প্রশ্নমালা–4(A)

1(i). $I=\int_0^{\pi/2}\frac{\sin x\,dx}{\sin x+\cos x}\cdots(1)$

বা $I=\int_0^{\pi/2}\frac{\sin(\pi/2-x)dx}{\sin(\pi/2-x)+\cos(\pi/2-x)}$

বা $I=\int_0^{\pi/2}\frac{\cos x\,dx}{\cos x+\sin x}\cdots(2)$

$\therefore (1)+(2)\Rightarrow 2I=\int_0^{\pi/2}\frac{\sin x+\cos x}{\sin x+\cos x}dx$

বা $2I=\int_0^{\pi/2}dx=[x]_0^{\pi/2}=\frac{\pi}{2}$

$\Rightarrow I=\frac{\pi}{4}.$

(ii). $I=\int_0^{\pi/2}\frac{\cos x\,dx}{\sin x+\cos x}\cdots(1)$

বা $I=\int_0^{\pi/2}\frac{\cos(\pi/2-x)dx}{\sin(\pi/2-x)+\cos(\pi/2-x)}$

বা $I=\int_0^{\pi/2}\frac{\sin x\,dx}{\cos x+\sin x}\cdots(2)$

$\therefore (1)+(2)\Rightarrow 2I=\int_0^{\pi/2}\frac{\cos x+\sin x}{\cos x+\sin x}dx$

বা $2I=\int_0^{\pi/2}dx=[x]_0^{\pi/2}=\frac{\pi}{2}.$

$\Rightarrow I=\frac{\pi}{4}.$

(iii). $I=\int_0^{\pi/2}\frac{\sqrt{\cos x}\,dx}{\sqrt{\sin x}+\sqrt{\cos x}}\cdots(1)$

বা $I=\int_0^{\pi/2}\frac{\sqrt{\cos(\pi/2-x)}dx}{\sqrt{\sin(\pi/2-x)}+\sqrt{\cos(\pi/2-x)}}$

বা $I=\int_0^{\pi/2}\frac{\sqrt{\sin x}\,dx}{\sqrt{\cos x}+\sqrt{\sin x}}\cdots(2)$

$\therefore (1)+(2)\Rightarrow 2I=\int_0^{\pi/2}\frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$

বা $2I = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}$

$\therefore I = \frac{\pi}{4}$.

(iv). $I = \int_0^{\pi/2} \frac{(\sin x)^{3/2}\, dx}{(\sin x)^{3/2} + (\cos x)^{3/2}} \cdots (1)$

বা $I = \int_0^{\pi/2} \frac{\{\sin(\pi/2 - x)\}^{3/2}\, dx}{\{\sin(\pi/2 - x)\}^{3/2} + \{\cos(\pi/2 - x)\}^{3/2}}$

বা $I = \int_0^{\pi/2} \frac{(\cos x)^{3/2}\, dx}{(\cos x)^{3/2} + (\sin x)^{3/2}} \cdots (2)$

$\therefore (1) + (2) \Rightarrow 2I = \int_0^{\pi/2} \frac{(\sin x)^{3/2} + (\cos x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}} dx$

বা $2I = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}$

$\therefore I = \frac{\pi}{4}$

(v). $I = \int_0^{\pi/2} \frac{\sin^n x\, dx}{\sin^n x + \cos^n x} \cdots (1)$

বা $I = \int_0^{\pi/2} \frac{\{\sin(\pi/2 - x)\}^n\, dx}{\{\sin(\pi/2 - x)\}^n + \{\cos(\pi/2 - x)\}^n}$

বা $I = \int_0^{\pi/2} \frac{\cos^n x\, dx}{\cos^n x + \sin^n x} \cdots (2)$

$\therefore (1) + (2) \Rightarrow 2I = \int_0^{\pi/2} \frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x} dx = \int_0^{\pi/2} dx$

বা $2I = [x]_0^{\pi/2} = \frac{\pi}{2}$

$\therefore I = \frac{\pi}{4}$.

2(i). $I = \int_0^{\pi/2} \frac{dx}{1 + \frac{\sin x}{\cos x}} = \int_0^{\pi/2} \frac{\cos x\, dx}{\cos x + \sin x} = \frac{\pi}{4}$ [1(ii) নং দেখুন।]

(ii). $I = \int_0^{\pi/2} \frac{dx}{1 + \sqrt{\cot x}} = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} = \frac{\pi}{4}$ [উদা–1 ঃ নং দেখুন

(iii). $I = \int_0^{\pi/2} \frac{dx}{1 + \sqrt{\sin x}/\sqrt{\cos x}} = \int_0^{\pi/2} \frac{\sqrt{\cos x}\,dx}{\sqrt{\cos x} + \sqrt{\sin x}} = \frac{\pi}{4}$

[উদা 1(iii) নং দেখুন ।]

(iv). $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}/\sqrt{\cos x}}{1 + \sqrt{\sin x}/\sqrt{\cos x}}dx = \int_0^{\pi/2} \frac{\sqrt{\sin x}\,dx}{\sqrt{\cos x} + \sqrt{\sin x}}$

$= \frac{\pi}{4}$; [উদা–1 দেখুন ।]

(v). $I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}} \cdots (1)$

বা $I = \int_0^{\pi/2} \frac{\sqrt{\tan(\pi/2 - x)}dx}{\sqrt{\tan(\pi/2 - \pi)} + \sqrt{\cot(\pi/2 - x)}}$

বা $I = \int_0^{\pi/2} \frac{\sqrt{\cot x}\,dx}{\sqrt{\cot x} + \sqrt{\tan x}} \cdots (2)$

$\therefore (1) + (2) \Rightarrow 2I = \int_0^{\pi/2} \frac{\sqrt{\tan x} + \sqrt{\cot x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx = \int_0^{\pi/2} dx$

বা $2I = [x]_0^{\pi/2} = \frac{\pi}{2}$.

$\therefore I = \frac{\pi}{4}$.

