Proving the Existence of an Infinite Number of Primitive Heronian Triangles with Equivalent Rectangles

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Abstract

This paper explores how results from computational resources used to generate example cases of primitive Heronian triangles with equivalent rectangles can be used to examine relationships between the side lengths of these triangles. From these relationships, equations are derived for the side lengths of these triangles to prove that there are infinitely many of these special triangles.

1 Introduction

As described in problem 1230 in the College Mathematics Journal, Heronian triangles are a special family of triangles that have integer side lengths as well as an integer area and integer perimeter. A Heronian triangle is defined as a primitive Heronian triangle if the greatest common divisor between its three side lengths is one. We define a primitive Heronian triangle to have an equivalent rectangle, if there exists a corresponding rectangle with integer dimensions that has equivalent area and perimeter to the primitive Heronian triangle.

2 Results

To examine relationships between cases of primitive Heronian triangles and equivalent rectangles, a python notebook cell, which is shown below, was written to generate a few cases. Since it used a brute force search algorithm to generate a few examples of both primitive and non-primitive Heronian triangles with equivalent triangles, the primitive cases and duplicate cases were filtered out manually.

```
for test_a in range(1, 100):
  for test_b in range(1, 100):
    for test_c in range(1, 100):
      for test_base in range(1, 100):
        for test_height in range(1, 100):
          s = (test_a + test_b + test_c) / 2
          if (s*(s-test_a)*(s-test_b)*(s-test_c))**(1/2) == test_base * test_height:
            if 2*s == 2 * test_height + 2 * test_base:
              print(test_a, test_b, test_c, test_base, test_height)
```

A table containing the filtered generated cases from the code is shown below.

Triangle Side Lengths	Equivalent Rectangle Side Lengths	Perimeter	Area
$(5, 5, 6)$	$(2, 6)$	16	12
$(13, 20, 21)$	$(6, 21)$	54	126
$(25, 51, 52)$	$(12, 52)$	128	624

From these examples cases, the following generalizations were developed:

The longest sides of a primitive Heronian triangle and its equivalent rectangle are of the same length.

The longest side of a primitive Heronian triangle with an equivalent rectangle will be one unit longer than the second longest side. The side lengths can be described as $(a, b, b+1)$, where $a, b \in Z$ and $b \geq a$ which implies that one of the sides of the equivalent rectangle will be $(b+1)$. Because the side lengths b and $(b+1)$ are relatively prime, the GCD of a triangle in this form will be 1, which satisfies the condition that the GCD of the side lengths of a primitive Heronian triangle must be one.

Using these generalizations, the other side length of the rectangle can be solved for using the property that the perimeters of the triangle and rectangle are the same, as shown below. The variable x represents the unknown side length of the rectangle.

$$a + b + (b+1) = 2(b+1) + 2x$$
$$a + 2b + 1 = 2b + 2 + 2x$$
$$a + 1 = 2 + 2x$$
$$a - 1 = 2x$$
$$x = \frac{a-1}{2}$$

For any primitive Heronian triangle in the form $(a, b, b+1), a, b \in Z$, the equivalent rectangle will have sides $\left(b+1, \frac{a-1}{2}\right)$. These dimensions satisfy the condition of the perimeter of the triangle and rectangle being equivalent. Along with the perimeter, the area of both shapes must be integer and equivalent, which implies that the equation below, which equates the areas of the two shapes, must be true for a primitive Heronian triangle with these dimensions. Heron's formula for the triangle area is used on the right side, and the base times height formula for the area of the equivalent rectangle is used on the left side. Let S, be the semi-perimeter of the triangle.

$$c = (b+1)$$

$$S = \frac{a+b+(b+1)}{2} = \frac{2b+a+1}{2}.$$

Using Heron's Formula, the areas of the triangle and rectangle in terms of a and b are equated.

$$(b+1)\left(\frac{a-1}{2}\right) = \sqrt{(S)\,(S-a)\,(S-b)\,(S-c)}$$

We substitute for S in terms of a and b which gives us the following equation.

$$(b+1)\left(\frac{a-1}{2}\right) = \sqrt{\left(\frac{2b+a+1}{2}\right)\left(\frac{2b-a+1}{2}\right)\left(\frac{a+1}{2}\right)\left(\frac{a-1}{2}\right)}$$

After deriving these equations, a Python script, which is shown below, was written to generate more example cases of primitive Heronian triangles with equivalent rectangles by searching for values of a and b that satisfy the equation above using a brute force search.

