2

Equations of First Order and First Degree

2·1. Differential equation of the first order and first degree. A differential equation of the type

$$M+N\frac{dy}{dx}=0,$$

where M and N are functions of x and y or constants, is called *a differential equation of the first order and first degree.*

We give below some methods of solving such equations.

2·2. Solution of the differential equation when variables are separable.

If an equation can be written in such a way that dx and all the terms containing x are on one side and dy and all the terms containing y on the other side, then this is an equation in which variables are separable. Such equations can therefore be written as $f_1(x)\,dx=f_2(y)\,dy$ and can be solved by integrating directly and adding a constant on either side.

Ex. 1. *Solve* $\dfrac{dy}{dx}=\dfrac{1+y^2}{1+x^2}.$

Solution Separating the variables the equation becomes

$$\frac{dy}{1+y^2}=\frac{dy}{1+x^2}$$

Integrating, we get $\tan^{-1} y=\tan^{-1} x+A$

or $\tan^{-1} y-\tan^{-1} x=A$ *i.e.*, $\tan^{-1}\dfrac{y-x}{1+xy}=A=\tan^{-1} C$ (say).

$\therefore\quad y-x=C(1+xy)$

which is the solution.

Ex. 2. *Solve* $\dfrac{dy}{dx}=e^{x-y}+x^2e^{-y}.$

[Gorakhpur 59 ; Andhra 60 : Sagar 54]

Solution. The given equation can be written as

$$e^y\,dy=(e^x+x^2)\,dx.$$

Integrating, $\quad e^y=e^x+\frac{1}{3}x^3+C.$

Ex. 3. *Solve* $\sec^2 x\tan y\,dx+\sec^2 y\tan x\,dy=0$

[Nagpur T.D.C. 61 ; Delhi 51]

Solution. Separating the variables, we get

$$\frac{\sec^2 x}{\tan x}dx+\frac{\sec^2 y}{\tan y}dy=0.$$

Integrating, $\log \tan x+\log \tan y=A$

or $\tan x \tan y=e^A=C$.

Ex. 4. *Solve* $(y-px)\,x=y$. **[Saugar 62]**

Solution. Equation is $px=y\,(x-1)$, *i.e.*, $\dfrac{dy}{dx}=\dfrac{y\,(x-1)}{x^2}$,

i.e., $$\frac{dy}{y}=\frac{x-1}{x^2}dx=\left(\frac{1}{x}-\frac{1}{x^2}\right)dx.$$

Integrating, $\log y=\log x+\dfrac{1}{x}+\log A$ or $\dfrac{y}{x}=Ae^{1/x}$.

problem

Ex. 5. *Solve* $y-x\dfrac{dy}{dx}=a\left(y^2+\dfrac{dy}{dx}\right)$. **[Saugar 63]**

Solution. The equation can be written as

$$\frac{dx}{x+a}=\frac{dy}{y\,(1-ay)}=\left(\frac{1}{y}+\frac{a}{1-ay}\right)dy.$$

Integrating, $x+a=C\dfrac{y}{1-ay}$. problem

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Ex. 6. *Solve*

(i) $(3+2\sin x+\cos x)\,dy=(1+2\sin y+\cos y)\,dx$.

(ii) $(e^y+1)\cos x\,dx+e^y \sin x\,dy=0$. **[Poona 64]**

2·3. Equations reducible to the form in which variables are separable

Equations of the form

$$\frac{dv}{dx}=f(ax+cy+c)$$

can be reduced to an equation in which variables can be separated. What is required is that we put

$$ax+by+c=v,$$

so that $a+b\dfrac{dy}{dx}=\dfrac{dv}{dx}$, *i.e.*, $\dfrac{dy}{dx}=\dfrac{1}{b}\left[\dfrac{dv}{dx}-a\right]$.

Then the equation becomes

$$\frac{1}{b}\left(\frac{dv}{dx}-a\right)=f(v) \text{ or } \frac{dv}{dx}=a+bf(v),$$

in which variables are separable.

Ex. 1. *Solve* $\dfrac{dy}{dx}=(4x+y+1)^2$.

[Raj. 61 ; Agra 54 ; Gujrat 65, 58]

C/S

Solution. Put $4x+y+1=v$, so that $4+\dfrac{dy}{dx}=\dfrac{dv}{dx}$.

The equation then reduces to

$$\frac{dv}{dx}-4=v^2 \quad \text{or} \quad \frac{dv}{dx}=v^2+4.$$

The variables are now separable and we can write $\dfrac{dv}{v^2+4}=dx.$

Integrating $\frac{1}{2}\tan^{-1}\left(\dfrac{v}{2}\right)=x+C$

or $\frac{1}{2}\tan^{-1}\left(\dfrac{4x+y+1}{2}\right)=x+C$ is the solution.

