18.100B - Problem Set 6

Rishad Rahman

6. (a) $s_k = \sum_{n=1}^{k} a_n = \sum_{n=1}^{k} \sqrt{n+1} - \sqrt{n} = \sqrt{k+1} - 1$. Note s_k is monotonically increasing and does not have an upper bound since we can always find k such that $\sqrt{k+1} - 1 > N$ for $N \in \mathbb{R}$. Therefore s_n diverges.

   (b) $0 < a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n} = \frac{1}{n(\sqrt{n+1} + \sqrt{n})} < \frac{1}{n \cdot \sqrt{n}} = \frac{1}{n^{\frac{3}{2}}}$. $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges so s_k converges as well by comparison.

   (c) $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{n} - 1 = 0 < 1$ so by the root test s_k converges. **Note that $\limsup_{n\to\infty} \sqrt[n]{n} - 1 = 0$ since $\lim_{n\to\infty} \sqrt[n]{n} - 1 = 0$ so the set of all subsequential limits is 0.

   (d) $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{\left|\frac{1}{1+z^n}\right|} < \limsup_{n\to\infty} \frac{1}{|z|} = \frac{1}{|z|}$. The series diverges if $\frac{1}{|z|} > 1 \to |z| < 1$. The series converges otherwise if $\frac{1}{|z|} < 1 \to |z| > 1$. If $|z| = 1$ then $a_n = \frac{1+\bar{z}^n}{|1+z^n|^2} > \frac{1+\bar{z}^n}{4}$ since $|1+z^n|^2 < (1+|z|^n)^2 = 4$ by Triangle Inequality. If $z = \cos\theta + i\sin\theta$ then $\bar{z}^n = \cos n\theta - i \sin n\theta$. So $\Re(a_n) > \frac{1+\cos n\theta}{4} = \frac{\cos^2 \frac{n\theta}{2}}{2} > 0$ but $\lim_{n\to\infty} \frac{\cos^2 \frac{n\theta}{2}}{2} \neq 0$ so the series must diverge and therefore $\Re(a_n)$ diverges by comparison so a_n also diverges.

7. By Schwarz Inequality $\left(\sum_{n=1}^{k} \frac{1}{n^2}\right)\left(\sum_{n=1}^{k} a_n\right) = \left(\sum_{n=1}^{k} \frac{1}{n^2}\right)\left(\sum_{n=1}^{k} \sqrt{a_n}^2\right) \geq \sum_{n=1}^{k} \frac{\sqrt{a_n}}{n}$. But the left hand side is the Cauchy product of two series with one of them absolutely convergent, specifically $\sum_{n=1}^{k} \frac{1}{n^2}$, therefore the product is convergent. Note that the LHS monotonically increases (since the terms of the series are all positive) to its limit so $s_k = \sum_{n=1}^{k} \frac{\sqrt{a_n}}{n}$ is bounded. But s_k is also monotonically increasing and therefore has a limit and thus the series converges.

9. (a) $\limsup_{n\to\infty} \sqrt[n]{|n^3|} = \limsup_{n\to\infty} \sqrt[n]{n}^3 = 1$ as $\lim_{n\to\infty} \sqrt[n]{n}^3 = 1^3 = 1$, so $R = 1$.

   (b) $\limsup_{n\to\infty} \sqrt[n]{\left|\frac{2^n}{n!}\right|} = \limsup_{n\to\infty} \frac{2}{\sqrt[n]{n!}} = 0$ since $\sqrt[n]{n!} \to \infty$. To show this: $\log \sqrt[n]{n!} = \frac{1}{n}\sum_{k=1}^{n} \log(k)$. Suppose $n = 2^j - 1$. Then $\frac{1}{n}\sum_{k=1}^{n}\log(k) > \frac{1}{2^j - 1}\sum_{k=1}^{j-1} 2^k \log(2^k) = \frac{1}{2^j-1}\sum_{k=1}^{j-1} k 2^k \log(2) > \frac{(j-1)2^{j-1}}{2^j - 1}\log 2 >$ $\frac{j-1}{2}\log 2 \to \infty$ as $j \to \infty$. Note the first inequality was established by taking the highest power of 2 lower than each term of the sum, i.e. $\log(2^j - k) \geq \log 2^{j-1}$ for $0 < k \leq 2^{j-1}$, and then summing. We have shown a_{2^k} goes to infinity. But $\sqrt[n+1]{(n+1)!} > \sqrt[n]{n!} \Leftrightarrow (n+1)!^n > n!^{n+1} \Leftrightarrow$

