Sat Sun Mon Tue Wed Thu Fri Date:

THEME

Ex. $(y^4+4x^3y+3x)\,dx+(x^4+4xy^3+y+1)\,dy=0$

Here $M = y^4+4x^3y+3x$

$$\frac{\partial M}{\partial y} = 4y^3+4x^3$$

and $N = x^4+4xy^3+y+1$

$$\frac{\partial N}{\partial x} = 4x^3+4y^3$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The equation is Exact

Solⁿ. becomes,

$\int$ (M keeping y as constant) dx + $\int$ In N terms free from x dy = C

$$\Rightarrow \int y^4+4x^3y+3x\,dx+\int y+1\,dy = \text{constant}$$

$$\Rightarrow xy^4+x^4y+\tfrac{3}{2}x^2+\tfrac{y^2}{2}+y = C.$$

In x^4+4xy^3+y+1, terms free from x are $y+1$ whose integral with respect to y is $\frac{1}{2}y^2+y$.

Therefore the general solution is

$$y^4x+x^4y+\tfrac{2}{5}x^5+\tfrac{1}{2}y^2+y=C.$$

Ex. 2. *Solve* $x(x^2+y^2-a^2)\,dx+y(x^2-y^2-b^2)\,dy=0.$ [Nag. 63; Poona 61]

Solution. Here $M=x^3+xy^2-a^2x$, $N=yx^2-y^3-b^2y$.

$$\frac{\partial M}{\partial y}=2xy \text{ and } \frac{\partial N}{\partial x}=2xy.$$

Since these are equal, the equation is exact,

Integrating M w.r.t. x keeping y as constant, we get

$$\tfrac{1}{4}x^4+\tfrac{1}{2}x^2y^2-\tfrac{1}{2}a^2x^2.$$

In N, terms free from x are $-y^3-b^2y$ whose integral is

$$-\tfrac{1}{4}y^4-\tfrac{1}{2}b^2y^2.$$

Hence the general solution is

$$\tfrac{1}{4}x^4+\tfrac{1}{2}x^2y^2-\tfrac{1}{2}a^2x^2-\tfrac{1}{4}y^4-\tfrac{1}{2}b^2y^2=\text{const.}$$

or $\quad x^4-y^4+2x^2y^2-2a^2x^2-2b^2y^2=C.$

Ex. 3. *Solve* $(x^2-2xy+3y^2)\,dx+(4y^3+6xy-x^2)\,dy=0.$ [Delhi Hons. 55]

Solution. Here $\dfrac{\partial M}{\partial y}=-2x+6y$, $\dfrac{\partial N}{\partial x}=6y-2x$.

Since these are equal the equation is exact.

Integrating M, *i.e.* $x^2-2xy+3y^2$ w.r.t. x keeping y as constant, we get

$$\tfrac{1}{3}x^3-x^2y+3y^2x$$

In N, term free from x is $+4y^3$ whose integral is y^4.

Hence the solution is $\frac{1}{3}x^3-x^2y+3y^2x+y^4=C.$

Ex. 4. *Solve* $(x-2e^y)\,dy+(y+x\sin x)\,dx=0.$ [Gujrat 61]

Solution. Here $M=y+x\sin x$. $N=x-2e^y$.

$\therefore \quad \dfrac{\partial M}{\partial y}=1, \dfrac{\partial N}{\partial x}=1$; therefore equation is exact.

Integrating $y+x\sin x$ with respect to x keeping y as constant, we get

$$xy+\int x\sin x\,dx=xy-x\cos x+\sin x.$$

In N, term free from x is $-2e^y$ whose integral with respect to y is $-2e^y$.

Hence the complete solution is

$$xy-x\cos x+\sin x-2e^y=C.$$

*Ex. 5. (a) *Solve* $x\,dx+y\,dy=\dfrac{a^2(x\,dy-y\,dx)}{x^2+y^2}.$

[Delhi Hons. 62]

Solution. The equation can be put as

$$\left(x+\frac{a^2y}{x^2+y^2}\right)dx+\left(y-\frac{a^2x}{x^2+y^2}\right)dy=0.$$

(ii) $(x^2-4xy-2y^2)\,dx+(y^2-4xy-2x^2)\,dy.$

Ans. $x^3+y^3-6xy\,(x+y)=C.$

(iii) $\cos x\,(\cos x-\sin a \sin y)\,dx$
$+\cos y\,(\cos y-\sin a \sin x)\,dy=0.$

Ans. $2(x+y)\sin 2x+\sin 2y-4\sin a \sin x \sin y=C.$

(iv) $(2xy+y-\tan y)\,dx+(x^2-x\tan^2 y+\sec^2 y)\,dy=0.$ **[Poona 1964]**

Ans. $x^2y+xy-x\tan y+\tan y=C.$

(v) $(2x^2y+4x^3-12xy^2+3y^2-xe^y+e^{2x})\,dy$
$+(12x^2y+2yx^2+4x^3-4y^3+2ye^{2x}-e^y)\,dx=0.$ **[Poona 64]**

Ans. $4x^3y+x^2y^2+x^4-4y^3x+ye^{2x}-xe^y+y^3=C.$

3·4. Integrating factors.

