CMJ 1230 SUBMISSION

PRANAV NARALA, SRIPAD GANTI, RISHYENDRA MEDAMANURI

To examine relationships between cases of primitive Heronian triangles with equivalent rectangles, a script was written to generate a few cases, which are shown below:

Triangle Side Lengths	Equivalent Rectangle Side Lengths
$(5, 5, 6)$	$(2, 6)$
$(13, 20, 21)$	$(6, 21)$
$(25, 51, 52)$	$(12, 52)$

From these examples cases, the following conjectures were developed:

Conjecture 0.1. *The longest side of a primitive Heronian triangle with rectangular properties will be a dimension of the corresponding rectangle.*

Conjecture 0.2. *The longest side of a primitive Heronian triangle with rectangular properties will be one unit longer than the second longest side. The side lengths can be described as* $(a, b, b+1)$*, where* $a, b \in \mathbb{Z}$*.*

Using these conjectures, the other side length of the rectangle can be solved for using the property that the perimeters of the triangle and rectangle are the same, as shown below. The variable x represents the unknown side length of the rectangle.

$$\begin{aligned} a + b + (b+1) &= 2 * (b+1) + 2 * x \\ a + 2b + 1 &= 2b + 2 + 2 * x \\ a + 1 &= 2 + 2 * x \\ a - 1 &= 2 * x \\ x &= \frac{a-1}{2} \end{aligned}$$

For any primitive heronian triangle in the form $(a, b, b+1) a, b \in \mathbb{Z}$, the equivalent rectangle will have sides $(b+1, \frac{a-1}{2})$. These dimensions satisfy the condition of the perimeter of the triangle and rectangle being equivalent. Along with the perimeter, the area of both shapes must be integer and equivalent $\implies$ the equation below, which equates the areas of the two shapes, must be true for a primitive heronian triangle with these dimensions.

Date: 6/21/2022.

$$(b+1)(\frac{a-1}{2}) = \sqrt{\left(\frac{2b+a+1}{2}\right)\left(\frac{2b-a+1}{2}\right)\left(\frac{a+1}{2}\right)\left(\frac{a-1}{2}\right)}$$

After deriving these equations, a Python script was written to generate more example cases of primitive Heronian triangles with equivalent rectangles by searching for values of a and b that satisfy the equation above. These new values are shown below.

Triangle Side Lengths	*Equivalent Rectangle Side Lengths*
$(5,5,6)$	$(2,6)$
$(13,20,21)$	$(6,21)$
$(25,51,52)$	$(12,52)$
$(41,104,105)$	$(20,105)$
$(61,185,186)$	$(30,186)$
$(85,300,301)$	$(42,301)$
$(113,455,456)$	$(56,456)$
$(145,656,657)$	$(72,657)$
$(181,909,910)$	$(90,910)$

Using these generated values, the following conjectures were developed.

Conjecture 0.3. *The following expression for a_n, where $n \in \mathbb{Z}$, gives values for the side length a. This equation was derived using a quadratic regression model, which fit perfectly to the example cases shown in the table above.*

$$a_n = 5 + 6n + 2n^2$$

Conjecture 0.4. *The following expression for b_n, where $n \in \mathbb{Z}$, gives values for the side length b. This equation was derived using a cubic regression model, which fit perfectly to the example cases shown in the table above.*

$$b_n = 5 + 9n + 5n^2 + n^3$$

To prove the two conjectures above, the expressions for a_n and b_n in terms of n can be plugged back into the equation equating the areas of the rectangle and triangle and still hold true, as shown below.

$$(b+1)(\frac{a-1}{2}) = \sqrt{(S)\,(S-a)\,(S-b)\,(S-c)}$$

$$S = \frac{a+2b+1}{2} = (n+2)^3$$

$$S - a = \frac{2b + 1 - a}{2}$$

$$S - b = \frac{a + 1}{2}$$

$$S - c = \frac{a - 1}{2}$$

$$(b+1)^2(\frac{a-1}{2})^2 = (S)\,(S-a)\,(S-b)\,(S-c)$$

$$(b+1)^2(\frac{a-1}{2})^2 = \frac{1}{8}\,(n+2)^3\,(2b+1-a)\,(a+1)\,(a-1)$$

$$(b+1)^2(\frac{a-1}{2})^2 = \frac{1}{8}\,(n+2)^3\left(11 + 18n + 10n^2 + 2n^3 - 5 - 6n - 2n^2\right)(a+1)\,(a-1)$$

$$(b+1)^2(\frac{a-1}{2})^2 = \frac{1}{4}\,(n+2)^3\left(n^3 + 4n^2 + 6n + 3\right)(a+1)\,(a-1)$$

$$(b+1)^2(\frac{a-1}{2})^2 = (n+2)^4\,(n+1)^2\left(n^2 + 3n + 3\right)^2$$

$$\frac{1}{4}(6 + 9n + 5n^2 + n^3)^2(2n^2 + 6n + 4)^2 = (n+2)^4\,(n+1)^2\left(n^2 + 3n + 3\right)^2$$

$$\frac{1}{4} * (n+2)^2(n^2 + 3n + 3)^2 * 4(n+1)^2(n+2)^2 = (n+2)^4\,(n+1)^2\left(n^2 + 3n + 3\right)^2$$

$$(n+2)^4(n+1)^2(n^2 + 3n + 3)^2 = (n+2)^4\,(n+1)^2\left(n^2 + 3n + 3\right)^2$$

For any $n \in \mathbb{Z}$, the two expressions for a_n and b_n described in the conjectures above can be used to generate a and b values for the triangle, and the last side length is $(b+1)$. The corresponding equivalent rectangles have dimensions $(b+1, \frac{a-1}{2})$. For an infinite domain of $n \geq 0$, where $n \in \mathbb{Z}$, infinite cases of primitive Heronian triangles can be generated using the expressions for a and b in terms of n, showing that infinitely many primitive Heronian triangles with equivalent rectangles exist.