18.100B - Problem Set 9

Rishad Rahman

19. (a) Since $f'(0)$ exists, $\forall\epsilon\ \exists\delta_1,\delta_2>0$ s.t. $\left|\dfrac{f(\beta_n)-f(0)}{\beta_n}-f'(0)\right|<\epsilon$ when $\beta_n<\delta_1$ and $\left|\dfrac{f(\alpha_n)-f(0)}{\alpha_n}-f'(0)\right|<\epsilon$ when $\alpha_n>-\delta_2$. Pick $\delta=\min(\delta_1,\delta_2)$. Similarly pick $N=\min(N_1,N_2)$ so that this is true for all α_n,β_n for $n\geq N$ which is guaranteed since the sequences go to 0. So we have $|f(\beta_n)-f(0)-\beta_n f'(0)|<\beta_n\epsilon$ and $|\alpha_n f'(0)-f(\alpha_n)+f(0)|<-\alpha_n\epsilon$. Adding the two inequalities and using Triangle Inequality gives $|f(\beta_n)-f(\alpha_n)-f'(0)(\beta_n-\alpha_n)|<(\beta_n-\alpha_n)\epsilon$ so $\left|\dfrac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}-f'(0)\right|<\epsilon$. This is true for any arbitrary ϵ for $n\geq N$ so the limit is $f'(0)$.

    (b) Following the same steps as the a) but instead since $\alpha_n>0$ instead we would have $|f(\beta_n)-f(0)-\beta_n f'(0)|<\beta_n\epsilon$ and $|\alpha_n f'(0)-f(\alpha_n)+f(0)|<\alpha_n\epsilon$ so $|f(\beta_n)-f(\alpha_n)-f'(0)(\beta_n-\alpha_n)|<(\beta_n+\alpha_n)\epsilon$. Now $\exists M\ \forall n\ \dfrac{\beta_n}{\beta_n-\alpha_n}<M$ so $\beta_n<M(\beta_n-\alpha_n)$ but $\alpha_n<\beta_n$ so $\alpha_n<M(\beta_n-\alpha_n)$ and adding the two gives $\beta_n+\alpha_n<2M(\beta_n-\alpha_n)$. Therefore $|f(\beta_n)-f(\alpha_n)-f'(0)(\beta_n-\alpha_n)|<2M\epsilon(\beta-\alpha_n)$. This is true for any arbitrary $2M\epsilon$ so by the same steps of the previous problem the limit is also $f'(0)$.

    (c) f' is continuous so by MVT, $\exists a$ s.t. $|a|<|\alpha_n|$, $\exists b$ s.t. $|b|<|\beta_n|$ $f(\alpha_n)=\alpha_n f'(a)+f(0)$ and $f(\beta_n)=\beta_n f'(b)+f(0)$ and subtracting the two and dividing by $\beta_n-\alpha_n$ gives $\dfrac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}=\dfrac{\beta_n f'(b)-\alpha_n f'(a)}{\beta_n-\alpha_n}$ But $a,b\to 0$ as $\beta_n,\alpha_n\to 0$ so taking $n\to\infty$ gives the limit as $\frac{\beta_n-\alpha_n}{\beta_n-\alpha_n}f'(0)=f'(0)$.
    Let $f(x)=x^2\sin\left(\frac{1}{x}\right)$ for $x\neq 0$ and $f(0)=0$ so $f'(0)=0$ but $f'(x)=2x\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right)$ which obviously isn't continuous at $x=0$.

25. (a) x_{n+1} is where the tangent to $f(x_n)$ hits the x-axis as $y-f(x_n)=f'(x_n)(x_{n+1}-x_n)$ so if $y=0$ $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$.

    (b) $x_1>\xi$, assume $x_n>\xi$. $\frac{f(x_n)-f(\xi)}{x_n-\xi}\leq f'(x_n)$ by MVT and since f' is monotonically increasing. So $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}\geq\xi$ which is valid since $x_n-\xi>0$ so this shows $x_n>\xi$ for all n. So $f(x_n)>0$ and since $f'>0$, $x_{n+1}<x_n$. We have showed x_n is monotonically decreasing and bounded so a limit exists. Taking the limit of both sides yields $L=L-\frac{f(L)}{f'(L)}\Rightarrow f(L)=0\Rightarrow L=\xi$.

    (c) $f(\xi)=0=f(x_n)+f'(x_n)(\xi-x_n)+\frac{f''(t_n)}{2}(x_n-\xi)^2$ so $(x_n-x_{n+1})f'(x_n)+f'(x_n)(\xi-x_n)+\frac{f''(t_n)}{2}(x_n-\xi)^2=0\Rightarrow f'(x_n)(x_{n+1}-\xi)=\frac{f''(t_n)}{2}(x_n-\xi)^2$ which leads to the conclusion.

    (d) Repeat c) and use the fact $f''\leq M$ to get $x_{n+1}-\xi\leq\frac{M^{1+2+4+\ldots+2^{n-1}}(x_1-\xi)^{2^n}}{(2\delta)^{1+2+4+\ldots+2^{n-1}}}=\frac{2\delta}{M}\left[\frac{M(x_1-\xi)}{2\delta}\right]^{2^n}=\frac{1}{A}[A(x_1-\xi)]^{2^n}$

    (e) $g(x)=x\Rightarrow f(x)=0\Rightarrow x=\xi$

    (f) $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-3x_n^{\frac{1}{3}}\cdot x_n^{\frac{2}{3}}=-2x_n=(-2)^{n-1}x_1$ so x_n does not converge and this does not allow us to find ξ.

