প্রশ্নমালা–4(B)

1(i). $I=\int_0^{\pi/4}\dfrac{dx}{1+\dfrac{2(1-\tan^2x/2)}{1+\tan^2x/2}}=\int_0^{\pi/4}\dfrac{\sec^2x/2\,dx}{3-\tan^2x/2}$

ধরি $\tan x/2=t$, তবে $\dfrac{1}{2}\sec^2 x/2\,dx=dt$

সীমা : যদি $x=0$ হয়, তবে $t=0$

এবং যদি $x=\dfrac{\pi}{4}$ হয়, তবে $t=\tan\dfrac{\pi}{8}$

$$\therefore I=2\int_0^{\tan\pi/3}\frac{dt}{(\sqrt{3})^2-t^2}=\frac{2}{2\sqrt{3}}\left[\ln\frac{\sqrt{3}+t}{\sqrt{3}-t}\right]_0^{\tan\pi/8}$$

(ii). $I=\int_0^{\pi}\dfrac{dx}{a-\dfrac{1-\tan^2x/2}{1+\tan^2x/2}}=\int_0^{\pi}\dfrac{\sec^2x/2\,dx}{(a-1)+(a+1)\tan^2x/2}$

ধরি $\sqrt{a+1}\tan\dfrac{x}{2}=t$ তবে $\dfrac{1}{2}\sqrt{a+1}\sec^2\dfrac{x}{2}dx=dt$

সীমা : যদি $x=0$ হয়, তবে $t=0$

এবং যদি $x=\pi$ হয়, তবে $t=\infty$

$$I=\frac{2}{\sqrt{a+1}}\int_0^{\infty}\frac{dt}{(\sqrt{a-1})^2+t^2}=\frac{2}{\sqrt{a+1}}\frac{1}{\sqrt{a-1}}\left[\tan^{-1}\frac{t}{\sqrt{a-1}}\right]_0^{\infty}$$

$$=\frac{2}{\sqrt{a^2-1}}[\tan^{-1}\infty-0]=\frac{2}{\sqrt{a^2-1}}\cdot\frac{\pi}{2}=\frac{\pi}{\sqrt{a^2-1}}$$

(iii). $I=\int_0^{\pi/2}\dfrac{dx}{3+\dfrac{5(1-\tan^2x/2)}{1+\tan^2x/2}}=\int_0^{\pi/2}\dfrac{\sec^2x/2\,dx}{8-2\tan^2x/2}$

$$=\frac{1}{2}\int_0^{\pi/2}\frac{\sec^2x/2\,dx}{4-\tan^2x/2}$$

ধরি $\tan x/2=t$ তবে $\dfrac{1}{2}\sec^2x/2\,dx=dt$

সীমা : যদি $x=0$ হয়, তবে $t=0$

এবং যদি $x=\dfrac{\pi}{2}$ হয়, তবে $t=1$

$$I=\int_0^1\frac{dt}{2^2-t^2}=\frac{1}{2.2}\left[\ln\frac{2+t}{2-t}\right]_0^1=\frac{1}{4}\left[\ln\frac{2+1}{2-1}-\ln\frac{2}{2}\right].$$

(iv). $I = \int_0^{\pi/2} \frac{dx}{a + \frac{b(1 - \tan^2 x/2)}{1 + \tan^2 x/2}} = \int_0^{\pi/2} \frac{\sec^2 x/2\, dx}{a + b + (a - b)\tan^2 x/2}$

ধরি $\sqrt{a-b}\tan x/2 = t$ তবে $\frac{1}{2}\sqrt{a-b}\sec^2 x/2\, dx = dt$

সীমা ঃ যদি $x = 0$ হয়, তবে $t = 0$

এবং যদি $x = \frac{\pi}{2}$ হয়, তবে $t = \sqrt{a-b}$

$$I = \frac{2}{\sqrt{a-b}} \int_0^{\sqrt{a-b}} \frac{dt}{(\sqrt{a+b})^2 + t^2}$$

$$= \frac{2}{\sqrt{a-b}} \cdot \frac{1}{\sqrt{a+b}} \left[\tan^{-1} \frac{t}{\sqrt{a+b}}\right]_0^{\sqrt{a-b}}$$

$$= \frac{2}{\sqrt{a^2-b^2}} \left[\tan^{-1} \frac{\sqrt{a-b}}{\sqrt{a+b}} - \tan^{-1} 0\right]$$

(v). $I = \int_0^{\pi} \frac{dx}{2 + \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}} = \int_0^{\pi} \frac{\sec^2 x/2\, dx}{3 + \tan^2 x/2}$

ধরি $\tan x/2 = t$ তবে $\frac{1}{2}\sec^2 x/2\, dx = dt$

সীমা ঃ যদি $x = 0$ হয়, তবে $t = 0$

এবং যদি $x = \pi$ হয়, তবে $t = \infty$

$$I = 2\int_0^{\infty} \frac{dt}{(\sqrt{3})^2 + t^2} = \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^{\infty} = \frac{2}{\sqrt{3}}\left[\frac{\pi}{2} - 0\right].$$

(vi). $I = \int_0^{\pi/2} \frac{dx}{5 + \frac{4(1 - \tan^2 x/2)}{1 + \tan^2 x/2}} = \int_0^{\pi/2} \frac{\sec^2 x/2\, dx}{9 + \tan^2 x/2}$

ধরি $\tan x/2 = t$ তবে $\frac{1}{2}\sec^2 x/2\, dx = dt$

সীমা ঃ যদি $x = 0$ হয়, তবে $t = 0$

এবং যদি $x = \frac{\pi}{2}$ হয়, তবে $t = 1$

$$I = 2\int_0^{1} \frac{dt}{3^2 + t^2} = \frac{2}{3}\left[\tan^{-1}\frac{t}{3}\right]_0^{1} = \frac{2}{3}\left[\tan^{-1}\frac{1}{3} - 0\right]$$

(vii). $I = \int_0^{\pi/2} \frac{dx}{1 + \frac{2(1 - \tan^2 x/2)}{1 + \tan^2 x/2}} = \int_0^{\pi/2} \frac{\sec^2 x/2\, dx}{3 - \tan^2 x/2}$

ধরি $\tan x/2 = t$ তবে $\frac{1}{2} \sec^2 x/2\, dx = dt$

সীমা ঃ যদি $x = 0$, তবে $t = 0$

এবং যদি $x = \frac{\pi}{2}$, তবে $t = 1$

$I = 2\int_0^1 \frac{dt}{(\sqrt{3})^2 - t^2} = \frac{2}{2\sqrt{3}}\left[ln \frac{\sqrt{3} + t}{\sqrt{3} - t}\right]_0^1 = \frac{1}{\sqrt{3}}\left[ln \frac{\sqrt{3} + 1}{\sqrt{3} - 1} - 0\right]$

$= \frac{1}{\sqrt{3}} ln \frac{(\sqrt{3} + 1)(\sqrt{3} + 1)}{(\sqrt{3} - 1)(\sqrt{3} + 1)} = \frac{1}{\sqrt{3}} ln \frac{(\sqrt{3} + 1)^2}{3 - 1}$

$= \frac{1}{\sqrt{3}} ln\left(\frac{3 + 1 + 2\sqrt{3}}{2}\right) = \frac{1}{\sqrt{3}} ln\,(2 + \sqrt{3}).$

(viii). $I = \int_0^{\pi/2} \frac{dx}{3 + \frac{2(1 - \tan^2 x/2)}{1 + \tan^2 x/2}} = \int_0^{\pi/2} \frac{\sec^2 x/2\, dx}{5 + \tan^2 x/2}$

ধরি $\tan x/2 = t$ তবে $\frac{1}{2} \sec^2 x/2\, dx = dt$

সীমা ঃ যদি $x = 0$ হয়, তবে $t = 0$

এবং যদি $x = \frac{\pi}{2}$ হয়, তবে $t = 1$

$I = 2\int_0^1 \frac{dt}{(\sqrt{5})^2 + t^2} = \frac{2}{\sqrt{5}}\left[\tan^{-1} \frac{t}{\sqrt{5}}\right]_0^{\infty} = \frac{2}{\sqrt{5}}\left[\tan^{-1} \frac{1}{\sqrt{5}} - 0\right].$

(ix). $I = \int_0^{\pi} \frac{dx}{5 + \frac{3(1 - \tan^2 x/2)}{1 + \tan^2 x/2}} = \int_0^{\pi} \frac{\sec^2 x/2\, dx}{8 + 2\tan^2 x/2}$