(vi). $I = \int_0^{\pi/2} \frac{(\cot x)^{1/2}}{1 + (\cot x)^{1/2}} dx = \int_0^{\pi/2} \frac{\sqrt{\cos x}/\sqrt{\sin x}\,dx}{1 + \sqrt{\cos x}/\sqrt{\sin x}}$

বা $I = \int_0^{\pi/2} \frac{\sqrt{\cos x}\,dx}{\sqrt{\sin x} + \sqrt{\cos x}} \cdots (1)$

$\Rightarrow I = \frac{\pi}{4}$; [1(iii) নং দেখুন ।]

3(i). $I = \int_0^{\pi} \frac{(x\sin x)/\cos x.\ dx}{1/\cos x + (\sin x)/\cos x} = \int_0^{\pi} \frac{x\sin x\,dx}{1 + \sin x} \cdots (1)$

$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)dx}{1 + \sin(\pi - x)} = \int_0^{\pi} \frac{(\pi - x)\sin x\,dx}{1 + \sin x} \cdots (2)$

$\therefore (1) + (2) \Rightarrow 2I = \int_0^{\pi} \frac{(x + \pi - x)\sin x\,dx}{1 + \sin x} = \pi \int_0^{\pi} \frac{\sin x}{1 + \sin x} dx$

বা $2I = \pi \int_0^{\pi} \frac{(1 + \sin x) - 1}{1 + \sin x} dx = \pi \int_0^{\pi} dx - \pi \int_0^{\pi} \frac{dx}{1 + \sin x}$

বা $2I=\pi[x]_0^{\pi}-2\pi=\pi^2-2\pi;$ [উদা-3 দেখুন।]

$$\Rightarrow I=\frac{1}{2}\pi(\pi-2)=\pi\left(\frac{1}{2}\pi-1\right)$$

(ii). $I=\int_0^{\pi}\frac{x\sin x}{1+\sin x}dx.$ [3(i) নং দেখুন।]

(iii). $I=\int_0^{\pi/2}\frac{\sin^2x\,dx}{1+\sin x\cos x}\ \cdots (1)$

বা $I=\int_0^{\pi/2}\frac{\{\sin(\pi/2-x)\}^2\,dx}{1+\sin(\pi/2-x)\cos(\pi/2-x)}$

বা $I=\int_0^{\pi/2}\frac{\cos^2x\,dx}{1+\cos x\sin x}\ \cdots (2)$

এখন $(1)+(2)\Rightarrow 2I=\int_0^{\pi/2}\frac{\sin^2x+\cos^2x}{1+\sin x\cos x}dx=\int_0^{\pi/2}\frac{dx}{1+\frac{1}{2}\sin 2x}$

বা $2I=\int_0^{\pi/2}\frac{2dx}{2+\sin 2x}=\int_0^{\pi/2}\frac{2dx}{2+\frac{2\tan x}{1+\tan^2x}}$

$$=2\int_0^{\pi/2}\frac{\sec^2x\,dx}{2\tan^2x+2\tan x+2}$$

ধরি $\tan x=z$ তবে $\sec^2x\,dx=dz$

সীমাঃ যদি $x=\frac{\pi}{2}$ হয়, তবে $z=\infty$

যদি $x=0$ হয়, তবে $z=0$

$2I=\int_0^{\infty}\frac{dz}{z^2+z+1}=\int_0^{\infty}\frac{dz}{(z+1/2)^2+(\sqrt{3}/2)^2}$

$$=\frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{z+1/2}{\sqrt{3}/2}\right]_0^{\infty}$$

বা $2I=\frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{2z+1}{\sqrt{3}}\right]_0^{\infty}=\frac{2}{\sqrt{3}}\left[\tan^{-1}\infty-\tan^{-1}\frac{1}{\sqrt{3}}\right]$

বা $2I=\frac{2}{\sqrt{3}}\left[\frac{\pi}{2}-\frac{\pi}{6}\right]=\frac{2}{\sqrt{3}}\cdot\frac{\pi}{3}\Rightarrow I=\frac{\pi}{3\sqrt{3}}.$

(iv). $I = \int_0^{\pi/2} \frac{(\sin x - \cos x)}{1 + \sin x \cos x} dx \cdots (1)$

বা $I = \int_0^{\pi/2} \frac{\sin(\pi/2 - x) - \cos(\pi/2 - x)}{1 + \sin(\pi/2 - x) \cdot \cos(\pi/2 - x)} dx$

বা $I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \cos x \sin x} dx \cdots (2)$

এখন (1) + (2) $\Rightarrow 2I = \int_0^{\pi/2} \frac{(\sin x - \cos x + \cos x - \sin x)}{1 + \sin x \cos x} dx$

বা $2I = \int_0^{\pi/2} 0 \, dx = 0$

$\therefore I = 0.$

(v). $I = \int_0^{\pi} \frac{x \sin x \, dx}{1 + \cos^2 x} \cdots (1)$

বা $I = \int_0^{\pi} \frac{(\pi - x) \sin(\pi - x)}{1 + \{\cos(\pi - x)\}^2} dx = \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \cos^2 x} dx \cdots (2)$

$\therefore$ (1) + (2) $\Rightarrow 2I = \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx$

$= 2\pi \int_0^{\pi/2} \frac{\sin x \, dx}{1 + \cos^2 x}$; $f(\pi - x) = f(x)$ ধর্ম দ্বারা।

ধরি $\cos x = z$ তবে $-\sin x \, dx = dz$

সীমা ঃ যদি $x = 0$ হয়, তবে $z = \cos 0 = 1$

যদি $x = \frac{\pi}{2}$ হয়, তবে $z = \cos\frac{\pi}{2} = 0$

$\therefore I = -\pi \int_1^0 \frac{dz}{1 + z^2} = \pi \int_0^1 \frac{dz}{1 + z^2} = \pi[\tan^{-1} z]_0^1 = \frac{\pi^2}{4}.$

