18.310A Problem Set #2

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1. (a) $\frac{23}{32} \cdot \frac{9}{31} + \frac{9}{32} \cdot \frac{23}{31} = \frac{207}{496}$

   (b) $\frac{22}{30} \cdot \frac{8}{29} \cdot 2 = \frac{176}{435}$ since there are 22 red and 8 blue balls left.

   (c) $P(\text{Anne same color}) = \frac{23 \cdot 22 + 9 \cdot 8}{32 \cdot 31} = \frac{289}{496}$. So the probability we want is $\frac{\frac{23 \cdot 22 \cdot 21 \cdot 9 \cdot 2 + 9 \cdot 8 \cdot 7 \cdot 23 \cdot 2}{32 \cdot 31 \cdot 30 \cdot 29}}{\frac{289}{496}} = \frac{17871}{41905}$ where we got the numerator from the two different cases where Anne picked two blue balls or two red balls.

   (d) We have to take into account what Anne might've gotten, same colors or different colors. This is equivalent to b)×P(Anne different colors) + c)×P(Anne same color) $= \frac{176}{435} \cdot \frac{207}{496} + \frac{17871}{41905} \cdot \frac{289}{496} = \frac{207}{496}$

   (e) The equality between a) and d) is an interesting one... The drawing of marbles can be represented by sequences with 23 of r and 9 of b. There are a total of $\binom{32}{9}$ possible of sequences. We are looking at the scenario where we have a rb or a br in a certain spot in the sequence. Then there are 22 red and 8 blue balls left for a total of $2 \cdot \binom{30}{8}$ desired sequences. Therefore the probability is $\frac{2 \cdot \binom{30}{8}}{\binom{32}{9}} = \frac{207}{496}$ and this works for any person in the sequence picking up two different balls since we can fix the two positions anywhere.

2. Let I_k be the indicator variable for A_k. Note $1 - \prod(1 - I_k) \leq \sum I_k$ which follows from $1 \leq \prod(1 - I_k) + \sum I_k$ since each variable is either 0 or 1 and we can only get 0 from the product if we have a variable equal to 1 but that makes the sum at least 1. Taking the expected value of both sides tells us

$$\mathbb{E}(1 - \prod(1 - I_k)) \leq \mathbb{E}(\sum I_k)$$
$$\mathbb{E}(\sum I_k - \sum I_j I_k + ... + (-1)^{n+1} \prod I_k) \leq \mathbb{E}(\sum I_k)$$
$$\sum \mathbb{E}(I_k) - \sum \mathbb{E}(I_j I_k) + ... + \mathbb{E}((-1)^{n+1} \prod I_k) \leq \sum \mathbb{E}(\sum I_k)$$
$$\sum \mathbb{P}(A_k) - \sum(\mathbb{P}(A_j \wedge A_k)) + ... + (-1)^{n+1} \mathbb{P}(\bigwedge_{k=1}^{n} A_k) \leq \sum \mathbb{P}(A_k)$$
$$\mathbb{P}\left(\bigvee_{k=1}^{n} A_k\right) \leq \sum_{k=1}^{n} \mathbb{P}(A_k)$$

3. Divvy up [0,1] into $[0, 1/n], [1/n, 2/n], [2/n, 3/n], ...[(n-1)/n, 1]$. Note two of $\{x\}, \{2x\}, \{3x\}, ...\{nx\}, \{(n+1)x\}$ have to fall in one of those intervals by Pigeonhole. Let the two numbers for which this happens be ax and bx, then $k \leq |b - a|x \leq k + \frac{1}{n}$ where k is an integer since $ax = n_1 + \{ax\}$, $bx = n_2 + \{bx\}$, and $|\{ax\} - \{bx\}| \leq \frac{1}{n}$ from our Pigeonhole argument. It follows that $0 \leq |b - a|x - k \leq \frac{1}{n}$ and note $|b - a|$ is an integer in $[1, n]$ so we are done.

4. $\sigma(X + X + ... + X) = 2\sqrt{2}\sigma(X) =$ where X is a die roll and the sum is taken over 8 independent die rolls which is why the standard deviation is changed by a factor of $\sqrt{8}$. $\mu(X) = \frac{7}{2}$ and $\mu(X^2) = \frac{1}{6} \cdot \frac{(6)(7)(13)}{6} = \frac{91}{6}$ so $\sigma(X) = \sqrt{\frac{35}{12}}$ and $\sigma(X + X... + X) = \sqrt{\frac{70}{3}} = 4.8305$. Note $\mu(X + X... + X) = 8 \cdot \frac{7}{2} = 28$. Note $38 - 28 = 10$ so the number of standard deviations it is away is $\frac{10}{4.8305} = 2.07$ and $30 - 28 = 2$ so the number of standard deviations it is away is $\frac{2}{4.8305} = 0.414$. So by Chebyshev's inequality

$$P(x \text{ is between 18 and 38}) = P(|x - 28| \leq 2.07\sigma) \geq 1 - \frac{1}{2.07^2} = 76.66\%$$
$$P(x \text{ is between 26 and 30}) = P(|x - 28| \leq 0.414\sigma) \geq 1 - \frac{1}{0.414^2} = -483.44\%$$

The 2nd result occurs because Chebyshev's Inequality does not tell us good information about numbers near the mean which is kind of expected since distributions are large near the mean.