18.310A Problem Set 1

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1. Let A: sum is 9 and B: red die is 4. Then $P(A \cup B) = \frac{1}{6} \cdot \frac{1}{6}$ and $P(B) = \frac{1}{6}$ so $P(A|B) = \frac{1}{6}$.
   Let A: sum is 9 and B: at least one die is 4. Then $P(A \cup B) = 2 \cdot \frac{1}{6} \cdot \frac{1}{6}$ and $P(B) = 1 - \left(\frac{5}{6}\right)^2 = \frac{11}{6^2}$ so $P(A|B) = \frac{2}{11}$.
2. Listing out the possibilities for A gives HHTT, HTHH, THHT, TTHH, HHHT, THHH, HHTH, and HHHH so $P(A) = \frac{8}{16} = \frac{1}{2}$. $P(B) = \frac{1}{2}$ by inspection as half the time the last flip will be different from the first. $P(C) = \frac{1}{2}$, it is pretty obvious half the time the sequence will start with H. $P(B|A) = \frac{1}{2} = P(B)$ via looking at A's sample space, so A and B are independent. But note $P(C|A) = \frac{5}{8} \neq \frac{1}{2} = P(C)$ so A and C are not independent therefore A, B, C are not independent.
3. $P(\neg A|\neg B) = P(\neg A) - P(\neg A|B) = P(\neg A) - \frac{P(\neg A)P(B|\neg A)}{P(B)} = P(\neg A) - \frac{P(\neg A)(P(B) - P(B|A))}{P(B)} = P(\neg A)$ since $P(B|A) = P(B)$. So we have showed $\neg A, \neg B, \neg C$ are pairwise independent (the other pairs follow without loss of generality). Now $P(\neg A \cap \neg B \cap \neg C) = \neg P(A \cup B \cup C) = \neg[P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)] = \neg[P(A) + P(B) + P(C) - P(A)P(B) - P(B)P(C) - P(C)P(A) + P(A)P(B)P(C)] = \neg[(P(A) - 1)(P(B) - 1)(P(C) - 1) + 1] = (1 - P(A))(1 - P(B))(1 - P(C)) = P(\neg A)P(\neg B)P(\neg C)$ so $\neg A, \neg B, \neg C$ are independent.
4. We can use linearity of expectation. Note that there are $n - 1$ possible places where a consecutive pair of heads can appear; indices $1 \to 2$ to $n - 1 \to n$. Each of these places has an expected value of HH of $\frac{1}{4}$ since all the other indices can be arbitrary and there are four possibilities for the two we are looking at (HH, HT, TH, TT) so $\mathbb{E}f = \frac{n-1}{4}$ which gives us the expected number of HHs in the entire sequence.
5. a) $\frac{3}{4}$ have at least one daughter while $\frac{1}{4}$ have two so the probability is $\frac{1}{3}$.
   b) The daughter that got dropped off must be in fifth grade while the other child could be a boy or a girl so the probability is $\frac{1}{2}$.