Partial Differentiation:

Given a function of two variables, ƒ (x, y), the derivative with respect to x only (treating y as a constant) is called the partial derivative of ƒ with respect to x and is denoted by either ∂ƒ / ∂ x or ƒ x. Similarly, the derivative of ƒ with respect to y only (treating x as a constant) is called the partial derivative of ƒ with respect to y and is denoted by either ∂ƒ / ∂ y or ƒ y.

The second partial dervatives of f come in four types:

Notations:

- Differentiate ƒ with respect to x twice. (That is, differentiate ƒ with respect to x; then differentiate the result with respect to x again.)

$$\frac{\partial^2 f}{\partial x^2} \quad \text{or} \quad f_{xx}$$

- Differentiate ƒ with respect to y twice. (That is, differentiate ƒ with respect to y; then differentiate the result with respect to y again.)

$$\frac{\partial^2 f}{\partial y^2} \quad \text{or} \quad f_{yy}$$

Mixed partials:

- First differentiate ƒ with respect to x; then differentiate the result with respect to y.

$$\frac{\partial^2 f}{\partial y \partial x} \quad \text{or} \quad f_{xy}$$

- First differentiate ƒ with respect to y; then differentiate the result with respect to x.

$$\frac{\partial^2 f}{\partial x \partial y} \quad \text{or} \quad f_{yx}$$

For virtually all functions ƒ (x, y) commonly encountered in practice, ƒ vx ; that is, the order in which the derivatives are taken in the mixed partials is immaterial.

Example 1: If ƒ (x, y) = 3 x 2 y + 5 x − 2 y 2 + 1, find ƒ x , ƒ y , ƒ xx , ƒ yy , ƒ xy 1, and ƒ yx .

First, differentiating ƒ with respect to x (while treating y as a constant) yields

$$f_x = 6xy + 5$$

Next, differentiating ƒ with respect to y (while treating x as a constant) yields

$$f_y = 3x^2 - 4y$$

The second partial derivative ƒ xx means the partial derivative of ƒ x with respect to x; therefore,

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(6xy + 5) = 6y$$

The second partial derivative ƒ yy means the partial derivative of ƒ y with respect to y; therefore,

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}(3x^2 - 4y) = -4$$

The mixed partial ƒ xy means the partial derivative of ƒ x with respect to y; therefore,

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(6xy + 5) = 6x$$

The mixed partial ƒ yx means the partial derivative of ƒ y with respect to x; therefore,

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(3x^2 - 4y) = 6x$$

Note that ƒ yx = ƒ xy, as expected.

Exercise Question:

Solutions to Examples on Partial Derivatives

1. (a) $f(x,y) = 3x + 4y;\quad \dfrac{\partial f}{\partial x} = 3;\quad \dfrac{\partial f}{\partial y} = 4.$

(b) $f(x,y) = xy^3 + x^2y^2;\quad \dfrac{\partial f}{\partial x} = y^3 + 2xy^2;\quad \dfrac{\partial f}{\partial y} = 3xy^2 + 2x^2y.$

(c) $f(x,y) = x^3y + e^x;\quad \dfrac{\partial f}{\partial x} = 3x^2y + e^x;\quad \dfrac{\partial f}{\partial y} = x^3.$

(d) $f(x,y) = xe^{2x+3y};\quad \dfrac{\partial f}{\partial x} = 2xe^{2x+3y} + e^{2x+3y};\quad \dfrac{\partial f}{\partial y} = 3xe^{2x+3y}.$

(e) $f(x,y) = \dfrac{x-y}{x+y}.$

$$\frac{\partial f}{\partial x} = \frac{x+y-(x-y)}{(x+y)^2} = \frac{2y}{(x+y)^2};$$
$$\frac{\partial f}{\partial y} = \frac{-(x+y)-(x-y)}{(x+y)^2} = -\frac{2x}{(x+y)^2}.$$

(f) $f(x,y) = 2x\sin(x^2y).$

$$\frac{\partial f}{\partial x} = 2x\,.\,\cos(x^2y)\,.\,2xy + 2\sin(x^2y) = 4x^2y\cos(x^2y) + 2\sin(x^2y);$$
$$\frac{\partial f}{\partial y} = 2x\,.\,\cos(x^2y)\,.\,x^2 = 2x^3\cos(x^2y).$$

2. $f(x,y,z) = x\cos z + x^2y^3e^z.$

$$\frac{\partial f}{\partial x} = \cos z + 2xy^3e^z,$$
$$\frac{\partial f}{\partial y} = 3x^2y^2e^z,$$
$$\frac{\partial f}{\partial z} = -x\sin z + x^2y^3e^z.$$