18.310A Problem Set 3

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1. a. Note that the generating function is just the product of the generating functions for the sequences made of $\{a, b, c\}$ and those made of $\{1, 2\}$. Note that there are 3^k ways to have a sequence of length k made of the former letters and 2^k for that of the latter. So our generating function is

$$\begin{aligned} C(x) &= \sum_{i=0}^{\infty} 3^i \sum_{j=0}^{\infty} 2^j \\ &= \frac{1}{1-3x} \cdot \frac{1}{1-2x} \\ &= \frac{3}{1-3x} - \frac{2}{1-2x} = \sum_{n=0}^{\infty} (3 \cdot 3^n - 2 \cdot 2^n) x^n \end{aligned}$$

   b. Reading from the above, we have $c_n = 3^{n+1} - 2^{n+1}$.

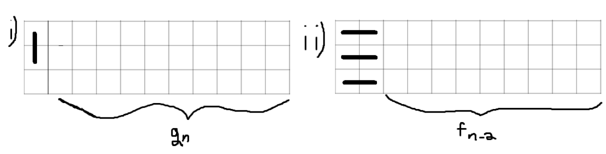
2. As we saw in lecture, solving a homogenous recurrence resolves to solving its characteristic equation which in this case is $\lambda^2 - \lambda - 6 = 0 \rightarrow \lambda_1 = -2, \lambda_2 = 3$. So our general solution is of the form $f_k = a \cdot (-2)^k + b \cdot 3^k$. Plugging in $k = 0, 1$ gives

$$\begin{aligned} a + b &= 1 \\ -2a + 3b &= 2 \\ \Rightarrow a &= \frac{1}{5}, b = \frac{4}{5} \end{aligned}$$

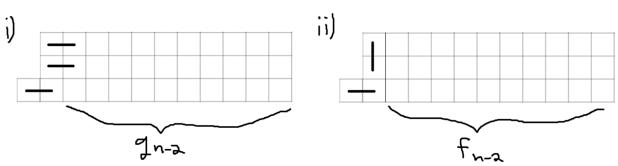
   so our general solution is $f_k = \frac{1}{5}(-2)^n + \frac{4}{5}(3)^n$ and this checks that $f_2 = 8$.

3. To find the recurrence, f_n, for the number of ways to tile a $3 \times n$ strip with tiles of size 2×1, we are going to have to divide the problem into cases which are shown in (a). Note that i) needs to be multiplied by two since there's a symmetric case and that we are going to need to know the number of ways, g_n to tile a $3 \times n - 1$ strip with a tile sticking out.. Looking at the diagram in b) tells us the recurrence for this which is in terms of f_n. Writing out the system of recurrences:

$$\begin{aligned} f_n &= f_{n-2} + 2g_n \\ g_n &= f_{n-2} + g_{n-2} \end{aligned}$$



(a) Recurrence for f



(b) Recurrence for g

We can eliminate g to get a recurrence solely in terms of f.

$$\frac{f_n - f_{n-2}}{2} = f_{n-2} + \frac{f_{n-2} - f_{n-4}}{2}$$
$$f_n = 4f_{n-2} - f_{n-4}$$

Note this is only valid for $n \geq 4$ which makes sense since the base cases are $f_0 = 1$ and $f_2 = 3$. So now we have to solve the characteristic equation $\lambda^2 - 4\lambda + 1 = 0 \rightarrow \lambda = 2 \pm \sqrt{3}$. So $f_n = a\lambda_1^n + b\lambda_2^n$.

$$1 = a + b$$
$$3 = a\lambda_1 + b\lambda_2$$
$$\Rightarrow a = \frac{\lambda_2 - 3}{\lambda_2 - \lambda_1} = \frac{-\sqrt{3} + 3}{6}, b = \frac{-\lambda_1 + 3}{\lambda_2 - \lambda_1} = \frac{\sqrt{3} + 3}{6}$$

So $f_{2n} = \frac{3-\sqrt{3}}{6} \cdot (2 - \sqrt{3})^n + \frac{3+\sqrt{3}}{6} \cdot (2 + \sqrt{3})^n$ which grows exponentially with $(2 + \sqrt{3})^n$.

4. Let's start going around the circle clockwise starting at 1. If the current vertex number is lower than that of the one it is connected to by a chord, then add U to a sequence, otherwise add D. The Us and Ds represent the direction of movement (UP and DOWN) along a Dyck walk. Note if we have two vertices then we get UD which is the Dyck walk of length 2. Assume a noncrossing matching of $2j$ points, $j \leq n$ can be represented by a Dyck walk and consider a noncrossing matching of $2n + 2$ points. Let's say 1 is connected to $2k$, $k \leq n + 1$. Then the noncrossing matching of the $2k - 2$ points between 1 and $2k$, exclusive, can be represented by a Dyck walk and the nnoncrossing matching of the $2n - 2k + 2$ points between $2k$ and 1, exclusive. We then have a Dyck walk through construction by U+(first sequence)+D+(second sequence) so noncrossing matching $\rightarrow$ Dyck walk follows by strong induction. Now we just have to show every Dyck walk corresponds to a noncrossing matching which we can also do by strong induction. Obviously UD corresponds to a noncrossing matching with two points. Assume this works for a Dyck walk of length $2j$ $j \leq n$ so we have to show it for $2n + 2$. Let the edges of a Dyck walk be labeled 1 to $2n + 2$. Let the edge before the first time the Dyck walk reaches the x-axis be $2k$. 1 must correspond to U while $2k$ corresponds to D. On the circle, connect 1 and $2k$ with a cord. However the points 2 through $2k - 1$ and $2k + 1$ to $2n + 2$ can form a noncrossing matching from a corresponding from a Dyck Walk by our induction hypothesis and obviously none of the chords can cross since the one connecting 1 and $2k$ divides the two smaller noncrossing matches, so all together we have a new noncrossing matching. This completes the induction which completes bijection and so we now know we can count the noncrossing matches of length $2n$ with the Catalan numbers $c_n = \frac{\binom{2n}{n}}{n+1}$.

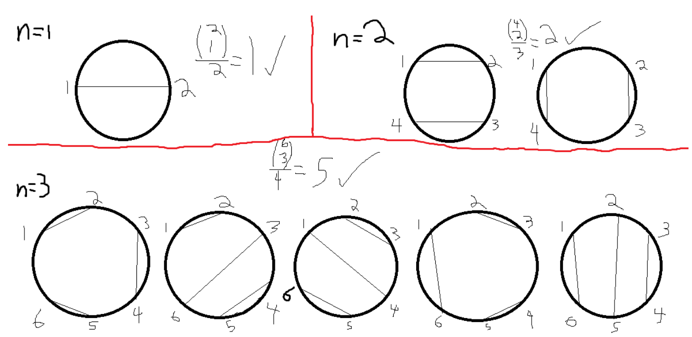


Figure 2: Noncrossing matchings for n=1,2,3