4(i). $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx \cdots (1)$

বা $I = \int_0^{\pi/2} \frac{\{\sin(\pi/2 - x)\}^2 \, dx}{\sin(\pi/2 - x) + \cos(\pi/2 - x)}$

বা $I = \int_0^{\pi/2} \frac{\cos^2 x \, dx}{\cos x + \sin x} \cdots (2)$

$\therefore$ (1) + (2) $\Rightarrow 2I = \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$

বা $2I = \frac{2}{\sqrt{2}} \ln(\sqrt{2} + 1) \Rightarrow I = \frac{1}{\sqrt{2}} \ln(\sqrt{2} + 1)$; [উদা–4 দেখুন।]

(ii). $I = \int_0^{\pi/2} \frac{\cos^2 x}{\sin x + \cos x} dx \cdots (1)$

বা $I = \int_0^{\pi/2} \frac{\{\cos(\pi/2 - x)\}^2 dx}{\sin(\pi/2 - x) + \cos(\pi/2 - x)}$

বা $I = \int_0^{\pi/2} \frac{\sin^2 x \, dx}{\cos x + \sin x} \cdots (2)$

$\therefore (1) + (2) \Rightarrow 2I = \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$

বা $2I = \frac{2}{\sqrt{2}} \ln(\sqrt{2} + 1) \Rightarrow I = \frac{1}{\sqrt{2}} \ln(\sqrt{2} + 1)$ [উদা–4 দেখুন

(iii). $I = \int_0^{\pi/2} \frac{x \, dx}{\sec x + \operatorname{cosec} x} \cdots (1)$

বা $I = \int_0^{\pi/2} \frac{(\pi/2 - x) dx}{\sec(\pi/2 - x) + \operatorname{cosec}(\pi/2 - x)}$

বা $I = \int_0^{\pi/2} \frac{(\pi/2 - x) \, dx}{\operatorname{cosec} x + \sec x} \cdots (2)$

$\therefore (1) + (2) \Rightarrow 2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{dx}{1/\cos x + 1/\sin x} = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin x . \cos x \, dx}{\sin x + \cos x}$

বা $2I = \frac{\pi}{4} \int_0^{\pi/2} \frac{(\sin x + \cos x)^2 - 1}{\sin x + \cos x} dx$

বা $2I = \frac{\pi}{4} \int_0^{\pi/2} \left[(\sin x + \cos x) - \frac{1}{\sin x + \cos x} \right] dx$

বা $2I = \frac{\pi}{4} [-\cos x + \sin x]_0^{\pi/2} - \frac{\pi}{4} \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$

বা $2I = \frac{\pi}{4} [(-0 + 1) - (-1 + 0)] - \frac{\pi}{4} \cdot \frac{1}{\sqrt{2}} \ln \frac{\sqrt{2} + 1}{\sqrt{2} - 1}$ [উদা–4 দেখু

বা $2I = \frac{\pi}{2} - \frac{\pi}{4\sqrt{2}} \ln \frac{(\sqrt{2} + 1)(\sqrt{2} - 1)}{(\sqrt{2} - 1)(\sqrt{2} - 1)}$

বা $2I = \frac{\pi}{2} - \frac{\pi}{4\sqrt{2}} \ln \frac{1}{(\sqrt{2} - 1)^2}$

বা $2I = \frac{\pi}{2} + \frac{\pi}{4\sqrt{2}} \ln(\sqrt{2} - 1)^2 = \frac{\pi}{2} + \frac{2\pi}{4\sqrt{2}} \ln(\sqrt{2} - 1)$

$\Rightarrow I = \frac{\pi}{4} \left[1 + \frac{1}{\sqrt{2}} \ln(\sqrt{2} - 1) \right]$.

5(i). $I = \int_0^{\pi} \frac{x\,dx}{a^2 - \cos^2 x} \cdots (1)$

বা $I = \int_0^{\pi} \frac{(\pi - x)\,dx}{a^2 - \{\cos(\pi - x)\}^2} = \int_0^{\pi} \frac{(\pi - x)\,dx}{a^2 - \cos^2 x} \cdots (2)$

$\therefore (1) + (2) \Rightarrow 2I = \int_0^{\pi} \frac{(x + \pi - x)dx}{a^2 - \cos^2 x} = \pi \int_0^{\pi} \frac{dx}{a^2 - \cos^2 x}$

বা $2I = 2\pi \int_0^{\pi/2} \frac{dx}{a^2 - \cos^2 x} = 2\pi \int_0^{\pi/2} \frac{\sec^2 x\,dx}{a^2 \sec^2 x - 1}$

বা $2I = 2\pi \int_0^{\pi/2} \frac{\sec^2 x\,dx}{a^2 - 1 + a^2 \tan^2 x}$

ধরি $a \tan x = t$ তবে $a \sec^2 x\,dx = dt$.

সীমা : যদি $x = \pi/2$ হয়, তবে $t = \infty$

যদি $x = 0$ হয়, তবে $t = 0$

$\therefore I = \frac{\pi}{a} \int_0^{\infty} \frac{dt}{(\sqrt{a^2 - 1})^2 + t^2} = \frac{\pi}{a\sqrt{a^2 - 1}} \left[\tan^{-1} \frac{t}{\sqrt{a^2 - 1}}\right]_0^{\infty}$

$= \frac{\pi}{a\sqrt{a^2 - 1}} \cdot \frac{\pi}{2} = \frac{\pi^2}{2a\sqrt{a^2 - 1}}$.

(ii). $I = \int_0^{\pi} \frac{x\,dx}{a^2 \cos^2 x + b^2 \sin^2 x} \cdots (1)$

বা $I = \int_0^{\pi} \frac{(\pi - x)\,dx}{a^2 \{\cos(\pi - x)\}^2 + b^2 \{\sin(\pi - x)\}^2}$

বা $I = \int_0^{\pi} \frac{(\pi - x)\,dx}{a^2 \cos^2 x + b^2 \sin^2 x} \cdots (2)$

$\therefore (1) + (2) \Rightarrow 2I = \int_0^{\pi} \frac{\pi\,dx}{a^2 \cos^2 x + b^2 \sin^2 x}$

বা $2I = 2\pi \int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$

বা $2I = 2\pi \int_0^{\pi/2} \frac{\sec^2 x\,dx}{a^2 + b^2 \tan^2 x}$

ধরি $b \tan x = t$ তবে $b \sec^2 x\,dx = dt$.