26. $|\frac{f(x)-f(a)}{x-a}| = |\frac{f(x)}{x-a}| \le M_1$ by MVT and bounds so $|f(x)| \le M_1(x-a) \le M_1(x_0-a) \le A(x_0-a)M_0$ since $M_1 \le AM_0$. But if $A(x_0-a) < 1$ then M_0 wouldn't be the supremum so for $a < x < x_0 < a+\frac{1}{A}$, the supremum must be 0 so $f = 0$ on that interval. We can easily split our interested interval into partitions smaller than $a + \frac{1}{A}$ so f must be 0 on the whole interval.

27. Let $f(x) = \phi(x, y_2) - \phi(x, y_1)$, then from 26, $f(x) = 0$ on the interval so $\phi(x, y_2) = \phi(x, y_1)$ so there cannot exist more than one solution. For the given initial-value problem those are the only solutions which can be easily verified through separation of variables and the case when $y = 0$.

1. It is integrable by Theorem 6.10. The integral is 0 since it is equal to the lower integral sum which is obviously 0 since every partition contains a point that is not x_0 so the infimum is always 0.

2. Assume $\exists c \in [a, b]$ s.t. $f(c) = d > 0$. By continuity $\exists\delta$ s.t. if $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$ where $\epsilon = \frac{d}{2}$. So on the interval $(c-\delta, c+\delta)$ f is bounded by $f(c) + \frac{d}{2}$ and $f(c) - \frac{d}{2} > 0$ so any partition containing that intersects with $(c-\delta, c+\delta)$ will result in a f having a positive supremum, M, on one of the Δx's. Since $f \ge 0$ we will have either 0 or a positive contribution to the integral sum but we have a positive integral sum since $\int f dx = \overline{\int} f dx > 0$ which is a contradiction so $f = 0$ on the interval.

3. (a) Take any partition of $[-1, 1]$ and add 0 into it if it is not which would result in a refinement. Let's say $x_{n-1} < 0$ and $x_n > 0$ (shift indices by 1 for $k > n$ if $x_n = 0$). Then $\alpha(0) - \alpha(x_{n-1}) = 0$ and $\alpha(x_n) - \alpha(0) = 1$. So $U(P, f, \alpha) = M(0, x_n)$ and $L(P, f, \alpha) = m(0, x_n)$. Let $0 < x_{n_1} < x_n$ so that we have a refinement of P, P_1^* when x_{n_1} is added. $\alpha(x_n) - \alpha(x_{n_1}) = 1 - 1 = 0$ so $U(P_1^*, f, \alpha) = M(0, x_{n_1})$ and $L(P, f, \alpha) = m(0, x_{n_1})$. By similar construction $U(P_k^*, f, \alpha) = M(0, x_{n_k})$ and $L(P, f, \alpha) = m(0, x_{n_k})$. So taking $k \to \infty$, $x_{n_k} \to 0+$ so $\overline{\int} f d\alpha = M(0, 0+)$ and $\underline{\int} f d\alpha = m(0, 0+)$. If $f(0) = f(0+)$ then $\overline{\int} f d\alpha = \underline{\int} f d\alpha = M(0, 0+) = m(0, 0+) = f(0)$ so $f \in \mathcal{R}(\beta_1)$ and $\int f d\alpha = f(0)$. If $f(0+) \ne f(0)$ then we would have one as a max which results in $\overline{\int} f d\alpha$ and the other one as a min which results in $\underline{\int} f d\alpha$ and they wouldn't be equal so f wouldn't be integrable and that proves the reverse direction.

   (b) $f \in \mathcal{R}(\beta_2) \Leftrightarrow f(0-) = f(0)$. The argument is almost identical to the previous solution except that we have $U(P_k^*, f, \alpha) = M(x_{n_k}, 0)$ and $L(P, f, \alpha) = m(x_{n_k}, 0)$ since $\Delta\alpha = 0$ if $x_{n-1}, x_n > 0$ but 1 if $x_{n-1} < 0$ and $x_n = 0$ so we would be dealing with $f(0)$ and $f(0-)$ instead and the reasoning is analogous.

   (c) Here we have $U(P_k^*, f, \alpha) = \frac{1}{2}(M(x_{n-1_k}, 0) + M(0, x_{n_k}))$ and $L(P_k^*, f, \alpha) = \frac{1}{2}(m(x_{n-1_k}, 0) + m(0, x_{n_k}))$. Taking limits we have $\overline{\int} f d\alpha = \frac{1}{2}(M(0-, 0) + M(0, 0+))$ and $\underline{\int} f d\alpha = \frac{1}{2}(m(0-, 0) + m(0, 0+))$. If f is continuous then $f(0) = f(0-) = f(0+)$ so $\overline{\int} f d\alpha = \underline{\int} f d\alpha = f(0)$ so $f \in \mathcal{R}(\beta_3)$. If f was not continuous then by the same argument as the previous parts we would have unique maximum and minimums so the upper and lower sums wouldn't agree. We would need $M(0-, 0) + M(0, 0+) = m(0-, 0) + m(0, 0+)$ or $M(0-, 0) - m(0-, 0) = m(0, 0+) - M(0, 0+)$ Obviously the left is non-negative and the right is non-positive so they both equal 0 so $M(0-, 0) = m(0-, 0)$. If $f(0-) \ne f(0)$ then this would not be possible and analogously for $M(0+, 0) = m(0+, 0)$ except with $f(0+)$ so f must be continuous.

   (d) If f is continuous, $f(0) = f(0-) = f(0+)$ so the conditions to a), b) are satisfied and we already showed in c) that the integral resolves to $f(0)$ so $\int f d\alpha = f(0)$ for $\alpha = \beta_1, \beta_2, \beta_3$.