27. Let $l_1x+m_1y+n_1=0$ and $l_2x+m_2y+n_2=0$ be the lines represented by the equation $ax^2+2hmy+by^2+2gx+2fy+c=0$.

Then $(l_1x+m_1y+n_1)(l_2x+m_2y+n_2)\equiv ax^2+2hxy+by^2+2gx+2fy+c$.

Comparing like terms from both sides,

$$\left.\begin{array}{l} l_1l_2=a,\quad m_1m_2=b,\quad n_1n_2=c,\quad l_1m_2+l_2m_1=2h,\\ l_1n_2+l_2n_1=2g,\quad m_1n_2+m_2n_1=2f,\end{array}\right\}\cdots(1)$$

The two lines will be equidistant from the origin, if

$$\frac{n_1}{\sqrt{l_1^2+m_1^2}}=\pm\frac{n_2}{\sqrt{l_2^2+m_2^2}}$$

or, $n_1^2(l_2^2+m_2^2)=n_2^2(l_1^2+m_1^2)$

or, $n_1^2l_2^2-n_2^2l_1^2=n_2^2m_1^2-n_1^2m_2^2$

or, $(n_1l_2+n_2l_1)(n_1l_2-n_2l_1)=(n_2m_1+n_1m_2)(n_2m_1-n_1m_2)$

or, $2g(n_1l_2-n_2l_1)=2f(n_2m_1-n_1m_2)$ [from (1)]

or, $g^2(n_1l_2-n_2l_1)^2=f^2(n_2m_1-n_1m_2)^2$

or, $g^2\{(n_1l_1+n_2l_1)^2-4n_1n_2.l_1l_2\}=f^2\{(n_2m_1+n_1m_2)^2-4n_1n_2m_1m_2\}$

or, $g^2\{4g^2-2c.2a\}=f^2\{4f^2-2c.2b\}$, [using (1)]

whence $f^4-g^4=c(bf^2-ag^2)$ (Proved).

Examples on Chapter V

1. (i) We know,

$$(x-\alpha)^2+(y-\beta)^2=\gamma^2,$$

here $(\alpha,\beta)\equiv(-1,5)$, $\gamma=3$;

Substituting them in the eqn. and simplifying,
we get $x^2+y^2+2x-10y+17=0$ which is the reqd. equ.

(ii) $2x^2+2y^2-2x+6y-45=0$

or, $x^2+y^2-x+3y=\frac{45}{2}$ [dividing by 2]

or, $(x-\frac{1}{2})^2+(y+\frac{3}{2})^2=\frac{45}{2}+\frac{1}{4}+\frac{9}{4}=5$

$\therefore$ centre is $(\frac{1}{2},-\frac{3}{2})$, and $r=5$.

(iii) Using *A. R. Khalifa's* method, we get

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=\lambda\{(x-x_1)(y-y_2)-(x-x_2)(y-y_1)\}$$

which represents a circle through (x_1,y_1) and (x_2,y_2) where λ is a parameter.

$\therefore$ the circle through (3,1) and (4,−3) is given by

$$(x-3)(x-4)+(y-1)(y+3)=\lambda\{(x-3)(y+3)-(x-4)(y-1)\}\cdots\cdots(1)$$

Then substituting (1,−1) for (x,y) in (1), $\lambda=-\frac{1}{5}$.

Now substituting for λ in (1) and simplifying,

$5x^2+5y^2-31x+11y+32=0$ which is reqd. equ.

(iv)
$x+y-5=0$ ... ... (1)
$13x+8y-35=0$ ... ... (2)
$2x-3y-35=0$ ... ... (3)

Solve (1) and (2), (2) and (3), (3) and (1) to get the vertices $A\equiv(-1,6)$, $B\equiv(7,-7)$, $C\equiv(10,-5)$ respectively.
Now proceed as in (iii) to get $5x^2+5y^2-43x-3y-210=0$.

2. The equ. of the given circle can be written as

$$(x-3)^2+(y+2)^2=5^2=a^2(\text{say}).$$

Transferring the origin to $(3,-2)$, that is, writing

$$X=x-3, \qquad x=X+3,$$
$$Y=y+2, \quad \text{or,} \quad y=Y-2.$$

The equ. of the circle becomes

$$X^2+Y^2=a^2=5^2.$$

Then the equ. of the line becomes

$$4(X+3)+\lambda(Y-2)+7=0$$

or, $$Y=-\frac{4}{\lambda}X-\frac{19-2\lambda}{\lambda}$$

and so we get, $m=-\dfrac{4}{\lambda}$, $C=-\dfrac{19-2\lambda}{\lambda}$.

The condition of tangency is

$$C^2=a^2(1+m^2)$$

or, $$\left(\frac{19-2\lambda}{\lambda}\right)^2=25\left(1+\frac{16}{\lambda^2}\right).$$

Now, simplifying and then factorising, we get

$$\lambda=-\frac{13}{21} \text{ or } -3.$$

3. The equ. of the given circle may be written as

$$(x-a)^2+y^2=a^2.$$

Writing $X=x-a$, $\qquad x=X+a$,

$Y=y \quad$ or, $\quad y=Y$.

The equ. of the circle becomes

$$X^2+Y^2=a^2=(\text{radius})^2.$$

Then the equ. of the line becomes

$$Y=mX+a\sqrt{1+m^2}$$

Here slope $=m$, $C=a\sqrt{1+m^2}$, or $C^2=a^2(1+m^2)$;

Condition of tangency is $C^2=(\text{radius})^2(1+m^2)$.

which obviously follows for all m.

$\therefore$ the given line touches the given circle for all m.

4. (i) $S\equiv x^2+y^2+2gx+2fy+d=0$

$L\equiv Ax+By-1=0.$

The equ. of the circle through the intersections of the given circle and the given line is

$$S+\lambda L=0$$

or, $$x^2+y^2+2gx+2fy+d+\lambda(Ax+By-1)=0 \quad \cdots(\text{i})$$

Now, the circle (i) passes through $(0,0)$.

$\therefore$ by substitution, we get $d=\lambda$.

Substituting this value of λ in (i)

$$x^2+y^2+2gx+2fy+dAx+dBy=0$$

or, $$x^2+y^2+(2g+Ad)x+(2f+Bd)y=0$$

(ii) $L\equiv Ax+By-1=0,$

$S\equiv x^2+y^2+2gx+2fy+d=0$

The circle through their intersections is $S+\lambda L=0$

or, $$x^2+y^2+(2g+\lambda A)x+(2f+\lambda B)y+(d-\lambda)=0 \ldots(P),$$

which has the centre $\equiv\left(-\frac{2g+\lambda A}{2}, -\frac{2f+\lambda B}{2}\right)$.

The co-ordinates of centre must satisfy the given line and by substituting and simplifying, we get

$\lambda=\frac{2(c-ag-bf)}{aA+bB}$. Substituting this value in (P),

we get the reqd. equ.

(iii) The circle (P) shown in (ii) will touch y-axis that is, the line $x=0$. Substituting this in (P), $y^2+(2f+\lambda B)y+(d-\lambda)=0$. This quadratic in y will have two coincident values at the pt. of contact. For this "b^2-4ac"$=0$

or, $(2f+\lambda B)^2-4(d-\lambda)=0$

or, $B^2\lambda^2+4(fB+1)\lambda+4(f^2-d)=0$ which will give two values of λ. $\therefore$ (P) will give two reqd. circles corresponding two values of λ.

5. The equ. of any circle passing through the intersections of the given circles is $S+\lambda S'=0$.

or, $x^2+y^2+2x+4y+1+\lambda(x^2+y^2-1)=0$... ...(i)

or, $\left(x+\frac{1}{1+\lambda}\right)^2+\left(y+\frac{2}{1+\lambda}\right)^2=\frac{5}{(1+\lambda)^2}-\frac{1-\lambda^2}{(1+\lambda)^2}$.

Writing $X=x+\frac{1}{1+\lambda}$, $Y=y+\frac{2}{1+\lambda}$,

or, $x=X-\frac{1}{1+\lambda}$, $y=Y-\frac{2}{1+\lambda}$.

The equ. of circle becomes

$$X^2+Y^2=\frac{4+\lambda^2}{(1+\lambda)^2}=a^2 \text{ (say)}.$$

Then the given line $x+2y+5=0$ reduces to

$$Y=-\tfrac{1}{2}X+\frac{-5\lambda}{2(1+\lambda)} \quad \left[\begin{array}{l}\text{by substituting}\\ \text{values of } x, y\end{array}\right].$$

The circle (i) will touch the given line

if $c^2=a^2(1+m^2)$

or, $\frac{5(5)\lambda^2}{4(1+\lambda)^2}=\frac{4+\lambda^2}{(1+\lambda)^2}\{1+(\tfrac{1}{2})^2\}$

or, $5\lambda^2=4+\lambda^2$. $\therefore$ $\lambda=\pm1$. The value $\lambda=-1$ is rejected, since this makes, equ. (i) linear.

$\therefore$ substituting $\lambda=1$ in (i), we get the reqd. result.

6. Worked out in the text book.

7. $S=0$ and $S'=0$ are the given circles.

The diameter of the 2nd circle is the common chord of the two circles. This chord has the equ. $S-S'=0$, that is, $2(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2=0$... ...$(P)$.

The equ. (P) must be satisfied by the centre of 2nd circle, that is, by $(-g_2, -f_2)$. Hence we get

$2(g_1-g_2)(-g_2)+2(f_1-f_2)(-f_2)+c_1-c_2=0$

or, $2(g_2^2+f_2^2-g_1g_2-f_1f_2)+c_1-c_2=0$ which is reqd. condition.

