18 April, 2023

Group Assignment of MAT-111

Patuakhali Science and Technology University

Submitted To:

Muhammad Masudur Rahman

- Associate Professor of Mathematics Department.
- Faculty of Computer Science and Engineering.

Submitted By:

Name	ID No.
Sadman Hafiz Shuvo	2102021
Noshin Nazia	2102022
Md. Senarul Islam	2102023
Md. Sharafat Karim	2102024
Seemanta Shill	2102025

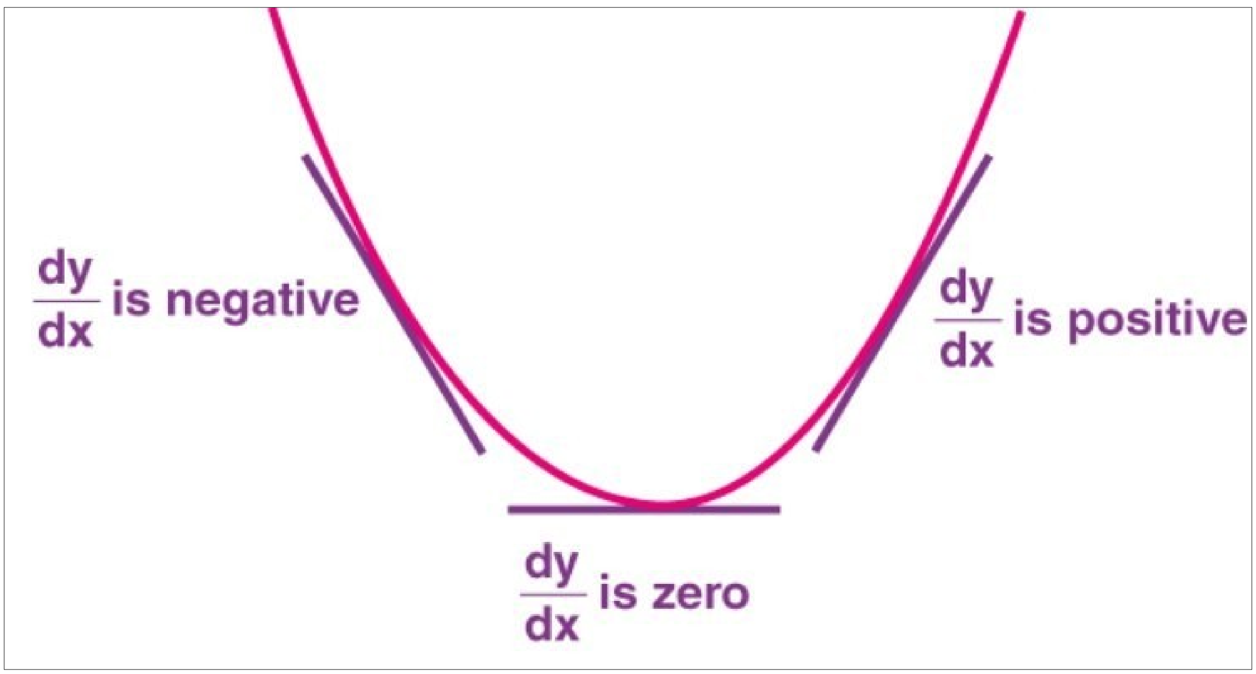
Topic Name:

Maxima and Minima. (differential calculus)

Maxima and Minima

Maxima and minima of calculus are found by using the concept of derivatives. As we know the concept of derivatives gives us the information regarding the gradient/slope of the function. We locate the points where the gradient is zero and these points are called turning points or stationary points. These are points associated with the largest or smallest values (locally) of the function.

Maxima and Minima Points:



From the figure one it gets zero between negative and positive values. So, it can be said dy/dx is -ve before stationary point and likewise dy/dx is +ve after stationary point. Hence it can be said d^2y/dx^2 is positive at the stationary point. So, whenever we'll see the double derivative is positive, it is the point of minima. Vice versa whenever the double derivative is negative, it is the point of maxima.

Stationary points vs Turning points:

Stationary points are the points where the slope of the graph becomes zero. In other words, the tangent of the function becomes horizontal dy/dx=0.

Let f be a function defined on an open interval I.

Let f be a continuous at critical point c in I.

If f'(x) doesn't change sign as x increases through c, then c is neither a point of local minima nor maxima and it is not a turning point. It is called point of inflection.

