Graph Theory

5.1 GRAPHS AND MULTIGRAPHS

The study of graph theory is introduced in this chapter and it will be continued in the next two chapters.

5.1 Define a graph.

A *graph* G consists of two parts:
(i) A set $V = V(G)$ whose elements are called *vertices, points,* or *nodes.*
(ii) A collection $E = E(G)$ of unordered pairs of distinct vertices called *edges.*
We write $G(V, E)$ when we want to emphasize the two parts of G.

5.2 Define a multigraph.

A *multigraph* $G = G(V, E)$ also consists of a set V of vertices and a set E of edges except that E may contain *multiple* edges, i.e., edges connecting the same endpoints, and E may contain one or more *loops,* i.e., an edge whose endpoints are the same vertex.

5.3 Describe a diagram of a graph (multigraph).

Graphs (multigraphs) $G = G(V, E)$ are pictured by diagrams in the plane as follows. Each vertex v in V is represented by a dot (or small circle) and each edge $e = \{u, v\}$ is represented by a curve which connects its *endpoints* u and v. (In fact, we usually denote a graph, when possible, by drawing its diagram rather than explicitly listing its vertices and edges.)

5.4 Describe formally the graph shown in Fig. 5-1.

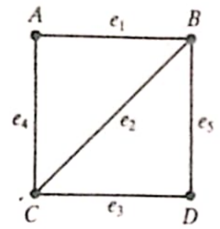


Fig. 5-1

Figure 5-1 shows the graph $G = G(V, E)$ where: (i) V consists of the vertices A, B, C, D; and (ii) E consists of the five edges $e_1 = \{A, B\}$, $e_2 = \{B, C\}$, $e_3 = \{C, D\}$, $e_4 = \{A, C\}$, $e_5 = \{B, D\}$.

5.5 The diagram in Fig. 5-2 shows a multigraph G. Why is G not a graph?

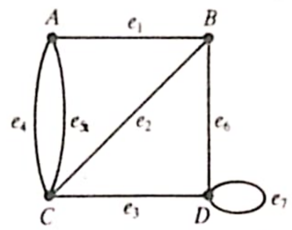


Fig. 5-2

G contains multiple edges, e_4 and e_5, which connect the same two vertices A and C. Also, G contains a loop e_7 whose endpoints are the same vertex D. A graph does not have multiple edges or loops.

5.6 Describe formally the graph shown in Fig. 5-3.

Figure 5-3 shows a graph $G = G(V, E)$ where (i) V consists of four vertices A, B, C, D; and (ii) E consists of five edges $e_1 = \{A, B\}$, $e_2 = \{B, C\}$, $e_3 = \{C, D\}$, $e_4 = \{A, C\}$, $e_5 = \{B, D\}$.

5.7 Consider the multigraph $G = G(V, E)$ shown in Fig. 5-4.
(a) Find the number of vertices and edges. **(b)** Are there any multiple edges or loops? If so, what are they?

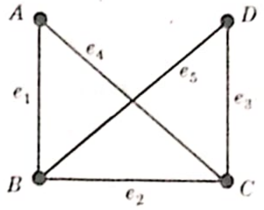


Fig. 5-3

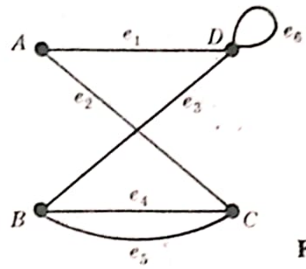


Fig. 5-4

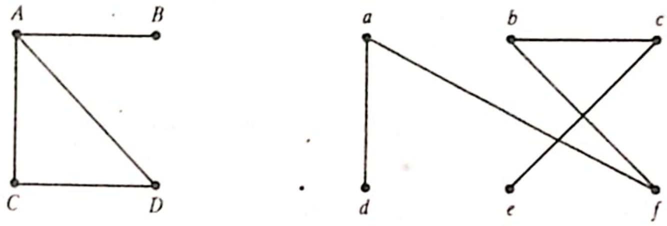
(a) G contains four vertices A, B, C, D; and six edges, $e_1, e_2, \ldots, e_6$. (Although the edges e_2 and e_3 cross at a point, the diagram does not indicate that the intersection point is a vertex of G.)

(b) The edges e_4 and e_5 are multiple edges since they both have the same endpoints B and C. The edge e_6 is a loop.

5.8 Draw a diagram for each of the following graphs $G = G(V, E)$:

(a) $V = \{A, B, C, D\}$, $E = [\{A, B\}, \{D, A\}, \{C, A\}, \{C, D\}]$

(b) $V = \{a, b, c, d, e, f\}$, $E = [\{a, d\}, \{a, f\}, \{b, c\}, \{b, f\}, \{c, e\}]$



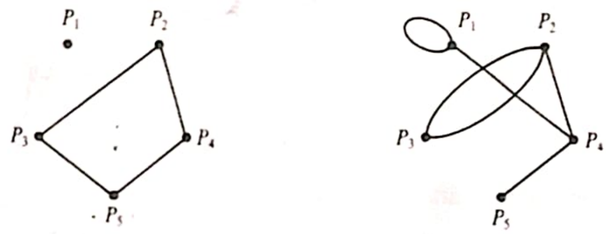
(a) (b) Fig. 5-5

First draw vertices of the graph, and then connect the appropriate vertices to indicate the edges of the graph, as shown in Fig. 5-5.

5.9 Draw a diagram of each of the following multigraphs $G(V, E)$ where $V = \{P_1, P_2, P_3, P_4, P_5\}$ and

(a) $E = [\{P_2, P_4\}, \{P_2, P_3\}, \{P_3, P_5\}, \{P_5, P_4\}]$

(b) $E = [\{P_1, P_1\}, \{P_2, P_3\}, \{P_2, P_4\}, \{P_3, P_2\}, \{P_4, P_1\}, \{P_5, P_4\}]$



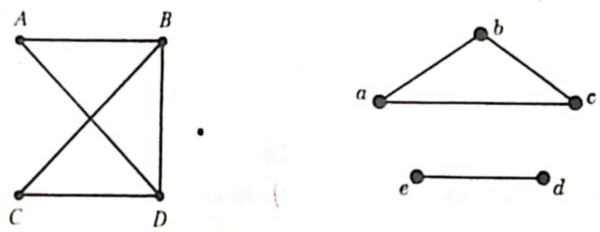
(a) (b) Fig. 5-6

As with graphs, draw the vertices and then indicate the edges by connecting the appropriate vertices, as in Fig. 5-6. [Note that **(a)** is a graph, besides being a multigraph.]

5.10 Draw the diagram of each of the following graphs $G(V, E)$:

(a) $V = \{A, B, C, D\}$, $E = [\{A, B\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}]$

(b) $V = \{a, b, c, d, e\}$, $E = [\{a, b\}, \{a, c\}, \{b, c\}, \{d, e\}]$

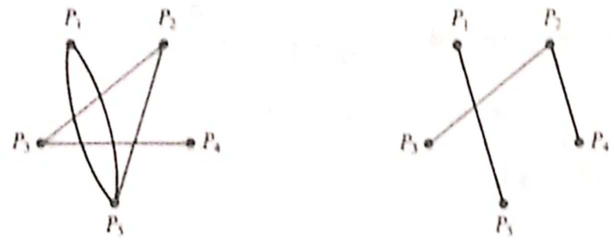


(a) (b) Fig. 5-7

Draw a dot for each vertex v in V, and for each edge $\{x, y\}$ in E draw a curve from the vertex x to the vertex y, as shown in Fig. 5-7.

5.11 Draw a diagram of each of the following multigraphs $G(V, E)$ where $V = \{P_1, P_2, P_3, P_4, P_5\}$ and

(a) $E = [\{P_1, P_5\}, \{P_3, P_4\}, \{P_2, P_3\}, \{P_2, P_5\}, \{P_1, P_5\}]$, **(b)** $E = [\{P_2, P_4\}, \{P_2, P_3\}, \{P_5, P_1\}]$



(a) (b) **Fig. 5-8**

Draw diagrams as in Fig. 5-8.

5.12 Determine whether or not each of the following multigraphs $G(V, E)$ is a graph where $V = \{A, B, C, D\}$ and

(a) $E = [\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{C, D\}]$
(b) $E = [\{A, B\}, \{B, B\}, \{A, D\}]$
(c) $E = [\{A, B\}, \{C, D\}, \{A, B\}, \{B, D\}]$
(d) $E = [\{A, B\}, \{B, C\}, \{C, B\}, \{B, B\}]$

Recall a multigraph $G(V, E)$ is a graph if it has neither multiple edges nor loops. Thus

(a) Yes.
(b) No, since $\{B, B\}$ is a loop.
(c) No, since $\{A, B\}$ and $\{A, B\}$ are multiple edges.
(d) No, since $\{B, C\}$ and $\{C, B\}$ are multiple edges, and, moreover, $\{B, B\}$ is a loop.

5.13 Describe formally the graph shown in Fig. 5-9.

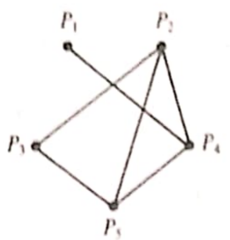


Fig. 5-9

There are five vertices, so

$$V = \{P_1, P_2, P_3, P_4, P_5\}$$

There are six edges and thus six pairs of vertices; hence

$$E = [\{P_1, P_4\}, \{P_2, P_3\}, \{P_2, P_4\}, \{P_2, P_5\}, \{P_4, P_5\}, \{P_3, P_5\}]$$

5.14 Describe formally the graph shown in Fig. 5-10.

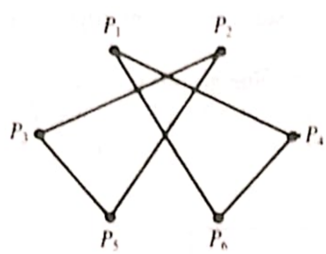


Fig. 5-10

There are six vertices, so

$$V = \{P_1, P_2, P_3, P_4, P_5, P_6\}$$

There are six edges and thus six pairs of vertices; hence

$$E = [\{P_1, P_4\}, \{P_1, P_6\}, \{P_4, P_6\}, \{P_3, P_2\}, \{P_3, P_5\}, \{P_2, P_5\}]$$

5.15 Describe formally the multigraph shown in Fig. 5-11.

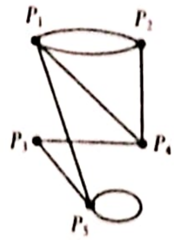


Fig. 5-11

▮ There are five vertices, so

$$V = \{P_1, P_2, P_3, P_4, P_5\}$$

There are eight edges (of which two are multiple edges and one is a loop) and thus eight pairs of vertices; hence

$$E = [\{P_1, P_2\}, \{P_1, P_2\}, \{P_1, P_4\}, \{P_1, P_5\}, \{P_2, P_4\}, \{P_3, P_4\}, \{P_3, P_5\}, \{P_5, P_5\}]$$

5.16 Describe formally the multigraph shown in Fig. 5-12.

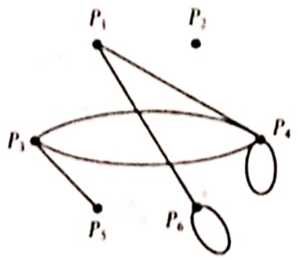


Fig. 5-12

▮ There are six vertices, so

$$V = \{P_1, P_2, P_3, P_4, P_5, P_6\}$$

There are seven edges (of which two are multiple edges and two are loops) and thus seven pairs of vertices; hence

$$E = [\{P_1, P_4\}, \{P_1, P_6\}, \{P_3, P_4\}, \{P_3, P_4\}, \{P_4, P_4\}, \{P_3, P_5\}, \{P_6, P_6\}]$$

5.17 Define a finite multigraph.

▮ A multigraph $G = G(V, E)$ is *finite* if both V is finite and E is finite. Note that a graph G with a finite number of vertices V must automatically have a finite number of edges and so must be finite.

5.18 What is the trivial graph? empty or null graph?

▮ The *trivial* graph is the graph with one vertex and no edges. The empty graph is the graph with no vertices and no edges.

5.19 What is an isolated vertex? Which vertex in Fig. 5-6 is isolated?

▮ A vertex V is *isolated* if it does not belong to any edge. The vertex P_1 in Fig. 5-6(*a*) is isolated.

5.20 Suppose $G = G(V, E)$ has five vertices. Find the maximum number m of edges in E if: **(*a*)** G is a graph, and **(*b*)** G is a multigraph.

▮ **(*a*)** There are $C(5, 2) = 10$ ways of choosing two vertices from V; hence $m = 10$.

(*b*) Since multiple edges are permitted, G can have any number of edges (and loops), finite or infinite; hence no such maximum number m exists.

5.2 DEGREE OF A VERTEX

5.21 Define the relation of adjacency and incidence in a graph G.

▮ Suppose $e = \{u, v\}$ is an edge in G, i.e., u and v are *endpoints* of e. Then the vertex u is said to be *adjacent* to the vertex v, and the edge e is said to be *incident* on u and on v.

5.22 Define the degree and parity (even or odd) of a vertex.

▮ The *degree* of a vertex v in a graph G, written $\deg(v)$, is equal to the number of edges which are incident on v or, in other words, the number of edges which contain v as an endpoint. The vertex v is said to be *even* or *odd* according as $\deg(v)$ is even or odd.

Theorem 5.1: The sum of the degrees of the vertices of a graph is equal to twice the number of edges.

5.23 Prove Theorem 5.1.

▮ Follows directly from the fact that each edge is counted twice in counting the degrees of the vertices of a graph G.

5.24 Does Theorem 5.1 hold for a multigraph?

▮ Yes. Note that a loop must be counted twice towards the degree of its endpoint.

