$$= \left| x \sin \frac{1}{x} \right| = | x | \cdot \left| \sin \frac{1}{x} \right|$$

$$\leq x, \text{ since } \left| \sin \frac{1}{x} \right| \leq 1$$

$$< \epsilon \text{ for } | x - 0 | < \epsilon.$$

The relations are satisfied, if $\delta = \epsilon$. So, $f(x)$ is continuous at $x = 0$.

(*b*) Here, $f(x) = x^2 \cos\left(\frac{1}{x}\right)$, when $x \neq 0$

$$f(0) = 0.$$

For continuity of $f(x)$ at $x = 0$, we are to find a δ depending upon ϵ, such that

$$| f(x) - f(0) | < \epsilon, \text{ for } | x - 0 | < \delta.$$

i.e., $\left| x^2 \cos \frac{1}{x} - 0 \right| < \epsilon$ for, $| x | < \delta$.

Since $\left| \cos \frac{1}{x} \right| \leq 1$, relations are satisfied, if we take $| x^2 | < \epsilon$ for $| x | < \delta$.

So, the relations are satisfied if $\delta = \sqrt{\epsilon}$.

Hence, $f(x)$ is continuous at $x = 0$.

EXAMPLES-IV

1. A function $f(x)$ is defined as follows :

   $f(x) = x^2$ when $x \neq 1$, $f(x) = 2$ when $x = 1$.
   Is continuous at $x = 1$?

2. Are the following functions continuous at the origin ?

   (i) $f(x) = \sin(1/x)$ when $x \neq 0$, $f(0) = 0$.

   (ii) $f(x) = x \cos(1/x)$ when $x \neq 0$, $f(0) = 0$.

   (iii) $f(x) = x \cos(1/x)$ when $x \neq 0$, $f(0) = 1$.

   (iv) $f(x) = \dfrac{\sin \frac{1}{x}}{\frac{1}{x}}$ when $x \neq 0$.

   $= 1$ when $x = 0$.

(iv) $f(x) = \sin x \cos \dfrac{1}{x}$ when $x \neq 0$.
$\quad = 0$ when $x = 0$.

3. A function $\phi(x)$ is defined as follows :

$$\begin{aligned} \phi(x) &= x^2 && \text{when } x < 1, \\ &= 2.5 && \text{when } x = 1, \\ &= x^2 + 2 && \text{when } x > 1. \end{aligned}$$

Is $\phi(x)$ continuous at $x = 1$?

4. A function $f(x)$ is defined in the following way ;

$$\begin{aligned} f(x) &= -x && \text{when } x \le 0, \\ &= x && \text{when } 0 < x < 1, \\ &= 2 - x && \text{when } x \ge 1. \end{aligned}$$

Show that it is continuous at $x = 0$ and $x = 1$. [C.P. 1942]

5. A function $f(x)$ is defined as follows :

$$f(x) = 1,\ 0 \text{ or } -1 \text{ according as } x >, = \text{ or } < 0,$$

Show that it is discontinuous at $x = 0$.

6. The function $f(x) = \dfrac{x^2 - 16}{x - 4}$ is undefined at $x = 4$.

What value must be assigned to $f(4)$, if $f(x)$ is to be continuous at $x = 4$?

7. Determine whether the following functions are continuous at $x = 0$.

(i) $f(x) = \left(x^4 + x^3 + 2x^2\right) / \sin x,\ f(0) = 0$.

(ii) $f(x) = \left(x^4 + 4x^3 + 2x\right) / \sin x,\ f(0) = 0$.

8. Find the points of discontinuity of the following functions :

(i) $\dfrac{x^3 + 2x + 5}{x^2 - 8x + 12}$ (ii) $\dfrac{x^3 + 2x + 5}{x^2 - 8x + 16}$

9. A function $f(x)$ is defined as follows

$$\begin{aligned} f(x) &= 3 + 2x && \text{for } -\tfrac{3}{2} \le x < 0 \\ &= 3 - 2x && \text{for } 0 \le x < \tfrac{3}{2} \\ &= -3 - 2x && \text{for } x \ge \tfrac{3}{2}. \end{aligned}$$

Show that $f(x)$ is continuous at $x = 0$ and discontinuous at $x = \frac{3}{2}$

10. The function $y = f(x)$ is defined as follows : $f(x) = 0$ when, $f(x) = 1$ when $x^2 < 1$, $f(x) = \frac{1}{2}$ when $x^2 = 1$. Draw a diagram of the function and discuss from diagram that, except at points $x = 1$ and $x = -1$, the function is continuous. Discuss also why the function is discontinuous at these two points although it has a value for every value of x.

Examine the continuity of the functions at $x = 0$ (Ex. 11-14)

11. $f(x) = (1 + x)^{1/x}$, when $x \neq 0$

$= 1$, when $x = 0$.

12. $f(x) = (1 + 2x)^{1/x}$, when $x \neq 0$

$= e^2$, when $x = 0$.

13. $f(x) = e^{-1/x^2}$, when $x \neq 0$

$= 1$, when $x = 0$.

14. $f(x) = \dfrac{e^{-1/x}}{1 + e^{1/x}}$, when $x \neq 0$

$= 1$, when $x = 0$.

15. The function f is defined by

$f(x) = 2x - [x] + \sin\dfrac{1}{x}$, for $x \neq 0$

$= 0$, for $x = 0$,

where $[x]$ denotes the greatest integer not greater than x.

Examine the continuity of $f(x)$ at $x = 0$ and $x = 2$.

ANSWERS

1. No
2. (i) No. (ii) Yes. (iii) No. (iv) No. (v) Yes
3. No. 6. 8. 7. (i) Continuous. (ii) Discontinuous
8. (i) 6, 2. (ii) 4. 11. Discontinuous. 12. Continuous.
13. Discontinuous. 14. Discontinuous. 15. Discontinuous.

