18.100B - Problem Set 12

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15. By equicontinuity $\forall n\ \forall \epsilon\ \exists \delta$ s.t. if $0 \leq \gamma < \delta$, $|f_n(\gamma) - f_n(0)| = |f(n\gamma) - f(0)| < \epsilon$. Suppose $n\gamma = x$ where x is arbitrary. Then $\gamma = \frac{x}{n}$ can be made less than δ for sufficiently large n. So $|f(x) - f(0)| < \epsilon$ $\forall \epsilon$ and therefore $f(x) = f(0)$ but note x was arbritrary but needs to be ≥ 0 for $\gamma \geq 0$ so f is constant on $[0, \infty)$.

16. Note that δ balls cover K and since it is compact there must be a finite k number of balls centered at $x_1, x_2, ..., x_k$ which cover K. Let's say δ is chosen such that $|f_n(x) - f_n(y)| < \epsilon\ \forall n$ and x, y s.t. $d(x, y) < \delta$ which is guaranteed by equicontinuity. Note this also applies to $|f_m(y) - f_m(x)| < \epsilon$. Now note that $f_n(y) \to f(y)$ so it is Cauchy and therefore $|f_n(y) - f_m(y)| < \epsilon$ for $m, n > N$. Now by Triangle Inequality we get $|f_n(x) - f_m(x)| < 3\epsilon$ for $m, n > N$. By taking $N = \max\{N_k\}$, where N_k is the corresponding N for each δ neighborhood, we have $|f_n(x) - f_m(x)| < 3\epsilon$ for $m, n > N$ for any $x \in K$. Note the maximimum exists because we have finite k.

1. Claim: $\lim\limits_{x \to 0} \dfrac{p(x)}{q(x)} e^{-\frac{1}{x^2}} = 0$ where $p(x)$ and $q(x)$ are polynomials. This is easily shown by dividing the lowest degree term out of $q(x)$ so that the denominator does not disappear at 0 but the numerator is now of the form $\sum a_n x^n e^{\frac{-1}{x^2}}$ which converges to 0 for n nonnegative (both terms go to 0) or n positive (exponential dominates as shown by 8.6).
Assume $f^{(n)}(x) = r_n(x) e^{-\frac{1}{x^2}}$ for $x \neq 0$ and where $r_n(x) = \frac{p_n(x)}{q_n(x)}$. $f^{(n+1)}(x) = e^{-\frac{1}{x^2}}(r'_n(x) + \frac{2}{x^3})$. It is trivial that $r'_n(x) + \frac{2}{x^3}$ is also a quotient of polynomials and so we can define p_{n+1}, q_{n+1}, and r_{n+1}. Thus it follows by induction, $r_0 = 1$, that $f^{(n)}(x) = r_n(x) e^{-\frac{1}{x^2}}$. Now assume $f^{(n)}(0) = 0$. By the definition of the derivative, $f^{(n+1)}(0) = \lim\limits_{x \to 0} \dfrac{r_n(x) e^{-\frac{1}{x^2}} - f^{(n)}(0)}{x - 0} = 0$ since $\dfrac{r_n(x)}{x}$ is a polynomial and we already know $f^{(0)}(0) = f(0) = 0$ so the result follows by induction.

2. $\sum\limits_i \sum\limits_j a_{ij} = -1 + \sum\limits_i (-1 + \sum\limits_{k=1}^{i} \frac{1}{2^k}) = -1 + \sum\limits_i (-1 + 1 - (\frac{1}{2})^i) = -1 + \sum\limits_i -(\frac{1}{2})^i = -1 + -1 = -2.$
$\sum\limits_j \sum\limits_i a_{ij} = \sum\limits_j (-1 + \sum\limits_i \frac{1}{2^i}) = \sum\limits_j 0 = 0.$

4. Note that because of continuity $x \to 0$ or $x \to \infty$ is the same as $cx \to 0$ or $cx \to \infty$ where c is a constant.