If an equation becomes exact after it has been multiplied by a function of x and y, then such a function is called an integrating factor **[Karnatak 61]**

3·5. Number of integrating factors.

To show that there is an infinite number of integrating factors for an equation. **[Karnatak 61]**

$$M\,dx+N\,dy=0.$$

To prove this let μ be an integrating factor; then

$$\mu\,(M\,dx+N\,dy)=du.$$

Integrating, $u=c$ is a solution.

Now multiplying both the sides by $f(u)$, a function of u, we get $\mu f(u)\,[M\,dx+N\,dy]=f(u)\,du.$

Expression on the right is directly integrable and therefore so must be the left hand side.

Hence $\mu f(u)$ is also an integrating factor. Since $f(u)$ is an arbitrary function of u, the number of integrating factors is infinite.

3·6. Integrating factor by inspection.

Sometimes an integrating factor can be found by inspection. For this the reader should study the following results :—

Group of terms	I. F.	Exact Differntial
$x\,dy-y\,dx$	$\dfrac{1}{x^2}$	$\dfrac{x\,dy-y\,dx}{x^2}=d\left(\dfrac{y}{x}\right)$
$x\,dy-y\,dx$	$\dfrac{1}{y^2}$	$\dfrac{y\,dx-x\,dy}{-y^2}=d\left(-\dfrac{x}{y}\right)$
$x\,dy-y\,dx$	$\dfrac{1}{xy}$	$\dfrac{dy}{y}-\dfrac{dx}{x}=d\left(\log\dfrac{y}{x}\right)$
$x\,dy-y\,dx$	$\dfrac{1}{x^2+y^2}$	$\dfrac{x\,dy-y\,dx}{x^2+y^2}=\dfrac{\dfrac{x\,dy-y\,dx}{x^2}}{1+\left(\dfrac{y}{x}\right)^2}=d\left[\tan^{-1}\dfrac{y}{x}\right]$

3·7. Rules for finding the integrating factor.

Rule. I. If $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=f(x)$, a function of x only, then $e^{\int f(x)\,dx}$ is an integrating factor. [Delhi Hons. 64]

Rule II. If $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{M}=g(y)$ is a function of y alone, then $e^{\int -g(y)\,dy}$ is an integrating factor.

We give below some examples to illustrate these rules.

Ex. 1. *Solve* $(x^2+y^2+x)\,dx+xy\,dy=0.$

Solution $M=x^2+y^2+x,\ N=xy.$

$\dfrac{\partial M}{\partial y}=2y,\ \dfrac{\partial N}{\partial x}=y$, equation is not exact.

However, $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\dfrac{2y-y}{xy}=\dfrac{1}{x}$, a function of x alone.

Hence I F. $=e^{\int \frac{1}{x}dx}=e^{\log x}=x.$

Multiplying by I F., the equation becomes

$(x^3+xy^2+x^2)\,dx+x^2y\,dy=0$, exact now (check up).

Integrating, $x^3+xy^2+x^2$ with regard to x, keeping y as constant, we get $\frac{1}{4}x^4+\frac{1}{2}x^2y^2+\frac{1}{3}x^3$

and in x^2y^2 there is no term free from x. Therefore the solution is

$\frac{1}{4}x^4+\frac{1}{2}x^2y^2+\frac{1}{3}x^3=C'$ or $3x^4+4x^3+6x^2y^2=C.$

Ex. 2. *Solve* $(x^2+y^2+1)\,dx-2xy\,dy=0.$

Solution. $\dfrac{\partial M}{\partial y}=2y,\ \dfrac{\partial N}{\partial x}=-2y$, not exact.

However, $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\dfrac{2y+2y}{-2xy}=-\dfrac{2}{x}$ function of x alone.