5.25 Consider the graph $G = G(V, E)$ in Fig. 5-13. **(a)** Describe G formally. **(b)** Find the degree and parity of each vertex of G. **(c)** Verify Theorem 5.1 for G.

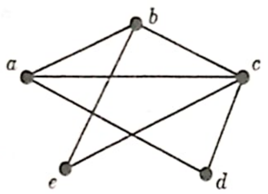


Fig. 5-13

▮ **(a)** There are five vertices, so $V = \{a, b, c, d, e\}$. There are seven pairs $\{x, y\}$ of vertices where the vertex x is connected with the vertex y: hence

$$E = [\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, e\}, \{c, d\}, \{c, e\}]$$

(b) The degree of a vertex is equal to the number of edges to which it belongs; e.g., $\deg(a) = 3$ since a belongs to $\{a, b\}$, $\{a, c\}$, $\{a, d\}$ or, equivalently, there are three edges leaving a in the diagram of G in Fig. 5-13. Similarly, $\deg(b) = 3$, $\deg(c) = 4$, $\deg(d) = 2$, $\deg(e) = 2$. Thus c, d, and e are even vertices and a and b are odd.

(c) The sum of the degrees of the vertices is $m = 3 + 3 + 4 + 2 + 2 = 14$ which does equal twice the number of edges.

5.26 Consider the graph G where

$$V(G) = \{A, B, C, D\} \quad \text{and} \quad E(G) = [\{A, B\}, \{B, C\}, \{B, D\}, \{C, D\}]$$

Find the degree and parity of each vertex in G.

▮ Count the number of edges to which each vertex belongs to obtain

$$\deg(A) = 1, \quad \deg(B) = 3, \quad \deg(C) = 2, \quad \deg(D) = 2$$

Thus C and D are even and A and B are odd.

5.27 Find the degree of each vertex in the multigraph in Fig. 5-11.

▮ Count the number of edges leaving each vertex to obtain

$$\deg(P_1) = 4, \quad \deg(P_2) = 3, \quad \deg(P_3) = 2, \quad \deg(P_4) = 3, \quad \deg(P_5) = 4$$

(Here the loop at P_5 is counted twice toward the degree of P_5.)

5.28 Find the degree of each vertex in the multigraph in Fig. 5-12.

▮ Count the number of edges leaving each vertex to obtain:

$$\deg(P_1) = 2, \quad \deg(P_2) = 0, \quad \deg(P_3) = 3, \quad \deg(P_4) = 5, \quad \deg(P_5) = 1, \quad \deg(P_6) = 3$$

(Here the loops at P_4 and P_6 are counted twice toward the degree of their corresponding vertices.)

5.29 Find the degree and parity of each vertex in the graph in Fig. 5-9.

Count the number of edges leaving each vertex to obtain

$$\deg(P_1)=1, \quad \deg(P_2)=3, \quad \deg(P_3)=2, \quad \deg(P_4)=3, \quad \deg(P_5)=3$$

Thus P_1, P_2, P_4, and P_5 are odd and P_3 is even.

5.30 Find the degree and parity of each vertex in the graph in Fig. 5-10.

Count the number of edges leaving each vertex to see that each vertex has degree 2 and hence each vertex is even.

5.31 Consider the multigraph G where $V(G)=\{A, B, C, D\}$ and

$$E(G)=[\{A, C\}, \{A, D\}, \{B, B\}, \{B, C\}, \{C, A\}, \{C, B\}, \{D, B\}, \{D, D\}]$$

(*a*) Find the degree and parity of each vertex in G.
(*b*) Verify Theorem 5.1 for the multigraph G.

(*a*) Count the number of edges to which each vertex belongs or, equivalently, count the number of times each vertex appears in $E(G)$ to obtain

$$\deg(A)=3, \quad \deg(B)=5, \quad \deg(C)=4, \quad \deg(D)=4$$

Thus A and B are odd, and C and D are even.

(*b*) The sum of the degrees of the vertices is $m=3+5+4+4=16$ which does equal twice the number (eight) of edges.

5.32 Find the sum m of the degrees of the vertices of G where $V(G)=\{A, B, C, D\}$ and
(*a*) $E(G)=[\{A, B\}, \{A, C\}, \{B, D\}, \{C, D\}]$
(*b*) $E(G)=[\{A, B\}, \{A, C\}, \{A, D\}, \{B, A\}, \{B, B\}, \{C, B\}, \{C, D\}]$

One way to determine m is to find the degree of each vertex, and sum the degrees over all vertices. However, a faster approach would be to apply Theorem 5.1, i.e., the required result is to double the number of edges. Hence
(*a*) There are 4 edges, so $m=2(4)=8$.
(*b*) There are 7 edges, so $m=2(7)=14$.

5.33 Suppose v is an isolated vertex in a graph (multigraph) G. What is its degree?

The vertex v is isolated if it does not belong to any edge. Thus v is isolated if and only if $\deg(v)=0$.

5.34 Consider $G=G(V, E)$ where $V=\{u, v, w\}$ and $\deg(v)=4$. (*a*) Does such a graph G exist? If not, why not? (*b*) Does such a multigraph G exist? If yes, give an example.

(*a*) No. Since multiple edges and loops are not permitted, there can only be one edge from v to each of the other two edges; hence $\deg(v)\le 2$.
(*b*) Yes. For example, $E=[\{u, v\}, \{u, v\}, \{v, w\}, \{v, w\}]$.

5.35 Consider $G=G(V, E)$ where $V=\{A, B, C, D\}$ and

$$\deg(A)=2, \quad \deg(B)=3, \quad \deg(C)=2, \quad \deg(D)=2$$

(*a*) Does such a graph G exist; If not, why not? (*b*) Does such a multigraph G exist?

(*a*) No. The sum m of the degrees of the vertices must be even, since m is twice the number of edges (Theorem 5.1). Here $m=7$, an odd number. Thus no such graph G exists.
(*b*) No, since Theorem 5.1 also holds for multigraphs.

5.3 PATHS, CONNECTIVITY

5.36 Define a path and its length in a graph (multigraph) G.

A *path* α in G with *origin* v_0 and *end* v_n is an alternating sequence of vertices and edges of the form

$$v_0, e_1, v_1, e_2, v_2, \ldots, e_{n-1}, v_{n-1}, e_n, v_n$$

where each edge e_i is incident on vertices v_{i-1} and v_i. The number n of edges is called the *length* of α. When there is no ambiguity, we denote α by its sequence of edges, $\alpha = (e_1, e_2, \ldots, e_n)$, or by its sequence of vertices, $\alpha = (v_0, v_1, \ldots, v_n)$.

5.37 Define a simple path and a trail in a graph (multigraph) G.

A path $\alpha = (v_0, v_1, \ldots, v_n)$ is *simple* if all the vertices are distinct. The path is a *trail* if all the edges are distinct.

5.38 Consider a graph (multigraph) G. Define a closed path and a cycle in G.

A path $\alpha = (v_0, v_1, \ldots, v_n)$ is *closed* if $v_0 = v_n$, that is, if origin (α) = end (α). The path α is a *cycle* if it is closed and if all vertices are distinct except $v_0 = v_n$. A cycle of length k is called a k-cycle. A cycle in a graph must therefore have length three or more.

5.39 Let u and v be vertices in a graph G. Define the distance between u and v, written $d(u, v)$.

If $u = v$, then $d(u, u) = 0$. Otherwise, $d(u, v)$ is equal to the length of a shortest path between u and v. If no path between u and v exists, then $d(u, v)$ is not defined.

5.40 Let G be the graph shown in Fig. 5-14. Consider the following paths in G:

(a) $\alpha = (e_1, e_4, e_6, e_5)$, **(b)** $\beta = (e_2, e_5, e_3, e_4, e_6, e_3, e_1)$

Convert each sequence of edges into the corresponding sequence of vertices.

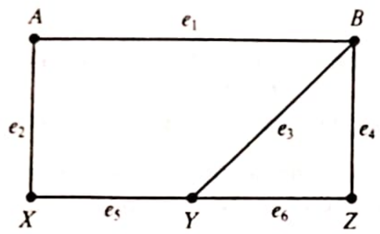


Fig. 5-14

List the initial vertex of the first edge followed by the terminal (end) vertex of each edge in the sequence to obtain

(a) $\alpha = (A, B, Z, Y, X)$, and **(b)** $\beta = (A, X, Y, B, Z, Y, B, A)$.

5.41 Let G be the graph in Fig. 5-14. Find: **(a)** all simple paths from vertex A to vertex Z, and **(b)** $d(A, Z)$.

(a) A path from A to Z is simple if no vertex is repeated. There are four such simple paths as follows:

$$(A, B, Z), \quad (A, B, Y, Z), \quad (A, X, Y, Z), \quad (A, X, Y, B, Z)$$

(b) $d(A, Z) = 2$ since the path $\alpha = (A, B, Z)$, of length 2, is the shortest path from A to Z.

5.42 Let G be the graph in Fig. 5-14. Find a k-cycle for: **(a)** $k = 3$, **(b)** $k = 4$, **(c)** $k = 5$, and **(d)** $k = 6$.

A k-cycle is a closed path of length k where all vertices are distinct (except $v_0 = v_n$). Thus **(a)** (B, Y, Z, B), **(b)** (A, B, Y, X, A), **(c)** (A, B, Z, Y, X, A), and **(d)** No 6-cycle exists.

5.43 Let G be the graph in Fig. 5-15. Determine whether or not each of the following sequences of edges forms a path:

(a) $(\{A, X\}, \{X, B\}, \{C, Y\}, \{Y, X\})$ **(c)** $(\{X, B\}, \{B, Y\}, \{Y, C\})$
(b) $(\{A, X\}, \{X, Y\}, \{Y, Z\}, \{Z, A\})$ **(d)** $(\{B, Y\}, \{X, Y\}, \{A, X\})$

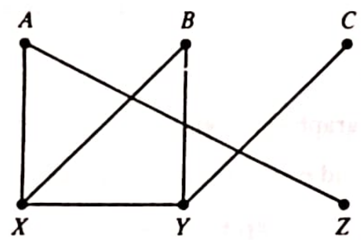


Fig. 5-15

▮ A sequence of edges is a path if the edges can be directed so that the end vertex of one edge is the initial vertex of the next edge.

(a) No. The edge $\{X, B\}$ is not followed by the edge $\{C, Y\}$.
(b) No. The pair $\{Y, Z\}$ is not an edge.
(c) Yes.
(d) Yes, since the sequence can be rewritten as $(\{B, Y\}, \{Y, X\}, \{X, A\})$.

5.44 Let G be the graph in Fig. 5-15. Find: **(a)** all simple paths from A to C, and **(b)** $d(A, C)$.

▮ **(a)** There are only two simple paths from A to C: (A, X, Y, C) and (A, X, B, Y, C).
(b) $d(A, C) = 3$ since 3 is the length of the shortest path from A to C.

5.45 Find all cycles in the graph G in Fig. 5-15.

▮ There is only one cycle in G, the 3-cycle $\alpha = (B, X, Y, B)$. [Here we identify α with the other cycles that have the same vertices as α, e.g., (X, Y, B, X), and those cycles obtained by reversing the order of the vertices, e.g., (B, Y, X, B)].

Theorem 5.2: There is a path from a vertex u to a vertex v if and only if there is a simple path from u to v.

5.46 Prove Theorem 5.2.

▮ Since every simple path is a path, we need only prove that if there is a path α from u to v, then there is a simple path from u to v. The proof is by induction on the length n of α. Suppose $n = 1$, i.e., $\alpha = (u, v)$. Then α is a simple path from u to v. Suppose $n > 1$, say

$$\alpha = (u = v_0, v_1, v_2, \ldots, v_{n-1}, v = v_n)$$

If no vertex is repeated, then α is a simple path from u to v. Suppose a vertex is repeated, say $v_i = v_j$ where $i < j$. Then

$$\beta = (v_0, v_1, \ldots, v_i, v_{j+1}, \ldots, v_n)$$

is a path from $u = v_0$ to $v = v_n$ of length less than n. By induction, there is a simple path from u to v.

5.47 Is there any inclusion relation between closed paths, trails, simple paths, and cycles?

▮ Yes. Every cycle is a closed path since, by definition, a cycle is a closed path with distinct vertices. Also, every simple path is a trail since a path with distinct vertices must have distinct edges. (A cycle is a trail, but not a simple path.)

5.48 Let G be the graph in Fig. 5-16. Determine whether each of the following is a closed path, trail, simple path, or cycle: **(a)** (B, A, X, C, B), **(b)** (X, A, B, Y), **(c)** (B, X, Y, B).

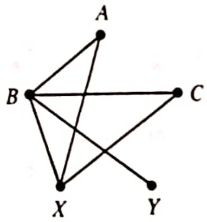


Fig. 5-16

▮ **(a)** This path is a cycle since it is closed and has distinct vertices.
(b) This path is simple since its vertices are distinct. It is not a cycle since it is not closed.
(c) This is not even a path since $\{X, Y\}$ is not an edge.

5.49 Repeat Problem 5.48 for each of the following: **(a)** (B, A, X, C, B, Y), **(b)** (X, C, A, B, Y), and **(c)** (X, B, A, X, C).

▮ **(a)** This path is a trail since its edges are distinct. It is not a simple path since the vertex B is repeated.
(b) This is not even a path since $\{C, A\}$ is not an edge.
(c) This path is a trail since the edges are distinct. It is not a simple path since the vertex X is repeated.

5.50 Repeat Problem 5.48 for each of the following: **(a)** (X, B, A, X, B), **(b)** (A, B, C, X, B, A), **(c)** (X, C, B, A).

(a) This path is neither closed nor a trail. [The edge $\{X, B\}$ is repeated.] Thus it is neither a cycle nor a simple path.
(b) This is a closed path. It is not a cycle since the vertex B is repeated.
(c) This is a simple path since the vertices are distinct.