14 TANGENT AND NORMAL

14.1. We shall now consider certain properties of curves represented by continuous functions. If the equation of the curve is given in the explicit form $y = f(x)$, we shall assume that $f(x)$ has a derivative at every point, except, in some cases, at isolated points. If the equation of the curve is given in the implicit form $f(x, y) = 0$, we shall assume that the function $f(x, y)$ possesses continuous partial derivatives f_x and f_y which are not simultaneously zero. When the equation of the curve is given in the parametric form $x = \phi(t)$, $y = \psi(t)$, we shall assume that $\phi'(t)$ and $\psi'(t)$ are not simultaneously zero.

14.2. Equation of the tangent.

Def. *The tangent at P to a given curve is defined as the limiting position of the secant* $\overline{PQ}$ (when such a limit exists) *as the point Q approaches P along the curve* (whether Q is taken on one side or the other of the point P).

(i) *Let the equation of the curve be* $y = f(x)$ and let the given point P on the curve be (x, y) and any other neighbouring point Q on the curve be $(x+\Delta x, + y+\Delta y)$

The equation of the secant $\overline{PQ}$ is, (X, Y denoting the current co-ordinates)

$$Y - y = \frac{y+\Delta y - y}{x+\Delta x - x}(X - x) = \frac{\Delta y}{\Delta x}(X - x)$$

$\therefore$ the equation of the tangent P is

$$Y - y = \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x}(X - x) = \frac{dy}{dx}(X - x)$$

provided dy/dx is finite.

Thus, *the tangent to the curve* $y = f(x)$ at (x, y) (not parallel to the y-axis) is

$$\mathbf{Y - y = \frac{dy}{dx}(X - x)}. \qquad \ldots (1)$$

(ii) *When the equation of the curve is* $f(x, y) = 0$.

Since $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$, $(f_y \neq 0)$, *the equation of the tangent to the curve at* (x, y) *is*

$$(\mathbf{X}-\mathbf{x})\mathbf{f}_{\mathbf{x}}+(\mathbf{Y}-\mathbf{y})\mathbf{f}_{\mathbf{y}}=0. \quad \ldots (2)$$

(iii) *When the equation of the curve is* $x=\phi(t),\ y=\psi(t)$

since $\dfrac{dy}{dx}=\dfrac{dy}{dt}\Big/\dfrac{dx}{dt}=\dfrac{\psi'(t)}{\phi'(t)}, \quad \phi'(t)\neq 0,$

the equation of the tangent at the point 't' is

$$\left.\begin{aligned} &Y-\psi(t)=\frac{\psi'(t)}{\phi'(t)}\{X-\phi(t)\}, \\ i.e.,\ &\psi'(t)X-\phi'(t)Y=\phi(t)\psi'(t)-\psi(t)\phi'(t), \end{aligned}\right\} \quad \ldots (3)$$

Note 1. When the left-hand and right-hand derivatives at (x, y) are infinite, with equal or opposite signs, the tangent at (x, y) can be conveniently obtained by using the *alternatives form of the equation of the tangent* $X-x=(Y-y)(dx/dy)$ which can be easily established as before.

[See *Ex. 32, Examples XIV(A)*]

Note 2. In the notation of Co-ordinate Geometry, the equation of the tangent to the curve $y=f(x)$ at (x_1, y_1) can be written as

$$y-y_1=f'(x_1)(x-x_1).$$

In the application of Differential Calculus to the theory of plane curves, for the sake of convenience, the current co-ordinates in the equation of the tangent and normal are usually denoted by (X, Y) while those of any particular point are denoted by (x, y). The current co-ordinates in the equation of the curve are however, as usual, denoted by (x_1, y_1).

14.3. Geometrical meaning of $\dfrac{dy}{dx}$.

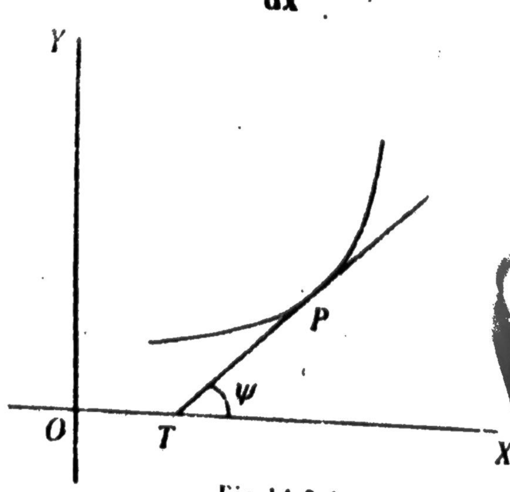

Fig 14.3.1

The equation (1) of the tangent can be written as

$$Y = \frac{dy}{dx} \cdot X + \left(y - x\frac{dy}{dx} \right)$$

which being of the form $y = mx + c$, the standard equation of a straight line, we conclude that

$\frac{dy}{dx}$ is the 'm' of the tangent at (x, y).

If ψ be the angle which the positive direction of the tangent at P makes with the positive direction of the x-axis, then $\tan\psi = m = \frac{dy}{dx}$.

Hence, *the direvative* $\frac{dy}{dx}$ *at* (x, y) *is equal to the trigonometrical tangent of the angle which the tangent to the curve at* (x, y) *makes with the positive direction of the* x-axis. [See Art. 7.14]

Note 1. It is customary to denote the angle which the tangent at any point on a curve makes with the x-axis by ψ.

Note 2. *The positive direction of the tangent* is the direction of the arc-length s increasing. Henceforth, this direciton will be spoken of as the direction of the tangent or simply as the tangent.

Note 3. $\tan\psi$, *i.e.*, $\frac{dy}{dx}$ is also called the **gradient of the curve at the point** $P(x, y)$.

Note 4. *The tangent at* (x, y) *is parallel to the x-axis* if $\psi = 0$, *i.e.*, if $\tan\psi = 0$, i.e., if $\frac{dy}{dx} = 0$.

The tangent at (x, y). *is perpendicular to the x-axis* (*i.e.*, parallel to the y-axis) if $\psi = \frac{1}{2}\pi$, *i.e.*, if $\cot\psi = 0$,

i.e., if $1 \Big/ \frac{dy}{dx} = 0$ or, $\frac{dx}{dy} = 0$.