$\therefore$ I.F. $=e^{-\int \frac{2}{x}dx}=e^{-2\log x}=\dfrac{1}{x^2}.$

Multiplying by $\dfrac{1}{x^2}$ the equation becomes

$\left(1+\dfrac{y^2}{x^2}+\dfrac{1}{x^2}\right)dx-\dfrac{2y}{x}\,dy=0$, exact now.

Integrating, $1+\dfrac{y^2}{x^2}+\dfrac{1}{x^2}$ with regard to x keeping y as constant.

we get $x-\dfrac{y^2}{x}-\dfrac{1}{x}$

Therefore the solution is $yx+\frac{2}{y^3}x+y^2=C$.

Ex. 9. *Solve* $(3x^2y^4+2xy)\,dx+(2x^3y^3-x^2)\,dy=0.$ **[Cal. Hons. 54, 53]**

Solution. Here $\frac{\partial M}{\partial y}=12x^2y^3+2x,\ \frac{\partial N}{\partial x}=6x^2y^3-2x.$

Now $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=\dfrac{6x^2y^3+4x}{y(3x^2y^3+2x)}=\dfrac{2}{y}$, unction of y alone.

$\therefore$ I. F. $=e^{\int -(2/y)\,dy}=e^{-2\log y}=\frac{1}{y^2}.$

Multiplying by $\frac{1}{y^2}$, the equation becomes

$$\left(3x^2y^2+\frac{2x}{y}\right)dx+\left(2x^3y-\frac{x^2}{y^2}\right)dy=0, \text{ exact now.}$$

Integrating $3x^2y^2+\frac{2x}{y}$ w.r.t. x keeping y as constant, we get

$x^3y^2+\frac{x^2}{y}.$

In $2x^3y-\frac{x^2}{y^2}$, there is no term free from x.

Hence the solution is $x^3y^2+\frac{x^2}{y}=C$

or $x^3y^3+x^2=Cy.$

Ex. 10. $(2xy^4e^y+2xy^3+y)\,dx+(x^2y^4e^y-x^2y^2-3x)\,dy=0.$

Solution. We have $\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{4}{y}.\quad\therefore\quad$ I.F. $=\frac{1}{y^4}.$

Solution is $\quad x^2e^y+\frac{x^2}{y}+\frac{x}{y^3}=C.$

3·8. Rule III.

If $M\,dx+N\,dy=0$ is homogeneous and $Mx+Ny\neq0$, then $\frac{1}{Mx+Ny}$ is an integrating factor.

Rule IV. **[Delhi Hons. 61]**

If the equation can be written in the form

$$yf(xy)\,dx+xg(xy)\,dy=0,\ f(xy)\neq g(xy),$$

then $\dfrac{1}{xy\,[f(xy)-g(xy)]}=\dfrac{1}{Mx-Ny}$ is an integrating factor.

Ex. 1. *Solve* $x^2y\,dx-(x^3+y^3)\,dy=0.$

Solution. The equation is homogeneous and

5

Linear Differential Equations with Constant Coefficients

5·1. Linear Differential Equation

A differential equation of the form

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\dots+P_n y=X$$

where $P_1, P_2, \dots, P_n$, and X are functions of x or constants, is called a *linear differential equation of n^{th} order.*

And if $P_1, P_2, \dots, P_n$ are all constants (not functions of x) and X is some function of x, then the equation is a *linear differential equation with constant coefficients.*

5.2. The Operator D. It is usual to write

$$D \text{ for } \frac{d}{dx},\ D^2 \text{ for } \frac{d^2}{dx^2},\dots,D^n \text{ for } \frac{d^n}{dx^n}.$$

And in terms of the operator D the differential equation (1) can be written as $[D^n+P_1D^{n-1}+P_2D^{n-2}+\dots+P_n]\ y=X.$

Note. It can be proved that D can be treated as an algebraic quantity in several respects.

5.3. A Theorem. *If $y=y_1, y=y_2, \dots, y=y_n$ are linearly independent solutions of*

$$(D^n+a_1D^{n-1}+a_2D^{n-2}+\dots+a^n)\ y=0, \qquad \dots(1)$$

then $y=C_1y_1+C_2y_2+\dots+C_ny_n$ is the general or complete solution of the differential equation, where $C_1, C_2, \dots, C_n$ are n arbitrary constants.