ধরি $\tan x/2 = t$ তবে $\frac{1}{2} \sec^2 x/2\, dx = dt$

সীমা ঃ যদি $x = 0$ হয়, তবে $t = 0$

এবং যদি $x = \pi$ হয়, তবে $t = \infty$

$I = \int_0^{\infty} \frac{dt}{2^2 + t^2} = \frac{1}{2}\left[\tan^{-1} \frac{t}{2}\right]_0^{\infty} = \frac{1}{2}\,[\tan^{-1} \infty - 0] = \frac{\pi}{4}$

(x). $I = \int_0^{\pi/2} \frac{dx}{4 + \frac{5.2\tan x/2}{1 + \tan^2 x/2}} = \int_0^{\pi/2} \frac{\sec^2 x/2\, dx}{4\tan^2 x/2 + 10\tan x/2 + 4}$

ধরি $\tan x/2 = t$ তবে $\frac{1}{2} \sec^2 x/2\, dx = dt$

সীমা ঃ যদি $x = 0$ হয়, তবে $t = 0$

এবং যদি $x = \frac{\pi}{2}$ হয়, তবে $t = 1$

$$I = 2\int_0^1 \frac{dt}{4t^2 + 10t + 4} = \frac{1}{2}\int_0^1 \frac{dt}{t^2 + 5t/2 + 1}$$

$$I = \frac{1}{2}\int_0^1 \frac{dt}{(t + 5/4)^2 - (3/4)^2} = \frac{1}{2}\cdot\frac{1}{2.3/4}\left[ln\ \frac{t + 5/4 - 3/4}{t + 5/4 + 3/4}\right]_0^1$$

$$= \frac{1}{3}\left[ln\ \frac{t + 1/2}{t + 2}\right]_0^1 = \left[ln\ \frac{2t + 1}{2t + 4}\right]_0^1$$

$$= \frac{1}{3}\left[ln\ \frac{3}{6} - ln\ \frac{1}{4}\right] = \frac{1}{3}\left[ln\ \frac{1}{2} - ln\ \frac{1}{4}\right] = \frac{1}{3}\ ln2.$$

(xi). $I = \int_0^{\pi/2} \frac{dx}{5 + \frac{4.2\tan x/2}{1 + \tan^2 x/2}} = \int_0^{\pi/2} \frac{\sec^2 x/2\ dx}{5\tan^2 x/2 + 8\tan x/2 + 5}$

ধরি $\tan x/2 = t$ তবে $\frac{1}{2}\sec^2 x/2\ dx = dt$

সীমা ঃ যদি $x = 0$ হয়, তবে $t = 0$

এবং যদি $x = \frac{\pi}{2}$ হয়, তবে $t = 1$

$$I = 2\int_0^1 \frac{dt}{5t^2 + 8t + 5} = \frac{2}{5}\int_0^1 \frac{dt}{t^2 + 8t/5 + 1}$$

$$= \frac{2}{5}\int_0^1 \frac{dt}{(t + 4/5)^2 + (3/5)^2} = \frac{2}{5}\cdot\frac{1}{3/5}\left[\tan^{-1}\frac{t + 4/5}{3/5}\right]_0^1$$

$$= \frac{2}{3}\left[\tan^{-1}\frac{5t + 4}{3}\right]_0^1 = \frac{2}{3}\left[\tan^{-1}3 - \tan^{-1}4/3\right] = \frac{2}{3}\tan^{-1}\frac{1}{3}.$$

(xii). $I = \int_0^{\pi/2} \frac{dx}{4 + \frac{5(1 - \tan^2 x/2)}{1 + \tan^2 x/2}} = \int_0^{\pi/2} \frac{\sec^2 x/2\ dx}{9 - \tan^2 x/2}$

ধরি $\tan x/2 = t$ তবে $\frac{1}{2}\sec^2 x/2\ dx = dt$

সীমা ঃ যদি $x = 0$ হয়, তবে $t = 0$

এবং যদি $x = \frac{\pi}{2}$ হয়, তবে $t = 1$

$$I = 2\int_0^1 \frac{dt}{3^2 - t^2} = \frac{2}{2.3}\left[ln\ \frac{3 + t}{3 - t}\right]_0^1 = \frac{1}{3}\left[ln\ \frac{4}{2} - ln1\right] = \frac{1}{3}\ ln2.$$

(xiii). $I = \int_0^{\pi/2} \frac{dx}{1 + \frac{\cos\alpha(1 - \tan^2 x/2)}{1 + \tan^2 x/2}}$

$= \int_0^{\pi/2} \frac{\sec^2 x/2\, dx}{1 + \cos\alpha + (1 - \cos\alpha)\tan^2 x/2}$

$= \frac{1}{2}\int_0^{\pi/2} \frac{\sec^2 x/2 . dx}{\cos^2\alpha/2 + \sin^2\alpha/2 \tan^2 x/2}$

ধরি $\sin\alpha/2.\tan x/2 = t$ তবে $\frac{1}{2}\sin\alpha/2.\sec^2 x/2\, dx = dt$

সীমা ঃ যদি $x = 0$ হয়, তবে $t = 0$

এবং যদি $x = \frac{\pi}{2}$ হয়, তবে $t = \sin\alpha/2$

$I = \frac{1}{\sin\alpha/2}\int_0^{\sin\alpha/2} \frac{dt}{(\cos\alpha/2)^2 + t^2}$

$= \frac{1}{\sin\alpha/2} \cdot \frac{1}{\cos\alpha/2}\left[\tan^{-1}\frac{t}{\cos\alpha/2}\right]_0^{\sin\alpha/2}$

$= \frac{2}{\sin\alpha}[\tan^{-1}(\tan\frac{\alpha}{2}) - 0] = \frac{2}{\sin\alpha} \cdot \frac{\alpha}{2} = \frac{\alpha}{\sin\alpha}$

(xiv). $I = \int_0^{\alpha} \frac{dx}{1 - \frac{\cos\alpha(1 - \tan^2 x/2)}{1 + \tan^2 x/2}}$

$= \int_0^{\alpha} \frac{\sec^2 x/2\, dx}{(1 - \cos\alpha) + (1 + \cos\alpha)\tan^2 x/2}$

$= \frac{1}{2}\int_0^{\alpha} \frac{\sec^2 x/2\, dx}{\sin^2\alpha/2 + \cos^2\alpha/2.\tan^2 x/2}$

ধরি $\cos\alpha/2.\ \tan x/2 = t$ তবে $\frac{1}{2}\cos\alpha/2.\ \sec^2 x/2\, dx = dt$

সীমা ঃ যদি $x = 0$ হয়, তবে $t = 0$

এবং যদি $x = \alpha$ হয়, তবে $t = \sin\alpha/2.$

$I = \frac{1}{\cos\alpha/2}\int_0^{\cos\alpha/2} \frac{dt}{(\sin\alpha/2)^2 + t^2}$

$= \frac{1}{\cos\alpha/2} \cdot \frac{1}{\sin\alpha/2}\left[\tan^{-1}\frac{t}{\cos\alpha/2}\right]_0^{\sin\alpha/2}$

$= \frac{2}{\sin\alpha}[\tan^{-1}1 - \tan^{-1}0] = \frac{2}{\sin\alpha} \cdot \frac{\pi}{4} = \frac{\pi}{2\sin\alpha}$

(xv). $I = \int_0^{\pi} \frac{dx}{1 - \frac{2a(1 - \tan^2 x/2)}{1 + \tan^2 x/2} + a^2}$

$= \int_0^{\pi} \frac{\sec^2 x/2\, dx}{1 - 2a + a^2 + (1 + 2a + a^2)\tan^2 x/2}$

$= \int_0^{\pi} \frac{\sec^2 x/2\, dx}{(1 - a)^2 + (1 + a)^2 \tan^2 x/2}$

ধরি $(1 + a) \tan x/2 = t$ তবে $\frac{1}{2}(1 + a) \sec^2 x/2\, dx = dt$

সীমা ঃ যদি $x = 0$ হয়, তবে $t = 0$

এবং যদি $x = \pi$ হয়, তবে $t = \infty$

$I = \frac{2}{1 + a} \int_0^{\infty} \frac{dt}{(1 - a)^2 + t^2} = \frac{2}{(1 + a)} \cdot \frac{1}{(1 - a)} \left[\tan^{-1} \frac{t}{1 - a}\right]_0^{\infty}$

$= \frac{2}{(1 - a^2)} [\tan^{-1}\infty - 0] = \frac{2}{(1 - a^2)} \cdot \frac{\pi}{2} = \frac{\pi}{1 - a^2}$.