5.51 Let G be the graph in Fig. 5-17. Find: (a) all simple paths from A to Z, (b) all trails from A to Z.

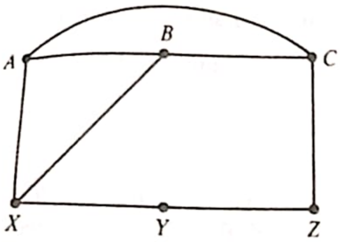


Fig. 5-17

(a) A path from A to Z is simple if no vertex (and hence no edge) is repeated. There are five such simple paths:

$$(A, C, Z), \quad (A, B, C, Z), \quad (A, X, Y, Z), \quad (A, B, X, Y, Z), \quad (A, X, B, C, Z)$$

(b) A path from A to Z is a trail if no edge is repeated. There are eight such trails, the five simple paths from (a) together with

$$(A, X, B, A, C, Z), \quad (A, C, B, A, X, Y, Z), \quad (A, B, C, A, X, Y, Z)$$

5.52 Find $d(A, Z)$ for the graph G in Fig. 5-17.

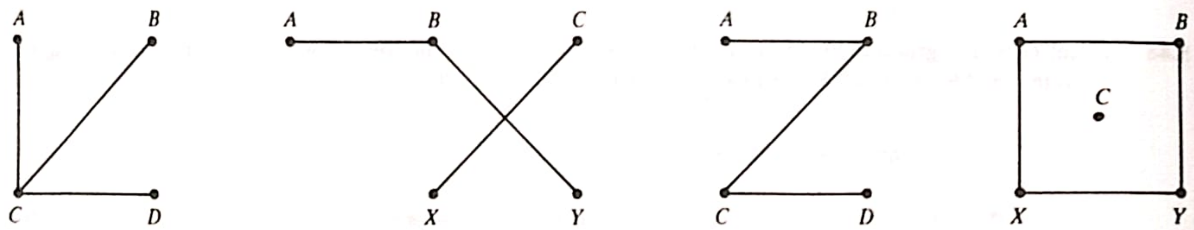
Here $d(A, Z) = 2$ since there is a path $\alpha = (A, C, Z)$ of length two from A to Z and none shorter. (Remember, length is the number of edges, not vertices, in a path.)

Connected Graphs

5.53 Define a connected graph (multigraph).

A graph (multigraph) G is *connected* if there is a path between any two of its vertices.

5.54 Determine whether or not each of the graphs in Fig. 5-18 is connected.



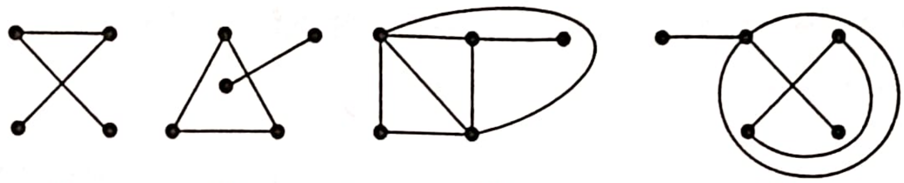
(a) (b) (c) (d)

Fig. 5-18

(a) Yes. There is a path between any two vertices of the graph.
(b) No. Here A, B, and Y are connected, and C and X are connected, but there is no path from A, B, or Y to either C or X.
(c) Yes. There is a path between any two vertices of the graph.
(d) No. There is no path from C to any other vertex of the graph.

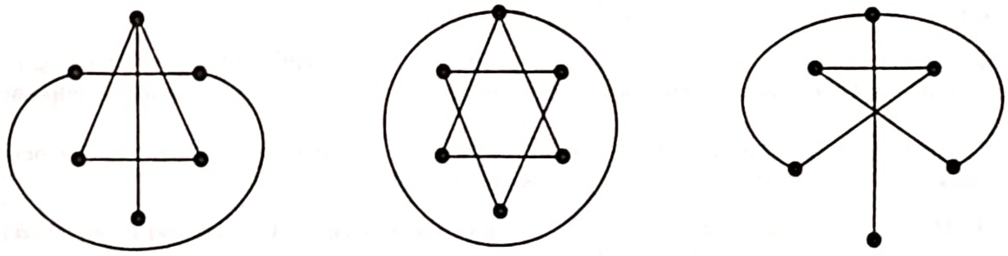
5.55 Consider the multigraphs in Fig. 5-19. Which of them are (a) connected, (b) loop-free (i.e., have no loops), (c) graphs?

(a) Only (i) and (iii) are connected.
(b) Only (iv) has a loop, i.e., an edge with the same endpoints.
(c) Only (i) and (ii) are graphs. The multigraph (iii) has multiple edges and (iv) has multiple edges and a loop.



(i) (ii) (iii) (iv) **Fig. 5-19**

5.56 Consider the multigraphs in Fig. 5-20. Which of them are (*a*) connected, (*b*) graphs?



(i) (ii) (iii)

Fig. 5-20

▮ (*a*) Only (iii) is connected. (*b*) They are all graphs.

5.57 Define the diameter of a connected graph G.

▮ The *diameter* of G, written diam (G), is the maximum distance between any two of its vertices.

5.58 Find the diameter of the connected graph in Fig. 5-17.

▮ Here diam $(G) = 2$ since it is the maximum distance between any two vertices.

5.59 Find the diameter of the connected graph G in Fig. 5-15.

▮ Note $d(C, Z) = 4$ and this is the maximum distance between any two vertices in G. Thus diam $(G) = 4$.

5.60 Find the diameter of the connected graph in Fig. 5-14.

▮ Here $d(A, Z) = 2$ and this is the maximum distance between any two vertices in G. Thus diam $(G) = 2$.

5.4 SUBGRAPHS, CONNECTED COMPONENTS, CUT POINTS, BRIDGES

5.61 Define the terms (*a*) subgraph and (*b*) full subgraph.

▮ (*a*) Let G be a graph. Then H is a *subgraph* of G if $V(H) \subseteq V(G)$, i.e., the vertices of H are also vertices of G, and $E(H) \subseteq E(G)$, i.e., the edges of H are also edges of G. In other words, $H(V', E')$ is a subgraph of $G(V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

(*b*) Suppose $H = H(V', E')$ is a subgraph of $G = G(V, E)$. Then H is called a *full subgraph* of G if E' contains all the edges of E whose endpoints lie in V'. In this case H is called the subgraph of G *generated* by V'.

5.62 Consider the graph $G = G(V, E)$ in Fig. 5-21. Determine whether or not $H = H(V', E')$ is a subgraph of G where

(*a*) $V' = \{A, B, F\}$ and $E' = [\{A, B\}, \{A, F\}]$,
(*b*) $V' = \{B, C, D\}$ and $E' = [\{B, C\}, \{B, D\}]$,
(*c*) $V' = \{A, B, C\}$ and $E' = [\{A, B\}, \{A, C\}]$.

▮ H is a subgraph of G if H is a graph and its vertices are contained in V and its edges are contained in E. (*a*) No, the vertex F is not a vertex in G. (*b*) Yes. (*c*) No, since $\{A, C\}$ is not an edge in G.

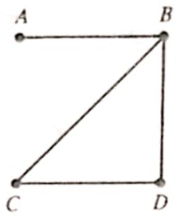


Fig. 5-21

5.63 Consider the graph $G = G(V, E)$ in Fig. 5-21. Determine whether or not $H = H(V', E')$ is a subgraph of G where
(*a*) $V' = \{A, B, D\}$ and $E' = [\{A, B\}, \{A, D\}]$,
(*b*) $V' = \{B\}$ and $E' = \emptyset$, the empty set,
(*c*) $V' = \{A, B, C\}$ and $E' = [\{A, B\}, \{B, C\}, \{B, D\}]$.

▌ (*a*) No, since $\{A, D\}$ is not an edge in G. (*b*) Yes. (*c*) No. Although $V' \subseteq V$ and $E' \subseteq E$, H is not a subgraph of G because H is not a graph. Specifically, $\{B, D\}$ in E' does not have its endpoints in V'.

5.64 Consider the graph $G = G(V, E)$ in Fig. 5-21. Find the (full) subgraph $H(V', E')$ of G generated by (*a*) $V' = \{A, B, C\}$, (*b*) $V' = \{A, C, D\}$, and (*c*) $V' = \{A, D\}$.

▌ Here E' will consist of all the edges in E whose endpoints lie in V'. Thus (*a*) $E' = [\{A, B\}, \{B, C\}]$, (*b*) $E' = [\{C, D\}]$, and (*c*) $E' = \emptyset$, the empty set.

5.65 Consider the graph $G = G(V, E)$ in Fig. 5-21. Find the number of full subgraphs of G.

▌ Each subset of $V = \{A, B, C, D\}$ determines a full subgraph of G. There are $m = 2^4 = 16$ subsets of V and hence there are $m = 16$ full subgraphs of G. (We are including the empty graph, the graph with no vertices and no edges.)

Connected Components

5.66 Let G be a graph (multigraph). Define a connected component of G. Illustrate with an example.

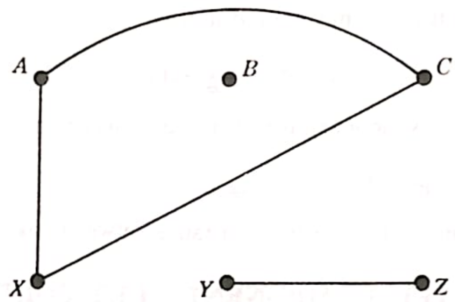


Fig. 5-22

▌ A *connected component* of G is a subgraph of G which is not contained in any larger connected subgraph of G. It is clear that a connected component is the full subgraph spanned by its vertices; hence we can designate a connected component by listing its vertices. It is also clear that G can be partitioned into its connected components. Figure 5-22 shows a graph with three connected components, $\{A, C, X\}$, $\{B\}$, and $\{Y, Z\}$.

5.67 Find the connected components of the graph G in Fig. 5-23.

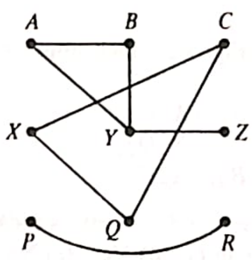


Fig. 5-23

▮ Start with any vertex, say A, and find all vertices connected to A; this gives the component $\{A, B, Y, Z\}$. Next select a vertex not included in this component and repeat the process to obtain another component. Continue in this way until all the components have been identified. For this graph, we obtain two additional components, $\{C, X, Q\}$ and $\{P, R\}$. Thus the components of G are $\{A, B, Y, Z\}$, $\{C, X, Q\}$, $\{P, R\}$.

5.68 Find the connected components of the graph G in Fig. 5-24.

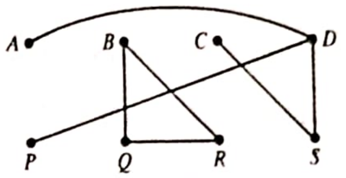


Fig. 5-24

▮ Proceed as in Problem 5.67 to obtain the connected components $\{A, D, P, S, C\}$ and $\{B, Q, R\}$.

5.69 Find the connected components of G where $V(G) = \{A, B, C, X, Y, Z\}$ and **(a)** $E(G) = [\{A, X\}, \{C, X\}]$, **(b)** $E(G) = [\{A, Y\}, \{B, C\}, \{Z, Y\}, \{X, Z\}]$.

▮ **(a)** Here A is connected to C and X; and B, Y, and Z are isolated vertices; hence $\{A, C, X\}$, $\{B\}$, $\{Y\}$, and $\{Z\}$ are the connected components of G.

(b) Here A, Y, Z, and X are connected; and B and C are connected. Thus $\{A, X, Y, Z\}$ and $\{B, C\}$ are the connected components of G.

5.70 Find the connected components of G where $V(G) = \{A, B, C, P, Q\}$ and **(a)** $E(G) = [\{A, C\}, \{B, Q\}, \{P, C\}, \{Q, A\}]$, **(b)** $E(G) = \emptyset$, the empty set.

▮ **(a)** Here G is connected, i.e., each vertex is connected to the other vertices. Thus G has one component $V(G) = \{A, B, C, P, Q\}$.

(b) Since $E(G)$ is empty, all the vertices are isolated; hence $\{A\}$, $\{B\}$, $\{C\}$, $\{P\}$, and $\{Q\}$ are the connected components of G.

5.71 Let G be a graph. For vertices u and v, define $u \sim v$ if $u = v$ or if there is a path from u to v. Show that $\sim$ is an equivalence relation on $V(G)$. How can one describe the equivalence classes induced by $\sim$?

▮ By definition, $u \sim u$ for every $u \in V(G)$, hence $\sim$ is reflexive. Suppose $u \sim v$. Then there is a path α from u to v. Reversing α gives a path from v back to u. Thus $v \sim u$ and therefore $\sim$ is symmetric. Lastly, suppose $u \sim v$ and $v \sim w$. Then there is a path α from u to v and a path β from v to w. However, end $(\alpha) = v =$ initial (β). Thus α may be continued by β to give a path $\alpha\beta$ from u to w. Thus $u \sim w$ and therefore $\sim$ is transitive. Accordingly, $\sim$ is an equivalence relation.

The equivalence classes determined by $\sim$ are the connected components of G.

Subgraph $G - v$, Cut Points

5.72 Define the subgraph $G - v$ where v is a vertex in G.

▮ $G - v$ is the subgraph of G obtained by deleting the vertex v from the vertex set $V(G)$ and deleting all edges in $E(G)$ which are incident on v. Alternately, $G - v$ is the full subgraph of G generated by the remaining vertices.