8. Let (x', y') be the pt., and circles are

$x^2+y^2+2gx+2fy+c=0$,

$x^2+y^2+2g'x+2f'y+c'=0$.

Then equs. of polar are

$xx'+yy'+g(x+x')+f(y+y')+c=0$ ... ...(1)

$xx'+yy'+g'(x+x')+f'(y+y')+c'=0$... ...(2)

(1) and (2) reduce to

$(x'+g)x+(y'+f)y+k=0$ (say),

$(x'+g')x+(y'+f')y+k'=0$ (say) respectively.

As the polars are perp. to each other, we have

$(x'+g)(x'+g')+(y'+f)(y'+f')=0$ [$\because a_1a_2+b_1b_2=0$]

Multiplying, we get,

$x'^2+y'^2+(g+g')x'+(f+f')y'+gg'+ff'=0$

which represents a circle on the join of the centres as diameter.

9. Worked out in the text book.

10. $3x-11y-13=0 \ldots \ldots (1)$

$8x+y-2=0 \ldots \ldots (2)$

$3x+2y+1=0 \ldots \ldots (3)$

Poles of lines (1), (2) and (3) are

(x_1,y_1), (x_2,y_2), (x_3,y_3) respectively (say).

The polars of (1), (2) and (3) with respect to the given circle $x^2+y^2-4x+3=0$, are

$(x_1-2)x+y_1y+(3-2x_1)=0 \ldots \ldots (1')$

$(x_2-2)x+y_2y+(3-2x_2)=0 \ldots \ldots (2')$

and $(x_3-2)x+y_3y+(3-2x_3)=0 \ldots \ldots (3')$ respectively.

Then (1) is identical with (1′), (2) with (2′) and (3) with (3′).

Now comparing coeffs, from (1) and (1′)

$$\frac{x_1-2}{3}=\frac{y_1}{-11}=\frac{3-2x_1}{-13} \text{ which gives } x_1=\frac{17}{7}, y_1=\frac{-11}{7}.$$

Similarly from (2) and (2′), (3) and (3′), we get

$$(x_2,y_2)\equiv\left(\frac{10}{7}, -\frac{1}{14}\right)$$

$$(x_3,y_3)\equiv\left(\frac{11}{7}, -\frac{2}{7}\right) \text{ respectively.}$$

Now these three poles will be collinear;

$$\text{if } D\equiv\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0.$$

Substituting the values and expanding the determinant, we get $D=0$.

$\therefore$ they are collinear.

11. The equ. of any circle touching both the axes is of the form $(x-g)^2+(y-g)^2=g^2$, where (g, g) is the centre of the circle,

or, $x^2+y^2-g(2x+2y)+g^2=0 \ldots \ldots (1)$

Let (x_1, y_1) be the pole of the fixed line

$\frac{x}{a}+\frac{y}{b}=1$, that is, on $bx+ay-ab=0 \ldots \ldots (2)$

with respect to the circle (1).

Now the polar of the point (x_1, y_1) with respect to the circle (1) is $xx_1+yy_1-g(x+y+x_1+y_1)+g^2=0$

or, $(x_1-g)x+(y_1-g)y-g(x_1+y_1-g)=0$ ... ...(3).

Now, the eqns. (2) and (3) represent the same st. line comparing coeffs. of x and y from (2) and (3),

$$\frac{b}{x_1-g}=\frac{a}{y_1-g}, \text{ that is, } g=\frac{ax_1-by_1}{a-b}.$$

Now, substituting the value of g in (3), and simplifying,

$(ax_1-by_1)(bx_1-ay_1)-ab(a-b)(x_1-y_1)=0.$

Dropping the suffixes in x and y, we get the reqd. locus.

For the 2nd part, simplify the equ. to reduce to the form $ax^2+by^2+2gx+2fy+2hxy+c=0$. Get the determinant which is of the form

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0.$$

Now expand the determinant remembering that both a and b are non-zero (since in that case the equ. does not remain quadratic).

You will get $a-b=0$ which is reqd. condition.

12. Worked out in the text book.

13. $S\equiv x^2+{}^2+ax+by+c=0$......(1)

$S'\equiv x^2+y^2+bx+ay+c=0.$

Then $S-S'=0$, that is, $(a-b)x-(a-b)y=0$

or, $x=y$...(3) is the equ. of radical axis.

This is also the equ. of common chord.

Circle (1) and line (3) have two common points.

$\therefore$ from (3) and (1),

$2x^2+(a+b)x+c=0$

This being a quadratic in x, we have

$$x_1+x_2=\frac{-(a+b)}{2},\ x_1x_2=\frac{c}{2}.$$

Similarly, (1) and (3) give $y_1+y_2=\dfrac{-(a+b)}{2},\ y_1y_2=\dfrac{c}{2}.$

$\therefore$ the reqd. length $=\{(x_1-x_2)^2+(y_1-y_2)^2\}^{\frac{1}{2}}.$

Substitute in terms of the values of x_1+x_2 etc. and get the length.

14. $S\equiv x^2+y^2+2x+3y+1=0,$

$S'\equiv x^2+y^2+4x+3y+2=0.$

Then $S+\lambda S'=0$ is the equ. of the circle through the intersections of the given circles.

Now, this is given by

$(1+\lambda)x^2+(1+\lambda)y^2+(2+4\lambda)x+(3+3\lambda)y+(1+2\lambda)=0$...(1)

or, $\left(x+\dfrac{1+2\lambda}{1+\lambda}\right)^2+\left(y+\dfrac{3}{2}\right)^2=R$ (say)... ... (2)

[dividing by $1+\lambda$ and etc.]

Equ. of common chord is given by

$S-S'=0$, that is, $2x+1=0$ ... ... (3)

The equ. (3) will be satisfied by the co-ordinates of the centre of (2); that is, we have

$$2.\frac{1+2\lambda}{1+\lambda}+1, \quad \text{or, } \lambda=-\tfrac{1}{3}.$$

Substituting this in (1), we have $2x^2+2y^2+2x+6y+1$,

15. The equ. $S+\lambda S'=0$, ($S=0$, $S'=0$ are given)

or, $x^2+y^2+\frac{2}{1+\lambda}x+\frac{2\lambda}{1+\lambda}y+\frac{5(1+\lambda)}{1+\lambda}=0$......(i),

where λ is a parameter, represents a system of co-axial circles which pass through the fixed points in which the given circles intersect.

The co-ordinates of centre of (1) are

$$\left(\frac{1}{1+\lambda}, -\frac{\lambda}{1+\lambda}\right)......(ii)$$

and $r=\frac{1}{(1+\lambda)^2}+\frac{\lambda^2}{(1+\lambda)^2}-5$;

now, for the limiting points $r=0$.

$\therefore$ $1+\lambda^2-5(1+\lambda)^2=0$, where $\lambda=-2$ or $-\frac{1}{2}$.

Substituting the values of λ in (ii), the limiting points are found to be $(1, -2)$ and $(-2, 1)$.

16. For their intersecting at right angles,

$2gg'+2ff'-C-C'=0$

or, $2d.(0)+2.0.d'-k^2+k^2=0$

or, $0+0-k^2+k^2=0$ which is true.

Hence the given circles are intersecting at rt. angles.

17. $S\equiv x^2+y^2-6=0$,

$S'\equiv x^2+y^2+3x+3y=0$.

Let the length of tangent from (f, g) to $S=0$ be t_1 and the length of tangent from (f, g) to $S'=0$ be t_2.

Then $t_1^2=f^2+g^2-6$.

$t_2^2=f^2+g^2+3f+3g$ [by substitution].

Then, by the questions, we have

$t_1=2t_2$

or, $f^2+g^2-6=2^2(f^2+g^2+3f+3g)$

or, $f^2+g^2+4f+4g+2=0$ [by simplifying].

or,

18. Proceed as in Ex. 13 above.

19. We have $S\equiv x^2+y^2-r^2=0$.

The middle pt. of chord AB be$\equiv(x_1, y_1)$. Then the equ. of chord in terms of its middle pt. is given by $xx_1+yy_1-(x_1^2+y_1^2)=0$......(1).

This must be identical with $ax+by-c=0$......(2)

Comparing (1) with (2).

$$\frac{x_1}{a}=\frac{y_1}{b}=\frac{x_1^2+y_1^2}{c}.$$

Now $x_1=\frac{a}{b}y_1$;

and $y_1=\frac{b}{c}(x_1^2+y_1^2)$.

$\therefore$ $y_1=\frac{b}{c}\left(\frac{a^2}{b^2}y_1^2+y_1^2\right)$

$$\text{or,}\quad 1=\frac{b}{c}y_1\left(\frac{a^2+b^2}{b^2}\right).$$

$$\therefore\quad y_1=\frac{bc}{a^2+b^2}.$$

Similarly, we can find x_1 to be $=\frac{ac}{a^2+b^2}$.

Hence the result.

20. Equ. of the circle described on AB as diameter is

$$x^2+y^2-r^2+\lambda(ax+by-c)=0$$

$$\text{or,}\quad x^2+y^2+a\lambda.x+b\lambda.y-(r^2+c\lambda)=0 \quad \cdots \cdots (1)$$

whose centre is at $C\left(-\frac{a\lambda}{2},\ -\frac{b\lambda}{2}\right)$...