14.4 Tangent at the origin.

If a curve passing through the origin be given by a rational integral algebraic equation, the equation of the tangent (or tangents) at the origin is obtained by equating to zero, the terms of the lowest degree in the equation.

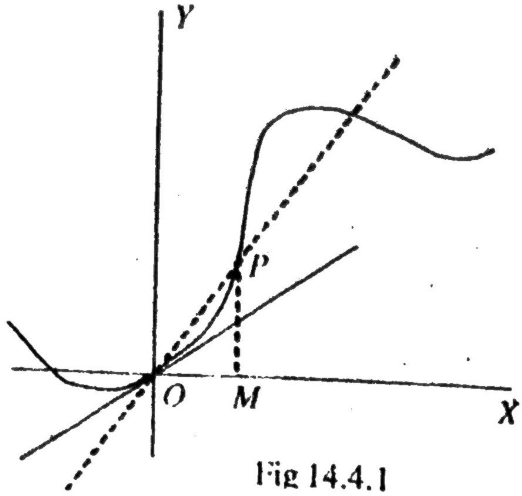

Fig 14.4.1

Let the equation of a curve of the n-th degree passing through the origin be

$$a_1 x + b_1 y + a_2 x^2 + b_2 xy + c_2 y^2 + \cdots + a_n x^n + \cdots + k_n y^n = 0 \qquad \ldots (1)$$

Let $P(x, y)$ be a point on the curve near the origin O. The equation of the secant $\overline{OP}$ is $Y = \frac{y}{x} X$.

$\therefore$ the equation of the tangent at O is

$$Y = \underset{\substack{x \to 0 \\ y \to 0}}{Lt} \frac{y}{x} \cdot X = mX \text{ (say)}. \qquad \ldots (2)$$

Thus, the 'm' of the tangent at the origin is $\underset{\substack{x \to 0 \\ y \to 0}}{Lt} \frac{y}{x}$.

CASE I. Let us suppose that m is finite, *i. e.*, the y-axis is not the tangent at the origin.

(i) Let us suppose $b_1 \neq 0$

Dividing (1) by x, we get

$$a_1 + b_1 \frac{y}{x} + a_2 x + b_2 y + c_2 y \cdot \frac{y}{x} + \cdots\cdots = 0$$

Now, let $x \to 0$, $y \to 0$, then $Lt\,(y/x) = m$.

$\therefore$ $a_1 + b_1 m = 0$, the other terms vanishing.

$\therefore$ $m = -a_1/b_1$. $\qquad \ldots (3)$

From (2) and (3), the equation of the tangent at the origin is $a_1 X + b_1 Y = 0$,

or, taking x and y as current co-ordinates, $a_1 x + b_1 y = 0$.

(ii) If $b_1 = 0$, then from (3) it follows that $a_1 = 0$; now in this case, let us suppose that b_2 and c_2 are not both zero. Then, the equation of the curve (1) can be written as

$$a_2x^2 + b_2xy + c_2y^2 + a_3x^3 + \cdots = 0 \qquad \ldots \qquad (4)$$

Dividing by x^2, $\quad a_2 + b_2\dfrac{y}{x} + c_2\left(\dfrac{y}{x}\right)^2 + a_3x + \cdots\cdots = 0$.

When $x \to 0$, $y \to 0$, we have

$a_2 + b_2m + c_2m^2 = 0$, the other terms vanishing. $\quad \ldots \quad$ (5)

From (5) it is clear that there are two values of m and hence, there are two tangents at the origin and their equation, which is obtained by eliminating m between (2) and (5), is

$$a_2X^2 + b_2XY + c_2Y^2 = 0$$

or, taking x and y as current co-ordinates,

$$a_2x^2 + b_2xy + c_2y^2 = 0.$$

If $a_1 = b_1 = a_2 = b_2 = c_2 = 0$, it can be shown similarly that the rule holds good then also; and so on.

Case II. When the tangent at the origin is the y-axis, then $Lt\,(x/y)$, as x and y both $\to 0$, being the tangent of the inclination of the tangent at the origin to the y-axis, is zero. Hence, dividing throughout the equation of the curve by y, and assuming $a_1 \neq 0$, and making x and y both approach zero, we find $b_1 = 0$. Hence, the equation of the curve now being

$$a_1x + a_2x^2 + b_2xy + c_2y^2 + \ldots\ldots = 0$$

we see that the theorem is still true in this case.

Illustration : If the equation of a curve be $x^2 - y^2 + x^3 + 3x^2y - y^3 = 0$ the tangents at the origin are given by $x^2 - y^2 = 0$, *i.e.*, $x + y = 0$ and $x - y = 0$.

14.5. Equation of the normal.

Definition. *The normal at any point of a curve is the straight line through that point drawn perpendicular to the tangent at that point.*

Let any line (not parallel to the co-ordinate axes) through the point be (x, y) be

$$Y - y = m(X - x).$$

This will be perpendicular to the tangent (not parallel to the co-ordinate axes) to the curve $y = f(x)$ at (x, y).

i.e., to $Y - y = \dfrac{dy}{dx}(X - x)$ if $m \cdot \dfrac{dy}{dx} = -1$, i.e., if $m = -1 \Big/ \dfrac{dy}{dx}$.

Substituting this value of m in the above equation, we see that *the normal to the curve $y = f(x)$ at (x, y)* (when not parallel to the co-ordinate axes) is

$$\frac{dy}{dx}(Y - y) + (X - x) = 0 \qquad \dots (1)$$

Similarly, if the equation of the curve is $f(x, y) = 0$, the equation of the normal at is

$$\frac{X - x}{f_x} = \frac{Y - y}{f_y} \qquad \dots (2)$$

and if the parametric equations of the curve are $x = \phi(t)$, $y = \psi(t)$, the equation of the normal at the point 't' is

$$\phi'(t)X + \psi'(t)Y = \phi(t)\phi'(t) + \psi(t)\psi'(t) \qquad \dots (3)$$

Note 1. When the tangents are parallel to $\overleftrightarrow{OX}$ and $\overleftrightarrow{OY}$ the nor mals are $X = x$ and $Y = y$ respectively.