সীমা : যদি $x = \frac{\pi}{2}$ হয়, তবে $t = \infty$ এবং যদি $x = 0$ হয়, তবে $t = 0$.

$\therefore I = \frac{\pi}{b} \int_0^{\infty} \frac{dt}{a^2 + t^2} = \frac{\pi}{ba} \left[\tan^{-1} \frac{t}{a}\right]_0^{\infty} = \frac{\pi}{ba}\left[\frac{\pi}{2} - 0\right] = \frac{\pi^2}{2ab}$.

(iii). $I = \int_0^{\pi} \frac{x\, dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \cdots (1)$

বা $I = \int_0^{\pi} \frac{(\pi - x)\, dx}{[a^2 \{\cos(\pi - x)\}^2 + b^2 \{\sin(\pi - x)\}^2]^2}$

বা $I = \int_0^{\pi} \frac{(\pi - x)\, dx}{[a^2 \cos^2 x + b^2 \sin^2 x]^2} \cdots (2)$

$\therefore (1) + (2) \Rightarrow 2I = \pi \int_0^{\pi} \frac{dx}{[a^2 \cos^2 x + b^2 \sin^2 x]^2}$

বা $2I = 2\pi \int_0^{\pi/2} \frac{\sec^4 x\, dx}{(a^2 + b^2 \tan^2 x)^2}$

বা $2I = 2\pi \int_0^{\pi/2} \frac{(1 + \tan^2 x) \sec^2 x\, dx}{(a^2 + b^2 \tan^2 x)^2}$

ধরি $b\tan x = a\tan\varphi$, তবে $b\sec^2 x\, dx = a\sec^2\varphi\, d\varphi$

সীমা ঃ যদি $x = \frac{\pi}{2}$ হয়, তবে $a\tan\varphi = \infty \Rightarrow \varphi = \frac{\pi}{2}$

যদি $x = 0$ হয়, তবে $\varphi = 0$.

$$I = \frac{\pi a}{b} \int_0^{\pi/2} \frac{\left[1 + \left(\frac{a^2}{b^2}\right) \tan^2\varphi\right] \sec^2\varphi\, d\varphi}{(a^2 \sec^2\varphi)^2}$$

$$= \frac{\pi a}{b^3 a^4} \int_0^{\pi/2} \frac{(b^2 + a^2 \tan^2\varphi)\, d\varphi}{\sec^2\varphi}$$

বা $I = \frac{\pi}{a^3 b^3} \int_0^{\pi/2} \left(b^2 + a^2 \frac{\sin^2\varphi}{\cos^2\varphi}\right) \cos^2\varphi\, d\varphi$

$$= \frac{\pi}{a^3 b^3} \int_0^{\pi/2} (b^2 \cos^2\varphi + a^2 \sin^2\varphi)\, d\varphi$$

$$= \frac{\pi}{a^3 b^3} \left[\frac{b^2\, \Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{1}{2}\right)}{2\, \Gamma(2)} + a^2 \frac{\Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{1}{2}\right)}{2\Gamma(2)}\right]$$

$$= \frac{\pi}{a^3 b^3} \frac{(b^2 + a^2) \frac{1}{2} \sqrt{\pi} \sqrt{\pi}}{2.1} = \frac{\pi^2 (a^2 + b^2)}{4a^3 b^3}.$$

6(i). $I = \int_0^{\pi/2} \ln(\tan x)\, dx \cdots (1)$

বা $I = \int_0^{\pi/2} \ln\left\{\tan\left(\frac{\pi}{2} - x\right)\right\} dx$

$= \int_0^{\pi/2} \ln(\cot x)\, dx \cdots (2)$

$\therefore (1) + (2) \Rightarrow 2I = \int_0^{\pi/2} [\ln(\tan x) + \ln(\cot x)]\, dx$

$= \int_0^{\pi/2} \ln(\tan x \cot x)\, dx$

$= \int_0^{\pi/2} \ln 1 .\, dx = \int_0^{\pi/2} 0\, dx = 0 \Rightarrow I = 0.$

(ii). ধরি $I = \int_0^{\pi/2} \ln(\tan x + \cot x)\, dx$

বা $I = \int_0^{\pi/2} \ln\left[\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right] dx = \int_0^{\pi/2} \ln\left[\frac{\sin^2 x + \cos^2 x}{\cos x \sin x}\right] dx$

বা $I = \int_0^{\pi/2} \ln\left[\frac{2}{\sin 2x}\right] dx = \int_0^{\pi/2} [\ln 2 - \ln(\sin 2x)]\, dx$

বা $I = \ln 2 \int_0^{\pi/2} dx - \int_0^{\pi/2} \ln(\sin 2x)\, dx$

বা $I = \ln 2\, [x]_0^{\pi/2} - I_1 \cdots (1)$

যখন $I_1 = \int_0^{\pi/2} \ln(\sin 2x)\, dx$

ধরি $2x = z$ তবে $dx = \frac{1}{2} dz$

সীমা : যদি $x = 0$ হয়, তবে $z = 0$

যদি $x = \frac{\pi}{2}$ হয়, তবে $z = \pi$.

$\therefore I_1 = \frac{1}{2} \int_0^{\pi} \ln(\sin z)\, dz = \frac{1}{2} \cdot \int_0^{\pi} \ln(\sin x)\, dx$

$= \frac{1}{2} \cdot 2 \int_0^{\pi/2} \ln(\sin x)\, dx = \frac{\pi}{2} \ln\frac{1}{2}$; [উদা–6(i) দেখুন।]

$\therefore (1) \Rightarrow I = \frac{\pi}{2} \ln 2 - \frac{\pi}{2} \ln\frac{1}{2} = \frac{\pi}{2} \ln 2 + \frac{\pi}{2} \ln 2$

অর্থাৎ $\int_0^{\pi/2} ln\,(\tan x + \cot x)\,dx = \pi ln2$.

