Lecture 1, 2

Function, Limit and Continuity

Lecture 3

Derivative:

The derivative is **the instantaneous rate of change of a function with respect to one of its variables**. This is equivalent to finding the slope of the tangent line to the function at a

Point.

## What is the Derivative of a Function

In very simple words, the derivative of a function f(x) represents its rate of change and is denoted by either f'(x) or df/dx. Let's first look at its definition and a pictorial illustration of the derivative.

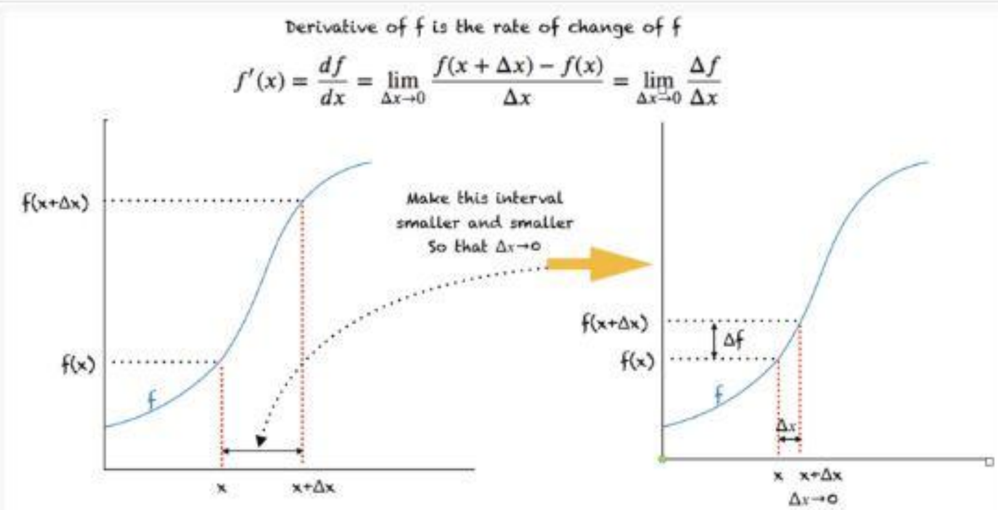



Illustration of the Definition of a Function Derivative

In the figure, Δx represents a change in the value of x. We keep making the interval between x and (x+Δx) smaller and smaller until it is infinitesimal. Hence, we have the limit ($\Delta x \to 0$). The numerator f(x+Δx)-f(x) represents the corresponding change in the value of the function f over the interval Δx. This makes the derivative of a function f at a point x, the rate of change of f at that point.

Find the derivative of the following

i. $y = f(x) = x^n$

$$\frac{dy}{dx} = f'(x) = \frac{d}{dx}x^n = nx^{n-1}$$

ii. Example $y = f(x) = x^5$

$$\frac{dy}{dx} = f'(x) = \frac{d}{dx}x^5 = 5x^{5-1} = 5x^4$$

iii. $y = f(x) = c$

$$\frac{dy}{dx} = f'(x) = \frac{d}{dx}c = 0$$

iv. $y = f(x) = 4$

$$\frac{dy}{dx} = f'(x) = \frac{d}{dx}4 = 0$$

v. $y = f(x) = cx^n$

$$\frac{dy}{dx} = f'(x) = \frac{d}{dx}cx^n = c\frac{d}{dx}x^n = cnx^{n-1}$$

vi. $y = f(x) = 3x^6$

$$\frac{dy}{dx} = f'(x) = \frac{d}{dx}3x^6 = 3\frac{d}{dx}x^6 = 3\times 6x^{6-1} = 18x^5$$

vii. $y = f(x) = \log x$

$$\frac{dy}{dx} = f'(x) = \frac{d}{dx}\log x = \frac{1}{x}$$

viii. $y = f(x) = e^x$

$$\frac{dy}{dx} = f'(x) = \frac{d}{dx}e^x = e^x$$

ix. $y = f(x) = uv$

$$\frac{dy}{dx} = f'(x) = \frac{d}{dx}uv = u\frac{d}{dx}v + v\frac{d}{dx}u$$

x. $y = f(x) = x^3 e^x$

$$\frac{dy}{dx} = f'(x) = \frac{d}{dx}x^3 e^x = x^3\frac{d}{dx}e^x + e^x\frac{d}{dx}x^3 = x^3 e^x + 3e^x x^2$$

