 matrices. The spinor indices are suppressed, but these quantities must always be written in the correct order of matrix algebra.

The x_1 and x_2 integrations in Eq. (7.34) give

$$\int d^4x_1 \exp[ix_1(p' + k' - q)] \int d^4x_2 \exp[ix_2(q - p - k)]$$
$$= (2\pi)^4\delta^{(4)}(p' + k' - q)(2\pi)^4\delta^{(4)}(q - p - k)$$
$$= (2\pi)^4\delta^{(4)}(p' + k' - p - k)(2\pi)^4\delta^{(4)}(q - p - k). \tag{7.35}$$

Hence energy and momentum are conserved at each vertex and overall for the process. In particular, the energy–momentum q of the virtual intermediate electron is fixed:

$$q = p + k = p' + k'. \tag{7.36}$$

Substituting Eq. (7.35) in (7.34) and carrying out the q integration, one obtains

$$\langle f|S_a|i\rangle = \Big[(2\pi)^4\delta^{(4)}(p' + k' - p - k)$$
$$\times \left(\frac{m}{VE_{\mathbf{p}}}\right)^{1/2}\left(\frac{m}{VE_{\mathbf{p}'}}\right)^{1/2}\left(\frac{1}{2V\omega_{\mathbf{k}}}\right)^{1/2}\left(\frac{1}{2V\omega_{\mathbf{k}'}}\right)^{1/2}\Big]\mathcal{M}_a \tag{7.37}$$

where $\mathcal{M}_a$, the Feynman amplitude associated with Fig. 7.12(a), is given by

$$\mathcal{M}_a = -e^2 u(\mathbf{p}')\not{\epsilon}(\mathbf{k}')iS_F(q = p + k)\not{\epsilon}(\mathbf{k})u(\mathbf{p}). \tag{7.38a}$$

It is left as an exercise for the reader to show that the second contribution to Compton scattering, $\langle f|S_b|i\rangle$, is given by the same equation (7.37), with $\mathcal{M}_a$ replaced by the Feynman amplitude for Fig. 7.12(b):

$$\mathcal{M}_b = -e^2\bar{u}(\mathbf{p}')\not{\epsilon}(\mathbf{k})iS_F(q = p - k')\not{\epsilon}(\mathbf{k}')u(\mathbf{p}). \tag{7.38b}$$

Our result, Eqs. (7.37) and (7.38), displays some general features which always occur in calculating S-matrix elements by these methods.

Firstly, the factors in Eqs. (7.38a) and (7.38b) are in the correct spinor order. Comparing these expressions with the Feynman graphs, Figs. 7.12(a) and (b), we can describe this order as: following a fermion line *in the sense of its arrows*, corresponds to writing the spinor factors *from right to left*.

Secondly, comparing these results with Eqs. (7.31) and (7.32), we note many common features. The square brackets in Eqs. (7.31) and (7.37) each contain a δ-function for overall energy–momentum conservation (multiplied by $(2\pi)^4$), and factors $(1/2V\omega_{\mathbf{k}})^{1/2}$ and $(m/VE_{\mathbf{p}})^{1/2}$ for each external photon and fermion line respectively. The Feynman amplitudes (7.32) and (7.38) contain a factor (ie), associated with each vertex in the related Feynman graphs, and factors $\bar{u}$, u and $\not{\epsilon}$, associated in an obvious manner with external electron and photon lines. The one additional feature in Eqs. (7.38) is the presence of the factors $iS_F(q)$ which correspond to the intermediate fermion lines in diagrams 7.12(a) and (b). These common features are examples of Feynman rules, which will be fully discussed in the next section.

Finally, we see that, for both Figs. 7.12(a) and (b), the intermediate particle cannot be a real particle: $q^2 \neq m^2$, since we cannot have energy–momentum conservation for three real particles at a vertex. This is in contrast to the non-covariant perturbation theory of non-relativistic quantum mechanics, where time and space coordinates (and consequently energy and three-momentum) are treated on different footings: particles in intermediate states satisfy the energy–momentum conditions of real particles (i.e. $p^2 = m^2$, $k^2 = 0$), but energy is not conserved in intermediate states although three-momentum is.

We briefly consider Compton scattering by positrons in order to establish some differences of detail which occur for positrons. The Feynman graphs in momentum space for this process are shown in Fig. 7.13. We leave it as exercises for the reader to show from first principles that the Feynman diagram 7.13(a) again leads to Eq. (7.37), with $\mathscr{M}_a$ replaced by

$$\mathscr{M}'_a = e^2 v(\mathbf{p}) \not{\epsilon}(\mathbf{k}) iS_F(q = -p - k) \not{\epsilon}(\mathbf{k}') v(\mathbf{p}') \tag{7.39}$$

and to obtain the corresponding result for diagram 7.13(b).

In Eq. (7.39), the spinor $v(\mathbf{p}')$ relates to the final-state positron, and the spinor $\bar{v}(\mathbf{p})$ to the initial-state positron. The order of the spinor factors in this equation corresponds to writing these factors *from right to left* as one follows the fermion line *in the sense of its arrows*. This is the same prescription as for electrons. Care is also needed in interpreting the momentum labels on Feynman diagrams. For *external* lines, the momenta shown are the *actual* four-momenta of the particles present initially and finally. This applies to electrons, positrons and photons. This means that on external *electron* lines, the flow of four-momentum is in the *same* sense as that of the arrows on the lines; on external *positron* lines it is in the sense *opposite* to that of the arrows. On *internal* fermion lines, on the other

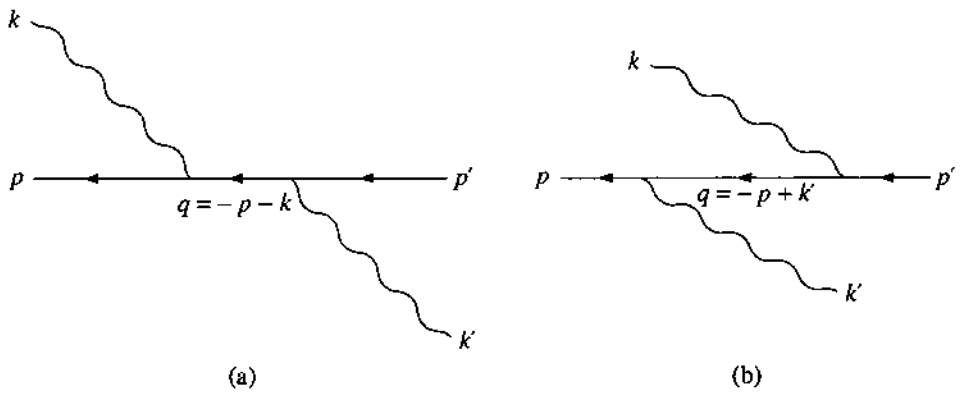


Figure 7.13 *Compton scattering by positrons*

hand, the four-momentum labels on Feynman graphs *always* represent energy–momentum flow in the *same* direction as the arrows.

This completes our detailed analysis of Compton scattering. In the following examples, the detailed derivations will be left as exercises for the reader, and we shall concentrate on the remaining features of Feynman graphs not yet encountered.

7.2.3 Electron–electron scattering

The Feynman diagrams in configuration space for Møller scattering were shown in Fig. 7.6. The corresponding momentum space graphs are shown in Fig. 7.14. The S-matrix element for the transition

$$|i\rangle = c^\dagger(\mathbf{p}_2)c^\dagger(\mathbf{p}_1)|0\rangle \rightarrow |f\rangle = c^\dagger(\mathbf{p}'_2)c^\dagger(\mathbf{p}'_1)|0\rangle \tag{7.40}$$

is obtained from Eqs. (7.17). One finds

$$\begin{aligned}&\langle f|S^{(2)}(2e^- \rightarrow 2e^-)|i\rangle \\ &= \left[(2\pi)^4\delta^{(4)}(p'_1 + p'_2 - p_1 - p_2)\Pi\left(\frac{m}{VE_\mathbf{p}}\right)^{1/2}\right](\mathcal{M}_a + \mathcal{M}_b)\end{aligned} \tag{7.41a}$$

where the meaning of $\Pi(m/VE_\mathbf{p})^{1/2}$ should be clear to the reader,[2] and where the Feynman amplitudes corresponding to Figs. 7.14(a) and (b) are given by

$$\mathcal{M}_a = -e^2\bar{u}(\mathbf{p}'_1)\gamma^\alpha u(\mathbf{p}_1)\mathrm{i}D_{\mathrm{F}\alpha\beta}(k = p_2 - p'_1)\bar{u}(\mathbf{p}'_2)\gamma^\beta u(\mathbf{p}_2) \tag{7.41b}$$

$$\mathcal{M}_b = +e^2\bar{u}(\mathbf{p}'_2)\gamma^\alpha u(\mathbf{p}_1)\mathrm{i}D_{\mathrm{F}\alpha\beta}(k = p_2 - p'_1)\bar{u}(\mathbf{p}'_1)\gamma^\beta u(\mathbf{p}_2). \tag{7.41c}$$

The last two equations exhibit explicitly the relative minus sign of the direct and exchange amplitudes, which reflects the exclusion principle, as discussed in the last section. The new feature in these equations is the appearance of the factors $\mathrm{i}D_{\mathrm{F}\alpha\beta}(k)$, corresponding to the

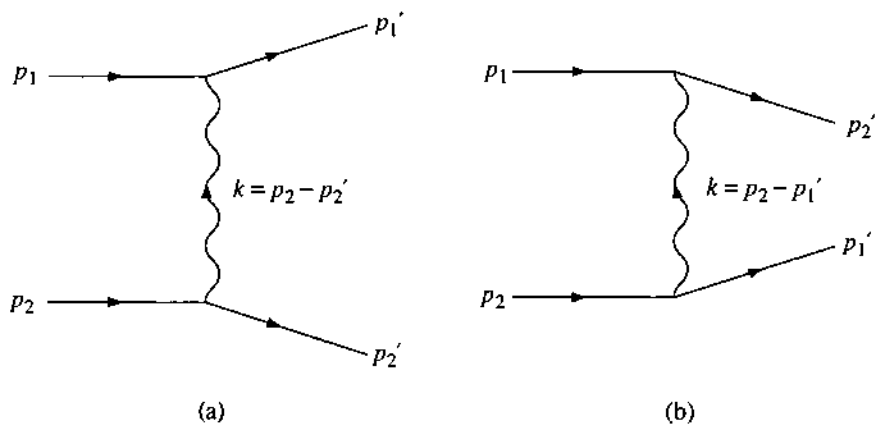


Figure 7.14 *Electron–electron scattering (Møller scattering)*

[2] i.e. even *without* detailed derivation, but Eqs. (7.41) should of course be derived by the reader.

internal photon lines in the Feynman graphs 7.14. Since from Eq. (7.24b) $D_{F\alpha\beta}(k) = D_{F\alpha\beta}(-k)$, the sense of k along an internal photon line is arbitrary. However, a definite direction must be chosen for k in order to assign consistent signs to k in the δ-functions associated with the vertices at the two ends of an internal photon line. For example, with the choice for k in Fig. 7.14(a), i.e. from the bottom to the top vertex, $p_2 = p'_2 + k$ and $p_1 + k = p'_1$, giving the correct overall energy–momentum conservation for the process.

7.2.4 Closed loops

A new feature occurs for Feynman diagrams containing closed loops of internal lines, such as the electron and photon self-energy diagrams, Figs. 7.8 and 7.9. For diagrams without loops, such as we have been considering in this section so far, energy–momentum conservation at the vertices determines the four-momenta of all internal lines completely. For loop diagrams this is not the case. Consider, as a typical example, the electron self-energy. Its Feynman graph in momentum space is shown in Fig. 7.15. Conservation of energy and momentum at the two vertices gives

$$p = q + k = p', \tag{7.42}$$

but this does not determine the internal momenta k and q separately. The intuitive response to this is that one must sum over all allowed values of k and q to find the total amplitude.

To see that this conjecture is correct, we require the matrix element of the electron self-energy operator $S^{(2)}(e^- \to e^-)$, Eq. (7.20), for the transition

$$|i\rangle = c^\dagger(\mathbf{p})|0\rangle \to |f\rangle = c^\dagger(\mathbf{p}')|0\rangle. \tag{7.43}$$

We leave it to the reader to obtain the result

$$\begin{aligned}
&\langle f|S^{(2)}(e^- \to e^-)|i\rangle \\
&= -e^2\left(\frac{m}{VE_{\mathbf{p}}}\right)^{1/2}\left(\frac{m}{VE_{\mathbf{p}'}}\right)^{1/2}\int \mathrm{d}^4q\ \mathrm{d}^4k\,\delta^{(4)}(p'-k-q)\delta^{(4)}(k+q-p) \\
&\quad \times\ \mathrm{i}D_{F\alpha\beta}(k)\bar{u}(\mathbf{p}')\gamma^\alpha \mathrm{i}S_F(q)\gamma^\beta u(\mathbf{p}) \\
&= \left[(2\pi)^4\delta^{(4)}(p'-p)\left(\frac{m}{VE_{\mathbf{p}}}\right)^{1/2}\left(\frac{m}{VE_{\mathbf{p}'}}\right)^{1/2}\right]\mathscr{M}
\end{aligned} \tag{7.44a}$$

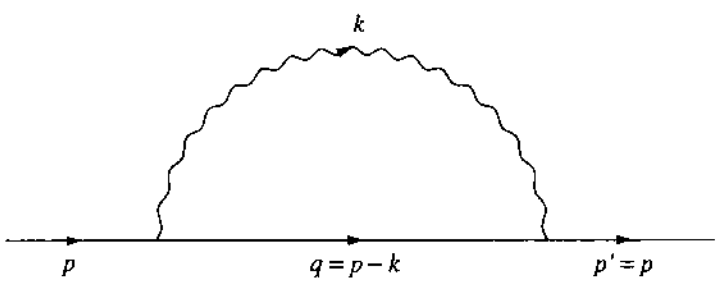


Figure 7.15 *The electron self-energy*

where

$$\mathcal{M} = \frac{-e^2}{(2\pi)^4} \int \mathrm{d}^4 k \mathrm{i} D_{\mathrm{F}\alpha\beta}(k) \overline{u}(\mathbf{p}) \gamma^\alpha \mathrm{i} S_{\mathrm{F}}(p-k) \gamma^\beta u(\mathbf{p}). \tag{7.44b}$$

As expected, this Feynman amplitude contains an integration over all internal photon momenta k, while for each value of k the internal fermion momentum has the value $q = p - k$, which corresponds to energy–momentum conservation at the vertices. This integration over an internal momentum is typical of a closed loop. (For another example, see Problem 7.2 on the photon self-energy.)

