18.100B - Problem Set 5

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1. $\{s_n\}$ converges $\to$ $\forall\epsilon$ $\exists N$ such that $|s - s_n| < \epsilon$ for $n \geq N$. $|s| - |s_n| \leq |s - s_n| < \epsilon$ and $|s_n| - |s| \leq |s_n - s| < \epsilon$ by the Triangle Inequality. Depending on the sign of $|s| - |s_n|$ we can take the absolute value of the corresponding inequality and achieve $||s| - |s_n|| < \epsilon$. So $\{|s_n|\}$ does converge and it converges to $|s|$. The converse is not true; take the sequence $s_n = (-1)^n$ for example, $|s_n| \to 1$ but s_n has no limit.

2. $\lim\limits_{n\to\infty} \sqrt{n^2+n} - n = \lim\limits_{n\to\infty} \sqrt{n^2+n} - n \cdot \dfrac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} = \lim\limits_{n\to\infty} \dfrac{n}{\sqrt{n^2+n}+n} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{1+\frac{1}{n}}+1} = \dfrac{1}{2}.$

   To verify this, note $1 + \frac{1}{n}$ is bounded below by 1 and is monotonically decreasing so $x_n = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$ is bounded above by $\frac{1}{1+1} = \frac{1}{2}$ and is monotonically increasing. This is also the least upper bound since if $x > 2$ we can always find n such that $\sqrt{1+\frac{1}{n}} + 1 < x$.

3. First of all $\{s_n\}$ is bounded above by 2. This is shown by induction. $s_1 = \sqrt{2} < 2$ checks. Now assume $s_k < 2 \to s_{k+1} = \sqrt{2+\sqrt{s_k}} < \sqrt{2+\sqrt{2}} < 2$.
   Now we must show that the sequence is monotonically increasing which we can also do by induction. $s_2 = \sqrt{2+\sqrt{s_1}} > \sqrt{2} = s_1$ checks. Now assume $s_k < s_{k+1}$. Then $s_{k+1} = \sqrt{2+\sqrt{s_k}} < \sqrt{2+\sqrt{s_{k+1}}}$. But $s_{k+2} = \sqrt{2+\sqrt{s_{k+1}}}$ so $s_{k+2} > s_{k+1}$ and the inductive step is complete.
   Since the sequence is monotonic and bounded, it therefore converges.

4. Note $s_2 = 0$ and $s_{2m} = \dfrac{\frac{1}{2} + s_{2m-2}}{2} = \dfrac{1}{4} + \dfrac{1}{2}s_{2m-2}$. It is not difficult to see $s_{2m} = \dfrac{1}{4} + \dfrac{1}{8} + \dfrac{1}{16} + \ldots + \dfrac{1}{2^m} = \dfrac{1}{2} \cdot \left(1 - \left(\dfrac{1}{2}\right)^m\right) = \dfrac{1}{2} - \left(\dfrac{1}{2}\right)^{m+1}$. Then $s_{2m+1} = 1 - \left(\dfrac{1}{2}\right)^{m+1}$. Assume $x \neq \frac{1}{2}, 1$. Take $r < \min(|x - \frac{1}{2}|, |x-1|)$. If $x > 1$ then $s_n \notin N_r(x)$ since $s_n < 1$ so there cannot be any subsequential limits here. Let us take $\{x < 1\} \setminus \{\frac{1}{2}\}$. By the Archimedean property we can find m such that $\frac{1}{2} - \left(\frac{1}{2}\right)^{m+1} > x + r$ or $1 - \left(\frac{1}{2}\right)^{m+1} > x + r$ with the left inequality corresponding to $x < \frac{1}{2}$ and the right one to $\frac{1}{2} < x < 1$. Thus in both these regions the chosen $N_r(x)$ will contain a finite number of points of the sequence since s_{2m} and s_{2m+1} are monotonically increasing and therefore x cannot be a subsequential limit in those regions. $x = \frac{1}{2}$ and $x = 1$ are the limits of s_{2m} and s_{2m+1} so they are the subsequential limits. Therefore $\limsup\limits_{n\to\infty} s_n = 1$ and $\liminf\limits_{n\to\infty} s_n = \dfrac{1}{2}$.

20. The convergence of p_{n_l} implies $\forall\epsilon$ $\exists N_1$ such that $d(p_{n_l}, p) < \frac{\epsilon}{2}$ for all $n_l > N_1$. Since $\{p_n\}$ is Cauchy, $\forall\epsilon$ $\exists N_2$ such that $d(p_n, p_{n_l}) < \frac{\epsilon}{2}$ for all $n, n_l > N_2$. Take $N = \max(N_1, N_2)$, then $d(p_n, p) < d(p_n, p_{n_l}) + d(p_{n_l}, p) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for all $n > N$. Since ϵ was arbitrary, this means $p_n \to p$.

21. If $E = \bigcap_1^\infty E_n$ had more than 1 points, the proof would be a direct copy of the latter half of the proof of 3.10(b) [diam $E_n \geq$ diam $E > 0$ which contradicits the limit] so we really only need to show that E is nonempty. Take a sequence $\{x_n\}$ such that $x_n \in E_n$. $\{x_n\}$ is Cauchy (since it is contained within E_n whose diameter goes to 0) and therefore convergent as well since it is in a complete metric space.

Let's say $x_n \to x$, then x must be a limit point of $E = \bigcap_1^\infty E_n$ since $d(x, x_n) < \epsilon$ and $x_n \in E_n\ \forall n$. But E_n is closed so $E = \bigcap_1^\infty E_n$ is also closed and therefore $x \in E$.

23. $\forall \epsilon\ \exists N_1$ such that $d(p_m, p_n) < \frac{\epsilon}{2}$ for $m, n \geq N_1$. Similar for $d(q_m, q_n)$ and let the determining number for that $\frac{\epsilon}{2}$ be N_2. Let $N = \max(N_1, N_2)$. By Triangle Inequality $d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n) \to d(p_n, q_n) - d(p_m, q_m) < \epsilon$ for $m, n \geq N$. If we computed the inequality focused on $d(p_m, q_m)$ instead we would have gotten $d(p_m, q_m) - d(p_n, q_n) < \epsilon$ so this means $|d(p_n, q_n) - d(p_m, q_m)| < \epsilon$ and therefore we have a Cauchy sequence in R so it converges.