প্রশ্নমালা–6(B)

(i). $S = \lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{n+n}\right]$

$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n(1+r/n)}$

$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+r/n} = \int_0^1\frac{dx}{1+x} = [ln(1+x)]_0^1$

$= ln2 - ln1 = ln2.$

(ii). $S = \lim_{n\to\infty}\left[\frac{1}{n+0}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+2n}\right]$

$= \lim_{n\to\infty}\sum_{r=0}^{2n}\frac{1}{n+r} = \lim_{n\to\infty}\sum_{r=0}^{2n}\frac{1}{n(1+r/n)}$

$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{2n}\frac{1}{1+r/n} = \int_0^2\frac{dx}{1+x}$

$= [ln(1+x)]_0^2 = ln3 - ln1 = ln3.$

(iii). $S = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{3n+r} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n(3+r/n)}$

$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{3+r/n} = \int_0^1\frac{dx}{3+x}$

$= [ln(3+x)]_0^1 = ln4 - ln3 = ln\frac{4}{3}.$

(iv). $S = \lim_{n\to\infty}\left[\frac{1}{na+0}+\frac{1}{na+1}+\frac{1}{na+2}+\cdots+\frac{1}{na+n(b-a)}\right]$

$= \lim_{n\to\infty}\sum_{r=0}^{n(b-a)}\frac{1}{na+r} = \lim_{n\to\infty}\sum_{r=0}^{n(b-a)}\frac{1}{n(a+r/n)}$

$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n(b-a)}\frac{1}{a+r/n} = \int_0^{b-a}\frac{1}{a+x}dx$

$= [ln(a+x)]_0^{a+b} = lnb - lna = ln\frac{b}{a}.$

$$(v).\ S = \lim_{n\to\infty}\left[\frac{1}{\sqrt{(2n.1-1)^2}}+\frac{1}{\sqrt{(2n.2-2^2)}}+\cdots+\frac{1}{\sqrt{(2n.n-n^2)}}\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{(2nr-r^2)}} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{n^2(2r/n-r^2/n^2)}}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\sqrt{2r/n-r^2/n^2}}$$

$$= \int_0^1\frac{dx}{\sqrt{2x-x^2}} = \int_0^1\frac{dx}{\sqrt{1-(x-1)^2}} = \left[\sin^{-1}(x-1)\right]_0^1$$

$$= \sin^{-1}0 - \sin^{-1}(-1) = \sin^{-1}1 = \frac{\pi}{2}.$$

$$(vi).\ S = \lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{1}{\sqrt{n^2-r^2}} = \lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{1}{\sqrt{n^2(1-r^2/n^2)}}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n-1}\frac{1}{\sqrt{1-(r/n)^2}} = \int_0^1\frac{dx}{\sqrt{1-x^2}}$$

$$= \left[\sin^{-1}x\right]_0^1 = \sin^{-1}1 - 0 = \frac{\pi}{2}.$$

$$(vii).\ S = \lim_{n\to\infty}\left[\frac{1}{\sqrt{(n^2+2n.1)}}+\frac{1}{\sqrt{(n^2+2n.2)}}+\cdots+\frac{1}{\sqrt{(n^2+2n.n)}}\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{(n^2+2nr)}} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n\sqrt{1+2r/n}}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\sqrt{1+2r/n}} = \int_0^1\frac{dx}{\sqrt{1+2x}} = \frac{1}{2}\int_0^1\frac{2dx}{\sqrt{1+2x}}$$

$$= \frac{1}{2}.2\left[\sqrt{1+2x}\right]_0^1 = \sqrt{3}-\sqrt{1} = \sqrt{3}-1.$$

$$(viii).\ S = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{n^2(1-r^2/n^2)}} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\sqrt{1-r^2/n^2}}$$

$$= \int_0^1\frac{dx}{\sqrt{1-x^2}} = \left[\sin^{-1}x\right]_0^1 = \sin^{-1}1 - 0 = \frac{\pi}{2}.$$