Let us denote the given equation (1) by $f(D)\ y=0$,

where $\quad f(D)=D^n+a_1D^{n-1}+a_2D^{n-2}+\dots+a_n.$

Since $y=y_1, y=y_2, \dots, y=y_n$ are solutions of the equation,

$$\therefore\ f(D)\ y_1=0, f(D)\ y_2=0, \dots, f(D)\ y_n=0. \qquad \dots(2)$$

Now putting $y=C_1y_1+C_2y_2+\dots+C_ny_n$ in (1), we have

$$D^n(C_1y_1+\dots+C_ny_n)+a_1D^{n-1}(C_1y_1+C_2y_2+\dots+C_ny_n)$$
$$+\dots+a_n(C_1y_1+C_2y_2+\dots+C_ny_n)=0$$

or $\quad C_1(D^ny_1+a_1D^{n-1}y_1+\dots+a_n)+C_2(D^ny_2+a_1D^{n-1}y_2+\dots+a_n)$
$$+\dots+C_n(D^ny_n+a_1D^{n-1}y_n+\dots+a_n)=0$$

or $\quad C_1f(D)\ y_1+C_2f(D)\ y_2+\dots+C_nf(D)\ y_n=0$

or $\quad C_1.0+C_2.0+\dots+C_n.0=0$ by (2).

Since (1) is satisfied by $y=C_1y_1+C_2y_2+\ldots+C_ny_n$, it is a solution of (1). Also since it contains n arbitrary constants, it is the general or complete solution of the equation.

5·4. Auxiliary Equation. Consider the differential equation

$$(D^n+a_1D^{n-1}+a_2D^{n-2}+\ldots+a_n)y=0 \qquad \ldots(1)$$

where $a_1, a_2, \ldots, a_n$ are all constants.

Let $y=e^x$ be a solution of this equation. Then putting

$$y=e^{mx},\ Dy=me^{mx},\ D^2y=m^2e^{mx},\ \ldots,\ D^ny=m^ne^{mx},$$

the equation becomes

$$(m^n+a_1m^{n-1}+a_2m^{n-2}+\ldots+a_n)\,e^{mx}=0.$$

Hence e^{mx} will be a solution of (1) if m is a root of the algebraic equation

$$m^n+a_1m^{n-1}+a_2m^{n-2}+\ldots+a_n=0 \qquad \ldots(2)$$

This equation in m is called the *Auxiliary equation.*

Note It is observed that the auxiliary equation $f(m)=0$ gives the same values of m as the equation $f(D)=0$ gives of D.

Hence $f(D)=0$, *i.e.*, $D^n+a_1D^{n-1}+\ldots+a_n=0$ can in general be regarded as the auxiliary equation.

Therefore in practice we do not replace D by m to form the auxiliary equation. The equation in D may be regarded as auxiliary equation.

5·5. Solution of equation (1) of the above article.

[Gujrat B.Sc. (Prin.) 58 ; Gujrat B.Sc. (Subsi.) 65]

Case I. When all the roots of auxiliary equation are real and different.

If $m_1, m_2, \ldots, m_n$ be the n **different** roots of (2), then $y=e^{m_1x}, y=e^{m_2x}, \ldots, y=e^{m_nx}$ are all independent solutions of (1). Therefore the general solution of (1) is

$$y=C_1e^{m_1x}+C_2e^{m_2x}+C_3e^{m_3x}+\ldots+C_ne^{m_nx}.$$

Ex. 1. *Solve* $\dfrac{d^3y}{dx^3}-13\dfrac{dy}{dx}-12y=0.$

Solution Equation is $(D^3-13D-12)\,y=0$.

The auxiliary equation is $(D^3-13D-12=0,$

i.e., $(D+1)(D+3)(D-4)=0,\ D=-1, -3, 4$

Hence the complete solution is

$$y=C_1e^{-x}+C_2e^{-3x}+C_3e^{4x}.$$

Ex. 2. *Solve* $(D^3+6D^2+11D+6)\,y=0.$ **[Delhi Pass 67]**

Solution. A.E. is $(D+1)(D+2)(D+3)=0,\ D=-1, -2, -3.$

The complete solution is

$$y=c_1e^{-x}+c_2e^{-2x}+c_3e^{-3x}.$$

5·6. Case II. Auxiliary equation having equal roots.
[Gujrat B. Sc (Princ.) 59; Poona T.D.C. 61 (S)]

$$\frac{d^3y}{dx^3}+6\frac{d^2y}{dx^2}+11\frac{dy}{dx}+6y=0$$

$$(D^3+6D^2+11D+6)y=0,$$

Let $y=e^{mx}$, then $\frac{dy}{dx}=me^{mx}$, $\frac{d^2y}{dx^2}=m^2e^{mx}$, $\frac{d^3y}{dx^3}=m^3e^{mx}$,