(iii). ধরি $I = \int_0^{\pi/4} ln(1 + \tan\theta)\,d\theta \cdots (1)$

বা $I = \int_0^{\pi/4} ln\left[1 + \tan\left(\frac{\pi}{4} - \theta\right)\right] d\theta$

বা $I = \int_0^{\pi/4} ln\left[1 + \frac{\tan \pi/4 - \tan\theta}{1 + \tan \pi/4.\tan\theta}\right] d\theta$

বা $I = \int_0^{\pi/4} ln\left[1 + \frac{1 - \tan\theta}{1 + \tan\theta}\right] d\theta$

বা $I = \int_0^{\pi/4} ln\left[\frac{2}{1 + \tan\theta}\right] d\theta = \int_0^{\pi/4} [ln2 - ln(1 + \tan\theta)]\,d\theta$

বা $I = ln2 \int_0^{\pi/4} d\theta - \int_0^{\pi/4} ln\,(1 + \tan\theta)\,d\theta$

বা $I = ln2\,[\theta]_0^{\pi/4} - I$

বা $2I = \frac{\pi}{4} ln2$

$\therefore\ I = \frac{\pi}{8} ln2.$

(iv). ধরি $I = \int_0^{\pi/2} \sin 2x.\ ln\,(\tan x)\,dx \cdots (1)$

বা $I = \int_0^{\pi/2} \sin(\pi - 2x).\ ln \tan\left(\frac{\pi}{2} - x\right) dx$

$\therefore I = \int_0^{\pi/2} \sin 2x\ ln\,(\cot x)\,dx \cdots (2)$

এখন $(1) + (2) \Rightarrow 2I = \int_0^{\pi/2} \sin 2x\,[ln \tan x + ln \cot x]\,dx$

বা $2I = \int_0^{\pi/2} \sin 2x.\ ln\,(\tan x.\ \cot x)\,dx = \int_0^{\pi/2} \sin 2x.\ ln1\,dx$

বা $2I = \int_0^{\pi/2} \sin 2x.\ 0 dx = 0 \Rightarrow I = 0.$

(v). ধরি $I=\int_0^{\pi/4} ln(\sin 2\theta)\,d\theta$; ধরি $2\theta = x$ তবে $d\theta = \frac{1}{2}dx$

সীমা : যদি $\theta = 0$ হয়, তবে $x = 0$

যদি $\theta = \frac{\pi}{4}$ হয়, তবে $x = \frac{\pi}{2}$

$\therefore I = \frac{1}{2}\int_0^{\pi/2} ln\,(\sin x)\,dx = \frac{1}{2}\cdot\frac{\pi}{2}\,ln\frac{1}{2}$. [উদাহরণ 6(i) দেখুন।]

(vi). $I = \int_0^{\pi} x\,ln\,(\sin x)\,dx \cdots (1)$

বা $I = \int_0^{\pi}(\pi - x)\,ln\,\sin(\pi - x)\,dx = \int_0^{\pi}(\pi - x)\,ln\,(\sin x)\,dx \cdots (2)$

এখন $(1)+(2) \Rightarrow 2I = \pi\int_0^{\pi} ln\,(\sin x)\,dx = 2\pi\int_0^{\pi/2} ln\,(\sin x)\,dx$

$\Rightarrow I = \pi\int_0^{\pi/2} ln\,(\sin x)\,dx = \pi\frac{\pi}{2}\;ln\frac{1}{2}$.

[উদা–6(i) দেখুন।]

(vii). $I = \int_0^1 ln\,\sin\left(\frac{\pi\theta}{2}\right)d\theta$; ধরি $\frac{\pi\theta}{2} = x$ তবে $\frac{\pi}{2}\,d\theta = dx$.

সীমা : যদি $\theta = 0$ হয়, তবে $x = 0$

যদি $\theta = 1$ হয়, তবে $x = \frac{\pi}{2}$.

$\therefore I = \frac{2}{\pi}\int_0^{\pi/2} ln(\sin x)\,dx = \frac{2}{\pi}\cdot\frac{\pi}{2}\,ln\frac{1}{2} = ln\frac{1}{2}$ [উদা–6(i) দেখুন।]

(viii). $I = \int_0^{\pi/2}\sin\theta\,ln(\sin\theta)\,d\theta = \frac{1}{2}\int_0^{\pi/2}\sin\theta\,ln(\sin^2\theta)\,d\theta$

$= \frac{1}{2}\int_0^{\pi/2}\sin\theta\,ln(1-\cos^2\theta)\,d\theta$

ধরি $\cos\theta = t$ তবে $\sin\theta\,d\theta = -\,dt$

সীমা : যদি $\theta = 0$ হয়, তবে $t = 1$

যদি $\theta = \pi/2$ হয়, তবে $t = 0$

$\therefore I = -\frac{1}{2}\int_1^0 ln\,(1-t^2)\,dt = -\frac{1}{2}\int_1^0\left(-t^2 - \frac{t^4}{2} - \frac{t^6}{3} - \ldots\right)dt$

