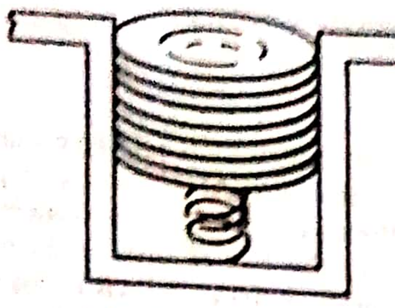
Stacks, Queues, and Priority Queues

There are data structures other than arrays and linked lists which will occur in our graph algorithms. These structures, stacks, queues, and priority queues, are briefly described below.

(a) *Stack*: A *stack*, also called a *last-in first-out* (LIFO) system, is a linear list in which insertions and deletions can take place only at one end, called the "top" of the list. This structure is similar in its operation to a stack of dishes on a spring system, as pictured in Fig. 8-3(a). Note that new dishes are inserted only at the top of the stack and dishes can be deleted only from the top of the stack.



(a) Stack of dishes.

(b) Queue waiting for a bus.

Fig. 8-3

(b) *Queue*: A *queue*, also called a *first-in first-out* (FIFO) system, is a linear list in which deletions can only take place at one end of the list, the "front" of the list, and insertions can only take place at the other end of the list, the "rear" of the list. The structure operates in much the same way as a line of people waiting at a bus stop, as pictured in Fig. 8-3(b). That is, the first person in line is the first person to board the bus, and a new person goes to the end of the line.

(c) *Priority queue*: Let S be a set of elements where new elements may be periodically inserted, but where the current largest element (element with the "highest priority") is always deleted. Then S is called a *priority queue*. The rules "women and children first" and "age before beauty" are examples of priority queues. Stacks and ordinary queues are special kinds of priority queues. Specifically, the element with the highest priority in a stack is the last element inserted, but the element with the highest priority in a queue is the first element inserted.

8.2 GRAPHS AND MULTIGRAPHS

A *graph G* consists of two things:

(i) A set $V = V(G)$ whose elements are called *vertices*, *points*, or *nodes* of G.

(ii) A set $E = E(G)$ of unordered pairs of distinct vertices called *edges* of G.

We denote such a graph by $G(V, E)$ when we want to emphasize the two parts of G.

Vertices u and v are said to be *adjacent* if there is an edge $e = \{u, v\}$. In such a case, u and v are called the *endpoints* of e, and e is said to *connect* u and v. Also, the edge e is said to be *incident* on each of its endpoints u and v.

Graphs are pictured by diagrams in the plane in a natural way. Specifically, each vertex v in V is represented by a dot (or small circle), and each edge $e = \{v_1, v_2\}$ is represented by a curve which connects its endpoints v_1 and v_2. For example, Fig. 8-4(a) represents the graph $G(V, E)$ where:

(i) V consists of vertices A, B, C, D.

(ii) E consists of edges $e_1 = \{A, B\}$, $e_2 = \{B, C\}$, $e_3 = \{C, D\}$, $e_4 = \{A, C\}$, $e_5 = \{B, D\}$.

In fact, we will usually denote a graph by drawing its diagram rather than explicitly listing its vertices and edges.

(a) Graph (b) Multigraph

Fig. 8-4

Multigraphs

Consider the diagram in Fig. 8-4(*b*). The edges e_4 and e_5 are called *multiple edges* since they connect the same endpoints, and the edge e_6 is called a *loop* since its endpoints are the same vertex. Such a diagram is called a *multigraph*; the formal definition of a graph permits neither multiple edges nor loops. Thus a graph may be defined to be a multigraph without multiple edges or loops.

Remark: Some texts use the term *graph* to include multigraphs and use the term *simple graph* to mean a graph without multiple edges and loops.

Degree of a Vertex

The *degree* of a vertex v in a graph G, written deg (v), is equal to the number of edges in G which contain v, that is, which are incident on v. Since each edge is counted twice in counting the degrees of the vertices of G, we have the following simple by important result.

Theorem 8.1: The sum of the degrees of the vertices of a graph G is equal to twice the number of edges in G.

Consider, for example, the graph in Fig. 8-4(*a*). We have

$$\deg(A) = 2, \qquad \deg(B) = 3, \qquad \deg(C) = 3, \qquad \deg(D) = 2$$

The sum of the degrees equals 10 which, as expected, is twice the number of edges. A vertex is said to be *even* or *odd* according as its degree is an even or an odd number. Thus A and D are even vertices whereas B and C are odd vertices.

Theorem 8.1 also holds for multigraphs where a loop is counted twice toward the degree of its endpoint. For example, in Fig. 8-4(*b*) we have deg $(D) = 4$ since the edge e_6 is counted twice; hence D is an even vertex.

A vertex of degree zero is called an *isolated vertex*.