5.73 Define a cut point for a connected graph G.

▮ A vertex v is called a *cut point* for G if $G - v$ is disconnected. (More generally, v is a cut point for any graph G if $G - v$ has more connected components than G.)

5.74 Let G be the graph in Fig. 5-25. Find: **(a)** $G - A$, **(b)** $G - B$, and **(c)** $G - C$.

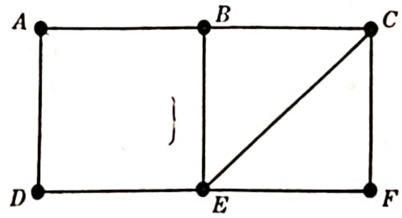


Fig. 5-25

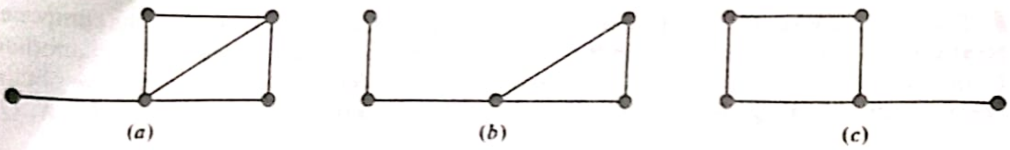


Fig. 5-26

▮ Delete the given vertex from G and all edges incident on that vertex to obtain the graphs in Fig. 5-26.

5.75 Let G be the graph in Fig. 5-25. Find: **(a)** $G - X$, **(b)** $G - Y$, and **(c)** $G - Z$.

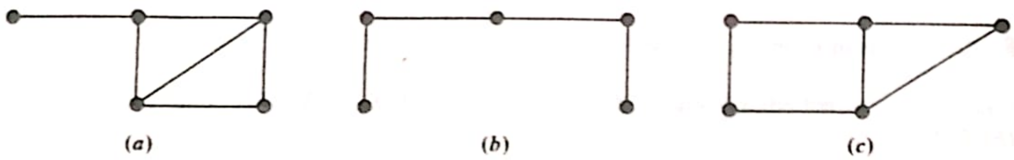


Fig. 5-27

▮ See Fig. 5-27.

5.76 Let G be the graph in Fig. 5-25. Does G have any cut points?

▮ Figures 5-26 and 5-27 show that $G - v$ is connected for any vertex v of G. Thus G has no cut points.

5.77 Let G be the graph in Fig. 5-28. Find: **(a)** $G - A$, **(b)** $G - B$, and **(c)** $G - C$.

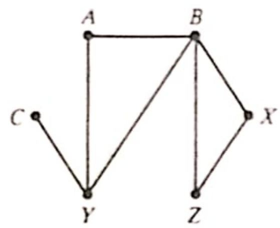


Fig. 5-28

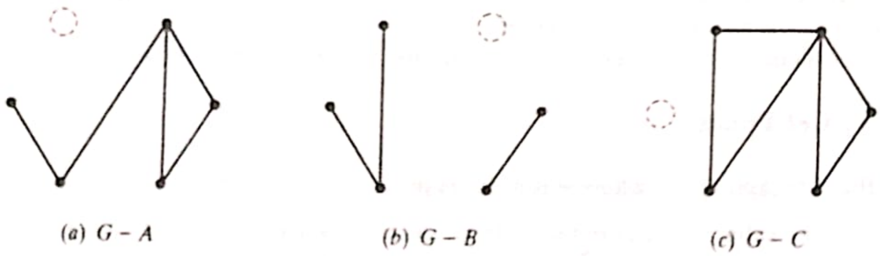


Fig. 5-29

▮ Delete the given vertex from the diagram and all edges incident on that vertex to obtain the graphs in Fig. 5-29.

5.78 Let G be the graph in Fig. 5-28. Find: **(a)** $G - X$, **(b)** $G - Y$, and **(c)** $G - Z$.

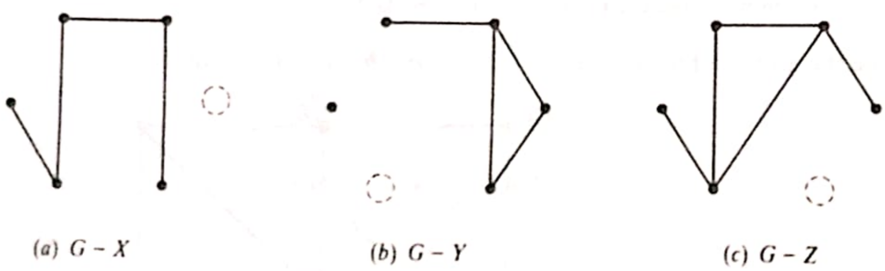


Fig. 5-30

▮ See Fig. 5-30.

5.79 Let G be the graph in Fig. 5-28. Does G have any cut points?

▮ Figures 5-29(b) and 5-30(b) show that only $G - B$ and $G - Y$ are disconnected. Thus B and Y are cut points of G.

5.80 Let G be the graph where $V(G) = \{A, B, C, X, Y, Z\}$ and $E(G) = [\{A, C\}, \{A, X\}, \{A, Y\}, \{B, Y\}, \{B, Z\}]$. **(a)** Find $G - A$. **(b)** Determine the number of connected components of $G - A$.

▮ **(a)** Delete A from $V(G)$ and delete all edges incident on A from $E(G)$ to obtain

$$V(G - A) = \{B, C, X, Y, Z\} \qquad \text{and} \qquad E(G - A) = [\{B, Y\}, \{B, Z\}]$$

(b) Here B, Y, and Z are connected and C and X are isolated vertices (in $G - A$). Thus $\{B, Y, Z\}$, $\{C\}$, and $\{X\}$ are the connected components of $G - A$.

5.81 Let G be the graph in Fig. 5-15. Does G have any cut points?

▮ Deleting A, X, or Y (and the edges incident on these vertices) disconnects G; hence A, X, and Y are cut points.

5.82 Let G be the graph in Fig. 5-16. Does G have any cut points?

▮ Only removal of B (and the edges incident on B) disconnects G; hence only B is a cut point.

Subgraph $G - e$, Bridges

5.83 Define the subgraph $G - e$ where e is an edge in G.

▮ $G - e$ is the graph obtained by simply deleting e from the edge set of G. Thus $V(G - e) = V(G)$ and $E(G - e) = E(G)\backslash\{e\}$.

5.84 Define a bridge for a connected graph G.

▮ An edge e is a bridge for G if $G - e$ is disconnected. (In general, e is a bridge for any graph G if $G - e$ has more connected components than G has.)

5.85 Let G be the graph in Fig. 5-21. Find **(a)** $G - \{A, B\}$, **(b)** $G - \{B, C\}$, **(c)** $G - \{B, D\}$, **(d)** $G - \{C, D\}$.

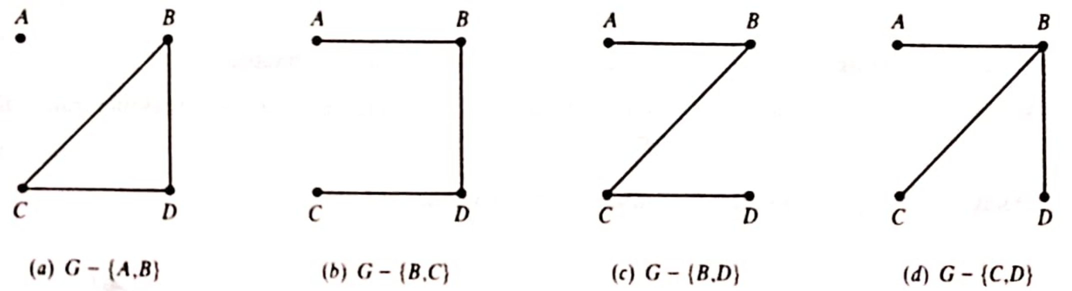


Fig. 5-31

▮ Simply delete the given edge from the graph to obtain the corresponding graphs in Fig. 5-31.

5.86 Let G be the graph in Fig. 5-21. Does G have any bridges?

▮ Figure 5-31 shows that only $G - \{A, B\}$ is disconnected; hence $\{A, B\}$ is a bridge and the only bridge for G.

5.87 Let G be the graph in Fig. 5-15. Does G have any bridges?

▮ G has three bridges $\{A, X\}$, $\{A, Z\}$, and $\{Y, C\}$. Deleting any other edge of G does not disconnect G.

5.88 Let G be the graph in Fig. 5-16. Does G have any bridges?

▮ Only $\{B, Y\}$ disconnects G and hence $\{B, Y\}$ is the only bridge for G.

5.89 Let G be the graph in Fig. 5-24. Does G have any bridges?

Here G has two connected components. Deleting $\{C, S\}$ or $\{D, S\}$ partitions $V(G)$ into three (more than two) connected components; hence $\{C, S\}$ and $\{D, S\}$ are bridges.

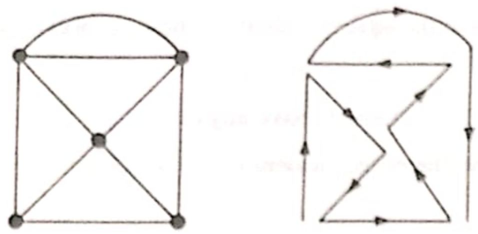
5.90 Let G be the connected graph in Problem 5.80. Does G have any bridges?

Each one of the edges disconnects G. Thus G has five bridges.

5.5 TRAVERSABLE MULTIGRAPHS

This section discusses traversable multigraphs. Unless the distinction is vital and implied by the discussion, the term graph may be used when referring to both graphs and multigraphs.

5.91 Define a traversable multigraph with an example.



(a) (b) **Fig. 5-32**

A multigraph G is said to be *traversable* if it "can be drawn without any breaks in the curve and without repeating any edge", that is, if there is a path which includes all vertices and uses each edge exactly once. Such a path must be a trail (since no edge is used twice) and it will be called a traversable trail. Clearly a traversable multigraph must be connected. Figure 5-32(b) shows a traversable trail of the multigraph in Fig. 5-32(a). (To indicate the direction of the trail, the diagram misses touching vertices which are actually traversed.)

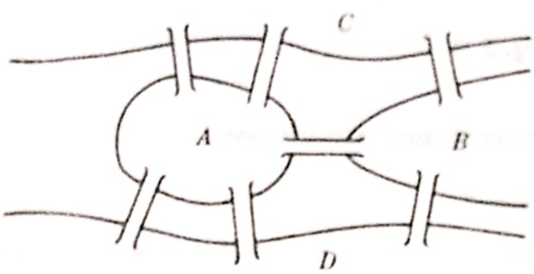
5.92 Suppose a multigraph G is traversable and that a traversable trail does not begin or end at a vertex P. Show that P is an even vertex.

Whenever the traversable trail enters P by an edge, there must always be an edge not previously used by which the trail can leave P. Thus the trail exhausts the edges incident on P in pairs, and so P has even degree, as claimed.

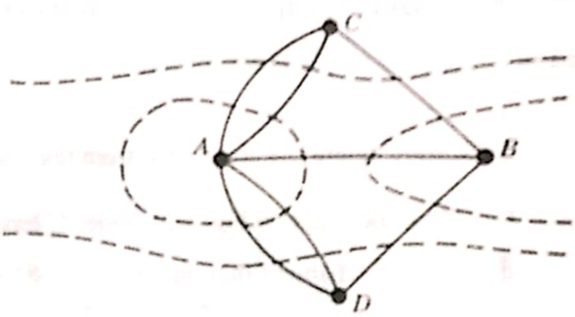
5.93 Show that a multigraph G with more than two odd vertices is not traversable.

Suppose G is traversable and Q is an odd vertex of G. By Problem 5.92, a traversable trail must either begin or end at Q. Thus G cannot have more than two odd vertices.

5.94 Discuss the Bridges of Königsberg problem and its solution.



(a) Königsberg in 1736



(b) Euler's graphical representation

Fig. 5-33

The eighteenth-century East Prussian town of Königsberg included two islands and seven bridges as shown in Fig. 5-33(a). Question: Beginning anywhere and ending anywhere, can a person walk through town crossing all seven bridges but not crossing any bridge twice? The people of Königsberg wrote to the celebrated Swiss mathematician L. Euler about this question. Euler proved in 1736 that such a walk is impossible. He replaced the islands and the two sides of the river by points and the bridges by curves, thus obtaining Fig. 5-33(b). It is not difficult to see that the walk in Königsberg is possible if and only if the multigraph in Fig. 5-33(b) is traversable. But this multigraph has four odd vertices, and hence it is not traversable. Thus one cannot walk through Königsberg so that each bridge is crossed exactly once.

5.95 Define an eulerian graph and an eulerian trail.

▌ A graph (multigraph) G is an *eulerian* graph if there exists a closed traversable trail, called an *eulerian* trail.

Theorem 5.3 (Euler): A finite connected graph G is eulerian if and only if each vertex has even degree.

Corollary 5.4: Any finite connected graph with two odd vertices is traversable. A traversable trail may begin at either odd vertex and will end at the other odd vertex.

5.96 Prove Theorem 5.3.

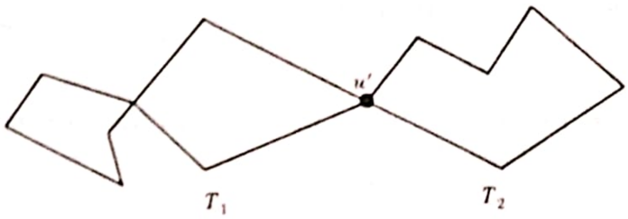


Fig. 5-34

▌ Suppose G is eulerian and T is a closed eulerian trail. For any vertex v of G, the trail T enters and leaves v the same number of times without repeating any edge. Hence v has even degree.

