18.100B - Problem Set 8

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2. Suppose $f(x) \geq f(y)$ with $b > y > x > a$. Then by the Mean Value Theorem there must be an $c \in (x, y)$ s.t. $f'(c) = \frac{f(y)-f(x)}{y-x} \leq 0$, a contradiction since $f'(c) > 0$, so f must be strictly increasing. Let $f(x) = y$ so that $g(y) = x$. Note the inverse is unique since f is strictly increasing. Then $g'(f(x)) = g'(y) = \lim_{t\to y} \dfrac{g(y)-g(t)}{y-t} = \lim_{t\to y} \dfrac{x-g(t)}{f(x)-f(g(t))} = \lim_{t\to y} \dfrac{1}{\frac{f(x)-f(g(t))}{x-g(t)}}$. Now note that as $t \to y$, $g(t) \to x$. This is easy to show since $t_1 < t_2 < t_3... < t$ iff $g(t_1) < g(t_2) < g(t_3)... < g(t)$ since $t_k = f(g(t_k))$ is increasing and 1-1. So $g'(f(x)) = \dfrac{1}{f'(g(y))} = \dfrac{1}{f'(x)}$.

5. By the Mean Value Theorem, there must be $t \in (x, x+1)$ s.t. $f'(t) = f(x+1) - f(x) = g(x)$. Note as $x \to +\infty$, $t \to +\infty$ so $\lim_{t\to\infty} f'(t) = \lim_{x\to\infty} g(x)$ but since t is a subsequence of $\mathbb{R}$ that goes to infinity $\lim_{t\to\infty} f'(t) = \lim_{x\to\infty} f'(x) = 0$ so $\lim_{x\to\infty} g(x) = 0$.

14. Forward direction: if f is convex, we showed in the last chapter/problem set $\dfrac{f(t)-f(s)}{t-s} \leq \dfrac{f(u)-f(t)}{u-t}$ when $t < s < u$. Taking $\lim_{t\to s}$ we have $f'(s) \leq \dfrac{f(u)-f(s)}{u-s}$ but taking $\lim_{t\to u}$ we have $\dfrac{f(u)-f(s)}{u-s} \leq f'(u)$ so $f'(s) \leq f'(u)$ when $s < u$ so f' is monotonically increasing.
Backward direction: If $a < x < t < y < b$ By MVT, there must be $\alpha \in (x, t)$ s.t. $f'(\alpha) = \dfrac{f(t)-f(x)}{t-x}$ and $\beta \in (t, y)$ s.t. $f'(\beta) = \dfrac{f(y)-f(t)}{y-t}$. Note $\alpha < \beta$ and since f is monotonically increasing, $\dfrac{f(t)-f(x)}{t-x} \leq \dfrac{f(y)-f(t)}{y-t}$, rearranging we get $f(t)\left(\dfrac{y-x}{(t-x)(y-t)}\right) \leq \dfrac{f(x)}{t-x} + \dfrac{f(y)}{y-t} \Rightarrow f(t) \leq \dfrac{y-t}{y-x} f(x) + \dfrac{t-x}{y-x} f(y)$. Letting $\lambda = \dfrac{y-t}{y-x} \Rightarrow t = y + (x-y)\lambda = \lambda x + (1-\lambda)y \Rightarrow f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$. Note λ indeed ranges from 0 to 1 when t varies in (x, y).
If $f''(x) \geq 0$ for all $x \in (a, b)$, then $f'(x)$ is monotonically increasing and the above follows. If f is convex, then f' is monotonically increasing so $f''(x) \geq 0$.

15. By Taylor's Theorem,

$$f(x+2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{2} f''(\xi) = f(x) + 2hf'(x) + 2h^2 f''(\xi) \text{ for some } \xi \in (x, x+2h)$$
$$\Rightarrow f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] - hf''(\xi)$$
$$\Rightarrow |f'(x)| = \left|\frac{1}{2h}[f(x+2h) - f(x)] - hf''(\xi)\right| \leq \frac{1}{2h}[|f(x+2h)| + |f(x)|] + h|f''(\xi)| \leq \frac{M_0}{h} + hM_2$$

Since M_1 is the least upper bound of $f'(x)$,
$M_1 \leq \dfrac{M_0}{h} + hM_2 \Rightarrow h^2 M_2 - hM_1 + M_0 \geq 0$. This should work for all h, so we want the discriminant of the quadratic to be ≤ 0 otherwise we will have a lower bound for $h > 0$. Therefore $M_1^2 - 4M_0M_2 \leq 0 \Rightarrow M_1^2 \leq 4M_0M_2$. For the given example $M_0 = 1$ since $\left|\frac{x^2-1}{x^2+1}\right| = \left|1 - \frac{2}{x^2+1}\right|$ and

$|2x^2 - 1| < 1$ on $(-1, 0)$. $f'(x) = 4x, \frac{4x}{(x^2+1)^2}$ for $(-1,0), [0,\infty)$, the former will obviously have larger magnitude since the latter is being divided by a number greater than 1 so $M_1 = \sup|4x| = 4$ on $(-1, 0)$. $f''(x) = 4, \frac{4-12x^2}{(x^2+1)^3}$. $|1 - 3x^2| \leq 1 + 3x^2 < (x^2+1)^3$ for $x > 0$ since $(x^2+1)^3 - 3x^2 - 1 = x^6 + 3x^4 > 0$. So $M_2 = \sup|f''(x)| = 4$. Note that for all these, the derivatives match at 0.

16. From the previous problem $M_1^2 \leq 4M_0M_2$. As $a \to \infty$, $M_0 \to 0$ since $f \to 0$, and M_2 is bounded so $M_1^2 \to 0 \Rightarrow M_1 \to 0$ so $\sup|f'(x)| \to 0 \Rightarrow f'(x) \to 0$.