Solution. The equation is of the above type and can be written as

$$(3x+4y-1)\,dy+(2x+3y+1)\,dx=0,$$

i.e., $3(x\,dy+y\,dx)+(4y-1)\,dy+(2x+1)\,dx=0.$

Integrating. $3xy+2y^2-y+x^2+x=C$ is the solution.

2.7. Linear Differential Equations

[Poona 63, 61; Nagpur 62, 61; Guj. 61]

A differential equation of the form

$$\frac{dy}{dx}+Py=Q,$$

where P, Q are functions of x or constants, is called the *linear differential equation of the first order.*

To solve this equation, multiply both the sides by $e^{\int P\,dx}$

Then it becomes $e^{\int P\,dx}\dfrac{dy}{dx}+Py\,e^{\int P\,dx}=Qe^{\int P\,dx}$.

or $\quad \dfrac{d}{dx}\left[ye^{\int P\,dx}\right]=Q\,e^{\int P\,dx}$.

Integrating both the sides, w.r.t. x, we get

$$ye^{\int P\,dx}=\int\left[Qe^{\int P\,dx}\right]dx+C,$$

which is the required solution.

Integrating factor (I.F.). It will be noticed that for solving (1), we multiplied it by a factor $e^{\int P\,dx}$ and the equation became readily (directly) integrable. Such a factor is called the integrating factor.

Note. Sometimes a differential equation takes linear form if we *regard x as dependent variable and y as independent variable.* The equation can then be put as $\dfrac{dx}{dy}+Px=Q$, where P, Q are functions of y or constants.

The integrating factor in this case is $e^{\int P\,dy}$ and solution is

$$xe^{\int P\,dy}=\int\left[Qe^{\int P\,dy}\right]dy+C.$$

(See Ex. 1 to 4 pages 21 and 22).

Ex. 1. *Solve* $(1-x^2)\dfrac{dy}{dx}-xy=1.$

[Delhi 68 : Nag. 61]

Solution. The equation can be written as

$$\frac{dy}{dx}-\frac{x}{1-x^2}\,y=\frac{1}{1-x^2}$$

This is now expressed in the linear form

$$P = -\frac{x}{1-x^2}, \quad \text{I.F.} = e^{\int P\,dx} = e^{\int \frac{-x}{1-x^2}dx} = e^{\frac{1}{2}\log(1-x^2)}$$
$$= \sqrt{(1-x^2)}.$$

Hence the solution is

$$y.\sqrt{(1-x^2)} = \int \frac{1}{1-x^2}\sqrt{(1-x^2)}\,dx + C.$$

Ex. 2. (a) *Solve* $x\frac{dy}{dx} + 2y = x^2 \log x.$ **[Lucknow 52]**

Solution. The equation is $\frac{dy}{dx} + \frac{2}{x}y = x \log x.$

$$\text{I.F.} = e^{\int (2/x)\,dx} = e^{2\log x} = x^2.$$

Hence the solution is

$$y.x^2 = C + \int x^2.x \log x\,dx = C + \int x^3 \log x\,dx$$
$$= C + \log x.\frac{x^4}{4} - \int \frac{1}{x}.\frac{x^4}{4}dx$$
$$= C + \tfrac{1}{4}x^4 \log x - \tfrac{1}{16}x^4$$

or $\quad y = Cx^{-2} + \frac{1}{4}x^2 (\log x - \frac{1}{4}).$

Ex. 2. (b) *Solve* $x\frac{dy}{dx} + 2y = x^4.$

[Bombay B.Sc. 61]

Solution. Equation is $\frac{dy}{dx} + \frac{2}{x}y = x^3$. **I.F.** $= x^2$ **as above.**

Solution is $\quad y.x^2 = C + \int x^3.x^2\,dx = C + \frac{1}{6}x^6.$

Ex. 3. *Solve* $(x^3 - x)\frac{dy}{dx} - (3x^2 - 1)\,y = x^5 - 2x^3 + x.$

[Gujrat B.Sc. (Sub.) 1961]

Solution. The equation is

$$\frac{dy}{dx} - \frac{3x^2-1}{x^3-x}\,y = (x^2 - 1).$$

$$\text{I.F.} = e^{-\int \frac{(3x^2-1)}{x^3-x}} = e^{-\log(x^3-x)} = \frac{1}{x^3-x}.$$

$\therefore$ Solution is $y.\frac{1}{x^3-x} = C + \int \frac{x^2-1}{x^3-1}dx$

$$= C + \int \frac{1}{x}\,dx = C + \log x.$$

Ex 4. *Sol.e* $\quad xp + y = ax^2 + bx + c, \; p = \frac{dy}{dx}.$

[Delhi Hons. 1957]

where P and Q are functions of x or constants.

