প্রশ্নমালা–5

1(i). ধরি $I_n = \int x^n \sin mx\,dx$

বা $I_n = x^n \int \sin mx\,dx - \int nx^{n-1}\left(\frac{-\cos mx}{m}\right)dx$

বা $I_n = x^n\left(\frac{-\cos mx}{m}\right) + \frac{n}{m}\int x^{n-1}\cos mx\,dx$

বা $I_n = -\frac{x^n \cos mx}{m} + \frac{n}{m}\left[x^{n-1}\int \cos mx\,dx\right) - \int (n-1)\,x^{n-2}\frac{\sin mx}{m}\,dx\Big]$

বা $I_n = -\frac{x^n \cos mx}{m} + \frac{n}{m}\left[x^{n-1}\frac{\sin mx}{m} - \frac{(n-1)}{m}\int x^{n-2}\sin mx\,dx\right]$

বা $I_n = -\frac{x^n \cos mx}{m} + \frac{nx^{n-1}\sin mx}{m^2} - \frac{n(n-1)}{m^2} I_{n-2}$

ইহাই নির্ণেয় লঘুকরণ সূত্র।

(ii). ধরি $I_n = \int x^n \cos mx\,dx$

বা $I_n = x^n \int \cos mx\,dx - \int nx^{n-1}.\frac{\sin mx}{m}\,dx$

বা $I_n = \frac{x^n \sin mx}{m} - \frac{n}{m}\int x^{n-1}\sin mx\,dx$

বা $I_n = \frac{x^n \sin mx}{m} - \frac{n}{m}\left[x^{n-1}\int \sin mx\,dx - \int (n-1)x^{n-2}\left(\frac{-\cos mx}{m}\right)dx\right]$

$\therefore \int x^n \cos mx\,dx = \frac{x^n \sin mx}{m} - \frac{n}{m}\left[-x^{n-1}\frac{\cos mx}{m} + \frac{(n-1)}{m}\int x^{n-2}\cos mx\,dx\right]$

ইহাই নির্ণেয় লঘুকরণ সূত্র।

2. ধরি $I_n = \int x^n \sin x\,dx$

বা $I_n = x^n \int \sin x\,dx - \int nx^{n-1}(-\cos x)\,dx$

বা $I_n = -x^n \cos x + n\int x^{n-1}\cos x\, dx$

বা $I_n = -x^n \cos x + n\left[x^{n-1}\int \cos x\, dx - \int (n-1)x^{n-2} \sin dx\right]$

বা $I_n = -x^n \cos x + n\left[x^{n-1} \sin x - (n-1)\int x^{n-1} \sin x\, dx\right]$

$\int x^n \sin x\, dx = -x^n \cos x + nx^{n-1} \sin x - n(n-1)\int x^{n-2} \sin x\, dx$

ইহাই নির্ণেয় লঘুকরণ সূত্র।

দ্বিতীয় অংশ ঃ এখন উপরের লঘুকরণ সূত্রে 0 হইতে $\pi/2$ সীমা গ্রহণ করিয়া পর্যায়ক্রমে $n = 4, 2$ স্থাপন করিয়া পাই

$$\int_0^{\pi/2} x^4 \sin x\, dx = \left[-x^4 \cos x + 4x^3 \sin x\right]_0^{\pi/2} - 4.3\int_0^{\pi/2} x^2 \sin x\, dx$$

বা $\int_0^{\pi/2} x^4 \sin x\, dx = 0 + 4\left(\frac{\pi}{2}\right)^3 1 - 12\int_0^{\pi/2} x^2 \sin x\, dx \cdots (1)$

এবং $\int_0^{\pi/2} x^2 \sin x\, dx = \left[-x^2 \cos x + 2x \sin x\right]_0^{\pi/2} - 2.1\int_0^{\pi/2} 1 \sin x\, dx$

বা $\int_0^{\pi/2} x^2 \sin x\, dx = 0 + 2.\frac{\pi}{2}1 + 2\left[\cos x\right]_0^{\pi/2}$

বা $\int_0^{\pi/2} x^2 \sin x\, dx = \pi + 2(0-1) = \pi - 2 \cdots (2)$

এখন (1) নং এবং (2) নং হইতে পাই

$$\int_0^{\pi/2} x^4 \sin x\, dx = \frac{1}{2}\pi^3 - 12(\pi - 2)$$

$$= \frac{1}{2}\pi^3 - 12\pi + 24$$

3. $U_n = \int_0^{\pi/2} x^n \sin mx\, dx$

$$= \left[x^n \frac{(-\cos mx)}{m}\right]_0^{\pi/2} - \int_0^{\pi/2} nx^{n-1} \frac{(-\cos mx)}{m} dx$$