Note 2. The *positive direction of the normal* makes an angle $+\frac{1}{2}\pi$ with the tangent, or $\frac{1}{2}\pi + \psi$ with the x-axis.

14.6. Angle of intersection of two curves.

The *angle of intersection of two curves* is the angle between the tangents to the two curves at their common point of intersection.

Suppose the two curves $f(x, y) = 0$, $\phi(x, y) = 0$ intersect at the point (x, y).

The tangents to the curves at (x, y) are

$$Xf_x + Yf_y - (xf_x + yf_y) = 0. \qquad [\text{by § } 14.2(2)]$$

$$X\phi_x + Y\phi_y - (x\phi_x + y\phi_y) = 0.$$

The angle α at which these lines cut is given by

$$\tan\alpha = \frac{f_x\phi_y - \phi_x f_y}{f_x\phi_x + f_y\phi_y}.$$

Hence, *if the curves touch at (x, y)*, $\alpha = 0$, i.e., $\tan\alpha = 0$

i.e., $f_x\phi_y = \phi_x f_y$ i.e., $f_x/\phi_x = f_y/\phi_y$, and if they cut orthogonally at at (x, y), $\alpha = \frac{1}{2}\pi$, i.e., $\cot\alpha = 0$.

i.e., $f_x\phi_x + f_y\phi_y = 0$.

Note. If the equation of the curves are given in the forms $y = f(x)$, $y = \phi(x)$ the angle of their intersection is given by $\tan^{-1}\dfrac{f'(x) \sim \phi'(x)}{1 + f'(x)\phi'(x)}$.

Hence, the curves cut orthogonally if $f'(x)\phi'(x) = -1$.

14.7. Cartesian Subtangent and Subnormal.

Let the tangent and normal at any point $P(x, y)$ on a curve meet the x-axis in T and N respectively and let $\overline{PM}$ be drawn perpendicular to $\overrightarrow{OX}$.

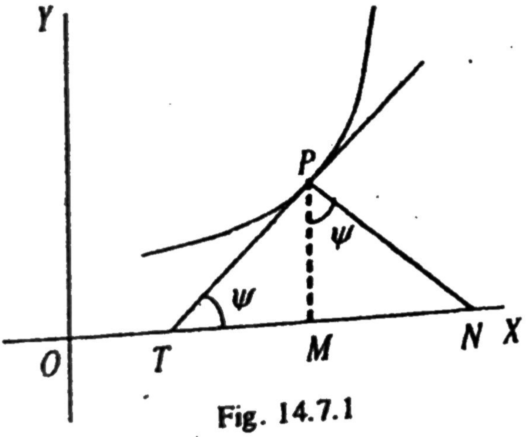

Fig. 14.7.1

Then, TM is called the *subtangent*, and MN the *subnormal* at P.

In the right-angled triangles PTM, PNM,

since $\angle NPM = \angle PTM = \psi$, and $PM = y$,

$$\textit{subtangent} = TM = y \cot\psi = y\Big/\left(\frac{dy}{dx}\right),$$

$$\textit{subnormal} = MN = y \tan\psi = y\,\frac{dy}{dx}.$$

Note. PT and PN are often called as the *length of the tangent* and the *length of the normal* (or sometimes simply tangent and normal) respectively.

Thus, from Δ^s PTM and PNM,

$$PT = y \operatorname{cosec}\psi = y\sqrt{1+\cot^2\psi} = y\sqrt{1+(1/y_1)^2} = \frac{1}{y_1}\left(y\sqrt{1+y_1^2}\right)$$

$$PN = y \sec\psi = y\sqrt{1+\tan^2\psi} = y\sqrt{1+y_1^2}.$$

14.8. Proof of $\underset{Q \to P}{\text{Lt}} \dfrac{\text{chord}PQ}{\text{arc}PQ} = 1$.

Let $\overline{PP_1}, \overline{P_1P_2}, \ldots, \overline{P_{n-1}Q}$ be the sides of an open polygon inscribed in arc PQ of the curve $y = f(x)$. If the sum of the n sides $\Sigma \overleftrightarrow{PP_1}$ tends to a definite limit when $n \to \infty$ and the length of each side tends to zero, *that limit is defined as the length of the arc PQ.*

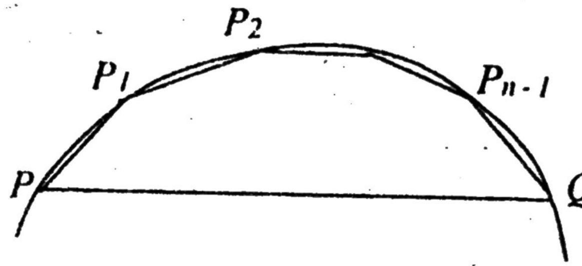

Fig. 14.8.1

Let $\theta_1, \theta_2, \ldots, \theta_n$ be the angles which the sides make with the chord $\overline{PQ}$, and let $f'(x)$ *be continuous* throughout PQ.