(xvi). ধরি $I = \int_0^{\pi} \frac{dx}{a^2 + b^2 - 2ab \cos x}$

$= \int_0^{\pi} \frac{dx}{a^2 + b^2 - 2ab \frac{(1 - \tan^2 x/2)}{1 + \tan^2 x/2}}$

$= \int_0^{\pi} \frac{(1 + \tan^2 x/2)\, dx}{a^2 + b^2 + (a^2 + b^2) \tan^2 x/2 - 2ab + 2ab \tan^2 x/2}$

$= \int_0^{\pi} \frac{\sec^2 x/2\, dx}{(a - b)^2 + (a + b)^2 \tan^2 x/2}$

ধরি $(a + b) \tan x/2 = z$ তবে $\frac{1}{2}(a + b) \sec^2 x/2\, dx = dz$

সীমা ঃ যদি $x = 0$ হয়, তবে $z = 0$

যদি $x = \pi$ হয়, তবে $z = \infty$

$\therefore I = \frac{2}{a + b} \int_0^{\infty} \frac{dz}{(a - b)^2 + z^2} = \frac{2}{(a + b)} \cdot \frac{1}{(a - b)} \left[\tan^{-1} \frac{z}{a - b}\right]_0^{\infty}$

$= \frac{2}{a^2 - b^2} [\tan^{-1}\infty - \tan^{-1}0] = \frac{2}{(a^2 - b^2)} \left[\frac{\pi}{2}\right] = \frac{\pi}{a^2 - b^2}$

(xvii). ধরি $I = \int_0^{\pi/3} \frac{\cos x\, dx}{3 + 4\sin x} = \frac{1}{4} \int_0^{\pi/3} \frac{4\cos x\, dx}{3 + 4\sin x}$

$= \frac{1}{4} [\ln(3 + 4 \sin x)]_0^{\pi/3} = \frac{1}{4}\left[\ln\left(3 + 4 \sin \frac{\pi}{3}\right) - \ln 3\right]$

$$=\frac{1}{4}\left[ln\left(3+4\frac{\sqrt{3}}{2}\right)-ln3\right]=\frac{1}{4}\,[ln(3+2\sqrt{3})-ln3]$$

$$=\frac{1}{4}\,ln\left(\frac{3+2\sqrt{3}}{3}\right)=\frac{1}{4}\,ln\left(1+\frac{2}{\sqrt{3}}\right).$$

(xviii). ধরি $I=\int_0^\pi\frac{dx}{(2+\cos x)^2}=\int_0^\pi\frac{dx}{\left(2+\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)^2}$

বা $I=\int_0^\pi\frac{(1+\tan^2 x/2)^2\,dx}{(3+\tan^2 x/2)^2}=\int_0^\pi\frac{(1+\tan^2 x/2)\sec^2 x/2\,dx}{(3+\tan^2 x/2)^2}$

ধরি $\tan x/2=t$, তবে $\frac{1}{2}\sec^2 x/2\,dx=dt$

সীমা : যদি $x=0$ হয়, তবে $t=0$

যদি $x=\pi$ হয়, তবে $t=\infty$

$$\therefore I=2\int_0^\infty\frac{(1+t^2)\,dt}{(3+t^2)^2}=2\int_0^\infty\frac{(3+t^2)-2}{(3+t^2)^2}\,dt$$

$$=2\int_0^\infty\frac{dt}{3+t^2}-4\int\frac{dt}{(3+t^2)^2}=\frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^\infty-I_1$$

যখন $I_1=4\int_0^\infty\frac{dt}{(3+t^2)^2}$

ধরি $t=\sqrt{3}\tan\theta$ তবে $dt=\sqrt{3}\sec^2\theta\,d\theta$

সীমা : যদি $t=0$ হয়, তবে $0=\sqrt{3}\tan\theta\Rightarrow\theta=0$

যদি $t=\infty$ হয়, তবে $\infty=\sqrt{3}\tan\theta\Rightarrow\theta=\frac{\pi}{2}$

$$\therefore I_1=4\int_0^{\pi/2}\frac{\sqrt{3}\sec^2\theta\,d\theta}{(3\sec^2\theta)^2}=\frac{4}{3\sqrt{3}}\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$=\frac{2}{3\sqrt{3}}\int_0^{\pi/2}[1+\cos 2\theta]\,d\theta=\frac{2}{3\sqrt{3}}\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$=\frac{2}{3\sqrt{3}}\left[\frac{\pi}{2}+0\right]=\frac{\pi}{3\sqrt{3}}$$

$$\therefore I=\frac{2}{\sqrt{3}}\,[\tan^{-1}\infty-\tan^{-1}0]-\frac{\pi}{3\sqrt{3}}=\frac{2}{\sqrt{3}}\left[\frac{\pi}{2}-0\right]-\frac{\pi}{3\sqrt{3}}=\frac{2\pi}{3\sqrt{3}}.$$

(xix). ধরি $I = \int_0^{\pi/2} \frac{dx}{1+\cos x} = \int_0^{\pi/2} \frac{dx}{2\cos^2 x/2} = \frac{1}{2}\int_0^{\pi/2} \sec^2 x/2\, dx$

$= \frac{1}{2}\left[2\tan\frac{x}{2}\right]_0^{\pi/2} = \tan\frac{\pi}{4} - 0 = 1$

b(i). $I = \int_0^{\pi/2} \frac{dx}{1 + \frac{2.2\tan x/2}{1+\tan^2 x/2} + \frac{1-\tan^2 x/2}{1+\tan^2 x/2}}$

$= \int_0^{\pi/2} \frac{\sec^2 x/2\, dx}{2 + 4\tan x/2}$

ধরি $2 + 4\tan x/2 = t$ তবে $2\sec^2 x/2\, dx = dt$

সীমা ঃ যদি $x = 0$ হয়, তবে $t = 2$

এবং যদি $x = \frac{\pi}{2}$ হয়, তবে $t = 6$

$I = \frac{1}{2}\int_2^6 \frac{dt}{t} = \frac{1}{2}[\ln t]_2^6 = \frac{1}{2}[\ln 6 - \ln 2] = \frac{1}{2}\ln\frac{6}{2} = \frac{1}{2}\ln 3.$

(ii). $I = \int_0^{\pi/2} \frac{dx}{1 + \frac{2\tan x/2}{1+\tan^2 x/2} + \frac{1-\tan^2 x/2}{1+\tan^2 x/2}}$

$= \int_0^{\pi/2} \frac{\sec^2 x/2\, dx}{2 + 2\tan x/2}$

ধরি $2 + 2\tan\frac{x}{2} = t$ তবে $\sec^2\frac{x}{2}\, dx = dt$

সীমা ঃ যদি $x = 0$ হয়, তবে $t = 2$

এবং যদি $t = \frac{\pi}{2}$ হয়, তবে $t = 4$

$I = \int_2^4 \frac{dt}{t} = [\ln t]_2^4 = \ln 4 - \ln 2 = \ln\frac{4}{2} = \ln 2$

(iii). $I = \int_{-\pi/2}^{\pi/2} \frac{dx}{5 + \frac{7(1-\tan^2 x/2)}{1+\tan^2 x/2} + \frac{2\tan x/2}{1+\tan^2 x/2}}$

$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \frac{\sec^2 x/2\, dx}{6 + \tan x/2 - \tan^2 x/2}$

ধরি $\tan x/2 = t$ তবে $\frac{1}{2}\sec^2 x/2\, dx = dt$

সীমা ঃ যদি $x = -\frac{\pi}{2}$ হয়, তবে $t = -1$

এবং যদি $x = \frac{\pi}{2}$ হয়, তবে $t = 1$

$$I = \int_{-1}^{1} \frac{dt}{6+t-t^2} = \int_{-1}^{1} \frac{dt}{(5/2)^2 - (t-1/2)^2}$$

$$= \frac{1}{2.5/2}\left[ln \frac{5/2 + (t-1/2)}{5/2 - (t-1/2)}\right]_{-1}^{1} = \frac{1}{5}\left[ln \frac{2+t}{3-t}\right]_{-1}^{1}$$

$$= \frac{1}{5}\left[ln \frac{3}{2} - ln \frac{1}{4}\right] = \frac{1}{5} ln \left(\frac{3}{2} \cdot \frac{4}{1}\right) = \frac{1}{5} ln 6$$