$= 0 + \frac{n}{m}\int_0^{\pi/2} x^{n-1} \cos mx\, dx$, যেহেতু m এর আকার 4r + 1

কাজেই $\cos\left(m\frac{\pi}{2}\right)$ 0 হ

$$=\frac{n}{m}\left\{\left[x^{n-1}\frac{\sin mx}{m}\right]_0^{\pi/2}-\int_0^{\pi/2}(n-1)x^{n-2}\frac{\sin mx}{m}dx\right\}$$

$$=\frac{n}{m}\left\{\frac{1}{m}\left(\frac{\pi}{2}\right)^{n-1}1-0-\frac{(n-1)}{m}\int_0^{\pi/2}x^{n-2}\sin mx\,dx\right\}$$

$$=\frac{n}{m^2}\left(\frac{\pi}{2}\right)^{n-1}-\frac{n(n-1)}{m^2}U_{n-2}=\frac{n\pi^{n-1}}{m^2 2^{n-1}}-\frac{n(n-1)}{m^2}U_{n-2}$$

4. $U_n=\int x^n(a-x)^{1/2}dx$

$$=\frac{x^n(a-x)^{3/2}}{(-1).3/2}-\int nx^{n-1}\frac{(a-x)^{3/2}}{(-1).3/2}dx$$

$$=-\frac{2}{3}x^n(a-x)^{3/2}+\frac{2n}{3}\int x^{n-1}(a-x)(a-x)^{1/2}dx.$$

$$=-\frac{2}{3}x^n(a-x)^{3/2}+\frac{2n}{3}\int[ax^{n-1}(a-x)^{1/2}-x^n(a-x)^{1/2}]dx$$

$$U_n=-\frac{2}{3}x^n(a-x)^{3/2}+\frac{2n}{3}[aU_{n-1}-U_n]$$

বা $3U_n=-2x^n(a-x)^{3/2}+2an\,U_{n-1}-2n\,U_n$

বা $(2n+3)U_n=2anU_{n-1}-2x^n(a-x)^{3/2}.$

5. ধরি $I_{m,n}=\int\frac{\sin^m x}{\cos^n x}dx\cdots(1)\ n\neq 1$

এখন $I_{p,q}=\int\sin^p x\cos^q x\,dx$ বিবেচনা করি

বা $I_{p,q}=\frac{\cos^{q-1}x\sin^{p+1}x}{p+q}+\frac{q-1}{p+q}I_{p,q-2}$

(মূল বইয়ের 5.4(i) দেখুন।)

এখন q এর স্থলে q + 2 স্থাপন করিয়া পাই

$$I_{p,q+2}=\frac{\cos^{q+1}x\sin^{p+1}x}{p+q+2}+\frac{q+1}{p+q+2}I_{p,q}$$

বা $\frac{q+1}{p+q+2}I_{p,q}=I_{p,q+2}-\frac{\cos^{q+1}x\sin^{p+1}x}{p+q+2}$

বা $I_{p,q}=\frac{\cos^{q+1}x\sin^{p+1}x}{q+1}+\frac{p+q+2}{q+1}I_{p,q+2}$

বা $\int\sin^p x\cos^q x\,dx=\frac{-\cos^{q+1}x\sin^{p+1}x}{q+1}$

$$+\frac{p+q+2}{q+1}\int\sin^p x\cos^{q+2}x\,dx$$

এখন $p = m$ এবং $q = -n$ স্থাপন করিয়া পাই,

$$\int \sin^m x \cos^{-n} x\, dx = \frac{-\cos n^{-n+1} x \sin^{m+1} x}{-n+1}$$

$$+ \frac{m-n+2}{-n+1} \int \operatorname{sim}^m x \cos^{-n+2} x\, dx$$

বা $$\int \frac{\sin^m x}{\cos^n x} dx = \frac{\sin^{m+1} x}{(n-1)\cos^{n-1} x} - \frac{m-n+2}{n-1} \int \frac{\sin^m x}{\cos^{n-2} x} dx$$

$$\therefore I_{m,n} = \frac{\sin^{m+1} x}{(n-1)\cos^{n-1} x} - \frac{m-n+2}{n-1} I_{m,n-2},$$ (1) নং দ্বারা।