Eqs. (7.44) display the same structure which occurred in all our other examples. The origins of the individual factors in Eq. (7.44a) should be clear to the reader. The Feynman amplitude $\mathcal{M}$ contains factors $\mathrm{i}S_{\mathrm{F}}$ and $\mathrm{i}D_{\mathrm{F}}$ for fermion and photon propagators, spinors u and $\overline{u}$ for the initial and final external electron lines, and a γ-factor for each vertex. The spinor quantities are in the expected order as one follows the fermion line in the direction of its arrows. The remaining factor $(-e^2) = (\mathrm{i}e)^2$ has its origin in the form of $S^{(2)}$ for QED, Eqs. (7.1) and (7.2).

7.3 Feynman Rules for QED

The S-matrix elements $\langle f|S|i\rangle$, which we have calculated for various processes, exhibit a definite structure, which allows one to identify individual factors and features with different aspects of the corresponding Feynman graphs. The same identification between the mathematical expressions and Feynman graphs is possible for all processes. Furthermore, no new features occur for other processes. This enables one to construct a set of rules for writing $\langle f|S|i\rangle$ down directly from the Feynman graphs.

In this section we shall state these rules for QED. They represent a generalization and a tidying-up of earlier results. Their origins should be clear to the reader. Where appropriate, we shall give explanations and cross-references, but we do not repeat everything from scratch; for example, we are assuming the conventions about arrows and momentum labels for Feynman graphs.

The expression for the S-matrix element for a transition can at once be written down by generalizing our earlier results. For the transition $|i\rangle \to |f\rangle$, where the initial and final states are specified by the momenta (and spin and polarization variables) of the particles present, the S-matrix element is given by

$$\langle f|S|i\rangle = \delta_{fi} + \left[(2\pi)^4 \delta^{(4)}(P_f - P_i) \prod_{\text{ext.}} \left(\frac{m}{VE}\right)^{1/2} \prod_{\text{ext.}} \left(\frac{1}{2V\omega}\right)^{1/2}\right] \mathcal{M}. \tag{7.45}$$

Here P_i and P_f are the total four-momenta in the initial and final states, and the products extend over all external fermions (e^- and e^+) and photons, E and ω being the energies of the individual external fermions and photons respectively.

The Feynman amplitude $\mathcal{M}$ is given by

$$\mathcal{M} = \sum_{n=1}^{\infty} \mathcal{M}^{(n)} \tag{7.46}$$

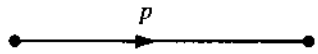
where the contribution $\mathcal{M}^{(n)}$ comes from the nth order perturbation term $S^{(n)}$. The Feynman amplitude $\mathcal{M}^{(n)}$ is obtained by drawing all topologically different, connected Feynman graphs in momentum space which contain n vertices and the correct external lines. The contribution to $\mathcal{M}^{(n)}$ from each graph is obtained from the following Feynman rules.

1. For each vertex, write a factor $ie\,\gamma^{\alpha}$ [see Eqs. (7.1) and (7.2)].
2. For each internal photon line, labelled by the momentum k, write a factor [see Eq. (7.24b)]

$$iD_{F\alpha\beta}(k) = i\frac{-g_{\alpha\beta}}{k^2 + i\varepsilon}. \qquad (7.47)$$

3. For each internal fermion line, labelled by the momentum p, write a factor [see Eq. (7.24a)]

$$iS_F(p) = i\frac{1}{\not{p} - m + i\varepsilon}. \qquad (7.48)$$



4. For each external line, write one of the following factors [see Eqs. (7.25) and (7.26)]:

   (a) for each initial electron:

$$u_r(\mathbf{p}) \qquad (7.49a)$$

   (b) for each final electron:

$$\bar{u}_r(\mathbf{p}) \qquad (7.49b)$$

   (c) for each initial positron:

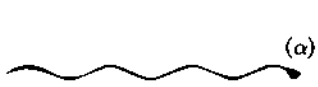
$$\bar{v}_r(\mathbf{p}) \qquad (7.49c)$$

   (d) for each final positron:

$$v_r(\mathbf{p}) \qquad (7.49d)$$

   (e) for each initial photon:

$$\varepsilon_{r\alpha}(\mathbf{k}) \qquad (7.49e)$$



   (f) for each final photon[3]:

$$\varepsilon_{r\alpha}(\mathbf{k}) \qquad (7.49f)$$

   In Eqs. (7.49) $\mathbf{p}$ and $\mathbf{k}$ denote the three-momenta of the external particles, and $r\,(=1, 2)$ labels their spin and polarization states.

5. The spinor factors (γ-matrices, S_F-functions, four-spinors) for each fermion line are ordered so that, reading from right to left, they occur in the same sequence as following the fermion line in the direction of its arrows.
6. For each closed fermion loop, take the trace and multiply by a factor (-1).

[3] For linear polarization states, which we are using, $\varepsilon_{r\alpha}(\mathbf{k})$ is real. In general it is complex (e.g. for circular polarization), and we must then replace $\varepsilon_{r\alpha}(\mathbf{k})$ by $\varepsilon^{*}_{r\alpha}(\mathbf{k})$ for a final-state photon.

This rule follows directly from the corresponding result in configuration space, derived in Section 7.1 [see Eq. (7.22)].

7. The four-momenta associated with the three lines meeting at each vertex satisfy energy–momentum conservation. For each four-momentum q which is not fixed by energy–momentum conservation, carry out the integration $(2\pi)^{-4}\int \mathrm{d}^4q$. One such integration with respect to an internal momentum variable q occurs for each closed loop.

We had an example of this rule in Eq. (7.44b) for the electron self-energy. Inclusion of the factors $(2\pi)^{-4}$ in this rule is convenient, since all numerical factors (except for the phase factor of rule 8) are accounted for in this way, as will be shown below.

8. Multiply the expression by a phase factor δ_p, which is equal to $+1$ (-1) if an even (odd) number of interchanges of neighbouring fermion operators is required to write the fermion operators in the correct normal order.

In general this phase factor is only of significance when the contributions of several Feynman graphs are added, and only the relative signs matter. The situation most frequently met is the one we discussed for (e^-e^-)- and (e^-e^+)- scattering, involving contributions from diagrams which differ only by the interchange of external fermion lines associated with identical fermion operators. This corresponds to the interchange of either (i) two initial e^- (e^+) lines, or (ii) two final e^- (e^+) lines, or (iii) an initial $e^-(e^+)$ line with a final e^+ (e^-) line.

It remains to justify our assertion, made when discussing rule 7, that the above rules allow for all numerical factors. The only factors not taken into account so far are factors $(2\pi)^4$ which occur together with δ-functions or result from propagators. The x-integration at each vertex gives a factor $(2\pi)^4$ [see Eq. (7.30)], and the Fourier transform of each propagator gives a factor $(2\pi)^{-4}$ [see Eqs. (7.23)]. For a Feynman diagram containing n vertices and $f_i(b_i)$ internal fermion (photon) lines, the Feynman amplitude contains a factor

$$\left[(2\pi)^4\right]^{n-f_i-b_i-1} \tag{7.50}$$

where the exponent -1 allows for the factor $(2\pi)^4$, which was separated out in Eq. (7.45). It is left as a problem for the reader to show that for a Feynman diagram containing l closed loops

$$n - f_i - b_i - 1 = -l. \tag{7.51}$$

Since one momentum integral $\int \mathrm{d}^4q$ occurs for each loop, we may omit the factor (7.50) from the Feynman amplitude expression, provided we replace each loop integral $\int \mathrm{d}^4q$ by $(2\pi)^{-4}\int d^4q$.

This completes our discussion of Feynman rules for QED. The reader is recommended to use Feynman rules to re-derive the matrix elements $\langle f|S|i\rangle$, which we obtained from first principles earlier in this chapter. After a little practice, Feynman rules provide an extraordinarily simple method for obtaining even very complicated matrix elements, and for this reason they form the basis of most practical calculations. Similar diagrammatic techniques are also of great importance in many other fields, e.g. weak interactions (to be studied later in this book) and condensed matter physics, where analogous rules can be developed. In Appendix B, at the end of this book, we give a summary of these rules for QED, as well as those for the standard electro-weak theory, which will be derived later.

7.4 Leptons

So-far we have treated QED as the interaction of electrons and positrons with the electromagnetic field. More generally, QED is usually understood to include the interactions of all charged leptons with the electromagnetic field. In addition to electrons,[4] these are the muons ($\mu^{\pm}$) and the tauons ($\tau^{\pm}$). Both muons and tauons have spin $\frac{1}{2}$ and charge $\pm e$. Furthermore, within experimental accuracy, which is very high, they exhibit all the properties of particles whose interactions are identical with those of electrons, except for their masses: $m_\mu = 105.7$ MeV, $m_\tau = 1776.84 \pm 0.17$ MeV. This is referred to as e–μ–τ *universality*. This extended QED, which we shall now study, displays a new richness: processes involving more than one kind of lepton.

Assuming universality, the extension of the theory is almost trivial. Like the electron, we describe each kind of lepton by a Dirac spinor field: $\psi_l(x)$, where l labels the kind of lepton: $l = e, \mu, \tau$. The generalization of the free-field Lagrangian density for electrons, Eq. (4.67), is

$$\mathscr{L}_0 = \sum_l \overline{\psi}_l(x)(\mathrm{i}\gamma^\alpha \partial_\alpha - m_l)\psi_l(x), \tag{7.52a}$$

and making the minimal substitution (4.64b) (with $q = -e$) leads to interaction Hamiltonian density

$$\mathscr{H}_{\mathrm{I}}(x) = - \quad _{\mathrm{I}}(x) = -e \sum_l \mathrm{N}\left[\overline{\psi}_l(x) \not{A}(x) \psi_l(x)\right]. \tag{7.52b}$$

This equation describes a *local* interaction, since all field operators are evaluated at a single space–time point. This is appropriate for the interaction of the electromagnetic field with a point particle. While, within the limits of current experiment, leptons are point-like, hadrons have finite size. For example, the experimental value of the proton radius is of the order[5] 0.8×10^{-15} m. For this reason the electromagnetic interactions of charged hadrons cannot be described by expressions like (7.52b).

The second point to note about the interaction (7.52b) is that it consists of a sum of terms each of which involves *one* kind of lepton only. Hence, the interaction is described by basic vertex parts like those of Fig. 7.1, with *both* fermion lines at a vertex referring to the *same* kind of lepton. Instead of two electrons, as in Fig. 7.1, they could both be muons or both tauons. But we could not, for example, have one electron and one muon. The vertex part in Fig. 7.16 conserves charge, but it does not occur with the interaction (7.52b), since it would require an interaction term of the form $-e\overline{\psi}_\mu \not{A} \psi_e$. Consequently, for any non-vanishing matrix element $\langle j|\mathscr{H}_{\mathrm{I}}|i\rangle$ the *electron number* $N(e)$, defined by

$$N(e) = N(e^-) - N(e^+) \tag{7.53a}$$

[4] Just as the names muon and tauon refer to both positively and negatively charged particles, so it is convenient to have a single word for electron and positron. It is usual to use electron for this purpose as well as for the negatively charged member of this pair. We shall follow this practice, adding the appropriate qualification when it is required to avoid ambiguity.

[5] The point-like nature of leptons has been tested to much shorter distances. (See Sections 8.4 and 8.5.)

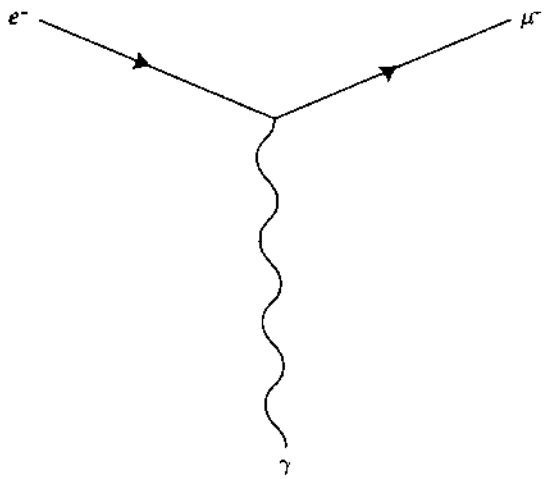


Figure 7.16 *A basic vertex part that does NOT occur with the interaction (7.52b), which conserves electron and muon numbers at each vertex*

in an obvious notation, is conserved, as are the *muon and tauon numbers*[6]

$$N(\mu) = N(\mu^-) - N(\mu^+) \tag{7.53b}$$

$$N(\tau) = N(\tau^-) - N(\tau^+). \tag{7.53c}$$

Consequently processes like

$$e^- + \mu^+ \rightarrow e^+ + \mu^-, \tag{7.54}$$

although they conserve charge, are forbidden, and indeed are not observed.