Note that the C is also the middle point of AB. Since C is point on the line $ax+by=c$, we have

$$-(a^2+b^2)\frac{\lambda}{2}=c \qquad \text{or,}\ \lambda=-\frac{2c}{a^2+b^2}.$$

$\therefore$ substituting for λ, the co-ordinates of C are

$$\left(\frac{ac}{a^2+b^2},\ \frac{bc}{a^2+b^2}\right)$$

Hence the result.

22. $S\equiv x^2+y^2-r^2 \cdots \cdots (i)$

Tangents are drawn from $P(x_1, y_1)$ to the circle $S=0$. A and B are the points of contact.

$\therefore$ equ. of AB is $xx_1+yy_1-r^2=0\ldots \ldots(2)$

Let the circle circumscribing the triangle PAB be $S'=0$. en AB is the common chord of the circles $S=0$ and $S'=0$, and then the equ. of common chord AB is $S-S'=0\ldots(2')$

From (2) and (2'),

$$S-S'=xx_1+yy_1-r^2\ldots(3)$$

(1) − (3) gives $S'\equiv x^2+y^2-xx_1-yy_1=0$

$\therefore$ $x^2+y^2=xx_1+yy_1$ is the reqd. equ.

23. We have,

$$x^2+y^2-9x+14y-7=0 \cdots \cdots (1)$$

$$x^2+y^2-15x+14=0 \cdots \cdots (2)$$

Let $x^2+y^2+2gx+2fy+c=0 \cdots \cdots (3)$

be the reqd. circle.

Condition of orthogonality of two circles is

$$2gg'+2ff'-c-c'=0\ ;$$

(1) and (3) are orthogonal.

$$\therefore\quad -9g+14f-c+7=0 \cdots (1').$$

Again, (2) & (3) being orthogonal, we have

$$15g-c-14=0 \qquad (2').$$

Circle (3) passes then (2,5),

$$4g+10f+c+29=0 \quad (3').$$

Now, solve equs. (1'), (2') and (3') for the values of g, f and c.

We find $g=0,\ f=-\frac{3}{2},\quad c=-14.$

Substituting them in (3), we get the regd. equ.

24. The equ. of circle through $(b,0)$, $(-b,0)$ is given by $S=AL$ [*Khalifa's* method]

or, $(x-b)(x+b)+y(y)=A\{(x-b)(y)-(x+b)y\}$ ... (1).

Now, (1) passes through $(0, c)$.

$\therefore$ we have,

by substitution $-b^2+c^2=A\{-2bc\}$, or, $A=\dfrac{b^2+c^2}{2bc}$.

Substituting the value of A in (1) and simplifying,

$$x^2+y^2+\frac{b-c^2}{c}\cdot y-b^2=0 \quad \ldots \ldots (1')$$

Also, we have $x^2+y^2-ax+b^2=0$ ... ... (2).

Circles (1′) and (2) will be orthogonal

if $2gg'+2ff'-c-c'=0$

or, if $2.(0).\left(-\dfrac{a}{2}\right)+2.\dfrac{b-c^2}{2c}\cdot(0)+b^2-b^2=0$

or, if $0+0+b^2-b^2=0$ which is true.

Hence the proof.

25. We are given that $S+\lambda S'=0$.

The equ. may be written as

$$x^2+y^2+\frac{2g}{1+\lambda}x+\frac{2f\lambda}{1+\lambda}y+\frac{c+c'\lambda}{1+\lambda}=0 \ldots \ldots (1)$$

which where λ is a varying const., represents a system of co-axial circles passing through the intersections of circles $S=0$ and $S'=0$.

Centre of (1) is $\equiv\left\{-\dfrac{g}{1+\lambda}, -\dfrac{f\lambda}{1+\lambda}\right\}$ ... ...(2)

and $r^2=\dfrac{g^2+f^2\lambda^2}{(1+\lambda)^2}-\dfrac{c+c'\lambda}{1+\lambda}$ ...(3)

For the limiting points $r=0$,

that is, relation (3) reduces to

$$\lambda^2+(f^2-c')+\lambda(-c-c')+(g^2-c)=0 \ldots (4)$$

where λ_1 and λ_2 are two values of λ.

Now (2) gives two points, that is,

$$\left(-\frac{g}{1+\lambda_1}, -\frac{g}{1+\lambda_2}\right) \text{ and } \left(-\frac{f}{1+\lambda_2}, -\frac{f}{1+\lambda_2}\right)$$

and the square of dist. between the points is

$$=\left(\frac{g}{1+\lambda_2}-\frac{g}{1+\lambda_1}\right)^2+\left(\frac{f}{1+\lambda_2}-\frac{f}{1+\lambda_1}\right)^2$$

$$=(g^2+f^2)\left\{\frac{\lambda_1-\lambda_2}{1+\lambda_1+\lambda_2+\lambda_1\lambda_2}\right\}^2=(g^2+f^2)\frac{N^2}{D^2} \text{ (say).}$$

Now, from (4),

$$D=1+(\lambda_1+\lambda_2)+(\lambda_1\lambda_2)=1+\frac{c+c'}{f^2-c'}+\frac{g^2-c}{f^2\times c'}=\frac{f^2+g^2}{f^2-c^2}$$

$$N^2=(\lambda_1-\lambda_2)^2=(\lambda_1+\lambda_2)^2-4\lambda_1\lambda_1$$

$$=\frac{(c-c')^2-4f^2g^2+4f^2c+4g^2c'}{(f^2-c')^2}$$

Substituting the values of N^2 and D^2, we have the reqd. result.

26. $S_1\equiv(x-a)^2+(y-b)^2-b^2=0$ ... ...(1)

$S_2\equiv(x-b)^2+(y-a)-a^2=0$ ... ...(2)

$S_3\equiv(x-a-b-c)^2+y^2-ab-c^2=0$... ...(3)

Then the radical axis of (1) and (2) is

$$S_1 - S_2 = 0 \quad \dots \quad (1')$$

or, $(x-a)^2+(y-b)^2-b^2-\{(x-b)^2+(y-a)^2-a^2\}=0$

Simplifying, $2x-2y=a+b$.

Similarly, the radical axis of (2) and (3) is

$$S_2 - S_3 = \dots \quad \dots (2')$$

that is $2(b+c)x-2by=(b+2c)(a+b)$.

Again radical axis of (3) and (1) is

$$S_3 - S_1 = 0 \quad \dots (3')$$

That is $2(c+a)x-2ay=(a+2c)(a+b)$.

Now $(1')+(2')+(3')$ gives

$$(S_1-S_2)+(S_2-S_3)+S_3-S_1)=0 \text{ identically.}$$

$\therefore$ the eques. $(1')$, $(2')$, $(3')$ hold simulteneously, that is, the radical axes are concurrent.

Let the equation of the circle cutting the given circles orthogonally be

$$(x-\alpha)^2+(y-\beta)^2=\gamma^2.$$

Then

$$(a-a)^2+(\beta-b)^2=\gamma^2+b^2 \dots \quad \dots (i)$$

$$(\alpha-b)^2+(\beta-a)^2=\gamma^2+a^2 \dots \quad \dots (ii)$$

$$\{\alpha-(a+b+c)\}^2+\beta^2=\gamma^2+(ab+c^2) \dots (iii)$$

Solving (i), (ii) and (iii), we get, $\alpha=a+b$,

$$\beta=\frac{a+b}{2} \text{ and } \gamma=\frac{a-b}{2}.$$

Hence the equation of the reqd. circle is

$$\{x-(a+b)\}^2+\left(y-\frac{a+b}{2}\right)^2=\frac{1}{4}(a-b)^2.$$

27. $S \equiv (a^2+b^2)(x^2+y^2)-2c^2=0$

$L \equiv ax+by+c=0$

The eqn. through the intersections of the given line and the given circle is

$$S+\lambda L=0.$$

or, $(a^2+b^2)(x^2+y^2)-2c^2+\lambda(ax+by+c)=0 \quad \dots \dots (1)$

or, $x^2+y^2+\frac{\lambda a}{a^2+b^2} \cdot x+\frac{\lambda b}{a^2+b^2} \cdot y=R$ (say).

The centre of this circle is $\equiv \left\{-\frac{\lambda a}{2(a^2+b^2)}, -\frac{\lambda b}{2(a^2+b^2)}\right\}$

If the portion of given line be the diameter of this circle, then the co-ordinates of centre must satisfy the eqn. of the line. $\therefore$ by substitution, we have

$$\frac{\lambda a^2}{2(a^2+b^2)}+\frac{\lambda b^2}{2(a^2+b^2)}-c=0$$

or, $\lambda=\frac{2c(a^2+b^2)}{a^2+b^2}=2c$

Substituting the value of λ in (1), we get

$(a^2+b^2)(x^2+y^2)+2(ax+by)c=0$ which is the reqd. eqn.

28. Let (x_1, y_1) be the pole of the line $y-k=0$.

Then the polar of (x_1, y_1) with respect to the given circles is $xx_1+yy_1+\lambda(x+x_1)+c=0$

or, $(x_1+\lambda)x+y_1y+\lambda x_1+c=0$ ... ... (1)

Line (1) is same as $y-k=0$ ... ... (2)

$\therefore$ comparing (1) with (2), $\dfrac{0}{x_1+\lambda}=\dfrac{-k}{\lambda x_1+c}$

or, $\lambda=-\dfrac{ky_1+c}{x_1}$

Substituting the value of λ in (1), and simplifying, we get $x_1^2=ky_1+c$.

Dropping the suffixes in x and y, we get the reqd. eqn.

29. $S\equiv x^2+y^2-r^2=0$ ... ... (1).