ইহাই নির্ণেয় লঘুকরণ সূত্র।

6. $$I_{m,n} = \int \cos^m x \sin nx\, dx$$

$$= \cos^m x \int \sin nx\, dx - \int m \cos^{m-1} x (-\sin x) . \frac{(-\cos nx)}{n} dx$$

$$= \frac{-\cos^m x \cos nx}{n} - \frac{m}{n} \int \cos^{m-1} x \sin x \cos nx\, dx \cdots (1)$$

এখন $\sin(n-1)x = \sin(nx - x)$

বা $\sin(n-1)x = \sin nx \cos x - \cos nx \sin x$

$\Rightarrow \cos nx . \sin x = \sin nx . \cos x - \sin(n-1)x \cdots (2)$

এখন (2) নং হইতে $\cos nx \sin x$ এর মান (1) নং এ স্থাপন করি

$$I_{m,n} = \frac{-\cos^m x \cos nx}{n} - \frac{m}{n} \int \cos^{m-1} x\, [\sin nx \cos x - \sin(n-1)x]\, dx$$

বা $$I_{m,n} = \frac{-\cos^m x \cos nx}{n} - \frac{m}{n} \int \cos^m x \sin nx\, dx$$

$$+ \frac{m}{n} \int \cos^{m-1} x \sin(n-1)x\, dx$$

বা $I_{m,n} = (-1/n)\cos^m x \cos nx - (m/n)\, I_{m,n} + (m/n)\, I_{m-1,\, n-1}$

বা $$\left(1 + \frac{m}{n}\right) I_{m,n} = \frac{-1}{n} \cos^m x \cos nx + \frac{m}{n} I_{m-1,\, n-1}$$

$$\frac{(m+n)}{n} I_{m,n} = -\frac{1}{n} \cos^m x \cos nx + \frac{m}{n} I_{m-1,\, n-1}$$

$$\therefore I_{m,n} = \frac{-\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,\, n-1}.$$ প্রমাণিত।

অর্থাৎ $\int \cos^m x \sin nx \, dx = \dfrac{-\cos^m x \cos nx}{m+n}$

$$+ \frac{m}{m+n} \int \cos^{m-1} x \sin(n-1)x \, dx$$

$$\Rightarrow \int_0^{\pi/2} \cos^m x \sin nx \, dx = -\left[\frac{\cos^m x \ \cos nx}{m+n}\right]_0^{\pi/2}$$

$$+ \frac{m}{m+n} \int_0^{\pi/2} \cos^{m-1} x \sin(n-1) \, dx$$

দ্বিতীয় অংশ ঃ

উপরের এই সূত্রে পর্যায়ক্রমে $m = 5, n = 3, m = 4, n = 2$ এবং $m = 3, n = 1$ স্থাপন করিয়া পাই

$$\int_0^{\pi/2} \cos^5 x \sin 3x \, dx = -\left[\frac{\cos^5 x \cos 3x}{5+3}\right]_0^{\pi/2}$$

$$+ \frac{5}{5+3} \int_0^{\pi/2} \cos^4 x \sin 2x \, dx,$$

বা $\displaystyle\int_0^{\pi/2} \cos^5 x \sin 3x \, dx = -\frac{1}{8}(0-1) + \frac{5}{8}\int_0^{\pi/2} \cos^4 x \sin 2x \, dx \cdots (1)$

বা $\displaystyle\int_0^{\pi/2} \cos^4 x \sin 2x \, dx = -\left[\frac{\cos^4 x \cos 2x}{4+2}\right]_0^{\pi/2}$

$$+ \frac{4}{4+2} \int_0^{\pi/2} \cos^3 x \sin x \, dx$$

বা $\displaystyle\int_0^{\pi/2} \cos^4 x \sin 2x \, dx = -\frac{1}{6}(0-1) + \frac{2}{3}\int_0^{\pi/2} \cos^3 x \sin x \, dx \cdots (2)$

এবং $\displaystyle\int_0^{\pi/2} \cos^3 x \sin x \, dx = -\left[\frac{\cos^3 x \cos x}{3+1}\right]_0^{\pi/2}$

$$+ \frac{3}{3+1} \int_0^{\pi/2} \cos^2 x \sin 0x \, dx$$

বা $\displaystyle\int_0^{\pi/2} \cos^3 x \sin x \, dx = -\frac{1}{4}(0-1) + 0 \cdots (3)$