Lecture 4, 5

Find the differential coefficient of the following

i) $y = x^{\tan x} + (\sin x)^{\cos x}$

ii) $y = x^{\cos^{-1} x} + (\sin x)^{\log x}$

Solution: (i)

$$y = x^{\tan x} + (\sin x)^{\cos x}$$

$$y = u + v$$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \text{.....................................................................(1)}$$

$u = x^{\tan x}$ and $v = \sin x^{\cos x}$

$\log u = \tan x \log x$ and $\log v = \cos x \log \sin x$

$$\frac{1}{u}\frac{du}{dx} = \tan x\frac{1}{x} + \log x \sec^2 x$$

$$\frac{du}{dx} = u[\tan x\frac{1}{x} + \log x \sec^2 x]$$

$$\frac{du}{dx} = x^{\tan x}[\tan x\frac{1}{x} + \log x \sec^2 x]$$

Again $\frac{1}{v}\frac{dv}{dx} = \cos x \frac{\cos x}{\sin x} - \sin x \log \sin x$

$$\frac{dv}{dx} = (\sin x)^{\cos x}[\cos x \frac{\cos x}{\sin x} - \sin x \log \sin x]$$

From (1), We get

$$\frac{dy}{dx} = x^{\tan x}[(\tan x)\frac{1}{x} + \log x \sec^2 x] + (\sin x)^{\cos x}[\cos x \frac{\cos x}{\sin x} - \sin x \log \sin x]$$

$$\frac{dy}{dx} = x^{\tan x}[(\tan x)\frac{1}{x} + \log x \sec^2 x] + (\sin x)^{\cos x}[\frac{\cos^2 x}{\sin x} - \sin x \log \sin x]$$

Find the differential coefficient of the following

i) $y = \tan^{-1} \frac{\sqrt{1+x^2} - 1}{x}$

ii) $y = \cos^{-1} \frac{1-x^2}{1+x^2}$

Solution: $x = \tan\theta, \quad \tan^{-1} x = \theta$

Find the differential coefficient of the following

i) $\tan^{-1} \frac{2x}{1-x^2}$ with respect to $\sin^{-1} \frac{2x}{1+x^2}$

Solution: $y = \tan^{-1} \frac{2x}{1-x^2}$ with respect to $z = \sin^{-1} \frac{2x}{1+x^2}$