Finite Graphs, Trivial Graph

A multigraph is said to be *finite* if it has a finite number of vertices and a finite number of edges. Observe that a graph with a finite number of vertices must automatically have a finite number of edges and so must be finite. The finite graph with one vertex and no edges, i.e., a single point, is called the *trivial graph*. Unless otherwise specified, the multigraphs in this book shall be finite.

8.3 SUBGRAPHS, ISOMORPHIC AND HOMEOMORPHIC GRAPHS

This section will discuss important relationships between graphs.

Subgraphs

Consider a graph $G = G(V, E)$. A graph $H = H(V', E')$ is called a *subgraph* of G if the vertices and edges of H are contained in the vertices and edges of G, that is, if $V' \subseteq V$ and $E' \subseteq E$. In particular:

(i) A subgraph $H(V', E')$ of $G(V, E)$ is called the subgraph *induced* by its vertices V' if its edge set E' contains all edges in G whose endpoints belong to vertices in H.

(ii) If v is a vertex in G, then $G - v$ is the subgraph of G obtained by deleting v from G and deleting all edges in G which contain v.

(iii) If e is an edge in G, then $G - e$ is the subgraph of G obtained by simply deleting the edge e from G.

Isomorphic Graphs

Graphs $G(V, E)$ and $G^*(V^*, E^*)$ are said to be *isomorphic* if there exists a one-to-one correspondence $f: V \to V^*$ such that $\{u, v\}$ is an edge of G if and only if $\{f(u), f(v)\}$ is an edge of G^*. Normally, we do not distinguish between isomorphic graphs (even though their diagrams may "look different"). Fig. 8-5 gives ten graphs pictured as letters. We note that A and R are isomorphic graphs. Also, F and T, K and X, and M, S, V, and Z are isomorphic graphs.

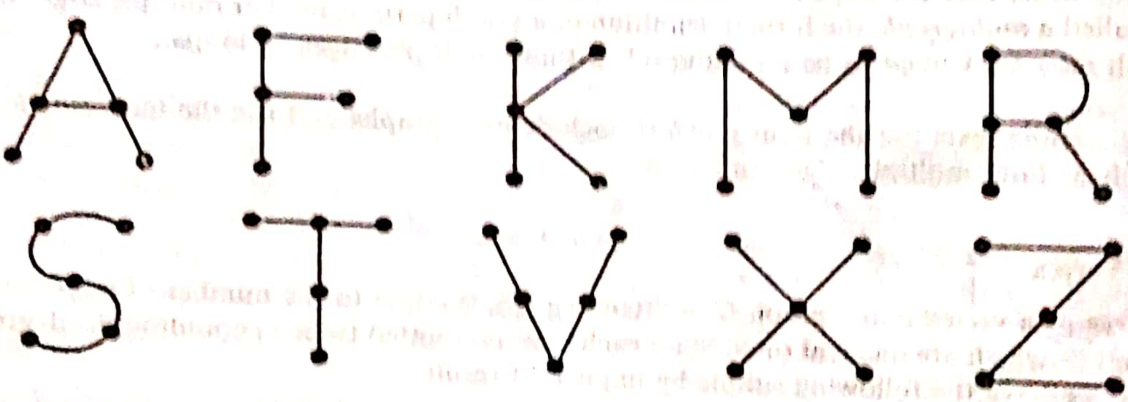
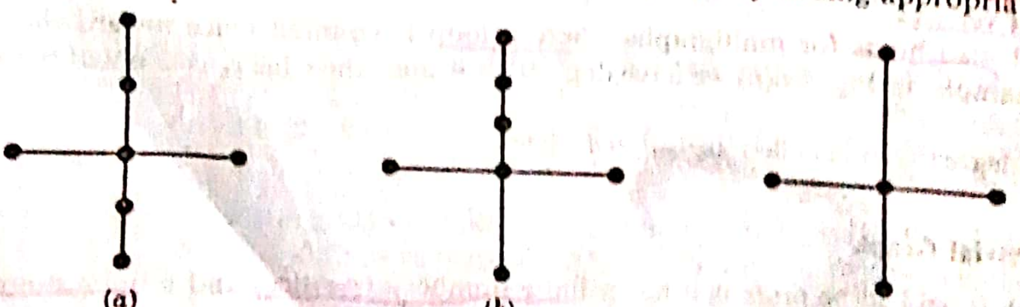


Fig. 8-5

Homeomorphic Graphs

Given any graph G, we can obtain a new graph by dividing an edge of G with additional vertices. Two graphs G and G^* are said to *homeomorphic* if they can be obtained from the same graph or isomorphic graphs by this method. The graphs (*a*) and (*b*) in Fig. 8-6 are not isomorphic, but they are homeomorphic since they can be obtained from the graph (*c*) by adding appropriate vertices.