$(n+1)^n > n! \Leftarrow (n+1)^n > n^n > n!$ so a_n is monotonically increasing and therefore a_n must go infinity since a_{2^k} goes to infinity. Back to our series, $R = \infty$.

(c) $\limsup_{n\to\infty} \sqrt[n]{\left|\frac{2^n}{n^2}\right|} = \limsup_{n\to\infty} \frac{2}{\sqrt[n]{n}^2} = 2$ since $\lim_{n\to\infty} \frac{2}{\sqrt[n]{n}^2} = \frac{2}{1^2} = 2$ so $R = \frac{1}{2}$.

(d) $\limsup_{n\to\infty} \sqrt[n]{\left|\frac{n^3}{3^n}\right|} = \limsup_{n\to\infty} \frac{\sqrt[n]{n}^3}{3} = \frac{1}{3}$ for similar reasons as above and so $R = 3$.

13. Let the two series be $\sum a_n$ and $\sum b_n$, so $\sum |a_n| = A$ and $\sum |b_n| = B$.

$$\begin{aligned}\sum |c_n| &= |a_0b_0| + |a_0b_1 + b_1a_0| + |a_0b_2 + a_1b_1 + a_2b_0| + ...\\ &\leq |a_0b_0| + |a_0b_1| + |a_1b_0| + |a_0b_2| + |a_1b_1| + |a_2b_0| + ...\\ &= \sum |a_n| \sum |b_n| = AB.\end{aligned}$$

Therefore $\sum |c_n|$ is bounded and obviously it is monotonically increasing so it converges.

16. (a) Note $x_n > \sqrt{\alpha}$ or $\alpha < x_n^2$ for all n which follows by induction ($x_1 > \sqrt{a} > 0$, assume $x_n > \sqrt{a} > 0$, then $x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) \geq \sqrt{\alpha}$ by AM-GM). Now to show x_n is monotonically decreasing... $x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) < \frac{1}{2}\left(x_n + \frac{x_n^2}{x_n}\right) = x_n$, therefore x_n is monotonically decreasing and bounded by $\sqrt{\alpha}$. To show that $\sqrt{\alpha}$ is the limit of the sequence, so we must show $\exists N\ \forall\epsilon : |x_n - \sqrt{\alpha}| < \epsilon$ for $n \geq N$. We can say $x_n < \sqrt{\alpha} + \epsilon$ since $x_n > \sqrt{\alpha}$. Suppose $x_1 < \sqrt{\alpha} + \beta$ for some β. Then $x_2 = \frac{1}{2}\left(x_1 + \frac{\alpha}{x_1}\right) < \frac{1}{2}(\sqrt{\alpha} + \beta + \sqrt{\alpha}) < \frac{1}{2}(2\sqrt{\alpha} + \beta) = \sqrt{\alpha} + \frac{\beta}{2}$. Repeating this n times leads to $x_n < \sqrt{\alpha} + \frac{\beta}{2^{n-1}}$. By the Archimedean property we can find N such that $\frac{\beta}{2^{N-1}} < \epsilon$ so $x_N < \sqrt{\alpha} + \frac{\beta}{2^{N-1}} < \epsilon + \sqrt{\alpha}$. And since x_n is monotonically decreasing, this is true for $n \geq N$ and so the limit is $\sqrt{\alpha}$.

