18.100B - Problem Set 11

Rishad Rahman

2. $\lim_{n\to\infty} \sup|f - f_n| = 0$ and $\lim_{n\to\infty} \sup|g - g_n| = 0$ where the sups are taken over E. $|f + g - f_n - g_n| \leq |f - f_n| + |g - g_n| \Rightarrow \lim_{n\to\infty} \sup|f + g - f_n - g_n| \leq \lim_{n\to\infty} \sup|f - f_n| + |g - g_n| = 0$. So $f_n + g_n$ uniformly converges to $f + g$.
$|fg - f_n g_n| \leq |f||g - g_n| + |g_n||f - f_n| \Rightarrow \lim_{n\to\infty} \sup|fg - f_n g_n| \leq \lim_{n\to\infty} \sup|f||g - g_n| + |g_n||f - f_n| = 0$ since g_n is bounded and note f must be bounded as well otherwise we can find x s.t. $|f - f_n| \geq |f| - |f_n| \geq |f| - M_n \geq \epsilon$. So $f_n g_n$ uniformly converges to fg.

3. Take $f_n = x$ and $g_n = \frac{1}{n}$ and $E = \mathbb{R}$. Note f_n uniformly converges to x and that g_n to 0. $f_n g_n = \frac{x}{n}$. Note $f_n g_n \to 0$ as $n \to 0$. But $\lim_{n\to\infty} \sup\left|\frac{x}{n}\right| \neq 0$.

4. $\frac{1}{|1+n^2x|} < \frac{1}{n^2|x|}$ so it converges by comparison for all $x > 0$. It obviously diverges if $x = 0$ since then we have an infinite summation of 1's or when $x = -\frac{1}{n^2}$, $n \in \mathbb{N}$, since then the series wouldn't be defined. Now consider all other values when $x < 0$. Let N be chosen s.t. $N^2x < -1$ so in this case we can really just test the convergence of $\sum_{n=N}^{\infty} \frac{1}{n^2x - 1}$ where $x > 0$ (change of variables from x to $-x$) and now $N^2x > 1$. But this converges by comparison to $\frac{2}{n^2x}$. Therefore the series converges absolutely on $\mathbb{R}\backslash(\{0\} \cup \{-\frac{1}{n^2}\})$, $n \in \mathbb{N}$.
Let's say the interval is $[a, b]$. Obviously $(\{0\} \cup \{-\frac{1}{n^2}\}) \cap [a, b] = \emptyset$, $n \in \mathbb{N}$, otherwise f wouldn't converge at all x. Otherwise if $b > a > 0$, $|a_n| = \frac{1}{|1+n^2x|} < \frac{1}{an^2}$ so by comparison f uniformly converges. If $a < b < 0$, $\frac{1}{|1+n^2x|} \to \frac{1}{n^2x-1} < \frac{2}{n^2b}$ which converges, when switching to nonnegative terms i.e. $x \to -x$, $[a, b] \to [-b, -a]$ and when $n^2x > 1$. But we can guarantee this by choosing $n > \sqrt{\frac{1}{b}}$ then $n^2x > \frac{1}{b} \cdot b = 1$. For $n \leq \sqrt{\frac{1}{b}}$ we have discrete terms which are obviously bounded so by comparison we have f uniformly convergent. So any interval $[a, b]$ s.t. $(\{0\} \cup \{-\frac{1}{n^2}\}) \cap [a, b] = \emptyset$, $n \in \mathbb{N}$.
Obviously if any of the aforementioned points are contained within the interval we wouldn't have a converging function but let us check intervals of the form $(a, b]$ where $a = -\frac{1}{n^2}$, $n \in \mathbb{N}$, or $a = 0$. Well notice that $\sup|f - f_n| = \sup|\sum \frac{1}{1+n^2x}| = \infty$ when x approaches any of those forbidden numbers so thThere is no way for f to uniformly converge.
f converges uniformly precisely at the same x when the series converges and uniform convergence implies continuity so f is continuous on the same numbers.
f is obviously not bounded since we can get arbitrarily get close to ∞ when $x \to 0$.