(ix). $S = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{r=1}^{n} \frac{1}{\sqrt{r}} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{r=1}^{n} \frac{\sqrt{n}}{\sqrt{n}\sqrt{r}}$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \sqrt{\frac{n}{r}} = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{\sqrt{r/n}}$

$= \int_0^1 \frac{dx}{\sqrt{x}} = 2\left[\sqrt{x}\right]_0^1 = 2(\sqrt{1} - 0) = 2.$

2(i). $S = \lim_{n\to\infty} \left[\frac{n}{n^2 + 0^2} + \frac{n}{n^2 + 1^2} + \cdots + \frac{n}{n^2 + (n-1)^2}\right]$

$= \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{n}{n^2 + r^2} = \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{n}{n^2(1 + r^2/n^2)}$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n-1} \frac{1}{1 + r^2/n^2} = \int_0^1 \frac{dx}{1 + x^2} = \left[\tan^{-1} x\right]_0^1$

$= \tan^{-1}1 - \tan^{-1}0 = \frac{\pi}{4}.$

(ii). $S = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{n^2 + r^2} = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{n^2(1 + r^2/n^2)}$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{1 + r^2/n^2} = \int_0^1 \frac{dx}{1 + x^2} = \left[\tan^{-1} x\right]_0^1$

$= \tan^{-1}1 - \tan^{-1}0 = \frac{\pi}{4}.$

(iii). $S = \lim_{n\to\infty} \sum_{r=1}^{n+1} \frac{n^2}{(n^2 + r^2)^{3/2}} = \lim_{n\to\infty} \sum_{r=1}^{n+1} \frac{n^2}{n^3(1 + r^2/n^2)^{3/2}}$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n+1} \frac{1}{(1 + r^2/n^2)^{3/2}} = \int_0^1 \frac{dx}{(1 + x^2)^{3/2}}$

ধরি $x = \tan\theta$, তবে $dx = \sec^2\theta \, d\theta$

সীমা ঃ যদি $x = 1$ হয়, তবে $\theta = \frac{\pi}{4}$.

এবং যদি $x = 0$ হয়, তবে $\theta = 0$

$$S=\int_0^{\pi/4}\frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^{3/2}}=\int_0^{\pi/4}\frac{\sec^2\theta\,d\theta}{\sec^3\theta}=\int_0^{\pi/4}\cos\theta\,d\theta$$

$$=[\sin\theta]_0^{\pi/4}=\sin\frac{\pi}{4}-\sin 0=\frac{1}{\sqrt{2}}$$

(iv). $$S=\lim_{n\to\infty}\left[\frac{\sqrt{n}}{(n+3.0)^{3/2}}+\frac{\sqrt{n}}{(n+3.1)^{3/2}}+\cdots+\frac{\sqrt{n}}{\{n+3(n-1)\}^{3/2}}\right]$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{\sqrt{n}}{(n+3r)^{3/2}}=\lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{\sqrt{n}}{n^{3/2}(1+3r/n)^{3/2}}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{(1+3r/n)^{3/2}}=\int_0^1\frac{dx}{(1+3x)^{3/2}}=\left[\frac{(1+3x)^{-1/2}}{3(-1/2)}\right]_0^1$$

$$=-\frac{2}{3}\left[\frac{1}{(1+3x)^{1/2}}\right]_0^1=\frac{-2}{3}\left[\frac{1}{2}-1\right]=-\frac{2}{3}\left(-\frac{1}{2}\right)=\frac{1}{3}.$$

(v). $$S=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{(n+r)^2}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{n^2(1+r/n)^2}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{(1+r/n)^2}=\int_0^1\frac{dx}{(1+x)^2}=-\left[\frac{1}{1+x}\right]_0^1=\frac{1}{2}$$

