MATRICES

A matrix is an ordered system of numbers arranged in formation of rows and columns, describing various aspects of a phenomenon inter-related to each other. For example, it is used in the study of dominance within a group by sociologists, in the study of births and survivals, marriage and decent, class structure and mobility by demographers, etc.

11.6 DEFINITION AND NOTATIONS OF MATRIX

order

Definition: A rectangular arrangement of numbers (real or complex) in m rows and n columns is called matrix of order (or size) m by n, denoted by $m \times n$. A general form of the matrix of order $m \times n$ is usually written as

$$A = \begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \begin{array}{l} \to \text{Row} \\ \\ \\ \\ \\ \\ \end{array} \\ \begin{array}{l} \uparrow \\ \text{Column} \end{array} \end{array}$$

Matrices

The number $a_{ij}\,(i = 1,2,\ldots,m \text{ and } j = 1,2,\ldots,n)$ in this array (rectangular or square) of mn numbers is called an element of the matrix A. In compact form, the matrix of order $m \times n$ is also written as :

$$A = (a_{ij})_{m \times n},\ i = 1,2,\ldots,m;\ j = 1,2,\ldots,n.$$

In this double subscript notation, the first subscript indicates the row and the second subscript indicates the column in which the element lies. For example, a_{ij} is an element of matrix appearing in i^{th} row and j^{th} column.

Remarks: A matrix is not a number. It has got no numerical value. For example, 5 is simply a number, but in the notation of matrix [5] is a matrix of order 1×1.

Notation: A matrix is some times represented by pairs of parentheses ( ) or a pair of double bars $\|\ \|$ or a pair of brackets [ ]. Matrices are generally denoted by capital letter of English alphabet such as
A, B, C, X, Y, etc., and their elements by corresponding small letters a, b, c etc. For example, the following matrices

$$A = \begin{bmatrix} 1 & 3 & 6 \\ 4 & -2 & 2 \end{bmatrix} \text{ or } \begin{pmatrix} 1 & 3 & 6 \\ 4 & -2 & 2 \end{pmatrix} \text{ or } \begin{Vmatrix} 1 & 3 & 6 \\ 4 & -2 & 2 \end{Vmatrix},\ B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

have two rows and three columns. Therefore, A and B are matrices of order 2×3.

11.7 TYPES OF MATRICES

Row and column matrices: A matrix $(a_{ij})_{1\times n}$ in which there is only one row is called a row matrix. A matrix $(a_{ij})_{m\times 1}$ in which there is only one column is called a column matrix. A row matrix is also called row vector and a column matrix is also called column vector.

Example: (i) The matrix $A = [3 \quad 6 \quad 2]$ is a 1×3 matrix or row matrix (or row vector).

(ii) The matrix $A = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a 2×1 matrix or column matrix (or column vector).

Square matrix: An $m\times n$ matrix A is said to be a square matrix if $m = n$, i.e., number of rows is equal to the number of columns.

Thus, the matrix $A = (a_{ij})_{n\times n}$, $i = 1,2,\ldots,n; j = 1,2,\ldots,n$. is a $n\times n$ square matrix.

For example, the following matrices:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}, \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}, \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$$ are square matrices.

In a square matrix of order n, elements a_{ij} for which $i = j$, i.e. the elements $a_{11}, a_{22}, \ldots, a_{nn}$ are known as diagonal elements, and the line along which the above elements lie is called the principal diagonal or the diagonal of the matrix. For example, the matrix

$$\begin{bmatrix} 3 & 4 & 5 \\ 6 & 2 & 3 \\ -2 & 5 & 1 \end{bmatrix}_{3\times 3}$$

is the square matrix of order 3, in which the diagonal elements are 3,2, and 1.

Null Matrix: A matrix in which every element is zero is called a zero (or null) matrix and is denoted by 0.

For example: $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a null matrix.

Equal Matrices: Two matrices are said to be equal if and only if

(a) they are of the same order, i.e. they have the same number of rows and columns and

(b) each element of one is equal to the corresponding element of the other.