A and B are the points of contact of tangents drawn from (h,k). The eqn. of chord of contact AB is given by

$$xh+yk-r^2=0.$$

AB subtends a rt. angle at $P\equiv(h',k')$, that is, AB is the diameter of a circle $S'=0$ through P, A and B.

Now, AB is the common chord of circles $S=0$ and $S'=0$ and its eqn. is $S-S'=0$.

$\therefore$ $S-S'=xh+yk-r^2=0$ ... ...(3)

(1)—(3) gives $S'\equiv x^2+y^2-r^2-(xh+yk-r^2)=0$...(4)

or, $S'\equiv\dfrac{x^2+y^2-r^2}{xh+yk-r^2}-1=0$ ... ... (4')

Now, circle (4') passes through (h',k').

$\therefore$ $\dfrac{h'^2+k'^2-r^2}{hh'+kk'-r^2}=1$ ... ... (5)

The centre $\left(\dfrac{h}{2},k/2\right)$ of circle (4) is a pt. on AB, that is, we get by substitution in (3), $h^2+k^2=2r^2$.

or, $\dfrac{2r^2}{h^2+k^2}=1$ ... (5')

From (5) and (5'), the reqd. result follows.

30. $x^2+y^2+\lambda(x-a)=0$ ... (1)

$x^2+y^2+\mu(y-b)=0$ ... (2).

If (x_1,y_1) be their pt. of contact, it will satisfy the circles, therefore,

$x_1^2+y_1^2+\lambda(x_1-a)=0$ which implies $\lambda=-\dfrac{x_1^2+y_1^2}{x_1-a}$;

and $x_1^2+y_1^2+\mu(y_1-b)=0$ which gives $\mu=-\dfrac{x_1^2+y_1^2}{y_1-b}$.

Now, the tangents at (x_1,y_1) to (1) and (2) are

$xx_1+yy_1+\frac{1}{2}\lambda(x+x_1)-\lambda a=0$,

and $xx_1+yy_1+\frac{1}{2}\mu(y+y_1)-\mu b=0$ respectively

i.e., $x\left(x_1+\dfrac{\lambda}{2}\right)+yy_1+\dfrac{\lambda}{2}(x_1-2a)=0$ ... (3)

and $xx_1+y\left(y_1+\dfrac{\mu}{2}\right)+\dfrac{\mu}{2}(y_1-2b)=0$ ... (4).

Since these are tangents at the same pt., (3) and (4) are identical. Then comparing coeffs., we get

$$\frac{x_1+\frac{\lambda}{2}}{x_1}=\frac{y_1}{y_1+\frac{\mu}{2}}=\frac{\frac{\lambda}{2}(x_1-2a)}{\frac{\mu}{2}(y_1-2b)}$$

$$\text{or,}\quad \frac{x_1 - \dfrac{x_1^2+y_1^2}{2(x_1-a)}}{x_1} = \frac{y_1}{y_1 - \dfrac{x_1^2+y_1^2}{2(y_1-b)}} \quad [\text{substituting for } \lambda, \mu]$$

Now, simplifying and rearranging the terms, we get

$$(x_1-a)^2+(y_1-b)^2=a^2+b^2.$$

Hence the locus of (x_1, y_1) is

$(x-a)^2+(y-b)^2=a^2+b^2$ which is a circle.

31. The eqn. of circle through $(0, a)$ and $(0, -a)$ is given by $S=AL$

or, $x.x+(y-a)(y+a)=A\{x(y+a)-x(y-a)\}$

or, $x^2+y^2-2aAx-a^2=0$ ... ... (1)

or, $(x-aA)^2+y^2=a^2+a^2A^2=(a^2)(1+A^2)$

Now, writing $X=x-aA$

$Y=y$, or $x=X+aA$.

The eqn. of the circle reduces to $X^2+X^2=a^2(1+A^2)$

then the eqn. of the line $y=mx+c$ reduces to

$Y=m(X+aA)+c=mY+(aAm+c)$.

Here slope$=m$, const,$=aAm+c$.

Now, by the condition of tangency,

$(aAm+c)^2=a^2(1+A^2)(1+m^2)$.

Simplifying, $a^2A^2-2amcA+(a^2+a^2m^2-c^2)=0$

where A has two values A_1 and A_2.

The eqn. (1) becomes

$$x^2+y^2-2aA_1x-a^2=0,$$
$$x^2+y^2-2aA_2x-a^2=0,$$

and these two circles will be orthogonal.

if $+2aA_1 . aA_2+a^2+a^2=0$

or, if $A_1A_2+1=0$

or, if $a^2+a^2m^2-c^2+a^2=0$ $\left[\because A_1A_2=\frac{a^2+a^2m^2-c^2}{a^2}\right]$

or, if $c^2=a^2(2+m^2)$.

Examples on Chapter VI

1. (i) $x^2-6xy+9y^2+4x+8y+15=0$.

Here $a=1$, $b=9$, $h=-3$; $\therefore$ $ab-h^2=9-9=0$.

Also show $\Delta\neq0$.

Hence the eqn. represents a parabola.

The given eqn. can be written as

$$(x-3y)^2+4x+8y+15=0$$

or, $(x-3y+k)^2=2k(x-3y)-4x-8y+k^2-15$

$=2(k-2)x-2(3k+4)y+k^2-15$ ... (1)

where k is a constant.

Now, k is taken such that the lines

$x-3y+k=0$ and $2(k-2)x-2(3k+4)y+k^2-15=0$ may be perp. to each other. This gives

1. $2(k-2)+3(2)(3k+4)=0$, i.e., $k=-1$.

... in (i), we get

$$(x-3y-1)^2=-6x-2y-14 \qquad \ldots \ldots \ldots (2)$$

Taking $x-3y-1=0$ as the axis of X

and $-6x-2y-14=0$ as the axis of Y, we get

$$Y=\frac{x-3y-1}{\sqrt{1^2+3^2}}=\frac{x-3y-1}{\sqrt{10}} \qquad \ldots \ldots \ldots (3)$$

and $$X=\frac{-6x-2y-14}{\sqrt{6^2+2^2}}=\frac{-6x-2y-14}{2\sqrt{10}} \qquad \ldots \ldots (4)$$

Then using (3) and (4) in (2), we have

$$\left(Y\sqrt{10}\right)^2=2\sqrt{10}.X, \text{ or } Y^2=\frac{\sqrt{10}}{5}X,$$

which is the standard form of eqn. of parabola

$$\left[\text{Here } 4A=\frac{\sqrt{10}}{5}\right]$$

(ii) $3x^2+2xy+3y^2+2x-6y+12\frac{1}{2}=0.$

Here $a=3, b=3, h=1$; $\therefore ab-h^2=9-1=8>0.$

Again,
$$\Delta\equiv\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=\begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & -3 \\ 1 & -3 & \frac{25}{2} \end{vmatrix}=64\neq 0.$$

Hence the eqn. represents an ellipse.

Then $$\frac{\Delta}{C}=\frac{64}{ab-h^2}=8 \qquad \ldots \ldots (1)$$

Let us then transfer the origin to the centre of the conic and then rotate the axes in order to remove the xy term. Let the finally reduced eqn. be

$$a'x^2+b'y^2+\frac{\Delta}{C}=0 \qquad \ldots \ldots (2)$$

By the invariants, $a'+b'=a+b=3+3=6,$

and $a'b'=ab-h^2=3\times 3-1=8.$

Solving these two, we get $a'=4, b'=2.$

Hence substituting for a', b' and $\frac{\Delta}{C}$ in (2), we have

$4x^2+2y^2+8=0$, or $\frac{x^2}{2}+\frac{y^2}{4}=-1$ which is the reqd. standard form of eqn. Length of semi-axes are $\sqrt{2}$, 2.

(iii) $4x^2-24xy-6y^2+4x-12y+1=0.$

Here $a=4$, $b=-6$, $c=1$, $g=2$, $f=-6$, $h=-12$,

and $$\Delta=\begin{vmatrix} 4 & -12 & 2 \\ -12 & -6 & -6 \\ 2 & -6 & 1 \end{vmatrix}=0.$$

Hence the eqn. represents a pair of st. lines. Let the finally reduced eqn. be

$$a'x^2+b'y^2+\frac{\Delta}{C}=0, \text{ i.e., } a'x^2+b'y^2=0 \ldots (i) \quad \left[\because \frac{\Delta}{C}=0\right].$$

By the invariants,

$a'+b'=a+b=4-6=-2,$

$a'b'=ab-h^2=-24-144=-168$

Solving these two, $a'=-14, b'=12.$

Hence substituting for a', b' in (i), we get

$-14x^2+12y^2=0$, i.e., $7x^2-6y^2=0$, which is the pair of st. lines.

(iv) Proceed as in (ii).

(v) $x^2-4xy-2y^2+10x+4y=0$.

Here $a=1$, $b=-2$, $h=-2$; $\therefore$ $ab-h^2=-2-4$

$=-6<0$.

Also, see $\Delta\neq 0$.

$\therefore$ the eqn. represents a hyperbola.

Then proceed as in (ii).

For (vi), proceed as in (ii) ; and for (vii) as in (i).

2. (i) $x^2-4xy+y^2+8x+2y-5=0$.

Here $a=1, b=1, h=-2, g=4, f=1, c=-5$.

Let (x_1, y_1) be the centre of the conic.

Then this pt. is given by the two eqns.

$ax_1+hy_1+g=0$ and $hx_1+by_1+f=0$

i.e., $x_1-2y_1+4=0$ ...(1), and $2x_1-y_1-1=0$ ... ...(2).

Solving (1) and (2), $(x_1, y_1)\equiv(2,3)$ which is the reqd. centre.