The extension of the S-matrix formalism and of the Feynman rules to the QED interaction (7.52b) is now straightforward. Each term $(\overline{\psi}\not{A}\psi)$ in our original interaction (7.2), which allowed for electrons only, is replaced by a sum $\Sigma_l(\overline{\psi}_l\not{A}\psi_l)$, and the S-matrix expansion (7.1) leads to

$$S = \sum_{n=0}^{\infty} \frac{(ie)^n}{n!} \int \ldots \int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n \sum_{l_1} \ldots \sum_{l_n}$$

$$\times \mathrm{T}\{\mathrm{N}(\overline{\psi}_{l_1}\not{A}\psi_{l_1})_{x_1} \ldots \mathrm{N}(\overline{\psi}_{l_n}\not{A}\psi_{l_n})_{x_n}\}. \tag{7.55}$$

This expansion first of all contains terms involving one kind of lepton only. These are the terms we considered in the last two sections, but we could now be considering muons or tauons instead of electrons. $S^{(1)}$ is of this type and so are the terms in $S^{(2)}$ with $l_1 = l_2$ (= e, μ, ...). The more interesting terms, for which new processes occur, are those involving more than one kind of lepton.

[6] These definitions will be modified when we consider weak interactions.

Consider, for example, the $l_1 = \mu$, $l_2 = e$ term in $S^{(2)}$, given by

$$S^{(2)}_{\mu e} = -e^2 \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 \mathrm{T}\{\mathrm{N}(\overline{\psi}_\mu A\!\!\!/ \psi_\mu)_{x_1} \mathrm{N}(\overline{\psi}_e A\!\!\!/ \psi_e)_{x_2}\}. \tag{7.56}$$

Using Wick's theorem [Eqs. (6.35) and (6.38)] to expand the T-product in terms of normal products, we obtain

$$\begin{aligned} S^{(2)}_{\mu e} = & -e^2 \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 \mathrm{N}[(\overline{\psi}_\mu A\!\!\!/ \psi_\mu)_{x_1} (\overline{\psi}_e A\!\!\!/ \psi_e)_{x_2}] \\ & - e^2 \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 \mathrm{N}[(\overline{\psi}_\mu A\!\!\!/ \, \psi_\mu)_{x_1} (\overline{\psi}_e A\!\!\!/ \, \psi_e)_{x_2}]. \end{aligned} \tag{7.57}$$

All other unequal-time contractions vanish, as follows from the definition (6.31) of a contraction as a vacuum expectation value.[7]

The first term in Eq. (7.57), like the term $S^{(2)}_{\mathrm{A}}$ Eq. (7.5a), corresponds to two independent unphysical processes of the kind shown in Fig. 7.1, except that now one refers to a muon instead of an electron.

The second term in Eq. (7.57) gives rise to processes which involve two external muons and two external electrons, and which must conserve charge, electron number and muon number. These include electron–muon scattering and, more interestingly, the process

$$e^+ + e^- \rightarrow \mu^+ + \mu^-, \tag{7.58}$$

i.e. the annihilation of an (e^+e^-) pair leading to the creation of a $(\mu^+\mu^-)$ pair. The term in $S^{(2)}_{\mu e}$ responsible for this process is

$$\begin{aligned} & S^{(2)}(e^+e^- \rightarrow \mu^+\mu^-) \\ & = -e^2 \int \mathrm{d}^4x_1 \ \mathrm{d}^4x_2 \mathrm{N}[(\overline{\psi}^-_\mu \gamma^\alpha \psi^-_\mu)_{x_1} (\overline{\psi}^+_e \gamma^\beta \psi^+_e)_{x_2}] \mathrm{i} D_{\mathrm{F}\alpha\beta}(x_1 - x_2), \end{aligned} \tag{7.59}$$

and from this operator one can calculate the transition matrix elements. For the transition shown in the Feynman graph in Fig. 7.17, i.e.

$$\begin{aligned} |i\rangle = |e^-\mathbf{p}_2; e^+\mathbf{p}_1\rangle &= c_e{}^\dagger(\mathbf{p}_2) d_e^\dagger(\mathbf{p}_1)|0\rangle \\ \rightarrow |f\rangle = |\mu^-\mathbf{p}'_2; \mu^+\mathbf{p}'_1\rangle &= c^\dagger_\mu(\mathbf{p}'_2) d^\dagger_\mu(\mathbf{p}'_1)|0\rangle, \end{aligned} \tag{7.60}$$

one would in this way obtain the Feynman amplitude

$$\mathscr{M}^{(2)}(e^+e^- \rightarrow \mu^+\mu^-) = -\mathrm{i}e^2 \overline{u}_\mu(\mathbf{p}'_2)\gamma^\alpha v_\mu(\mathbf{p}'_1) D_{\mathrm{F}\alpha\beta}(p_1 + p_2) \overline{v}_e(\mathbf{p}_1)\gamma^\beta u_e(\mathbf{p}_2). \tag{7.61}$$

Here the labels e and μ, attached to the fermion lines in Fig. 7.17, to creation and absorption operators in Eq. (7.60), and to spinors in Eq. (7.61), distinguish electrons and muons.

We do not advocate deriving the Feynman amplitude (7.61) from first principles (other than as an exercise), since it is trivial to extend the rules for calculating amplitudes, which were given in the last section, to QED involving several leptons. The S-matrix operator

[7] When dealing with several fermion fields, we must assume that the field operators for different fermion fields anticommute. We continue to assume that fermion and boson field operators always commute with each other. (See J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields*, McGraw-Hill, New York, 1965, p. 98.)

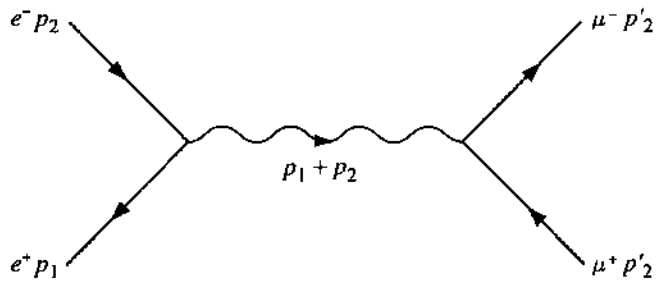


Figure 7.17 *The process* $e^+ + e^- \rightarrow \mu^+ + \mu^-$

(7.59) differs from the operator S_b, Eq. (7.19c), for the annihilation diagram, Fig. 7.7(b), for (e^+e^-) scattering in that in the final state the electrons are replaced by muons. Hence the amplitude (7.61) can be derived in two steps.

Firstly, obtain the Feynman amplitude $\mathcal{M}_b$ corresponding to the operator S_b, Eq. (7.19c), for the transition analogous to (7.60), but with all particles electrons. $\mathcal{M}_b$ can be written down directly, using the Feynman rules of Section 7.3.

Secondly, in the expression for $\mathcal{M}_b$, replace all quantities referring to the final state electrons by the corresponding quantities for muons.

It is left as an exercise for the reader to verify that this procedure leads to Eq. (7.61).

There is one important difference between the $e^+ + e^- \rightarrow \mu^+ + \mu^-$ process and (e^+e^-) scattering. For the latter, a second contribution stems from S_a, Eq. (7.19b), corresponding to Fig. 7.7(a). For the former, as we have seen, Eq. (7.57) gives no such contribution. It would correspond to replacing the final electron lines in Fig. 7.7(a) by muon lines, so that each vertex would involve one electron line and one muon line (e.g. the vertex at the top would look like Fig. 7.16), violating the conservation of both electron number $N(e)$ and of muon number $N(\mu)$ at each vertex.

From this example, it is easy to see how to extend the rules of Section 7.3. For any process, one must draw all relevant Feynman diagrams which conserve $N(e)$, $N(\mu)$,... at each vertex, i.e. the two lepton lines entering and leaving a vertex must be of the same kind (both e or both μ, etc.). The Feynman amplitude corresponding to each of these diagrams is then written down directly using the Feynman rules of the last section.

Problems

7.1. Derive the lowest-order non-vanishing S-matrix element (7.19) and hence the corresponding Feynman amplitude for Bhabha scattering, i.e. the process

$$e^+(\mathbf{p}_1, r_1) + e^-(\mathbf{p}_2, r_2) \rightarrow e^+(\mathbf{p}'_1, s_1) + e^-(\mathbf{p}'_2, s_2).$$

7.2. Show that the Feynman amplitude for the photon self-energy diagram in Fig. 7.9 is given by

$$\mathcal{M} = \frac{-e^2}{(2\pi)^4} \int d^4p \mathrm{Tr}[\not{\epsilon}_r(\mathbf{k}) S_F(p+k) \not{\epsilon}_r(\mathbf{k}) S_F(p)],$$

where $\mathbf{k}$ and $\varepsilon_r(\mathbf{k})$ are the momentum and polarization vectors of the photon.

7.3. A real scalar field $\phi(x)$, associated with a spin-zero boson B, is described by the Lagrangian density

$$\mathscr{L}(x) = \mathscr{L}_0(x) + \mathscr{L}_1(x)$$

where $\mathscr{L}_0$ is the free-field density (3.4), and

$$\mathscr{L}_1(x) = g[\phi(x)]^4/4!$$

describes an interaction of the field with itself, with g a real coupling constant. (Normal ordering of operators is assumed throughout.)

Write down the S-matrix expansion, and pick out the normal ordered term that gives rise to the BB scattering process

$$B(\mathbf{k}_1) + B(\mathbf{k}_2) \rightarrow B(\mathbf{k}_3) + B(\mathbf{k}_4)$$

in first-order perturbation theory. Draw the Feynman diagram representing this term, and show that the corresponding S-matrix element is given by

$$\langle k_3, k_4 | S^{(1)} | k_1, k_2 \rangle = (2\pi)^4 \delta^{(4)}(k_3 + k_4 - k_1 - k_2) \prod_i \left(\frac{1}{2V\omega_i} \right)^{1/2} \mathscr{M}$$

with the Feynman amplitude $\mathscr{M} = \mathrm{i}g$. [Note that $\mathscr{M}$ is independent of the boson four-momenta $k_i^\alpha \equiv (\omega_i, \mathbf{k}_i)$.]

7.4. Pseudo-scalar meson theory is defined by the Lagrangian density

$$\mathscr{L}(x) = \mathscr{L}_0(x) + \mathscr{L}_1(x)$$

where

$$\mathscr{L}_0(x) = \frac{1}{2}\left[\partial_\alpha \phi(x) \partial^\alpha \phi(x) - \mu^2 \phi^2(x)\right] + \overline{\psi}(x)(\mathrm{i}\gamma^\alpha \partial_\alpha - m)\psi(x)$$

represents a free real spin 0 field $\phi(x)$ and a free fermion field $\psi(x)$, and

$$\mathscr{L}_1(x) = -\mathrm{i}g\overline{\psi}(x)\gamma_5 \psi(x)\phi(x)$$

describes their interaction.

The interaction Lagrangian density $\mathscr{L}_1(x)$ is similar to that of QED, except that $e\gamma^\alpha$ is replaced by $(-\mathrm{i}g\gamma_5)$, and the photon field $A_\alpha(x)$ is replaced by the meson field $\phi(x)$. Exploit this similarity to write down the Feynman rules for pseudo-scalar meson theory.

7.5. A real scalar field $\phi(x)$ is described by the Lagrangian density

$$\mathscr{L}(x) = \mathscr{L}_0(x) + \mu U(\mathbf{x})\phi^2(x),$$

where $\mathscr{L}_0$ is the free-field Lagrangian density (3.4), and $U(\mathbf{x})$ *is* a static external potential.

Derive the equation of motion

$$(\Box + \mu^2)\phi(x) = 2\mu U(\mathbf{x})\phi(x).$$

Show that, in lowest order, the S-matrix element for an incoming boson, with momentum $k_i = (\omega_i, \mathbf{k}_i)$, to be scattered to a state with momentum $k_f = (\omega_f, \mathbf{k}_f)$, is given by

$$\left\langle \mathbf{k}_f \left| S^{(1)} \right| \mathbf{k}_i \right\rangle = \frac{\mathrm{i}2\pi\delta(\omega_f - \omega_i)}{(2V\omega_i)^{1/2}(2V\omega_f)^{1/2}} 2\mu\, \tilde{U}(\mathbf{k}_f - \mathbf{k}_i)$$

where

$$\tilde{U}(\mathbf{q}) = \int \mathrm{d}^3\mathbf{x} U(\mathbf{x}) \mathrm{e}^{-\mathrm{i}q\cdot x}.$$

This type of problem, with a static external potential, will be considered further in Chapter 8.

8

QED Processes in Lowest Order

In the last chapter we established the Feynman rules for obtaining the matrix element S_{fi} for any collision process in QED. In this chapter we shall start by deriving from S_{fi} the experimentally observable quantities, i.e. the cross-sections. This is a straightforward generalization of the corresponding kinematical and phase–space arguments of non-relativistic collision theory.

The cross-sections obtained in this way are fully polarized, i.e. the photons and leptons present initially and finally are in definite polarization states. (As is customary, we use the term 'polarization state' for both photons and fermions, meaning a spin state in the latter case.) In most practical situations, the beams of colliding particles are unpolarized, and the polarizations of the particles produced in the collision are not observed. It then becomes necessary to average and sum over polarization states of initial and final particles, respectively. The very powerful and elegant techniques for performing these spin and polarization sums are developed in Sections 8.2 and 8.3. The corresponding formalism for analysing polarization properties is more complex, and we shall consider a simple example only.

In Sections 8.4–8.6, we shall illustrate our results by deriving the cross-sections, in lowest non-vanishing order of perturbation theory, for some of the processes considered in the previous chapter. By the end of this chapter, the reader should be able to deal in a similar way with any collision problem in QED. (A reader who tires of these applications should not be tempted also to omit Sections 8.7–8.9, which introduce some fundamental new ideas.)

We shall extend the S-matrix formalism to allow for the presence of an external electromagnetic field, i.e. of a field whose quantum fluctuations are negligible, so that it can be described by an unquantized classical field. As an application of these ideas, we shall consider the scattering of electrons by the Coulomb field of a nucleus, both elastic

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scattering (Section 8.7) and inelastic scattering, accompanied by emission of radiation, i.e. bremsstrahlung (Section 8.8).

In studying these Coulomb scattering processes, we shall encounter a new feature. There exists the possibility of the emission by a charged particle of one or more *very soft* photons (i.e. with very little energy). Experimentally, because of finite energy resolution, the distinction between elastic and inelastic scattering becomes blurred. This unrealistic separation into elastic and inelastic scattering events leads to the infrared divergence. In the last section of this chapter we shall see how this difficulty is resolved.