2(i). $$I = \int_0^{\pi/3}\left[\frac{2}{\cos^2 x} + \frac{3\sin x}{\cos^2 x}\right] dx = \int_0^{\pi/3} [2\sec^2 x + 3\sec x \tan x]\, dx$$

$$= [2\tan x + 3\sec x]_0^{\pi/3} = \left(2\tan\frac{\pi}{3} + 3\sec\frac{\pi}{3}\right) - (0+3)$$

$$= 2\sqrt{3} + 3.2 - 3 = 2\sqrt{3} + 3.$$

(ii). $$I = \int_0^{\pi} \sqrt{2(1+\cos\theta)}\, d\theta = \int_0^{\pi} \sqrt{2.2\cos^2\theta}\, d\theta$$

$$= \int_0^{\pi} 2\cos\frac{\theta}{2}\, d\theta = 2\left[\frac{\sin\theta/2}{1/2}\right]_0^{\pi} = 4\left[\sin\frac{\pi}{2} - 0\right] = 4[1-0]$$

(iii). $$I = \int_0^{\pi/4} \sec x \sec^2 x\, dx = \int_0^{\pi/4} \sqrt{1+\tan^2 x}\, \sec^2 x\, dx$$

ধরি $\tan x = z$ তবে $\sec^2 x\, dx = dz$

সীমা ঃ যদি $x = 0$ হয়, তবে $z = 0$

এবং যদি $x = \frac{\pi}{4}$ হয়, তবে $z = 1$

$$I = \int_0^{1} \sqrt{1+z^2}\, dz = \left[\frac{z\sqrt{1+z^2}}{2} + \frac{1}{2} ln\,(z + \sqrt{1+z^2})\right]_0^{1}$$

$$= \frac{\sqrt{1+1}}{2} + \frac{1}{2} ln\,(1+\sqrt{2}) - 0 = \frac{1}{\sqrt{2}} + \frac{1}{2} ln\,(1+\sqrt{2})$$

(iv). $$I = \int_0^{\pi/2} \frac{\cos x\, dx}{(1+\sin x)(2+\sin x)}$$

ধরি $\sin x = z$ তবে $\cos x\, dx = dz$

সীমা ঃ যদি $x = 0$ হয়, তবে $z = 0$

এবং যদি $x = \frac{\pi}{2}$ হয়, তবে $t = 1$

$$I=\int_0^1\frac{dz}{(1+z)(2+z)}=\int_0^1\left[\frac{1}{(1+z)(2-1)}+\frac{1}{(1-2)(2+z)}\right]dz$$

$$=\int_0^1\left[\frac{1}{1+z}-\frac{1}{2+z}\right]dz=[ln(1+z)-ln(2+z)]_0^1$$

$$=[(ln2-ln3)-(ln1-ln2)=2ln2-ln3=ln4-ln3.$$

(v). $I=\displaystyle\int_0^{\pi/2}\cos^2x\,(\sin x)^{1/4}\cos x\,dx$

$$=\int_0^{\pi/2}(1-\sin^2x)(\sin x)^{1/4}\cos x\,dx$$

ধরি $\sin x=z$ তবে $\cos x\,dx=dz$

সীমা ঃ যদি $x=0$ হয়, তবে $z=0$

এবং যদি $x=\frac{\pi}{2}$ হয়, তবে $z=1$

$$I=\int_0^1(1-z^2)\,z^{1/4}\,dz=\int_0^1[z^{1/4}-z^{9/4}]\,dz=\left[\frac{z^{5/4}}{5/4}-\frac{z^{13/4}}{13/4}\right]_0^1$$

$$=\frac{4}{5}-\frac{4}{13}=\frac{32}{65}$$

3(i). $I=\displaystyle\int_0^{\pi/4}\frac{\sin^2x\cos^2x\,dx}{\cos^6x(1+\tan^3x)^2}=\int_0^{\pi/4}\frac{\tan^2x\sec^2x\,dx}{(1+\tan^3x)^2}$

ধরি $1+\tan^3x=z$ তবে $3\tan^2x.\sec^2x\,dx=dz$

সীমা ঃ যদি $x=0$ হয়, তবে $z=1$

এবং যদি $x=\frac{\pi}{4}$ হয়; তবে $z=1+1=2$

$$I=\frac{1}{3}\int_1^2\frac{dz}{z^2}=\left[\frac{-1}{3z}\right]_1^2=-\frac{1}{3}\left(\frac{1}{2}-1\right)=\frac{1}{6}$$

(ii). ধরি $a^2\cos^2x+b^2\sin^2x=t$

তবে $2(b^2-a^2)\sin x\cos x\,dx=dt$

$$\therefore\int\frac{\sin x\cos x\,dx}{(a^2\cos^2x+b^2\sin^2x)^2}=\frac{1}{2(b^2-a^2)}\int\frac{dt}{t^2}=\frac{-1}{2(b^2-a^2)t}$$

$$\Rightarrow\int\frac{\sin x\cos x\,dx}{(a^2\cos^2x+b^2\sin^2x)^2}=\frac{-1}{2(b^2-a^2)(a^2\cos^2x+b^2\sin^2x)}$$

$$\therefore I = \int_0^{\pi/2} \frac{x \sin x \cos x\, dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2}$$

$$= \left[x \frac{(-1)}{2(b^2 - a^2)(a^2 \cos^2 x + b^2 \sin^2 x)}\right]_0^{\pi/2}$$

$$- \int_0^{\pi/2} 1 . \frac{(-1)\, dx}{2(b^2 - a^2)(a^2 \cos^2 x + b^2 \sin^2 x)}$$ [(i) নং দ্বারা।]

$$= \frac{-\pi/2}{2(b^2 - a^2)(0 + b^2)} + \frac{1}{2(b^2 - a^2)} I_1$$

এখন $I_1 = \int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x}$

ধরি $b \tan x = z$ তবে $b \sec^2 x\, dx = dz$

সীমা ঃ যদি $x = 0$ হয়, তবে $z = 0$

এবং যদি $x = \frac{\pi}{2}$ হয়, তবে $z = \infty$

$$I_1 = \frac{1}{b} \int_0^{\infty} \frac{dz}{a^2 + z^2} = \frac{1}{ab}\left[\tan^{-1} \frac{z}{a}\right]_0^{\infty} = \frac{1}{ab}\left[\frac{\pi}{2} - 0\right]$$

$$\therefore I = \frac{-\pi}{4(b^2 - a^2)\, b^2} + \frac{1}{2(b^2 - a^2)} \cdot \frac{\pi}{2ab}$$

$$= \frac{\pi}{4b(b^2 - a^2)}\left[-\frac{1}{b} + \frac{1}{a}\right] = \frac{\pi(b - a)}{4b(b^2 - a^2)\, ab}.$$

(iii). $I = \int_0^{\pi/4} \frac{2\sin x \cos x\, dx}{\cos^4 x\,(1 + \tan^4 x)} = \int_0^{\pi/4} \frac{2\tan x . \sec^2 x\, dx}{1 + \tan^4 x}$

ধরি $\tan^2 x = z$ তবে $2\tan x \sec^2 x\, dx = dz$

সীমা ঃ যদি $x = 0$ হয়, তবে $z = 0$

এবং যদি $x = \frac{\pi}{4}$ হয়, তবে $z = 1$

$$\therefore I = \int_0^1 \frac{dz}{1 + z^2} = [\tan^{-1} z]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}.$$