Otherwise : The centre of the conic is given by $\left(\dfrac{G}{C}, \dfrac{F}{C}\right)$

where G, F, C are the co-factors of g, f, c in the det.

$$\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=\begin{vmatrix} 1 & -2 & 4 \\ -2 & 1 & 1 \\ 4 & 1 & -5 \end{vmatrix}$$

$$\therefore \text{Then } \frac{G}{C}=\frac{\begin{vmatrix} -2 & 1 \\ 4 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix}}=\frac{-6}{-3}=2$$

Similarly $\dfrac{F}{C}=3$; hence (2,3) is the centre.

For (ii), (iii), (iv), proceed as in 2 (i).

3. $3x^2-2xy-y^2+2x+y-1=0$ ... ...(1)

Here $a=3$, $h=-1$, $g=1$, $b=-1$, $f=\frac{1}{2}$.

Then the centre (x_1, y_1) of the conic is given by

ax_1+hy_1+g and hx_1+by_1+f

i.e., by $\qquad 3x_1-y_1+1=0$

$\qquad -x_1-y_1+\frac{1}{2}=0$

Solving these two, $\quad x_1=-\frac{1}{8}, y_1=\frac{5}{8}$.

Now, $(-\frac{1}{8}, \frac{5}{8})$ is the centre of (1).

If the centre satisfies the eqn. of line $x-3y+2=0$ ...(2) then this line is a diameter of (1).

L. H. S. of (2) $=-\frac{1}{8}-3(\frac{5}{8})+2$ [substituting for x, y]

$=0=$ R. H. S.

Hence $x-3y+2=0$ is a diameter of (1)

and 'm' of this diameter is $\frac{1}{3}$.

Then the conjugate diameter is given by

$ax+hy+g+m(hx+by+f)=0$

i.e., $3x-y+1+\frac{1}{3}(-x-y+\frac{1}{2})=0$ $\quad$ [$\because$ $m=$

or, $16x-8y+7=0$.

4. $ax^2+2hxy+by^2+c=0$... ... (1)

$\lambda x^2+2\mu xy+\nu y^2=0$ ... ... (2).

Let $y-m_1x=0$ and $y-m_2x=0$ be eqns. of the two conjugate diameters given by (2) w. r. t. the conic (1).

Then $\lambda x^2+2\mu xy+\nu y^2 \equiv \nu(y-m_1x)(y-m_2x)$.

$\therefore \quad m_1+m_2=-\dfrac{2\mu}{\nu}, \quad m_1m_2=\dfrac{\lambda}{\nu}.$

The condition that two diameters of gradients m_1 and m_2 will be conjugate w. r. t (1) is

$$a+h(m_1+m_2)+b(m_1m_2)=0$$

or, $\quad a+h\left(-\dfrac{2\mu}{\nu}\right)+b\left(\dfrac{\lambda}{\nu}\right)=0$

or, $\quad a\nu-2h\mu+b\lambda=0 \quad \ldots$

Let $\lambda x^2+2\mu xy+\nu y^2=0 \quad \ldots \quad$ (3)

Represent two diameters which are conjugate w. r. to both the conics $2x^2+4xy-3y^2=1$ and $6x^2-10xy+7y^2=1$. $\quad \ldots$ (4)

Then from the condition (3), we should have,

$$2\nu-2\mu-3\lambda=0,$$

$$6\nu+5\mu+7\lambda=0'$$

whence by the method of cross-multiplication,

$$\frac{\nu}{-14+15}=\frac{\mu}{-18-14}=\frac{\lambda}{10+12}$$

or, $\quad \dfrac{\lambda}{22}=\dfrac{\mu}{-32}=\dfrac{\nu}{1} \quad \ldots \quad \ldots$ (5)

Eliminating λ, μ, ν from (4) and (5), we get

$$22x^2+32xy+y^2=0 \quad \text{(Ans.)}$$

5. Worked out in the book.

6. The gen. eqn. of 2nd degree is $ax^2+2hxy+by^2+2gx+2fy+c=0$

If it passes through $(\alpha,0)$, $(\beta,0)$, $(0,\gamma)$, $(0,\delta)$, then

$c\alpha^2+2g\alpha+c=0 \ldots (1)$, $\qquad b\gamma^2+2f\gamma+c=0 \ldots (3)$,

$a\beta^2+2g\beta+c=0 \ldots (2)$, $\qquad b\delta^2+2f\delta+c=0 \ldots (4)$.

Solving (1) and (2), we get $g=-\dfrac{c(\alpha+\beta)}{2\alpha\beta}$ and $a=\dfrac{c}{\alpha\beta}$.

Solving (3) and (4), we have $f=-\dfrac{c(\gamma+\delta)}{2\gamma\delta}$ and $b=\dfrac{c}{\gamma\delta}$.

Substituting these values of f, g, a, b in the general eqn. we get

$$\frac{c}{\alpha\beta}x^2+2h'xy+\frac{c}{\gamma\delta}y^2-\frac{c(\alpha+\beta)}{\alpha\beta}x-\frac{c(\gamma+\delta)}{\gamma\delta}y+c=0$$

or, $\quad \dfrac{1}{\alpha\beta}(x^2-\alpha x-\beta y+\alpha\beta)+\dfrac{1}{\gamma\delta}(y^2-c\gamma+\gamma\delta+\gamma\delta)-1+\dfrac{h'}{c}\cdot 2xy=0$

or, $\quad \dfrac{1}{\alpha\beta}(x-\alpha)(x-\beta)+\dfrac{1}{\gamma\delta}(y-\gamma)(y-\delta)-1+2hxy=0$,

where h is variable.

The eqn. of conic through the pts. (1,0), (2,0), (0,1), (0,2) and (2,2) is

$a+2g+c=0 \ldots (5)$, $\qquad b+2f+c=0 \ldots (7)$,

$4a+4g+c=0 \ldots (6)$, $\qquad 4b+4f+c=0 \ldots (8)$,

$4a+8h+4b+4g+4f+c=0 \ldots (9)$.

Solving (5) and (6), $a=\dfrac{c}{2}$, $g=-\dfrac{3}{4}c$.

Solving (7) and (8), $b=\dfrac{c}{2}$, $f=-\dfrac{3}{4}c$.

Substituting these values in (9), we get

$$2c+8h+2c-3c-3c+c=0, \quad \text{i.e., } h=\frac{c}{8}.$$

Now, putting these values in the gen. eqn., we get

$$\frac{c}{2}(x^2+y^2)+\frac{c}{4}(xy)-\frac{3}{2}c(x+y)+c=0$$

or, $2(x^2+y^2)+xy-6(x+y)+4=0.$

Here $a=b=2,\ h=\frac{1}{2},\ g=f=-3,\ c=4.$

The eccentricity is then given by,

$$e^4+\frac{(a-b)^2+4h^2}{ab-h^2}(e^2-1)=0$$

i.e., $e^4+\frac{0+1}{4-\frac{1}{4}}(e^2-1)=0$ [substituting for a, b, h]

or, $15e^4+4e^2-4=0$ which is a quadratic in e^2.

$$\therefore\ e^2=\frac{-4\pm\sqrt{4^2+4(15)4}}{2\times 15}=\frac{-4\pm 4(4)}{2(15)}$$

We consider the +ve sign only, since the –ve sign makes $e^2<0$ which is impossible.

$$\therefore\ e^2=\frac{12}{30}=\frac{2}{5}; \quad \therefore\ e=\sqrt{\frac{2}{5}}.$$

7. Here $a=b=1+\lambda^2,\ h=-2\lambda,\ g=f=\lambda,\ c=2.$

Now $\Delta=abc+2fgh-af^2-bg^2-ch^2$

$=-4\lambda^3-6\lambda^2+2$ [substituting the values and simplifying]

Let us now find out the values of λ for which $\Delta=0$,

i.e., $2\lambda^3+3\lambda^2-1=0$ [dividing both sides by -2]

or, $(\lambda+1)(\lambda+1)(2\lambda-1)=0.$

$\therefore\ \lambda=-1,-1,\frac{1}{2}$ for which Δ vanishes.

Now, Δ may be taken to be non-zero, if $\lambda>\frac{1}{2}$, and

then $ab-h^2=(1+\lambda^2)^2-4\lambda^2$

$=(1-\lambda^2)^2>0,$ [except $\lambda=1$]

In that case, given eqn. will represent an ellipse if $e<1$.

The eccentricity of the conic is given by

$$e^4+\frac{(a-b)^2+4h^2}{ab-h^2}(e^2-1)=0$$

i.e., $e^4+\frac{16\lambda^2}{(1-\lambda^2)^2}(e^2-1)=0$

or, $(1-\lambda^2)^2e^4+16\lambda^2.\,e^2-16\lambda^2=0.$

$$\therefore\ e^2=\frac{-16\lambda^2\pm\sqrt{(16)^2e^4+4(1-\lambda^2)^2(16\lambda^2)}}{2(1-\lambda^2)^2}$$

$$=\frac{-16\lambda^2\pm 8\lambda\sqrt{4\lambda^2+(1-\lambda^2)^2}}{2(1-\lambda^2)^2}.$$

$$\therefore\ e^2=\frac{-16\lambda^2+8\lambda(1+\lambda^2)}{2(1+\lambda^2)^2}$$ [$\because$ the other sign makes $e<0$ which is not true

or. $e^2=\frac{4\lambda}{(1+\lambda)^2}$ [simplifying]

or, $e=\frac{2\sqrt{\lambda}}{1+\lambda}$. For an ellipse $e<1$, i.e., $\frac{2\sqrt{\lambda}}{1+\lambda}<1$

or, $2\sqrt{\lambda}<1+\lambda$ or, $4\lambda<(1+\lambda)^2$

or, $0<(1+\lambda)^2-4\lambda$

or, $0<(1-\lambda)^2$ or, $0<1-\lambda$ or, $\lambda<1$.