Ex. 2. *Solve* $\dfrac{dy}{dx}=\sin(x+y)+\cos(x+y).$ [Agra B Sc. 67]

Solution. Put $x+y=v,\quad 1+\dfrac{dy}{dx}=\dfrac{dv}{dx}.$

$\therefore$ equation is $\dfrac{dv}{dx}-1=\sin v+\cos v$ or $\dfrac{dv}{dx}=1+\sin v+\cos v$

or $dx=\dfrac{dv}{1+\sin v+\cos v}=\dfrac{dv}{2\cos^2\frac{1}{2}v+2\sin\frac{1}{2}v\cos\frac{1}{2}v}$ $\quad 1+\dfrac{1-\tan^2\frac{v}{2}}{1+\tan^2\frac{v}{2}}+$

or $\dfrac{dv}{2\cos^2\frac{1}{2}v\,(1+\tan\frac{1}{2}v)}=dx$ or $\dfrac{\frac{1}{2}\sec^2\frac{1}{2}v\,dv}{1+\tan\frac{1}{2}v}=dx.$

Integrating, $\log(1+\tan\frac{1}{2}v)=x+C,$ where $v=x+y.$

$\therefore \log[1+\tan\frac{1}{2}(x+y)]=x+C$ is the required solution.

Ex. 3. *Solve* $(x-y)^2\dfrac{dy}{dx}=a^2.$

[Calcutta Hons. 63; Bihar 61; Vikram 65]

Solution. Put $x-y=v$, so that $1-\dfrac{dy}{dx}=\dfrac{dv}{dx}.$

$\therefore$ equation is $v^2\left[1+\dfrac{dv}{dx}\right]=a^2$ or $\dfrac{dv}{dx}=\dfrac{v^2-a^2}{v^2}.$

or $dx=\dfrac{v^2}{v^2-a^2}\,dv=\left(1+\dfrac{a^2}{v^2-a^2}\right)dv.$

Integrating, $x+C=v+a^2\dfrac{1}{2a}\log\dfrac{v-a}{v+a}$

or $x+C=(x-y)+\frac{1}{2}a\log\dfrac{x-y-a}{x-y+a}$ is the solution.

Ex. 4. *Solve* $(x+y)^2\dfrac{dy}{dx}=a^2$

[Poona 64; Raj. 63; Delhi Hons. 60; Alld 60]

Solution. Put $x+y=v$, so that $1+\dfrac{dy}{dx}=\dfrac{dv}{dx}.$

$\therefore v^2\left(\dfrac{dv}{dx}-1\right)=a^2,\ \dfrac{dv}{dx}=1+\dfrac{a^2}{v^2}=\dfrac{a^2+v^2}{v^2}$

$\therefore \quad dx=\frac{v^2}{a^2+v^2}\,dv=\left(1-\frac{a^2}{a^2+v^2}\right)dv.$

Integrating, $x+C=v-a\tan^{-1}\frac{v}{a}$

or $x+C=(x+y)-a\tan^{-1}\frac{x+y}{a}$

or $y=C+a\tan^{-1}\frac{x+y}{a}$ is the solution.

***Ex. 5.** *Solve* $\frac{x\,dx+y\,dy}{x\,dy-y\,dx}=\sqrt{\left(\frac{a^2-x^2-y^2}{x^2+y^2}\right)}.$

[Delhi Hons. 62; Agra B.Sc. 55]

Solution. Here we change to polar co-ordinates by putting

$x=r\cos\theta,\ y=r\sin\theta,\ x^2+y^2=r^2,\ x\,dx+y\,dy=r\,dr.$

$\frac{y}{x}=\tan\theta, \quad \therefore \quad \frac{x\,dy-y\,dx}{x^2}=\sec^2\theta\,d\theta$ or $x\,dy-y\,dx=r^2\,d\theta.$

$\therefore$ the equation becomes $\frac{1}{r}\frac{dr}{d\theta}=\sqrt{\left(\frac{a^2-r^2}{r^2}\right)}.$

Separating the variables, $\frac{dr}{\sqrt{(a^2-r^2)}}=d\theta.$

Integrating, $\sin^{-1}(r/a)=\theta+C$ or $r=a\sin(\theta+C)$,

i.e., $\sqrt{(x^2+y^2)}=a\sin[\tan^{-1}(y/x)+C].$

Ex. 6. *Solve* $x\frac{dy}{dx}-y=x\sqrt{(x^2+y^2)}.$ [Bombay 61; Agra 56]

Solution. The equation can be put as

$x\,dy-y\,dx=x\sqrt{(x^2+y^2)}\,dx$ or $\frac{x\,dy-y\,dx}{x^2}=\sec^2\theta\,d\theta.$

Changing to polars as above, the equation becomes

$x^2\sec^2\theta\,d\theta=xr\,dx$

or $x\sec^2\theta\,d\theta=r\,dx$ or $r\cos\theta\sec^2\theta\,d\theta=r\,dx$

or $\sec\theta\,d\theta=dx$, variables separated.