8.1 The Cross-Section

We consider a scattering process in which two particles, they may be leptons or photons, with four-momenta $p_i = (E_i, \mathbf{p}_i)$, $i = 1, 2$, collide and produce N final particles with momenta $p'_f = \left(E'_f, \mathbf{p}'_f\right)$, $f = 1, \ldots, N$. Initial and final particles are assumed to be in definite polarization states. As in Chapter 7, the indices labelling these states will in general be suppressed. Eq. (7.45), defining the Feynman amplitude $\mathscr{M}$ for this process, can now be written

$$\begin{aligned} S_{fi} = \ & \delta_{fi} + (2\pi)^4 \delta^{(4)}\Big(\sum p'_f - \sum p_i\Big) \prod_i \left(\frac{1}{2VE_i}\right)^{1/2} \\ & \times \prod_f \left(\frac{1}{2VE'_f}\right)^{1/2} \prod_l (2m_l)^{1/2} \mathscr{M} \end{aligned} \tag{8.1}$$

where the index l runs over all external leptons in the process.

Eq. (8.1) corresponds to the limit of an infinite time interval, $T \to \infty$, and an infinite volume, $V \to \infty$. For finite T and V, we would have obtained the same expression (8.1) with

$$\begin{aligned} (2\pi)^4 \delta^{(4)}\Big(\sum p'_f - \sum p_i\Big) &= \lim_{\substack{T\to\infty \\ V\to\infty}} \delta_{TV}\Big(\sum p'_f - \sum p_i\Big) \\ &\equiv \lim_{\substack{T\to\infty \\ V\to\infty}} \int_{-T/2}^{T/2} \mathrm{d}t \int_V \mathrm{d}^3\mathbf{x} \exp\Big[\mathrm{i}x\Big(\sum p'_f - \sum p_i\Big)\Big] \end{aligned} \tag{8.2}$$

replaced by $\delta_{TV}\left(\Sigma p'_f - \Sigma p_i\right)$. In deriving the cross-section, it will help to take T and V finite, to begin with. In this case the transition probability per unit time

$$w = |S_{fi}|^2 / T \tag{8.3}$$

involves the factor $\left[\delta_{TV}\left(\Sigma p'_f - \Sigma p_i\right)\right]^2$. For large values of T and V, we can then take

$$\delta_{TV}\Big(\sum p'_f - \sum p_i\Big) = (2\pi)^4 \delta^{(4)}\Big(\sum p'_f - \sum p_i\Big) \tag{8.4}$$

and

$$\left[\delta_{TV}\left(\sum p'_f - \sum p_i\right)\right]^2 = TV(2\pi)^4\delta^{(4)}\left(\sum p'_f - \sum p_i\right) \tag{8.5}$$

with errors which tend to zero as $T \to \infty$ and $V \to \infty$. Hence Eq. (8.3) becomes

$$w = V(2\pi)^4\delta^{(4)}\left(\sum p'_f - \sum p_i\right)\left(\prod_i \frac{1}{2VE_i}\right)\left(\prod_f \frac{1}{2VE'_f}\right)\left(\prod_l (2m_l)\right)|\mathcal{M}|^2. \tag{8.6}$$

Eq. (8.6) is the transition rate to one definite final state. To obtain the transition rate to a group of final states with momenta in the intervals $\left(\mathbf{p}'_f,\ \mathbf{p}'_f + \mathrm{d}\mathbf{p}'_f\right)$, $f = 1, \ldots, N$, we must multiply w by the number of these states which is

$$\prod_f \frac{V\mathrm{d}^3\mathbf{p}'_f}{(2\pi)^3}. \tag{8.7}$$

The differential cross-section is the transition rate into this group of final states for one scattering centre and unit incident flux. With our choice of normalization for the states, the volume V which we are considering contains one scattering centre, and the incident flux is v_{rel}/V, where v_{rel} is the relative velocity of the colliding particles.

Combining these results with Eq. (8.6), we obtain the required expression for the differential cross-section

$$\begin{aligned}\mathrm{d}\sigma &= w\frac{V}{v_{\mathrm{rel}}}\prod_f \frac{V\mathrm{d}^3\mathbf{p}'_f}{(2\pi)^3} \\ &= (2\pi)^4\delta^{(4)}\left(\sum p'_f - \sum p_i\right)\frac{1}{4E_1E_2v_{\mathrm{rel}}}\left(\prod_l (2m_l)\right)\left(\prod_f \frac{\mathrm{d}^3\mathbf{p}'_f}{(2\pi)^3 2E'_f}\right)|\mathcal{M}|^2.\end{aligned} \tag{8.8}$$

Eq. (8.8) holds in any Lorentz frame in which the colliding particles move collinearly. In such a frame the relative velocity v_{rel} is given by the expression

$$E_1E_2v_{\mathrm{rel}} = \left[(p_1p_2)^2 - m_1^2m_2^2\right]^{1/2}, \tag{8.9}$$

where m_1 and m_2 are the rest masses of the colliding particles. Two important examples of such frames are the centre-of-mass (CoM) system, and the laboratory (Lab) system. In the CoM system we have $\mathbf{p}_1 = -\mathbf{p}_2$, and hence

$$v_{\mathrm{rel}} = \frac{|\mathbf{p}_1|}{E_1} + \frac{|\mathbf{p}_2|}{E_2} = |\mathbf{p}_1|\frac{E_1 + E_2}{E_1E_2} \qquad \text{(CoM)}. \tag{8.10a}$$

In the laboratory system, the target particle (particle 2, say) is at rest, $\mathbf{p}_2 = 0$, and

$$v_{\mathrm{rel}} = \frac{|\mathbf{p}_1|}{E_1} \qquad \text{(Lab)}. \tag{8.10b}$$

Eqs. (8.10) of course also follow from the general result (8.9).

The relativistic invariance of the cross-section formula (8.8) follows from Eq. (8.9) and from the Lorentz invariance of $d^3\mathbf{p}/2E$ for any four-vector $p = (E, \mathbf{p})$.[1]

Because of conservation of energy and momentum, the final-state momenta $\mathbf{p}'_1, \ldots, \mathbf{p}'_N$ are not all independent variables. In order to obtain a differential cross-section in the independent variables appropriate to a given situation, we integrate Eq. (8.8) with respect to the remaining variables. We illustrate this for the frequently occurring case of a process leading to a two-body final state. Eq. (8.8) now becomes

$$d\sigma = f(p'_1, p'_2)\delta^{(4)}(p'_1 + p'_2 - p_1 - p_2)d^3\mathbf{p}'_1\, d^3\mathbf{p}'_2, \tag{8.12a}$$

where

$$f(p'_1, p'_2) \equiv \frac{1}{64\pi^2 v_{\text{rel}} E_1 E_2 E'_1 E'_2}\left(\prod_l (2m_l)\right)|\mathcal{M}|^2. \tag{8.12b}$$

Integration of Eq. (8.12a) with respect to $\mathbf{p}'_2$ gives

$$d\sigma = f(p'_1, p'_2)\ \delta(E'_1 + E'_2 - E_1 - E_2)|\mathbf{p}'_1|^2 d|\mathbf{p}'_1| d\Omega'_1, \tag{8.13}$$

where $\mathbf{p}'_2 = \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}'_1$, and integrating Eq. (8.13) over $|\mathbf{p}'_1|$ we obtain[2]

$$d\sigma = f(p'_1, p'_2)|\mathbf{p}'_1|^2 d\Omega'_1 \left[\frac{\partial(E'_1 + E'_2)}{\partial|\mathbf{p}'_1|}\right]^{-1}, \tag{8.15}$$

where $p'_2 = p_1 + p_2 - p'_1$, and the partial derivative is evaluated with the polar angles θ'_1, ϕ'_1 of the vector $\mathbf{p}'_1$ constant.

To obtain the differential cross-section in the CoM system, we note that in the CoM system, $\mathbf{p}'_1 = -\mathbf{p}'_2$. From

$$(E'_f)^2 = (m'_f)^2 + |\mathbf{p}'_f|^2, \qquad f = 1, 2, \tag{8.16}$$

we find

$$\frac{\partial(E'_1 + E'_2)}{\partial|\mathbf{p}'_1|} = |\mathbf{p}'_1|\frac{E_1 + E_2}{E'_1 E'_2}, \tag{8.17}$$

---

[1] $d^3\mathbf{p}/2E$ can be written in the explicitly invariant form

$$\frac{d^3\mathbf{p}}{2E} = \int d^4p\ \delta(p^2 - m^2)\theta(p^0) \tag{8.11}$$

where $m^2 = E^2 - \mathbf{p}^2$, $\theta(p^0)$ is the step function (3.53), and the integration is with respect to p^0 over the range $-\infty < p^0 < \infty$.

[2] We are here using the general relation

$$\begin{aligned}\int f(x, y)\delta[g(x, y)]\, dx &= \int f(x, y)\delta[g(x, y)]\left(\frac{\partial x}{\partial g}\right)_y dg \\ &= \left[\frac{f(x, y)}{(\partial g/\partial x)_y}\right]_{g=0}\end{aligned} \tag{8.14}$$

and, combining Eqs. (8.15), (8.12b), (8.10a) and (8.17), we obtain the CoM differential cross-section

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_1'}\right)_{\mathrm{CoM}} = \frac{1}{64\pi^2(E_1+E_2)^2}\frac{|\mathbf{p}_1'|}{|\mathbf{p}_1|}\left(\prod_l 2m_l\right)|\mathscr{M}|^2. \tag{8.18}$$

Finally, we note that all the cross-section formulae which we have derived apply irrespective of whether identical particles are present or not. However, on calculating total cross-sections in cases where two or more final-state particles are identical, one must integrate only over those ranges of angles which correspond to physically distinguishable events. For example, if the CoM cross-section (8.18) refers to a process with two identical particles in the final state, then the scattering angles $(\theta_1', \phi_1') = (\alpha, \beta)$ and $(\theta_1', \phi_1') = (\pi - \alpha, \pi + \beta)$ describe the same process. Hence the total CoM cross-section is obtained by integrating Eq. (8.18) only over the forward hemisphere $0 \leq \theta_1' \leq \frac{1}{2}\pi$, i.e.

$$\sigma_{\mathrm{CoM}}^{\mathrm{tot}} = \int_0^1 \mathrm{d}(\cos\theta_1')\int_0^{2\pi}\mathrm{d}\phi_1'\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_1'}\right)_{\mathrm{CoM}} = \frac{1}{2}\int_{4\pi}\mathrm{d}\Omega_1'\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_1'}\right)_{\mathrm{CoM}}, \tag{8.19}$$

where the last integral is over the complete solid angle 4π, as indicated.

8.2 Spin Sums

In the last section we considered a reaction in which the initial and final states are completely specified, including the polarization states of the leptons and photons present initially and finally. In many experiments, the colliding particles are unpolarized and the polarizations of the final-state particles are not detected. To obtain the corresponding unpolarized cross-section from Eq. (8.8), we must *average* $|\mathscr{M}|^2$ over all initial polarization states, and we must *sum* it over all final polarization states. In this section we shall show how to obtain these averages and sums over initial and final lepton spins. We shall find that the unpolarized cross-section can always be expressed in terms of traces of products of γ-matrices.

Consider a Feynman amplitude of the form

$$\mathscr{M} = \bar{u}_s(\mathbf{p}')\Gamma u_r(\mathbf{p}). \tag{8.20}$$

This occurs, for example, for Compton scattering [see Fig. 7.12 and Eqs. (7.38)]. Here the spinors $u_r(\mathbf{p})$ and $\bar{u}_s(\mathbf{p}')$ completely specify the momenta and spins of the electron in the initial and final states, and the operator Γ is a 4×4 matrix built up out of γ-matrices. Eq. (8.20) gives rise to an unpolarized cross-section proportional to

$$X \equiv \frac{1}{2}\sum_{r=1}^{2}\sum_{s=1}^{2}|\mathscr{M}|^2 \tag{8.21}$$

where we have averaged over initial spins $\left(\frac{1}{2}\Sigma_r\right)$ and summed over final spins (Σ_s). Defining

$$\tilde{\Gamma} \equiv \gamma^0\Gamma^\dagger\gamma^0, \tag{8.22}$$

we can write Eq. (8.21) as

$$X = \frac{1}{2}\sum_r \sum_s (\bar{u}_s(\mathbf{p}')\Gamma u_r(\mathbf{p}))(\bar{u}_r(\mathbf{p})\tilde{\Gamma} u_s(\mathbf{p}')). \tag{8.23}$$

Writing out the spinor indices explicitly, this can be written

$$X = \frac{1}{2}\left(\sum_s u_{s\delta}(\mathbf{p}')\bar{u}_{s\alpha}(\mathbf{p}')\right)\Gamma_{\alpha\beta}\left(\sum_r u_{r\beta}(\mathbf{p})\bar{u}_{r\gamma}(\mathbf{p})\right)\tilde{\Gamma}_{\gamma\delta}.$$

We introduce the positive energy projection operator [Eqs. (A.31) and (A.35)],

$$\Lambda^+_{\alpha\beta}(\mathbf{p}) = \left(\frac{\not{p}+m}{2m}\right)_{\alpha\beta} = \sum_{r=1}^{2} u_{r\alpha}(\mathbf{p})\bar{u}_{r\beta}(\mathbf{p}) \tag{8.24a}$$

in order to eliminate the sums over positive energy states.[3] This leads to our final result

$$\begin{aligned} X &= \frac{1}{2}\Lambda^+_{\delta\alpha}(\mathbf{p}')\Gamma_{\alpha\beta}\Lambda^+_{\beta\gamma}(\mathbf{p})\tilde{\Gamma}_{\gamma\delta} \\ &= \frac{1}{2}\mathrm{Tr}\left[\Lambda^+(\mathbf{p}')\Gamma\Lambda^+(\mathbf{p})\tilde{\Gamma}\right] \\ &= \frac{1}{2}\mathrm{Tr}\left[\frac{\not{p}'+m}{2m}\Gamma\frac{\not{p}+m}{2m}\tilde{\Gamma}\right]. \end{aligned} \tag{8.25}$$

In addition to the amplitude (8.20), involving the absorption and emission of an external negative lepton, there are also Feynman amplitudes of the form

$$\mathscr{M} = \bar{v}_s(\mathbf{p}')\Gamma v_r(\mathbf{p}) \tag{8.26a}$$

$$\mathscr{M} = \bar{u}_s(\mathbf{p}')\Gamma v_r(\mathbf{p}) \tag{8.26b}$$

$$\mathscr{M} = \bar{v}_s(\mathbf{p}')\Gamma u_r(\mathbf{p}). \tag{8.26c}$$

These represent: (a) absorption and emission of a positive lepton, as in Compton scattering by positrons [Eq. (7.39) and Fig. 7.13]; (b) creation of a lepton pair, as in $2\gamma \to e^+e^-$ (Fig. 7.5); and (c) annihilation of a lepton pair, as in $e^+e^- \to 2\gamma$ (Fig. 7.4).