The equation becomes

$$(m^3+6m^2+11m+6)e^{mx}=0$$

Auxiliary equation is

$$(m^3+6m^2+11m+6)=0$$

Roots are $m_1=-1$, $m_2=-2$, $m_3=-3$

The complete solution is

$$y=c_1e^{-x}+c_2e^{-2x}+c_3e^{-3x}$$

We have shown in case I § 5·5, that when $m_1, m_2, \ldots, m_n$ are all different, the general solution is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \ldots + C_n e^{m_n x}.$$

But if $m_1 = m_2$ (two roots equal) then this becomes

$$y = (C_1 + C_2) e^{m_1 x} + C_3 e^{m_3 x} + \ldots + C_n e^{m_n x},$$

which clearly contains only $n-1$ arbitrary constants (since $C_1 + C_2$ is equivalent to only one arbitrary constant)

Therefore this is no longer a general solution.

Consider an equation $(D - m_1)^2 y = 0$, ...(1)

a differential equation of second order having both the roots equal.

Put $(D - m_1) y = v$, then (1) becomes

$$(D - m_1) v = 0 \quad \text{or} \quad \frac{dv}{dx} = m_1 v.$$

Separating the variables, $\dfrac{dv}{v} = m_1\, dx$.

Integrating, $\log v = \log C + m_1 x$ or $v = Ce^{m_1 x}$

or $(D - m_1) y = Ce^{m_1 x}$ as $v = (D - m_1) y$

or $\dfrac{dy}{dx} - m_1 y = Ce^{m_1 x}$

which is a linear equation of the first order, its I.F. $= e^{-mx}$

$$\therefore \quad ye^{-m_1 x} = \int Ce^{m_1 x} e^{-m_1 x}\, dx + C_2$$

or $y = (Cx + C_2) e^{m_1 x}$.

Therefore the most general solution of

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n) y = 0,$$

when two roots of A.E. are equal, is

$$y = (C_1 + C_2 x) e^{m_1 x} + C_3 e^{m_3 x} \ldots + C_n e^{m_n x}.$$

Cor. In case three roots are equal, *i.e.*, $m_1 = m_2 = m_3$, the general solution is

$$y = (C_1 + C_2 x + C_3 x^2) e^{m_1 x} + C_4 e^{m_4 x} + \ldots \quad C_n e^{m_n x}.$$

Ex. 1. *Solve* $\dfrac{d^4 y}{dx^4} - \dfrac{d^3 y}{dx^3} - 9\dfrac{d^2 y}{dx^2} - 11\dfrac{dy}{dx} - 4y = 0.$

Solution. A.E. is $D^4 - D^3 - 9D^2 - 11D - 4 = 0$,

i.e., $(D+1)^3 (D-4) = 0$, $D = -1, -1, -1, 4$.

Hence the general solution is

$$y = (C_1 + C_2 x + C_3 x^2) e^{-x} + C_4 e^{4x}.$$

Ex. 2. *Solve* $(D^3 - 2D^2 - 4D + 8) y = 0$. **(Delhi Pass 1968)**

Solution. Auxiliary equation is

$D^3 - 2D^2 - 4D + 8 = 0$ or $(D+2)(D-2)^2 = 0$,

$D = -2, 2, 2.$

$\therefore \; y = (C_1 + C_2 x) e^{2x} + C_3 e^{-2x}$

$y = (C_1 + C_2 x) e^{2x} + C_3 e^{-2x}$

5·7 Case III. Auxiliary equation having imaginary roots.

Let $\alpha \pm i\beta$ be the imaginary roots of an equation of second order (since imaginary roots occur in pairs).

Then its general solution is $y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$

$$y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$$
$$= e^{\alpha x}[C_1 e^{i\beta x} + C_2 e^{-i\beta x}]$$
$$= e^{\alpha x}[C_1(\cos\beta x + i\sin\beta x) + C_2(\cos\beta x - i\sin\beta x)]$$
$$= e^{\alpha x}[(C_1+C_2)\cos\beta x + (C_1-C_2)\, i\sin\beta x]$$
$$= e^{\alpha x}[A\cos\beta x + B\sin\beta x].$$

Note. The above result after suitably adjusting constants may also be written as

$$y = e^{\alpha x}.A\cos(\beta x + B) \quad \text{or} \quad y = e^{\alpha x}.A\sin(\beta x + B).$$

