18.100B - Problem Set 7

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1. No f need not be continuous. Take $f(x) = x$ except when $x = 0$, otherwise $f(0) = 1$. It is not hard to see $\forall x \lim\limits_{h\to 0} f(x+h) - f(x-h) = \lim\limits_{h\to 0} 2h = 0$ since we can always find a ball around $x \neq 0$ with $r < d(x, 0)$ and in that ball $f(x) = x$ and if $x = 0$ then $x + h \neq 0$ and $x - h \neq 0$. So this function satisfies the condition but is obviously not continuous at $x = 0$ since $\lim\limits_{x\to 0} f(x) = 0$.

2. If $x \in E$ then $f(x) \in f(E) \subset \overline{f(E)}$. Suppose $x \in E'$, then $\forall \epsilon_1 > 0\ \exists y \in E$ s.t $d(x, y) < \epsilon_1$. Note $f(y) \in f(E)$. Since f is continuous, $\forall \epsilon_2 > 0\ \exists \delta$ s.t $d(f(x), f(x^*)) < \epsilon_2\ \forall x^*$ s.t. $d(x, x^*) < \delta$. Take $\epsilon_1 = \delta$, then $\exists y \in f(E)$ and δ s.t $d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon_2\ \forall \epsilon_2 > 0 \Rightarrow f(x)$ is a limit point of $f(E)$ so $f(x) \in f(E)' \subset \overline{f(E)}$. Therefore $f(\overline{E}) \subset \overline{f(E)}$. Let $f(x) = e^{-x}$ where $x \in X = (0, +\infty)$ and $E = \mathbb{N}$. Then $\overline{E} = \mathbb{N}$ so $f(E) = f(\overline{E}) = e^{-n}$, $n \in \mathbb{N}$, but $\overline{f(E)} = e^{-n} \cup \{0\}$for $n \in \mathbb{N}$. Therefore $f(\overline{E})$ is a proper subset of $\overline{f(E)}$.

4. If $x \in E$ then $f(x) \in f(E)$. Otherwise suppose $x \in X$ is a limit point of E. Note that $f(x) \in f(X)$ and we showed in problem 2 that $f(x)$ is a limit point of $f(E)$. So now we have established $f(E)$ is dense in $f(X)$. Note everything for f goes as well for g. Suppose $p \in E^c$. Since $f(E)$ is dense in $f(X)$, $f(p)$ is a limit point of $f(E)$ and so $\forall \epsilon > 0\ \exists f(x) \in f(E)$ s.t. $d(f(p), f(x)) < \epsilon$. So $d(g(x), g(p)) < \epsilon$ as well. Note $x \in E$ so $f(x) = g(x)$. Therefore $d(f(x), g(p)) < \epsilon$, adding the two triangles together and then using the Triangle Inequality gives $d(f(p), g(p)) < 2\epsilon\ \forall \epsilon > 0$ which forces $d(f(p), g(p)) = 0$ so $f(p) = g(p)$.

5. Note E^c is open so it can be formed by the union of a most countable number of disjoint segments, $\cup(a_k, b_k)$. Let $g(x) = f(x)$ if $x \in E$, otherwise $g(x) = f(a_i) + \frac{f(b_i) - f(a_i)}{b_i - a_i} \cdot (x - a_i)$ where $a_i < x < b_i$. g is obviously continuous on the line segments or on an interior point of E. Otherwise $\lim\limits_{x\to b_i^-} g(x) = f(a_i) + f(b_i) - f(a_i) = f(b_i) = \lim\limits_{x\to b_i^+} g(x)$ so $g(x)$ is continuous on R^1. Take $f(x) = \frac{1}{x}$ on $(0, +\infty)$. That set is open but there is no way to assign a value to $g(0)$ to create a continuous extension in R^1. If $f(x) =< f_1(x), f_2(x), ..., f_n(x) >$ is continuous, then each component, $f_k(x)$, must be continuous as well. Let a continuous extension of $f_k(x)$ be $g_k(x)$, then the continuous extension of $f(x)$ is $< g_1(x), g_2(x), ..., g_n(x) >$.

8. Lemma: If $p, q \in [c, c+\delta]$ where $p, q \in E$ and $|f(p) - f(q)| < \epsilon\ \forall p, q$ then f is bounded on $[c, c+\delta]$. Proof: Let $E \cap [c, c+\delta] = F$. Assume f was unbounded on F. This means $\forall p \in F$ and $\forall \epsilon$ we can find $q \in F$ s.t. $|f(p)| > \epsilon + |f(q)|$. But $|f(p) - f(q)| \geq |f(p)| - |f(q)| > \epsilon$, a contradiction so f is bounded on F.
   Suppose $\inf E = M$ and $\sup E = N$ where $M, N \in \mathbb{R}$ which exist since E is bounded in $\mathbb{R}$. Since f is uniformly continuous, $\exists$ a δ neighborhood such that for all p, q in E if $|p - q| < \delta$ then $|f(p) - f(q)| < \epsilon$. Note that this satisfies the conditions of the lemma. Divide $[M, N]$ into the sub-intervals $[M, M+\delta], [M+\delta, M+2\delta], ..., [M+(n-1)\delta, M+n\delta]$ where $N \in [M+(n-1)\delta, M+n\delta]$. By the lemma, f is bounded on each of those sub-intervals and therefore f must be bounded on the union of those intervals, E, as well by taking the maximum upper bound. If E was not bounded, we could just take $f(x) = x$ on $E = \mathbb{R}$ which is uniformly continuous but not bounded.