Suppose conversely that each vertex of G has even degree. We construct an eulerian trail. We begin a trail T_1 at any edge e. We extend T_1 by adding one edge after the other. If T_1 is not closed at any step, say T_1 begins at u but ends at $v \neq u$, then only an odd number of the edges incident on v appear in T_1; hence we can extend T_1 by another edge incident on v. Thus we can continue to extend T_1 until T_1 returns to its initial vertex u, i.e., until T_1 is closed. If T_1 includes all the edges of G, then T_1 is our eulerian trail.

Suppose T_1 does not include all edges of G. Consider the graph H obtained by deleting all edges of T_1 from G. H may not be connected, but each vertex of H has even degree since T_1 contains an even number of the edges incident on any vertex. Since G is connected, there is an edge e' of H which has an endpoint u' in T_1. We construct a trail T_2 in H beginning at u' and using e'. Since all vertices in H have even degree, we can continue to extend T_2 in H until T_2 returns to u' as pictured in Fig. 5-34. We can clearly put T_1 and T_2 together to form a larger closed trail in G. We continue this process until all the edges of G are used. We finally obtain an eulerian trail, and so G is eulerian.

5.97 Prove Corollary 5.4.

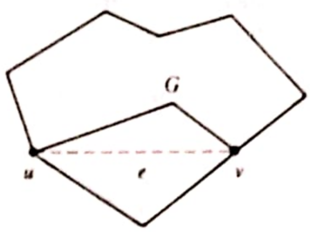


Fig. 5-35

▌ Suppose G is the finite connected graph with exactly two odd vertices, say u and v. Add another edge e from u to v to the graph G to form the new graph $G' = G \cup \{e\}$, as shown in Fig. 5-35. Then all the vertices of the graph G' are even. By Theorem 5.3, there is a closed traversable trail α of G'. Since α is closed, we can assume, without loss of generality, that α begins with e. Let β be the path α without its first edge e. Then β is a traversable trail of G beginning at v and ending at u, as required.

5.98 Determine whether or not each of the graphs in Fig. 5-36 is traversable.

▌ Find the degree of each vertex and then determine whether all the vertices are of even degree or exactly two are of odd degree. If either condition is met, the graph is traversable.

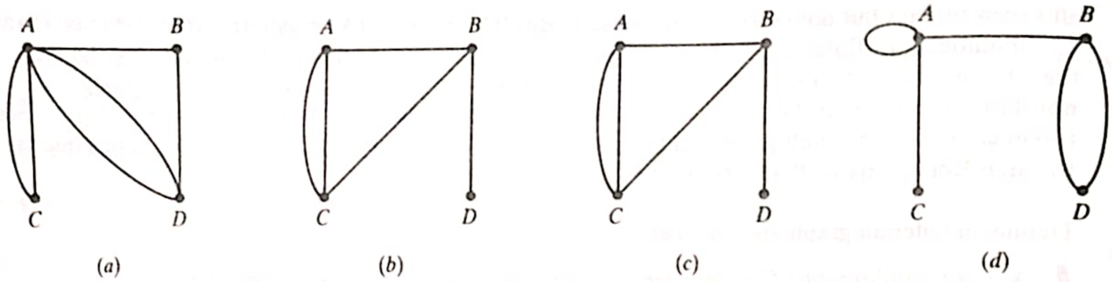


Fig. 5-36

(a) Yes, since exactly two of its vertices, A and D, are of odd degree.
(b) Yes, since all vertices are of even degree.
(c) No, since all four vertices are of odd degree.
(d) Yes, since exactly two of its vertices, B and C, are of odd degree.

5.99 Determine which of the following graphs G are traversable where: $V(G) = \{A, B, C, D\}$ and
(a) $E(G) = [\{A, B\}, \{B, C\}, \{C, D\}, \{D, A\}]$
(b) $E(G) = [\{A, B\}, \{A, C\}, \{B, C\}, \{B, D\}, \{C, D\}, \{D, A\}]$
(c) $E(G) = [\{A, B\}, \{C, D\}, \{B, A\}, \{C, C\}, \{D, C\}]$

As in Problem 5.98, find the degree of each vertex and then determine whether all vertices are of even degree or whether exactly two are of odd degree.
(a) Yes, since all vertices are of degree two.
(b) No, since all four vertices are of degree three.
(c) No. Although all vertices are of even degree, the graph is not connected.

5.100 Determine which of the graphs in Fig. 5-37 are traversable.

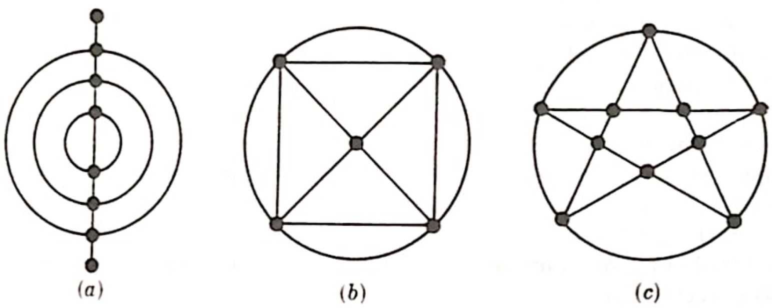


Fig. 5-37

(a) Traversable since six vertices are even and two are odd.
(b) Not traversable since there are four odd vertices.
(c) Traversable since all ten vertices are even.

5.101 Find a traversable trail for the multigraph in Fig. 5-38(*a*).

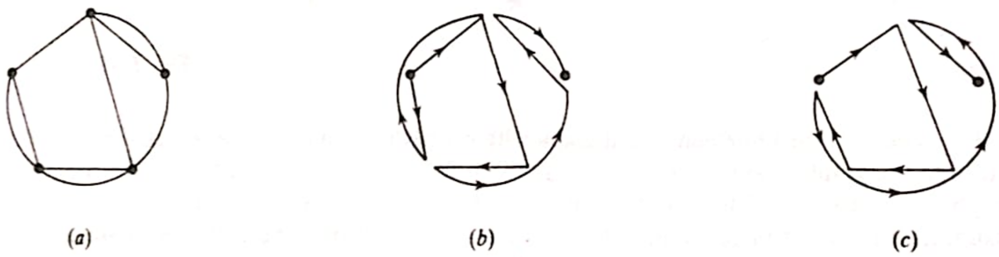
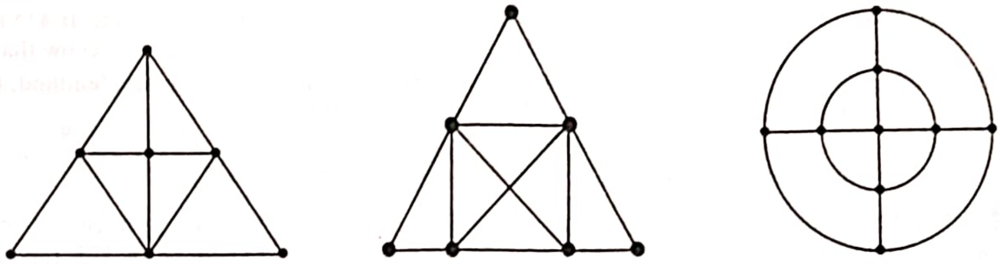


Fig. 5-38

There are many possible solutions, but all of them will start at one of the odd vertices and end at the other. Figures 5-38(*b*) and 5-38(*c*) give two possible solutions.

5.102 Determine which of the graphs in Fig. 5-39 are traversable.



(a) (b) (c)

Fig. 5-39

▌ (*a*) Traversable since five vertices are even and two are odd.
(*b*) Traversable since five vertices are even and two are odd.
(*c*) Not traversable since the four outer vertices are odd.

5.103 Find a traversable trail for the multigraph in Fig. 5-40(*a*).

(a) (b)

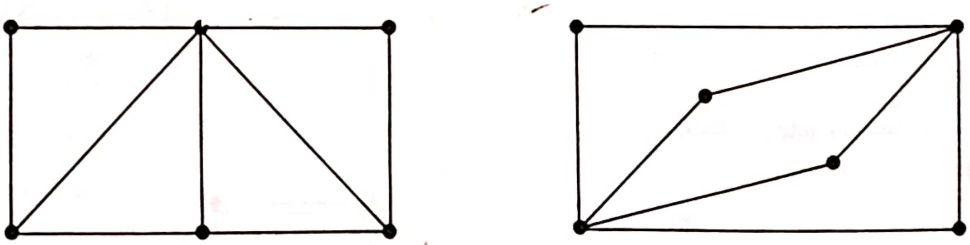
Fig. 5-40

▌ There are many possible solutions, but all of them must begin at one of the odd vertices and end at the other odd vertex. Figure 5-40(*b*) gives one such solution.

5.104 Define a hamiltonian graph.

▌ A *hamiltonian graph* is a graph with a closed path that includes every vertex exactly once. Such a path is a cycle and is called a *hamiltonian cycle*. Note that an eulerian cycle uses every edge exactly once but may repeat vertices, while a hamiltonian cycle uses each vertex exactly once (except for the first and last) but may skip edges.

5.105 Draw a graph with six vertices which is hamiltonian but not eulerian.



(*a*) Hamiltonian and noneulerian (*b*) Eulerian and nonhamiltonian

Fig. 5-41

▌ There are many possible solutions to this problem and one of these is shown in Fig. 5-41(*a*). Every solution, however, must have a cycle that includes every vertex exactly once (hamiltonian), but must not have a closed trail that uses every edge exactly once. (eulerian). Note that when a candidate hamiltonian graph has been identified, one can easily determine if it is eulerian by looking for vertices of odd degree. Should at least one such vertex exist, the graph is not eulerian.

5.106 Draw a graph with six vertices which is eulerian but not hamiltonian.

▮ As in Problem 5.105, there are many possible solutions, one of which is shown in Fig. 5-41(b). Every solution, however, must have a closed trail that uses every edge exactly once (eulerian), but must not have a cycle that includes every vertex exactly once (hamiltonian). From Euler's theorem we know that any graph with all vertices of even degree is eulerian. But once a candidate eulerian graph has been identified, there is no simple criterion for determining whether or not the graph is hamiltonian.

5.107 Let G be a connected graph with three vertices. Show that G is traversable.

▮ Since the sum of the degree of the vertices must be even, G cannot have one or three odd vertices. Thus G must have only even vertices or two odd vertices and one even vertex. In either case, G is traversable.

5.108 Find a traversable trail α for the graph G where

$$V(G) = \{A, B, C, D\} \qquad \text{and} \qquad E(G) = [\{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}]$$

▮ Here C and D are odd vertices; hence one must begin at C and end at D or vice versa. One such trail is $\alpha = (C, A, D, B, C, D)$.

5.109 Show that one can add or delete loops from a multigraph G and the graph G remains traversable or nontraversable.

▮ The degree of a vertex v in G is increased or decreased by two according as one adds or deletes a loop at v. Thus the parity (evenness or oddness) of v is not changed. Accordingly, the condition that G has zero or two odd vertices is not changed by adding or deleting loops.

5.6 SPECIAL GRAPHS

There are many different types of graphs. This section defines four of them: complete, regular, bipartite, and tree graphs. (Here the term graph does not include multigraphs.)

5.110 Define a complete graph.

▮ A graph G is *complete* if each vertex is connected to every other vertex. The complete graph with n vertices is denoted by K_n.

5.111 Draw a diagram of the complete graphs K_1, K_2, K_3, and K_4.

▮ First draw the appropriate number n of vertices. Then draw an edge from each vertex to every other vertex. The required diagrams appear in Fig. 5-42(a).

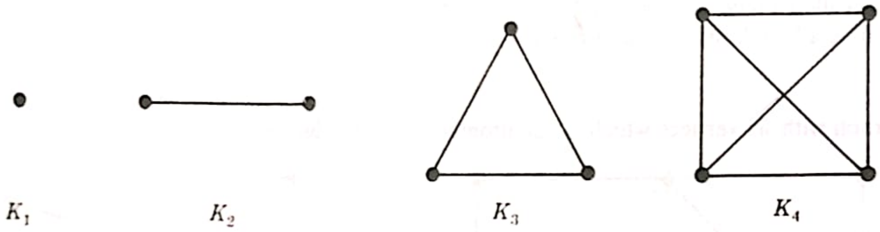


Fig. 5-42(a)

5.112 Draw the complete graphs K_5 and K_6.

▮ See Fig. 5-42(b).

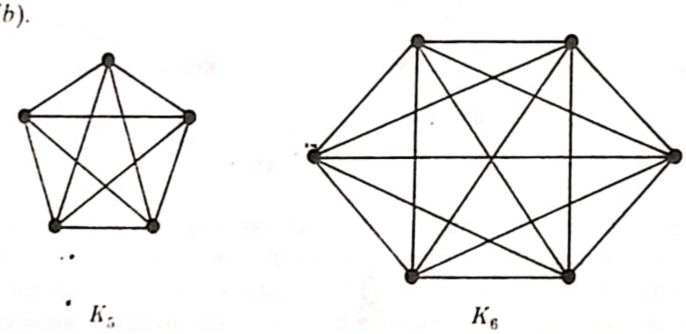


Fig. 5-42(b)

5.113 Find the number m of edges in the complete graph K_n.

▮ Each pair of vertices determines an edge. Thus $m = C(n, 2) = n(n-1)/2$ since there are $C(n, 2)$ ways of selecting two vertices out of n vertices.

5.114 Find the number m of edges in the graphs (*a*) K_8, (*b*) K_{12}, and (*c*) K_{15}.

▮ (*a*) $m = \dfrac{8 \cdot 7}{2} = 28$, (*b*) $m = \dfrac{12 \cdot 11}{2} = 66$, (*c*) $m = \dfrac{15 \cdot 14}{2} = 105$.

5.115 The complete graph K_n is connected since each vertex is connected to every other vertex. Find diam (K_n).

▮ Here $d(u, v) = 1$ for any two distinct vertices u and v in K_n; hence diam $(K_n) = 1$.