We have to find $\frac{dy}{dz} = \frac{dy}{dx} / \frac{dz}{dx}$

$$x = \tan\theta \ , \quad \tan^{-1} x = \theta$$

$$y = \tan^{-1} \frac{2x}{1-x^2}$$

Lecture 6

If $y = f(x)$, the succssive derivative are also denoted by

$$y_1, y_2, y_3, \ldots\ldots\ldots\ldots\ldots\ldots, y_n$$

$$y', y'', y''', \ldots\ldots\ldots\ldots\ldots\ldots, y^{(n)}$$

$$f'(x), f''(x), f'''(x), \ldots\ldots\ldots\ldots\ldots\ldots, f^{(n)}(x)$$

$$Df(x), D^2 f(x), D^3 f(x), \ldots\ldots\ldots\ldots\ldots\ldots, D^n f(x)$$

$D$ standing for the symbol $\frac{d}{dx}$

1. The nth derivative of some special functions

$$y = x^n$$

$$y_1 = nx^{n-1}$$

$$y_2 = n(n-1)x^{n-2}$$

$$y_3 = n(n-1)(n-2)x^{n-3}$$

$$y_3 = n(n-1)\{n-(3-1)\}x^{n-3}$$

and proceeding in a similar mannner

$$y_r = n(n-1)(n-2)\ldots\ldots\ldots\ldots\ldots\ldots\{(n-(r-1)\}x^{n-r}$$

$$y_n = n(n-1)(n-2)\ldots\ldots\ldots\ldots\ldots\ldots\{(n-(n-1)\}x^{n-n}$$

$$y_n = n(n-1)(n-2)\ldots\ldots\ldots\ldots 3,2,1 = n!$$

2. $y = (ax+b)^m$

$$y_1 = ma(ax+b)^{m-1}$$

$$y_2 = a^2 m(m-1)(ax+b)^{m-2}$$

$$y_3 = a^3 m(m-1)(m-2)(ax+b)^{m-3}$$

$$y_3 = a^3 m(m-1)\{m-(3-1)\}(ax+b)^{m-3}$$

and proceeding in a similar manner

$$y_n = a^n m(m-1)(m-2)\ldots\ldots\ldots\ldots\ldots\{(m-(n-1)\}(ax+b)^{m-n}$$

$$y_n = a^n \frac{m!}{(m-n)!}(ax+b)^{m-n}$$

**Leibnitz's theorem:** (nth derivative of the product of two functions)

If $u$ and $v$ are two functions of $x$, then the nth derivative of their product i.e.,

$$(uv)_n = u_n v + {}^n c_1 u_{n-1} v_1 + {}^n c_2 u_{n-2} v_2 + \ldots\ldots\ldots + {}^n c_r u_{n-r} v_r + \ldots\ldots\ldots + u v_n$$

where the suffixes in $u$ and $v$ denote the order of differentiations of $u$ and $v$ with respect to $x$

Let $y = uv.$ By actual differentiation, we have $y_1 = u_1 v + u v_1$

$$y_2 = u_2 v + 2u_1 v_1 + u v_2 = u_2 v + {}^2 c_1 u_1 v_1 + u v_2$$

$$y_3 = u_3 v + 3u_2 v_1 + 3u_1 v_2 + u v_3 = u_3 v + {}^3 c_1 u_2 v_1 + {}^3 c_2 u_1 v_2 + u v_3$$

The theorem is thus seen to be true when n=2 and n=3.

Let us assume therefore that

$$y_n = u_n v + {}^n c_1 u_{n-1} v_1 + {}^n c_2 u_{n-2} v_2 + \ldots\ldots\ldots + {}^n c_r u_{n-r} v_r + \ldots\ldots\ldots + u v_n$$

Where n has any particular value.

Differentiating,

$$y_{n+1} = u_{n+1} v + ({}^n c_1 + 1) u_n v_1 + ({}^n c_2 + {}^n c_1) u_{n-1} v_2 + \ldots\ldots\ldots + ({}^n c_r + {}^n c_{r-1}) u_{n-r+1} v_r + \ldots\ldots\ldots + u v_{n+1}$$

Since $({}^n c_r + {}^n c_{r-1}) = {}^{n+1} c_r$ and $({}^n c_1 + 1) = {}^{n+1} c_1$

$$y_{n+1} = u_{n+1} v + {}^{n+1} c_1 u_n v_1 + {}^{n+1} c_2 u_{n-1} v_2 + \ldots\ldots\ldots + {}^{n+1} c_r u_{n-r+1} v_r + \ldots\ldots\ldots + u v_{n+1}$$

Thus, if thetheorem holdsfor n differentiations, it also holds for n+1. But it was proved to hold for 2 and 3 differentiations. Hence it holds for four, and so on, and thus the theorem is true for every positive integral value of n.