(vi). $$S=\lim_{n\to\infty}n\left[\frac{1}{n^2+2^2.0^2}+\frac{1}{n^2+2^2.1^2}+\cdots+\frac{1}{n^2+2^2(n-1)^2}\right]$$

$$=\lim_{n\to\infty}n\sum_{r=0}^{n-1}\frac{1}{n^2+2^2.r^2}=\lim_{n\to\infty}n\sum_{r=0}^{n-1}\frac{1}{n^2(1+4r^2/n^2)}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{1+4(r/n)^2}=\int_0^1\frac{dx}{1+4x^2}=\frac{1}{4}\int_0^1\frac{dx}{(1/2)^2+x^2}$$

$$=\frac{1}{4}2[\tan^{-1}2x]_0^1=\frac{1}{2}[\tan^{-1}2-0]=\frac{1}{2}\tan^{-1}2.$$

(vii). $$\lim_{n\to\infty}\left[\frac{\sqrt{n}}{\sqrt{(n+4.0)^3}}+\frac{\sqrt{n}}{\sqrt{(n+4.1)^3}}+\frac{\sqrt{n}}{\sqrt{(n+4.2)^3}}+\cdots+\frac{\sqrt{n}}{\sqrt{\{n+4(n-1)\}^3}}\right]$$

$$= \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{\sqrt{n}}{\sqrt{(n+4r)^3}} = \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{\sqrt{n}}{n^{3/2}(1+4r/n)^{3/2}}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n-1} \frac{1}{(1+4r/n)^{3/2}} = \int_0^1 \frac{dx}{(1+4x)^{3/2}}$$

$$= \left[\frac{(1+4x)^{-1/2}}{4(-1/2)}\right]_0^1 = -\frac{1}{2}\left[\frac{1}{(1+4x)^{1/2}}\right]_0^1 = -\frac{1}{2}\left[\frac{1}{\sqrt{5}} - 1\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{\sqrt{5}}\right] = \frac{5}{10}\left[1 - \frac{1}{\sqrt{5}}\right] = \frac{1}{10}[5 - \sqrt{5}].$$

(viii). $S = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{n^3}{n^2(1+r^2/n^2)n^2(1+2r^2/n^2)}$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{(1+r^2/n^2)(1+2r^2/n^2)} = \int_0^1 \frac{dx}{(1+x^2)(1+2x^2)}$$

$$= \int_0^1 \left[\frac{-1}{1+x^2} + \frac{2}{1+2x^2}\right]dx = \int_0^1 \left[\frac{-1}{1+x^2} + \frac{2}{2\{(1/\sqrt{2})^2 + r^2\}}\right]dx$$

$$= \left[-\tan^{-1} x + \sqrt{2}\tan^{-1}\sqrt{2}x\right]_0^1 = -\tan^{-1}1 + \sqrt{2}\tan^{-1}\sqrt{2}.$$

(ix). $S = \lim_{n\to\infty} \left[\frac{n}{(n+1)(n+2.1)} + \frac{n}{(n+2)(n+2.2)} + \cdots + \frac{n}{(n+n)(n+2n)}\right]$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{(n+r)(n+2r)} = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{n(1+r/n)\,n(1+2r/n)}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{(1+r/n)(1+2r/n)} = \int_0^1 \frac{dx}{(1+x)(1+2x)}$$

$$= \int_0^1 \left[\frac{-1}{1+x} + \frac{2}{1+2x}\right]dx = [-ln(1+x) + ln(1+2x)]_0^1$$

$$= -ln2 + ln3 = ln\frac{3}{2}.$$

(x). $S = \lim_{n\to\infty}\left[\frac{n}{(n+1)\sqrt{\{1(2n+1)\}}} + \frac{n}{(n+2)\sqrt{\{2(2n+2)\}}} + \cdots + \frac{n}{(n+n)\sqrt{\{n(2n+n)\}}}\right]$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{(n+r)\sqrt{\{r(2n+r)\}}} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{(n+r)\sqrt{(2nr+r^2)}}$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{n(1+r/n)\sqrt{n^2(2r/n+r^2/n^2)}}$$