এখন (1), (2) & (3) নং হইতে পাই,

$$\int_0^{\pi/2} \cos^5 x \sin 3x \, dx = \frac{1}{8} + \frac{5}{8}\left[\frac{1}{6} + \frac{2}{3}\left(\frac{1}{4} + 0\right)\right] = \frac{1}{3}.$$

7. $I_n = \displaystyle\int_0^{\infty} e^{-ax} \cos^n x \, dx$

বা $I_n = \left[\cos^n x \dfrac{e^{-ax}}{-a}\right]_0^{\infty} - \displaystyle\int_0^{\infty} n\cos^{n-1} x \, (-\sin x) \frac{e^{-ax}}{-a} \, dx$

বা $I_n = -\frac{1}{a}(0-1) - \frac{n}{a}\int_0^\infty (\cos^{n-1}x \sin x)\, e^{-ax}\, dx$

বা $I_n = \frac{1}{a} - \frac{n}{a}\left\{\left[(\cos^{n-1}x \sin x)\frac{e^{-ax}}{-a}\right]_0^\infty\right.$

$\left. - \int_0^\infty \left[\{\cos^{n-1}x \cos x + (n-1)\cos^{n-2}x(-\sin x)\sin x\}\frac{e^{-ax}}{-a}\, dx\right]\right\}$

বা $I_n = \frac{1}{a} - \frac{n}{a}\left\{-\frac{1}{a}(0-0) + \frac{1}{a}\int_0^\infty \cos^n x\, e^{-ax}\, dx\right.$

$\left. - \frac{(n-1)}{a}\int_0^\infty \cos^{n-2}x\, e^{-ax}(1-\cos^2 x)\, dx\right\}$

বা $I_n = \frac{1}{a} - \frac{n}{a}\left[\frac{1}{a}\int_0^\infty e^{-ax}\cos^n x\, dx\right.$

$\left. - \frac{(n-1)}{a}\int_0^\infty \cos^{n-2}x\, e^{-ax}\, dx + \frac{(n-1)}{a}\int_0^\infty \cos^n x\, e^{-ax}\, dx\right]$

বা $I_n = \frac{1}{a} - \frac{n}{a}\left[\frac{1}{a}I_n - \frac{(n-1)}{a}I_{n-2} + \frac{(n-1)}{a}I_n\right]$

বা $I_n = \frac{1}{a} - \frac{n}{a^2}I_n + \frac{n(n-1)}{a^2}I_{n-2} - \frac{n(n-1)}{a^2}I_n$

বা $\left[1 + \frac{n}{a^2} + \frac{n^2-n}{a^2}\right]I_n = \frac{1}{a} + \frac{n(n-1)}{a^2}I_{n-2}$

বা $\left[\frac{a^2+n^2}{a^2}\right]I_n = \frac{a}{a^2} + \frac{n(n-1)}{a^2}I_{n-2}$

$\therefore I_n = \frac{a}{a^2+n^2} + \frac{n(n-1)}{a^2+n^2}I_{n-2}$

দ্বিতীয় অংশ ঃ প্রথম অংশ হইতে পাই

$$\int_0^\infty e^{-ax}\cos^n x\, dx = \frac{a}{a^2+n^2} + \frac{n(n-1)}{a^2+n^2}\int_0^\infty e^{-ax}\cos^{n-2}x\, dx$$

এই লঘুকরণ সূত্রে পর্যায় ক্রমে $n = 5, 3, 1$ স্থাপন করিয়া পাই

$$\int_0^\infty e^{-ax}\cos^5 x\, dx = \frac{a}{a^2+25} + \frac{5.4}{a^2+25}\int_0^\infty e^{-ax}\cos^3 x\, dx \cdots (1)$$

$$\int_0^\infty e^{-ax}\cos^3 x\, dx = \frac{a}{a^2+9} + \frac{3.2}{a^2+9}\int_0^\infty e^{-ax}\cos x\, dx \cdots (2)$$

এবং $\int_0^\infty e^{-ax}\cos x\, dx = \frac{a}{a^2+1} + 0 \cdots (3)$

এখন (1), (2) & (3) নং হইতে পাই,

$$\int_0^{\infty} e^{-ax}\cos^5 x\,dx = \frac{a}{a^2+25} + \frac{20}{a^2+25}\left[\frac{a}{a^2+9} + \frac{6}{a^2+9}\cdot\frac{a}{a^2+1}\right]$$

$$= \frac{a}{a^2+25} + \frac{20a}{(a^2+25)(a^2+9)} + \frac{5.4.3.2a}{(a^2+25)(a^2+9)(a^2+1)}$$