Lecture 8

Example: If $y = e^{\tan^{-1} x}$ then (i) $(1+x^2)y_2 + (2x-1)y_1 = 0$

(ii) $(1+x^2)y_{n+2} + \{2(n+1)x-1\}y_{n+1} + n(n+1)y_n = 0$

Solution:

$$y = e^{\tan^{-1} x}$$
$$\log y = \tan^{-1} x$$
$$\frac{1}{y} y_1 = \frac{1}{1+x^2}$$
$$(1+x^2)y_1 = y$$
$$(1+x^2)y_2 + 2xy_1 = y_1$$
$$(1+x^2)y_2 + (2x-1)y_1 = 0$$
$$y_2(1+x^2) + y_1(2x-1) = 0$$

By leibnitz's theorem

$$y_{n+2}(1+x^2) + {}^n c_1 y_{n+1}(2x) + {}^n c_2 y_n 2 + y_{n+1}(2x-1) + {}^n c_1 y_n 2 = 0$$

$$(1+x^2)y_{n+2} + ny_{n+1}(2x) + \frac{n(n-1)}{2} y_n 2 + (2x-1)y_{n+1} + ny_n 2 = 0$$

$$(1+x^2)y_{n+2} + (2xn + 2x - 1)y_{n+1} + (n^2 - n + 2n)y_n = 0$$

$$(1+x^2)y_{n+2} + \{2(n+1)x-1\}y_{n+1} + n(n+1)y_n = 0$$

Example: If $y = \tan^{-1} x$ then

(i) $(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$

Example: If $y = \sin(m \sin^{-1} x)$ then

(i) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$

Maxima and Minima

CHAPTER VII

# MAXIMA AND MINIMA

## ( *Functions of a Single Variable* )

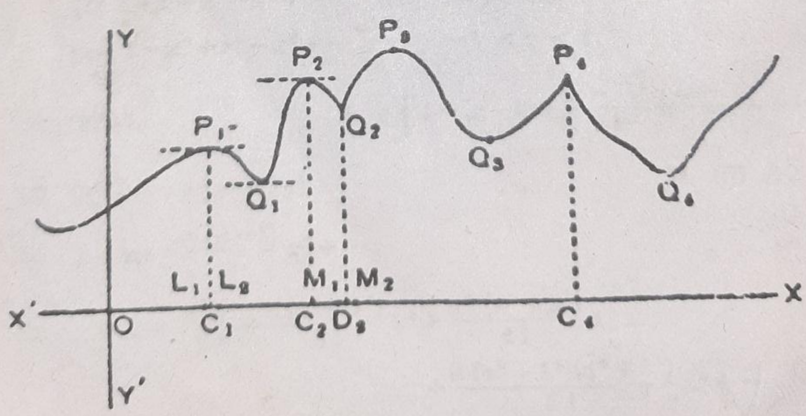
7·1. By the maximum value of a function $f(x)$ in Calculus we do not necessarily mean the absolutely greatest value attainable by the function. A function $f(x)$ is said to be maximum for a value $c$ of $x$, provided $f(c)$ is greater than every other value assumed by $f(x)$ in the immediate neighbourhood of $x=c$. Similarly, a minimum value of $f(x)$ is defined to be the value which is less than other values in the immediate neighbourhood. A formal definition is as follows :

*A function $f(x)$ is said to have a maximum value for $x=c$, provided we can get a positive quantity $\delta$ such that for all values of $x$ in the interval $c-\delta < x < c+\delta$, $(x \neq c)$ $f(c) > f(x)$ ;*

*i.e.*, if $f(c+h)-f(c) < 0$, for $|h|$ sufficiently small.

Similarly, *the function $f(x)$ has a minimum value for $x=d$, provided we can get an interval $d-\delta' < x < d+\delta'$ within which $f(d) < f(x)$ $(x \neq d)$ ;*

*i.e.*, if $f(d+h)-f(d) > 0$, for $|h|$ sufficiently small.