(iv). $I = \int_0^{\pi/2} \frac{dx}{\cos^4 x (a^2 + b^2 \tan^2 x)^2} = \int_0^{\pi/2} \frac{\sec^4 x\, dx}{(a^2 + b^2 \tan^2 x)^2}$

$$= \int_0^{\pi/2} \frac{(1 + \tan^2 x) \sec^2 x\, dx}{(a^2 + b^2 \tan^2 x)^2}$$

ধরি $b \tan x = a \tan t$, তবে $b \sec^2 x\, dx = a \sec^2 t\, dt$

সীমা ঃ যদি $x = 0$ হয়, তবে $a \tan t = 0 \Rightarrow t = 0$

যদি $x = \frac{\pi}{2}$ হয়, তবে $a \tan t = \infty \Rightarrow t = \frac{\pi}{2}$

$$I = \int_0^{\pi/2} \frac{\{1 + (a^2/b^2) \tan^2 t\} . (a/b) \sec^2 t\, dt}{(a^2 + a^2 \tan^2 t)^2}$$

$$= \int_0^{\pi/2} \frac{(b^2 + a^2 \tan^2 t)\, a \sec^2 t\, dt}{b^2 (a^2 \sec^2 t)^2\, b}$$

$$= \frac{a}{a^4 b^3} \int_0^{\pi/2} \left(b^2 + \frac{a^2 \sin^2 t}{\cos^2 t}\right) \cos^2 t\, dt$$

$$= \frac{1}{a^3 b^3} \int_0^{\pi/2} (b^2 \cos^2 t + a^2 \sin^2 t)\, dt$$

$$= \frac{1}{a^3 b^3} \left[b^2 \frac{\Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{1}{2}\right)}{2\Gamma(2)} + \frac{a^2\, \Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{1}{2}\right)}{2\Gamma(2)} \right] = \frac{\pi(a^2 + b^2)}{4 a^3 b^3}$$

4(i). $I = \int_0^1 x \sin^{-1} x\, dx = \left[\frac{x^2}{2} . \sin^{-1} x \right]_0^1 - \int_0^1 \frac{1}{\sqrt{1 - x^2}} . \frac{x^2}{2} dx$

$$= \frac{1}{2} \sin^{-1} 1 - 0 - \frac{1}{2} \int_0^1 \frac{-(1 - x^2) + 1}{\sqrt{1 - x^2}} dx$$

$$= \frac{1}{2} . \frac{\pi}{2} - \frac{1}{2} \int_0^1 \left[-\sqrt{1 - x^2} + \frac{1}{\sqrt{1 - x^2}} \right] dx$$

$$= \frac{\pi}{4} - \frac{1}{2} \left[\frac{-x \sqrt{1 - x^2}}{2} - \frac{1}{2} \sin^{-1} x + \sin^{-1} x \right]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2} \left[0 + \frac{1}{2} \sin^{-1} 1 - 0 \right] = \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8}.$$

(ii). ধরি $x = z^2$ তবে $dx = 2z\, dz$

সীমা ঃ যদি $x = 0$ হয়, তবে $z = 0$

এবং যদি $x = 1$ হয়, তবে $z = 1$

$$I = 2 \int_0^1 \tan^{-1} z . z dz = 2 \left\{ \left[\tan^{-1} z . \frac{z^2}{2} \right]_0^1 - \int_0^1 \frac{1}{(1 + z^2)} . \frac{z^2}{2} dz \right\}$$

$$= 2 \left\{ \left(\frac{1}{2} \tan^{-1} 1 - 0 \right) - \frac{1}{2} \int_0^1 \frac{(1 + z^2) - 1}{1 + z^2} dz \right\}$$

বা $I = 2\left\{\frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\int_0^1\left[1 - \frac{1}{1+z^2}\right]dz\right\}$

$= 2\left\{\frac{\pi}{8} - \frac{1}{2}\left[z - \tan^{-1} z\right]_0^1\right\}$

$= 2\left\{\frac{\pi}{8} - \frac{1}{2}(1 - \tan^{-1}1) - 0\right\}$

$= 2\left[\frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8}\right] = \frac{\pi}{2} - 1.$

(iii). $I = \int_0^1 \tan^{-1}x.\ 1\,dx = \left[\tan^{-1} x.x\right]_0^1 - \int_0^1 \frac{1}{1+x^2}x\,dx$

$= (\tan^{-1}1.\ 1 - 0) - \frac{1}{2}\left[ln(1+x^2)\right]_0^1 = \frac{\pi}{4} - \frac{1}{2}(ln2 - ln1)$

(iv). $I = \int_0^1 x(\tan^{-1}x)^2\,dx$

$= \left[(\tan^{-1}x)^2.\frac{x^2}{2}\right]_0^1 - 2\int_0^1 \frac{\tan^{-1}x}{1+x^2}\cdot\frac{x^2}{2}dx$

$= \frac{1}{2}(\tan^{-1}1)^2 - 0 - \int_0^1 \tan^{-1}x\left[\frac{(x^2+1)-1}{1+x^2}\right]dx$

$I = \frac{1}{2}\left(\frac{\pi}{4}\right)^2 - \int_0^1 \tan^{-1}x\,dx + \int_0^1 \frac{\tan{-}x}{1+x^2}dx \cdots (1)$

এখন $\int_0^1 \tan^{-1}x\,dx = \left[\tan^{-1} x.x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}dx$

$= \tan^{-1}1 - \frac{1}{2}\left[ln(1+x^2)\right]_0^1 = \frac{\pi}{4} - \frac{1}{2}ln2$

এবং $\int_0^1 \frac{\tan^{-1}x}{1+x^2}dx = \int_0^{\pi/4} z\,dz$, যখন $\tan^{-1}x = z$, তখন $\frac{dx}{1+x^2} = dz$

$= \left[\frac{z^2}{2}\right]_0^{\pi/4} = \frac{1}{2}\left(\frac{\pi^2}{16}\right) = \frac{\pi^2}{32}$

(1) $\Rightarrow I = \frac{\pi^2}{32} - \frac{\pi}{4} + \frac{1}{2}ln2 + \frac{\pi^2}{32}$

$= \frac{\pi^2}{16} - \frac{\pi}{4} + ln\,2^{1/2} = \frac{\pi^2}{16} - \frac{\pi}{4} + ln\sqrt{2}.$

(v). ধরি $\tan^{-1}x = z$ তবে $\frac{dx}{1+x^2} = dz$

সীমা ঃ যদি $x = 0$ হয়, তবে $z = 0$

এবং যদি $x = 1$ হয়, তবে $t = \tan^{-1}1 = \frac{\pi}{4}$

$$I = \int_0^{\pi/4} z^{1/2}\,dz = \left[\frac{z^{3/2}}{3/2}\right]_0^{\pi/4} = \frac{2}{3}\left[\left(\frac{\pi}{4}\right)^{3/2} - 0\right] = \frac{\pi\sqrt{\pi}}{12}.$$

(vi). ধরি $\tan^{-1}x = z$ তবে $\frac{dx}{1+x^2} = dz$

সীমা ঃ যদি $x = 0$ হয়, তবে $z = 0$

এবং যদি $x = \infty$ হয়, তবে $z = \frac{\pi}{2}$

$$I = \int_0^{\pi/2} z^2\,dz = \left[\frac{z^3}{3}\right]_0^{\pi/2} = \frac{1}{3}\left[\left(\frac{\pi}{2}\right)^3 - 0\right] = \frac{\pi^3}{24}.$$

(vii). ধরি $x = \tan\theta$ তবে $dx = \sec^2\theta\,d\theta$

সীমা ঃ যদি $x = 0$ হয়, তবে $\theta = 0$

যদি $x = a$ হয়, তবে $\tan\theta = a$

$$\therefore I = \int_0^{\theta} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta\,d\theta = \int_0^{\theta} \sin^{-1}(\sin 2\theta)\sec^2\theta\,d\theta$$

$$= 2\int_0^{\theta} \theta\sec^2\theta\,d\theta = 2\left\{[\theta\tan\theta]_0^{\pi/2} - \int_0^{\theta} 1\,\tan\theta\,d\theta\right\}$$

$$= 2\left\{\theta\tan\theta - 0 - [ln(\sec\theta)]_0^{\theta}\right\} = 2\theta\tan\theta - 2ln(\sec\theta)$$

$$= 2\theta\tan\theta - ln(1+\tan^2\theta) = 2a\tan^{-1}a - ln(1+a^2).$$

(viii). ধরি $x = \cos\theta$ তবে $dx = -\sin\theta\,d\theta$

সীমা ঃ যদি $x = 0$ হয়, তবে $0 = \cos\theta \Rightarrow \theta = \frac{\pi}{2}$

এবং যদি $x = 1$ হয়, তবে $1 = \cos\theta \Rightarrow \theta = 0$

$$I = -\int_{\pi/2}^{0} (\cos^{-1}\cos\theta)^2\sin\theta\,d\theta = \int_0^{\pi/2} \theta^2\sin\theta\,d\theta$$

$$= [\theta^2(-\cos\theta)]_0^{\pi/2} - 2\int_0^{\pi/2} \theta(-\cos\theta)\,d\theta$$

$$= 0 + 2\int_0^{\pi/2} \theta\cos\theta\,d\theta = 2\left\{[\theta\sin\theta]_0^{\pi/2} - \int_0^{\pi/2} 1\sin\theta\,d\theta\right\}$$

$$= 2\left\{\left(\frac{\pi}{2}\cdot 1 - 0\right) + [\cos\theta]_0^{\pi/2}\right\} = 2\left\{\frac{\pi}{2} + (0-1)\right\} = \pi - 2.$$