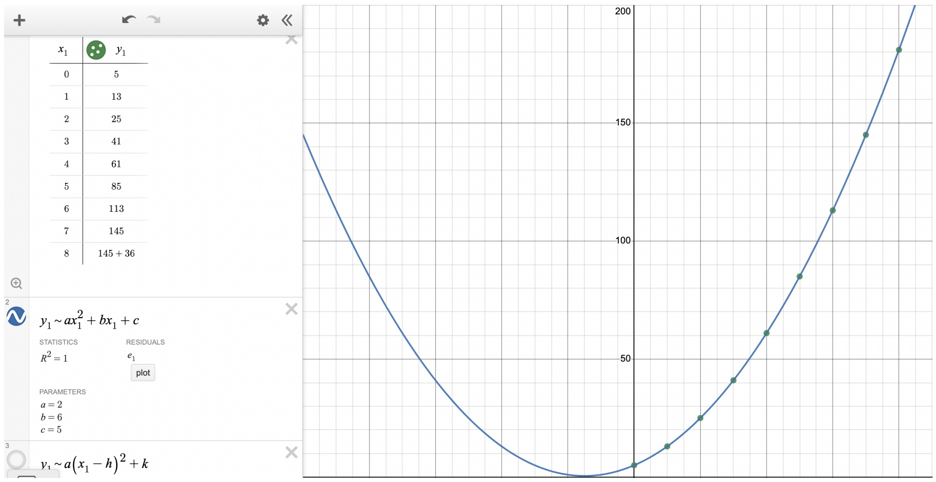
```
a_values = []
b_values = []

for a in range(5, 1000):
  for b in range(5, 1000):
    if (b+1) * ((a - 1) / 2) == (((2*b+a+1)/2)*((2*b-a+1)/2)*((a+1)/2)*((a-1)/2))**(0.5):
      print(a, b, b + 1, b+1, (a-1) / 2)
      a_values.append(a)
      b_values.append(b)
```

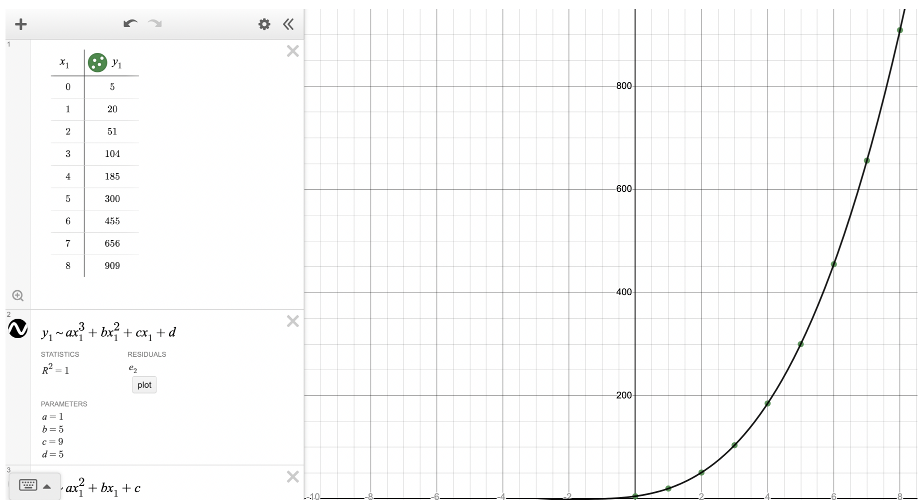
A table containing the new values generated by the code is shown below.

Triangle Side Lengths	Equivalent Rectangle Side Lengths	Perimeter	Area
$(5, 5, 6)$	$(2, 6)$	16	12
$(13, 20, 21)$	$(6, 21)$	54	126
$(25, 51, 52)$	$(12, 52)$	128	624
$(41, 104, 105)$	$(20, 105)$	250	2100
$(61, 185, 186)$	$(30, 186)$	432	5580
$(85, 300, 301)$	$(42, 301)$	686	12642
$(113, 455, 456)$	$(56, 456)$	1024	25536
$(145, 656, 657)$	$(72, 657)$	1458	47304
$(181, 909, 910)$	$(90, 910)$	2000	81900

Shown below is the graph for $a_n$vs.n.



Shown below is the graph for $b_n$vs.n.



The following conjectures were developed through running quadratic and cubic regression models that perfectly fit to these generated values.