Thus, in the above figure which represents graphically the function $f(x)$ (a continuous function here), the function has a maximum value at $P_1$, as also at $P_2$, $P_3$, $P_4$, etc. and has minimum values at $Q_1$, $Q_2$, $Q_3$, $Q_4$, etc. At $P_1$, for instance, corresponding to $x = OC_1$ $(= c_1$ say), the value of the function, namely, the ordinate $P_1C_1$ is not necessarily bigger than the value $Q_2D_2$ at $x = OD_2$, but we can get a range, say $L_1C_1L_2$ in the neighbourhood of $C_1$ on either side of it, (*i.e.*, we can find a $\delta = L_1C_1 = C_1L_2$ say) such that for every value of $x$ within $L_1C_1L_2$ (except at $C_1$), the value of the function (represented by the corresponding ordinate) is less than $P_1C_1$ (the value at $C_1$). Hence, by definition, the function is maximum at $x = OC_1$. Similarly, we can find out an interval $M_1D_2M_2$ $(M_1D_2 = D_2M_2 = \delta'$ say) in the neighbourhood of $D_2$ within which for every other value of $x$ the function is greater than that at $D_2$. Hence, the function at $D_2$ (represented by $Q_2D_2$) is a minimum.

From the figure the following features regarding maxima and minima of a continuous function will be apparent :

(i) that the function may have several maxima and minima in an interval ; (ii) that a maximum value of the function at some point may be less than a minimum value of it at another point $(C_1P_1 < D_2Q_2)$ ; (iii) maximum and minimum values of the function occur alternately, *i.e.*, between any two consecutive maximum values there is a minimum value, and *vice versa*.

**7·2. A necessary condition for maximum and minimum.**

*If $f(x)$ be a maximum, or a minimum at $x = c$, and if $f'(c)$ exists, then* $f'(c) = 0$.

By definition, $f(x)$ is a maximum at $x = c$, provided we can find a positive number $\delta$, such that

$$f(c+h) - f(c) < 0 \text{ whenever } -\delta < h < \delta,\ (h \neq 0).$$

$$\therefore \quad \frac{f(c+h)-f(c)}{h} < 0 \text{ if } h \text{ be positive and sufficiently}$$

small, and $> 0$ if $h$ be negative and numerically sufficiently small.

Thus, $\underset{h\to 0+}{Lt}\ \dfrac{f(c+h)-f(c)}{h} < 0$, [ See *Ex.* 5, § 2·11 ]

and similarly, $\underset{h\to 0-}{Lt}\ \dfrac{f(c+h)-f(c)}{h} > 0$.

Now, if $f'(c)$ exists, the above two limits, which represent the right-hand and left-hand derivatives respectively of $f(x)$ at $x=c$, must be equal. Hence, the only common value of the limit is zero. Thus, $f'(c)=0$.

Exactly similar is the proof when $f(c)$ is a minimum.

Note. *In case* $f'(c)$ *does not exist*, $f(c)$ *may be a maximum or a minimum*, as is apparent from the figure for points $Q_2$ and $P_4$. At the former point $f(x)$ is a minimum, and at the latter it is a maximum. $f'(x)$ is however not zero at these points, for $f'(x)$ does not exist at all at these points.

## 7·3. Determination of Maxima and Minima.

(A) *If* c *be a point in the interval in which the function* $f(x)$ *is defined, and if* $f'(c)=0$, *and* $f''(c)\neq 0$, *then* $f(c)$ is (i) **a maximum** *if* **$f''(c)$ is negative** *and* (ii) **a minimum** *if* **$f''(c)$ is positive.**

*Proof*: Suppose $f'(c)=0$, and $f''(c)$ exists, and $\neq 0$.