বা $S = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{(1+r/n)\sqrt{2r/n+r^2/n^2}} = \int_0^1\frac{dx}{(1+x)\sqrt{2x+x^2}}$

$$= \int_0^1\frac{dx}{(1+x)\sqrt{(1+x)^2-1}} = [\sec^{-1}(x+1)]_0^1$$

$$= \sec^{-1}2 - \sec^{-1}1 = \frac{\pi}{3}.$$

(xi) $S = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{n(1+r/n)\sqrt{n^2(2r/n+r^2/n^2)}}$

$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{(1+r/n)\sqrt{2r/n+r^2/n^2}} = \frac{\pi}{3}$; (x) নং দেখুন।

(xii). $S = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{n}}{n^{3/2}(9+40r/n)^{3/2}}$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{(9+40r/n)^{3/2}} = \int_0^1\frac{dx}{(9+40x)^{3/2}}$$

$$= \left[\frac{(9+40x)^{-1/2}}{40(-1/2)}\right]_0^1 = -\frac{1}{20}\left[\frac{1}{(9+40x)^{1/2}}\right]_0^1$$

$$= -\frac{1}{20}\left[\frac{1}{7}-\frac{1}{3}\right] = \frac{1}{105}.$$

3(i). $S = \lim_{n\to\infty}\left[\frac{n+1}{n^2+1^2} + \frac{n+2}{n^2+2^2} + \cdots + \frac{n+n}{n^2+n^2}\right]$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{n+r}{n^2+r^2} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{n(1+r/n)}{n^2(1+r^2/n^2)}$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \frac{1 + r/n}{1 + r^2/n^2} = \int_0^1 \frac{(1 + x)}{1 + x^2}\,dx$$

$$= \int_0^1 \frac{dx}{1 + x^2} + \int_0^1 \frac{x.dx}{1 + x^2}$$

$$= [\tan^{-1} x]_0^1 + \frac{1}{2}[ln(1 + x^2)]_0^1$$

$$= \tan^{-1}1 - 0 + \frac{1}{2}[ln2 - ln1] = \frac{\pi}{4} + \frac{1}{2}ln2.$$

(ii). $$S = \lim_{n\to\infty}\left[\frac{\sqrt{n^2 - 0^2}}{n^2} + \frac{\sqrt{(n^2 - 1^2)}}{n^2} + \dots + \frac{\sqrt{\{n^2 - (n-1)^2\}}}{n^2}\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{\sqrt{(n^2 - r^2)}}{n^2} = \lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{n\sqrt{1 - r^2/n^2}}{n^2}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n-1}\sqrt{1 - (r/n)^2} = \int_0^1 \sqrt{1 - x^2}\,dx$$

$$= \left[\frac{x\sqrt{1 - x^2}}{2} + \frac{1}{2}\sin^{-1}x\right]_0^1$$

$$= 0 + \frac{1}{2}\sin^{-1}1 - 0 = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}.$$

(iii). $$S = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{n + r}}{n\sqrt{n}} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{n}\sqrt{1 + r/n}}{n\sqrt{n}}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\sqrt{\left(1 + \frac{r}{n}\right)} = \int_0^1 \sqrt{(1 + x)}\,dx = \left[\frac{(1 + x)^{3/2}}{3/2}\right]_0^1$$

$$= \frac{2}{3}[2^{3/2} - 1] = \frac{2}{3}(2\sqrt{2} - 1).$$

(iv). $$S = \lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{\sqrt{n(1 + r/n)}}{n\sqrt{n(1 - r/n)}} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n-1}\frac{\sqrt{1 + r/n}}{\sqrt{1 - r/n}}$$

$$= \int_0^1 \frac{\sqrt{1 + x}}{\sqrt{1 - x}}\,dx = \int_0^1 \frac{(1 + x)}{\sqrt{1 - x^2}}\,dx$$

$$= \int_0^1 \frac{dx}{\sqrt{1 - x^2}} + \int_0^1 \frac{x\,dx}{\sqrt{1 - x^2}}$$

$$= \int_0^1 \frac{dx}{\sqrt{1-x^2}} - \frac{1}{2}\int_0^1 \frac{-2x\,dx}{\sqrt{1-x^2}}$$