Integrating, $\log(\sec\theta+\tan\theta)=x+\log C.$

$\therefore \quad \sec\theta+\tan\theta=ce^x$ or $\sqrt{(1+y^2/x^2)}+y/x=ce^x.$

Ex. 7. *Solve* $\left(\frac{x+y-a}{x+y-b}\right)\frac{dy}{dx}=\left(\frac{x+y+a}{x+y+b}\right)$

[Delhi Hons. 63; Nagpur 55]

Solution. Put $x+y=v$, so that $1+\frac{dy}{dx}=\frac{dv}{dx}.$

i.e., $\frac{dv}{dx}=1+\left(\frac{v+a}{v-b}\right)\left(\frac{v-b}{v-a}\right)=\frac{2(v^2-ab)}{v^2+(b-a)v-ab}$

or $2\,dx=\left(1+\frac{b-a}{2}\,\frac{2v}{v^2-ab}\right)dv.$

Problem

Integrating, $2x+C=v+\frac{b-a}{2}\log(v^2-ab)$

or $2x+C=x+y+\frac{1}{2}(b-a)\log[(x+y)^2-ab]$ etc.

Ex. 8. $\frac{dy}{dx}=(x+y)^2$. [Gauhati 62; Delhi 62; Raj. 62]

Hint. Put $x+y=v$ etc.

2·4. Homogeneous Differential Equations. [Poona 61 (S)]

An equation of the form $\frac{dy}{dx}=\frac{f_1(x,y)}{f_2(x,y)}$ in which $f_1(x,y)$ and $f_2(x,y)$ are homogeneous functions* of x and y of the same degree can be reduced to an equation in which variables are separable by putting $y=vx$, $\frac{dy}{dx}=v+x\frac{dv}{dx}$.

The following few examples will illustrate the method.

Ex. 1. *Solve* $(x^2+y^2)\,dx+2xy\,dy=0$.

Solution. We have $\frac{dy}{dx}=-\frac{x^2+y^2}{2xy}$ (homogeneous).

Putting $y=vx$, $\frac{dy}{dx}=v+x\frac{dv}{dx}$, the equation becomes

$$v+x\frac{dv}{dx}=-\frac{x^2+x^2v^2}{2x.vx}=-\frac{1+v^2}{2v}$$

or $$x\frac{dv}{dx}=-\frac{1+v^2}{2v}-v=-\frac{1+3v^2}{2v}\text{ (variable separable).}$$

$$\therefore\ \frac{dx}{x}=-\frac{2v}{1+3v^2}\,dv.$$

Integrating, $\log x+\frac{1}{3}\log(1+3v^2)=\log C$

or $x(1+3v^2)^{1/3}=C$ or $x(1+3y^2/x^2)^{1/3}=C$.

Ex. 2. *Solve* $x^2y\,dx-(x^3+y^3)\,dy=0$. [Agra B.Sc. 54]

Solution. We have $\frac{dy}{dx}=\frac{x^2y}{x^3+y^3}$ (homogeneous).

Putting $y=vx$, $\frac{dy}{dx}=v+x\frac{dv}{dx}$, the equation becomes

$$v+x\frac{dv}{dx}=\frac{v}{1+v^3}\text{ or }x\frac{dv}{dx}=\frac{v}{1+v^3}-v=-\frac{v^4}{1+v^3}$$

or $$\frac{dx}{x}=-\frac{1+v^3}{v^4}\,dv=-\left[\frac{1}{v^4}+\frac{1}{v}\right]dv.$$

Integrating, $\log x=\frac{1}{3v^3}-\log v+C$; $\log vx=\frac{1}{3v^3}+C$

*A function $f(x,y)$ is called homogeneous of degree n, if $f(tx,ty)=t^nf(x,y)$.