The spin sums for these cases are performed as for the case which we considered in detail, but using the negative energy projection operator [Eqs. (A.31) and (A.35)]

$$\Lambda^-_{\alpha\beta}(\mathbf{p}) = -\left(\frac{\not{p}-m}{2m}\right)_{\alpha\beta} = -\sum_{r=1}^{2} v_{r\alpha}(\mathbf{p})\bar{v}_{r\beta}(\mathbf{p}) \tag{8.24b}$$

to eliminate sums over negative energy states. For example, Eq. (8.26b) leads to

$$\begin{aligned} \frac{1}{2}\sum_r\sum_s |\mathscr{M}|^2 &= -\frac{1}{2}\mathrm{Tr}\left[\Lambda^+(\mathbf{p}')\Gamma\Lambda^-(\mathbf{p})\tilde{\Gamma}\right] \\ &= \frac{1}{2}\mathrm{Tr}\left[\frac{\not{p}'+m}{2m}\Gamma\ \ \frac{\not{p}-m}{2m}\tilde{\Gamma}\right]. \end{aligned} \tag{8.27}$$

[3] Equation numbers (A. X) refer to equations in Appendix A at the end of the book. This appendix gives a self-contained account of the properties of Dirac spinors, etc. which we here require, and a reader not familiar with these is advised to study the appendix.

Spin sums, and consequently traces like Eqs. (8.25) and (8.27), frequently occur in practice. There exist simple techniques for calculating such traces. These use algebraic identities for γ-matrices (see Appendix A, Section A.2) and some general rules for calculating the traces of products of γ-matrices (see Section A.3). Later in this chapter we shall repeatedly apply these results and methods in calculating the unpolarized cross-sections for various processes in QED.

To conclude this section, we briefly discuss how to calculate the spin polarization properties of a process. This involves evaluating $|\mathscr{M}|^2$ for specific initial and final spin states. This can be done either by using a specific matrix representation for the spinors or by employing helicity or spin-projection operators to select the appropriate spin states. The latter technique again leads to traces and is usually the more convenient one.

We shall illustrate this method for the particular process resulting from the Feynman amplitude (8.20), in which the incident electron has positive helicity and the outgoing electron has negative helicity. The cross-section for this helicity flip process is proportional to

$$\begin{aligned} X &= |\bar{u}_2(\mathbf{p}')\Gamma u_1(\mathbf{p})|^2 \\ &= (\bar{u}_2(\mathbf{p}')\Gamma u_1(\mathbf{p}))(\bar{u}_1(\mathbf{p})\tilde{\Gamma}u_2(\mathbf{p}')). \end{aligned} \tag{8.28}$$

We introduce the helicity projection operators

$$\Pi^{\pm}(\mathbf{p}) = \frac{1}{2}(1 \pm \sigma_{\mathbf{p}}), \tag{A.37}$$

which have the properties

$$\Pi^{+}(\mathbf{p})u_r(\mathbf{p}) = \delta_{1r}u_r(\mathbf{p}), \qquad \Pi^{-}(\mathbf{p})u_r(\mathbf{p}) = \delta_{2r}u_r(\mathbf{p}). \tag{A.40}$$

Eq. (8.28) then becomes

$$\begin{aligned} X &= (\bar{u}_2(\mathbf{p}')\Gamma\Pi^{+}(\mathbf{p})u_1(\mathbf{p}))(\bar{u}_1(\mathbf{p})\tilde{\Gamma}\Pi^{-}(\mathbf{p}')u_2(\mathbf{p}')) \\ &= \sum_r \sum_s (\bar{u}_s(\mathbf{p}')\Gamma\Pi^{+}(\mathbf{p})u_r(\mathbf{p}))(\bar{u}_r(\mathbf{p})\tilde{\Gamma}\Pi^{-}(\mathbf{p}')u_s(\mathbf{p}')) \\ &= \mathrm{Tr}[\Lambda^{+}(\mathbf{p}')\Gamma\Pi^{+}(\mathbf{p})\Lambda^{+}(\mathbf{p})\tilde{\Gamma}\Pi^{-}(\mathbf{p}')], \end{aligned} \tag{8.29}$$

where the last line follows from Eqs. (8.23) and (8.25) with Γ and $\tilde{\Gamma}$ replaced by $\Gamma\Pi^{+}(\mathbf{p})$ and $\tilde{\Gamma}\Pi^{-}(\mathbf{p}')$.

In the relativistic limit $E \gg m$, the helicity projection operators (A.40) simplify to

$$\Pi^{\pm}(\mathbf{p}) = \frac{1}{2}(1 \pm \gamma^5) \quad (E \gg m), \tag{A.43}$$

which also leads to a considerable simplification of Eq. (8.29) in the relativistic limit $E \gg m$, $E' \gg m$.

8.3 Photon Polarization Sums

In the last section, we showed how to perform spin sums in order to obtain unpolarized cross-sections. We now consider the corresponding photon polarization sums. We met an

example of this for Thomson scattering in Section 1.4.4, where we first obtained the fully polarized cross-section, Eq. (1.69), and then explicitly performed the summing and averaging over final and initial polarizations by means of Eq. (1.71). An alternative covariant formalism exists for obtaining the unpolarized cross-section directly. This formalism depends on the gauge invariance of the theory, the consequences of which we shall now consider in more detail.

Gauge invariance of the theory implies the gauge invariance of the matrix elements, i.e. of the Feynman amplitudes. It is, of course, only the matrix element itself, corresponding to the sum of all possible Feynman graphs in a given order of perturbation theory, which must be gauge invariant. The contributions to the amplitude from individual Feynman graphs are, in general, not gauge invariant. For example, for Compton scattering, the individual amplitudes $\mathscr{M}_a$ and $\mathscr{M}_b$, Eqs. (7.38a) and (7.38b), are not gauge invariant, but their sum, $\lambda(\mathscr{M}_a + \mathscr{M}_b)$, is. (The verification of this statement, using the method to be developed in this section, is left as a problem for the reader (see Problem 8.7).)

For any process involving external photons, the Feynman amplitude $\mathscr{M}$ is of the form

$$\mathscr{M} = \varepsilon^{\alpha}_{r_1}(\mathbf{k}_1)\varepsilon^{\beta}_{r_2}(\mathbf{k}_2)\ldots\ \mathscr{M}_{\alpha\beta\ldots}(\mathbf{k}_1,\ \mathbf{k}_2,\ \ldots), \tag{8.30}$$

with one polarization vector $\varepsilon(\mathbf{k})$ for each external photon, and the tensor amplitude $\mathscr{M}_{\alpha\beta\ldots}(\mathbf{k}_1,\ \mathbf{k}_2,\ \ldots)$ independent of these polarization vectors. [This follows from our fourth Feynman rule, Eqs. (7.49e) and (7.49f); we are again using real polarization vectors.]

The polarization vectors are of course gauge dependent. For example, for a free photon, described in a Lorentz gauge by the plane wave

$$A^{\mu}(x) = \text{ const. } \varepsilon^{\mu}_{r}(\mathbf{k})\ \mathrm{e}^{\pm ikx},$$

the gauge transformation

$$A^{\mu}(x) \to A^{\mu}(x) + \partial^{\mu} f(x), \qquad \text{with } f(x) = \tilde{f}(k)\mathrm{e}^{\pm ikx},$$

implies

$$\varepsilon^{\mu}_{r}(\mathbf{k})\mathrm{e}^{\pm ikx} \to [\varepsilon^{\mu}_{r}(\mathbf{k}) \pm \mathrm{i}k^{\mu}\,\tilde{\mathrm{f}}(k)]\ \mathrm{e}^{\pm ikx}. \tag{8.31}$$

Invariance of the amplitude (8.30) under this transformation requires

$$k^{\alpha}_1\mathscr{M}_{\alpha\beta\ldots}(\mathbf{k}_1,\ \mathbf{k}_2,\ \ldots) = k^{\beta}_2\mathscr{M}_{\alpha\beta\ldots}(\mathbf{k}_1, \mathbf{k}_2,\ \ldots) = \ \cdots = 0, \tag{8.32}$$

i.e. when any external photon polarization vector is replaced by the corresponding four-momentum, the amplitude must vanish.

To illustrate how Eq. (8.32) is used to calculate photon polarization sums, we consider, as a simple example, the matrix element

$$\mathscr{M}_r(\mathbf{k}) = \varepsilon^{\alpha}_{r}(\mathbf{k})\mathscr{M}_{\alpha}(\mathbf{k}),$$

corresponding to a process involving one external photon. The gauge invariance now implies

$$k^{\alpha}\mathscr{M}_{\alpha}(\mathbf{k}) = 0. \tag{8.33}$$

The unpolarized cross-section for the process is proportional to

$$X = \sum_{r=1}^{2} |\mathcal{M}_r(\mathbf{k})|^2 = \mathcal{M}_\alpha(\mathbf{k})\mathcal{M}^*_\beta(\mathbf{k}) \sum_{r=1}^{2} \varepsilon_r^\alpha(\mathbf{k})\varepsilon_r^\beta(\mathbf{k}). \tag{8.34}$$

Using the relation

$$\sum_{r=1}^{2} \varepsilon_r^\alpha(\mathbf{k})\varepsilon_r^\beta(\mathbf{k}) = -g^{\alpha\beta} - \frac{1}{(kn)^2}[k^\alpha k^\beta - (kn)(k^\alpha n^\beta + k^\beta n^\alpha)], \tag{8.35}$$

which follows from Eqs. (5.39) and (5.40) for a real photon ($k^2 = 0$), and the gauge condition (8.33), we at once obtain from Eq. (8.34)

$$\sum_{r=1}^{2} |\mathcal{M}_r(\mathbf{k})|^2 = -\mathcal{M}^\alpha(\mathbf{k})\mathcal{M}^*_\alpha(\mathbf{k}). \tag{8.36}$$

Eq. (8.36) is our desired result, and it is easily extended to transitions involving several external photons. This formalism necessitates working in a general Lorentz gauge, as the explicit gauge invariance of the matrix element may be lost in a particular gauge. (We shall meet an example of this when discussing Compton scattering in Section 8.6.) However, in practice, it may be advantageous to choose a particular gauge, which simplifies the algebra of the trace sums, and to carry out the photon polarization sums explicitly, as was done for Thomson scattering in Section 1.4.4. The use of both techniques will be illustrated later in this chapter (see Sections 8.6 and 8.8 on Compton scattering and on bremsstrahlung).

8.4 Lepton Pair Production in (e^+e^-) Collisions

As a first illustration of the use of the above methods in calculating processes to lowest non-vanishing order of perturbation theory, we shall consider the processes in which an electron–positron pair annihilates in collision, producing a charged lepton pair $(l^+ l^-)$. These processes are of considerable interest and have been studied experimentally over a wide range of energies. In this section, we shall take the final lepton pair to be muons or tauons, but not electrons. The case of Bhabha scattering (i.e. $e^+e^- \to e^+e^-$) will be considered in the next section.

We already considered the process

$$e^+(\mathbf{p}_1, r_1) + e^-(\mathbf{p}_2, r_2) \to l^+(\mathbf{p}'_1, s_1) + l^-(\mathbf{p}'_2, s_2) \tag{8.37}$$

(where $l = \mu, \tau, \ldots$) in Section 7.4. Its Feynman amplitude, corresponding to the Feynman graph of Fig. 7.17, is given by Eq. (7.61), which we now write in slightly modified notation as

$$\mathcal{M}(r_1, r_2, s_1, s_2) = ie^2 [\bar{u}_{s_2}(\mathbf{p}'_2)\gamma_\alpha v_{s_1}(\mathbf{p}'_1)]_{(l)} \frac{1}{(p_1 + p_2)^2} [\bar{v}_{r_1}(\mathbf{p}_1)\gamma^\alpha u_{r_2}(\mathbf{p}_2)]_{(e)}. \tag{8.38}$$

The labels (l) and (e) distinguish quantities referring to leptons and to electrons. In Eq. (8.38) we have dropped the term $(+ i\varepsilon)$ in the photon propagator. This term is only of

significance at the pole of the propagator, and in the present case $(p_1 + p_2)^2 \geq 4m_e^2$ cannot vanish.