14. Either $f(x) > x$ or $f(x) < x$ for all $x \in [0, 1]$, otherwise we could guarantee $f(x) = x$ by the Intermediate Value Theorem (Let $g(x) = f(x) - x$, if $g(a) > 0$ and $g(b) < 0$ there must be a c in between such that $g(c) = 0 \Rightarrow f(x) = x$). If $f(x) > x > 0$ for $x > 0$ and $f(0) \neq 0 \Rightarrow 0 \notin I$. If $f(x) < x < 1$ for $x < 1$ and $f(1) \neq 1 \Rightarrow 1 \notin I$. This is a contradiction so we must have $f(x) = x$ for some $x \in [0, 1]$.

15. Suppose f was not monotonic. Then there exists an interval (a, b) with $c \in (a, b)$ s.t. f is monotonically increasing on $(a, c]$ and decreasing on $[c, b)$ or vice versa but the proof for the other case would be analogous. So $f(c)$, which exists since f is continuous, would be a maximum on (a, b) so the range of f on (a, b) would be of the form $(k, f(c)]$. But this set is not open even though (a, b) is open so this contradicts the fact f is a continuous open mapping, therefore f must be monotonic.

16. $f(x) = [x]$ has discontinuities of the first kind on $\mathbb{Z}$. Let's first show it is continuous on $\mathbb{R}\backslash\mathbb{Z}$. Suppose $n < p < n + 1$ where $n \in \mathbb{Z}$ so $f(p) = n$. Pick $\delta = \min(d(p, n), d(p, n + 1))$. Then if $d(x, p) < \delta \Rightarrow x \in (n, n + 1) \Rightarrow f(x) = n \Rightarrow d(f(p), f(x)) = 0 < \epsilon \; \forall \epsilon > 0$ so f is continuous in $\mathbb{R}\backslash\mathbb{Z}$. However if $p = n$ then $f(p) = n$ but $\lim\limits_{x \to p^-} f(x) = n - 1$ so f is discontinuous there and since it is a jump, it is of the first kind. $f(x) = (x)$ also has discontinuities of the first kind on $\mathbb{Z}$, and is continuous everywhere else. However this is easy to show since $f(x) = (x) = x - [x]$ is a sum of two continuous functions if $x \notin \mathbb{Z}$ and is a sum of a continuous and discontinuous function if $x \in \mathbb{Z}$. The properties of limits will guarantee continuity in the former but the left and right hand limits will still disagree in the latter.

22. We will need the results of 20.
20a): $\rho_E(x) = \inf\limits_{z \in E} d(x, z) \geq 0$. If $x \in E$ then $d(x, x) = 0$ so $\rho_E(x) = 0$. If $x \in E'$, then $\forall \epsilon \; \exists z \in E$ s.t. $d(x, z) < \epsilon$. So $0 \leq \inf d(x, z) \leq \inf \epsilon = 0$ so $\rho_E(x) = 0$. Now suppose $\inf\limits_{z \in E} d(x, z) = 0$ so $\forall \epsilon \; \exists z \in E$ s.t $0 \leq d(x, z) < \epsilon \Rightarrow x \in E' \subset \overline{E}$.
20b): From the hint, which follows from taking the infimum of both sides of the Triangle Inequality, $\rho_E(x) \leq d(x, y) + \rho_E(y)$ and by a symmetric argument $\rho_E(y) \leq d(x, y) + \rho_E(x)$ so $|\rho_E(x) - \rho_E(y)| \leq d(x, y)$. If $d(x, y) < \delta = \epsilon$ then $d(\rho_E(x), \rho_E(y)) < \epsilon$ which implies $\rho_E(x)$ is uniformly continuous.
Back to 22, if $f(p) = 0$ then $\rho_A(p) = 0 \implies p \in \overline{A} = A$ by 20a. Note if $\rho_A(p) + \rho_B(p) == 0$, then $\rho_A(p) = \rho_B(p) = 0 \implies p \in A \cap B = \phi$ so this is impossible. If $f(p) = 1$ then $\rho_B(p) = 0$ so $p \in B$ for precisely the same reason. f is continuous since it is a quotient of two continuous functions, implied by 20b, where the denominator is never 0. $0 \leq \rho_A(p), \rho_B(p)$ so $0 \leq f(p) \leq 1 \Rightarrow$ range of f is in $[0, 1]$. Since f is continuous and $[0, \frac{1}{2})$, $(\frac{1}{2}, 1]$ are open in $[0, 1]$, their preimages are open as well so V and W are open. They are also disjoint since $f(p)$ cannot be in both intervals at the same time by the definition of a function. $p \in A \Rightarrow f(p) = 0 \Rightarrow p \in f^{-1}(0) \subset V$ and similarly $p \in B \Rightarrow f(p) = 1 \Rightarrow p \in f^{-1}(1) \subset W$.