$\log xv = \frac{1}{3v^3} + c$

$= \log . x . \frac{y}{x} = \frac{1}{3\left(\frac{y^3}{x^3}\right)}$

or $\log y = \frac{x^3}{3y^3} + C$ as $v = \frac{y}{x}$. $\Rightarrow \log y = \frac{x^3}{3y^3} + c$

Ex. 3. *Solve* $\frac{dy}{dx} = \frac{y^3 + 3x^2y}{x^3 + 3xy^2}$. [Lucknow Pass 60]

Solution. Putting $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we get

$$x\frac{dv}{dx} = \frac{dy}{dx} - v = \frac{v^3 + 3v}{1 + 3v^2} - v = \frac{2v(1 - v^2)}{1 + 3v^2}$$

or $\frac{2\,dx}{x} = \frac{1 + 3v^2}{2v(1 - v^2)}\,dv = \left(\frac{1}{v} - \frac{2}{1+v} + \frac{2}{1-v}\right)dv$. → problem

Integrating,

$$2\log x = \log v - 2\log(1 - v) - 2\log(1 + v) + \log C \quad \text{— problem}$$

or $x^2(1 - v)^2(1 + v)^2 = Cv$. Put $v = y/x$ etc.

Ex. 4. *Solve* $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$.

[Delhi Hons. 66; Cal. Hons. 61, 56; Osmania 60; Gujrat 61]

Solution. The equation is $\frac{dy}{dx} = \frac{y^2}{xy - x^2}$ [homogeneous].

Putting $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we get

$$v + x\frac{dv}{dx} = \frac{v^2}{v - 1} \quad \text{or} \quad x\frac{dv}{dx} = \frac{v^2}{v - 1} - v$$

or $x\frac{dv}{dx} = \frac{v}{v - 1}$ or $\frac{dx}{x} = \frac{v - 1}{v}\,dv$

or $\frac{dx}{x} = \left(1 - \frac{1}{v}\right)dv$.

Integrating, $\log x = v - \log v + \log c$

or $\log xv = v + \log c$ or $xv = ce^v$

or $y = ce^{y/x}$ as $y = vx$.

$y = ce^{y/x}$

$\log xv = v + \log c$

$\Rightarrow$

Ex. 5. *Solve* $(x^2 + y^2)\,dy = xy\,dx$. [Nagpur T.D.C. 1961]

Hint. Homogeneous. Put $y = vx$. Ans. $y = Ce^{x^2/2y^2}$

Ex. 6. *Solve the following homogeneous equations :*

(i) $y(y^2 - 2x^2)\,dx + x(2y^2 - x^2)\,dy = 0$. → Wrong math

[Karnatak B.Sc. (Sub.) 1960]

(ii) $\frac{1}{2x}\frac{dy}{dx} + \frac{x + y}{x^2 + y^2} = 0$. [Lucknow Pass 1955]

(iii) $\frac{dy}{dx} + \frac{y(x + y)}{x^2} = 0$. Ans. $x^2y = c^2(y + 2x)$

[Poona 1964; Nag 58; Kerala 61; Vikram 61]

(iv) $x^2y\,dx - x^3\,dy = y^3\,dy$. Ans. $\log y = \frac{x^3}{3y^3} + C$.

(v) $(x^2-y^2)\dfrac{dy}{dx}=xy.$

(vi) $(x+y)^2=xy\dfrac{dy}{dx}.$ [Poona 1964]

(vii) $x\dfrac{dy}{dx}-y=\sqrt{(x^2+y^2)}.$ [Sagar 1963; Cal. Hons. 62; Raj. 56]

(Cf. Ex. 6 P. 10) Ans. $x^2+y^2=(Cx^2-y)^2.$

Ex. 7. $\left(x\cos\dfrac{y}{x}+y\sin\dfrac{y}{x}\right)y=\left(y\sin\dfrac{y}{x}-x\cos\dfrac{y}{x}\right)x\dfrac{dy}{dx}$

[Cal. Hons 1962]

or $x\cos\dfrac{y}{x}(y\,dx+x\,dy)=y\sin\dfrac{y}{x}(x\,dy-y\,dx).$

[Raj. 1959; Cal. Hons. 61, 55; Delhi 68, 61]

Solution. The equation is $\dfrac{dy}{dx}=\dfrac{y\,(y\sin y/x+x\cos y/x)}{x\,(y\sin y/x-x\cos y/x)}.$

Putting $y=vx$, $x\dfrac{dv}{dx}=\dfrac{dy}{dx}-v=\dfrac{2v\cos v}{v\sin v-\cos v}$

or $\left(\tan v-\dfrac{1}{v}\right)dv=2\dfrac{dx}{x}$, *i.e.*, $\log\dfrac{\sec v}{v}=\log C+2\log x$

or $\sec(y/x)=Cxy$ is the solution.

