APPLYING THE INEQUALITY FOR ARITHMETIC AND GEOMETRIC MEANS TO SOLVE PROBLEM 1226 IN THE COLLEGE MATHEMATICS JOURNAL

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Abstract. The Inequality for Arithmetic and Geometric Means, also known as the AM-GM Inequality, states that for a list of positive real numbers, the arithmetic mean is greater than or equal to the geometric mean. This theorem produces powerful results and will play a pivotal role in proving the Inequality posed in problem 1226 in the Mathematical Association of America Journal.

1. Introduction

The Mathematical Association of America's Problem 1226 in the College Math Journal is stated as the following. Let a, b, and c be positive real numbers. Prove that

$$\ln\left(\frac{27abc}{(a+b+c)^3}\right) \leq \frac{(a-b)^2+(b-c)^2+(c-a)^2}{3}$$

2. Methods

Using the inequality of arithmetic and geometric means (AM-GM inequality), stated below[**?**], the denominator of the expression within the natural logarithm can be proven to always be greater than the numerator.

$$\frac{x_1+x_2+...+x_n}{n} \geq \sqrt[n]{x_1x_2...x_n}$$

Since $c, b, a \geq 0$, as stated in the problem statement, by the AM-GM inequality,

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}$$

$$\left(\frac{a+b+c}{3}\right)^3 \geq \left(\sqrt[3]{abc}\right)^3$$

Date: June 2022.

$$\frac{(a+b+c)^3}{27} \geq abc$$

$$(a+b+c)^3 \geq 27abc$$

In the inequality derived from the AM-GM inequality, as shown above, the right hand side is the numerator, and the left hand side is the denominator of the expression within the natural logarithm from the problem statement. This shows that the denominator of this expression will always be greater than or equal to the the numerator $\implies$ the entire expression will yield a value less than or equal to one and greater than 0 for all values of a, b, and c.

The value of the expression within the logarithm will yield a value within (0,1] $\implies$ the value of the entire expression on the left hand side will be ≤ 0 after the natural logarithm is applied to the expression's value. The value of the expression on the right hand side is always ≥ 0 because the expressions within the numerator are squared, yielding positive values added to each other, and the denominator always has a value of 3 $\implies$ the right hand side will always be ≥ 0. The case in which it evaluates to 0 is when $a = b = c$. Because the right hand side ≥ 0, and the left hand side is ≤ 0, the right hand side is always $\geq$ left hand side.

3. Acknowledgments

We would like to thank Dr. Philip Benge from the North Carolina School of Science and Mathematics, as well as Dr.Todd Lee from Elon University for their invaluable contribution, support, and advice throughout this project.

References