Given two matrices A and B.

$$A = \begin{bmatrix} a & b \\ d & e \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$$

The matrix A is of order 2×2 and B is also of the order 2×2, so A and B are equal if

$a = 1,\ b = -1$

$d = 3,\ c = 2$

Diagonal matrix: A square matrix $(a_{ij})_{n\times n}$ is called a diagonal matrix if each of its non-diagonal element is zero i.e. if $a_{ij} = 0$ when $i \neq j$. Thus, the matrix

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

is a diagonal matrix of order 3, and it can be written as diag. $[a_{11} \quad a_{22} \cdots a_{33}]$

Scalar matrix: A diagonal matrix whose diagonal elements are all equal is called a scalar matrix. i.e. a square matrix where diagonal elements are all equal and the remaining elements are zero is a scalar matrix. Thus, the matrix

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$ is a scalar matrix of order 3.

Unit (or Identity) matrix: A square matrix where diagonal elements are unity (or one) and remaining elements are zero is called a unit (or identity) matrix

A square matrix $A = (a_{ij})_{n\times n}$ is a unit matrix if (i) $a_{ij} = 1$ when $i = j$, and

(ii) $a_{ij} = 0$ when $i \neq j$.

Such a matrix is denoted by I_n. For example,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2},\ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$ are unit matrices of order two and three, respectively.

Symmetric matrix: A matrix is said to be symmetric, if it is

(i) a square matrix and

(ii) $a_{ij} = a_{ji}$, i.e. (i,j)th element is the same as the (j,i)th element.

In other words, a symmetric matrix does not change if we interchange rows and columns. For example, the following two matrices

$$\begin{bmatrix} a & h \\ h & b \end{bmatrix}, \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

are symmetric matrices because $a_{12} = a_{21}, a_{13} = a_{31}$ and so on.

Skew symmetric matrix: A square matrix $A = (a_{ij})_{n\times n}$ is said to be skew symmetric, if $a_{ij} = -a_{ji}$, i.e. (i,j)th element is the negative of (j,i)th element.

If we put $j = i$, we have $a_{ii} = -a_{ii} = 0$ which implies that the elements of the diagonal of skew symmetric are all zero. For example, matrices

$$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}, \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

are skew symmetric matrices because $a_{12} = -a_{21}, a_{11} = a_{22} = a_{33} = 0, a_{13} = -a_{31}$.

Sub Matrix: A matrix which is obtained from a given matrix by deleting any number of rows and any number of columns is called a sub matrix of the given matrix.

For example, if we delete first row and first column from the matrix $\begin{bmatrix} 5 & 3 & 2 \\ 1 & 1 & 2 \\ 7 & 3 & 4 \end{bmatrix}$ of order 3×3, then the matrix so obtained $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is the sub-matrix of the given matrix.

Upper and lower triangular matrices: Upper triangular matrix is a square matrix in which all the elements below the principal diagonal are zero, i.e., $a_{ij} = 0$ for all $i > j$.

Lower triangular matrix is a square matrix in which all the elements above the principal diagonal are zero, i.e. $a_{ij} = 0$ for all $i < j$.

For example, the following two matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ 0 & a_{22} & a_{23} \cdots & a_{2n} \\ \cdots & \cdots & \cdots \cdots & \cdots \end{bmatrix}_{n\times n} \text{ and } \begin{bmatrix} a_{11} & 0 & 0 \cdots & 0 \\ a_{21} & a_{22} & 0 \cdots & 0 \\ \cdots & \cdots & \cdots \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots \cdots & a_{nn} \end{bmatrix}_{n\times n}$$

are upper and lower triangular matrices, respectively.

A triangular matrix, $A = (a_{ij})_{n\times n}$ is called a strictly triangular if $a_{ij} = 0$ for all $i = 1, 2, \ldots, n$.

11.8 ALGEBRA OF MATRICES

In the algebra of matrices, we shall define mathematical operations on matrices, which enable us to combine matrices so as to produce new matrix. The main operations are:

(i) Addition and subtraction of matrices

(ii) Scalar multiplication i.e., multiplication of a matrix by a scalar.

(iii) Multiplication of matrices.

Addition and Subtraction of Matrices

Addition: The sum of two matrices of the same order is obtained by adding their corresponding elements and their sum is said to be defined as the matrix of the same order.

If $A=[a_{ij}]$ and $B=[b_{ij}]$ are two matrices of the same order $m\times n$, then their sum is another matrix $C=[c_{ij}]_{m\times n}$, where $c_{ij}=a_{ij}+b_{ij}$, i.e. each element of C is the sum of the corresponding elements of A and B and is denoted by $A+B$.