Ex. 8. *Solve* $\left(x\sin\dfrac{y}{x}\right)\dfrac{dy}{dx}=\left(y\sin\dfrac{y}{x}-x\right)$ [Delhi Pass 67]

Solution. Equation is $\dfrac{dy}{dx}=\dfrac{y}{x}-\operatorname{cosec}\dfrac{y}{x}.$

Putting $y=vx$. $\dfrac{dy}{dx}=x\dfrac{dv}{dx}+v.$

Equation reduces to $\sin v\;dv=-\dfrac{dx}{x}.$

Integrating, $-\cos v=-\log Cx$

or $\cos\dfrac{y}{x}=\log Cx$ is the solution.

Ex. 9. *Solve* $(x^2+2xy-y^2)\,dx+(y^2+2xy-x^2)\,dy=0.$

[Gujrat B.Sc. (Prin.) 1961]

Solution. $\dfrac{dy}{dx}=-\dfrac{x^2+2xy-y^2}{y^2+2xy-x^2}.$ Put $y=vx$.

$\therefore\quad v+x\dfrac{dv}{dx}=-\dfrac{1+2v-v^2}{v^2+2v-1}.$

$x\dfrac{dv}{dx}=-\dfrac{1+2v-v^2}{v^2+2v-1}-v=-\dfrac{v^3+v^2+v+1}{v^2+2v-1}.$

$\therefore\quad\dfrac{dx}{x}=-\dfrac{v^2+2v-1}{v^3+v^2+v+1}\,dv=-\dfrac{v^2+2v-1}{(v+1)(v^2+1)}\,dv$

$$=\left(\frac{1}{v+1}-\frac{2v}{v^2+1}\right)dv.$$

Integrating, $\log x=\log(v+1)-\log(v^2+1)+\log C$

or $\dfrac{x}{v^2+1}=C(v+1)$ or $\dfrac{x}{y^2/x^2+1}=C\left(\dfrac{y}{x}+1\right)$

Ex. 10. *Solve* $2y^3\,dx+(x^2-3y^2)\,x\,dy=0.$

[Bombay B.Sc. (Sub.) 1962]

Solution. Proceed yourself.

2·5. Equation Reducible to Homogeneous Form.

An equation of the type $\boxed{\dfrac{dy}{dx}=\dfrac{ax+by+c}{a'x+b'y+c'}}$, when $\boxed{\dfrac{a}{a'}\neq\dfrac{b}{b'}}$ can be reduced to homogeneous form as follows :

Put $x=X+h$, $y=Y+k$; then $\dfrac{dy}{dx}=\dfrac{dY}{dX}$, where X, Y are new variables and h, k are arbitrary constants. The equation now becomes

$$\frac{dY}{dX}=\frac{aX+bY+(ah+bk+c)}{a'X+b'Y+(a'h+b'k+c')}.$$

We choose the constants h and k in such a way that

$$ah+bk+c=0,\quad a'h+b'k+c'=0.$$

With this substitution the differential equation reduces to $\dfrac{dY}{dX}=\dfrac{aX+bY}{a'X+b'Y}$ which is a homogeneous equation in X, Y and can be solved by putting $Y=vX$ as earlier.

Special Case. When $\dfrac{a}{a'}=\dfrac{b}{b'}=\dfrac{1}{m}$ (say), then the differential equation can be written as

$$\frac{dy}{dx}=\frac{ax+by+c}{m(ax+by)+c} \quad \ldots(1)$$

Put $ax+by=v$, so that $a+b\dfrac{dy}{dx}=\dfrac{dv}{dx}$.

(1) then becomes $\dfrac{1}{b}\left(\dfrac{dv}{dx}-a\right)=\dfrac{v+c}{mv+c}$ in which variables can be separated.

Ex. 1. *Solve* $\dfrac{dy}{dx}=\dfrac{x+2y-3}{2x+y-3}$. [Vikram 60]

Solution. Put $x=X+h$, $y=Y+k$, where h, k are some constants; then $\dfrac{dy}{dx}=\dfrac{dY}{dX}$. The given equation then becomes

$$\frac{dY}{dX}=\frac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)}.$$

Now choose h, k such that $h+2k-3=0$ and $2h+k-3=0$. Solving these we get $h=1$, $k=1$.

$$\therefore \quad \frac{dY}{dX}=\frac{X+2Y}{2X+Y} \text{ homogeneous in } X \text{ and } Y.$$

Put $Y=vX$, so that $\dfrac{dY}{dX}=v+X\dfrac{dv}{dX}$.

$$\therefore \quad v+X\frac{dv}{dX}=\frac{X+2vX}{2X+vX}=\frac{1+2v}{2+v}, \text{ i.e., } X\frac{dv}{dX}=\frac{1+2v}{2+v}-v$$

or $$\frac{dX}{X}=\frac{2+v}{1-v^2}\,dv=\left(\frac{1}{1-v^2}+\frac{v}{1-v^2}\right)dv.$$