[Nag. T.D.C. 1961; Poona T.D.C. 61 ; Gujrat B.Sc. (Prin.) 58; Poona B.A. (Gen.) 60]

Dividing both the sides by y^n we have

$$y^{-n}\frac{dy}{dx}+Py^{-n+1}=Q. \quad ...(1)$$

Now put $y^{-n+1}=v$ so that $(1-n)\,y^{-n}\dfrac{dy}{dx}=\dfrac{dv}{dx}$.

Then (1) becomes $\dfrac{1}{1-n}\dfrac{dv}{dx}+Pv=Q$

or $\dfrac{dv}{dx}+P\,(1-n)\,v=(1-n)\,Q$

which is a linear equation in v and x.

II. Equation $f'(y)\dfrac{dy}{dx}+Pf(y)=Q$,

where P and Q are functions of x or constants.

Put $f(y)=v$ so that $f'(y)\dfrac{dy}{dx}=\dfrac{dv}{dx}$.

$\therefore$ equation becomes $\dfrac{dv}{dx}+Pv=Q$,

which is a linear equation in v and x.

Note. In each of these equations, single out Q (function o. .. on the right) and then make suitable substitution to reduce the equation in linear form.

Ex. 1. *Solve* $\dfrac{dy}{dx}=x^3y^3-xy$.

[Karnatak B.Sc. (Prin.) 1960, 62; Agra 61; Bihar 62; Gujrat B.Sc. (Sub.) 61]

Solution. The equation is $\dfrac{dy}{dx}+xy=x^3y^3$.

Dividing by y^3; $\dfrac{1}{y^3}\dfrac{dy}{dx}+x.\dfrac{1}{y^2}=x^3$.

Put $\dfrac{1}{y^2}=v$, so that $-\dfrac{2}{y^3}\dfrac{dy}{dx}=\dfrac{dv}{dx}$, *i.e.*, $\dfrac{1}{y^3}\dfrac{dy}{ax}=-\frac{1}{2}\dfrac{dv}{dx}$.

$\therefore$ equation becomes $-\frac{1}{2}\dfrac{dv}{dx}+xv=x^3$

or $\dfrac{dv}{dx}-2x.v=-2x^3$.

Linear, I. F. $=e^{\int -2x\,dx}=e^{-x^2}$.

Hence $ve^{-x^2}=\int -2x^3e^{-x^2}\,dx+C$

$=\int x^2\,(-2x)\,e^{-x^2}\,dx+C$

3
Equations of First Order and First Degree

Exact Differential Equations and Reduction to Exact Equations

3·1. Exact Differential Equations. **[Bombay 61 ; Karnatak 60]**

Study the following two differential eqnations :

1. $x\,dy+y\,dx=0$. Solution is $xy=C$.
2. $\sin x\cos y\,dy+\cos x\sin y\,dx=0$.

   Solution is $\sin x\sin y=C$.

We see that these differential equations can be obtained by directly differentiating their solutions. Differential equations of this type are called exact equations and bear the following property :

An exact differential equation can always be obtained from its primitive directly by differentiation, without any subsequent multiplication, elimination etc.

***3·2. Necessarry and Sufficient Condition**

To find the necessary and sufficient condition for a differential equation of first degree being exact.

[Poona 63, 61 ; Delhi Hons. 57, 55 ; Nag. 63 ; Gujrat 59; Bombay 61]

Let the equation be $M+N\dfrac{dy}{dx}=0$. ...(1)

Let $u=C$ be its primitive. ...(2)

If (1) is exact, it can be obtained by directly differentiating its primitive.

Differentiating (2), we have $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}\cdot\dfrac{dy}{dx}=0$. ...(3)

Comparing (1) and (3) we get $M=\dfrac{\partial u}{\partial x}$ and $N=\dfrac{\partial u}{\partial y}$, so that

$$\frac{\partial M}{\partial y}=\frac{\partial^2 u}{\partial y\partial x},\quad \frac{\partial N}{\partial x}=\frac{\partial^2 u}{\partial x\partial y}.$$

Hence the condition is $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$.

That the condition is necessary has been proved. Now we prove that it is sufficient also, *i.e.* if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then we show that $M+N\frac{dy}{dx}=0$ or $M\,dx+N dy=0$ is an exact equation.