8. $I_n = \int_0^{\pi/2} e^{2x}\sin^n x\,dx$

বা $I_n = \left[\sin^n x\,\frac{e^{2x}}{2}\right]_0^{\pi/2} - n\int_0^{\pi/2}\sin^{n-1}x\cos x\,\frac{e^{2x}}{2}\,dx$

বা $I_n = \frac{1}{2}\left(\sin\frac{\pi}{2}\right)^n e^{\pi} - 0 - \frac{n}{2}\int_0^{\pi/2}\sin^{n-1}\cos x\,e^{2x}\,dx$

বা $2I_n = e^{\pi} - n\left\{\left[\sin^{n-1}x\cos x\,\frac{e^{2x}}{2}\right]_0^{\pi/2} - \int_0^{\pi/2}\frac{d}{dx}(\sin^{n-1}x\cos x)\,\frac{e^{2x}}{2}\,dx\right\}$

বা $2I_n = e^{\pi} - n\left\{0 - \int_0^{\pi/2}[(n-1)\sin^{n-2}x\cos^2 x - \sin^{n-1}x\sin x]\,\frac{e^{2x}}{2}\,dx\right\}$

বা $2I_n = e^{\pi} + \frac{n}{2}\int_0^{\pi/2}[(n-1)\sin^{n-2}x\,(1-\sin^2 x) - \sin^n x]\,e^{2x}\,dx$

বা $4I_n = 2e^{\pi} + n\int_0^{\pi/2}[(n-1)\sin^{n-2}x - (n-1)\sin^n x - \sin^n x]\,e^{2x}\,dx$

বা $4I_n = 2e^{\pi} + n(n-1)\int_0^{\pi/2} e^{2x}\sin^{n-2}x\,dx - n^2\int_0^{\pi/2} e^{2x}\sin^n x\,dx$

বা $4I_n = 2e^{\pi} + n(n-1)\,I_{n-2} - n^2 I_n$

বা $(n^2+4)\,I_n = n(n-1)\,I_{n-2} + 2e^{\pi}$. প্রমানিত।

দ্বিতীয় অংশ ঃ প্রথম অংশ হইতে পাই

$$(n^2+4)\int_0^{\pi/2} e^{2x}\sin^n x\,dx = n(n-1)\int_0^{\pi/2} e^{2x}\sin^{n-2}x\,dx + 2e^{\pi}$$

উপরের লঘুকরণ সূত্রে $n = 3$ স্থাপন করিয়া পাই

$$(3^2+4)\int_0^{\pi/2} e^{2x}\sin^3 x\,dx = 3.2\int_0^{\pi/2} e^{2x}\sin x\,dx + 2e^{\pi}$$

বা $\int_0^{\pi/2} e^{2x}\sin^3 x\,dx = \frac{6}{13}\int_0^{\pi/2} e^{2x}\sin x\,dx + \frac{2}{13}e^{\pi}$ ... (1)

এখন $\int_0^{\pi/2} e^{2x}\sin x\,dx = \left[\frac{e^{2x}(2\sin x - \cos x)}{2^2+1^2}\right]_0^{\pi/2}$

$$= \frac{1}{5}[e^{\pi}(2.1-0) - e^0(0-1)]$$

$$= \frac{2e^{\pi}+1}{5} \cdots (2)$$

(1) নং এবং (2) নং হইতে পাই

$$\int_0^{\pi/2} e^{2x}\sin^3 x\,dx = \frac{6}{13}\cdot\frac{1}{5}(2e^{\pi}+1) + \frac{2e^{\pi}}{13}$$

$$= \frac{22e^{\pi}}{65} + \frac{6}{65}.$$

9. $I_n = \int_0^{\pi/2} \tan^n x\,dx$

বা $I_n = \int_0^{\pi/4} \tan^{n-2}x\tan^2 x\,dx = \int_0^{\pi/4} \tan^{n-2}x(\sec^2 x - 1)\,dx$

বা $I_n = \int_0^{\pi/4} \tan^{n-2}x\sec^2 x\,dx - \int_0^{\pi/4} \tan^{n-2}x\,dx$

বা $I_n = \int_0^{\pi/4} (\tan x)^{n-2}\,d(\tan x) - I_{n-2}$

বা $I_n + I_{n-2} = \left[\frac{(\tan x)^{n-1}}{n-1}\right]_0^{\pi/4} = \frac{1}{n-1}\left[\left(\tan\frac{\pi}{4}\right)^{n-1} - 0\right]$