By the Mean Value Theorem*,

$$f(c+h)-f(c)=hf'(c+\theta h),\ 0<\theta<1,$$

$$=\theta h^2 \cdot \frac{f'(c+\theta h)-f'(c)}{\theta h}.$$

Since $0<\theta<1$, $\theta h\to 0$ as $h\to 0$, and writing $\theta h=k$, the coefficient of $\theta h^2$ on the right side $\to \underset{k\to 0}{Lt}\ \dfrac{f'(c+k)-f'(c)}{k}=f''(c)$. Accordingly, since $\theta h^2$ is positive, $f(c+h)-f(c)$ has the same sign as that of $f''(c)$ when $|h|$ is sufficiently small.

* Since, $f''(c)$ exists, $f'(x)$ also exists in the neighbourhood of c.

$\therefore$ if $f''(c)$ is positive, $f(c+h)-f(c)$ is positive, whether $h$ is positive or negative, provided $|h|$ is small. Hence $f(c)$ is a *minimum*, by definition.

Similarly, if $f''(c)$ is negative, $f(c+h)-f(c)$ is negative, whether $h$ is positive or negative, when $|h|$ is small, and so $f(c)$ is a *maximum*.

(B) *Let c be an interior point of the interval of definition of the function* $f(x)$, *and let*

$$f'(c)=f''(c)=\cdots=f^{n-1}(c)=0, \text{ and } f^n(c)\neq 0\ ;$$

*then* (i) *if* **n be even, f(c)** *is a* **maximum** *or a* **minimum** *according as* **$f^n(c)$** *is* **negative** *or* **positive,**
*and* (ii) *if* **n be odd, f(c)** *is* **neither a minimum, nor a maximum.**

*Proof*: By the Mean Value Theorem of Higher order, here

$$f(c+h)-f(c)=\frac{h^{n-1}}{(n-1)!}f^{n-1}(c+\theta h),\ 0<\theta<1$$

$$=\frac{\theta h^n}{(n-1)!}\cdot\frac{f^{n-1}(c+\theta h)-f^{n-1}(c)}{\theta h}.$$

Since $0<\theta<1$, as $h\to 0$, $\theta h\to 0$ and the coefficient of $\theta h^n/(n-1)!$, on the right side $\to f^n(c)$.

Now, suppose $n$ is *even*; then, $\theta h^n/(n-1)!$ is positive.

$\therefore$ $f(c+h)-f(c)$ has the same sign as of $f^n(c)$, whether $h$ is positive or negative, provided $|h|$ is sufficiently small. Hence, *if* $f^n(c)$ *be positive*, $f(c+h)-f(c)$ is positive for either sign of $h$, when $|h|$ is small, and so $f(c)$ *is a minimum*. Similarly, *if* $f^n(c)$ *is negative*, $f(c)$ is *a maximum*.

Next suppose $n$ *is odd*; then $\theta h^n/(n-1)!$ is positive or negative according as $h$ is positive or negative. Hence, $f(c+h)-f(c)$ changes in sign with the change of $h$ whatever the sign of $f^n(c)$ may be, and so $f(c)$ cannot be either a maximum or a minimum at $x=c$.

Hence to determine maxima and minima of $f(x)$, we proceed with the following **working rule :**

## 7·5. Illustrative Examples.

**Ex. 1.** *Find for what values of $x$, the following expression is maximum and minimum respectively :*

$$2x^3 - 21x^2 + 36x - 20.$$

*Find also the maximum and minimum values of the expression.*

[ *C. P. 1936* ]

Let $f(x) = 2x^3 - 21x^2 + 36x - 20.$

$\therefore$ $f'(x) = 6x^2 - 42x + 36$, which exists for all values of $x$.

Now, when $f(x)$ is a maximum or a minimum, $f'(x) = 0$.

$\therefore$ we should have $6x^2 - 42x + 36 = 0$, *i.e.*, $x^2 - 7x + 6 = 0$,

or, $(x-1)(x-6) = 0$; $\therefore$ $x = 1$ or $6$.

Again, $f''(x) = 12x - 42 = 6(2x - 7)$.