$$= \left[\sin^{-1}x - \frac{1}{2}\,2\sqrt{1-x^2}\right]_0^1$$

$$= (\sin^{-1}1 - 0) - (0 - \sqrt{1}) = \frac{\pi}{2} + 1.$$

(v). $$S = \lim_{n\to\infty}\left[\frac{(1^2n-m)^{1/3}}{1.n} + \frac{(2^2n-m)^{1/3}}{2n} + \frac{(3^2n-m)^{1/3}}{3n} + \dots + \frac{(n^2n-m)^{1/3}}{nn}\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{(r^2n-m)^{1/3}}{rn} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{\{n^3(r^2/n^2 - m/n^3)\}^{1/3}}{n^2(r/n)}$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{n(r^2/n^2 - m/n^3)^{1/3}}{n^2(r/n)} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{(r^2/n^2 - m/n^3)^{1/3}}{r/n}$$

$$= \int_0^1 \frac{(x^2-0)^{1/3}}{x}dx = \int_0^1 x^{-1/3}dx = \left[\frac{x^{2/3}}{2/3}\right]_0^1 = \frac{3}{2}(1-0) = \frac{3}{2}.$$

4(i). $$S = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^{10}}{n^1.n^{10}} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{10}$$

$$= \int_0^1 x^{10}dx = \left[\frac{x^{11}}{11}\right]_0^1 = \frac{1}{11}(1-0) = \frac{1}{11}.$$

(ii). $$S = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{r}{r^2+3n^2} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{r}{n^2(r^2/n^2+3)}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{(r/n)}{(r/n)^2+3} = \int_0^1\frac{x.\,dx}{x^2+3} = \frac{1}{2}[\ln(x^2+3)]_0^1$$

$$= \frac{1}{2}(\ln 4 - \ln 3) = \frac{1}{2}\ln\frac{4}{3}.$$

(iii). $$S = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^m}{n^{m+1}} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^m$$

$$= \int_0^1 x^m dx = \left[\frac{x^{m+1}}{m+1}\right]_0^1 = \frac{1}{m+1}[1-0] = \frac{1}{m+1}.$$

(iv). $S = \lim_{n\to\infty} \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n}{n}\right)\right]$

$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x).\,dx$, ইহাই নির্ণেয় নির্দিষ্ট ইন্টিগ্রাল

2য় অংশ ঃ $S = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{\sqrt{r}}{n.n^{1/2}} = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \sqrt{\frac{r}{n}}$

$= \int_0^1 \sqrt{x}\,dx = \left[\frac{x^{3/2}}{3/2}\right]_0^1 = \frac{2}{3}(1-0) = \frac{2}{3}$

(v). $S = \lim_{n\to\infty}\left[\frac{1}{n^2}\sec^2\frac{1^2}{n^2} + \frac{2}{n^2}\sec^2\frac{2^2}{n^2} + \frac{3}{n^2}\sec^2\frac{3^2}{n^2} + \cdots + \frac{n}{n^2}\sec^2\frac{n^2}{n^2}\right]$

$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{r}{n^2}\sec^2\frac{r^2}{n^2} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{r}{n}\sec^2\left(\frac{r}{n}\right)^2$

$= \int_0^1 x.\sec^2 x^2\,dx = \frac{1}{2}\int_0^1 \sec^2 x^2\,d(x^2) = \frac{1}{2}[\tan x^2]_0^1$

$= \frac{1}{2}\tan 1.$

(vi). $S = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\sin^{2k}\frac{r\pi}{2n} = \int_0^1 \sin^{2k}\frac{\pi x}{2}\,dx$

ধরি $\frac{1}{2}\pi x = z$, তবে $\frac{1}{2}\pi dx = dz \Rightarrow dx = \frac{2}{\pi}dz$

সীমা ঃ যদি $x = 0$ হয়, তবে $z = 0$

এবং যদি $x = 1$ হয়, তবে $z = \frac{\pi}{2}$

$S = \frac{2}{\pi}\int_0^{\pi/2}\sin^{2k}z\,dz$; ওয়েলীর সূত্রের সাহায্যে

$= \frac{2}{\pi}.\frac{(2k-1)}{2k}.\frac{(2k-3)}{(2k-2)}\cdots\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$; যেহেতু 2k জোড়।