5.116 Find the degree of each vertex in K_n.

▮ Each vertex v is connected to the other $n - 1$ vertices; hence deg $(v) = n - 1$ for every v in K_n.

5.117 Find those values of n for which K_n is traversable.

▮ If n is odd, then every vertex v is even since deg $(v) = n - 1$. Thus K_n is traversable for n odd. Also, K_2 is traversable since it has only one edge connecting the two vertices. However, for $n > 2$ and n even, the complete graph will have n (more than two) odd vertices and hence will not be traversable.

5.118 Define a regular graph.

▮ A graph G is *regular of degree* k or k-*regular* if every vertex has degree k. In other words, a graph is regular if every vertex has the same degree.

5.119 Describe and draw the connected regular graphs of degrees 0, 1, and 2.

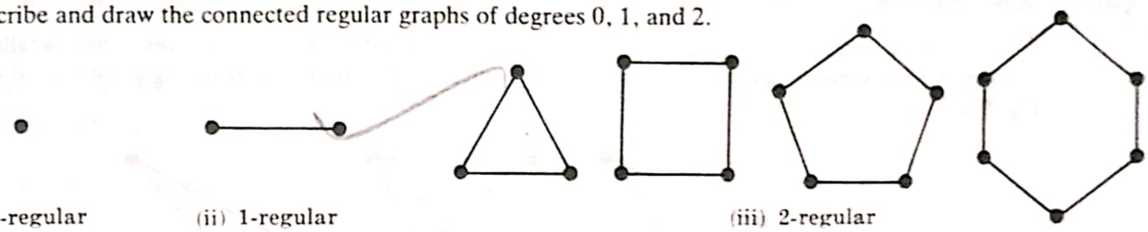


Fig. 5-43

▮ The connected 0-regular graph is the trivial graph with one vertex and no edges. The connected 1-regular graph is the graph with two vertices and one edge connecting them. The connected 2-regular graph with n vertices is the graph which consists of a single n-cycle. Figure 5-43 shows the connected 0-regular and 1-regular graphs and some of the connected 2-regular graphs.

5.120 Suppose r is an odd integer. Show that an r-regular graph must have an even number n of vertices.

▮ Let S be the sum of the degrees of an r-regular graph with n vertices. Then $S = rn$. By Theorem 5.1, the sum S must be even. If r is odd, then n must be even.

5.121 Find those values of n for which the complete graph K_n is regular.

▮ Every vertex in K_n has degree $n - 1$. Thus, for every n, the graph K_n is regular of degree $n - 1$.

5.122 Draw two 3-regular graphs with six vertices.

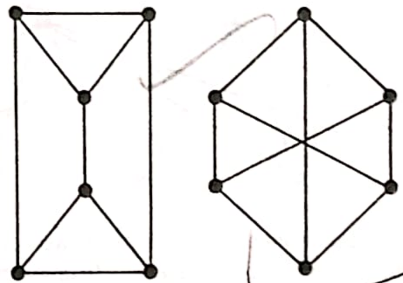


Fig. 5-44

▮ See Fig. 5-44.

5.123 Draw two 3-regular graphs with seven vertices.

▮ No such graphs exist since a 3-regular graph must have an even number of vertices. (See Problem 5.120.)

5.124 Draw two 3-regular graphs with eight vertices.

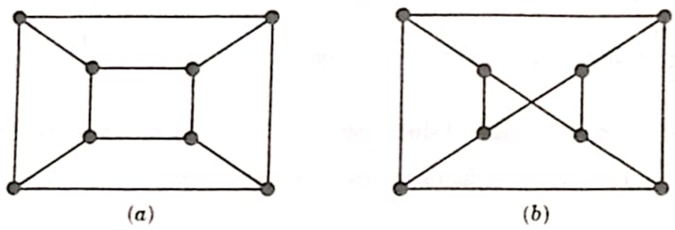


Fig. 5-45

▮ See Fig. 5.45. The two graphs are distinct since only graph (*b*) has a 5-cycle.

5.125 Define a bipartite graph and a complete bipartite graph.

▮ A graph G is said to be *bipartite* if its vertices V can be partitioned into two subsets M and N such that each edge of G connects a vertex of M to a vertex of N. By a complete bipartite graph, we mean that each vertex of M is connected to each vertex of N; this graph is denoted by $K_{m,n}$ where m is the number of vertices in M and n is the number of vertices in N, and for standardization, we assume $m \leq n$.

5.126 Find the number of edges in the complete bipartite graph $K_{m,n}$.

▮ Each of m vertices is connected to each of n vertices; hence $K_{m,n}$ has mn edges.

5.127 Draw the complete bipartite graphs $K_{2,3}$, $K_{3,3}$, and $K_{2,4}$.

▮ To draw a complete bipartite graph, just place the appropriate number of vertices in two parallel columns and connect the vertices in one group with all the vertices in the other. The resulting graphs are shown in Fig. 5-46.

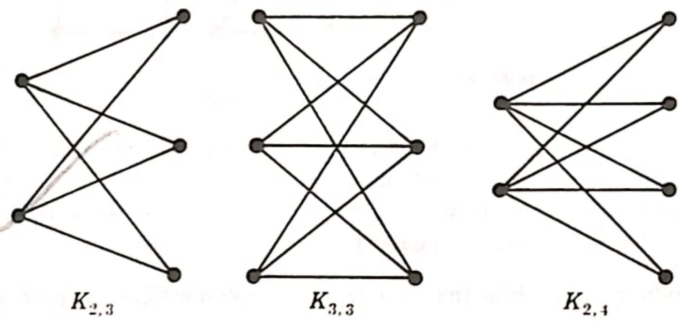


Fig. 5-46

5.128 Determine the diameter of any complete bipartite graph.

▮ The diameter of $K_{1,1}$ will be one since there are only two vertices and the shortest path between them is length one. All other bipartite graphs will have diameter two since any two points in either M or N will be exactly distance 2 apart. (One edge to reach the other subgroup of vertices and one to return.)

5.129 Draw the graph $K_{2,5}$.

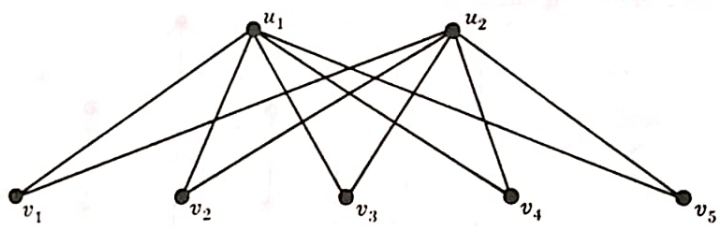


Fig. 5-47

▮ $K_{2,5}$ consists of seven vertices partitioned into a set M of two vertices, say u_1 and u_2, and a set N of five vertices, say v_1, v_2, v_3, v_4, and v_5, and all possible edges from a vertex u_i to a vertex v_j. The required graph is shown in Fig. 5-47.

5.130 Which connected graphs can be both regular and bipartite?

Fig. 5-48

▮ The bipartite graph $K_{m,m}$ is regular of degree m since each vertex is connected to m other vertices and hence its degree is m. Subgraphs of $K_{m,m}$ can also be regular if m disjoint edges are deleted. For example, the subgraph of $K_{4,4}$ shown in Fig. 5-48 is 3-regular. We can continue to delete m disjoint edges and each time obtain a regular graph of one less degree. These graphs may be disconnected, but in any case their connected components have the desired properties.

Trees

This subsection introduces the notion of a tree graph. Such tree graphs will be covered more thoroughly in the next two chapters. Here we simply give its definition and some examples.

5.131 Define a cycle-free graph and a tree graph.

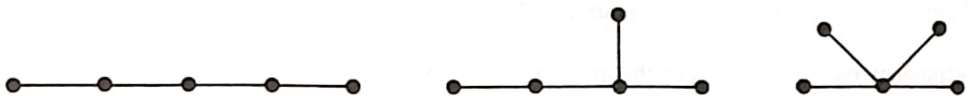
▮ A graph G is said to be *cycle-free* or *acyclic* if it has no cycles. If G has no cycles and is connected, then G is called a *tree*.

5.132 Draw all trees with four or fewer vertices.

(a) (b) (c) (d) (e) Fig. 5-49

▮ See Fig. 5-49. Note there are two trees with four vertices. The graph with one vertex and no edge is called the trivial tree [Fig. 5-49(*a*)].

5.133 Draw all trees with five vertices.



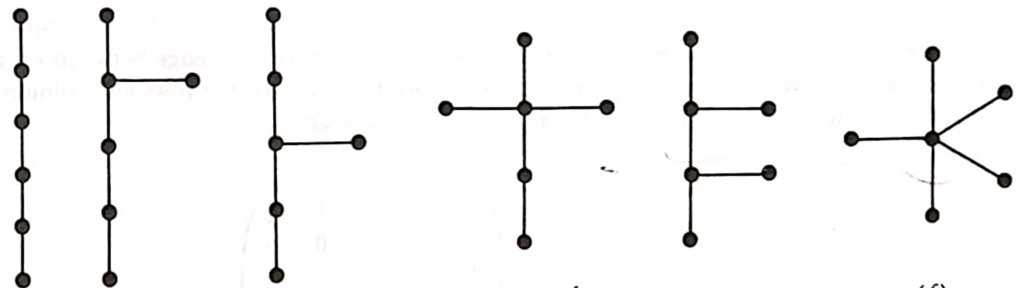
(a) (b) (c) Fig. 5-50

▮ First draw five vertices. Then connect them so that no cycles are created. In this exercise, we must be careful not to repeat trees since two trees which appear different may just be drawn differently. There are three trees with five vertices as shown in Fig. 5-50.

5.134 Draw all trees with six vertices.

▮ As in Problem 5.133, we must be careful not to repeat any trees. There are six trees with six vertices as shown in Fig. 5-51.

5.135 Find the diameters of the trees in Fig. 5-51.



(a) (b) (c) (d) (e) (f)

Fig. 5-51

Recall that diam (G) is the maximum distance between any two points in G. Thus the diameters of the graphs (a), (b), $\ldots$, (f) are 5, 4, 4, 3, 3, and 2 respectively.

5.136 Show that the trees in Fig. 5-51 are all different.

By Problem 5.135, we need only show that (b) and (c) are different and that (d) and (e) are different. The tree (d) has a vertex with degree four, but (e) does not; hence (d) and (e) are different trees. Deleting the vertex of degree three in either (b) or (c) results in three connected components. However, the connected components have different numbers of vertices, that is, 1, 1, and 3 in (b) but 1, 2, and 2 in (c). Thus the trees (b) and (c) are also different.

5.137 Prove that a finite tree G (with at least one edge) has at least two vertices of degree 1.

Let $\alpha = (u = v_0, v_1, \ldots, v_n = v)$ be a simple path of maximum length in G. If $\deg(u) > 1$, then there is an edge $e = \{u, w\}$. If w is not one of the vertices in α, then α does not have maximum length. If w is one of the vertices in α, say $w = v_k$, then $(w, v_0, \ldots, v_k)$ is a cycle which cannot exist in a tree. Thus $\deg(u) = 1$. Similarly, $\deg(v) = 1$.

5.7 MATRICES AND GRAPHS, LINKED REPRESENTATION

This section considers two important matrices associated with a graph G.

5.138 Let G be a graph with vertices $v_1, v_2, \ldots, v_m$ and edges $e_1, e_2, \ldots, e_n$. Define: **(a)** the adjacency matrix of G; **(b)** the incidence matrix of G.

(a) Adjacency matrix. Let $A = (a_{ij})$ be the $m \times m$ matrix defined by

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge, i.e., if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

Then A is called the *adjacency matrix* of G. Observe that $a_{ij} = a_{ji}$; hence A is a symmetric matrix. (We define an adjacency matrix for a multigraph by letting a_{ij} denote the number of edges $\{v_i, v_j\}$.)

(b) Incidence matrix. Let $M = (m_{ij})$ be the $m \times n$ matrix defined by

$$m = \begin{cases} 1 & \text{if the vertex } v_i \text{ is incident on the edge } e_j \\ 0 & \text{otherwise} \end{cases}$$

Then M is called the *incidence matrix* of G.