(ix). ধরি $I = \int_0^1 \frac{\sin^{-1}x\, dx}{\sqrt{1-x^2}}$ এবং $\sin^{-1}x = z$ তবে $\frac{dx}{\sqrt{1-x^2}} = dz$

সীমা ঃ যদি $x = 0$ হয়, তবে $z = \sin^{-1}0 = 0$

যদি $x = 1$ হয়, তবে $z = \sin^{-1}1 = \frac{\pi}{2}$

$$I = \int_0^{\pi/2} z\, dz = \left[\frac{z^2}{2}\right]_0^{\pi/2} = \frac{1}{2}\frac{\pi^2}{4} = \frac{\pi^2}{8}.$$

5(i). ধরি $ln x = z$ তবে $\frac{dx}{x} = dz$

সীমা ঃ যদি $x = a$ হয়, তবে $z = ln a$

এবং যদি $x = b$ হয়, তবে $z = ln b$

$$I = \int_{\log b}^{\log a} z\, dz = \left[\frac{z^2}{2}\right]_{\log a}^{\log b} = \frac{1}{2}[(ln b)^2 - (ln a)^2]$$

$$= \frac{1}{2}[(ln b + ln a)(ln b - ln a)] = \frac{1}{2} ln(ab)\, ln\left(\frac{b}{a}\right).$$

(ii). ধরি $ln x = z$ তবে $\frac{dx}{x} = dz$

সীমা ঃ যদি $x = 1$ হয়, তবে $z = 0$

এবং যদি $x = 3$ হয়, তবে $z = ln 3$

$$I = \int_0^{\log 3} \cos z\, dz = [\sin z]_0^{ln 3} = \sin(ln 3) - 0.$$

(iii). ধরি $1 + ln x = z$ তবে $\frac{dx}{x} = dz$

সীমা ঃ যদি $x = 1$ হয়, তবে $z = 1 + ln 1 = 1$

যদি $x = e^2$ হয়, তবে $z = 1 + ln e^2 = 1 + 2 ln e = 1 + 2 = 3$

$$= \int_0^3 \frac{dz}{z^2} = -\left[\frac{1}{z}\right]_1^3 = -\left(\frac{1}{3} - 1\right) = \frac{2}{3}.$$

(iv). $$I = \int_0^1 \frac{ln(1-x)}{x} dx = \int_0^1 \left[\frac{-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots}{x}\right] dx$$

$$= -\int_0^1 \left[1 + \frac{1}{2}x + \frac{1}{3}x^2 + \cdots\right] dx$$

$$= -\left[x + \frac{1}{2^2}x^2 + \frac{1}{3^2}x^3 + \cdots\right]_0^1$$

$$= -\left\{\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) - (0)\right\} = -\frac{\pi^2}{6}.$$

(v). ধরি $lnx = z$, বা $x = e^z$ তবে $dx = e^z dz$

সীমা ঃ যদি $x = 2$ হয়, তবে $z = ln2$

এবং যদি $x = e$ হয়, তবে $z = 1$

$$I = \int_{\log 2}^{1}\left[\frac{1}{z} - \frac{1}{z^2}\right] e^z dz = \int_{\log 2}^{1}\frac{1}{z} e^z dz - \int_{\log 2}^{1}\frac{1}{z^2} e^z dz$$

$$= \left[\frac{1}{z} . e^z\right]_{\log 2}^{1} - \int_{\log 2}^{1}\left(-\frac{1}{z^2}\right) e^z dz - \int_{\log 2}^{1}\frac{1}{z^2} e^z dz$$

$$= \frac{e}{1} - \frac{e^{ln2}}{ln2} + \int_{\log 2}^{1}\frac{e^z}{z^2} dz - \int_{\log 2}^{1}\frac{e^z}{z^2} dz$$

$$= e - \frac{e^{ln2}}{ln2}.$$

(vi). ধরি $I = \int_0^3 ln(x + \sqrt{x^2 - 1}).1\ dx$

$$= \left[ln(x + \sqrt{x^2 - 1}).x\right]_0^3 - \int_0^3 \frac{\left(1 + \frac{2x}{2\sqrt{x^2 - 1}}\right)}{x + \sqrt{x^2 - 1}} x\, dx$$

$$= ln(3 + \sqrt{8}).\ 3 - 0 - \int_0^3 \frac{(\sqrt{x^2 - 1} + x)\ x\, dx}{(x + \sqrt{x^2 - 1})\ \sqrt{x^2 - 1}}$$

$$= 3\ ln(3 + 2\sqrt{2}) - \frac{1}{2}\int_0^3 \frac{2x\, dx}{\sqrt{x^2 - 1}}$$

$$= 3ln(3 + 2\sqrt{2}) - \frac{1}{2}2\left[\sqrt{x^2 - 1}\right]_0^3$$

$$= 3ln(3 + 2\sqrt{2}) - \sqrt{8} - 0 = 3ln(3 + 2\sqrt{3}) - 2\sqrt{2}.$$

6(i). মনেকরি $1 + e^x = z$ তবে $e^x dx = dz$

সীমা ঃ যদি $x = 0$ হয়, তবে $z = 1 + e^0 = 1 + 1 = 2$

এবং যদি $x = ln2$ হয়, তবে $z = 1 + e^{ln2} = 1 + 2 = 3$

$$I = \int_2^3 \frac{dt}{t} = [lnt]_2^3 = ln3 - ln2 = ln\frac{3}{2}$$

(ii). ধরি $I = \int_0^{\pi/6} e^{2\theta} \sin 3\theta\, d\theta = \left[\frac{e^{2\theta}(2\sin 3\theta - 3\cos 3\theta)}{2^2 + 3^2}\right]_0^{\pi/6}$

$$= \frac{1}{13}\left[e^{\pi/3}\left(2\sin\frac{\pi}{2} - 3\cos\frac{\pi}{2}\right) - e^0(2\sin 0 - 3\cos 0)\right]$$

$$= \frac{1}{13}[e^{\pi/3}(2 - 0) - (0 - 3)] = \frac{1}{13}[2e^{\pi/3} + 3].$$

7(i). প্রশ্ন সংশোধন ঃ $x - 1$ এর স্থলে $x + 1$ হইবে।

ধরি $x + 1 = z^2$, বা $x = z^2 - 1$ তবে $dx = 2z\,dz$

সীমা ঃ যদি $x = 8$ হয়, তবে $z = 3$

এবং যদি $x = 15$ হয়, তবে $z = 4$

$$I = \int_3^4 \frac{2z\,dz}{(z^2 - 1 - 3)z} = 2\int_3^4 \frac{dz}{z^2 - 2^2} = \frac{2}{2.2}\left[ln\,\frac{z-2}{z+2}\right]_3^4$$

$$= \frac{1}{2}\left[ln\,\frac{2}{6} - ln\,\frac{1}{5}\right] = \frac{1}{2}\,ln\left(\frac{2}{6}\cdot\frac{5}{1}\right) = \frac{1}{2}\,ln\,\frac{5}{3}.$$

(ii). $$I = \int_0^1 \frac{dx}{\sqrt{2x - x^2}} = \int_0^1 \frac{dx}{\sqrt{1 - (x-1)^2}} = [\sin^{-1}(x-1)]_0^1$$

$$= \sin^{-1}0 - \sin^{-1}(-1) = 0 + \sin^{-1}1 = \frac{\pi}{2}.$$

(iii). ধরি $x = \sin\theta$ তবে $dx = \cos\theta\,d\theta$.