Subtraction: Let $A=[a_{ij}]$ and $B=[b_{ij}]$ be two matrices of same order $m\times n$. Then $C=A-B=A+(-B)=[a_{ij}]_{m\times n}+[-b_{ij}]_{m\times n}=[a_{ij}-b_{ij}]_{m\times n}$ i.e. the difference $A-B$ is obtained by subtracting each element of B from the corresponding element of A.

Example: If $A=\begin{bmatrix}3 & 2 & 0\\ 2 & 5 & -1\end{bmatrix}$ and $B=\begin{bmatrix}1 & -1 & 0\\ 2 & 0 & 3\end{bmatrix}$ are two matrices of same order. then

$$A+B=\begin{bmatrix}3+1 & 2-1 & 0+0\\ 2+2 & 5+0 & -1+3\end{bmatrix}=\begin{bmatrix}4 & 1 & 0\\ 4 & 5 & 2\end{bmatrix}$$

and $A-B=\begin{bmatrix}3-1 & 2-(-1) & 0-0\\ 2-2 & 5-0 & -1-3\end{bmatrix}=\begin{bmatrix}2 & 3 & 0\\ 0 & 5 & -4\end{bmatrix}$ are two matrices, of order

Properties of Matrix Addition:

Let $A=[a_{ij}]$, $B=[b_{ij}]$ and $C=[c_{ij}]$ be three matrices of order $m\times n$ each. Then

(i) Matrix addition is commutative: $A+B=B+A$

(ii) Matrix addition is associative: $(A+B)+C=A+(B+C)$.

(iii) Distributive law for scalar multiplication: $\alpha(A+B)=\alpha A+\alpha B$ where α is a scalar.

(iv) Existence of additive identity: For any matrix, say $A=[a_{ij}]$ of order $m\times n$, there is a nu (or zero) matrix 0 of the same order, such that $A+0=0+A=A$

This is because $a_{ij}+0=0+a_{ij}=a_{ij}$ for each (i,j).

(v) Additive inverse: Any matrix A of order $m\times n$ is called the additive inverse of a matı $B(=-A)$ of the same order if $A+(-A)=0=(-A)+A$, where 0 is the additive identit

(vi) Cancellation law: If A, B, and C are matrices of the same order, then $A+C=B+C$ implies $A=B$.

For example: Let $A=\begin{bmatrix}2 & 3 & 5\\ 6 & 7 & 8\end{bmatrix}$, $B=\begin{bmatrix}7 & 6 & 3\\ 1 & 4 & 5\end{bmatrix}$, $C=\begin{bmatrix}6 & -2 & 7\\ 6 & 0 & 9\end{bmatrix}$ and $0=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

(a) Addition is commutative $A+B=\begin{bmatrix}9 & 9 & 8\\ 7 & 11 & 13\end{bmatrix}$, $B+A=\begin{bmatrix}9 & 9 & 8\\ 7 & 11 & 13\end{bmatrix}$

(b) Addition is associative

$$A+(B+C)=\begin{bmatrix}2 & 3 & 5\\ 6 & 7 & 8\end{bmatrix}+\begin{bmatrix}13 & 4 & 10\\ 9 & 4 & 14\end{bmatrix}=\begin{bmatrix}15 & 7 & 15\\ 15 & 11 & 22\end{bmatrix}$$

$$(A+B)+C=\begin{bmatrix}9 & 9 & 8\\7 & 11 & 13\end{bmatrix}+\begin{bmatrix}6 & -2 & 7\\8 & 0 & 9\end{bmatrix}=\begin{bmatrix}15 & 7 & 15\\15 & 11 & 22\end{bmatrix}$$

(c) Additive identity

$$A+0=\begin{bmatrix}2 & 3 & 5\\6 & 7 & 8\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\end{bmatrix}=\begin{bmatrix}2 & 3 & 5\\6 & 7 & 8\end{bmatrix}=A$$

(d) Additive inverse

$$A+(-A)=\begin{bmatrix}2 & 3 & 5\\6 & 7 & 8\end{bmatrix}+\begin{bmatrix}-2 & -3 & -5\\-6 & -7 & -8\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\end{bmatrix}=0$$

Scalar Multiplication of a Matrix

Definition: If $A=[a_{ij}]$ is an $m\times n$ matrix and k is a real number (also called a scalar), then kA is defined as the matrix each element of which is k times the corresponding element of the matrix A, i.e. $kA=[ka_{ij}]_{m\times n}$,

For example, if $A=\begin{bmatrix}2 & 3 & 5\\6 & 7 & 8\end{bmatrix}$, then $2A=\begin{bmatrix}4 & 6 & 10\\12 & 14 & 16\end{bmatrix}$.