5.139 Find the adjacency matrix $A = (a_{ij})$ of the graph G in Fig. 5-52.

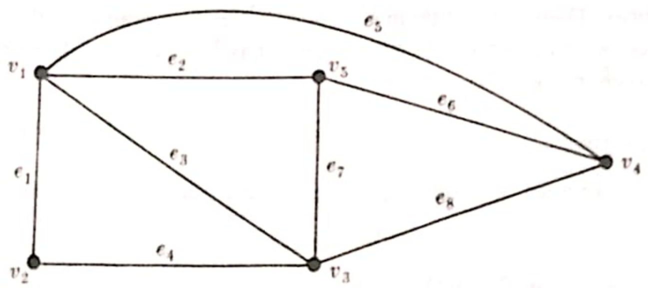


Fig. 5-52

Since G has five vertices, a will be a 5×5 matrix. Set $a_{ij} = 1$ if there is an edge between v_i and v_j, and set $a_{ij} = 0$ otherwise. This yields the following matrix (where we have labeled the rows and columns by the corresponding vertices for easier reading though this is not necessary):

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

5.140 Find the incidence matrix $M = (m_{ij})$ of the graph G in Fig. 5-52.

▮ Since G has five vertices and eight edges, M will be a 5×8 matrix. Set $m_{ij} = 1$ if vertex v_i belongs to the edge e_j, and set $m_{ij} = 0$ otherwise. This yields the following matrix (where we have labeled the rows and columns by the corresponding vertices and edges for easier reading though this is not necessary):

$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

5.141 Find the adjacency matrix $A = (a_{ij})$ of the multigraph G in Fig. 5-53.

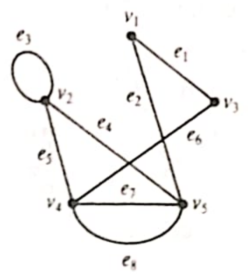


Fig. 5-53

▮ Since G has five vertices, A will be a 5×5 matrix. Set $a_{ij} = n$ where n is the number of edges between v_i and v_j, and set $a_{ij} = 0$ otherwise. This yields the following matrix:

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 2 & 0 \end{pmatrix} \end{array}$$

5.142 Find the incidence matrix $M = (m_{ij})$ of the multigraph G in Fig. 5-53.

▮ Since G has five vertices and eight edges, m will be a 5×8 matrix. Set $m_{ij} = 1$ if vertex v_i belongs to the edge e_j, and set $m_{ij} = 0$ otherwise. This yields

$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

5.143 Find the adjacency matrix A of the graph G in Fig. 5-54.

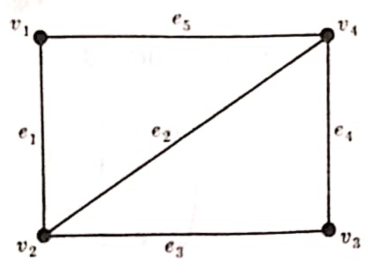


Fig. 5-54

The adjacency matrix $A = (a_{ij})$ is defined by $a_{ij} = 1$ if there is an edge $\{v_i, v_j\}$ and $a_{ij} = 0$ otherwise. Hence

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

5.144 Find the incidence matrix M of the graph G in Fig. 5-54.

The incidence matrix $M = (m_{ij})$ is defined by $m_{ij} = 1$ if edge e_j is incident on vertex v_i and $m_{ij} = 0$ otherwise. Hence

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

5.145 Find the adjacency matrix A of the graph G in Fig. 5-55.

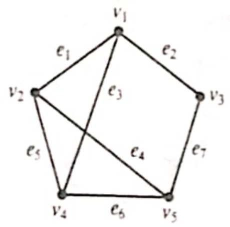


Fig. 5-55

The adjacency matrix $A = (a_{ij})$ is defined by $a_{ij} = 1$ if there is an edge $\{v_i, v_j\}$ and $a_{ij} = 0$ otherwise. Hence

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

5.146 Find the incidence matrix M of the graph G in Fig. 5-55.

The incidence matrix $M = (m_{ij})$ is defined by $m_{ij} = 1$ if edge e_j is incident on vertex v_i and $m_{ii} = 0$ otherwise. Hence

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

5.147 Find the adjacency matrix A for the multigraph in Fig. 5-56.

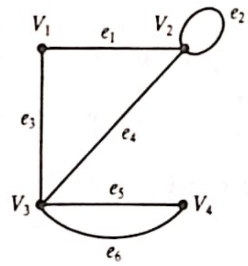


Fig. 5-56

For a multigraph, the adjacency matrix $A = (a_{ij})$ is defined by $a_{ij} = n$ where $n \geq 0$ is the number of edges $\{v_i, v_j\}$

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

5.148 Find the incidence matrix M of the multigraph G in Fig. 5-56.

The incidence matrix $M = (m_{ij})$ is defined by $m_{ij} = 1$ if vertex v_i is incident on edge e_j and $m_{ij} = 0$ otherwise. Hence

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

5.149 Draw the graph G whose adjacency matrix $A = (a_{ij})$ follows:

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

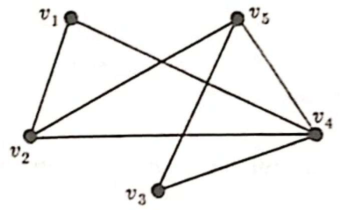


Fig. 5-57

Since A is a 5-square matrix, G has five vertices, say $v_1, \ldots, v_5$. Draw an edge from v_i to v_j if $a_{ij} = 1$. The graph is shown in Fig. 5-57.

5.150 Draw the multigraph G whose adjacency matrix $A = (a_{ij})$ follows:

$$A = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

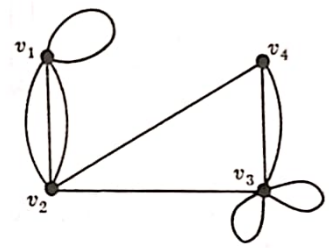


Fig. 5-58

Since A is a 4-square matrix, G has four vertices, say $v_1, \ldots, v_4$. Draw n edges from v_i to v_j if $a_{ij} = n$. Note that v_i has n loops if $a_{ii} = n$. The multigraph is shown in Fig. 5-58.

5.151 Draw the graph G whose adjacency matrix A is

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

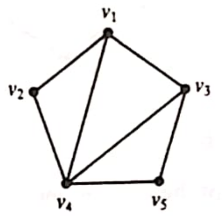


Fig. 5-59

Since A is a 5-square matrix, G has five vertices, say $v_1, \ldots, v_5$. We draw an edge from v_i to v_j if $a_{ij} = 1$. The resulting graph is shown in Fig. 5-59.

5.152 Draw the multigraph G whose adjacency matrix A is

$$A = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

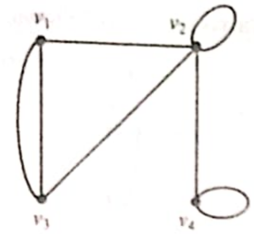


Fig. 5-60

Since A is a 4-square matrix, G has four vertices, say $v_1, \ldots, v_4$. For each $a_{ij} = n$ we draw n edges from v_i to v_j. The resulting multigraph is shown in Fig. 5-60.

5.153 Determine the number of loops and multiple edges in a multigraph G from its adjacency matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & 2 & 1 & 3 \\ 2 & 1 & 0 & 1 \\ 0 & 3 & 1 & 0 \end{pmatrix}$$

Draw the graph G and check your answer.

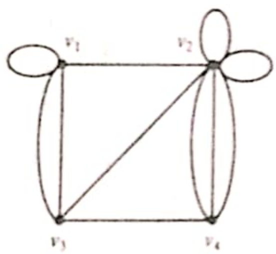


Fig. 5-61

Since A is a 4-square matrix, g has four vertices, say $v_1, \ldots, v_4$. We can find the number of loops by looking along the main diagonal of A since these entries indicate the number of edges originating and terminating at the same vertex. Thus there are three loops: one at vertex v_1 and two at vertex v_2.

To find the number of multiple edges, we simply sum the number of entries greater than one below the main diagonal and add the number of entries greater than one along the main diagonal. We include the main diagonal since multiple loops are also multiple edges. We exclude the entries above the main diagonal since the matrix is symmetric and thus all the off diagonal entries are repeated. Thus there are seven multiple edges: two edges from v_3 to v_1, three edges from v_4 to v_2, and two loops at v_2.

The diagram of G is shown in Fig. 5-61. The number of multiple edges and loops are as predicted.

5.154 Find the adjacency matrix A and the incidence matrix M for the graph G in Fig. 5-62.

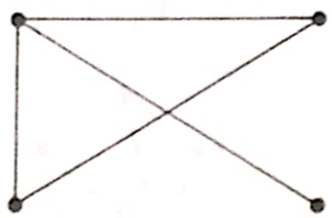


Fig. 5-62

The matrices depend on the ordering of the vertices and the edges. One such ordering yields the following matrices:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

5.155 Draw the multigraph corresponding to each of the following adjacency matrices:

$$(a)\quad A = \begin{pmatrix} 0 & 2 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \qquad (b)\quad A = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 2 & 0 & 2 & 2 \end{pmatrix}$$

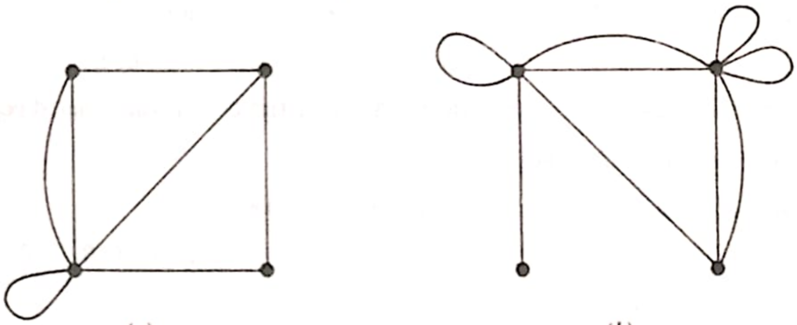
▌ See Fig. 5-63.

5.156 Suppose a graph G has m vertices. Define the connection matrix C of G. Characterize C when G is a connected graph.

▌ The connection matrix of G is the $m \times m$ matrix $C = (c_{ij})$ where

$$c_{ij} = \begin{cases} 1 & \text{if } i = j \text{ or there is a path from } v_i \text{ to } v_j \\ 0 & \text{otherwise} \end{cases}$$

Thus G is connected if and only if C has no zero entry.



(a) (b) **Fig. 5-63**

Theorem 5.5: Let A be the adjacency matrix of a graph G with m vertices where $m > 1$. Then the ij entry of the matrix A^n gives the number of paths of length n from the vertex v_i to the vertex v_j.

5.157 Show the relationship between the adjacency matrix A and the connection matrix C of a graph G with m vertices.

▌ Since G has m vertices, any simple path from v_i to v_j must have length $m - 1$ or less. Thus the matrix

$$A + A^2 + \cdots + A^{m-1}$$

can have a zero ij entry only if there is no path from v_i to v_j. [By Theorem 5.2, no simple path means no path.] Hence the matrix C and the matrix $A + A^2 + \cdots + A^{m-1}$ have the same zero entries off the main diagonal.

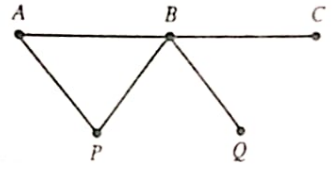
5.158 Let G be a graph with m vertices. Describe two major drawbacks in the computer storage of G as its adjacency matrix A.

▌ First of all, if the vertex and/or edge set of G is subject to change, it may be difficult to effect the required alterations of A. Secondly, the number of edges in G may be of order m or of order $m \log m$, so the adjacency matrix A will be sparse (will contain many zeros). Accordingly, a great deal of space will be wasted when A is stored in memory. (Therefore, G will usually be represented in memory by an AS representation discussed in Problem 5.159.)

Adjacency-Structure (Linked) Representation of a Graph

5.159 Define the adjacency-structure (AS) or linked representation of a graph using, as an example, the graph G in Fig. 5-64(a).

▌ The *adjacency-structure* (AS) *representation* of G shows each vertex u of G followed by its set of adjacent



(a)

Adjacency Lists
A: B, P B: A, C, P, Q C: B P: A, B Q: B

(b)

Fig. 5-64

vertices. Such a representation may be presented as in Fig. 5-64(b) or in the compact form:

$$G = [A\colon B, P; B\colon A, C, P, Q; C\colon B; P\colon A, B; Q\colon B]$$

Notice that a colon ":" separates a vertex from its list of adjacent vertices, and that a semicolon ";" separates the different lists. This representation is sometimes called the *linked* representation of G since the lists of adjacency vertices are frequently represented in memory by means of linked lists.

5.160 Find the AS representation of the graph G in Fig. 5-21.

List each vertex of G followed by its list of adjacent nodes, that is,

$$G = [A\colon B; B\colon A, C, D; C\colon B, D; D\colon B, C]$$

5.161 Find the AS representation of the graph G in Fig. 5-22.

List each vertex of G followed by its list of adjacent nodes, that is,

$$G = [A\colon C, X; B; C\colon A, X; Y\colon Z; Z\colon Y]$$

Note that the list following B is empty which reflects the fact that B is an isolated vertex.

5.162 Find the AS representation of the graph G in Fig. 5-28.

List each vertex of G followed by its list of adjacent nodes, that is,

$$G = [A\colon B, Y; B\colon A, X, Y, Z; C\colon Y; X\colon B, Z; Y\colon A, B, C; Z\colon B, X]$$

5.163 Find the AS representation of the graph G in Fig. 5-1.

List each vertex of G followed by its list of adjacent nodes, that is,

$$G = [A\colon B, C; B\colon A, C, D; C\colon A, B, D; D\colon C, B]$$

5.164 Find the AS representation of the multigraph G in Fig. 5-2.

List each vertex of G followed by its list of adjacent nodes, including multiplicity. Thus

$$G = [A\colon B, C, C; B\colon A, C, D; C\colon A, A, B, D; D\colon B, C, D]$$

5.165 Find the AS representation of the graph G in Fig. 5-15.

List each vertex of G followed by its list of adjacent nodes, that is,

$$G = [A\colon X, Z; B\colon X, Y; C\colon Y; X\colon A, B, Y; Y\colon B, C, X; Z\colon A]$$

5.8 LABELED GRAPHS

5.166 Explain the meaning of a labeled graph with an example.

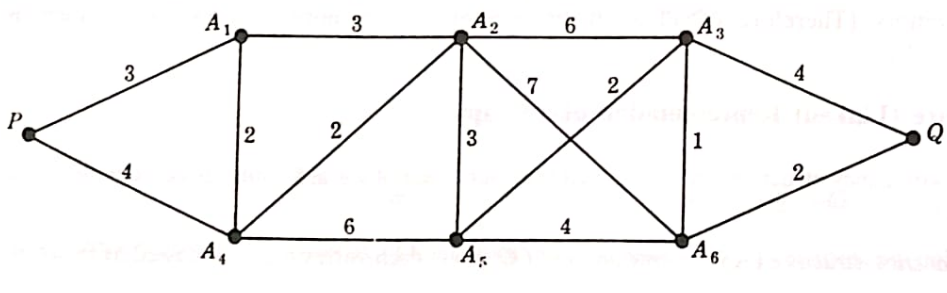


Fig. 5-65

A graph G is called a *labeled graph* if its edges and/or vertices are assigned data of one kind or another. In particular, if each edge e of G is assigned a nonnegative number $\ell(e)$ then $\ell(e)$ is called the *weight* or *length* of e. Figure 5-65 shows a labeled graph where the length of each edge is given in the obvious way.