Conjecture 2.1 *The following expression for a_n, where $n \in Z$, gives values for the triangle side length a.*

$$a_n = 5 + 6n + 2n^2$$

Conjecture 2.2 *The following expression for b_n, where $n \in Z$, gives values for the triangle side length b.*

$$b_n = 5 + 9n + 5n^2 + n^3$$

To prove the two conjectures above, the expressions for a_n and b_n in terms of n can be plugged back into the equation equating the areas of the rectangle and triangle and still hold true

$$(b+1)\left(\frac{a-1}{2}\right) = \sqrt{(S)(S-a)(S-b)(S-c)}$$

We can first rewrite the perimeter $2S$ in terms of n

$$2S = a + 2b + 1 = \left(5 + 6n + 2n^2\right) + 2\left(5 + 9n + 5n^2 + n^3\right) + 1 = 2n^3 + 12n^2 + 24n + 16 = 2\left(n+2\right)^3$$

Now, let's rewrite each component of Heron's Formula in terms of n

$$\begin{aligned} S &= (n+2)^3 \\ S - a &= \frac{2b + 1 - a}{2} \\ S - b &= \frac{a+1}{2} \\ S - c &= \frac{a-1}{2} \end{aligned}$$

We now multiply the above terms resulting in the expression below

$$\begin{aligned} (S)(S-a)(S-b)(S-c) &= \frac{1}{8}(n+2)^3(2b+1-a)(a+1)(a-1) \\ &= \frac{1}{8}(n+2)^3\left(11 + 18n + 10n^2 + 2n^3 - 5 - 6n - 2n^2\right)(a+1)(a-1) \\ &= \frac{1}{8}(n+2)^3\left(2n^3 + 8n^2 + 12n + 6\right)(a+1)(a-1) \\ &= \frac{1}{4}(n+2)^3(n+1)\left(n^2 + 3n + 3\right)(a+1)(a-1) \\ &= \frac{1}{4}(n+2)^3(n+1)\left(n^2 + 3n + 3\right)\left(5 + 6n + 2n^2 + 1\right)\left(5 + 6n + 2n^2 - 1\right) \\ &= \frac{1}{2}(n+2)^3(n+1)\left(n^2 + 3n + 3\right)^2\left(4 + 6n + 2n^2\right) \\ &= (n+2)^4(n+1)^2\left(n^2 + 3n + 3\right)^2 \end{aligned}$$

We set this equal to the square of the area, which we rewrite in terms of n

$$\begin{aligned} \frac{1}{4}(b+1)^2(a-1)^2 &= (n+2)^4(n+1)^2\left(n^2+3n+3\right)^2 \\ \frac{1}{4}\left(5 + 9n + 5n^2 + n^3 + 1\right)^2(a-1)^2 &= (n+2)^4(n+1)^2\left(n^2+3n+3\right)^2 \\ \frac{1}{4}(n+2)^2\left(n^2+3n+3\right)(a-1)^2 &= (n+2)^4(n+1)^2\left(n^2+3n+3\right)^2 \\ \frac{1}{4}(n+2)^2\left(n^2+3n+3\right)\left(5 + 6n + 2n^2 - 1\right)^2 &= (n+2)^4(n+1)^2\left(n^2+3n+3\right)^2 \\ \frac{1}{4} * 4(n+2)^4(n+1)^2\left(n^2+3n+3\right)^2 &= (n+2)^4(n+1)^2\left(n^2+3n+3\right)^2 \\ (n+2)^4(n+1)^2\left(n^2+3n+3\right)^2 &= (n+2)^4(n+1)^2\left(n^2+3n+3\right)^2 \end{aligned}$$

Therefore, the equation holds true, so the areas of the triangle and rectangle are the same, satisfying the condition.

For any $n \in Z$, the two expressions for a_n and b_n described in the conjectures above can be used to generate a and b values for the triangle, with the last side length being $(b+1)$. The corresponding equivalent rectangles have dimensions $\left(b+1, \frac{a-1}{2}\right)$. For an infinite domain of $n \geq 0$, where $n \in Z$, infinite cases of primitive Heronian triangles can be generated using the expressions for a and b in terms of n, proving that infinitely many primitive Heronian triangles with equivalent rectangles exist.

3 Conjectures

Based on the generalizations that we observed

4 Future Research

While the initial question of whether an infinite number of primitive Heronian triangles with equivalent rectangles exist was proven in this paper, some conjectures used in this proof have not been proven. The first course of action would be to prove conjecture 2.1 and conjecture 2.2 because they are the basis for our proof, as they helped discover more primitive Heronian triangles. Next, we would potentially like to deduce an equation relating a to b which we believe exists from some research that is not included in this paper. We found a few fits with a R^2 value of 1, so using the equation, we could essentially generalize the primitive Heronian triangles and their equivalent rectangles to where everything has just one variable (a or b). In this research, we have only looked into triangles in the form $(a, b, b+1)$, so we can also attempt to find primitive Heronian triangles of different "triangle families" and study those triangles.

5 Acknowledgements

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