বা $S = \frac{2k(2k-1)(2k-2)(2k-3)\cdots 4.3.2.1}{[(2k)(2k-2)\cdots 4.2]^2}$

$= \frac{(2k)!}{[2^k k(k-1)\cdots 2.1]^2} = \frac{(2k)!}{2^{2k}(k!)^2}$

$$5(i).\ S = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]^{1/n}$$

$$ln S = \lim_{n\to\infty}\frac{1}{n}\left[ln\left(1+\frac{1}{n}\right)+ln\left(1+\frac{2}{n}\right)+\cdots+ln\left(1+\frac{n}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} ln\left(1+\frac{r}{n}\right)$$

$$= \int_0^1 ln(1+x)\,dx.$$

ধরি $1+x=t$, তবে $dx=dt$

সীমা ঃ যদি $x=0$ হয়, তবে $t=1$

এবং যদি $x=1$ হয়, তবে $t=2$

$$\therefore ln S = \int_1^2 lnz\,dz = [zlnz - z]_1^2 = 2ln2 - 2 - 1.\ ln\,1 + 1$$

$$lnS = 2ln2 - 1 = ln2^2 - lne = ln4/e \Rightarrow S = 4/e$$

$$(ii).\ S = \lim_{n\to\infty}\left[\left\{1+\left(\frac{1}{n}\right)^4\right\}^{1/1}\left\{1+\left(\frac{2}{n}\right)^4\right\}^{1/2}\cdots\left\{1+\left(\frac{n}{n}\right)^4\right\}^{1/n}\right]$$

$$\Rightarrow lnS = \lim_{n\to\infty}\left[\frac{1}{1}ln\left\{1+\left(\frac{1}{n}\right)^4\right\}+\frac{1}{2}ln\left\{1+\left(\frac{2}{n}\right)^4\right\}+\right.$$

$$\left.\cdots+\frac{1}{n}ln\left\{1+\left(\frac{n}{n}\right)^4\right\}\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{r}ln\left\{1+\left(\frac{r}{n}\right)^4\right\} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{r/n}ln\left\{1+\left(\frac{r}{n}\right)^4\right\}$$

$$\text{বা } lnS = \int_0^1\frac{1}{x}ln(1+x^4)\,dx = \int_0^1\frac{1}{x}\left[x^4-\frac{1}{2}x^8+\frac{1}{3}x^{12}-\cdots\right]dx$$

$$= \int_0^1\left[x^3-\frac{1}{2}x^7+\frac{1}{3}x^{11}-\cdots\right]dx$$

$$= \left[\frac{x^4}{4}-\frac{x^8}{2.8}+\frac{x^{12}}{3.12}-\cdots\right]_0^1$$

$$=\frac{1}{4}-\frac{1}{16}+\frac{1}{36}-\cdots=\frac{1}{2^2}-\frac{1}{4^2}+\frac{1}{6^2}-\cdots$$

$$=\frac{1}{2^2}\left[1-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\right]$$

$$=\frac{1}{4}\cdot\frac{\pi^2}{12}=\frac{\pi^2}{48}\Rightarrow S=e^{\pi^2/48}.$$

(iii). $S=\lim\limits_{n\to\infty}\left[\left(2+\frac{1^2}{n^2}\right)^{1/n^2}\left(2+\frac{2^2}{n^2}\right)^{2/n^2}\cdots\left(2+\frac{n^2}{n^2}\right)^{n/n^2}\right]$

$$\Rightarrow \ln S=\lim_{n\to\infty}\left[\frac{1}{n^2}\ln\left(2+\frac{1^2}{n^2}\right)+\frac{2}{n^2}\ln\left(2+\frac{2^2}{n^2}\right)+\cdots\right.$$

$$\left.\cdots+\frac{n}{n^2}\ln\left(2+\frac{n^2}{n^2}\right)\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r}{n^2}\ln\left(2+\frac{r^2}{n^2}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{r}{n}\ln\left(2+\frac{r^2}{n^2}\right)$$