Let C be an interior point of the interval of the definitions of the function *f(x)* and let,

$$f'(c) = f''(c) = \ldots = f^{n-1}(c) = 0, \text{ and } f^{n}(c) \neq 0$$

then, (I) If n be even, *f(c)* is a maximum or a minimum according as f^n is negative or positive,

or, (II) If n be odd, *f(c)* is neither a maximum nor a minimum.

Derivative Test:

Let f be the function defined on an interval I and it is two times differentiable at c.

a) $x=c$ will be a point of local maxima if $f'(c)=0$ and $f''(c) <0$.
b) $x=c$ will be a point of local minima if $f'(c)=0$ and $f''(c) >0$. Then f(c) will have local minimum value.
c) When both $f'(c)=0$ and $f''(c)=0$,the test fails. And that first derivative test will give you the value of local maxima and minima.

Examples:

Example 1: Find the turning points of the function $y = 4x^3 + 12x^2 + 12x + 10$.

Solution: For turning points $dy/dx = 0$.

$dy/dx = 12x^2 + 24x + 12 = 0$

$\Rightarrow 3x^2 + 6x + 3 = 0$

$\Rightarrow (x + 1)(3x + 1) = 0$

$\Rightarrow x= -1$ and $x = (-1)/3$

Second derivative test:

At $x = -1$:

$d^2 y/dx^2 = 24x + 24 = 24(-1) + 24 = -24 + 24 = 0.$

Hence $x = -1$ is the point of inflection, it is a non-turning point.

At $x = (-1)/3$:

$d^2y/dx^2 = 24x + 24 = 24((-1)/3) + 24 = -8 + 24 = 16.$

Hence $x = (-1)/3$ is a point of minima, it is a turning point.

Example 2: Find the local maxima and minima of the function $f(x) = 3x^4 + 4x^3 – 12x^2 + 12.$

Solution:

For stationary points, $f'(x) = 0$.

$f'(x)= 12x^3 + 12x^2 – 24x = 0$

$\Rightarrow 12x(x^2 + x – 2) = 0$

$\Rightarrow 12x(x – 1)(x + 2) = 0$

=> Hence, $x = 0$, $x = 1$ and $x = -2$

Second derivative test:

$f''(x) = 36x^2 + 24x – 24$

$f''(x) = 12(3x^2 + 2x – 2)$

At $x = -2$

$f''(-2) = 12(3(-2)^2 + 2(-2) – 2) = 12 (12 – 4 – 2) = 12(6) = 72 > 0$

At $x = 0$

$f''(0) = 12(3(0)^2 + 2(0) – 2) = 12(-2) = -24 < 0$

At $x = 1$

$f''(1) = 12(3(1)^2 + 2(1) – 2) = 12(3+2-2) = 12(3) = 36 > 0$

Therefore, by the second derivative test, $x=0$ is the point of local maxima, while $x = -2$ and $x = 1$ are the points of local minima.

Example 3: Find the extrema of the given function: $f(x) = -3x^2 + 4x + 7$ and the extremum value using maxima and minima formulas.

Solution:

Using second order derivative test for the maxima and minima of a function:

Given function: $f(x) = -3x^2 + 4x + 7$ ----------------(eq 1)

Differentiate on both sides of (eq 1), w.r.t x.

$\Rightarrow dy/dx = d(-3x^2)/dx + d(4x)/dx + d(7)/dx$

$\Rightarrow dy/dx = - 6x + 4$

Putting $dy/dx = 0$ to find critical points.

$\Rightarrow -6x + 4 = 0$ ----------------(eq 2)

$\Rightarrow x = 2/3$

The critical point is 2/3.

Differentiate both sides of (eq 2), w.r.t x.

$\Rightarrow d^2y/dx^2 = d(-6x)/dx + d(4)/dx$

$\Rightarrow d^2y/dx^2 = -6$

Since $d^2y/dx^2 < 0$, the given curve will have maxima at $x = 2/3$.

The maxima value of $f(x)$ at $x = 2/3$ is,

$f(2/3) = -3(2/3)^2 + 4(2/3) + 7 = -4/3 + 8/3 + 7 = 25/3$

Example 4: A stone is thrown in the air. Its height at any time t is given by $h = -5t^2+10t+4$.

Find its maximum height.