$\therefore I_n + I_{n-2} = 1/(n-1)$

দ্বিতীয় অংশ ঃ উপরের লঘুকরণ সূত্রে $n = 3$ স্থাপন করিয়া পাই

$$I_3 + I_1 = \frac{1}{2} \cdots (1)$$

কিন্তু আমাদের আছে $I_n = \int_0^{\pi/4} \tan^n x\,dx$

$\Rightarrow I_1 = \int_0^{\pi/4} \tan x\,dx = [\ln(\sec x)]_0^{\pi/4} = \ln\left(\sec\frac{\pi}{4}\right) - \ln(\sec 0)$

বা $I_1 = \ln\sqrt{2} - \ln 1 = \ln\sqrt{2}$

এখন I_1 এর মান (1) নং এ স্থাপন করি

$$I_3 + \ln\sqrt{2} = \frac{1}{2} \Rightarrow I_3 = \frac{1}{2} - \ln\sqrt{2}.$$

10. $U_n = \int_0^{\pi} \frac{1-\cos nx}{1-\cos x}\,dx \Rightarrow U_{n+1} = \int_0^{\pi} \frac{1-\cos(n+1)x}{1-\cos x}\,dx$

$$\therefore U_n - U_{n+1} = \int_0^{\pi} \frac{\cos(n+1)x - \cos nx}{1-\cos x} dx$$

$$= \int_0^{\pi} \frac{2\sin(n+1/2)x.\sin(-x/2)}{2\sin^2 x/2} dx$$

$$= -\int_0^{\pi} \frac{\sin(2n+1)x/2}{\sin x/2} dx, \text{ ধরি } \frac{x}{2} = z$$

তবে $dx = 2\, dz$

সীমা ঃ যদি $x = 0$ হয়, তবে $z = 0$, এবং যদি $x = \pi$ হয়, তবে $z = \pi/2$.

$$\therefore U_n - U_{n+1} = -2\int_0^{\pi/2} \frac{\sin(2n+1)z}{\sin z} dz$$

বা $U_n - U_{n+1} = -2.\pi/2$; মূল বইয়ের 269 পৃষ্ঠা দেখুন।

বা $U_{n+1} - U_n = \pi \cdots (1)$

এখন (1) নং এ n এর স্থলে n + 1 স্থাপন করিয়া পাই

বা $U_{n+2} - U_{n+1} = \pi$

$U_{n+2} - U_{n+1} = U_{n+1} - U_n$; (1) নং দ্বারা।

বা $U_{n+2} + U_n = 2U_{n+1}$

11. $U_n = \int \frac{\cos n\theta}{\sin\theta} d\theta \Rightarrow U_{n-2} = \int \frac{\cos(n-2)\theta}{\sin\theta} d\theta$

$$\therefore U_n - U_{n-2} = \int \frac{\cos n\theta - \cos(n-2)\theta}{\sin\theta} d\theta$$

বা $U_n - U_{n-2} = \int \frac{2\sin(n-1)\theta \sin(-\theta)}{\sin\theta} d\theta$

বা $U_n - U_{n-2} = -2\int \sin(n-1)\theta\, d\theta$

$$\therefore U_n - U_{n-2} = \frac{2\cos(n-1)\theta}{(n-1)} \cdots (1)$$

দ্বিতীয় অংশ ঃ $\int \frac{\sin 5\theta \sin 3\theta}{\sin\theta} d\theta = \frac{1}{2}\int \frac{\cos 2\theta - \cos 8\theta}{\sin\theta} d\theta$

বা $\int \frac{\sin 5\theta \sin 3\theta}{\sin\theta} d\theta = \frac{1}{2}\left[\int \frac{\cos 2\theta}{\sin\theta} d\theta - \int \frac{\cos 8\theta}{\sin\theta} d\theta\right]$

বা $\int \frac{\sin 5\theta \sin 3\theta}{\sin\theta} d\theta = \frac{1}{2}[U_2 - U_8] = -\frac{1}{2}[U_8 - U_2] \cdots (2)$