Integrating, $\log X=2.\frac{1}{2}\log\dfrac{1+v}{1-v}-\frac{1}{2}\log(1-v^2)+\log C$

or $$X=C\,\frac{1+v}{1-v}\cdot\frac{1}{\sqrt{(1-v^2)}}=\frac{C\sqrt{(1+v)}}{(1-v)^{3/2}}$$

or $$X^2(1-v)^3=C^2(1+v)$$

or $$X^2\left(1-\frac{Y}{X}\right)^3=C^2\left(1+\frac{Y}{X}\right) \text{ as } v=\frac{Y}{X}$$

or $$(X-Y)^3=C^2(X+Y) \text{ but } x=X+1,\ y=Y+1.$$

$\therefore \quad (x-y)^3=C^2(x+y-2)$ is the required solution.

ex **Ex. 2.** Solve $(3x-7y-3)\dfrac{dy}{dx}=3y-7x+7$. **[Raj. M.Sc. 61]**

Solution. $\dfrac{dy}{dx}=\dfrac{3y-7x+7}{3x-7y-3}$.

Put $x=X+h$, $y=Y+k$, where h, k are some constants. Then $\dfrac{dy}{dx}=\dfrac{dY}{dX}$. And the given equation becomes

$$\frac{dY}{dX}=\frac{3Y-7X+(3k-7h+7)}{3X-7Y+(3h-7k-3)}.$$

Choose h, k such that $3h-7k-3=0$ and $3k-7h+7=0$, which give $h=1$, $k=0$.

$$\therefore \quad \frac{dY}{dX}=\frac{3Y-7X}{3X-7Y} \text{ [homogeneous].}$$

Put $Y=vX$, $\dfrac{dY}{dX}=v+X\dfrac{dv}{dX}$.

$$\therefore \quad v+X\frac{dv}{dX}=\frac{3vX-7X}{3X-7vX}=\frac{3v-7}{3-7v}$$

or $$X\frac{dv}{dX}=\frac{3v-7}{3-7v}-v=\frac{7(v^2-1)}{3-7v}$$

or $$\frac{7\,dX}{X}=\frac{3-7v}{(v^2-1)}\,dv=-\left(\frac{2}{v-1}+\frac{5}{v+1}\right)dv.$$

Integrating, $7 \log X = -2 \log (v-1) - 5 \log (v+1) + \log C$

or $\quad X^7 (v-1)^2 (v+1)^5 = C$

or $\quad X^7 \left(\dfrac{Y}{X}-1\right)^2 \left(\dfrac{Y}{X}+1\right)^5 = C \quad$ as $\; Y = vX$

or $\quad (Y-X)^2 (Y+X)^5 = C$

or $\quad (y-x+1)^2 (y+x-1)^5 = C \quad$ as $\; x = X+1,\; y = Y+0.$

Ex. 3. *Solve* $(2x+y+3) \dfrac{dy}{dx} = x+2y+3.$

[Karnatak B.Sc. (Princ.) 60]

Solution. $\dfrac{dy}{dx} = \dfrac{x+2y+3}{2x+y+3}.$

Put $x = X+h,\; y = Y+k$, where h, k are constants.

$\therefore \quad dx = dX, \quad dy = dY; \quad \therefore \quad \dfrac{dy}{dx} = \dfrac{dY}{dX}$

$\therefore \quad \dfrac{dY}{dX} = \dfrac{X+2Y+(h+2k+3)}{2X+Y+(2h+k+3)}.$

Choose h, k such that $h+2k+3=0,\; 2h+k+3=0.$ Solving these, we get $h=-1,\; k=-1.$

$\therefore \quad \dfrac{dY}{dX} = \dfrac{X+2Y}{2X+Y}.$ Put $Y = vX,\; \dfrac{dY}{dX} = v + X\dfrac{dv}{dX}.$

$\therefore \quad v + X \dfrac{dv}{dX} = \dfrac{X+2vX}{2X+vX} \quad$ or $\quad X \dfrac{dv}{dX} = \dfrac{1+2v}{2+v} - v$

or $\quad \dfrac{dX}{X} = \dfrac{2+v}{1-v^2}\, dv = \left(\dfrac{\frac{3}{2}}{1-v} + \dfrac{\frac{1}{2}}{1+v}\right) dv.$

Integrating, $2 \log X = -3 \log (1-v) + \log (1+v) + \log C$

or $\quad X^2 \dfrac{(1-v)^3}{1+v} = C \quad$ or $\quad X^2 \dfrac{(1-Y/X)^3}{(1+Y/X)} = C$

or $\quad (X-Y)^3 = C\,(X+Y)\;$; where $x = X-1,\; y = Y-1$

or $\quad (X-y)^3 = C\,(x+y-2)$ is the solution.