Projecting the sides on $\overline{PQ}$, we have

$$PQ = \text{proj.}\, PP_1 + \text{proj.}\, P_1P_2 + \ldots + \text{proj.}\, P_{n-1}Q$$

$$= PP_1 \cos\theta_1 + P_1P_2 \cos\theta_2 + \ldots + P_{n-1}Q \cos\theta_n$$

$\therefore$ it follows that $PQ < PP_1 + P_1P_2 + \ldots + P_{n-1}Q$

and $> \left(PP_1 + P_1P_2 + \ldots + P_{n-1}Q\right)\cos\theta$

where θ is numerically the greatest of the angles $\theta_1, \theta_2, \ldots, \theta_n$.

$$\text{Hence, } \cos\theta < \frac{PQ}{PP_1 + P_1P_2 + \ldots + P_{n-1}Q} < 1.$$

Since the chords $\overline{PP_1}, \overline{P_1P_2}, \ldots, \overline{P_{n-1}Q}$ as well as $\overline{PQ}$ are parallel to the tangents to the arcs at points between their respective extremities (by the Mean Value Theorem), it follows from the continuity of $f'(x)$ that the numerical value of θ can be made as small as we please by taking Q sufficiently near to P, and, in the limiting position, $\cos\theta \to 1$ and $\Sigma PP_1 \to \text{arc } PQ$.

$$\therefore \quad \underset{Q \to P}{Lt} \frac{\text{chord } PQ}{\text{arc } PQ} = 1$$

14.9. Derivative of arc-length (Cartesian).

Let $P(x, y)$ be the given point, and $Q(x + \Delta x, \ y + \Delta y)$ be any point near P on the curve.

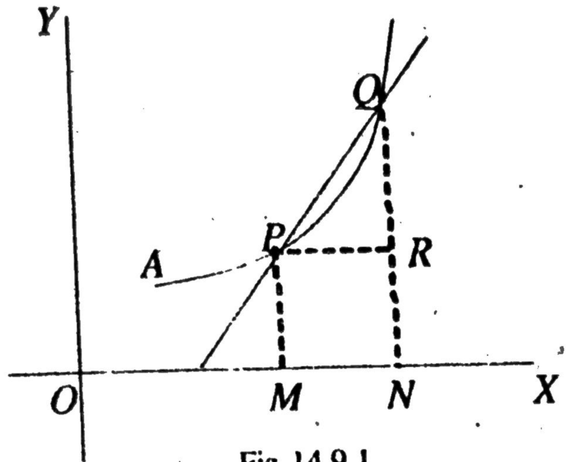

Fig 14.9.1

Let s denote the length of the arc AP measured from a fixed point A on the curve, and let $s + \Delta s$ denote the arc AQ, so that arc $PQ = \Delta s$. Here, s is obviously a function of x, and hence of y. We shall *assume* the fundamental limit

$$\underset{Q \to P}{Lt} \frac{\text{chord } PQ}{\text{arc } PQ} = 1$$

From the figure, $(\text{chord } PQ)^2 = PR^2 + QR^2 = (\Delta x)^2 + (\Delta y)^2$.

$$\therefore \quad \left(\frac{\text{chord } PQ}{\Delta s}\right)^2 \cdot \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2.$$

Now let $Q \to P$ as a limiting position; then $\Delta x \to 0$ and we have

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2, \qquad \ldots (1)$$

$$\text{or} \quad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \qquad \ldots (2)$$

Since $\frac{ds}{dy} = \frac{ds}{dx} \cdot \frac{dx}{dy}$, we get, on multiplying both sides of (2) by $\frac{dx}{dy}$,

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \qquad (3)$$

Cor: Multiplying both sides of (1), (2) and (3) by dx^2, dx, dy, we get the corresponding differential form

$$ds^2 = dx^2 + dy^2;$$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx; \quad ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\cdot dy.$$

14.10 Values of $\sin\psi$, $\cos\psi$.

From ΔPQR [See *Fig.*, § *14.9*], $\sin QPR = \dfrac{RQ}{PQ} = \dfrac{\Delta y}{\Delta s}\cdot\dfrac{\Delta s}{PQ}$.

In the limiting position when $Q \to P$, the secant PQ becomes the tangent at P, $\angle QPR \to \psi$ and $\Delta s \to 0$ and $\dfrac{\Delta s}{PQ} = \dfrac{arc\ PQ}{chord\ PQ} \to 1$;

$$\therefore \quad \sin\psi = \underset{\Delta s \to 0}{Lt}\frac{\Delta y}{\Delta s} = \frac{dy}{ds}. \qquad \ldots (1)$$

Similarly, $\cos\psi = \underset{\Delta s \to 0}{Lt}\dfrac{\Delta x}{\Delta s} = \dfrac{dx}{ds}$. $\qquad \ldots (2)$

Since $\tan\psi = \dfrac{dy}{dx}$ and $\cot\psi = \dfrac{dx}{dy}$, we get, from (2) and (3) of Art. 14.9, $\dfrac{ds}{dx} = \sec\psi . \dfrac{ds}{dy} = \operatorname{cosec}\psi$,

whence also $\cos\psi$, $\sin\psi$ are obtained.

$$\therefore \quad \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1. \qquad \ldots (3)$$

Cor. If $x = \phi(t)$, $y = \psi(t)$, $\dfrac{dx}{dt} = \dfrac{dx}{ds}\cdot\dfrac{ds}{dt}$; $\dfrac{dy}{dt} = \dfrac{dy}{ds}\cdot\dfrac{ds}{dt}$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left\{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right\}\left(\frac{ds}{dt}\right)^2$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2 \qquad \ldots (4)$$

Note. Relations (2) and (3) of Art. 14.9 can also be deduced from the values of $\sin\psi$, $\cos\psi$, $\tan\psi$.

14.11. Illustrative Examples.

Ex. 1. *Find the equation of the tangent at (x, y) to the curve*

$$(x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} = 1.$$

Here the equation of the curve is $f(x, y) \equiv (x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} - 1 = 0$.

The equation of the tangent is

$$(X - x) f_x + (Y - y) f_y = 0,$$

i.e., $(X - x) . \frac{2}{3} x^{-\frac{1}{3}} / a^{\frac{2}{3}} + (Y - y) \frac{2}{3} y^{-\frac{1}{3}} / b^{\frac{2}{3}} = 0,$

i.e., $X x^{-\frac{1}{3}} / a^{\frac{2}{3}} + Y y^{-\frac{1}{3}} / b^{\frac{2}{3}} = (x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}},$

i.e., $X x^{-\frac{1}{3}} / a^{\frac{2}{3}} + Y y^{-\frac{1}{3}} / b^{\frac{2}{3}} = 1.$

Note. The equation of the tangent should be simplified as much as possible as in the above example.

