18.100B - Problem Set 13

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10. The hint basically shows everything necessary so I will just verify each step. The first step follows from the Fundamental Theorem of Arithmetic and the fact there will exist a term, $p = p_1^{e_1} p_2^{e_2} p_3^{e_3} \ldots$ in the product expansion such that $\frac{1}{N} < \frac{1}{p}$. The second step follows from the formula for the sum of a geometric series. The third step follows from showing $(1-x)^{-1} \leq e^{2x}$ for $0 \leq x \leq \frac{1}{2}$. Notice that $e^{2x} \geq 1 + 2x$ so $e^{2x} - \frac{1}{1-x} \geq 1 + 2x - \frac{1}{1-x} = \frac{x(1-2x)}{1-x} \geq 0$ when $0 \leq x \leq \frac{1}{2}$. Then the result follows from comparison.

12. (a) $c_n = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \dfrac{1}{2\pi}\int_{-\delta}^{\delta} e^{-inx}dx = -\dfrac{e^{-inx}}{2\pi in}\Bigg|_{-\delta}^{\delta} = \dfrac{e^{\delta ni} - e^{-\delta ni}}{2\pi ni} = \dfrac{\sin n\delta}{\pi n}$ for $n \neq 0$. $c_0 = \frac{\delta}{\pi}$ obviously.

    (b) Note that $c_n = c_{-n}$ so $\displaystyle\sum_{n\geq 1} c_n = \sum_{n=1}^{\infty} \frac{\sin n\delta}{\pi n} = \frac{f(0) - c_0}{2} = \frac{1 - \frac{\delta}{\pi}}{2}$ and multiplying across by π gives the claim.

    (c) Note that $|f(x)|^2 = f(x)$. By Parseval's theorem, $\displaystyle\frac{\delta^2}{\pi^2} + \sum_{n\neq 0} \frac{\sin^2 n\delta}{\pi^2 n^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = c_0 = \frac{\delta}{\pi}$. So similarly we have $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n\delta}{\pi^2 n^2} = \frac{\frac{\delta}{\pi} - \frac{\delta^2}{\pi^2}}{2}$. Multiplying by $\dfrac{\pi^2}{\delta}$ gives the claim.

    (d) Note that the integral converges since it exists from $x = 0$ to $x = 1$, as there is a discontinuity of the first kind at $x = 0$, and from $x = 1$ to ∞ as we can compare to $\int \frac{1}{x^2}$ which does exist. Notice that the sum in (c) is uniformly convergent by comparison and so continuous so we can take $\delta \to 0$ using $\delta_n = \frac{1}{n}$ and it converges to $\frac{\pi}{2}$ (Theorems 7.11 + 7.12). So after substitution and taking limits, $\displaystyle\frac{\pi}{2} = \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{\sin^2(\frac{k}{n})}{\frac{k^2}{n^2}} \frac{1}{n} = \int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx$ since it is a Riemann sum.

    (e) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n\frac{\pi}{2}}{n^2 \frac{\pi}{2}} = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi}{4} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

13. Let $f(x+2\pi) = f(x)$. Note $\displaystyle 2\pi c_n = \int_{-\pi}^{\pi} f(x)e^{-inx}dx = \int_0^{2\pi} xe^{-inx}dx = -\frac{xe^{-inx}}{in}\Bigg|_0^{2\pi} + \int_0^{2\pi} \frac{e^{-inx}}{in}dx = \frac{i2\pi e^{-i2n\pi}}{n} + \frac{e^{-inx}}{n^2}\Bigg|_0^{2\pi} = \frac{i2\pi e^{-i2n\pi}}{n} + \frac{e^{-i2n\pi}}{n^2} - \frac{1}{n^2} = \frac{i2\pi}{n}$, since $e^{-i2n\pi} = 1$. So $|c_n|^2 = \frac{1}{n^2}$ for $n \neq 0$. Note $c_0 = \pi$, the average value of f, and that $|c_n^2| = |c_{-n}^2|$. So $\displaystyle\pi^2 + 2\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{2\pi}\int_0^{2\pi} x^2 dx = \frac{4\pi^2}{3}$ and thus $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