Properties of scalar multiplication: If A, B and O are three matrices of same order, then for any scalar k, k_1 and k_2, we have the following properties of scalar multiplication:

(i) $k(A+B)=kA+kB$ (ii) $(k_1+k_2)A=k_1A+k_2A$

(iii) $k_1(k_2A)=(k_1k_2)A$ (iv) $0A=0$

(v) $k0=0$ (vi) $(-k)A=-(kA)=k(-A)$

For example If $A=\begin{bmatrix}0 & 2 & 3\\2 & 1 & 4\end{bmatrix}$ and $B=\begin{bmatrix}7 & 6 & 3\\1 & 4 & 5\end{bmatrix}$, find the value of $2A+3B$.

Solution: $2A=2\begin{bmatrix}0 & 2 & 3\\2 & 1 & 4\end{bmatrix}=\begin{bmatrix}0 & 4 & 6\\4 & 2 & 8\end{bmatrix}$, $3B=3\begin{bmatrix}7 & 6 & 3\\1 & 4 & 5\end{bmatrix}=\begin{bmatrix}21 & 18 & 9\\3 & 12 & 15\end{bmatrix}$

$$\therefore 2A+3B=\begin{bmatrix}0 & 4 & 6\\4 & 2 & 8\end{bmatrix}+\begin{bmatrix}21 & 18 & 9\\3 & 12 & 15\end{bmatrix}=\begin{bmatrix}21 & 22 & 15\\7 & 14 & 23\end{bmatrix}.$$

11.9 MULTPLICATION OF MATRICES

If A and B are two matrices such that the number of columns in A is equal to the number of rows in B, i.e. if $A=[a_{ij}]_{m\times n}$ and $B=[b_{jk}]_{n\times p}$, then the product of A and B denoted by AB is defined as $C=[c_{ik}]m\times p$, For example, consider the matrices

$$A=\begin{bmatrix}2 & 3 & 1\\3 & 1 & 0\end{bmatrix}_{2\times 3}, B=\begin{bmatrix}4\\6\\7\end{bmatrix}_{3\times 1}$$

Theorem-02: Let A and B be symmetric matrices of the same order. Then, show that

(i) $A+B$ is a symmetric matrix

(ii) $AB-BA$ is a skew-symmetric matrix

Solution: Since A and B are symmetric matrices, therefore $A^T=A$ and $B^T=B$.

(i) $(A+B)^T=A^T+B^T=A+B \quad (\because A^T=A, B^T=B)$

Thus, $A+B$ is symmetric

(ii) $(AB-BA)^T=(AB)^T-(BA)^T$

$=B^TA^T-A^TB^T$ (By reversal law)

$=BA-AB=-(AB-BA) \quad [\because B^T=B, A^T=A]$

Thus $AB-BA$ is skew-symmetric.

11.11 ADJOINT OF A SQUARE MATRIX

Let $A=|a_{ij}|$ be a square matrix of order n and let A_{ij} be the co-factor of a_{ij} in determinant A. Then the transpose of the co-factor matrix $B=|A_{ij}|$ of elements of A is called the adjoint of A and is denoted by adj A. Thus, if $A=|a_{ij}|$ and $B=|A_{ij}|$, then

$$\text{adj } A=[A_{ij}]^T=\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

PROPERTIES OF AD-JOINT

(i) If $A=|a_{ij}|$ is a square matrix of order n and I is an identity matrix of order n then $A(adj A)=(adj\, A)A=|A|I$.

(ii) $|\text{adj} A|=|A|^{n-1}$.

(iii) adj $(A^T)=(\text{adj} A)^T$.

(iv) If A and B are two square matrices of same order, then $\text{adj}(AB)=\text{adj} B\times \text{adj} A$

For example: Find the ad-joint of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 4 \end{bmatrix}$ and verify the result:

$A(\text{adj} A)=(\text{adj} A).A=|A|.I_3$.