Ex. 2. *Find the angle of intersection of the curves* $x^2 - y^2 = a^2$ *and* $x^2 + y^2 = a^2 \sqrt{2}$. [*Patna, 1940*]

Adding and subtracting the equations of the two curves, we find their common points of intersection given by $2x^2 = a^2 (\sqrt{2} + 1)$, *i.e.*, $x = \pm a \sqrt{(\sqrt{2} + 1)} / \sqrt{2}$ and $2y^2 = a^2 (\sqrt{2} - 1)$, *i.e.*, $y = \pm a \sqrt{(\sqrt{2} - 1)} / \sqrt{2}$.

Since the equations of the curves can be written as

$$f(x, y) \equiv x^2 - y^2 - a^2 = 0 \text{ and } \phi(x, y) \equiv x^2 + y^2 - a^2 \sqrt{2} = 0$$

hence if α be the angle of intersection of the curves at (x, y), we have, by Art. 14.6,

$$\tan \alpha = \frac{2x.2y - (2x)(-2y)}{2x.2x + (-2y)(2y)} = \frac{\pm 2xy}{x^2 - y^2} = 1,$$

on substituting the values of x and y found above. Hence, $\alpha = \frac{1}{4}\pi$.

$\therefore$ the curves intersect at an angle of 45°.

Ex. 3. *Find the condition that the conics*

$$ax^2 + by^2 = 1 \text{ and } a_1 x^2 + b_1 y^2 = 1$$

shall cut orthogonally.

The equations of the conics are

$$f(x, y) \equiv ax^2 + by^2 - 1 = 0, \quad \ldots (1)$$

$$\phi(x, y) \equiv a_1 x^2 + b_1 y^2 - 1 = 0 \quad \ldots (2)$$

Now, the condition that they should cut orthogonally at (x, y) is, by §14.6.

$$f_x \phi_x + f_y \phi_y = 0,$$

i.e., $2ax.2a_1x + 2by.2b_1y = 0,$

i.e., $aa_1x^2 + bb_1y^2 = 0$ ... (3)

Since the point (x, y) is common to both (1) and (2), the required condition is obtained by eliminating x, y from (1), (2) and (3).

Subtracting (2) from (1), $(a - a_1)x^2 + (b - b_1)y^2 = 0$ ... (4)

Comparing (3) and (4), we get

$$\frac{a - a_1}{aa_1} = \frac{b - b_1}{bb_1}, \text{ or, } \frac{1}{a_1} - \frac{1}{a} = \frac{1}{b_1} - \frac{1}{b}$$

which is the required condition.

Ex. 4. *If* $x \cos \alpha + y \sin \alpha = p$ *touches the curve*

$$\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1,$$

show that $(a \cos \alpha)^{\frac{m}{m-1}} + (b \sin \alpha)^{\frac{m}{m-1}} = p^{\frac{m}{m-1}}$. [C.P. 1996]

The equation of the tangent to the given curve at (x, y) by formula (2) of Art. 14.2 is

$$(X - x)\frac{mx^{m-1}}{a^m} + (Y - y)\frac{my^{m-1}}{b^m} = 0,$$

i.e., $X x^{m-1}/a^m + Y y^{m-1}/b^m = x^m/a^m + y^m/b^m = 1$ ... (1)

If $X \cos \alpha + Y \sin \alpha = p$ ... (2)

touches the given curve, equations (1) and (2) must be identical.

Hence, $$\frac{x^{m-1}/a^m}{\cos \alpha} = \frac{y^{m-1}/b^m}{\sin \alpha} = \frac{1}{p},$$

i.e., $$\frac{x^{m-1}/a^{m-1}}{a \cos \alpha} = \frac{y^{m-1}/b^{m-1}}{b \sin \alpha} = \frac{1}{p},$$

$$\therefore \left(\frac{x}{a}\right)^{m-1} = \frac{a \cos \alpha}{p}, \left(\frac{y}{b}\right)^{m-1} = \frac{b \sin \alpha}{p}.$$

$$\therefore \left(\frac{a \cos \alpha}{p}\right)^{\frac{m}{m-1}} + \left(\frac{b \sin \alpha}{p}\right)^{\frac{m}{m-1}} = \left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^m = 1$$

i.e., $(a \cos \alpha)^{\frac{m}{m-1}} + (b \sin \alpha)^{\frac{m}{m-1}} = p^{\frac{m}{m-1}}$.

Ex.5. *If x_1, y_1 be the parts of the axes of x and y intercepted by the tangent at any point (x, y) to the curve $(x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} = 1$, show that $x_1^2/a^2 + y_1^2/b^2 = 1$.*

The equation of the tangent at (x, y) to the given curve is, as in Ex.1.

$$X x^{-\frac{1}{3}} / a^{\frac{2}{3}} + Y y^{-\frac{1}{3}} / b^{\frac{2}{3}} = 1.$$

Where it meets the x-axis, $Y = 0$, hence $X = a^{\frac{2}{3}} x^{\frac{1}{3}}$, *i.e.*, $x_1 = a^{\frac{2}{3}} x^{\frac{1}{3}}$, and where it meets the y-axis, $X = 0$, hence $Y = a^{\frac{2}{3}} y^{\frac{1}{3}}$, *i.e.*, $y_1 = b^{\frac{2}{3}} y^{\frac{1}{3}}$.

$$\therefore x_1^2/a^2 + y_1^2/b^2 = a^{\frac{4}{3}} x^{\frac{2}{3}} / a^2 + b^{\frac{4}{3}} y^{\frac{2}{3}} / b^2 = (x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} = 1.$$

EXAMPLES - XIV(A)

1. Find the equation of the tangent at the point (x, y) on each of the following curves :

(i) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$ (ii) $\dfrac{x^m}{a^m} + \dfrac{y^m}{b^m} = 1.$

(iii) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. (iv) $x^3 - 3axy + y^3 = 0.$

(v) $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

2. (i) Find the equation of the tangent at the point θ on each of the following curves :

(a) $x = a\cos\theta, \quad y = b\sin\theta$.