(b) $\epsilon_{n+1} = x_{n+1} - \sqrt{\alpha} = \frac{x_n}{2} + \frac{\alpha}{2x_n} - \sqrt{\alpha} = \frac{x_n^2 + \alpha - 2x_n\sqrt{\alpha}}{2x_n} = \frac{(x_n - \sqrt{a})^2}{2x_n} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}$ (since $x_n > \sqrt{\alpha}$)$= \frac{\epsilon_n^2}{\beta}$. By repeating the inequality $\epsilon_{n+1} < \frac{\epsilon_1^{2^n}}{\beta^{1+2+4+...2^{n-1}}} = \frac{\epsilon_1^{2^n}}{\beta^{2^n-1}} = \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^n}$.

(c) $\epsilon_1 = x_1 - \sqrt{\alpha} = 2 - \sqrt{3}$ and $\beta = 2\sqrt{3}$ so $\frac{\epsilon_1}{\beta} = \frac{2-\sqrt{3}}{2\sqrt{3}} = \frac{2\sqrt{3}-3}{6} < \frac{2\sqrt{3.24}-3}{6} = \frac{2\cdot 1.8 - 3}{6} = \frac{1}{10}$. So $\epsilon_5 < 2\sqrt{3}\left(\frac{1}{10}\right)^{16} < 2\sqrt{4}\cdot 10^{-16} = 4\cdot 10^{-16}$ and $\epsilon_6 < 2\sqrt{3}\left(\frac{1}{10}\right)^{32} < 4\cdot 10^{-32}$ by similar algebra.

18. Taking the limits gives $\lim x_{n+1} = \lim \frac{p-1}{p}x_n + \lim \frac{\alpha}{p}x_n^{-p+1} \Rightarrow L = \frac{p-1}{p}L + \frac{\alpha}{p}L^{-p+1} \Rightarrow p = p - 1 + \alpha L^{-p} \Rightarrow L = \sqrt[p]{\alpha}$ as long as the limit L exists and is not 0. Therefore we should verify it. Suppose $x_1 > \sqrt[p]{\alpha}$. By induction, $x_{n+1} = \frac{p-1}{p}x_n + \frac{\alpha}{p}\frac{1}{x_n^p}x_n < \frac{p-1}{p}x_n + \frac{\alpha}{p}\frac{1}{\alpha}x_n = x_n$ if $x_n > \sqrt[p]{\alpha} \leftrightarrow \alpha < x_n^p$ so x_n is monotonically decreasing. Now using AM-GM on $p-1$ x_n's and αx_n^{-p+1} gives $x_{n+1} \geq \sqrt[p]{x_n^{p-1}\alpha x_n^{-p+1}} = \sqrt[p]{\alpha}$ if $x_n > 0$ which follows by induction ($x_1 > 0$). So $|x_n - \sqrt[p]{\alpha}| < \epsilon \Rightarrow x_n < \sqrt[p]{\alpha} + \epsilon$. If $x_1 < \sqrt[p]{\alpha} + \beta$ for some $\beta \Rightarrow x_2 = \frac{p-1}{p}x_1 + \frac{\alpha}{p}\frac{1}{x_1^{p-1}} < \frac{p-1}{p}(\sqrt[p]{\alpha} + \beta) + \frac{\sqrt[p]{\alpha}}{p} = \sqrt[p]{\alpha} + \frac{p-1}{p}\beta$. Repeating this n times gives $x_n < \sqrt[p]{\alpha} + (\frac{p-1}{p})^{n-1}\beta$. By the Archimedean principle we can find an N such that $(\frac{p-1}{p})^{N-1}\beta < \epsilon$ so $x_N < \sqrt[p]{\alpha} + \epsilon$ and since this sequence is monotonically decreasing this is true for $n \geq N$ and so we have proved the limit is $\sqrt[p]{\alpha}$.