Ex. 4. *Solve* $(2x-2y+5) \dfrac{dy}{dx} = x-y+3.$

[Sagar 63; Agra B.Sc. 61, 52]

Solution. The equation is $\dfrac{dy}{dx} = \dfrac{x-y+3}{2(x-y)+5}.$

Put $x-y=v$, so that $1 - \dfrac{dy}{dx} = \dfrac{dv}{dx} \quad$ or $\quad \dfrac{dy}{dx} = 1 - \dfrac{dv}{dx}.$

$\therefore$ The equation becomes

$1 - \dfrac{dv}{dx} = \dfrac{v+3}{2v+5} \quad$ or $\quad \dfrac{dv}{dx} = 1 - \dfrac{v+3}{2v+5} = \dfrac{v+2}{2v+5}$

or $\quad dx = \dfrac{2v+5}{v+2}\, dv = \left(2 + \dfrac{1}{v+2}\right) dv$, separating the variables.

Integrating, $x=2v+\log(v+2)+C$,

$$x=2(x-y)+\log(x-y+2)+C \text{ as } v=x-y$$

or $$2y-x=\log(x-y+2)+C \text{ is the required solution.}$$

Ex. 5. *Solve* $\dfrac{dy}{dx}=\dfrac{6x-4y+3}{3x-2y+1}$.

[Poona 64; Karnatak B.Sc. (Princ.) 61]

Solution. Put $3x-2y=v$, *i.e.*, $3-2\dfrac{dy}{dx}=\dfrac{dv}{dx}$.

$$\therefore \quad \frac{dv}{dx}=3-2\frac{2v+3}{v+1}=-\frac{v+3}{v+1}.$$

$$\therefore \quad dx=-\frac{v+1}{v+3}\,dv=-\left(1-\frac{2}{v+3}\right)dv.$$

Integrating, $x=-v+2\log(v+3)+C$

or $$x=(2y-3x)+2\log(3x-2y+3)+C$$

or $$2x-y=\log(3x-2y+3)+\tfrac{1}{2}C \text{ is the solution.}$$

Ex. 6. *Solve* $(6x-4y+1)\dfrac{dy}{dx}=(3x-2y+1)$.

[Karnatak B.Sc. (Sub.) 61]

Solution. $\dfrac{dy}{dx}=\dfrac{3x-2y+1}{2(3x-2y)+1}$. Put $3x-2y=v$.

$$\therefore \quad \frac{dv}{dx}=3-2\frac{dy}{dx}=3-2\frac{v+1}{2v+1}=\frac{4v+1}{2v+1}$$

or $\quad dx=\dfrac{2v+1}{4v+1}\,dv \quad$ or $\quad 2\,dx=\left(1+\dfrac{1}{4v+1}\right)dv$ etc.

Ex. 7. *Solve the following equations :*

(i) $(2x+y+1)\,dx+(4x+2y-1)\,dy=0$.

[Gujrat B.Sc. (Princ.) 61]

(ii) $\dfrac{dy}{dx}=\dfrac{6x-2y-7}{3x-y+4}$.

[Luck. Pass 56]

(iii) $(2x-5y+3)\,dx-(2x+4y-6)\,dy=0$. [Delhi Hons. 61]

(iv) $\dfrac{dy}{dx}=\dfrac{y-x+1}{y-x-5}$

[Poona 62; Nag. 62]

(v) $\dfrac{dy}{dx}=\dfrac{3x-4y-2}{3x-4y-3}$.

[Cal. Hons 63]

(vi) $(3y+2x+4)\,dx-(4x+6y+5)\,dx=0$. [Karnatak 63]

(vii) $(2x-5y+3)\,dx-(2x+4y-6)\,dy=0$. [Delhi Hons. 66]

(viii) $(x-y-2)\,dx+(x-2y-3)\,dy=0$. [All. 66]

(ix) $(4x+2y+1)\,dy=(2x+y+3)\,dx$. [Delhi Pass 67]

Hint. In (*i*) put $2x+y=v$, in (*ii*) put $3x-y=v$ and (*iii*) can be reduced to homogeneous form as usual. In (*ix*) putting $v=2x+y$, variables can be separated.

Ex. 8. *Solve* $2y\dfrac{dy}{dx}=\dfrac{x+y^2}{x+4y^2}$ [Bombay B.Sc. 61]

Solution. Put $y^2=v$, $2y\dfrac{dy}{dx}=\dfrac{dv}{dx}$.