$= \frac{1}{2}\left[\frac{t^3}{3} + \frac{t^5}{2.5} + \frac{t^7}{3.7} + \ldots\right]_1^0$

$$= -\frac{1}{2}\left[\frac{1}{3}+\frac{1}{2.5}+\frac{1}{3.7}+\dots\right]$$

$$= -\left[\frac{1}{2.3}+\frac{1}{4.5}+\frac{1}{6.7}+\dots\right]$$

$$= -\left[\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)+\dots\right]$$

$$= -\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\dots$$

$$= \left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\dots\right)-1$$

$$= \ln(1+1)-1 = \ln 2-\ln e = \ln(2/e).$$

(ix). $I = \int_0^{\pi}\cos 2x.\ \ln(\sin x)\,dx = 2\int_0^{\pi/2}\cos 2x\ \ln(\sin x)\,dx$

$$= 2\left\{\left[\log(\sin x)\frac{\sin 2x}{2}\right]_0^{p/2} - \int_0^{\pi/2}\frac{\cos x}{\sin x}\cdot\frac{\sin 2x}{2}\,dx\right\}$$

$$= 2\left\{0-\frac{1}{2}\int_0^{\pi/2}\frac{\cos x}{\sin x}.\ 2\sin x\cos x\,dx\right\}$$

$$= -2\int_0^{\pi/2}\cos^2 x\,dx = -2.\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma(2)}$$

$$= \frac{-2.\frac{1}{2}\sqrt{\pi}\,.\,\sqrt{\pi}}{2.1} = -\frac{\pi}{2}.$$

7(i). ধরি $I = \int_0^1 \frac{\ln x}{\sqrt{1-x^2}}dx$, এবং $x = \sin\theta$ তবে $dx = \cos\theta\,d\theta$

সীমা ঃ যদি $x = 0$ হয়, তবে $\sin\theta = 0 \Rightarrow \theta = 0$

যদি $x = 1$ হয়, তবে $\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore I = \int_0^{\pi/2}\frac{\ln(\sin\theta)}{\sqrt{(1-\sin^2\theta)}}\cos\theta\,d\theta = \int_0^{\pi/2}\ln(\sin\theta)\frac{\cos\theta}{\cos\theta}d\theta$$

$$= \int_0^{\pi/2}\ln(\sin\theta)d\theta = \frac{\pi}{2}\ln\frac{1}{2}$$ [উদা 6(i) এর দ্বারা।]

(ii). ধরি $I = \int_1^0 \ln\left(x+\frac{1}{x}\right)\frac{dx}{1+x^2}$

এবং $x = \tan\theta$ তবে $dx = \sec^2\theta\,d\theta$.

সীমা ঃ যদি $x = 0$ হয়, তবে $\tan\theta = 0 \Rightarrow \theta = 0$

যদি $x = \infty$ হয়, তবে $\tan\theta = \infty \Rightarrow \theta = \frac{\pi}{2}$

$\therefore I = \int_0^{\pi/2} \ln\left(\tan\theta + \frac{1}{\tan\theta}\right) \frac{\sec^2\theta\, d\theta}{1 + \tan^2\theta}$

$= \int_0^{\pi/2} \ln(\tan\theta + \cot\theta)\, d\theta = \ln\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right) d\theta$

$= \int_0^{\pi/2} \ln\left(\frac{1}{\sin\theta\cos\theta}\right) d\theta = \int_0^{\pi/2} \ln\left(\frac{2}{\sin 2\theta}\right) d\theta$

$= \ln 2 \int_0^{\pi/2} d\theta - \int_0^{\pi/2} \ln(\sin 2\theta)\, d\theta$

$= \ln 2\ [\theta]_0^{\pi/2} - I_1.$

যখন $I_1 = \int_0^{\pi/2} \ln(\sin 2\theta)\, d\theta$; ধরি $2\theta = z$ তবে $2d\theta = dz$

সীমা ঃ যদি $\theta = 0$ হয়, তবে $z = 0$

যদি $\theta = \frac{\pi}{2}$ হয়, তবে $z = \pi$.

$\therefore I_1 = \frac{1}{2}\int_0^{\pi} \ln(\sin z)\, dz = \frac{1}{2} 2\int_0^{\pi/2} \ln(\sin z)\, dz.$

$= \int_0^{\pi/2} \ln(\sin x)\, dx = -\frac{\pi}{2} \ln 2$; [উদা–6(i) দ্বারা।]

$\therefore I = \ln 2.\left(\frac{\pi}{2} - 0\right) + \frac{\pi}{2} \ln 2 = \pi \ln 2$

$\therefore \int_1^0 \frac{\ln(x + 1/x)}{1 + x^2} dx = \pi \ln 2.$

(iii). ধরি $I = \int_0^1 \frac{\ln x}{1 + x} dx = \int_0^1 \ln x\, .\, (1 + x)^{-1} dx$

$= \int_0^1 \ln x\, .\, (1 - x + x^2 - x^3 + x^4 - \ldots)\, dx$

বা $I = \left[\log x.\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots\right)\right]_0^1$

$- \int_0^1 \frac{1}{x}\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots\right) dx$

$= \ln 1.\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\right) - 0 - \int_0^1 \left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots\right) dx$

$$= 0 - \left[x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \ldots\right]_0^1$$

$$= -\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right) - 0$$

$$= -\frac{\pi^2}{12}$$

$$\therefore \int_0^1 \frac{\ln x}{1+x}\, dx = -\frac{\pi^2}{12}.$$

(iv). $I = \int_1^0 \frac{\ln x.\, dx}{1+x^2}$

এবং $x = \tan\theta$ তবে $dx = \sec^2\theta\, d\theta$

সীমা ঃ যদি $x = 0$ হয়, তবে $0 = \tan\theta \Rightarrow \theta = 0$

যদি $x = \infty$ হয়, তবে $\infty = \tan\theta \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore I = \int_0^{\pi/2} \frac{\ln(\tan\theta).\, \sec^2\theta\, d\theta}{1+\tan^2\theta}$$

বা $I = \int_0^{\pi/2} \ln(\tan\theta)\, d\theta$

বা $I = 0$; [6(i) দেখুন।]

8(i). $I = \int_{-\pi/2}^{\pi/2} \sin^7 x\, dx = \int_{-\pi/2}^{\pi/2} f(x)\, dx$; যখন $f(x) = \sin^7 x$

বা $I = \int_0^{\pi/2} \{f(-x) + f(x)\}\, dx \cdots (1)$ যুগ্ম-অযুগ্ম ফাংশন দ্বারা।