   (a) $\lim\limits_{x \to 0} \dfrac{b^x - 1}{x} = \lim\limits_{x \to 0} \dfrac{e^{x \log b} - 1}{x} = \log b \lim\limits_{x \to 0} \dfrac{e^{x \log b} - e^0}{x \log b - 0} = (e^x)' \Big|_{x=0} \log b = \log b$

   (b) $\lim\limits_{x \to 0} \dfrac{\log(1 + x)}{x} = \lim\limits_{x \to 1} \dfrac{\log(x) - \log 1}{x - 1} = (\log(x))' \Big|_{x=1} = 1$

   (c) By (b) $\lim\limits_{x \to 0} \log(1 + x)^{\frac{1}{x}} = 1$ so $(1 + x)^{\frac{1}{x}} \to e$.

   (d) $\lim\limits_{n \to \infty} \left(1 + \dfrac{x}{n}\right)^n = \lim\limits_{n \to \infty} \left(\left(1 + \dfrac{1}{\frac{n}{x}}\right)^{\frac{n}{x}}\right)^x = e^x.$

5. (a) $\lim_{x\to 0}\frac{e-(1+x)^{1/x}}{x} = \lim_{x\to 0}\frac{e-e^{\frac{1}{x}\log(1+x)}}{x} = -\lim_{x\to 0} e^{\frac{1}{x}\log(1+x)}\left(\frac{\frac{x}{x+1}-\log(x+1)}{x^2}\right)$
$= -e\lim_{x\to 0}\left(\frac{\frac{1}{(x+1)^2}-\frac{1}{1+x}}{2x}\right) = e\lim_{x\to 0}\left(\frac{1}{2(x+1)^2}\right) = \frac{e}{2}$ where L'Hospital's rule was used repeatedly in the latter half of the calculation.

(b) $\lim_{n\to\infty}\frac{n}{\log n}(n^{1/n}-1) = \lim_{n\to\infty}\frac{e^{\frac{1}{n}\log n}-1}{\frac{1}{n}\log n} = \lim_{x\to 0}\frac{e^x-1}{x} = (e^x)'\Big|_{x=0} = 1.$

(c) $\lim_{x\to 0}\frac{\tan x - x}{x(1-\cos x)} = \lim_{x\to 0}\frac{\sin x - x\cos x}{x\cos x(1-\cos x)} = \lim_{x\to 0}\frac{\sin x - x\cos x}{x - x\cos x} = \lim_{x\to 0}\frac{x\sin x}{1-\cos x + x\sin x}$
$= \lim_{x\to 0}\frac{\sin x + x\cos x}{2\sin x + x\cos x} = \lim_{x\to 0}\frac{2\cos x - x\sin x}{3\cos x - x\sin x} = \frac{2}{3}$

(d) $\lim_{x\to 0}\frac{x-\sin x}{\tan x - x} = \lim_{x\to 0}\frac{x\cos x - \sin x\cos x}{\sin x - x\cos x} = \lim_{x\to 0}\frac{x-\sin x}{\sin x - x\cos x} = \lim_{x\to 0}\frac{1-\cos x}{x\sin x}$
$= \lim_{x\to 0}\frac{\sin x}{\sin x + x\cos x} = \lim_{x\to 0}\frac{\cos x}{2\cos x - x\sin x} = \frac{1}{2}.$

9. (a) Let $a_n = 1 + 1/2 + ... + 1/n - \log n$. $a_{n+1} \le a_n \Leftrightarrow 1 + 1/2 + ... + 1/(n+1) - \log(n+1) \le 1 + 1/2 + ... + 1/n - \log n \Leftrightarrow 1/(n+1) \le \log((n+1)/n) \Leftrightarrow \left(\frac{n+1}{n}\right)^{n+1} = (1+\frac{1}{n})(1+\frac{1}{n})^n \ge e$. It is easy to see that this is verified since the limit of the LHS is e and we can show that it is monotonically decreasing. Looking at $f(x) = (x+1)\log(1+\frac{1}{x})$, we have $f'(x) = \log(1+\frac{1}{x}) - \frac{1}{x} \le 0$ since $e^{\frac{1}{x}} \ge 1 + \frac{1}{x}$.

Now let's show a_n is bounded below. $a_n = 1 + 1/2 + ... + 1/n - \log n > \int_1^{n+1}\frac{1}{x}dx - \log n = \log(n+1) - \log n > 0$ where we used the upper bound for the integral.
Since a_n is bounded below and monotonically decreasing, its limit exists.

(b) We have shown $a_n > 0$ so $s_n > \log N$ and $\log N = m\log 10 > 100$ when $m > \frac{100}{\log 10} = 43.43$. Dat slow growth.