Imaginary roots repeated. If auxiliary equation has two equal pairs of imaginary roots, *i.e.*, if $\alpha + i\beta$ and $\alpha - i\beta$ occur twice, then general solution is obtained as

$$y = e^{\alpha x}[(C_1 + C_2 x)\cos\beta x + (C_3 + C_4 x]\sin\beta x].$$

Cor. If a pair of roots of the auxiliary equation occur in the form of quadratic surd $\alpha \pm \sqrt{\beta}$, where β is +ive, then the corresponding term in the solution may be written as

$$e^{\alpha x}[C_1\cosh x\sqrt{\beta} + C_2\sinh x\sqrt{\beta}]$$

or $\quad C_1 e^{\alpha x}\cosh(x\sqrt{\beta} + C_2) \quad$ or $\quad C_1 e^{\alpha x}\sinh(x\sqrt{\beta} + C_2).$

Ex. 1. *Solve* $(D^4 + 5D^2 + 6)\,y = 0.$ **(Karnatak M. A. 61)**

Solution Auxiliary equation is $(D^4 + 5D^2 + 6) = 0$,

i.e., $\quad (D^2+3)(D^2+2) = 0 \quad \therefore \quad D = \pm\sqrt{3}i, \pm\sqrt{2}i.$

Hence the complete solution is

$$y = C_1\cos\sqrt{3}x + C_2\sin\sqrt{3}x + C_3\cos\sqrt{2}x + C_4\sin\sqrt{2}x.$$

Ex. 2. *Solve* $(D^4 - D^3 - D + 1)\,x = 0$ **(Gujrat 58)**

$(D^3-1)(D-1)$ **Solution.** Auxiliary equation is $D^4 - D^3 - D + 1 = 0$

or $\quad (D^3-1)(D-1) = 0 \quad$ or $\quad (D-1)^2(D^2+D+1) = 0$

or $$D = 1, 1, -\tfrac{1}{2} \pm \frac{\sqrt{3}}{2}i.$$

Hence the complete solution is

$$y = (C_1 + C_2 x)\,e^x + e^{-x/2}\left[C_3\cos\frac{\sqrt{3}}{2}x + C_4\sin\frac{\sqrt{3}}{2}x\right].$$

Ex. 3. *Solve the differential equation*

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0,$$

a b being constants. **(Delhi Hons. 66)**

Solution. Proceed yourself.

5·8. Synopsis of the forms of solutions

To solve an equation of the from

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n)\,y = 0:$$

1. Find the roots of the auxiliary equation, *viz*. $D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n = 4$.
2. Put the *General Solution* as follows :

Roots of Auxi. Equation	Complete Solution
Case I	
All roots $m_1, m_2, m_3, \dots m_n$ real and different.	$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x}$
Case II	
$m_1 = m_2$ but other roots real and different.	$y = (C_1 + C_2 x) e^{m_1 x} + C_3 e^{m_3 x} + \dots + C_n e^{m_n x}$
Case III (Imag. Roots)	
1. $\alpha \pm i\beta$, a pair of imaginary roots.	Corresponding part of the general solution is $e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$ or $C_1 e^{\alpha x} \cos (\beta x + C_2)$ or $C_1 e^{\alpha x} \sin (\beta x + C_2)$.
2. $(\alpha \pm i\beta), (\alpha \pm i\beta)$ repeated twice.	Corresponding part of general solution is $y = e^{\alpha x}[(C_1 + C_2 x)\cos \beta x + (C_3 + C_4 x)\sin \beta x]$.

Ex. 1. *Solve* $\dfrac{d^4 y}{dx^4} - a^4 y = 0$.

Solution. The auxiliary equation is $(D^4 - a^4) = 0$

or $(D^2 - a^2)(D^2 + a^2) = 0, \; D = \pm a, \pm ai$.

$\therefore$ solution is $y = C_1 e^{ax} + C_2 e^{-ax} + (C_3 \cos ax + C_4 \sin ax)$.

Ex. 2 *Solve* $\dfrac{d^4 y}{dx^4} + m^4 y = 0$. **[Agra B. Sc. 55]**

Solution. Auxiliary equation is $D^4 + m^4 = 0$

or $(D^2 + m^2)^2 - 2m^2 D^2 = 0$

or $(D^2 - \sqrt{2} mD + m^2)(D^2 + \sqrt{2} mD + m^2) = 0$.

When $D^2 - \sqrt{2} mD + m^2 = 0$, $D = \dfrac{m \pm mi}{\sqrt{2}}$.

When $D^2 + \sqrt{2} mD + m^2 = 0$, $D = \dfrac{-m \pm mi}{\sqrt{2}}$.

i.e., roots of auxiliary equation are $\frac{m}{\sqrt{2}}\pm\frac{m}{\sqrt{2}}i, -\frac{m}{\sqrt{2}}\pm\frac{m}{\sqrt{2}}i$.