Now, when $x = 1$, $f''(x) = -30$ which is *negative*,

when $x = 6$, $f''(x) = 30$, which is *positive*.

Hence, the given expression is *maximum* for $x = 1$, and *minimum* for $x = 6$.

The maximum and minimum values of the given expression are respectively $f(1)$, *i.e.*, $-3$, and $f(6)$, *i.e.*, $-128$.

**Ex. 2.** *Investigate for what values of $x$,*

$$f(x) = 5x^6 - 18x^5 + 15x^4 - 10$$

*is a maximum or minimum.*

Here, $f'(x) = 30(x^5 - 3x^4 + 2x^3)$.

Putting $f'(x) = 0$, we have $x^3(x^2 - 3x + 2) = 0$,

*i.e.*, $x^3(x-1)(x-2) = 0$, whence, $x = 0$, $1$ or $2$.

Again, $f''(x) = 30(5x^4 - 12x^3 + 6x^2)$.

When $x = 1$, $f''(x)$ is *negative*, and hence $f(x)$ is a *maximum* for $x = 1$.

When $x = 2$, $f''(x)$ is *positive*, and hence $f(x)$ is a *minimum* for $x = 2$.

When $x = 0$, $f''(x) = 0$; so the test fails, and we have to examine higher order derivatives.

$f'''(x) = 120(5x^3 - 9x^2 + 3x)$. $\quad\therefore$ $f'''(0) = 0$.

$f^{iv}(x) = 360(5x^2 - 6x + 1)$. $\quad\therefore$ $f^{iv}(0)$ is *positive*.

Since *even order* derivative is positive for $x=0$,

$\therefore$ for $x=0$, $f(x)$ is a *minimum*.

**Ex. 3.** *Show that* $f(x)=x^3-6x^2+24x+4$ *has neither a maximum nor a minimum.*

Here, $f'(x)=3(x^2-4x+8)=3\{(x-2)^2+4\}$

which is always positive and can never be zero.

$\therefore$ $f(x)$ has neither a maximum nor a minimum.

**Ex. 4.** *Examine* $f(x)=x^3-9x^2+24x-12$ *for maximum or minimum values.*

Here, $f'(x)=3(x^2-6x+8)=3(x-2)(x-4)$.

Putting $f'(x)=0$, we find $x=2$ or $4$.

Now, $f'(2-h)=3(-h)(-2-h)=+$,

and $f'(2+h)=3(h)(h-2)=-$, since, $h$ is positive and small.

$\therefore$ by § 7·3, Note 1, for $x=2$, $f(x)$ has a *maximum* value, and this is $f(2)=8$.

Again, $f'(4-h)=3.(2-h)(-h)=-$, since $h$ is positive and small,

$f'(4+h)=3.(2+h)(h)=+$.

$\therefore$ by § 7·3, Note 1, for $x=4$, $f(x)$ has a *minimum* value, and this is $f(4)=4$.

**Note.** In this case we could have easily applied rule of Art. 7·3.

**Ex. 5.** *Find the maxima and minima of*

$$1+2\sin x+3\cos^2 x. \qquad (0\leqslant x\leqslant \tfrac{1}{2}\pi).$$

Let $f(x)=1+2\sin x+3\cos^2 x$.

Then $f'(x)=2\cos x-6\cos x\sin x$.

$\therefore$ $f'(x)=0$ when $2\cos x(1-3\sin x)=0$, *i.e.*, when $\cos x=0$, and also when $\sin x=\frac{1}{3}$.

$$f''(x)=-2\sin x-6(\cos^2 x-\sin^2 x).$$

When $\cos x=0$, $x=\frac{1}{2}\pi$. $\therefore$ $\sin x=1$. $\therefore$ $f''(x)=-2+6=4$ (*positive*).

$\therefore$ for $\cos x=0$, $f(x)$ is a *minimum*, and the minimum value is 3.