(b) $x = a\cos^3\theta, \quad y = b\sin^3\theta$.

(c) $x = a(\theta + \sin\theta), \quad y = a(1 - \cos\theta)$.

(ii) Find the equation of the normal at 't' on the curve

$x = a(2\cos t + \cos 2t), \quad y = a(2\sin t - \sin 2t)$.

3. (i) Find the tangent at the point $(1, -1)$ to the curve

$x^3 + xy^2 - 3x^2 + 4x + 5y + 2 = 0$

(ii) Show that the tangent at (a, b) to the curve

$(x/a)^3 + (y/b)^3 = 2$ is $x/a + y/b = 2$. [C.P. 1943]

(iii) Show that the normal at the point $\theta = \frac{1}{4}\pi$ on the curve $x = 3\cos\theta - \cos^3\theta$, $y = 3\sin\theta - \sin^3\theta$ passes through the origin.

4. (i) Find the tangent and the normal to the curve

$$y(x-2)(x-3)-x+7=0$$

at the point where it cuts the x-axis.

(ii) Show that of the tangents at the points where the curve $y=(x-1)(x-2)(x-3)$ is met by the x-axis, two are parallel, and the third makes an angle of 135° with the x-axis.

(iii) Find the tangent to the curve $xy^2 = 4(4-x)$ at the point where it is cut by the line $y=x$.

5. (i) Find where the tangent is parallel to the x-axis for the curves:

(a) $y = x^3 - 3x^2 - 9x + 15$ (b) $ax^2 + 2hxy + by^2 = 1$.

(ii) Find where the tangent is perpendicular to the x-axis for the curves

(a) $y^2 = x^2(a-x)$. (b) $ax^2 + 2hxy + by^2 = 1$.

(c) $y = (x-3)^2(x-2)$. [C.P. 1935]

(iii) Show that the tangents to the curve

$$3x^2 + 4xy + 5y^2 - 4 = 0$$

at the points in which it is intersected by the lines

$$3x + 2y = 0 \text{ and } 2x + 5y = 0$$

are parallel to the axes of co-ordinates.

(iv) Find at what points on the curve

$$y = 2x^3 - 15x^2 + 34x - 20$$

the tangents are parallel to $y + 2x = 0$.

(v) Find the points on the curve $y = x^2 + 3x + 4$ the tangents at which pass through the origin.

6. Show that the tangent to the curve $x^3 + y^3 = 3axy$ at the point other than the origin, where it meets the parabola $y^2 = ax$, is parallel to the y-axis.

7. Prove that all points of the curve

$$y^2 = 4a\left\{x + a\sin(x/a)\right\}$$

at which the tangent is parallel to the x-axis lie on a parabola.

[C.P. 1998]

8. Tangents are drawn from the origin to the curve $y = \sin x$. Prove that their points of contact lie on $x^2y^2 = x^2 - y^2$.

9. (i) Show that the curve $(x/a)^n + (y/b)^n = 2$ touches the straight line $x/a + y/b = 2$ at the point (a, b), whatever be the value of n.

(ii) Prove that $\frac{x}{a}+\frac{y}{b}=1$ touches the curve $\frac{x}{a}+\log\left(\frac{y}{b}\right)=0$.

10. (i) If $lx+my=1$ touches the curve $(ax)^n+(by)^n=1$, show that $(l/a)^{\frac{n}{n-1}}+(m/b)^{\frac{n}{n-1}}=1$. [B.P. 1989]

(ii) If $lx+my=1$ is a normal to the parabola $y^2=4ax$, then show that $al^3+2alm^2=m^2$. [V.P. 1999]

11. Prove that the condition that $x\cos\alpha+y\sin\alpha=p$ should touch $x^m y^n=a^{m+n}$ is $p^{m+n}m^m n^n=(m+n)^{m+n}a^{m+n}\sin^n\alpha\cos^m\alpha$.

12. Find the angles of intersection of the following curves :

(i) $x^2-y^2=2a^2$ and $x^2+y^2=4a^2$.

(ii) $x^2=4y$ and $y(x^2+4)=8$.

(iii) $y=x^3$ and $6y=7-x^2$.

13. (i) Prove that the curves $\frac{x^2}{a}+\frac{y^2}{b}=1$ and $\frac{x^2}{a'}+\frac{y^2}{b'}=1$ will cut orthogonally if $a-b=a'-b'$. [C.P. 1980, '90, 2007 V.P. 2000]

(ii) Find the condition that the curves $ax^3+by^3=1$ and $a'x^3+b'y^3=1$ should cut orthogonally.

(iii) Show that the curves $x^3-3xy^2=-2$ and $3x^2y-y^3=2$ cut orthogonally. [C. P. 2006]

14. (i) Prove that the sum of the intercepts of the tangent to the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ upon the co-ordinate axes is constant. [B.P. 1993]

(ii) Find the abscissa of the point on each of the curves

(a) $ay^2=x^3$,

(b) $\sqrt{xy}=a+x$, the normal at which cuts off equal intercepts from the co-ordinate axes.

15. Show that the portion of the tangent at any point on the following curves intercepted between the axes is of constant length.

(i) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ [C.P. 1940]

(ii) $x=a\cos^3\theta$, $y=a\sin^3\theta$.

16. If the tangent at (x_1, y_1) to the curve $x^3+y^3=a^3$ meets the curve again in (x_2, y_2), show that $x_2/x_1+y_2/y_1=-1$.

17. (i) Show that at any point on the parabola $y^2 = 4ax$, the subnormal is constant and the subtangent varies as the abscissa of the point of contact.

(ii) Show that at any point on the hyperbola $xy = c^2$, the subtangent varies as the abscissa and the subnormal varies as the cube of the ordinate of the point of contact.

18. Prove that the subtangent is of constant length for the curve $\log y = x \log a$.

19. Show that for the curve $by^2 = (x + a)^3$ the square of the subtangent varies as the subnormal.

[*C. P. 2006*]

20. Show that at any point on the curve $x^{m+n} = k^{m-n} y^{2n}$ the m^{th} power of the subtangent varies as the n^{th} power of the subnormal.