Solution:

Given $h = -5t^2+10t+4$

$dh/dt = -10t+10$

Now find when $dh/dt = 0$

$dh/dt = 0 \Rightarrow -10t+10 = 0$

$\Rightarrow -10t = -10$

$t = 10/10 = 1$

Height at $t = 1$ is given by $h = -5\times1^2+10\times1+4$

$= -5+10+4$

$= 9$

Hence, the maximum height is 9 m.

Example 5: Find the maxima and minima for $f(x) = 2x^3-21x^2+36x-15$

Solution:

We have $f(x) = 2x^3-21x^2+36x-15$

$f'(x) = 6x^2-42x+36$

Now find the points where $f'(x) = 0$

$f'(x) = 0 \Rightarrow 6x^2-42x+36 = 0$

$\Rightarrow x^2-7x+6 = 0$

$\Rightarrow (x-6)(x-1) = 0$

$\Rightarrow$ x = 6 or x = 1 are the possible points of minima or maxima.

Let us test the function at each of these points.

$f''(x) = 12x-42$

At x = 1, $f''(1) = 12-42 = -30 < 0$

Therefore x = 1 is a point of the local maximum.

The maximum value is $f(1) = 2-21+36-15 = 2$

At x= 6, $f''(x) = 12\times6-42 = 30 > 0$

Therefore, x = 6 is a point of the local minimum.

The local minimum value is

$f(6) = 2(6)^3-21(6)^2+36(6)-15$

$= 2\times216-21\times36+216-15$

$= 432-756+216-15$

$= -123$

Example 6: Find the points of maxima and minima of a function:$y = 2x^3 - 3x^2 + 6$

Solution

Given function: $y = 2x^3 - 3x^2 + 6$

Using second order derivative test for the maxima and minima of a function:

Taking first order derivative of:

$y = 2x^3 - 3x^2 + 6$ ----- (eq 1)

Differentiate both of sides (eq 1), w.r.t x.

$\Rightarrow dy/dx = d(2x^3)/dx - d(3x^2)/dx + d(6)/dx$

$\Rightarrow dy/dx = 6x^2 - 6x + 0$

$\Rightarrow dy/dx = 6x^2 - 6x$ ------ (eq 2)

Putting $dy/dx = 0$ to find critical points,

$\Rightarrow 6x^2 - 6x = 0$

$\Rightarrow 6x (x - 1) = 0$

$\Rightarrow x = 0,1$

The critical points are 0 & 1.

Differentiate both of sides of (eq 2), w.r.t x.

$\Rightarrow d^2y/dx^2 = d(6x^2)/dx - d(6x)/dx$

$\Rightarrow d^2y/dx^2 = 12x - 6$

Now, put the values of x and find the max or min value.

At $x = 0$, $d^2y/dx^2 = 12(0) - 6 = -6 < 0$, hence $x = 0$ is a point of maxima

At $x = 1$, $d^2y/dx^2 = 12(1) - 6 = 6 > 0$, hence $x = 1$ is a point of minima

Example 7: What is the value of the function $(x - 1)(x - 2)^2$ at its maxima?

Solution:

Given $f(x) = (x - 1)(x - 2)^2$

$f(x) = (x - 1)(x^2 + 4 - 4x)$;

$f(x) = (x^3 - 5x^2 + 8x - 4)$

Now $f'(x) = 3x^2 - 10x + 8$, $f'(x) = 0$

$3x^2 - 10x + 8 = 0$

$(3x - 4)(x - 2) = 0$

$x = 4 / 3, 2$

Now $f''(x) = 6x - 10$

$f''(4 / 3) = 6 \times [4 / 3] - 10 < 0$

$f''(2) = 12 - 10 > 0$

Hence, at $x = 4 / 3$ the function will occupy maximum value.

$\therefore$ Maximum value = $f(4 / 3) = 4 / 27$

Example 8: Find the maximum value of function $x^3 - 12x^2 + 36x + 17$ in the interval $[1, 10]$.

Solution:

Let $f(x) = x^3 - 12x^2 + 36x + 17$

$\therefore f'(x) = 3x^2 - 24x + 36 = 0$ at $x = 2, 6$

Again $f''(x) = 6x - 24$ is negative at $x = 2$

So that $f(6) = 17$, $f(2) = 49$

At the end points = $f(1) = 42$, $f(10) = 177$

So, that $f(x)$ has its maximum value as 177.