For the unpolarized cross-section we require

$$X = \frac{1}{4}\sum_{r_1}\sum_{r_2}\sum_{s_1}\sum_{s_2}|\mathscr{M}(r_1, r_2, s_1, s_2)|^2. \tag{8.39}$$

Using the hermiticity condition $\gamma^{\alpha\dagger} = \gamma^0\gamma^\alpha\gamma^0$ [Eq. (A.6)], Eq. (8.38) gives

$$\mathscr{M}^*(r_1, r_2, s_1, s_2) = -ie^2\left[\bar{v}_{s_1}(\mathbf{p}'_1)\gamma_\beta u_{s_2}(\mathbf{p}'_2)\right]_{(l)}\frac{1}{(p_1+p_2)^2}\left[\bar{u}_{r_2}(\mathbf{p}_2)\gamma^\beta v_{r_1}(\mathbf{p}_1)\right]_{(e)} \tag{8.40}$$

and Eq. (8.39) becomes

$$X = \frac{e^4}{4\left[(p_1+p_2)^2\right]^2}A_{(l)\alpha\beta}B^{\alpha\beta}_{(e)}. \tag{8.41}$$

Here $A_{(l)\alpha\beta}$ is given by

$$\begin{aligned} A_{(l)\alpha\beta} &= \sum_{s_1}\sum_{s_2}\left[(\bar{u}_{s_2}(\mathbf{p}'_2)\gamma_\alpha v_{s_1}(\mathbf{p}'_1))(\bar{v}_{s_1}(\mathbf{p}'_1)\gamma_\beta u_{s_2}(\mathbf{p}'_2))\right]_{(l)} \\ &= \mathrm{Tr}\left[\frac{\not{p}'_2 + m_l}{2m_l}\gamma_\alpha\frac{\not{p}'_1 - m_l}{2m_l}\gamma_\beta\right], \end{aligned} \tag{8.41a}$$

where we used the energy projection operators (8.24a) and (8.24b). Similarly one obtains

$$B^{\alpha\beta}_{(e)} = \mathrm{Tr}\left[\frac{\not{p}_1 - m_e}{2m_e}\gamma^\alpha\frac{\not{p}_2 + m_e}{2m_e}\gamma^\beta\right]. \tag{8.41b}$$

The traces (8.41a) and (8.41b) are easily evaluated using the results of Appendix A, Sections A.2 and A.3. Since the trace of a product of an odd number of γ-matrices vanishes [Eq. (A. 16)], Eq. (8.41a) becomes

$$A_{(l)\alpha\beta} = \frac{1}{4m_l^2}\left[\mathrm{Tr}(\not{p}'_2\gamma_\alpha\not{p}'_1\gamma_\beta) - m_l^2\ \mathrm{Tr}\ (\gamma_\alpha\gamma_\beta)\right],$$

and from Eqs. (A.17) this gives

$$A_{(l)\alpha\beta} = \frac{1}{m_l^2}\left[p'_{1\alpha}p'_{2\beta} + p'_{2\alpha}p'_{1\beta} - (m_l^2 + p'_1p'_2)g_{\alpha\beta}\right]. \tag{8.42a}$$

Similarly one finds

$$B^{\alpha\beta}_{(e)} = \frac{1}{m_e^2}\left[p_1^\alpha p_2^\beta + p_2^\alpha p_1^\beta - (m_e^2 + p_1p_2)g^{\alpha\beta}\right]. \tag{8.42b}$$

Substituting Eqs. (8.42) into Eq. (8.41), one obtains

$$\begin{aligned} X = \frac{e^4}{2m_e^2m_l^2\left[(p_1+p_2)^2\right]^2}\{&(p_1p'_1)(p_2p'_2) + (p_1p'_2)(p_2p'_1) \\ &+ m_e^2(p'_1p'_2) + m_l^2(p_1p_2) + 2m_e^2m_l^2\}. \end{aligned} \tag{8.43}$$

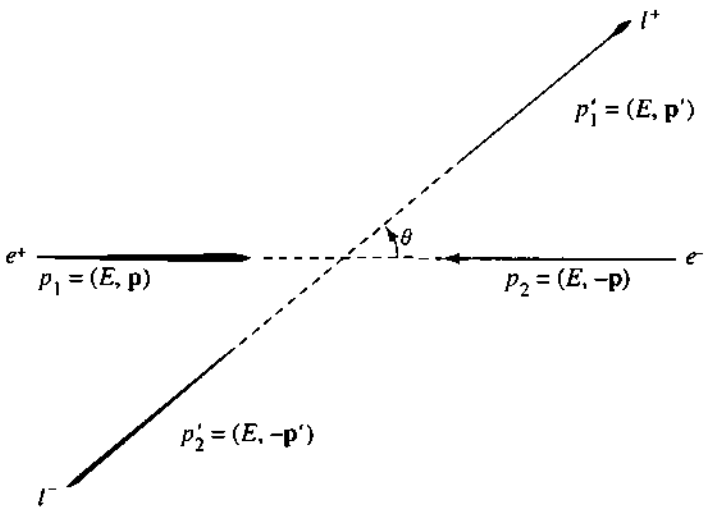

Figure 8.1 *Kinematics for the process $e^+e^- \to l^+l^-$ in the CoM system*

So far we have worked in an arbitrary reference frame. We now specialize to the CoM frame, as specified in Fig. 8.1. The kinematic factors occurring in Eq. (8.43) now take the form

$$\left.\begin{aligned} p_1p_1' = p_2p_2' &= E^2 - pp'\cos\theta, & p_1p_2' = p_2p_1' &= E^2 + pp'\cos\theta \\ p_1p_2 &= E^2 + p^2, & p_1'p_2' &= E^2 + p'^2 \\ & & (p_1 + p_2)^2 &= 4E^2 \end{aligned}\right\} \tag{8.44a}$$

where

$$p \equiv |\mathbf{p}|, \quad p' \equiv |\mathbf{p}'|. \tag{8.44b}$$

Furthermore, since $E \geq m_\mu \approx 207m_e$, it is a very good approximation to take $p \equiv |\mathbf{p}| = E$, and to neglect terms proportional to m_e^2 inside the curly brackets in Eq. (8.43). On making these approximations and substituting Eqs. (8.43) and (8.44) in the CoM cross-section formula (8.18), we finally obtain

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CoM}} = \frac{\alpha^2}{16E^4}\left(\frac{p'}{E}\right)\left(E^2 + m_l^2 + p'^2\cos^2\theta\right) \tag{8.45a}$$

for the differential cross-section, and

$$\sigma_{\mathrm{tot}} = \frac{\pi\alpha^2}{4E^4}\left(\frac{p'}{E}\right)\left[E^2 + m_l^2 + \frac{1}{3}p'^2\right] \tag{8.45b}$$

for the total cross-section. In the extreme relativistic limit, $E \gg m_l$, these formulae reduce to the much quoted

$$\left.\begin{aligned} \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CoM}} &= \frac{\alpha^2}{16E^2}\left(1 + \cos^2\theta\right) \\ \sigma_{\mathrm{tot}} &= \frac{\pi\alpha^2}{3E^2} \end{aligned}\right\} \quad (E \gg m_l). \tag{8.46}$$

Both the processes $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to \tau^+\tau^-$ have been extensively studied over a wide range of energies.

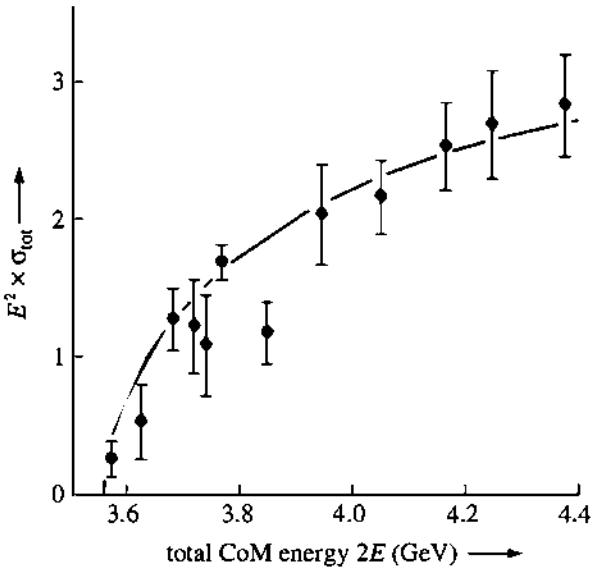

Figure 8.2 *$E^2 \times \sigma_{tot}$ (in arbitrary units) for the process $e^+e^- \to \tau^+\tau^-$ near threshold $2E = 2m_\tau$. [After W. Bacino et al., Phys. Rev. Lett. **41** (1978), 13.]. ⊥: experimental data; curve: theoretical prediction (8.45b)*

Typical results are shown for $\tau^+\tau^-$production near threshold and for both reactions at higher energies in Figs. 8.2 and 8.3 respectively. The latter results are of interest since they probe the interaction down to very small distances and so represent a severe test of QED. In the CoM system, the energy of the virtual photon in the process is $2E$, implying a timescale of order $\hbar/2E$ and a corresponding distance scale of order $\hbar c/2E$. For $E \approx 15$ GeV, this corresponds to a distance of the order 7×10^{-3} f. The agreement between theory and experiment implies that even at these very small distances, the electron, muon and tauon are adequately described as point charges. This small value should be compared with the experimental r.m.s. charge radius of the proton, which is of order 0.8 f.

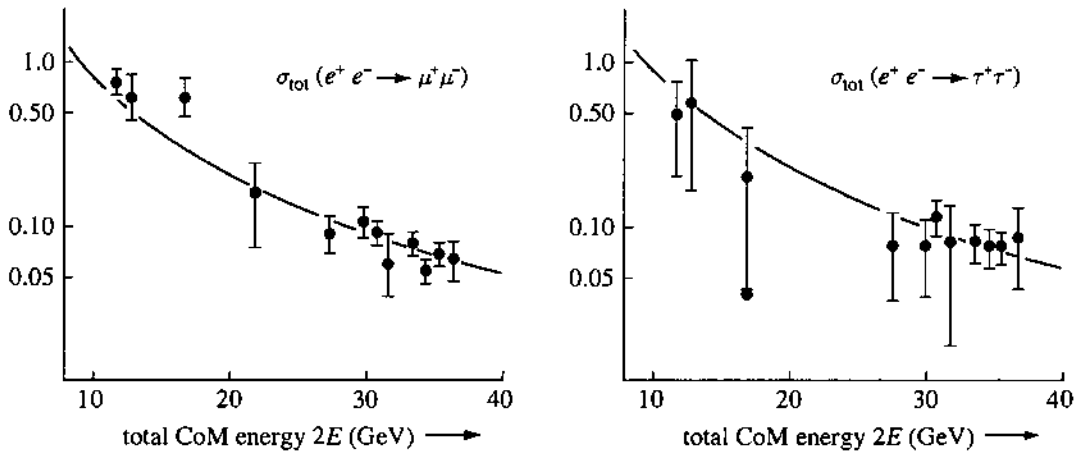

Figure 8.3 *The total cross-sections (in nb) for the processes $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to \tau^+\tau^-$ at relativistic energies. [After D. P. Barber et al., Phys. Rev. Lett. **43**, (1979) 1915.] ⊥: experimental data; curves: theretical cross-section formula (8.46)*

Finally we note that at still higher energies, the contribution of the weak interaction to the cross sections must be taken into account. This will be discussed in the context of the unified theory of electromagnetic and weak interactions in Section 19.4.

8.5 Bhabha Scattering

We now consider elastic e^+e^- scattering. This process is a little more complicated than those considered in the last section since, in addition to the annihilation diagram [Figs. 7.17 or 7.7(b)], the scattering diagram [Fig. 7.7(a)] also contributes. The Feynman amplitude for the process

$$e^+(\mathbf{p}_1, r_1) + e^-(\mathbf{p}_2, r_2) \to e^+(\mathbf{p}'_1, s_1) + e^-(\mathbf{p}'_2, s_2) \tag{8.47}$$

is

$$\mathcal{M} = \mathcal{M}_a + \mathcal{M}_b$$

where $\mathcal{M}_a$ and $\mathcal{M}_b$ correspond to the scattering and annihilation diagrams and are given by

$$\mathcal{M}_a = -ie^2\left[\bar{u}(\mathbf{p}'_2)\gamma_\alpha u(\mathbf{p}_2)\right]\frac{1}{(p_1 - p'_1)^2}\left[\bar{v}(\mathbf{p}_1)\gamma^\alpha v(\mathbf{p}'_1)\right] \tag{8.48a}$$

$$\mathcal{M}_b = ie^2\left[\bar{u}(\mathbf{p}'_2)\gamma_\alpha v(\mathbf{p}'_1)\right]\frac{1}{(p_1 + p_2)^2}\left[\bar{v}(\mathbf{p}_1)\gamma^\alpha u(\mathbf{p}_2)\right]. \tag{8.48b}$$

We have here suppressed the spin indices again. $\mathcal{M}_b$ is of course just the amplitude (8.38) with e^+e^- for the final state lepton pair. We again note the relative minus sign between these two terms corresponding to Feynman rule 8 (Section 7.3).

We shall evaluate the cross-section for this process in the CoM system, restricting ourselves for simplicity to the important case of the relativistic high-energy limit. The kinematics are now defined by Eqs. (8.44) with $p' = p$ and $E \gg m(\equiv m_e)$. The cross-section formula (8.18) now leads to

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CoM}} = \frac{m^4}{16\pi^2E^2}\left(X_{aa} + X_{bb} + X_{ab} + X^*_{ab}\right), \tag{8.49}$$

where

$$X_{aa} = \frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_a|^2 \tag{8.50a}$$

$$X_{bb} = \frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_b|^2 \tag{8.50b}$$

$$X_{ab} = \frac{1}{4}\sum_{\text{spins}}\mathcal{M}_a\mathcal{M}^*_b, \tag{8.50c}$$

the summations being over the spins of all four fermions.