(a) (b) (c)

graph

Fig. 8-6

From (c), graph (a) & (b) are produced so (a) & (b) are homomorphic

8.4 PATHS, CONNECTIVITY

A *path* in a multigraph G consists of an alternating sequence of vertices and edges of the form

$$v_0,\quad e_1,\quad v_1,\quad e_2,\quad v_2,\quad \ldots,\quad e_{n-1},\quad v_{n-1},\quad e_n,\quad v_n$$

where each edge e_i contains the vertices v_{i-1} and v_i (which appear on the sides of e_i in the sequence). The number n of edges is called the *length* of the path. When there is no ambiguity, we denote a path by its sequence of vertices $(v_0, v_1, \ldots, v_n)$. The path is said to be *closed* if $v_0 = v_n$. Otherwise, we say the path is from v_0 *to* v_n, or *between* v_0 and v_n, or *connects* v_0 to v_n.

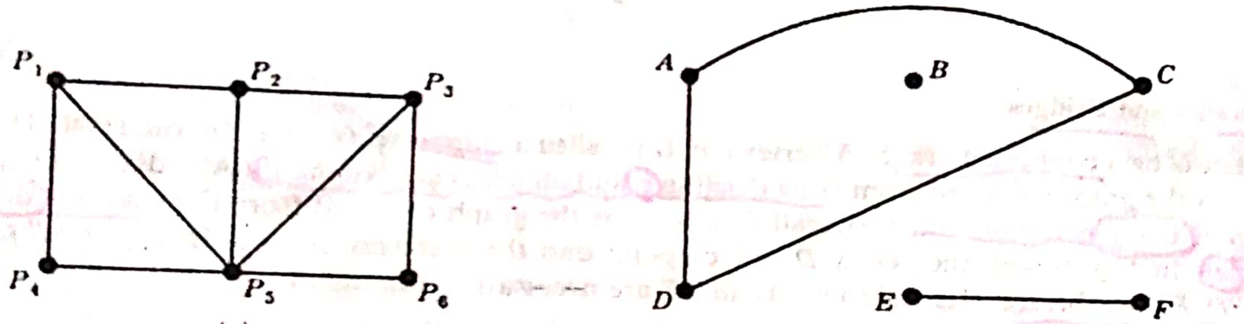
A *simple path* is a path in which all vertices are distinct. (A path in which all edges are distinct will be called a *trail*.) A *cycle* is a closed path in which all vertices are distinct except $v_0 = v_n$. A cycle of length k is called a *k-cycle*. In a graph, any cycle must have length 3 or more.

edge

EXAMPLE 8.1 Consider the graph G in Fig. 8-7(*a*). Consider the following sequences:

$$\alpha = (P_4, P_1, P_2, P_5, P_1, P_2, P_3, P_6), \qquad \beta = (P_4, P_1, P_5, P_2, P_6),$$
$$\gamma = (P_4, P_1, P_5, P_2, P_3, P_5, P_6), \qquad \delta = (P_4, P_1, P_5, P_3, P_6).$$

The sequence α is a path from P_4 to P_6; but it is not a trail since the ege $\{P_1, P_2\}$ is used twice. The sequence β is not a path since there is no edge $\{P_2, P_6\}$. The sequence γ is a trail since no edge is used twice; but it is not a single path since the vertex P_5 is used twice. The sequence δ is a simple path from P_4 to P_6; but it is not the shortest path (with respect to length) from P_4 to P_6. The shortest path from P_4 to P_6 is the simple path (P_4, P_5, P_6) which has length 2.



(*a*) (*b*)

Fig. 8-7

By eliminating unnecessary edges, it is not difficult to see that any path from a vertex u to a vertex v can be replaced by a simple path from u to v. We state this result formally.

Theorem 8.2: There is a path from a vertex u to a vertex v if and only if there exists a simple path from u to v.

Connectivity, Connected Components

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A graph G is *connected* if there is a path between any two of its vertices. The graph in Fig. 8-7(*a*) is connected, but the graph in Fig. 8-7(*b*) is not connected since, for example, there is no path between vertices D and E.

Suppose G is a graph. A connected subgraph H of G is called a *connected component* of G if H is not contained in any larger connected subgraph of G. It is intuitively clear that any graph G can be partitioned into its connected components. For example, the graph G in Fig. 8-7(*b*) has three connected components, the subgraphs induced by the vertex sets $\{A, C, D\}$, $\{E, F\}$, and $\{B\}$.

The vertex B in Fig. 8-7(*b*) is called an *isolated vertex* since B does not belong to any edge or, in other words, deg $(B) = 0$. Therefore, as noted, B itself forms a connected component of the graph.

Remark: Formally speaking, assuming any vertex u is connected to itself, the relation "u is connected to v" is an equivalence relation on the vertex set of a graph G and the equivalence classes of the relation form the connected components of G.