Hence the given eqn. represents a real ellipse, if $\frac{1}{2}<\lambda<1$ where λ is real.

Part *b*. Now, $\Delta=0$, if $\lambda=\frac{1}{2}$,

and then $ab-h^2=(1-\lambda^2)^2=\left(1-\frac{1}{4}\right)^2=\frac{9}{16}>0$.

Hence the given eqn. represents a real true-pair $\lambda=\frac{1}{2}$ and λ being real.

[Before doing this sum, see Ex. 1 of this Chap.

8. In finding the eqn. of directrix, we are to red. the given eqn. to its standard form.

The given eqn. can be written as

$$(\lambda x+\mu y+k)^2=2k(\lambda x+\mu y)+2\rho x+k^2$$

$$=2(k\lambda+\rho)x+2k\mu y+k^2 \quad \ldots (1),$$

where k is a const.

Now, k may be so chosen that the lines

$\lambda x+\mu y+k=0$ and $2(k\lambda+\rho)x+2k\mu y+k^2=0$

are prep. to each other.

Then $\lambda(k\lambda+\rho)2+\mu.k\mu.2=0$ [using $a_1a_2+b_1b_2=0$]

or, $k=\dfrac{-\rho\lambda}{\lambda^2+\mu^2}$.

Substituting for k in (1), we get

$$\left(\lambda x+\mu y-\frac{\rho\lambda}{\lambda^2+\mu^2}\right)^2=\frac{2\rho\mu^2}{\lambda^2+\mu^2}x-\frac{2\rho\mu\lambda}{\lambda^2+\mu^2}y+\frac{\rho^2\lambda^2}{(\lambda^2+\mu^2)^2}\ldots$$

Taking $\lambda x+\mu y-\dfrac{\rho\lambda}{\lambda^2+\mu^2}=0$ as the X-axis,

and $\dfrac{2\rho\mu\lambda^2}{\lambda^2+\mu^2}x-\dfrac{2\rho\lambda}{\lambda^2+\mu^2}\cdot y+\dfrac{\rho^2\lambda^3}{(\lambda^2+\mu^2)^2}=0$ as the Y-axis,

we get

$$Y=\{\lambda x+\mu y)(\lambda^2+\mu^2)-\rho\lambda\}/D \quad \ldots\ \ldots\ \ldots (3),$$

$$X=\{(2\rho\mu^2x-2\rho\mu\lambda y)(\lambda^2+\mu^2)+\rho^2\lambda^2\}/(2\rho\mu.D)$$

where $D=\sqrt{(\lambda^3+\lambda\mu^2)^2+(\mu^3+\lambda^2\mu)^2} \quad \ldots\ \ldots (4)$

Then using (3) and (4), (2) reduces to

$(Y, D)^2=2\rho\mu D.X$

or, $Y^2=\dfrac{2\rho\mu}{D}\cdot X=4A.X$ (say) $\quad \ldots\ \ldots\ \ldots (5)$.

which is the standard eqn. of the parabola.

Again, $A=\dfrac{\rho\mu}{2D} \quad \ldots\ \ldots\ \ldots (6)$, [by (5)]

The eqn. of the directrix is $X+A=0$

i.e., $\dfrac{(2\rho\mu^2x-\rho2\mu\lambda y)(\lambda^2+\mu^2)+\rho^2\lambda^2}{2\rho\mu.D}+\dfrac{\rho\mu}{2D}=0$

or, $2\mu(\mu x-\lambda y)\rho=0$ [simplifying].

The axis has the eqn. $Y=0$

i.e., $\lambda(\lambda^2+\mu^2)x+\mu(\lambda^2+\mu^2)y\ \ \rho x=0$ [by (3)] ... ... (7)

The focus of the parabola can be found the solutions eqns.

$X=A$ and $Y=0$. Now $X=A$ gives

$$2\mu^2(\lambda^2+\mu^2)x-2\lambda\mu(\lambda^2+\mu^2)y+\rho(\lambda^2-\mu^2)=0$$

Solving (7) and (8), the focus is

$$\left\{\frac{\rho}{2(\lambda^2+\mu^2)},\ \frac{\rho\lambda}{2\mu(\lambda^2+\mu^2)}\right\}.$$

9. $ax^2+2hxy+by^2+2gx+2fy+c=0$.

Here $\Delta=abc+2fgh-af^2-bg^2-ch^2$,

and $C=ab-h^2$.

Let us transfer the origin to the centre of the conic and then rotate the axes to remove the xy term. Let the finally reduced eqn. be

$$a'x^2+b'y^2+c'=0 \quad \ldots \ldots (1)$$

where $c'=\frac{\Delta}{C}$.

By the invariants, $a'+b'=a+b$,

and $a'b'=ab-h^2$.

Now (1) can be written as $a'x^2+b'y^2=-c'$.

or, $$\frac{x^2}{\frac{-c'}{a'}}+\frac{y^2}{\frac{-c'}{b'}}=1$$

Then lengths of semi-axes are

$$a_0=\sqrt{\left|\frac{-c'}{a'}\right|} \text{ and } b_0=\sqrt{\left|\frac{-c'}{b'}\right|}$$

The area of the ellipse is

$$\pi a_0 b_0=\frac{\pi c'}{\sqrt{a'b'}} \quad [\text{substituting for } a_0, b_0].$$

$$=\frac{\pi\Delta}{C\sqrt{ab-h^2}} \quad \left[\because c'=\frac{\Delta}{C},\ a'b'=ab-h^2\right]$$

$$=\frac{\pi\Delta}{(ab-h^2)\sqrt{ab-h^2}}$$

$$=\frac{\pi\Delta}{\delta^{3/2}} \quad [\because ab-h^2=\delta].$$

10. $x^2-xy+2y^2-2x-6y+7=0$

In finding the lengths and eqns. of the axes, we need not reduce the eqn. to its standard form. We transfer the origin to the centre of conic. If (x_1,y_1) be the centre of the conic, the eqn. then reduces to

$$X^2-XY+2Y^2+\frac{\Delta}{C}=0 \quad \ldots \ldots (1)$$

Now, the centre is given by the eqns.

$ax_1+hy_1+g=0$ and $hx_1+by_1+f=0$

i.e., $2x_1-y_1-2=0$,

$x_1-4y_1+6=0$

whence solving, $x_1=2$, $y_1=2$.

∴ the relation between the old co-ordinates (x,y) and the new co-ordinates (X,Y) of a point is given by

$x=x_1+X=2+X$,

$y=y_1+Y=2+Y$ i.e., $X=x-2$ and $Y=y-2$ ... (2)

Again, $$\Delta=\begin{vmatrix}1 & -\frac{1}{2} & -1\\ -\frac{1}{2} & 2 & -3\\ -1 & -3 & 7\end{vmatrix}=-\frac{7}{4}$$

and $C=ab-h^2=\frac{7}{4}$. ∴ $\frac{\Delta}{C}=-1$.

Hence (1) becomes $X^2-XY+2Y^2-1=0$ ... ... (3).

The eqn. of concentric circle of radius r is $X^2+Y^2=r^2$ ...

Making (3) homogeneous with the help of (4), we ...

$$X^2-XY+2Y^2-\frac{X^2+Y^2}{r^2}=0$$

or, $(r^2-1)X^2+(2r^2-1)Y^2-r^2XY=0$ ... ... (5).

(5) then represents a pair of st. lines through X, $Y=0$, i.e., through the common centre and the intersec... of (3) and (4).

When r^2 is equal to the radius of either semi-axes, circle touches the ellipse at the ends of that axis and lines represented by (5) become coincident. The cond... for this is

$r^4=4(2r^2-1)(r^2-1)$ [using the condition $h^2=ab$] ...

or, $7r^4-12r^2+4=0$,

which is a quadratic in r^2.

$$\therefore\ r^2=\{12\pm\sqrt{(12)^2-4(7).4)}\}/(14)=\frac{6\pm2\sqrt{2}}{7}.$$

Then $r_1^2=$ sq. of the semi-major axis$=\frac{6+2\sqrt{2}}{7}$,

and $r_2^2=$ sq. of the semi-minor axis$=\frac{6-2\sqrt{2}}{7}$.

$\therefore$ product of the two semi-axes

$$=\sqrt{\frac{6+2\sqrt{2}}{7}}.\ \sqrt{\frac{6-2\sqrt{2}}{7}}$$

$$=\frac{\sqrt{36-8}}{7}=\frac{2}{\sqrt{7}}\quad \text{(Proved).}$$

When two lines given by (5) coincide, (5) can be written as

$$(r^2-1)X^2-r^2XY+\frac{r^4}{4(r^2-1)}Y^2=0\ [\text{ by (6) }].$$

or, $4(r^2-1)^2X^2-4r^2(r^2-1)XY+r^4Y^2=0$

or, $\{2(r^2-1)X-r^2Y\}^2=0$.

Hence the eqn. of major axis is $2(r_1^2-1)X-r_1^2Y=0$, and that of the minor axis is $2(r_2^2-1)X-r_2^2Y=0$.