The term X_{bb} follows at once as a special case of X, Eq. (8.43), with $m_l = m$ and $E \gg m$:

$$X_{bb} = \frac{e^4}{16m^4}\left[1 + \cos^2\theta + O\left(\frac{m^2}{E^2}\right)\right]. \tag{8.51}$$

The evaluation of X_{aa} is essentially similar to that of X_{bb} and is left as an exercise for the reader. The result is

$$\begin{aligned} X_{aa} &= \frac{e^4}{2m^4\left[(p_1-p_1')^2\right]^2}\left\{(p_1p_2)(p_1'p_2') + (p_1p_2')(p_2p_1') + O(E^2m^2)\right\} \\ &= \frac{e^4}{8m^4\sin^4(\theta/2)}\left[1+\cos^4\frac{\theta}{2} + O\left(\frac{m^2}{E^2}\right)\right]. \end{aligned} \tag{8.52}$$

The interference term X_{ab}, Eq. (8.50c), is more complicated, and we shall give its evaluation in some detail in order to illustrate how such more complicated spin sums can be handled simply. From Eqs. (8.50c) and (8.48),

$$\begin{aligned} X_{ab} &= \frac{-e^4}{4(p_1-p_1')^2(p_1+p_2)^2}\sum_{\text{spins}}\left\{\left[\bar u(\mathbf{p}_2')\gamma_\alpha u(\mathbf{p}_2)\right]\left[\bar u(\mathbf{p}_2)\gamma_\beta v(\mathbf{p}_1)\right]\right. \\ &\qquad \left.\times\left[\bar v(\mathbf{p}_1)\gamma^\alpha v(\mathbf{p}_1')\right]\left[\bar v(\mathbf{p}_1')\gamma^\beta u(\mathbf{p}_2')\right]\right\} \\ &= \frac{-e^4}{4(p_1-p_1')^2(p_1+p_2)^2}\mathrm{Tr}\left\{\frac{\not p_2'+m}{2m}\gamma_\alpha\frac{\not p_2+m}{2m}\gamma_\beta\frac{\not p_1-m}{2m}\gamma^\alpha\frac{\not p_1'-m}{2m}\gamma^\beta\right\} \\ &= \frac{-e^4}{64m^4(p_1-p_1')^2(p_1+p_2)^2}\left[\mathrm{Tr}(\not p_2'\gamma_\alpha\not p_2\gamma_\beta\not p_1\gamma^\alpha\not p_1'\gamma^\beta) + O(E^2m^2)\right]. \end{aligned}$$

We must therefore evaluate the trace of a product of eight γ-matrices. It is usually possible and highly desirable to simplify such a product before taking the trace, rather than blindly use Eq. (A.18c). The contraction identities (Appendix A, Section A.2) are particularly useful in this respect. By means of

$$\gamma_\lambda\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma^\lambda = -2\gamma_\gamma\gamma_\beta\gamma_\alpha, \quad \gamma_\lambda\gamma_\alpha\gamma_\beta\gamma^\lambda = 4g_{\alpha\beta} \tag{A14a}$$

we find that the trace in X_{ab} equals

$$-2\,\mathrm{Tr}(\not p_2'p_1\gamma_\beta\not p_2\not p_1'\gamma^\beta) = -8(p_2p_1')\mathrm{Tr}(\not p_2'\not p_1) = -32(p_2p_1')(p_2'p_1),$$

whence X_{ab} becomes

$$\begin{aligned} X_{ab} &= \frac{-e^4}{2m^4(p_1-p_1')^2(p_1+p_2)^2}\left[(p_1p_2')(p_2p_1') + O(E^2m^2)\right] \\ &= \frac{-e^4}{8m^4\sin^2(\theta/2)}\left[\cos^4\frac{\theta}{2} + O\left(\frac{m^2}{E^2}\right)\right]. \end{aligned} \tag{8.53}$$

We see from this equation that X_{ab} is real. Hence substituting Eqs. (8.51)–(8.53) in Eq. (8.49), we obtain for the CoM differential cross-section in the high-energy limit ($E \gg m$)

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CoM}} = \frac{\alpha^2}{8E^2}\left[\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)} + \frac{1+\cos^2\theta}{2} - \frac{2\cos^4(\theta/2)}{\sin^2(\theta/2)}\right]. \tag{8.54}$$

The three terms in this equation correspond to the photon exchange diagram, Fig. 7.7(a), the annihilation diagram 7.7(b) and the interference term between them. It should be compared with the corresponding result (8.46) for the process $e^+e^- \to l^+l^-$ with $l \neq e$, when only the annihilation diagram is present.

At small angles, the exchange term dominates, giving rise to an infinite cross-section in the forward direction, $\theta = 0$, and an infinite total cross-section. These features are a consequence of the infinite range of the electromagnetic forces or, equivalently, of the zero mass of the photon. As $\theta \to 0$, the four-momentum $k^\alpha = (p_1 - p_1')^\alpha$ of the exchanged photon tends to zero, and the factor

$$\frac{1}{k^2 + i\varepsilon} = \frac{1}{(p_1 - p_1')^2 + i\varepsilon}$$

in the photon propagator diverges, from which the divergence of the amplitude (8.48a) and of the cross-section (8.54) follows.[4]

At large angles the photon-exchange term and the annihilation term are of comparable importance, and sensitive to the short-distance behaviour. For the annihilation term this was

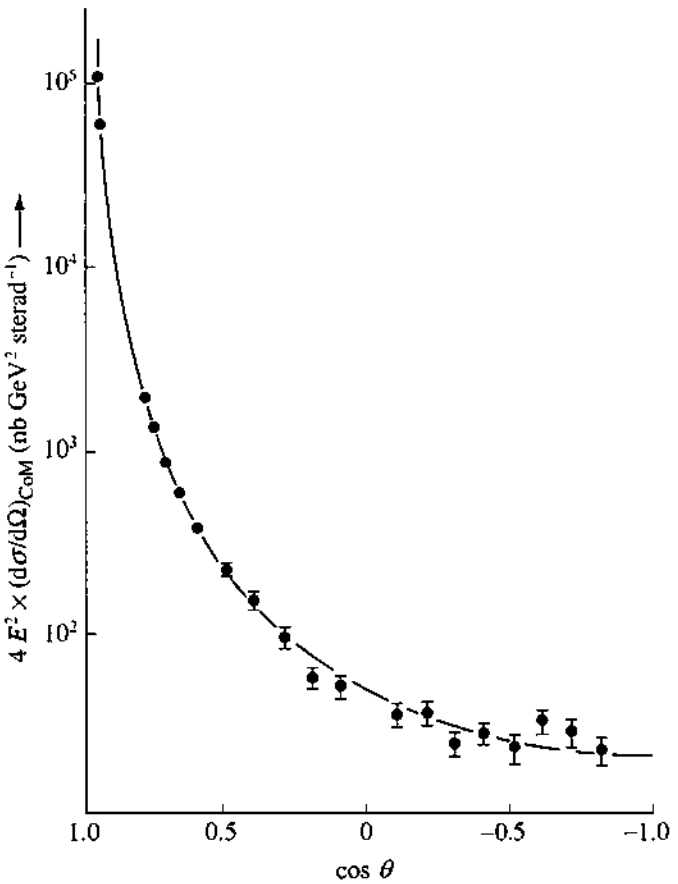

Figure 8.4 *The differential cross-section* $(d\sigma/d\Omega)_{CoM}$ *for Bhabha scattering,* $e^+e^- \to e^+e^-$, *at the total CoM energy* $2E = 34$ *GeV, [After H. J. Behrend et al., Phys. Lett,* ***103B*** *(1981), 148.]* ϯ*: experimental data; curve: QED cross-section formula (8.54)*

[4] As is often done, we suppressed the term $+ i\varepsilon$ in the photon propagator in Eq. (8.48a). This term is only relevant at the pole, $(p_1 - p_1')^2 = 0$.

discussed in the last section. For the exchange diagram, the exchanged photon has the wave number $|\mathbf{k}| = |\mathbf{p}_1 - \mathbf{p}_1'| = 2E\sin(\theta/2)$, with an associated wavelength $\lambda = 2\pi/|\mathbf{k}|$.

Experimentally, the predicted behaviour has been confirmed over a wide range of energies and angles, and the interaction has been tested down to very short distances, comparable to those probed in the $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to \tau^+\tau^-$ experiments, which were discussed in the last section. Typical results for Bhabha scattering are shown in Fig. 8.4.

8.6 Compton Scattering

We shall now derive the cross-section for Compton scattering. In this process a photon is present in both the initial and final states, and we shall apply our earlier results to carry out the photon polarization sums, as well as the electron spin sums.

Suppose that in the initial state we have an electron with momentum $p = (E, \mathbf{p})$ in the spin state $u \equiv u_r(\mathbf{p})$, and a photon with momentum $k = (\omega, \mathbf{k})$ and polarization vector $\varepsilon \equiv \varepsilon_s(\mathbf{k})$, and that the corresponding quantities for the final state are $p' = (E', \mathbf{p}')$, $u' \equiv u_{r'}(\mathbf{p}')$, and $k' = (\omega', \mathbf{k}')$, $\varepsilon' \equiv \varepsilon_{s'}(\mathbf{k}')$. The differential cross-section for this process is given by Eqs. (8.15) and (8.12b) as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{m^2\omega'}{16\pi^2 EE'\omega v_{\mathrm{rel}}}\left[\left(\frac{\partial(E'+\omega')}{\partial\omega'}\right)_{\theta\phi}\right]^{-1}|\mathscr{M}|^2 \tag{8.55}$$

where $\mathscr{M}$ is the Feynman amplitude for this transition, (θ, ϕ) are the polar angles of $\mathbf{k}'$ and $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ is the corresponding element of solid angle. We shall take $\mathbf{k}$ as polar coordinate axis, so that θ is the photon scattering angle: $\mathbf{k}\cdot\mathbf{k}' = \omega\omega'\cos\theta$. In Eq. (8.55), initial and final momenta are related by the conservation laws

$$p + k = p' + k'. \tag{8.56}$$

In lowest order, the Feynman amplitude $\mathscr{M}$ results from the two Feynman graphs in Figs. 7.12(a) and 7.12(b), and the corresponding contributions to $\mathscr{M}$ are given by Eqs. (7.38a) and (7.38b). Defining

$$f_1 \equiv p + k, \quad f_2 \equiv p - k', \tag{8.57}$$

we obtain $\mathscr{M}$ from these equations as

$$\mathscr{M} = \mathscr{M}_{\mathrm{a}} + \mathscr{M}_{\mathrm{b}}, \tag{8.58}$$

where

$$\mathscr{M}_{\mathrm{a}} = -\mathrm{i}e^2\frac{\bar{u}'\not{\varepsilon}'(\not{f}_1 + m)\not{\varepsilon}u}{2(pk)}, \quad \mathscr{M}_{\mathrm{b}} = \mathrm{i}e^2\frac{\bar{u}'\not{\varepsilon}(\not{f}_2 + m)\not{\varepsilon}'u}{2(pk')}. \tag{8.59}$$

These results refer to a general reference frame. In most experiments, the photon beam is incident on a target of nearly stationary electrons. We shall now specialize Eq. (8.55) to the laboratory frame in which $p = (m, 0, 0, 0)$ and

$$\mathbf{p}' = \mathbf{k} - \mathbf{k}', \tag{8.60a}$$

$$E' = \left[m^2 + (\mathbf{k} - \mathbf{k}')^2\right]^{1/2} = \left[m^2 + \omega^2 + \omega'^2 - 2\omega\omega'\cos\ \theta\right]^{1/2}. \tag{8.60b}$$

From Eq. (8.56) we have generally

$$pk = p'k + k'k = pk' + k'k,$$

which in the laboratory system reduces to

$$\omega' = \frac{m\omega}{m + \omega(1 - \cos\ \theta)}. \tag{8.61}$$

This equation gives the energy shift of the scattered photon due to the recoil of the target electron. From Eq. (8.60b) we find

$$\left(\frac{\partial(E' + \omega')}{\partial\omega'}\right)_{\theta\phi} = \frac{m\omega}{E'\omega'}, \tag{8.62}$$

so that Eq. (8.55) gives for the differential cross-section in the laboratory frame

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Lab}} = \frac{1}{(4\pi)^2}\left(\frac{\omega'}{\omega}\right)^2 |\mathscr{M}|^2. \tag{8.63}$$

The cross-sections (8.55) and (8.63) are fully polarized, i.e. both initial and final electrons and photons are in definite polarization states. To obtain the cross-section for Compton scattering by an unpolarized electron target, and with the spin of the final electron undetected, we must sum and average the above cross-section formulas over final and initial electron spins. To obtain the unpolarized cross-section, we must also sum and average over final and initial photon polarizations. We shall illustrate both the methods of Section 1.4.4 and of Section 8.3 for handling photon polarization.

To obtain the unpolarized cross-section directly, we use the covariant method of Section 8.3. Writing

$$\mathscr{M} \equiv \varepsilon_\alpha \varepsilon'_\beta \mathscr{M}^{\alpha\beta}, \tag{8.64}$$

we obtain

$$\frac{1}{4}\sum_{\text{pol}}\sum_{\text{spin}} |\mathscr{M}|^2 = \frac{1}{4}\sum_{\text{spin}} \mathscr{M}^{\alpha\beta}\mathscr{M}^*_{\alpha\beta}, \tag{8.65}$$

where the summations are over initial and final electron spins and photon polarizations. Eq. (8.65) is the analogue of Eq. (8.36) for the case of two external photons. Carrying out the spin summations, we obtain from Eqs. (8.65) and (8.58)–(8.59)

$$\begin{aligned}\frac{1}{4}\sum_{\text{pol}}\sum_{\text{spin}} |\mathscr{M}|^2 &= \frac{1}{4}\sum_{\text{pol}}\sum_{\text{spin}}\left\{|\mathscr{M}_{\mathrm{a}}|^2 + |\mathscr{M}_{\mathrm{b}}|^2 + \mathscr{M}_{\mathrm{a}}\mathscr{M}^*_{\mathrm{b}} + \mathscr{M}_{\mathrm{b}}\mathscr{M}^*_{\mathrm{a}}\right\} \\ &= \frac{e^4}{64m^2}\left\{\frac{X_{\mathrm{aa}}}{(pk)^2} + \frac{X_{\mathrm{bb}}}{(pk')^2} - \frac{X_{\mathrm{ab}} + X_{\mathrm{ba}}}{(pk)(pk')}\right\},\end{aligned} \tag{8.66}$$

where

$$X_{aa} = \mathrm{Tr}\{\gamma^\beta(\not f_1 + m)\gamma^\alpha(\not p + m)\gamma_\alpha(\not f_1 + m)\gamma_\beta(\not p' + m)\} \tag{8.67a}$$

$$X_{bb} = \mathrm{Tr}\{\gamma^\alpha(\not f_2 + m)\gamma^\beta(\not p + m)\gamma_\beta(\not f_2 + m)\gamma_\alpha(\not p' + m)\} \tag{8.67b}$$

$$X_{ab} = \mathrm{Tr}\{\gamma^\beta(\not f_1 + m)\gamma^\alpha(\not p + m)\gamma_\beta(\not f_2 + m)\gamma_\alpha(\not p' + m)\} \tag{8.67c}$$

$$X_{ba} = \mathrm{Tr}\{\gamma^\alpha(\not f_2 + m)\gamma^\beta(\not p + m)\gamma_\alpha(\not f_1 + m)\gamma_\beta(\not p' + m)\}. \tag{8.67d}$$

Note that the effect of the substitutions

$$k \leftrightarrow -k', \qquad \varepsilon \leftrightarrow \varepsilon', \tag{8.68a}$$

is to induce the transformations

$$f_1 \leftrightarrow f_2, \qquad \mathscr{M}_a \leftrightarrow \mathscr{M}_b \tag{8.68b}$$

and hence

$$X_{aa} \leftrightarrow X_{bb}, \qquad X_{ab} \leftrightarrow X_{ba}, \tag{8.68c}$$

We need therefore only calculate X_{aa} and X_{ab} from first principles. From Eq. (8.66), $X_{ba} = X_{ab}^*$. Furthermore, it follows from Eqs. (8.67c) and (8.67d) and the general property (A.20a) of γ-matrices, that $X_{ab} = X_{ba}$. Hence X_{ab} is real, and it is symmetric with respect to the transformation (8.68a), which provides two useful checks in its calculation.