সীমা ঃ যদি $x = 0$ হয়, তবে $\theta = 0$

এবং যদি $x = \frac{1}{\sqrt{3}}$ হয়, তবে $\sin\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \sin^{-1}\frac{1}{\sqrt{3}}$.

$$I = \int_0^\theta \frac{\cos d\theta}{(1 + \sin^2\theta)\sqrt{\cos^2\theta}} = \int_0^\theta \frac{d\theta}{1 + \sin^2\theta}$$

$$= \int_0^\theta \frac{\sec^2\theta\,d\theta}{1 + 2\tan^2\theta}$$

$$= \frac{1}{2}\int_0^\theta \frac{\sec^2\theta\,d\theta}{1/2 + \tan^2\theta} = \frac{1}{2}\int_0^\theta \frac{d(\tan\theta)}{(1/\sqrt{2})^2 + (\tan\theta)^2}$$

$$= \frac{1}{2}\cdot\frac{1}{1/\sqrt{2}}\left[\tan^{-1}\frac{\tan\theta}{1/\sqrt{2}}\right]_0^\theta = \frac{1}{\sqrt{2}}\{\tan^{-1}(\sqrt{2}\tan\theta) - 0\}$$

যেহেতু $\sin\theta = \frac{1}{\sqrt{3}}$, কাজেই $\tan\theta\,\frac{1}{\sqrt{3 - 1^2}} = \frac{1}{\sqrt{2}}$

$$\therefore I = \left(\frac{1}{\sqrt{2}}\right)\tan^{-1}\left(\frac{\sqrt{2}.1}{\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}}\right)\tan^{-1}.1 = \frac{\pi}{4\sqrt{2}}.$$

(iv). ধরি $x = b\tan\theta$ তবে $dx = b\sec^2\theta\,d\theta$

সীমা ঃ যদি $x = 0$ হয়, তবে $\theta = 0$

এবং যদি $x = \infty$ হয়, তবে $\theta = \frac{\pi}{2}$.

$$I = \int_0^{\pi/2} \frac{b\sec^2\theta\,d\theta}{(a^2 + b^2\tan^2\theta)\,b\sec\theta} = \int_0^{\pi/2} \frac{\sec\theta\,d\theta}{a^2 + b^2(\sin^2\theta)/\cos^2\theta}$$

$$= \int_0^{\pi/2} \frac{\cos\theta\,d\theta}{a^2\cos^2\theta + b^2\sin^2\theta} = \int_0^{\pi/2} \frac{\cos\theta\,d\theta}{a^2 + (b^2 - a^2)\sin^2\theta}$$

ধরি $\sqrt{b^2-a^2}\sin\theta = z$ তবে $\sqrt{b^2-a^2}\cos\theta\, d\theta = dz$

সীমা ঃ যদি $\theta = 0$ হয়, তবে $z = 0$

এবং যদি $\theta = \frac{\pi}{2}$ হয়, তবে $z = \sqrt{b^2-a^2}$.

$$I = \frac{1}{\sqrt{b^2-a^2}}\int_0^{\sqrt{a^2-b^2}} \frac{dz}{a^2+z^2} = \frac{1}{\sqrt{b^2-a^2}}\cdot\frac{1}{a}\left[\tan^{-1}\frac{z}{a}\right]_0^{\sqrt{b^2-a^2}}$$

$$= \frac{1}{a\sqrt{b^2-a^2}}\left[\tan^{-1}\frac{\sqrt{b^2-a^2}}{a} - 0\right]$$

$$= \frac{1}{a\sqrt{b^2-a^2}}\cdot\tan^{-1}\frac{\sqrt{b^2-a^2}}{a}$$

(v). $I = \int_3^3 \frac{2dx}{x\sqrt{1+x}} - \int_3^3 \frac{3dx}{\sqrt{1+x}} = I_1 - I_2$, ধরি।

এখন $I_1 = 2\int_3^3 \frac{dx}{x\sqrt{1+x}}$, ধরি $1+x = z^2$ তবে $dx = 2z\,dz$

সীমা ঃ যদি $x = 3$ হয়, তবে $z = 2$

এবং যদি $x = 8$ হয়, তবে $z = 3$.

$$I_1 = 2\int_2^3 \frac{2z\,dz}{(z^2-1)z} = 4\int_2^3 \frac{dz}{z^2-1} = \frac{4}{2.1}\left[\ln\frac{z-1}{z+1}\right]_2^3$$

$$= 2\left[\ln\frac{2}{4} - \ln\frac{1}{3}\right] = 2\ln\left(\frac{1}{2}\cdot\frac{3}{1}\right) = 2\ln\frac{3}{2},$$

এখন $I_2 = 3\int_3^3 \frac{dx}{\sqrt{1+x}} = 6\left[\sqrt{1+x}\right]_3^8 = 6(3-2) = 6$.

$$\therefore I = 2\ln\frac{3}{2} - 6 = 2\left[\ln\frac{3}{2} - 3\right] = 2\left[\ln\frac{3}{2} - 3\ln e\right]$$

$$= 2\left[\ln\frac{3}{2} - \ln e^3\right] = 2\ln\left(\frac{3}{2e^3}\right)$$

(vi). $I = \int_{1/4}^{1/2} \frac{dx}{\sqrt{(1/2)^2 - (x-1/2)^2}}$

$$= \left[\sin^{-1}\frac{x-1/2}{1/2}\right]_{1/4}^{1/2} = \left[\sin^{-1}(2x-1)\right]_{1/4}^{1/2}$$

$$= \sin^{-1}0 - \sin^{-1}\left(-\frac{1}{2}\right) = 0 + \sin\frac{1}{2} = \frac{\pi}{6}.$$

(vii). ধরি $x = \frac{1}{z}$, তবে $dx = \frac{-dz}{z^2}$

সীমা ঃ যদি $x = 1$ হয়, তবে $z = 1$

এবং যদি $x = 2$ হয়, তবে $z = \frac{1}{2}$

$$I = -\int_1^{1/2} \frac{dz}{z^2 \cdot 1/z\sqrt{1/z^2 + 5/z + 1}}$$

$$= \int_{1/2}^{1} \frac{dz}{\sqrt{1 + 5z + z^2}}$$

$$= \int_{1/2}^{1} \frac{dz}{\sqrt{(z + 5/2)^2 - (\sqrt{21}/2)^2}}$$

$$= \left[ln\left\{\left(z + \frac{5}{2}\right) + \sqrt{1 + 5z + z^2}\right\}\right]_{1/2}^{1}$$

$$= ln\left(\frac{7}{2} + \sqrt{7}\right) - ln\left(3 + \frac{\sqrt{15}}{2}\right).$$

(viii). $$I = \int_0^1 \frac{-(1 + x) + 2}{1 + x} dx = \int_0^1 \left[-1 + \frac{2}{1 + x}\right] dx$$

$$= [-x + 2ln(1 + x)]_0^1 = -1 + 2ln2 = -lne + ln2^2$$

$$= ln\frac{4}{e}.$$

(ix). ধরি $I = \int_0^1 \frac{(1 - \sqrt{x})\,dx}{1 + \sqrt{x}}$

$$= \int_0^1 \frac{-(1 + \sqrt{x}) + 2}{1 + \sqrt{x}} dx = -\int_0^1 dx + 2\int_0^1 \frac{dx}{1 + \sqrt{x}}$$

$$= -[x]_0^1 + 2\int_0^1 \frac{dx}{1 + \sqrt{x}} = -1 + I_1$$

যখন $I_1 = 2\int_0^1 \frac{dx}{1 + \sqrt{x}}$; ধরি $x = z^2$ তবে $dx = 2z\,dz$

সীমা ঃ যদি $x = 0$ হয়, তবে $z = 0$

এবং যদি $x = 1$ হয়, তবে $z = 1$

$$\therefore I_1 = 2\int_0^1 \frac{2z\,dz}{1 + z} = 4\int_0^1 \frac{(1 + z) - 1}{1 + z} dz$$

$$= 4\int_0^1 \left[1 - \frac{1}{1 + z}\right] dz = 4[z - ln(1 + z)]_0^1 = 4[1 - ln2]$$

$$\therefore I = -1 + 4 - 4ln2 = 3 - 4ln2.$$

(x). $I = \int_0^2 \frac{x^4+1}{x^2+1} dx = \int_0^2 \frac{x^2(x^2+1) - (x^2+1) + 2}{x^2+1} dx,$

$= \int_0^2 \left[x^2 - 1 + \frac{2}{x^2+1}\right] dx = \left[\frac{x^3}{3} - x + 2\tan^{-1}x\right]_0^2$

$= \frac{8}{3} - 2 + 2\tan^{-1}2 = \frac{2}{3} + 2\tan^{-1}2.$

(xi). $I = \int_0^1 \frac{(1-x)\,dx}{(3+x)\sqrt{1-x}} = \int_0^1 \frac{-(3+x)+4}{(3+x)\sqrt{1-x}} dx$