23. As $x \to p^-$, $f(\lambda p + (1 - \lambda)x) \to f^-(p)$, and $\lambda f(p) + (1 - \lambda)f(x) \to \lambda(f(p) - f^-(p)) + f^-(p)$ where f^- represents the left handed limit. So we have $f^-(p) \leq \lambda(f(p) - f^-(p)) + f^-(p) \Rightarrow 0 \leq \lambda((f(p) - f^-(p)))$ and $\lambda > 1$ so $f(p) \geq f^-(p)$. Similarly $f(p) \geq f^+(p)$. However if we let $\lambda x + (1 - \lambda)y = p$ so $f(p) \leq \lambda f(x) + (1 - \lambda)f(y)$. Note as $x \to p^-$, $y \to p^+$. So taking the left handed limit with respect to x, $f(p) \leq \lambda f^-(x) + (1 - \lambda)f^+(x)$. So similarly $f(p) \leq \lambda f^+(x) + (1 - \lambda)f^-(x)$. Adding the two inequalities gives $2f(p) \leq f^-(x) + f^+(x)$ but $f^-(p), f^+ \leq f(p)$ so this forces $f^-(p) = f^+(p) = f(p)$ and therefore the function is continuous.
Let this function be of the form $h(x) = g(f(x))$ where f and g are convex and g is increasing.

$$\begin{aligned} h(\lambda x + (1 - \lambda)y) = g(f(\lambda x + (1 - \lambda)y)) &\leq g(\lambda f(x) + (1 - \lambda)f(y)) \text{ since } f \text{ is convex and } g \text{ is increasing} \\ &\leq \lambda g(f(x)) + (1 - \lambda)g(f(y)) \text{ since } g \text{ is convex} \\ &= \lambda h(x) + (1 - \lambda)h(y) \text{ therefore } h \text{ is convex} \end{aligned}$$

Let $\lambda = \dfrac{t - s}{u - s}$, $x = u$, $y = s$, then

$$\begin{aligned} f\left(\frac{t - s}{u - s}u + \frac{u - t}{u - s}s\right) = f(t) &\leq \frac{t - s}{u - s}f(u) + \frac{u - t}{u - s}f(s) = \frac{t - s}{u - s}f(u) + \frac{s - t}{u - s}f(s) + f(s) \\ &= \frac{t - s}{u - s}(f(u) - f(s)) + f(s) \text{ so } \frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s}. \end{aligned}$$

Let $\lambda = \dfrac{u - t}{u - s}$, $x = s$, $y = u$, then

$$f\left(\frac{u-t}{u-s}s+\frac{t-s}{u-s}u\right)=f(t)\le\frac{u-t}{u-s}f(s)+\frac{t-s}{u-s}f(u)=\frac{u-t}{u-s}f(s)+\frac{t-u}{u-s}f(u)+f(u)$$
$$=\frac{u-t}{u-s}(f(s)-f(u))+f(u) \text{ so } \frac{f(u)-f(t)}{u-t}\ge\frac{f(u)-f(s)}{u-s}.$$

24. First of all note that the conditions for a convex function are satisfied for $\lambda=\frac{m}{2^k}$ which follows by induction. If $k=1$, $\lambda=0,1,\frac{1}{2}$ which hold by the given conditions. Now suppose $f(\frac{m}{2^n}x+\frac{2^n-m}{2^n}y)\le\frac{m}{2^n}f(x)+\frac{2^n-m}{2^n}f(y)$. Then $f(\frac{m}{2^{n+1}}x+\frac{2^{n+1}-m}{2^{n+1}}y)\le\frac{1}{2}(f(\frac{m}{2^n}x+\frac{2^n-m}{2^n}y)+f(y))\le\frac{m}{2^{n+1}}f(x)+\frac{2^{n+1}-m}{2^{n+1}}f(y)$ which completes the induction. Now f is continuous and all real numbers have a binary expansion. This means there is a sequence of λ_is which converge to any $\lambda\in(0,1)$ so by taking limits we can say $f(\lambda x+(1-\lambda)y)=\lim_{i\to\infty}f(\lambda_i x+(1-\lambda_i)y)\le\lim_{i\to\infty}\lambda_i f(x)+(1-\lambda_i)f(y)=\lambda f(x)+(1-\lambda)f(y)$.