The traces in Eqs. (8.67) involve products of up to eight γ-matrices. Their computation is much simplified by the use of the contraction identities (Appendix A, Section A.2) which eliminate four γ-matrices. We illustrate this for X_{aa}. The trace in Eq. (8.67a) contains the factor

$$\begin{aligned} Y &\equiv \gamma^\beta(\not f_1 + m)\gamma^\alpha(\not p + m)\gamma_\alpha(\not f_1 + m)\gamma_\beta \\ &= \gamma^\beta(\not f_1 + m)(-2\not p + 4m)(\not f_1 + m)\gamma_\beta \\ &= 4\not f_1 \not p \not f_1 + m[-16(pf_1) + 16f_1^2] + m^2(4\not p - 16\not f_1) + 16m^3. \end{aligned}$$

Hence, using Eqs. (A.16) and (A.18a), (A.18b), one obtains directly

$$\begin{aligned} X_{aa} &= \mathrm{Tr}\{Y(\not p' + m)\} \\ &= 16\{2(f_1p)(f_1p') - f_1^2(pp') + m^2[-4(pf_1) + 4f_1^2] \\ &\quad + m^2[(pp') - 4(f_1p')] + 4m^4\}. \end{aligned} \tag{8.69}$$

If we express all quantities in terms of the three linearly independent scalars

$$p^2 = p'^2 = m^2, \quad pk = p'k', \quad pk' = p'k, \tag{8.70}$$

X_{aa} simplifies to

$$X_{aa} = 32[m^4 + m^2(pk) + (pk)(pk')]. \tag{8.71a}$$

From Eqs. (8.68) we have at once

$$X_{bb} = 32[m^4 - m^2(pk') + (pk)(pk')]. \tag{8.71b}$$

The interference term X_{ab}, Eq. (8.67c), is similarly computed with the result

$$X_{ab} = 16m^2\left[2m^2 + (pk) - (pk')\right]. \tag{8.71c}$$

As expected, X_{ab} is real and symmetric, i.e. the substitution $k \leftrightarrow -k'$ transforms X_{ab} into

$$X_{ba} = X_{ab}. \tag{8.71d}$$

Substituting Eqs. (8.71a)–(8.71d) into Eq. (8.66), we obtain

$$\frac{1}{4}\sum_{\text{pol}}\sum_{\text{spin}}|\mathscr{M}|^2 = \frac{e^4}{2m^2}\left\{\left(\frac{pk}{pk'} + \frac{pk'}{pk}\right) + 2m^2\left(\frac{1}{pk} - \frac{1}{pk'}\right) + m^4\left(\frac{1}{pk} - \frac{1}{pk'}\right)^2\right\}. \tag{8.72}$$

In the laboratory system, $pk = m\omega$, $pk' = m\omega'$, and from Eq. (8.61)

$$\frac{1}{\omega} - \frac{1}{\omega'} = \frac{1}{m}(\cos\theta - 1).$$

Hence Eq. (8.72) reduces to

$$\left[\frac{1}{4}\sum_{\text{pol}}\sum_{\text{spin}}|\mathscr{M}|^2\right]_{\text{Lab}} = \frac{e^4}{2m^2}\left\{\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right\}, \tag{8.73}$$

and Eq. (8.63) gives the unpolarized cross-section

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Lab}} = \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2\left\{\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right\}. \tag{8.74}$$

By means of Eq. (8.61), ω' can of course be eliminated altogether from this equation. In the low-energy limit $\omega \ll m$,we have $\omega' \approx \omega$, i.e. the kinetic energy of the recoil electron is negligible, and Eq. (8.74) reduces to the Thomson scattering cross-section, Eq. (1.69a).

We shall now derive the cross-section for initial and final photons in states of definite polarization, i.e. summing and averaging over electron spins only. On also summing and averaging over photon polarizations, using the method of Section 1.4.4, we shall regain the result (8.74). In this case, the trace calculations cannot be simplified through use of the contraction identities. However, they are greatly facilitated by a suitable choice of gauge. In any reference frame, it is possible to find a Lorentz gauge in which the vacuum contains no longitudinal or scalar photons, and free photons are transverse (see Section 5.2). In this gauge, the polarization vectors of the external photons are of the form $\varepsilon = (0, \boldsymbol{\varepsilon})$, $\varepsilon' = (0, \boldsymbol{\varepsilon}')$, with

$$\varepsilon k = -\boldsymbol{\varepsilon}\cdot\mathbf{k} = 0, \qquad \varepsilon' k' = -\boldsymbol{\varepsilon}'\cdot\mathbf{k}' = 0. \tag{8.75a}$$

The analysis is further simplified if we work in the laboratory frame, $p = (m, 0, 0, 0)$, in which we also have

$$p\varepsilon = p\varepsilon' = 0. \tag{8.75b}$$

It follows from the anticommutation relations $\left[\gamma^\alpha, \gamma^\beta\right]_+ = 2g^{\alpha\beta}$ and the Dirac equation $(\not{p} - m)u(\mathbf{p}) = 0$, that

$$\not{p}\not{\varepsilon}u = -m\not{\varepsilon}u, \quad \not{p}\not{\varepsilon}'\not{k} = -m\not{\varepsilon}'u,$$

so that the matrix elements (8.59) simplify to

$$\mathcal{M}_a = -ie^2 \frac{\bar{u}'\not{\varepsilon}\not{k}'\not{\varepsilon}'u}{2(pk)}, \quad \mathcal{M}_b = -ie^2 \frac{\bar{u}'\not{\varepsilon}\not{k}'\not{\varepsilon}'u}{2(pk')}. \tag{8.76}$$

Note that Eq. (8.76) does not give a gauge-invariant expression for the matrix element $\mathcal{M} = \mathcal{M}_a + \mathcal{M}_b$. For example, the gauge transformation $\varepsilon \to \varepsilon + \lambda k$, where λ is a constant, leads to $\mathcal{M}_a \to \mathcal{M}_a$, (since $\not{k}\not{k} = k^2 = 0$) but $\mathcal{M}_b \not\to \mathcal{M}_b$. This is of course due to the fact that we dropped the terms $p\varepsilon$ and $p\varepsilon'$ which are zero in our gauge.

Summing and averaging over electron spins now gives

$$\frac{1}{2}\sum_{\text{spin}} |\mathcal{M}|^2 = \frac{e^4}{32m^2}\left\{\frac{Y_{aa}}{(pk)^2} + \frac{Y_{bb}}{(pk')^2} + \frac{Y_{ab} + Y_{ba}}{(pk)(pk')}\right\} \tag{8.77}$$

where

$$Y_{aa} = \mathrm{Tr}\left\{\not{\varepsilon}'\not{k}\not{\varepsilon}(\not{p} + m)\not{\varepsilon}\not{k}\not{\varepsilon}'(\not{p}' + m)\right\} \tag{8.78a}$$

$$Y_{bb} = \mathrm{Tr}\left\{\not{\varepsilon}\not{k}\not{\varepsilon}'(\not{p} + m)\not{\varepsilon}'\not{k}\not{\varepsilon}(\not{p}' + m)\right\} \tag{8.78b}$$

$$Y_{ab} = \mathrm{Tr}\left\{\not{\varepsilon}'\not{k}\not{\varepsilon}(\not{p} + m)\not{\varepsilon}'\not{k}\not{\varepsilon}(\not{p}' + m)\right\} \tag{8.78c}$$

$$Y_{ba} = \mathrm{Tr}\left\{\not{\varepsilon}\not{k}\not{\varepsilon}'(\not{p} + m)\not{\varepsilon}\not{k}\not{\varepsilon}'(\not{p}' + m)\right\}. \tag{8.78d}$$

Substituting $k \leftrightarrow -k'$, $\varepsilon \leftrightarrow \varepsilon'$, again leads to $\mathcal{M}_a \leftrightarrow \mathcal{M}_b$, and

$$Y_{aa} \leftrightarrow Y_{bb}, \qquad Y_{ab} \leftrightarrow Y_{ba}, \tag{8.79}$$

and $Y_{ab} = Y_{ba} = Y_{ab}^*$.

The traces in Eqs. (8.78) contain products of up to eight γ-matrices. We reduce this number using

$$\not{A}\not{B} = -\not{B}\not{A} + 2AB. \tag{8.80a}$$

For $A = B$, we have

$$\not{A}\not{A} = A^2, \tag{8.80b}$$

and, in particular,

$$\not{p}\not{p} = m^2, \quad \not{k}\not{k} = 0, \quad \not{\varepsilon}\not{\varepsilon} = \not{\varepsilon}'\not{\varepsilon}' = -1. \tag{8.80c}$$

For $AB = 0$, we have

$$\not{A}\not{B} = -\not{B}\not{A}, \tag{8.80d}$$

which will be particularly useful on account of Eqs. (8.75).

We illustrate the use of these tricks by computing Y_{aa}. Since $\not{k}\not{\varepsilon}\not{k} = -\not{k}\not{k} = 0$ Eq. (8.78a), reduces to

$$Y_{aa} = \mathrm{Tr}\{\not{\varepsilon}'\not{k}\not{\varepsilon}\not{p}\not{\varepsilon}\not{k}\not{\varepsilon}'\not{p}'\} = \mathrm{Tr}\{\not{\varepsilon}'\not{k}\not{p}\not{k}\ \not{\varepsilon}'\not{p}'\}$$

since $\not{\varepsilon}\not{p}\not{\varepsilon} = -\not{p}\not{\varepsilon}\not{\varepsilon} = \not{p}$. Permuting $\not{p}$ and $\not{k}$, and using $\not{k}\not{k} = 0$, we obtain

$$\begin{aligned} Y_{aa} &= 2(pk)\mathrm{Tr}\{\not{\varepsilon}'\not{k}\not{\varepsilon}'\not{p}'\} = 8(pk)[2(\varepsilon' k)(\varepsilon' p') + (kp')] \\ &= 8(pk)\left[2(\varepsilon' k)^2 + (pk')\right], \end{aligned} \tag{8.81a}$$

since $p' - k = p - k'$ implies $\varepsilon' p' = \varepsilon' k$ and $kp' = pk'$. From Eq. (8.79) we have

$$Y_{bb} = -8(pk')\left[2(\varepsilon k')^2 - (pk)\right]. \tag{8.81b}$$

The interference term Y_{ab}, Eq. (8.78c), is more complicated to evaluate. Its simplification depends essentially on writing $p' = p + k - k'$, so that the orthogonality relations (8.75) and Eqs. (8.80) can be used to the full. In this way one finds

$$Y_{ab} = 8(pk)(pk')\left[2(\varepsilon\varepsilon')^2 - 1\right] - 8(k\varepsilon')^2(pk') + 8(k'\varepsilon)^2(pk), \tag{8.81c}$$

which is real and symmetric (i.e. $Y_{ab} = Y_{ba}$). From Eqs. (8.81), (8.77) and (8.63), one obtains the differential cross-section for Compton scattering of polarized photons

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{Lab, pol}} = \frac{\alpha^2}{4m^2}\left(\frac{\omega'}{\omega}\right)^2\left\{\frac{\omega}{\omega'} + \frac{\omega'}{\omega} + 4(\varepsilon\varepsilon')^2 - 2\right\}. \tag{8.82}$$

Eq. (8.82) is known as the Klein–Nishina formula. From it one obtains the unpolarized cross-section by summing and averaging over final and initial photon polarizations. Since $\varepsilon\varepsilon' = -\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}'$, one can write Eq. (1.71)

$$\frac{1}{2}\sum_{\text{pol}}(\varepsilon\varepsilon')^2 = \frac{1}{2}(1 + \cos^2\theta), \tag{8.83}$$

and applying this equation to Eq. (8.82) one at once regains the unpolarized cross-section formula (8.74).

8.7 Scattering by an External Field

So far, the electromagnetic field has been described by a quantized field, involving photon creation and annihilation operators. In some problems, where the quantum fluctuations of the field are unimportant, it may be adequate to describe the field as a purely classical function of the space–time coordinates. An example would be the scattering of electrons or positrons by an applied 'external' electromagnetic field $A_e^\alpha(x)$, such as the Coulomb field of a heavy nucleus.[5] More generally, one may have to consider both types of field,

[5] The meaning of 'external' in the present context and in the description of Feynman graphs (where we talk of external lines, particles, etc.) should not be confused.