When $\sin x=\frac{1}{3}$,

$$f''(x)=-2\sin x-6(1-2\sin^2 x)=-\tfrac{2}{3}-6(1-\tfrac{2}{9})\ (\textit{negative}).$$

Therefore, for $\sin x = \frac{1}{3}$, $f(x)$ is a *maximum* and the maximum value is $1+2.\frac{1}{3}+3.(1-\frac{1}{9})=4\frac{1}{3}$.

**Ex. 6.** *Examine whether $x^{\frac{1}{x}}$ possesses a maximum or a minimum, and determine the same.* [ C. P. 1941, '45 ]

Let $\quad y = x^{\frac{1}{x}}$. $\quad \therefore \log y = \frac{1}{x} \log x$.

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2} - \frac{1}{x^2}\log x = \frac{1}{x^2}(1-\log x). \quad \cdots \quad (1)$$

$\therefore$ when $\frac{dy}{dx}=0$, $1-\log x = 0$. $\therefore \log x = 1 = \log e$. $\therefore x = e$.

Again, differentiating (1) with respect to $x$,

$$-\frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 + \frac{1}{y}\frac{d^2y}{dx^2} = \frac{x^2.(-1/x)-(1-\log x)\,2x}{x^4} = \frac{-3+2\log x}{x^3},$$

$\therefore$ when $x=e$, $\frac{d^2y}{dx^2} = e^{\frac{1}{e}}.\frac{-3+2}{e^3} = -\frac{e^{\frac{1}{e}}}{e^3}$, which is *negative*.

$$\left(\because \text{ for } x=e, \frac{dy}{dx}=0.\right)$$

$\therefore$ for $x=e$, the function is a *maximum*, and the maximum value is $e^{\frac{1}{e}}$.

**Ex. 7.** *Find the maximum and minimum values of $u$ where*

$$u = \frac{4}{x} + \frac{36}{y} \textit{ and } x+y=2.$$

Eliminating $y$ between the two given relations

$$u = \frac{4}{x} + \frac{36}{2-x}. \quad \therefore \frac{du}{dx} = -\frac{4}{x^2} + \frac{36}{(2-x)^2} = \frac{16\,(2x^2+x-1)}{x^2\,(2-x)^2}.$$

$\frac{du}{dx}=0$, gives $x=\frac{1}{2}$ or $-1$. Also, $\frac{d^2u}{dx^2} = \frac{8}{x^3} + \frac{72}{(2-x)^3}$.

When $x=\frac{1}{2}$, $\frac{d^2u}{dx^2} = \frac{8}{(\frac{1}{2})^3} + \frac{72}{(\frac{3}{2})^3}$ which is *positive*.

$\therefore$ for $x=\frac{1}{2}$, $u$ is a *minimum*.

$\therefore$ *minimum value* of $u = \frac{4}{\frac{1}{2}} + \frac{36}{2-\frac{1}{2}} = 32$.

When $x=-1$, $\frac{d^2u}{dx^2} = -8 + \frac{72}{27}$, which is *negative*.

$\therefore$ for $x=-1$, $u$ is a *maximum*.

$\therefore$ *maximum value* of $u = \frac{4}{-1} + \frac{36}{2+1} = 8$.

Define homogeneous function

**Euler's theorem ( FROM BOOK)**

If $f(x, y)$ be a homogeneous function of x and y of degree n, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$

Proof: Since $f(x, y)$ is a homogeneous function of x and y of degree n

Let

$$f(x, y) = x^n \phi(\frac{y}{x})$$

$$= x^n \phi(v)$$

$$\frac{\partial f}{\partial x} = nx^{n-1}\phi(\frac{y}{x}) + x^n \phi'(\frac{y}{x})\frac{-y}{x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{i}$$

$$\frac{\partial f}{\partial y} = x^n \phi'(\frac{y}{x})\frac{1}{x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{ii}$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^n \phi(\frac{y}{x}) = nf(x, y)$$

Page 227

Ex: 1