$= \int_0^1 \left[\frac{-1}{\sqrt{1-x}} + \frac{4}{(3+x)\sqrt{1-x}}\right] dx = I_1 + I_2$

এখন $I_1 = \int_0^1 \frac{-1\,dx}{\sqrt{1-x}} = 2\left[\sqrt{1-x}\right]_0^1 = 2(0 - \sqrt{1}) = -2$

এবং $I_2 = \int_0^1 \frac{4dx}{(3+x)\sqrt{1-x}}$

ধরি $1 - x = z^2$, বা $x = 1 - z^2$ তবে $dx = -2z\,dz$

সীমা : যদি $x = 0$ হয়, তবে $z = 1$

এবং যদি $x = 1$ হয়, তবে $z = 0$

$I_2 = -4\int_1^0 \frac{2z\,dz}{(3+1-z^2)z} = 8\int_0^1 \frac{dz}{2^2 - z^2} = \frac{8}{2.2}\left[ln\,\frac{2+z}{2-z}\right]_0^1$

$= 2\,[ln3 - ln1] = 2ln3$

$\therefore I = I_1 + I_2 = -2 + 2ln3 = 2(ln3 - 1).$

(xii). $I = \int_0^a \frac{-(a^2+x^2) + 2a^2}{(a^2+x^2)^{3/2}} dx$

$= -\int_0^a \frac{dx}{\sqrt{a^2+x^2}} + 2a^2\int_0^a \frac{dx}{(a^2+x^2)^{3/2}}$

$= I_1 + I_2$ ধরি।

এখন $I_1 = -\int_0^a \frac{dx}{\sqrt{a^2+x^2}} = -\left[ln(x + \sqrt{a^2+x^2})\right]_0^a$

$= -[ln(a + a\sqrt{2}) - lna] = -ln(1+\sqrt{2})$

এবং $I_2 = 2a^2\int_0^a \frac{dx}{(a^2+x^2)^{3/2}}$

ধরি $x = a\tan\theta$, তবে $dx = a\sec^2\theta\,d\theta$

সীমা : যদি $x = 0$ হয়, তবে $\theta = 0$

এবং যদি $x = a$ হয়, তবে $\theta = \frac{\pi}{4}$

$$I_2 = 2a^2 \int_0^{\pi/4} \frac{a\sec^2\theta\, d\theta}{(a^2\sec^2\theta)^{3/2}} = 2a^2 \int_0^{\pi/4} \frac{a\sec^2\theta\, d\theta}{a^3\sec^3\theta} = 2\int_0^{\pi/4} \cos\theta\, d\theta$$

$$= 2[\sin\theta]_0^{\pi/4} = 2\left[\sin\frac{\pi}{4} - \sin 0\right] = 2\left[\frac{1}{\sqrt{2}} - 0\right] = \sqrt{2}$$

$\therefore I = I_1 + I_2 = -\ln(1+\sqrt{2}) + \sqrt{2}$.

(xiii). ধরি $1 + 3x^4 = z^2$ তবে $12x^3\, dx = 2z\, dz$

সীমা ঃ যদি $x = 0$ হয়, তবে $z = 1$

এবং যদি $x = 1$ হয়, তবে $z = 2$

$$I = \frac{1}{6}\int_1^2 z.z\, dz = \frac{1}{6}\left[\frac{1}{3}\cdot z^3\right]_1^2 = \frac{1}{18}[8-1] = \frac{7}{18}$$

(xiv). $I = \int_0^4 \frac{x^2 x\, dx}{\sqrt{(x^2+9)}}$; ধরি $x^2 + 9 = z^2$, তবে $2x\, dx = 2z\, dz$

সীমা ঃ যদি $x = 0$ হয়, তবে $z = 3$

এবং যদি $x = 4$ হয়, তবে $z = 5$

$$\therefore I = \int_3^5 \frac{(z^2-9)\, zdz}{z} = \int_3^5 (z^2-9)\, dz = \left[\frac{1}{3}z^3 - 9z\right]_3^5$$

$$= \left(\frac{1}{3}.125 - 45\right) - \left(\frac{1}{3}.27 - 27\right)$$

$$= -\frac{10}{3} - (-18) = \frac{44}{3}.$$

(xv). ধরি $x = z^4$, তবে $dx = 4z^3\, dz$

সীমা ঃ যদি $x = 0$ হয়, তবে $z = 0$

এবং যদি $x = 16$ হয়, তবে $z = 2$

$$I = \int_0^2 \frac{z.4z^3\, dz}{1+z^2} = 4\int_0^2 \frac{z^2(z^2+1) - (z^2+1) + 1}{z^2+1}\, dz$$

$$= 4\int_0^2 \left[z^2 - 1 + \frac{1}{z^2+1}\right] dz = 4\left[\frac{1}{3}z^3 - z + \tan^{-1}z\right]_0^2$$

$$= 4\left[\frac{8}{3} - 2 + \tan^{-1}2 - 0\right] = 4\left[\frac{2}{3} + \tan^{-1}2\right].$$

(xvi). আংশিক ভগ্নাংশের কভার আপ নিয়মানুসারে পাই

$$\frac{x-3}{x^2(x+1)} = \frac{0-3}{x^2(0+1)} + \frac{A}{x} + \frac{-1-3}{(-1)^2(x+1)}$$

$$\text{বা } \frac{x-3}{x^2(x+1)} = \frac{-3}{x^2} + \frac{A}{x} - \frac{4}{x+1} \ldots (1)$$

উভয় পক্ষে $x = 1$ বসাইয়া পাই

$\frac{-2}{1.2} = \frac{-3}{1} + A - \frac{4}{2}$, বা $-1 + 3 + 2 = A \Rightarrow A = 4$

$$(1) \Rightarrow \frac{x-3}{x^3+x^2} = \frac{4}{x} - \frac{3}{x^2} - \frac{4}{x+1}$$

$$\Rightarrow \int_1^3 \frac{(x-3)dx}{x^3+x^2} = \int_1^3 \left[\frac{4}{x} - \frac{3}{x^2} - \frac{4}{x+1}\right] dx$$

$$= \left[4\ln x + \frac{3}{x} - 4\ln(x+1)\right]_1^3$$

$$= (4\ln 3 + 1 - 4\ln 4) - (0 + 3 - 4\ln 2)$$

$$= 4(\ln 3 - \ln 4 + \ln 2) - 2 = 4\ln\frac{6}{4} - 2.$$

(xvii). $$I = \int_2^3 \frac{(x^2+1)\,dx}{(2x+1)(x-1)(x+1)}$$

$$= \int_2^3 \left[\frac{1/4+1}{(2x+1)(-3/2)(1/2)} + \frac{1+1}{3(x-1)2} + \frac{1+1}{-1(-2)(x+1)}\right] dx$$

$$= \int_2^3 \left[\frac{-5}{3(2x+1)} + \frac{1}{3(x-1)} + \frac{1}{x+1}\right] dx$$

$$= \left[-\frac{5}{6}\ln(2x+1) + \frac{1}{3}\ln(x-1) + \ln(x+1)\right]_2^3$$

$$= -\frac{5}{6}(\ln 7 - \ln 5) + \frac{1}{3}(\ln 2 - \ln 1) + \ln 4 - \ln 3$$

$$= \frac{5}{6}\ln\frac{5}{7} + \frac{1}{3}\ln 2 + \ln\frac{4}{3}.$$

(xviii). $$I = \int_0^b \frac{(\sqrt{x+b} - \sqrt{x})\,dx}{(\sqrt{x+b} - \sqrt{x})(\sqrt{x+b} + \sqrt{x})}$$

$$= \int_0^b \frac{\{(x+b)^{1/2} - x^{1/2}\}\,dx}{(x+b) - x}$$

$$= \frac{1}{b}\int_0^b [(x+b)^{1/2} - x^{1/2}]\,dx = \frac{1}{b}\left[\frac{(x+b)^{3/2}}{3/2} - \frac{x^{3/2}}{3/2}\right]_0^b$$

$$= \frac{2}{3b}[(2b)^{3/2} - b^{3/2} - b^{3/2}] = \frac{2}{3b}[2b\sqrt{2b} - 2b^{3/2}]$$

$$= \frac{2}{3b}[2b\sqrt{2}\sqrt{b} - 2b\sqrt{b}] = \frac{2\sqrt{b}}{3}[2\sqrt{2} - 2].$$

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