Solution: $\text{adj}\,A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

where $A_{11} = (-1)^{1+1}\begin{vmatrix} 5 & 0 \\ 4 & 3 \end{vmatrix} = 15,\ A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix} = 0,\ A_{13} = (-1)^{1+3} = \begin{vmatrix} 0 & 5 \\ 2 & 4 \end{vmatrix} = -10.$

$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = 6,\ A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = -3,\ A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0.$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix} = -15,\ A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} = 0,\ A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} = 5.$

$$\therefore \text{adj}\,A = \begin{bmatrix} 15 & 0 & -10 \\ 6 & -3 & 0 \\ -15 & 0 & 5 \end{bmatrix}^T = \begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$$

Also, $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{vmatrix} = 1\begin{vmatrix} 5 & 0 \\ 4 & 3 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix} + 3\begin{vmatrix} 0 & 5 \\ 2 & 4 \end{vmatrix} = 1(15) - 2(0) + 3(-10) = 15 - 30 = -15$

$$\therefore A \times \text{adj}\,A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}\begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -15 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & -15 \end{bmatrix} = -15\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|I_3 \quad \text{...(i)}$$

Also $\quad \text{adj}A \times A = \begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -15 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & -15 \end{bmatrix} = -15\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|I_3 \quad \text{...(ii)}$$

Hence, from (i) and (ii), adj $A \times A = A \times \text{adj}\,A = |A|I$. [Proved]

11.12 INVERSE OF SQUARE MATRIX

Singular and non-singular matrices: A square matrix A is said to be non-singular if $|A| \neq 0$, otherwise singular.
Definition: Let A be any square matrix of order n. If there exists a square matrix B of order n which satisfies the relation: $AB = BA = I_n$, where I_n is the identity matrix of order n, then B is called the inverse of A and is denoted by A^{-1}, i.e., $B = A^{-1}$ so that, $AA^{-1} = A^{-1}A = I_n$.

Remarks: 1. A matrix, which has inverse, is also called invertible.

2. If A is a square matrix such that $|A| \neq 0$, then A is invertible and $A^{-1} = \dfrac{\text{adj } A}{|A|}$.

The concept of inverse of a matrix is useful in solving the system of linear simultaneous equations, input-output analysis and regression analysis. We shall discuss here following two methods of finding the inverse of a given square matrix:
(i) Adjoint matrix method
(ii) Row transformation method
Remark: The necessary and sufficient condition for a matrix to be invertible is that it must be non-singular and only square matrices can have inverse.

PROPERTIES OF INVERSE

The useful properties of matrix inverse are summarized as follows:

(i) The inverse of a non-singular matrix is unique, that is, there is only one matrix for which the relation, $AB = BA = I$ holds.

(ii) If two n-order matrices A, B are such that $AB = I$, then A and B are non-singular, $A^{-1} = B, B^{-1} = A$ and $BA = I$.

(iii) If A, B are two n-order non-singular matrices, then $(AB)^{-1} = B^{-1}A^{-1}\left(\text{not } A^{-1}B^{-1}\right)$.

(iv) The inverse of the inverse is the original matrix A, i.e. $\left(A^{-1}\right)^{-1} = A$.

(v) The inverse of the transpose of a matrix is the transpose of its inverse, i.e. $\left(A^{T}\right)^{-1} = \left(A^{-1}\right)^{T}$.

(vi) If A is non-singular and $AB = O$, then $B = O$.

(vii) The identity matrix is its own inverse, i.e. $I^{-1} = I$.

METHODS OF COMPUTING THE INVERSE OF A MATRIX

(a) Adjoint Matrix Method

Given a non-singular matrix A of order n. We already know that

$$A.(\text{adj } A) = |A|.I = (\text{adj } A).A$$

$$\text{or, } \frac{A.(\text{adj } A)}{|A|} = \frac{(\text{adj } A).A}{|A|} = I \text{ Hence, } A^{-1} = \frac{\text{adj } A}{|A|}, \text{ provided } |A| \neq 0.$$

11.13 SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS

Definition The system of n equations in n unknowns $x_1, x_2, ..., x_n$ of the form

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn} = b_n$$

is known as a system of simultaneous (non-homogenous) linear equations, where $b_1, b_2, ..., b_n$ are constants.