Let $\int M\,dx=U$, then $\frac{\partial U}{\partial x}=M$, so that

$$\frac{\partial^2 U}{\partial y\,\partial x}=\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \text{ as } \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}.$$

i.e. $\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right)$

Integrating, $N=\frac{\partial U}{\partial y}+f(y)$, where $f(y)$ is a function of y free from x.

$$\therefore \quad M+N\frac{dy}{dx}=\frac{\partial U}{\partial x}+\left[\frac{\partial U}{\partial y}+f(y)\right]\frac{dy}{dx}$$

$$=\frac{d}{dx}\left[U+\int f(y)\,\frac{dy}{dx}\,dx\right]$$

$$=\frac{d}{dx}\,[U+F(y)].$$

This shows that $M+N\frac{dy}{dx}=0$ is an exact equation.

3·3. Working Rule (Remember it).

If the equation $M\,dx+N\,dy=0$ satisfies the condition

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x},$$

then it is exact. To integrate it,

(*i*) integrate M with regard to x regarding y as constant;

(*ii*) find out those terms in N which are free from x and integrate them with regard to y;

(*iii*) add the two expressions so obtained and equate the sum to an arbitrary constant.

This gives the general solution of the given exact equation.

Ex. 1. $(y^4+4x^3y+3x)\,dx+(x^4+4xy^3+y+1)\,dy=0$ **[Karnatak 60]**

Solution Here $M=y^4+4x^3y+3x$ and $N=x^4+4xy^3+y+1$.

$$\frac{\partial M}{\partial y}=4y^3+4x^3 \text{ and } \frac{\partial N}{\partial x}=4x^3+4y^3.$$

Since these are equal, the equation is exact.

To find solution of the differential equation, integrating M *i.e.* y^4+4x^3y+3x w.r.t. x, keeping y as constant, we get $y^4x+x^4y+\frac{3}{2}x^2$.

$$= \int -te^t\,dt + C \text{ where } t = -x^2$$
$$= -te^t + e^t + C = -e^{-x^2}(x^2-1) + C$$

Hence $\quad v = 1 - x^2 + Ce^{x^2}$ or $\dfrac{1}{y^2} = 1 - x^2 + Ce^{x^2}$

Ex. 2. *Solve* $\dfrac{dy}{dx} + xy = xy^2$. [Karnatak 1960]

Solution. Dividing by y^2, $y^{-2}\dfrac{dy}{dx} + xy^{-1} = x$.

Put $y^{-1} = v$, so that $-y^{-2}\dfrac{dy}{dx} = \dfrac{dv}{dx}$.

$\therefore$ equation is $\dfrac{dv}{dx} - xv = -x$.

I. F. $= e^{\int -x\,dx} = e^{-\frac{1}{2}x^2}$.

$\therefore \quad ve^{-\frac{1}{2}x^2} = C - \int xe^{-\frac{1}{2}x^2}$

$\qquad = C + \int e^t\,dt$, where $-\frac{1}{2}x^2 = t$, $-x\,dx = dt$

or $\quad y^{-1}e^{-\frac{1}{2}x^2} = C + e^t = C + e^{-\frac{1}{2}x^2}$

or $\quad y^{-1} = Ce^{\frac{1}{2}x^2} + 1$ is the solution.

Ex 3. *Solve* $\dfrac{dy}{dx} + \dfrac{2}{x}y = \dfrac{y^3}{x^3}$. [Nag. 1958]

Solution. Dividing by y^3, $y^{-3}\dfrac{dy}{dx} + \dfrac{2}{x}y^{-2} = \dfrac{1}{x^3}$.

Put $y^{-2} = v$, so that $-2y^{-3}\dfrac{dy}{dx} = \dfrac{dv}{dx}$.

$\therefore$ equation becomes $-\frac{1}{2}\dfrac{dv}{dx} + \dfrac{2}{x}v = \dfrac{1}{x^3}$

or $\quad \dfrac{dv}{dx} - \dfrac{4}{x}v = -\dfrac{2}{x^3}$.

I. F. $= e^{\int(-4/x)\,dx} = e^{-4\log x} = \dfrac{1}{x^4}$.

$\therefore \quad v\dfrac{1}{x^4} = \int -\dfrac{2}{x^3}\cdot\dfrac{1}{x^4}\,dx + C = C + \dfrac{1}{3x^6}$

or $\quad \dfrac{1}{y^2}\cdot\dfrac{1}{x^4} = \dfrac{1}{3x^6} + C$ is the solution.

***Ex. 4.** *Solve* $\dfrac{dy}{dx}(x^2y^3 + xy) = 1$.

[Sagar 1962; Raj. 63; Cal. Hons. 62; Luck. 63]

Solution. The equation can be written as

$$\frac{dx}{dy} - xy = x^2y^3.$$