$=\int_0^1 x\ln(2+x^2)\,dx,$ ধরি $2+x^2=z$ তবে $2x\,dx=dz$

সীমা ঃ যদি $x=0$ হয়, তবে $z=2$

এবং যদি $x=1$ হয়, তবে $z=3$

$$\ln S=\frac{1}{2}\int_2^3\ln z\,dz=\frac{1}{2}[z\ln z-z]_2^3$$

বা $\ln S=\frac{1}{2}[(3\ln 3-3)-(2\ln 2-2)]$

বা $\ln S=\frac{1}{2}[\ln 27-\ln 4-\ln e]$

বা $\ln S=\frac{1}{2}\ln\left(\frac{27}{4e}\right)=\log\left(\frac{27}{4e}\right)^{1/2}$

$$\Rightarrow S=\left(\frac{27}{4e}\right)^{1/2}.$$

(iv). $S = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{1/1}\left(1+\frac{2}{n}\right)^{1/2}\left(1+\frac{3}{n}\right)^{1/3}\cdots\left(1+\frac{n}{n}\right)^{1/n}\right]$

$$\Rightarrow lnS = \lim_{n\to\infty}\left[\frac{1}{1}\,ln\left(1+\frac{1}{n}\right)+\frac{1}{2}\,ln\left(1+\frac{2}{n}\right)+\cdots+\frac{1}{n}\,ln\left(1+\frac{n}{n}\right)\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{r}\,ln\left(1+\frac{r}{n}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{r/n}\,ln\left(1+\frac{r}{n}\right)$$

$$= \int_0^1\frac{1}{x}\,ln(1+x)\,dx = \int_0^1\frac{1}{x}\left[x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots\right]dx$$

$$= \int_0^1\left[1-\frac{1}{2}x+\frac{1}{3}x^2-\frac{1}{4}x^3+\cdots\right]dx$$

$$= \left[x-\frac{x^2}{2^2}+\frac{x^3}{3^2}-\frac{x^4}{4^2}+\cdots\right]_0^1$$

$$= 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$$

$$lnS = \frac{\pi^2}{12} \Rightarrow S = e^{\pi^2/12}.$$

(v). $S = \lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)^{2/n^2}\left(1+\frac{2^2}{n^2}\right)^{4/n^2}\cdots\left(1+\frac{n^2}{n^2}\right)^{2n/n^2}\right]$

$$\Rightarrow lnS = \lim_{n\to\infty}\left[\frac{2.1}{n^2}\,ln\left(1+\frac{1^2}{n^2}\right)+\frac{2.2}{n^2}\,ln\left(1+\frac{2^2}{n^2}\right)+\cdots+\frac{2n}{n^2}\,ln\left(1+\frac{n^2}{n^2}\right)\right]$$

বা $lnS = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{2r}{n^2}\,ln\left(1+\frac{r^2}{n^2}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{2r}{n}\,ln\left(1+\frac{r^2}{n^2}\right)$

$= \int_0^1 2x\,ln(1+x^2)dx$, ধরি $1+x^2=z$, তবে $2x\,dx = dz$

সীমা ঃ যদি $x=0$ হয়, তবে $z=1$

এবং যদি $x=1$ হয়, তবে $z=2$

$$lnS = \int_1^2 lnz\,dz = [zlnz - z]_1^2 = 2ln2 - 2 - 1.\,ln1 + 1$$

বা $lnS = ln2^2 - 1 = ln4 - lne = ln4/e$

$\Rightarrow S = 4/e.$

(vi). $S = \lim\limits_{n\to\infty} \left[\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n}\right)\left(\frac{n+3}{n}\right)\cdots\left(\frac{n+n}{n}\right)\right]^{1/n}$

বা $S = \lim\limits_{n\to\infty} \left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]^{1/n}$