Hence the general solution is

$$y=e^{(m/\sqrt{2})x}C_1\cos\left(\frac{m}{\sqrt{2}}x+C_2\right)+e^{(-m/\sqrt{2})x}C_3\cos\left(\frac{m}{\sqrt{2}}x+C_4\right).$$

5·9. General solution of $(D^n+a_1D^{n-1}+...a_n)y=X$. ...(1)

[Bombay 61 : Gujrat 52]

To show that if $y=Y$ is a complete solution of

$$(D^n+a_1D^{n-1}+...+a_n)y=0 \quad ...(2)$$

and $y=u$ is a particular solution of (1); then $y=Y+u$ is a general solution of (1). **[Nagpur B.Sc. 55 (S)]**

Since $y=Y$ is a solution of (2), we have

$$(D^n+a_1D^{n-1}+...+a_n)(Y)=0. \quad ...(3)$$

Also since $y=u$ is a solution of (1), we have

$$(D^n+a_1D^{n-1}-...+a_n)u=X. \quad ...(4)$$

Adding (3) and (4), we have

$$(D^n+a_1D^{n-1}+...+a_n)(Y+u)=X.$$

This shows that $y=Y+u$ is a solution of (1). Now Y being a general solution of (2) contains n arbitrary constants and as such $Y+u$ also contains n arbitrary constants. Therefore $y=Y+u$ is a general solution of (1).

Note 1. In the general solution $y=Y+u$ of the equation (1), Y is called the **Complementary Function** (C.F.) and u is called the **Particular Integral (P. I.)** and thus

The General Solution=C.F.+P.I.

2. The solution Y of (2) can be determined by the methods discussed above. The problem is now to find the particular integral u of (1). We give below certain methods of finding u.

Ex. *Define the Complementary Function and Particular Integral for the linear differential equation with constant coefficients $f(D)y=X$.* **[Karnatak 62]**

5·10. Meaning of the symbol $\frac{1}{f(D)}$.

Def. *$\frac{1}{f(D)}X$ is that function of x, free from arbitrary constants, which when operated by $f(D)$ gives X.*

Thus $f(D).\frac{1}{f(D)}X=X.$

Therefore $f(D)$ and $\frac{1}{f(D)}$ are inverse operators (*i e.* they cancel each other's effect on the function on which they operate)

Thus the symbol $\frac{1}{D}$ stands for integration.

5.11. $\frac{1}{f(D)}$ X is the particular integral of f(D) y=X.

Clearly $\frac{1}{f(D)} X$ will be solution of (1) if it satisfies (1).

So putting $\frac{1}{f(D)} X$ for y in (1), we get

$$f(D) \frac{1}{f(D)} X = X \text{ i.e., } X = X, \text{ which is true.}$$

It means that $\frac{1}{f(D)} X$ is a particular solution of (1).

Therefore to find the particular solution of $f(D)\ y=X$, we should find the value of $\frac{1}{f(D)} X$.

Note. We know that in solving $f(D)\ y=0$, $f(D)=0$ forms the auxiliary equation, which can be resolved into linear factors (real or imaginary). Therefore $\frac{1}{f(D)}$ can be resolved into partial fractions. The partial fractions will be of the form $\frac{1}{D-\alpha}$ where α is real or imaginary.

5·12. To show that $\frac{1}{D-\alpha} X = e^{\alpha x} . \frac{1}{D} (e^{-\alpha x} X)$.

Suppose $y = \frac{1}{D-\alpha} X$; then $(D-\alpha)\ y = X$,

or $\frac{dy}{dx} - \alpha y = X$; this is linear in y, as $D \equiv \frac{d}{dx}$.

$\therefore$ Integrating factor $= e^{\int P\,dx} = e^{\int -\alpha\,dx} = e^{-\alpha x}$

and the solution is $ye^{-\alpha x} = \int e^{-\alpha x} X\,dx$.

(constant is not added as it is the particular solution)

or $$y = e^{\alpha x} \int e^{-\alpha x} . X\,dx$$

$$= e^{\alpha x} \frac{1}{D} (Xe^{-\alpha x}) \text{ as } \frac{1}{D} \equiv \text{integration.}$$

5·13. Working rule for finding the Particular integral of f(D) y=X.

Let $f(D) = (D-\alpha_1)(D+\alpha_2)\ldots(D-\alpha_n)$.