[*C.P. 1995, '97, 2002, 2004*]

21. For the curve $x^m y^n = a^{m+n}$, show that the subtangent at any point varies as the abscissa of the point.

22. Show that for any curve the rectangle contained by the subtangent and subnormal is equal to the square on the corresponding ordinate.

[*C. P. 2005*]

23. Find the lengths of the subtangent, subnormal, tangent and normal of the curve :

(i) $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at 'θ'

(ii) $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$ at 't'.

[*C. P. 2006*]

24. Find the value of n so that the subnormal at any point on the curve $xy^n = a^{n+1}$ may be constant.

25. Show that in any curve

$$\frac{\text{subnormal}}{\text{subtangent}} = \left(\frac{\text{length of normal}}{\text{length of tangent}}\right)^2$$

26. Show that the length of the tangent at any point on the following curves is constant :

(i) $x = \sqrt{a^2 - y^2} + \dfrac{a}{2}\log\dfrac{a - \sqrt{a^2 - y^2}}{a + \sqrt{a^2 - y^2}}$.

(ii) $x = a\left(\cos t + \log\tan\frac{1}{2}t\right)$, $y = a\sin t$.

(iii) $s = a\log(a/y)$.

27. (i) If p_1 and p_2 be the perpendiculars from the origin on the tangent and normal respectively at any point (x, y) on the curve, then show that

$$p_1 = x\sin\psi - y\cos\psi, \quad p_2 = x\cos\psi + y\sin\psi,$$

where, as usual, $\tan\psi = dy/dx$.

(ii) If, in the above case, the curve be $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ show that $4p_1^{\,2} + p_2^{\,2} = a^2$.

28. In the curve $x^m y^n = a^{m+n}$, show that the portion of the tangent intercepted between the axes is divided at its point of contact into segments which are in a constant ratio.

29. (i) In the catenary $y = c\cosh(x/c)$ show that the length of the perpendicular from the foot of the ordinate on the tangent is of constant length. [C. P. 1943]

(ii) Show that for the catenary $y = c\cosh(x/c)$ the length of the normal at any point is y^2/c.

30. Prove that the equation of the tangent to the curve $x = af(t)/\psi(t)$, $y = a\phi(t)/\psi(t)$ may be written in the form

$$\begin{vmatrix} x & y & a \\ f(t) & \phi(t) & \psi(t) \\ f'(t) & \phi'(t) & \psi'(t) \end{vmatrix} = 0$$

31. Find the equation of the tangent at the origin of the curve

$$\begin{aligned} y &= x^2\sin(1/x) \quad \text{for } x \neq 0 \\ &= 0 \qquad\qquad\quad \text{for } x = 0. \end{aligned}$$

32. Show that for the curve $y = x^{\frac{2}{3}}$ the tangent at the origin is $x = 0$, although dy/dx does not exist there.

33. If α and β be the intercepts on the axes of x and y cut off by the tangent to the curve $(x/a)^n + (y/b)^n = 1$ thén show that28

$$(a/\alpha)^{\frac{n}{n-1}} + (b/\beta)^{\frac{n}{n-1}} = 1$$

34. Find $\dfrac{ds}{dx}$ for the following curves :

(i) $y^2 = 4ax$. (ii) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

(iii) $y = \frac{1}{2}a\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$.

29. (iv) $x = a(1 - \cos\theta)$, $y = a(\theta + \sin\theta)$

35. If for the ellipse $x^2/a^2 + y^2/b^2 = 1$, $x = a\sin\phi$, show that

$$\frac{ds}{d\phi} = a\sqrt{1-e^2\sin^2\phi}.$$

36. Two curves are defined as follows :

(i) $x = t^3$,

$y = t^3 \sin(1/t)$, for $t \neq 0$

$= 0$ for $t = 0$.

(ii) $x = 2t + t^2\sin(1/t)$, $y = t^2\sin(1/t)$, for $t \neq 0$

$x = 0$, $y = 0$ for $t = 0$

show that, for the first curve, although dx/dt, dy/dt are continuous for $t = 0$, the *curve has no tangent* at the point; and for the second curve, although dx/dt, dy/dt *are not continuous* for $t = 0$, the curve *has a tangent* at the point.

ANSWERS

1. (i) $\dfrac{Xx}{a^2} + \dfrac{Yy}{b^2} = 1$. (ii) $\dfrac{Xx^{m-1}}{a^m} + \dfrac{Yy^{m-1}}{b^m} = 1$.

(iii) $Xx^{-\frac{1}{3}} + Yy^{-\frac{1}{3}} = a^{\frac{2}{3}}$. (iv) $X(x^2 - ay) + Y(y^2 - ax) = axy$.

(v) $X\{2x(x^2+y^2) - a^2x\} + Y\{2y(x^2+y^2) + a^2y\} = a^2(x^2-y^2)$.

2. (i) (a) $\dfrac{X}{a}\cos\theta + \dfrac{Y}{b}\sin\theta = 1$.

(b) $bX\sin\theta + aY\cos\theta = ab\sin\theta\cos\theta$.

(c) $X\sin\frac{1}{2}\theta - Y\cos\frac{1}{2}\theta = a\theta\sin\frac{1}{2}\theta$.

(ii) $X\cos\frac{1}{2}t - Y\sin\frac{1}{2}t = 3a\cos\frac{3}{2}t$.

3. (i) $2x + 3y + 1 = 0$.

4. (i) Tangent $x - 20y - 7 = 0$; normal $20x + y - 140 = 0$.

(iii) $x + y - 4 = 0$

5. (i) (a) $(3, -12)$, $(-1, 20)$.

(b) Where $ax + hy = 0$ intersects the curve.

(ii) (a) $(0, 0)$, $(a, 0)$.

(b) Where $hx + by = 0$ intersects the curve.

(c) No such point exists

(iv) $(2, 4)$; $(3, 1)$. (v) $(2, 14)$; $(-2, 2)$.