5.167 Let G be a labeled graph with lengths (weights) assigned to its edges. Explain the minimum path problem using Fig. 5-65 as an example.

Let P and Q be vertices in G. The minimum path problem refers to finding a path of minimum length between P and Q where the length of the path is the sum of the lengths of its edges. Clearly, such a minimum path must be a simple path. A minimum path between P and Q in Fig. 5-65 is

$$(P, A_1, A_2, A_5, A_3, A_6, Q)$$

which has length 15. (The reader can try to find another minimum path.)

5.168 Let G be the labeled graph in Fig. 5-66. Find a minimum path α between A and D.

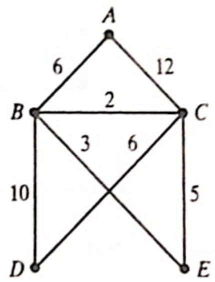


Fig. 5-66

There are six sample paths between A and D. These paths and their corresponding lengths follow:

(A, B, D):	16	(A, C, D):	18
(A, B, C, D):	14	(A, C, B, D):	24
(A, B, E, C, D):	20	(A, C, E, B, D):	30

Thus $\alpha = (A, B, C, D)$ and its length is 14.

5.169 Let G be the labeled graph in Fig. 5-67. Find: **(a)** all simple paths between A and F, and **(b)** a minimum path α between A and F.

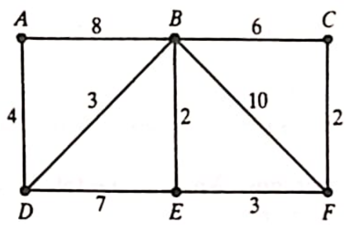


Fig. 5-67

(a) There are ten such paths:

(A, B, F)	(A, D, B, F)	(A, D, E, F)
(A, B, C, F)	(A, D, B, C, F)	(A, D, E, B, F)
(A, B, E, F)	(A, D, B, E, F)	(A, D, E, B, C, F)
(A, B, D, E, F)		

(b) Find the length of each of the simple paths in **(a)** to obtain $\alpha = (A, D, B, E, F)$ which has length 12.

5.170 Give a "real life" example of a graph G where both the vertices and edges are assigned data.

Let the vertices of G denote cities with their population, and let the edges of G denote the distances between the cities.

5.171 Show how every connected graph G may be viewed as a (weighted) labeled graph. What is a minimum path in such a graph?

▮ Here we can assume that every edge in G has length 1. Then a minimum path α between vertices P and Q is a path of minimum length in the original sense, i.e., a path with a minimum number of edges.

5.172 Consider the following shopping problem: You are refurnishing your house and have decided to purchase the following items, each from a different store: (A) matching sofa and chair, (B) coffee table, (C) rug, (D) T.V., and (E) floor lamp. There are a few restrictions on the order in which these items can be purchased. First, the rug must be purchased before the sofa and chair set and coffee table (because this furniture must match the rug). Second, the lamp must be purchased last because, if there is no money left, it is the most expendable item.

You have determined the traveling time (in minutes) between stores and have organized this information into the graph G in Fig. 5-68. Find the most efficient sequence α of purchases, that is, find the shopping order that will minimize the traveling time.

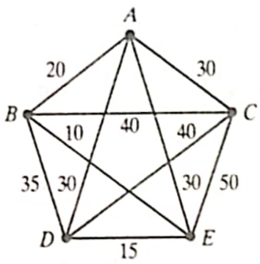


Fig. 5-68

▮ Here the required path α must include all five vertices. It must begin at C or begin at D followed by C. In all cases it must end at E. Assuming that back-tracking will not decrease the time, there are eight possible simple paths which meet the given conditions. These paths and their corresponding traveling times follow:

$CABDE$:	100	$CBADE$:	110	$CDABE$:	100	$DCABE$:	100
$CADBE$:	105	$CBDAE$:	135	$CDBAE$:	125	$DCBAE$:	130

Thus $\alpha = (D, C, A, B, E)$ or $\alpha = (C, A, B, D, E)$ or $\alpha = \{C, D, A, B, E\}$ and the minimum traveling time is 100 minutes.

5.9 ISOMORPHIC AND HOMEOMORPHIC GRAPHS

5.173 Define isomorphism of graphs.

▮ Suppose $G(V, E)$ and $G^*(V^*, E^*)$ are graphs and $f: V \to V^*$ is a one-to-one correspondence between the sets of vertices such that $\{u, v\}$ is an edge of G if and only if $\{f(u), f(v)\}$ is an edge of G^*. Then f is called an *isomorphism* between G and G^*, and G and G^* are said to be *isomorphic* graphs. Normally, we do not distinguish between isomorphic graphs (even though their diagrams may "look different").

5.174 Suppose G and G^* are isomorphic graphs. Which of the following two conditions must hold for corresponding vertices: (*a*) degree, (*b*) being a cut point?

▮ Both must hold for corresponding vertices.

5.175 Suppose G and G^* are isomorphic graphs. Find the number of connected components of G^* if G has eight connected components.

▮ The graph G^* must also have eight connected components.

7.176 Suppose G and G^* are isomorphic graphs and G is traversable. Is G^* traversable? If yes, what would be a traversable path for G^*?

▮ Yes, G^* is traversable. Also, if α is a traversable path for G, then the corresponding path, say α', would be a traversable path for G^*.

5.177 Figure 5-69 shows ten graphs pictured as letters. Which of the ten graphs are isomorphic to M?

▮ M consists of five vertices in a single line. Thus S, V, and Z (and M itself) are isomorphic to M.

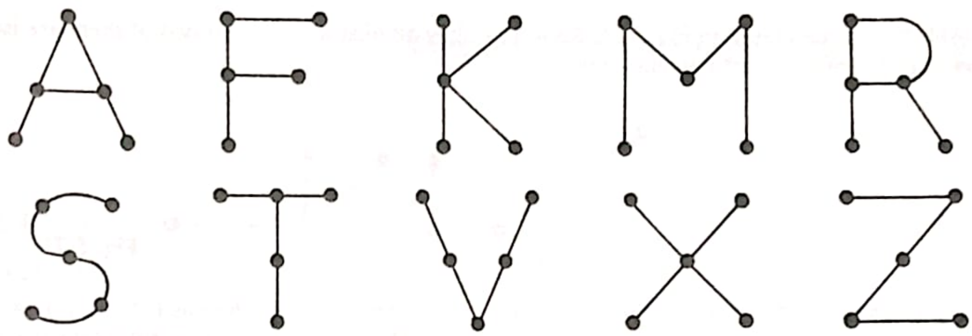


Fig. 5-69

5.178 Consider the "letters" A, F, K, R, T, and X in Fig. 5-69. Which of them are isomorphic?

▮ The letters A and R, F and T, and K and X are isomorphic.

5.179 Define homeomorphic graphs.

▮ Given any graph G, we can obtain a new graph by dividing an edge of G with additional vertices. Two graphs G and G^* are said to be *homeomorphic* if they can be obtained from isomorphic graphs by this method.

5.180 Give an example of graphs which are homeomorphic, but not isomorphic.

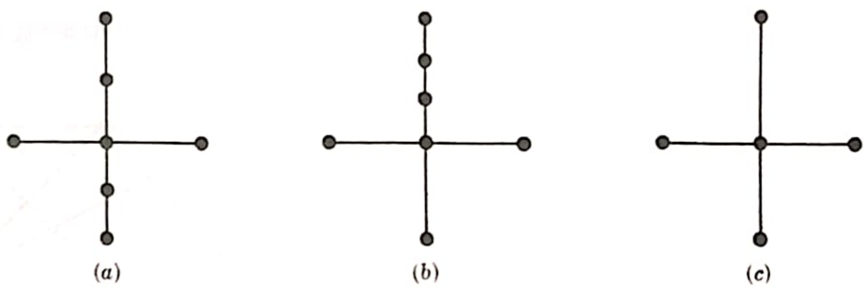


Fig. 5-70

▮ The graphs (a) and (b) in Fig. 5-70 are not isomorphic; but they are homeomorphic since each can be obtained from (c) by adding appropriate vertices.

5.181 Find all (nonisomorphic) connected graphs with four vertices.

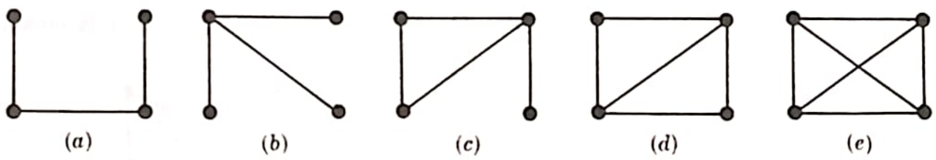


Fig. 5-71

▮ There are five of them, as shown in Fig. 5-71.

5.182 Can a finite graph G be isomorphic to one of its subgraphs (other than itself)?

▮ No since isomorphic graphs must have the same number of elements.

5.183 Can a finite graph G be homeomorphic to one of its subgraphs (other than itself)?

▮ Yes. For example, let G be the graph in Fig. 5-71(a). Deleting one or both of the vertices of degree 1 from G yields a subgraph which is homeomorphic to G itself.

5.184 Give an example of an infinite graph G which is isomorphic to one of its subgraphs (other than itself).

▮ Let $V(G) = \{1, 2, 3, \ldots\}$ and let $E(G) = [\{1, 2\}, \{2, 3\}, \ldots, \{n, n+1\}, \ldots]$. Consider the subgraph G' where $V(G') = \{2, 3, 4, \ldots\}$ and $E(G') = [\{2, 3\}, \{3, 4\}, \ldots]$. Then G is isomorphic to G' under the isomorphism $f(n) = n + 1$.

5.185 Consider the three graphs in Fig. 5-26. Show that they are distinct, i.e., no two of them are isomorphic. Also show that two of them are homeomorphic.

Fig. 5-72

▮ The graph (a) is not isomorphic to (b) or (c) since it has six edges whereas (b) and (c) each have five edges. Furthermore, if we delete the vertex of degree three in (b) and (c), we obtain different subgraphs. Thus (b) and (c) are not isomorphic.

On the other hand, (b) and (c) are homeomorphic since they can be obtained, respectively, from the isomorphic graphs in Fig. 5-72 by adding an appropriate "internal" vertex.

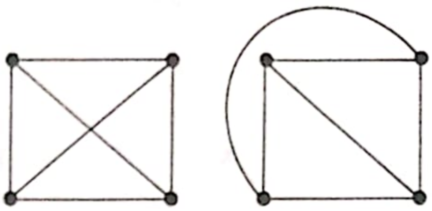
CHAPTER 6

Planar Graphs and Trees

6.1 PLANAR GRAPHS

A graph or multigraph that can be drawn in a plane or on a sphere so that its edges do not cross is said to be *planar*.

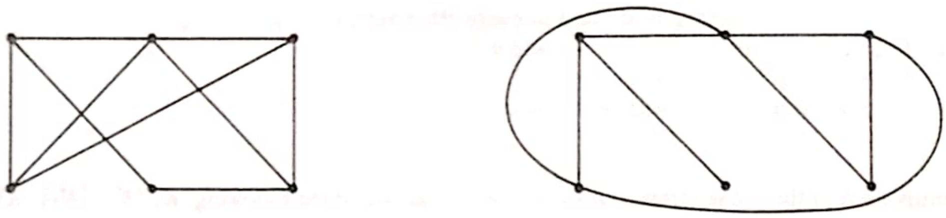
6.1 The graph K_4, which is a planar graph, is usually drawn with crossing edges as shown in Fig. 6-1(a). Draw this graph so that none of its edges cross.



(a) (b) **Fig. 6-1**

A drawing of K_4 without crossing edges is shown in Fig. 6-1(b).

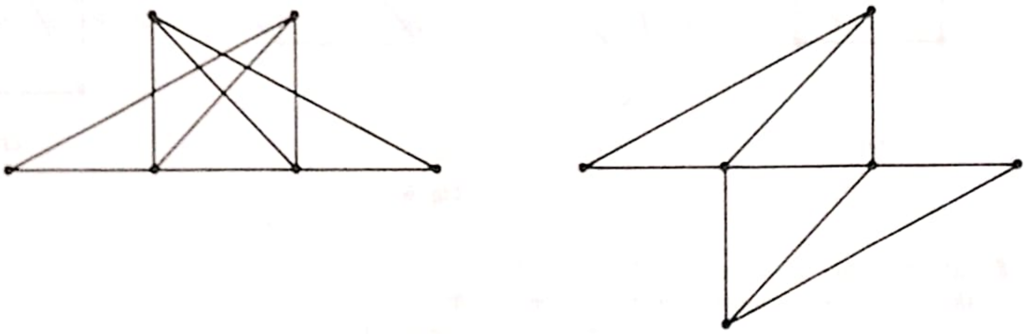
6.2 Draw the planar graph shown in Fig. 6-2(a) so that none of its edges cross.



(a) (b) **Fig. 6-2**

A solution is shown in Fig. 6-2(b).

6.3 Draw the planar graph shown in Fig. 6.3(a) so that none of its edges cross.



(a) (b)

Fig. 6-3

Rearrange the position of one of the vertices to obtain a solution as shown in Fig. 6-3(b).

6.4 Draw the planar graph shown in Fig. 6-4(a) so that none of its edges cross.

A solution is shown in Fig. 6-4(b).