Substituting in these two for r_1^2 and r_2^2, we get

$2(2\sqrt{2}-1)X=(6+2\sqrt{2})Y$,

i.e., $(2\sqrt{2}-1)X=(3+\sqrt{2})Y$

and $2(-2\sqrt{2}-1)X=(6-2\sqrt{2})Y$,

i.e., $(2\sqrt{2}+1)X=-(3-\sqrt{2})Y$,

or, referred to the old axes, i.e.. putting $X=x-2$, $Y=y-2$, we have the eqn. of axes

$(2\sqrt{2}-1)(x-2)=(3+\sqrt{2})(y-2)$ ... ... (9),

and $(2\sqrt{2}+1)(x-2)=-(3-\sqrt{2})(y-2)$ ... (10)

Now multiplying (9) by (10), we get

$(2^2.2-1)(x-2)^2=-(3^2-2)(y-2)^2$

or, $(x-2)^2+(y-2)^2=0$

or, $x^2-4x+y^2-4y+8=0$ which is the eqn. of the axes.

[Note : for this sum, see the fig. of p. 136]

11. The given eqn. can be written as

$$(x-2y+k)^2=2k(x-2y)-10x+8y-13+k^2$$
$$=2(k-5)x+4(2-k)y-13+k^2 \quad \cdots (1)$$

where k is const.

Now, k may be so chosen that the lines

$$x-2y+k=0 \text{ and } 2(k-5)x+4(2-k)y-13+k^2=0$$

are perp. to each other.

Then, $2(k-5)-4(2)(2-k)=0$ [using $a_1a_2+b_1b_2=0$]

or, $k=\frac{13}{5}$.

Substituting for k in (1), we get

$$\left(x-2y+\frac{13}{5}\right)^2=-\frac{24}{5}x-\frac{12}{5}y-13+\left(\frac{13}{5}\right) \quad \ldots \ldots (2)$$

Taking $x-2y+\frac{13}{5}=0$ as the X axis, and

$-\frac{24}{5}x-\frac{12}{5}y-13+\left(\frac{13}{5}\right)=0$ as the Y-axis, we get

$$Y=\frac{x-2y+13/5}{\sqrt{1^2+2^2}}=\frac{x-2y+\frac{13}{5}}{\sqrt{5}} \quad \ldots \ldots (3).$$

$$\text{Similarly, } X=\frac{-\frac{24}{5}x-\frac{12}{5}y-13+\left(\frac{13}{5}\right)^2}{\frac{12}{\sqrt{5}}} \quad \ldots \ldots (4)$$

Then, using (3) and (4), (2) reduces to

$$Y^2=\frac{12}{5\sqrt{5}}.X=4A.X \text{ (say)} \quad \ldots \ldots \ldots (5)$$

Hence $A=\frac{3}{5\sqrt{5}}$ [by (5)].

The co-ordinates of the focus can be found from the solutions of the two eqns. $X=A$ and $Y=0$,

Now, $X=A$ gives $10x+5y+16=0$, and $Y=0$ gives $5x-10y+13=0$ $\Big\}$ substituting the values of X, A, Y.

Solving these two, we get the focus $\left(-\frac{9}{5}, \frac{2}{5}\right)$.

The co-ordinates of the vertex is given by the solutions of $X=0$ and $Y=0$. Substitute for X, Y from (4) and (3), and then solve.

This is found to be $\left(-\frac{39}{25}, \frac{13}{25}\right)$.

12. Here $x^2-11y^2-16xy+10x+10y-7=0$.

Now, $a=1$, $b=-11$, $h=-8$, $g=5$, $f=5$, $c=-17$

$$\text{Then } \Delta=\begin{vmatrix} 1 & -8 & 5 \\ -8 & -11 & 5 \\ 5 & 5 & -7 \end{vmatrix}=375 \neq 0.$$

$C=ab-h^2=-75<0$.

Hence the eqn. represents a hyperbola.

The centre (x_1, y_1) of the conic is given by the eqns.

$$x_1-8y_1+5=0,$$

and $-8x_1-11y_1+5=0$. Solving them, $(x_1, y_1)\equiv\left(-\frac{1}{5}, \frac{3}{5}\right)$.

The eccentricity of the conic is given by

$$e^4+\frac{(a-b)^2+h^2}{ab-h^2}(e^2-1)=0$$

$$\text{or, } e^4+\frac{12^2+4.8^2}{-75}(e^2-1)=0$$

or, $3e^4-16e^2+16=0$, which is a ...

Find e^2 and hence e.

Now follow Example 10 for finding the ...

Thus we show :

(1) It represents a hyperbola,

(2) The centre of the conic,

(3) The value of eccentricity,

(4) The eqns. of axes.

Hence the answer.

13. (i) Here $\alpha=2$, $\beta=1$.

$$Ax+By+C\equiv x-2y+3=0,$$

and $e=\frac{1}{2}$.

Let $P(x,y)$ be any point on the conic. ... from P at M which is on the directrix $x-2y+3=0$ ...

If $S(\alpha,\beta)$ be the focus, then $SP^2=e^2.PM^2$

i.e., $(x-\alpha)^2+(y-\beta)^2=e^2.\frac{(Ax+By+C)^2}{A^2+B^2}$

i.e., $(x-2)^2+(y-1)^2=\frac{1}{4}.\frac{(x-2y+3)^2}{1^2+2^2}$

or, $19x^2+16y^2+4xy-86x-28y+91=0$,

which is the eqn. of the conic.

For (ii) and (iii), follow the same procedure ...

14. 1st Part—Here $ax^2+2hxy+by^2-1=0$ ...

(1) will have its centre at the origin if $\frac{\Delta}{C}$ ...

Now, $\Delta=abc+2fgh-af^2-bg^2-ch^2$

$=-ab+h^2$ [$\because$ $c=-1, f=0, g=0$],

and $C=ab-h^2$.

Then $\frac{\Delta}{C}=-1$. Hence centre of (1) is at (0,0).

[See p. 124, art. 8]

The eqn. of a concentric circle of radius r

is $x^2+y^2=r^2$

i.e., $(x^2+y^2)/r^2=1$ ... ... (2)

Making (1) homogeneous with the help of (2), we get

$$ax^2+2hxy+by^2-\frac{x^2+y^2}{r^2}=0$$

or, $(ar^2-1)x^2+2hxyr^2+(br^2-1)y^2=0$ ... ... (3)

(3) Then represents a pair of lines through the origin, that is, through the common centre and the intersections of (1) and (2). When r^2 is equal to the square of either semi-axes, the circle touches the ellipse at the ends of that axes and the lines represented by (3) become coincident. The condition for this is

$4r^4h^2=4(ar^2-1)(br^2-1)$ ... ... (4)

[using the condition $h^2=ab$]

or, $h^2r^4=abr^4-ar^2-br^2+1$

or, $(h^2-ab)r^4+(a+b)r^2-1=0$.

$\therefore$ $\frac{1}{r^4}-\frac{1}{r^2}(a+b)+ab-h^2=0$. [illegible]

2nd Part—

(3) gives $(ar^2-1)x^2+2hxyr^2+\frac{h^2r^4}{ar^2-1}y^2=0$ [by (4)]

or, $(ar^2-1)^2x^2+2hxyr^2(ar^2-1)+h^2r^4y^2=0$

or, $\{(ar^2-1)x+hr^2y\}^2=0$

or, $(ar^2-1)x+hr^2y=0$

or, $\left(a-\frac{1}{r^2}\right)x+hy=0$ [dividing by r^2]

which is the eqn. giving the position of the axes.

15. Worked out in the book.

16. (i) The conic is $3x^2+8xy-3y^2+6x+8y+4=0$.

Let the eqn. of the pair of asymptotes of the conic be

$$3x^2+8xy-3y^2+6x+8y+c=0 \quad \cdots \cdots (1).$$

Here $a=3, b=-3, h=4, g=3, f=4, c=c$.

Then (1) will represent a pair of lines, if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 3 & 4 & 3 \\ 4 & -3 & 4 \\ 3 & 4 & c \end{vmatrix} = 0$$

i.e., if $c=3$.

Hence the pair of asymptotes is

$$3x^2+8xy-3y^2+6x+8y+3=0 \quad \cdots \cdots (2)$$

Let the two lines given by (2), be

$$y-m_1x-c_1=0 \quad \cdots \cdots \cdots (3),$$

and $y-m_2x-c_2=0 \quad \cdots \cdots \cdots (4)$

$\because$ $m_1+m_2=\frac{-2h}{b}=\frac{8}{3}$, $m_1m_2=\frac{a}{b}=-1$;

we get $m_1=3, m_2=-\frac{1}{3}$.

Corresponding values of c_1 and c_2 are given by,

$$c_1=-\frac{g+fm_1}{h+bm_1}=-\frac{3+4.3}{4-3.3}=3,$$

and $c_2=-\frac{g+fm_2}{h+bm_2}=-\frac{1}{3}$.

Putting these values in (3) and (4), we get

$y-3x-3=0$, and $3y+x+1=0$

which are the reqd. asymptotes.

For (ii), (iii) and (iv), follow the same procedure.

17. Here $L\equiv 2x+3y-5=0$,

$L'\equiv 5x+3y-8=0$.

The joint eqn. of the asymptotes is $LL'=0$,

i.e., $(2x+3y-5)(5x+3y-8)=0$

or, $10x^2+21xy+9y^2-41x-39y+40=0 \ldots$ (1)

$\therefore$ the eqn. of the conic may be obtained by changing the const. term in (1). Thus,

$$10x^2+21xy+9y^2-41x-39y+C=0 \quad \cdots \cdots (2)$$

is the eqn. of the conic.