$\therefore\ f(-x) = \{\sin(-x)\}^7 = (-\sin x)^7 = -\sin^7 x$

এখন $f(x)$ এবং $f(-x)$ এর মান (1) নং এ স্থাপন করিয়া পাই

$$I = \int_0^{\pi/2} (-\sin^7 x + \sin^7 x)dx$$

$$= \int_0^{\pi/2} 0\, dx = 0.$$

(ii). $I = \int_{-\pi/2}^{\pi/2} \sin^{15} x\, dx = \int_{-\pi/2}^{\pi/2} f(x)\, dx$; যখন $f(x) = \sin^{15} x$

বা $I = \int_0^{\pi/2} \{f(-x) + f(x)\}\, dx \cdots (1)$

$\therefore\ f(-x) = \{\sin(-x)\}^{15} = (-\sin x)^{15} = -\sin^{15} x$

এখন f(x) এবং f(–x) এর মান (1) নং এ স্থাপন করিয়া পাই,

$$I=\int_0^{\pi/2}(-\sin^{15}x+\sin^{15}x)\,dx=\int_0^{\pi/2}0\,dx=0$$

(iii). ধরি $I=\int_{-a}^{a}x\sqrt{a^2-x^2}\,dx=\int_{-a}^{a}f(x)\,dx$

যখন $f(x)=x\sqrt{a^2-x^2}$

$$\therefore I=\int_0^a\{f(-x)+f(x)\}\,dx\cdots(1)$$

$\therefore f(-x)=-x\sqrt{a^2-(-x)^2}=-x\sqrt{a^2-x^2}$

এখন f(x) এবং f(–x) এর মান (1) নং এ স্থাপন করিয়া পাই

$$I=\int_0^a[-x\sqrt{a^2-x^2}+x\sqrt{a^2-x^2}]dx$$

$$=\int_0^a 0\,dx=0$$

(iv). ধরি $I=\int_{-a}^{a}x\,e^{-x^2}\,dx=\int_{-a}^{a}f(x)\,dx$; যখন $f(x)=xe^{-x^2}$

$$\text{বা } I=\int_0^a\{f(-x)+f(x)\}\,dx\cdots(1)$$

$\therefore f(-x)=-xe^{-(-x)^2}=-xe^{-x^2}$

এখন f(x) এবং f(–x) এর মান (1) নং এ বসাইয়া পাই,

$$I=\int_0^a(-xe^{-x^2}+xe^{-x^2})\,dx=\int_0^a 0\,dx=0.$$

(v). ধরি $I=\int_{-\infty}^{\infty}\frac{dx}{x(1+x^2)}\int_{-\infty}^{\infty}f(x)dx$; যখন $f(x)=\frac{1}{x(1+x^2)}$

$$\text{বা } I=\int_0^{\infty}\{f(-x)+f(x)\}\,dx\cdots(1)$$

$$\therefore f(-x)=\frac{1}{-x\{1+(-x)^2\}}=\frac{-1}{x(1+x^2)}$$

$$\therefore (1)\Rightarrow I=\int_0^{\infty}\left\{\frac{-1}{x(1+x^2)}+\frac{1}{x(1+x^2)}\right\}dx=\int_0^{\infty}0\,dx=0$$

9(i). $I=\int_0^{\pi/2}\frac{dx}{1+\tan^2x}=\int_0^{\pi/2}\cos^2x\,dx=\frac{1}{2}\int_0^{\pi/2}(1+\cos2x)\,dx$

$$=\frac{1}{2}\left[x+\frac{\sin 2x}{2}\right]_0^{\pi/2}=\frac{1}{2}\left[\frac{\pi}{2}+0\right]=\frac{\pi}{4}$$

(ii). $I = \int_0^{\pi} \frac{x^2 \sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x - \pi} dx \cdots (1)$

বা $I = \int_0^{\pi} \frac{(\pi - x)^2 \sin(2\pi - 2x) \sin\left\{\frac{1}{2}\cos(\pi - x)\right\}}{2(\pi - x) - \pi} dx$

বা $I = \int_0^{\pi} \frac{(\pi^2 - 2\pi x + x^2)(-\sin 2x) \sin\left(-\frac{\pi}{2}\cos x\right)}{\pi - 2x} dx$

বা $I = \int_0^{\pi} \frac{(-\pi^2 + 2\pi x - x^2) \sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x - \pi} dx \cdots (2)$

এখন (1)+(2) $\Rightarrow 2I = \int_0^{\pi} \frac{(x^2 - \pi^2 + 2\pi x - x^2)\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x - \pi}$ d

বা $2I = \int_0^{\pi} \frac{\pi(2x - \pi)\, 2\sin x \cos x \sin\left(\frac{\pi}{2}\cos x\right)}{(2x - \pi)} dx$

বা $I = \pi \int_0^{\pi} \sin x \cos x \sin\left(\frac{\pi}{2}\cos x\right) dx$

বা $I = 2\pi \int_0^{\pi/2} \sin x \cos x \sin\left(\frac{\pi}{2}\cos x\right) dx$

ধরি $\frac{\pi}{2}\cos x = t$ তবে $-\frac{\pi}{2}\sin x\, dx = dt$

সীমা ঃ যদি $x = 0$ হয়, তবে $t = \frac{\pi}{2}$

যদি $x = \frac{\pi}{2}$ হয়, তবে $t = 0$

$I = -2\pi \frac{2}{\pi} \int_{\pi/2}^{0} \frac{2}{\pi} t \sin t\, dt = \frac{8}{\pi} \int_0^{\pi/2} t \sin t\, dt$

$= \frac{8}{\pi}[-t\cos t + \sin t]_0^{\pi/2} = \frac{8}{\pi}[(0 + 1) - (0 + 0)] = \frac{8}{\pi}$