6. $\sup|f - f_n| = \sup\left|\sum_{k=n+1}^{\infty} (-1)^k \frac{x^2 + k}{k^2}\right|$. Note that the magnitude of the terms of the series monotonically decreases, since adding 1 to the numerator does not have as large of an effect as adding $2n + 1$ to the denominator. Therefore the sup is precisely the first term or $\frac{x^2+n+1}{n^2} \to 0$ as $n \to \infty$ so f uniformly converges. The series obviously does not converge absolutely since it splits into $\sum \frac{x^2}{n^2} + \frac{1}{n}$, which is a sum of a convergent and divergent series so the whole series diverges.

7. I claim f_n uniformly converges to $f = 0$. $\sup|f - f_n| = \sup\left|\frac{x}{1+nx^2}\right| = \sup\left|\frac{1}{\frac{1}{x}+nx}\right| = \frac{1}{2\sqrt{n}} \to 0$ by AM-GM.
$f_n'(x) = \frac{1-nx^2}{(nx^2+1)^2} \to 0$ as $n \to \infty$ when $x \neq 0$ and $f'(0) = 0$ which checks. But $f_n'(0) = 1$ so the statement is true for all $x \neq 0$.

10. Claim: f is discontinuous on $\mathbb{Q}$. First note f_n, the partial sums, is uniformly convergent since $0 \leq (nx) < 1$ so we just compare to $\frac{1}{n^2}$. Note when x is irrational we can never find an n such that $(nx) = 0$ so $\frac{(nx)}{n^2}$ is continuous and so the partial sums are continuous and by 7.12 f is therefore continuous as well. When $x \in \mathbb{Q}$ we can find n s.t. $(nx) = 0$ but note it is discontinuous at that x since it approaches 1 from the left. Because $f(x) = \lim\limits_{t\to x}\sum\limits_{n=1}^{\infty}\dfrac{(nt)}{n^2} = \lim\limits_{n\to\infty}\lim\limits_{t\to x}\sum\limits_{k=1}^{n}\dfrac{(kt)}{k^2}$ but the latter obviously does not exist since the right and left hand limits differ when $(nx) = 0$ so f is discontinuous on $\mathbb{Q}$ which we know is a countable dense set.
Note f_n is Riemann integrable since it is a finite sum of Riemann integrable functions (since they have finitely many jump discontinuities on any bounded interval) and so f must be Riemann integrable by Theorem 7.16.

12. $\lim\limits_{n\to\infty}\displaystyle\int_0^{\infty} f_n(x)dx = \lim\limits_{n\to\infty}\lim\limits_{c\to\infty}\int_0^{c} f_n(x)dx$. We just need to show that $\int f_n(x)dx$ converges uniformly to $\int f(x)dx$ on $(0,\infty)$ then we can exchange the order of the limits by 7.11 and we are done, note the process is similar when taking 0 as the limit point instead of ∞. Let us say we have a bounded interval $[a,b]$. Since f_n uniformly converges to f, $\forall\epsilon > 0\ \exists N$ s.t. for $n > N$ $\left|\displaystyle\int_a^b f - f_n dx\right| \leq \int_a^b |f - f_n|dx < (b-a)\epsilon$ and so we have shown the integral uniformly converges as well so we can obtain the result.

14. Φ is continuous since $x(t)$ and $y(t)$ are continuous by 7.10.
$3^k t_0 = \sum\limits_{i=1}^{\infty} 3^{k-i-1}(2a_i) \equiv \sum\limits_{i=1}^{\infty} 3^{-i}(2a_{k+i-1}) \mod 2$. If $a_k = 0$ then the sum is in the interval $[0, \frac{1}{3}]$ so $f(3^k t_0) = 0 = a_k$. Similarly if $a_k = 1$ then the sum is in the interval $[\frac{2}{3}, 1]$ and $f(3^k t_0) = 1 = a_k$. Substituting $2n-1$ and $2n$ for k and then substituting into the definitions for $x(t)$ and $y(t)$ gives the result.