$\therefore\ \dfrac{dv}{dx}=\dfrac{x+v}{x+4v}$ [homogeneous]. Now put $v=xz$ etc.

2·6. A particular case

A differential equation of the form

$$\frac{dy}{dx}=\frac{ax+by+c}{-bx+hy+k'}$$

in which coefficient of y in the numerator is equal to the coefficient of x in the denominator with sign changed, can be integrated as follows :

The equation (1) can be written as

$$-b\,(x\,dy+y\,dx)+(hy+k)\,dy-(ax+c)\,dx=0.$$

Integrating, we get $-bxy+(\frac{1}{2}hy^2+ky)-(\frac{1}{2}ax^2+cx)=A.$

Ex. 1. *Solve* $\dfrac{dy}{dx}+\dfrac{ax+hy+g}{hx+by+f}=0.$

[Raj. B.Sc. 66; Agra B.Sc. 57; Delhi B.A. 57; Raj. M.Sc. 62]

Solution. The equation can be written as

$$(hx+by+f)\,dy+(ax+hy+g)\,dx=0$$

or $h\,(x\,dy+y\,dx)+(by+f)\,dy+(ax+g)\,dx=0.$

Integrating, $hxy+\frac{1}{2}by^2+fy+\frac{1}{2}ax^2+gx=A$

or $ax^2+2hxy+by^2+2fy+2gx+c=0$, writing $c=-2A$.

Ex. 2. *Solve* $\dfrac{dy}{dx}=\dfrac{2x-y+1}{x+2y-3}$ [Agra B Sc. 59; Nag. 53 (S)]

Solution. Here coefficient of y in numerator is equal to coefficient of x in the denominator with sign changed. Hence write it as

$$(x+2y-3)\,dy-(2x-y+1)\,dx=0$$

or $(x\,dy+y\,dx)+(2y-3)\,dy-(2x+1)\,dx=0.$

Integrating, $xy+y^2-3y-x^2-x=C.$

Ex. 3. *Solve* $(2x-y+1)\,dx+(2y-x-1)\,dy=0.$

[Bombay B Sc. (Sub.) 61; Poona 61]

Solution. The equation is of above type. Hence after regrouping, we have

$$(2x+1)\,dx+(2y-1)\,dy-(y\,dx+x\,dy)=0.$$

Integrating, $(x^2+x)+(y^2-y)-xy=C$,

which is the solution.

Ex. 4. *Solve* $\dfrac{dy}{dx}+\dfrac{2x+3y+1}{3x+4y-1}=0.$ [Delhi Hons. 60]

Solution The equation is of the above type and can be written as

$$(3x+4y-1)\,dy+(2x+3y+1)\,dx=0,$$

i.e., $3(x\,dy+y\,dx)+(4y-1)\,dy+(2x+1)\,dx=0.$

Integrating, $3xy+2y^2-y+x^2+x=C$ is the solution.

2.7. Linear Differential Equations

[Poona 63, 61; Nagpur 62, 61; Guj 61]

A differential equation of the form

$$\frac{dy}{dx}+Py=Q,$$

where P, Q are functions of x or constants, is called the *linear differential equation of the first order.*

To solve this equation, multiply both the sides by $e^{\int P\,dx}$.

Then it becomes $e^{\int P\,dx}\dfrac{dy}{dx}+Py\,e^{\int P\,dx}=Qe^{\int P\,dx}$,

or $$\frac{d}{dx}\left[ye^{\int P\,dx}\right]=Q\,e^{\int P\,dx}.$$

Integrating both the sides, w.r.t. x, we get

$$ye^{\int P\,dx}=\int\left[Qe^{\int P\,dx}\right]dx+C,$$

which is the required solution.

Integrating factor (I.F.). It will be noticed that for solving (1), we multiplied it by a factor $e^{\int P\,dx}$ and the equation became readily (directly) integrable. Such a factor is called the integrating factor.

Note. Sometimes a differential equation takes linear form if we *regard x as dependent variable and y as independent variable.* The equation can then be put as $\dfrac{dx}{dy}+Px=Q$, where P, Q are functions of y or constants.

The integrating factor in this case is $e^{\int P\,dy}$ and solution is

$$xe^{\int P\,dy}=\int\left[Qe^{\int P\,dy}\right]dy+C.$$

(See Ex. 1 to 4 pages 21 and 22).

important

Ex. 1. *Solve* $(1-x^2)\dfrac{dy}{dx}-xy=1.$

[Delhi 68 : Nag. 61]

Solution. The equation can be written as

$$\frac{dy}{dx}-\frac{x}{1-x^2}\,y=\frac{1}{1-x^2}$$

This is now expressed in the linear form