Then resolving into partial fraction, we get

$$\frac{1}{f(D)}=\frac{A_1}{D-\alpha_1}+\frac{A_2}{D-\alpha_2}+\ldots+\frac{A_n}{D-\alpha_n} \text{ say.}$$

Now particular integral

$$=\frac{1}{f(D)}X=\left\{\frac{A_1}{D-\alpha_1}+\frac{A_2}{D-\alpha_2}+\ldots+\frac{A_n}{D-\alpha_n}\right\}X$$

$$=A_1\frac{1}{D-\alpha_1}X+A_2\frac{1}{D-\alpha_2}X+\ldots+A_n\frac{1}{D-\alpha_n}X$$

$$=A_1e^{\alpha_1x}\int e^{-\alpha_1x}X\,dx+A_2e^{\alpha_2x}\int e^{-\alpha_2x}X\,dx+\ldots$$

$$+A_ne^{\alpha_nx}\int e^{-\alpha_nx}X\,dx.$$

which can in general be evaluated and thus the particular integral can be found.

Particular Integral in some special cases.

5·14. Particular Integral when $X=e^{ax}$

[Nagpur 61 ; Poona 61 ; Karnatak 61 ; Gujrat 59 ; Bombay 61]

By successive differentiation, we find that

$$e^{ax}=e^{ax}, \quad \ldots(1)$$

$$De^{ax}=ae^{ax} \quad \ldots(2)$$

$$D^2e^{ax}=a^2e^{ax}, \quad \ldots(3)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \quad \ldots$$

$$D^ne^{ax}=a^ne^{ax}. \quad \ldots(n)$$

If $f(D)=(D^n+a_1D^{n-1}+a_2D^{n-2}+\ldots+a_{n-1}D+a_n)$, then multiplying (1), (2), (3)......(n) by $a_n, a_{n-1}, \ldots$, 1 respectively and adding, we obtain

$$f(D)\,e^{ax}=f(a)\,e^{ax}$$

Now operating on both the sides by $\frac{1}{f(D)}$,

$$\frac{1}{f(D)}f(D)\,e^{ax}=\frac{1}{f(D)}f(a)\,e^{ax}$$

or $e^{ax}=f(a)\frac{1}{f(D)}e^{ax}$ or $\frac{1}{f(a)}e^{ax}=\frac{1}{f(D)}e^{ax}$,

dividing by $f(a)\neq 0$

Therefore $\frac{1}{\mathbf{f(D)}}e^{ax}=\frac{1}{\mathbf{f(a)}}e^{ax}$, provided that $f(a)\neq 0$.

Ex. 1. *Solve* $\frac{d^2y}{dx^2}-2k\frac{dy}{dx}+k^2y=0.$ **[Nagpur 1957]**

Solution. Auxiliary equation is $D^2-2kD+k^2=0$,

i.e., $(D-k)^2=0$ or $D=k, k.$

$\therefore$ C F.$=(C_1+C_2x)\,e^{kx}$.

5.11. $\frac{1}{f(D)}X$ is the particular integral of $f(D)y = X$

5.14. $\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$ Provided that $f(a) \neq 0$

Ex. $(D^2 - 2kD + k^2)y = e^x$

Auxiliary equation is

$$(m^2 - 2km + k^2) = 0$$

Roots are $m_1 = k$, $m_2 = k$

$$CF = (c_1 + c_2 x)e^{kx}$$

$$P.I = \frac{1}{D^2 - 2kD + k^2}e^x = \frac{1}{1 - 2k + k^2}e^x = \frac{1}{(1-k)^2}e^x$$

Hence the general solution is

$$y = (c_1 + c_2 x)e^{kx} + \frac{1}{(1-k)^2}e^x \quad , k \neq 1$$

5.15. $\frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax$ if $f(-a^2) \neq 0$

Similarly: $\frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax$ if $f(-a^2) \neq 0$

Ex. $(D^2 + D + 1)y = \sin 2x$

Auxiliary equation is

$$(m^2 + m + 1) = 0$$

Roots are $m = -\frac{1}{2} \pm \frac{1}{2} i\sqrt{3}$

$$CF = e^{-\frac{1}{2}x} c_1 \cos(\frac{1}{2}\sqrt{3}x + c_2)$$

$$P.I = \frac{1}{D^2 + D + 1} \sin 2x = \frac{1}{-4 + D + 1} \sin 2x = \frac{1}{(D-3)} \sin 2x$$

$$= \frac{D+3}{D^2 - 9} \sin 2x = -\frac{1}{13}(2\cos 2x + 3\sin 2x)$$

Hence the general solution is

$$y = C.F + P.I$$