Distance and Diameter

Consider a connected graph G. The *distance* between vertices u and v in G, written $d(u, v)$, is the length of the shortest path between u and v. The *diameter* of G, written $\text{diam}(G)$, is the maximum distance between any two points in G. For example, in Fig. 8-8(a), $d(A, F) = 2$ and $\text{diam}(G) = 3$, whereas in Fig. 8-8(b), $d(A, F) = 3$ and $\text{diam}(G) = 4$.

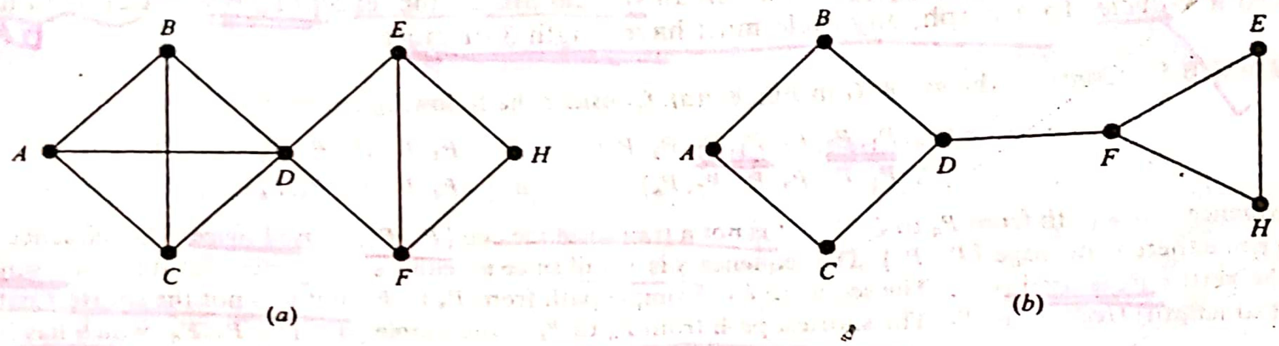


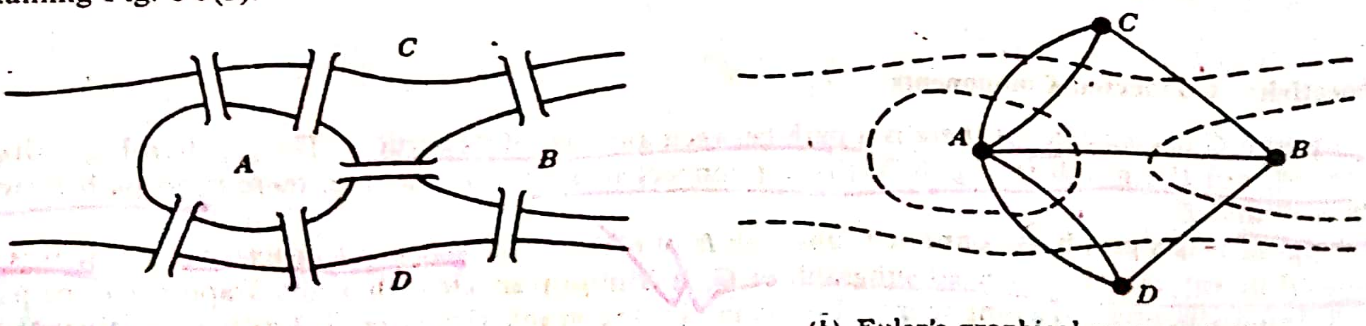
Fig. 8-8

Cutpoints and Bridges

Let G be a connected graph. A vertex v in G is called a *cutpoint* if $G - v$ is disconnected. (Recall that $G - v$ is the graph obtained from G by deleting v and all edges containing v.) An edge e of G is called a *bridge* if $G - e$ is disconnected. (Recall that $G - e$ is the graph obtained from G by simply deleting the edge e.) In Fig. 8-8(a), the vertex D is a cutpoint and there are no bridges. In Fig. 8-8(b), the edge $e = \{D, F\}$ is a bridge. (Its endpoints D and F are necessarily cutpoints.)

8.5 THE BRIDGES OF KÖNIGSBERG, TRAVERSABLE MULTIGRAPHS

The eighteenth-century East Prussian town of Königsberg included two islands and seven bridges as shown in Fig. 8-9(a). Question: Beginning anywhere and ending anywhere, can a person walk through town crossing all seven bridges but not crossing any bridge twice? The people of Königsberg wrote to the celebrated Swiss mathematician L. Euler about this question. Euler proved in 1736 that such a walk is impossible. He replaced the islands and the two sides of the river by points and the bridges by curves, obtaining Fig. 8-9(b).



(a) Königsberg in 1736 (b) Euler's graphical representation

Fig. 8-9