Definition: By the solution of this system of equations we mean a set of values of $x_1, x_2, ..., x_n$ which satisfies simultaneously these n equations. When a system of equations has one or more solutions, it is said to be consistent, otherwise the system of equations is said to be inconsistent. This system of equations can also be written in the matrix form as: $AX = B$

$$\text{where } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},\ B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

so that, A, X, B are matrices of order $n \times n, n \times 1, n \times 1$, respectively, and the matrix A is known as coefficient matrix.

Remarks:

1. Any set of values of variables $x_1, x_2, ..., x_n$ which simultaneously satisfy the system of equations $AX = B$ is called a solution of the system.
2. If $b_1 = b_2 = \cdots = b_n = 0$, the set of equations is said to be homogeneous. A system of linear homogeneous equations has a non-trivial solution if the number of equations is less than the number of variables.
3. If $|A| \neq 0$, then A is non-singular and hence its inverse, A^{-1} exists and unique. Take the system of equations $AX = B$. Pre-multiplying both sides by A^{-1}, we have

$$A^{-1}(AX) = A^{-1}B$$

or, $IX = A^{-1}B \quad (\because A^{-1}A = I)$

or, $X = A^{-1}B = \dfrac{\text{adj } A}{|A|} \times B$

which gives the solution of the system of simultaneous (non-homogenous) equations when the number of variables are equal to the number of equations.

CRITERION OF CONSISTENCY

If there is a system of n linear equations in n variables given by $AX = B$ and A is non-singular, then the system has unique solution given by $X = A^{-1}B$.
The following are criteria of consistency or inconsistency of a system of linear equations given $AX = B$, where A is square matrix.

(i) If $|A| \neq 0$, then system is consistent and has a unique solution given by $X = A^{-1}B$.

(ii) If $|A| = 0$, the system of equations has either no solution or an infinite number of solutions.

(a) If $(\text{adj } A)B \neq 0$, the system has no solution and is therefore inconsistent.

(b) If $(\text{adj } A)B = 0$, the system is consistent and has infinitely many solutions.

METHODS OF SOLVING NON-HOMOGENEOUS LINEAR EQUATIONS

The solution of linear non-homogeneous simultaneous equations can be obtained by applying following methods:

1. Matrix Inverse Method
2. Gauss Elimination Method
3. Cramer's Rule

1. Inverse matrix method:

If $|A| \neq 0$, then A is non-singular and hence its inverse, A^{-1} exists and unique. Take the system of equations $AX = B$. Pre-multiplying both sides by A^{-1}, we have

$$A^{-1}(AX) = A^{-1}B$$

$$\text{or, } IX = A^{-1}B \qquad (\because A^{-1}A = I)$$

$$\text{or, } X = A^{-1}B = \frac{\text{adj } A}{|A|} \times B$$

which gives the solution of the system of simultaneous (non-homogenous) equations when the number of variables are equal to the number of equations.

2. Gauss elimination method (or Triangular form reduction method)
This method is also known as **pivotal reduction method.** According to this method the co-efficient matrix $|a_{ij}|$ for a system of simultaneous equations (homogeneous or non-homogeneous) is reduced in such a way that $a_{ij} = 0$ for $j > i$.

This method can be used even if the co-efficient matrix is singular or non-singular, the number of equations equals to the number of unknowns or otherwise.

3. Cramer's Rule: Please see the determinant section, i.e. section 11.5

Illustration-29: Given, $A=\begin{bmatrix}2 & 4\\ -1 & 3\end{bmatrix}; B=\begin{bmatrix}1 & 0 & 9\\ 6 & 1 & 1\end{bmatrix}$ and $C=\begin{bmatrix}3 & 8\\ 0 & 1\end{bmatrix}$

Verify that (i) $(AB)^T = B^T A^T$ (ii) $(A+C)^T = C^T + A^T$

Solution: (Try yourself)

Illustration-30: Find the inverse of the matrix, $A=\begin{bmatrix}2 & 2 & 2\\ 2 & 1 & 1\\ -7 & 2 & -3\end{bmatrix}$