If (2) passes through $(1,-1)$, then

$10-21+9-41+39+C=0$,

[by substituting $(1,-1)$ in (2)]

i.e., $C=4$.

Hence (2) reduces $10x^2+21xy+9y^2-41x-39y+4=0$ which is the reqd. eqn. of the conic.

18. The conic is given by, $ax^2+by^2+2hxy+2gx+2fy+c=0$ ... (1)

Then the eqn. of the pair of asymptotes of the conic

$$ax^2+by^2+2hxy+2gx+2fy+c'=0 \quad \ldots \quad \ldots \quad (2)$$

[changing the const. term in (1)]

The const. $c'=0$, since (2) passes through (0,0).

Now, (2) represents a pair of straight lines, if

$\Delta \equiv abc'+2fgh-af^2-bg^2-c'h^2=0$

i.e., $2fgh-af^2-bg^2=0$ [$\because$ $c'=0$]

which is the reqd. condition.

19. Worked out in the book.

20. Here $S\equiv 16x^2+4xy+19y^2-56x-72y+84=0$...(1)

The polar of (x_1,y_1) w. r. t. the conic (1) is given by

$$16xx_1+2(xy_1+x_1y)+19yy_1-28(x+x_1)-36(y+y_1)+84=0 \quad \ldots (2)$$

Then the polar of (3, 0) w. r. t. (1) is $16x(3)+2(0+3y)+0-28(x+3)-36(y+0)+84=0$ [substituting (3,0) for (x_1,y_1) in (2)]

or, $2x-3y=0$ ... ... (3)

Similarly, the polars of (0, 2) and (6,2) w. r. t. (1), are

$16x+37y-78=0$ ... ... (4)

and $36x+7y-78=0$ ... ... (5) respectively

The lines (3), (4) and (5) will be concurrent, if

$$\begin{vmatrix} 2 & -3 & 0 \\ 16 & 37 & -78 \\ 36 & 7 & -78 \end{vmatrix}=0$$ [eliminating x and y from (3), (4), (5)]

Now, the det. $=2(-78)\begin{vmatrix} 1 & -3 & 0 \\ 8 & 37 & 1 \\ 18 & 7 & 1 \end{vmatrix}$

$=2(-78)\{1(37-7)-8(-3-0)+18(-3-0)\}$

[expanding w. r. t. col. 1]

$=2(-78)\{30+24-54\}=0.$

Hence they are concurrent.

21. Worked out in the book.

22. Let P (x_1, y_1) be the middle point of the chord of intersection of a circle with $C(x', y')$ as centre and the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$. Its equation is

$$axx_1+h(xy_1+x_1y)+byy_1+g(x+x_1)+f(y+y_1)+c$$
$$=ax_1^2+2hx_1y_1+by_1^2+2gx_1-2fy_1+c \quad \text{(i.e., } T=S_1\text{)}$$

or, $(ax_1+hy_1+g)x+(hx_1+by_1+f)y$
$$=ax_1^2+2hx_1y_1+by_1^2+gx_1+fy_1 \quad \ldots \quad \ldots \quad (1)$$

$\therefore$ its gradient is

$$m=-\frac{ax_1+hy_1+g}{hx_1+by_1+f} \quad \ldots \quad \ldots \quad (2)$$

Again the gradient of $C.P$ is $m^1=\dfrac{y_1-y'}{x_1-x'}$ ... ... (3)

Since the perpendicular bisector of any chord of a circle passes through its centre, we have that CP is perp. to the chord given by (1). Hence

$mm'=-1.$

$\therefore$ from (2) and (3),

$$\frac{(y_1-y')}{(x_1-x')}\cdot\frac{ax_1+hy_1+g}{hx_1+by_1+f}=-1$$

or $(x_1-x')(hx_1+by_1+f)-(y_1-y')(ax_1+hy_1+g)=0$

$\therefore$ the locus of (x_1, y_1) is

$$(x-x')(hx+by+f)-(y-y')(ax+hy+g)=0 \text{ (Proved)}$$

23. $ax^2+2\lambda xy+by^2+2gx+2fy=0 \quad \cdots \quad \cdots \text{ (1)},$

$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \cdots \quad \cdots \text{ (2)}.$

Then, the condition for (1) to represent a line-pair is $\Delta=abc+2fgh-af^2-bg^2-ch^2=0$

i.e., $2fg\lambda-af^2-bg^2=0$ [$\because$ $h=\lambda$, $c=0$]

i.e., $\lambda=\dfrac{af^2+bg^2}{2fg} \quad \cdots \text{ (3)}$

Now, $ax^2+\dfrac{af^2+bg^2}{fg}\cdot xy+by^2+2gx+2fy=0$

[substituting for λ in (i)

or, $afgx^2+(af^2+bg^2)xy+bfgy^2+2fg^2x+2f^2gy=0$

or, $(gx+fy)(afx+bgy+2fg)=0.$

Hence $gx+fy=0$ is one of the lines given by (1).

Now, the tangent to the curve (2) at (0, 0) is $gx+fy=0$

[$\because$ here $(x_1, y_1)\equiv(0,0)$; and $c=0$.

Since this passes through origin]

Therefore, one of the lines given by (i) is the tangent at the origin to the curve (2) for the value of λ given by (3).

24. Since m is the gradient of the tangent at P to the conic (as the gradient of a conic at a point means the gradient of its tangent there). CP bisects all chords which are parallel to the tangent at P.

$\therefore$ equation of CP is

$ax+hy+g+m(hx+by+f)=0$ [See Art. 82, eqn. (1)] $\quad \cdots \quad \cdots$ (1)

or, $(a+hm)x+(h+bm)y+g+mf=0$

$\therefore$ gradient of CP is $m^1=\dfrac{a+hm}{h+bm}$. $\quad \cdots \quad \cdots \quad \cdots$ (2)

If CP is an axis of the conic, then CP must be perpendicular to the tangent at P, the condition for which is

$$mm^1=-1$$

or, $m(a+hm)-(h+bm)=0$ [using (2)]

or, $hm^2+(a-b)m-h=0 \quad \cdots \quad \cdots \quad \cdots$ (3)

But for any point. (x, y) on CP, we have from (i).

$$m=-\frac{ax+hy+g}{hx+by+f}=-\frac{X}{Y}.$$

Hence from (3),

$$h\left(-\frac{X}{Y}\right)^2+(a-b)\left(-\frac{X}{Y}\right)-h=0$$

or $hX^2-(a-b)XY-hY^2=0$

giving the equation of the axes (Proved).

25. Proceed exactly as in Example No. 14.

Make $ax^2+2hxy+ay^2-d=0$ with respect to a concentric circle $x^2+y^2=r^2$ getting

$$ax^2+2hxy+ay^2-d\,\frac{x^2+y^2}{r^2}=0$$

$$\text{or,}\quad (ar^2-d)x^2+2hr^2\,xy+(ar^2-d)y^2=0 \quad \ldots\ \ldots\ (1)$$

For either axes (1) will represent a pair of coincident straight lines, the condition being

$$(hr^2)^2=(ar^2-d)^2$$

$$\text{or,}\quad ar^2-d=\pm hr^2$$

$$\text{or,}\quad (a\pm h)r^2=d$$

$\therefore\quad r=\sqrt{\dfrac{d}{a\pm h}}$, giving the lengths of the semi-axes.

Taking $r=\sqrt{\dfrac{d}{a+h}}$, we have from (1),

$$\left(\frac{ad}{a+h}-d\right)x^2+2h\frac{d}{a+h}xy+\left(\frac{ad}{a+h}-d\right)y^2=0$$

$$\text{or,}\quad -hd\,x^2+2hd\,xy-hd\,y^2=0$$

$$\text{or,}\quad -hd\,(x-y)^2=0$$

$$\text{or,}\quad x-y=0 \quad \ldots\ \ldots\ \ldots\ (2)$$

giving the equation of one axis.

Similarly taking $r=\sqrt{\dfrac{d}{a-h}}$, the equation of the other axis is found to be

$$x+y=0 \quad \ldots\ \ldots\ \ldots\ (3)$$

Hence from (2) and (3), the joint equation of the axis is

$$(x-y)(x+y)=0$$

$$\text{or,}\quad x^2-y^2=0 \text{ (Proved).}$$

Examples on Chapter VII

1. The tangent to the parabola $y^2=4ax$ at the point (x',y') is $yy'=2a(x+x')$,

$$\text{i.e.,}\quad y=\frac{2a}{y'}x+\frac{2ax'}{y'} \quad \ldots\ \ldots\ (1)$$

Again the tangent to the same parabola at $\left(\dfrac{a^2}{x'},\ -\dfrac{4a^2}{y'}\right)$

$$\text{is } y\left(-\frac{4a^2}{y'}\right)=2a\left(x+\frac{a^2}{x'}\right),$$

$$\text{i.e.,}\quad y=\frac{2a(y')}{-4a^2}x+\frac{a^2(y')}{-4a^2}$$

$$\text{or,}\quad y=-\frac{y'}{2a}x-\frac{y'}{4} \quad \ldots\ \ldots\ (2)$$

Tangent (1) and (2) will be perp. to each other if $m_1m_2=-1$.

$$\text{Here } m_1m_2=\frac{2a}{y'}\left(-\frac{y'}{2a}\right)=-1.$$

Hence the tangents are perp. to each other.

2. 1st Part—

Since the tangents $y=mx+\dfrac{a}{m}$ and $y=m'x+\dfrac{a}{m'}$ intersect at a point where

$$mx+\frac{a}{m}=m'x+\frac{a}{m'} \quad \ldots\ \ldots\ (1)$$