(1) নং হইতে পাই, $U_n - U_{n-2} = \frac{2\cos(n-1)\theta}{n-1}$

এখন উপরের সূত্রে পর্যায়ক্রমে $n = 8, 6, 4$ স্থাপন করিয়া পাই

$$U_8 - U_6 = \frac{2}{7}\cos 7\theta$$

$$U_6 - U_4 = \frac{2}{5}\cos 5\theta$$

$$U_4 - U_2 = \frac{2}{3}\cos 3\theta$$

যোগ করি, $U_8 - U_2 = 2\left[\frac{1}{7}\cos 7\theta + \frac{1}{5}\cos 5\theta + \frac{1}{3}\cos 3\theta\right]$

$$\Rightarrow \frac{1}{2}[U_8 - U_2] = \frac{1}{7}\cos 7\theta + \frac{1}{5}\cos 5\theta + \frac{1}{3}\cos 3\theta \quad \cdots (3)$$

এখন (3) নং হইতে $\frac{1}{2}[U_8 - U_2]$ এর মান (2) নং এ স্থাপন করি

$$\int \frac{\sin 5\theta \sin 3\theta}{\sin\theta} d\theta = -\left[\frac{\cos 7\theta}{7} + \frac{\cos 5\theta}{5} + \frac{\cos 3\theta}{3}\right]$$

$$\Rightarrow \int_0^{\pi/2} \frac{\sin 5\theta \sin 3\theta}{\sin\theta} d\theta = -\left[\frac{\cos 7\theta}{7} + \frac{\cos 5\theta}{5} + \frac{\cos 3\theta}{3}\right]_0^{\pi/2}$$

$$= -\left\{0 - \left(\frac{1}{7} + \frac{1}{5} + \frac{1}{3}\right)\right\} = \frac{71}{105}$$

12. $I_n = \int_0^a (a^2 - x^2)^n \, dx$

ধরি $x = a\sin\theta$, তবে $dx = a\cos\theta \, d\theta$

সীমা : যদি $x = 0$ হয়, তবে $\theta = 0$

এবং যদি $x = a$ হয়, তবে $\theta = \pi/2$

$$I_n = \int_0^{\pi/2} (a^2 \cos^2\theta)^n \, a\cos\theta \, d\theta = a^{2n+1} \int_0^{\pi/2} \cos^{2n+1}\theta \, d\theta$$

বা $I_n = \dfrac{a^{2n+1}\,\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{2n+3}{2}\right)} \quad \cdots (1)$

এখন n এর স্থলে $n - 1$ স্থাপন করিয়া পাই

$$I_{n-1} = \frac{a^{2n-1}\,\Gamma(n)\,\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{2n+1}{2}\right)} \quad \cdots (2)$$

$$\therefore (1) \div (2) \Rightarrow \frac{I_n}{I_{n-1}} = \frac{a^{2n+1}\,\Gamma(n+1)\,\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{2n+3}{2}\right)} \times \frac{2\Gamma\left(\frac{2n+1}{2}\right)}{a^{2n-1}\,\Gamma(n)\,\Gamma\left(\frac{1}{2}\right)}$$

$$\text{বা } \frac{I_n}{I_{n-1}} = \frac{a^2\,a^{2n-1} n\,\Gamma(n)\,\Gamma\left(\frac{2n+1}{2}\right)}{\left(\frac{2n+1}{2}\right)\Gamma\left(\frac{2n+1}{2}\right) a^{2n-1}\,\Gamma(n)}$$

$$\text{বা } \frac{I_n}{I_{n-1}} = \frac{na^2}{(2n+1)/2} = \frac{2na^2}{2n+1}$$

$$\therefore I_n = \frac{2na^2}{2n+1} I_{n-1}$$

13. $U_n = \int_0^1 x^n \tan^{-1}x\,dx$

$$\text{বা } U_n = \left[\tan^{-1}x.\,\frac{x^{n+1}}{n+1}\right]_0^1 - \int_0^1 \frac{1}{1+x^2}\cdot\frac{x^{n+1}}{n+1}\,dx$$

$$\text{বা } U_n = \frac{1}{n+1}(\tan^{-1}1 - 0) - \frac{1}{n+1}\int_0^1 x^{n-1}\frac{x^2}{1+x^2}\,dx$$

$$\text{বা } U_n = \frac{1}{(n+1)}\cdot\frac{\pi}{4} - \frac{1}{n+1}\int_0^1 x^{n-1}\frac{\{(1+x^2)-1\}}{1+x^2}\,dx$$

$$\text{বা } U_n = \frac{\pi}{4(n+1)} - \frac{1}{n+1}\int_0^1 x^{n-1}\,dx + \frac{1}{n+1}\int_0^1 x^{n-1}\frac{1}{1+x^2}\,dx$$