Solution: $|A| = 2\begin{vmatrix}1 & 1\\ 2 & -3\end{vmatrix} - 2\begin{vmatrix}2 & 1\\ -7 & -3\end{vmatrix} + 0\begin{vmatrix}2 & 1\\ -7 & 2\end{vmatrix} = 2(-3-2)-2(-6+7) = -12 \neq 0.$

Since $|A| \neq 0$, matrix A is non-singular. The co-factors A_{ij} of A are

$$A_{11}=(-1)^{1+1}\begin{vmatrix}1 & 1\\ 2 & -3\end{vmatrix}=-5,\ A_{12}=(-1)^{1+2}\begin{vmatrix}2 & 1\\ -7 & -3\end{vmatrix}=-1, A_{13}=(-1)^{1+3}\begin{vmatrix}2 & 1\\ -7 & 2\end{vmatrix}=11,$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}2 & 0\\ 2 & -3\end{vmatrix}=6,\ A_{22}=(-1)^{2+2}\begin{vmatrix}2 & 0\\ -7 & -3\end{vmatrix}=-6,\ A_{23}=(-1)^{2+3}\begin{vmatrix}2 & 2\\ -7 & 2\end{vmatrix}=-18$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}2 & 0\\ 1 & 1\end{vmatrix}=2,\ A_{32}=(-1)^{3+2}\begin{vmatrix}2 & 0\\ 2 & 1\end{vmatrix}=-2,\ A_{33}=(-1)^{3+3}\begin{vmatrix}2 & 2\\ 2 & 1\end{vmatrix}=-2.$$

The adjoint of A is:

$$\operatorname{adj}A=\begin{bmatrix}A_{11} & A_{21} & A_{31}\\ A_{12} & A_{22} & A_{32}\\ A_{13} & A_{23} & A_{33}\end{bmatrix}=\begin{bmatrix}-5 & 6 & 2\\ -1 & -6 & -2\\ 11 & -18 & -2\end{bmatrix}$$

$$A^{-1}=\frac{\operatorname{adj}A}{|A|}=-\frac{1}{12}\begin{bmatrix}-5 & 6 & 2\\ -1 & -6 & -2\\ 11 & -18 & -2\end{bmatrix}=\begin{bmatrix}5/12 & -1/2 & -1/6\\ 1/12 & 1/2 & 1/6\\ -11/12 & 3/2 & 1/6\end{bmatrix}$$

(b) Row Transformation Method

We consider the partitioned matrix $[A|I]$, where A is non-singular matrix. Applying a sequence of row transformations (operations) in such a manner that matrix A is changed to I and I is changed to A^{-1}.

From (i) and (ii), we get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1/10 & 3/10 \\ -4/10 & 2/10 \end{bmatrix}\begin{bmatrix} 3 \\ 11 \end{bmatrix} = \begin{bmatrix} -3/10+33/10 \\ -12/10+22/10 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Hence, the unique solution is given by; $x = 3$ and $y = 1$.

Illustration-34: By matrix method solve the following equations:

$$5x - 6y + 4z = 15$$

$$7x + 4y - 3z = 19$$

$$2x + y + 6z = 46.$$

Solution: The given system of equations may be represented in the matrix natations as:

$$\begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$$

or, $\quad AX=B \quad$ or $\quad X=A^{-1}B$

Since $\quad |A| = \begin{vmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{vmatrix} = 5\begin{vmatrix} 4 & -3 \\ 1 & 6 \end{vmatrix} + 6\begin{vmatrix} 7 & -3 \\ 2 & 6 \end{vmatrix} + 4\begin{vmatrix} 7 & 4 \\ 2 & 1 \end{vmatrix} = 135 + 288 - 4 = 419 \neq 0$

Hence, A is non-singular and A^{-1} exists. The system has the unique solution.

The matrix of co-factors of elements of A is given by

$A_{11} = 27, \qquad A_{12} = -48 \qquad A_{13} = -1$

$A_{21} = 40, \qquad A_{22} = 22, \qquad A_{23} = -17$

$A_{31} = 2, \qquad A_{32} = 43, \qquad A_{33} = 62.$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{31} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}$$

And $\quad A^{-1} = \dfrac{\text{Adj } A}{A} = \dfrac{1}{419}\begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{419}\begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}\begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

Hence, the unique solution of the system is: $x = 3,\ y = 4,\ z = 6$.