$\therefore S = \frac{4}{e}$; 5(i) নং দেখুন।

(vii). $S = \lim\limits_{n\to\infty} \left[\frac{\underline{|n}}{n^n}\right]^{1/n} = \lim\limits_{n\to\infty} \left[\frac{1.2.3.\cdots n}{n.n.n.\cdots n}\right]^{1/n}$

বা $S = \lim\limits_{n\to\infty} \left[\left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\left(\frac{3}{n}\right)\cdots\left(\frac{n}{n}\right)\right]^{1/n}$

$lnS = \lim\limits_{n\to\infty} \frac{1}{n}\left[ln\left(\frac{1}{n}\right) + ln\left(\frac{2}{n}\right) + ln\left(\frac{3}{n}\right) + \cdots + ln\left(\frac{n}{n}\right)\right]$

$\Rightarrow lnS = \lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{r=1}^{n} ln\left(\frac{r}{n}\right) = \int_0^1 lnx\,dx$

বা $lnS = [xlnx - x]_0^1 = (1.ln1 - 1) - 0 = -1$

বা $lnS = lne^{-1} \Rightarrow S = e^{-1} = \frac{1}{e}$

(viii). $S = \lim\limits_{n\to\infty} \left[\sin\frac{\pi}{2n}.\sin\frac{2\pi}{2n}.\sin\frac{3\pi}{2n}\cdots\sin\frac{n\pi}{2n}\right]^{1/n}$

$\Rightarrow lnS = \lim\limits_{n\to\infty} \frac{1}{n}\left[ln\left(\sin\frac{1\pi}{2n}\right) + ln\left(\sin\frac{2\pi}{2n}\right) + \cdots + ln\left(\sin\frac{n\pi}{2n}\right)\right]$

$= \lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{r=1}^{n} ln\left(\sin\frac{r\pi}{2n}\right) = \int_0^1 ln\left(\sin\frac{\pi x}{2}\right)dx$

ধরি, $\frac{1}{2}\pi x = z$ তবে $\frac{1}{2}\pi dx = dz \Rightarrow dx = \left(\frac{2}{\pi}\right)dz$

সীমা ঃ যদি $x = 0$ হয়, তবে $z = 0$

এবং যদি $x = 1$ হয়, তবে $z = \frac{\pi}{2}$

$$\ln S = \frac{2}{\pi}\int_0^{\pi/2} \ln(\sin z)\, dz = \frac{2}{\pi}\cdot\frac{\pi}{2}\ln\frac{1}{2}$$

বা $\ln S = \ln\frac{1}{2} \Rightarrow S = \frac{1}{2}$.

(ix). $$S = \lim_{n\to\infty}\left[\tan\frac{\pi}{2n}\cdot\tan\frac{2\pi}{2n}\cdot\tan\frac{3\pi}{2n}\cdots\tan\frac{n\pi}{2n}\right]^{1/n}$$

$$\Rightarrow \ln S = \lim_{n\to\infty}\frac{1}{n}\left[\ln\left(\tan\frac{\pi}{2n}\right) + \ln\left(\tan\frac{2\pi}{2n}\right) + \cdots + \ln\left(\tan\frac{n\pi}{2n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ln\left(\tan\frac{r\pi}{2n}\right)$$

$$= \int_0^1 \ln\left(\tan\frac{1}{2}\pi x\right)dx$$

ধরি $\frac{1}{2}\pi x = z$, তবে $\frac{1}{2}\pi dx = dz \Rightarrow dx = \left(\frac{2}{\pi}\right)dz$

সীমা ঃ যদি $x = 0$ হয়, তবে $z = 0$

এবং যদি $x = 1$ হয়, তবে $z = \frac{\pi}{2}$.

$$\therefore \ln S = \frac{2}{\pi}\int_0^{\pi/2}\ln(\tan z)\, dz = \frac{2}{\pi}0$$

বা $\ln S = 0 = \ln 1 \Rightarrow S = 1$. প্রশ্নমালা 4(A) এর 6(i) নং দেখুন।