14. $2\pi c_n = \int_{-\pi}^{\pi} (\pi - |x|)^2 e^{-inx} dx = \int_0^{\pi} (\pi - x)^2 e^{-inx} + (\pi - x)^2 e^{inx} dx = 2\int_0^{\pi} (\pi - x)^2 \cos nx dx = 2\int_0^{\pi} x^2 \cos(n\pi - nx) dx = 4x \frac{\cos(n\pi - nx)}{n^2}\Big|_0^{\pi} = \frac{4\pi}{n^2}$ so $c_n = \frac{2}{n^2}$ for $n \neq 0$. $c_0 = \frac{1}{\pi}\int_0^{\pi} x^2 = \frac{\pi^2}{3}$. Note $c_n = c_{-n}$ so $c_n e^{-inx} + c_{-n} e^{inx} = 2c_n \cos nx = \frac{4}{n^2}\cos nx$. So $f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{-inx} + c_{-n} e^{inx} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$. Plugging in $x = 0$ gives $\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$ which rearranges into the first result. By Parseval's Theorem and similar integration tricks, we have $\frac{\pi^4}{9} + 2\sum_{n=1}^{\infty} \frac{4}{n^4} = \frac{1}{\pi}\int_0^{\pi} x^4 dx = \frac{\pi^4}{5}$ which simplifies into the second result.

15. Note $\sum \sin(n+\frac{1}{2})x = \Im(\sum e^{i(n+\frac{1}{2})x}) = \Im(e^{\frac{ix}{2}} \sum e^{inx}) = \Im(e^{\frac{ix}{2}} \cdot \frac{1-e^{ix(N+1)}}{1-e^{ix}}) = \Im(\frac{1-\cos(N+1)x - i\sin(N+1)x}{e^{-\frac{ix}{2}} - e^{\frac{ix}{2}}}) = \frac{1-\cos(N+1)x}{2\sin\frac{x}{2}}$, so $\sum D_n(x) = \frac{1-\cos(N+1)x}{2\sin^2\frac{x}{2}} = \frac{1-\cos(N+1)x}{1-\cos x}$ and K_N follows.

(a) The fact that $\cos x \leq 1$ makes it clear $K_N \geq 0$.

(b) $\int_{-\pi}^{\pi} D_N(x) dx = \sum_{n=-N}^{N} \int_{-\pi}^{\pi} e^{inx} dx = \sum_{n=-N}^{N} \frac{2\sin \pi n}{n} = 2\pi$ since it is 0 everywhere except at $n = 0$ which is calculated easily. So $\int_{-\pi}^{\pi} \sum_{n=0}^{N} D_n(x) dx = \sum_{n=0}^{N} \int_{-\pi}^{\pi} D_n(x) dx = \sum_{n=0}^{N} 2\pi = 2\pi(N+1)$. The integral involving $K_N(x)$ follows easily.

(c) $1 - \cos(N+1)x \leq 2$ and $1 - \cos x \geq 1 - \cos\delta$ so $\frac{1-\cos(N+1)x}{1-\cos x} \leq \frac{2}{1-\cos\delta}$ and the inequality involving $K_N(x)$ follows.

By 8.13, $s_N(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t) D_N(t) dt$. So $\sigma_N(f;x) = \frac{1}{N+1}\sum_{n=0}^{N} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t) D_n(t) dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(n-t) \sum_{n=0}^{N} \frac{D_n(t)}{N+1} dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(n-t) K_N(x) dx$.

$|\sigma_n(x) - f(x)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-t) - f(x)| K_n(t) dt$. $\forall\epsilon\ \exists\delta$ s.t. $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$ since f is continuous on a compact set, hence uniformly continuous. Note f must also be bounded by a number M. So splitting our inequality gives us $|\sigma_n(x) - f(x)| \leq \frac{1}{2\pi}\int_{|t|<\delta} |f(x-t) - f(x)| K_n(t) dt + \frac{1}{2\pi}\int_{|t|\geq\delta} |f(x-t) - f(x)| K_n(t) dt \leq \frac{\epsilon}{2\pi}\int_{-\pi}^{\pi} K_n(t) + \frac{1}{2\pi}\int_{|t|\geq\delta} 2M \cdot \frac{1}{N+1} \cdot \frac{2}{1-\cos\delta} dt < \epsilon + 2M \cdot \frac{1}{n+1} \cdot \frac{2}{1-\cos\delta} < \epsilon + \epsilon = 2\epsilon$ where we can choose n large enough to guarantee the second ϵ, hence $\sigma_N(f;x) \to f(x)$ uniformly on $[-\pi, \pi]$.