$$\text{বা } U_n = \frac{\pi}{4(n+1)} - \frac{1}{n+1}\left[\frac{x^n}{n}\right]_0^1 + \frac{1}{n+1}\left\{\left[x^{n-1}\tan^{-1}x\right]_0^1 - \int_0^1 (n-1)x^{n-2}\tan^{-1}x\,dx\right\}$$

$$\text{বা } U_n = \frac{\pi}{4(n+1)} - \frac{1}{n(n+1)} + \frac{1}{n+1}\{1\tan^{-1}1 - 0 - (n-1)\int_0^1 x^{n-2}\tan^{-1}x\,dx\}$$

$$\text{বা } U_n = \frac{\pi}{4(n+1)} - \frac{1}{n(n+1)} + \frac{1}{n+1}\left\{\frac{\pi}{4} - (n-1)U_{n-2}\right\}$$

$$\text{বা } U_n = \frac{\pi}{2(n+1)} - \frac{1}{n(n+1)} - \frac{n-1}{n+1}U_{n-2}$$

$$\text{বা } (n+1)U_n = \frac{\pi}{2} - \frac{1}{n} - (n-1)U_{n-2}$$

$$\therefore (n+1)U_n + (n-1)U_{n-2} = \pi/2 - 1/n.$$

14. যেহেতু $I_n = \int \frac{\sin 2n\theta \, d\theta}{\sin\theta}$, কাজেই $I_{n-1} = \int \frac{\sin 2(n-1)\theta \, d\theta}{\sin\theta}$

এখন $I_n - I_{n-1} = \int \frac{\sin 2n\theta}{\sin\theta} d\theta - \int \frac{\sin 2(n-1)\theta}{\sin\theta} d\theta$

বা $I_n - I_{n-1} = \int \frac{\sin 2n\theta - \sin(2n-2)\theta}{\sin\theta} d\theta$

বা $I_n - I_{n-1} = \int \frac{2\cos(2n\theta + 2n\theta - 2\theta)/2 . \sin(2n\theta - 2n\theta + 2\theta)/2 . d\theta}{\sin\theta}$

বা $I_n - I_{n-1} = 2\int \frac{\cos(2n-1)\theta \sin\theta}{\sin\theta} d\theta$

বা $I_n - I_{n-1} = 2\int \cos(2n-1)\theta \, d\theta$

বা $I_n = I_{n-1} + \frac{2\sin(2n-1)\theta}{2n-1}$. প্রমাণিত।

দ্বিতীয় অংশ ঃ আমরা জানি

$I_n = I_{n-1} + \frac{2\sin(2n-1)\theta}{2n-1}$

$\Rightarrow \int_0^{\pi/2} \frac{\sin 2n\theta}{\sin\theta} d\theta = \int_0^{\pi/2} \frac{\sin(2n-2)\theta \, d\theta}{\sin\theta} + 2\left[\frac{\sin(2n-1)\theta}{2n-1}\right]_0^{\pi/2} \cdots (1)$

এখন (1) নং এ পর্যায়ক্রমে 2n এর স্থলে 5, 3 বসাইয়া পাই,

$$\int_0^{\pi/2} \frac{\sin 5\theta}{\sin\theta} d\theta = \int_0^{\pi/2} \frac{\sin 3\theta}{\sin\theta} d\theta + 2\left[\frac{\sin 4\theta}{4}\right]_0^{\pi/2}$$

বা $\int_0^{\pi/2} \frac{\sin 5\theta}{\sin\theta} d\theta = \int_0^{\pi/2} \frac{\sin 3\theta}{\sin\theta} d\theta + 0 \cdots (2)$

এবং $\int_0^{\pi/2} \frac{\sin 3\theta \, d\theta}{\sin\theta} = \int_0^{\pi/2} \frac{\sin\theta \, d\theta}{\sin\theta} + 2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$

বা $\int_0^{\pi/2} \frac{\sin 3\theta \, d\theta}{\sin\theta} = \int_0^{\pi/2} d\theta + 0$

বা $\int_0^{\pi/2} \frac{\sin 3\theta \, d\theta}{\sin\theta} = [\theta]_0^{\pi/2}$

$\Rightarrow \int_0^{\pi/2} \frac{\sin 3\theta \, d\theta}{\sin\theta} = \frac{\pi}{2} \cdots (3)$

এখন (2) নং এবং (3) নং হইতে পাই

$$\int_0^{\pi/2} \frac{\sin 5\theta}{\sin\theta} d\theta = \frac{\pi}{2}.$$

তবে