(ii). $I = \int_0^1 \cot^{-1}(1 - x + x^2) dx$

বা $I = \int_0^1 \tan^{-1} \frac{1}{1 - x + x^2} dx = \int_0^1 \frac{1}{1 + x(x - 1)} dx$

$$= \int_0^1 \tan^{-1} \frac{x - (x - 1)}{1 + x(x - 1)} dx$$

$$= \int_0^1 [\tan^{-1} x - \tan^{-1} (x - 1)] dx$$

$$বা\ I = \int_0^1 \tan^{-1}x\, dx - \int_0^1 \tan^{-1} (x - 1)\, dx$$

$$= \int_0^1 \tan^{-1}x\, dx - \int_0^1 \tan^{-1} \{(1 - x) - 1\}\, dx$$

$$= \int_0^1 \tan^{-1}x\, dx + \int_0^1 \tan^{-1}x\, dx$$

$$= 2 \int_0^1 \tan^{-1}x\, dx = 2\left[x \tan^{-1} x - \frac{1}{2} \log(1 + x^2)\right]_0^1$$

$$= 2\left[1 \tan^{-1} 1 - \frac{1}{2} \log(1 + 1) - 0\right] = 2\left[\frac{\pi}{4} - \frac{1}{2} \log 2\right]$$

$$\therefore I = \frac{\pi}{2} - \ln 2.$$

$\int 1x \tan^{-1}x$

$- \int \partial \tan x\, dx$

$x \tan^{-1}x$

$- \int \log \frac{1+x^2}{2}$

(iv). $I = \int_0^1 \ln \left(\frac{1 - x}{x}\right) dx \cdots (1)$

$$বা\ I = \int_0^1 \ln \left\{\frac{1 - (1 - x)}{1 - x}\right\} dx = \int_0^1 \ln \frac{x}{1 - x} dx$$

$$\therefore I = - \int_0^1 \ln \left(\frac{1 - x}{x}\right) dx \cdots (2)$$

$$\therefore (1) + (2) \Rightarrow 2I = \int_0^1 \ln \left(\frac{1 - x}{x}\right) dx - \int_0^1 \ln \left(\frac{1 - x}{x}\right) dx$$

বা $2I = 0 \Rightarrow I = 0.$

(v). $I = \int_0^{\pi/2} x \cot x\, dx = [x \log(\sin x)]_0^{\pi/2} - \int_0^{\pi/2} 1.\ \ln (\sin x)\, dx$

$$= 0 - \int_0^{\pi/2} \ln(\sin x)\, dx = - \frac{\pi}{2} \ln \frac{1}{2}$$ [উদা–6(i) দেখুন।]

$$\therefore I = \frac{\pi}{2} \ln 2.$$

(vi). $I = \int_0^{\pi/2} \frac{\theta^2}{\sin^2\theta} d\theta = \int_0^{\pi/2} \theta^2 \operatorname{cosec}^2\theta\, d\theta$

$$= [\theta^2(- \cot \theta]_0^{\pi/2} - \int_0^{\pi/2} [2\theta\, (-\cot\theta)]\, d\theta$$

$$= 0 + 2\int_0^{\pi/2} \theta \cot\theta \, d\theta = 2.\frac{\pi}{2} \ln 2;$$ [9(v) নং দেখুন ।]

$\therefore I = \pi \ln 2.$

(vii). $I = \int_0^{\infty} \frac{\tan^{-1}x \, dx}{x(1 + x^2)}$

ধরি $x = \tan\theta$ তবে $dx = \sec^2\theta \, d\theta$

সীমা ঃ যদি $x = 0$ হয়, তবে $\theta = 0$

যদি $x = \infty$ হয়, তবে $\theta = \frac{\pi}{2}$

$$\therefore I = \int_0^{\pi/2} \frac{\tan^{-1}(\tan\theta) \sec^2\theta \, d\theta}{\tan\theta \, (1 + \tan^2\theta)} = \int_0^{\pi/2} \theta \cot\theta \, d\theta = \frac{\pi}{2} \ln 2;$$

[9(v) নং দেখুন]

(viii). $I = \int_0^{\infty} (\cot^{-1}x)^2 \, dx;$ ধরি $x = \cot\theta$ তবে $dx = -\operatorname{cosec}^2\theta \, d\theta$

সীমা ঃ যদি $x = 0$ হয়, তবে $\theta = \frac{\pi}{2}$

যদি $x = \infty$ হয়, তবে $\theta = 0$

$$\therefore I = -\int_{\pi/2}^{0} (\cot^{-1}\cot\theta)^2 \operatorname{cosec}^2\theta \, d\theta$$

$$= \int_0^{\pi/2} \theta^2 \operatorname{cosec}^2\theta \, d\theta = \pi \ln 2;$$ [9(vi) নং দেখুন ।]

(ix). $I = \int_0^{\pi/4} \sqrt[3]{\tan x} \, dx$

ধরি $\tan x = z^3$, তবে $\sec^2 x \, dx = 3z^2 \, dz$

বা $dx = \frac{3z^2 dz}{\sec^2 x} = \frac{3z^2 dz}{1 + \tan^2 x} = \frac{3z^2 \, dz}{1 + z^6}$

সীমা ঃ যদি $x = 0$ হয়, তবে $0 = z^3 \Rightarrow z = 0$

যদি $x = \frac{\pi}{4}$ হয়, তবে $\tan\frac{\pi}{4} = z^3 \Rightarrow z = 1$

$$\therefore I = 3\int_0^1 \frac{z.z^2 \, dz}{1 + z^6} = 3\int_0^1 \frac{zz^2 \, dz}{(z^2 + 1)(z^4 - z^2 + 1)}$$

আবার ধরি $z^2 = t$ তবে $z \, dz = \frac{dt}{2}$

সীমা ঃ যদি $z = 0$ হয়, তবে $t = 0$

যদি $z = 1$ হয়, তবে $t = 1$

$$\therefore I = \frac{3}{2}\int_0^1 \frac{t \, dt}{(t + 1)(t^2 - t + 1)}$$

ধরি $\dfrac{t}{(t+1)(t^2-t+1)} = \dfrac{A}{t+1} + \dfrac{Bt+C}{t^2-t+1} \cdots (1)$

$\Rightarrow A = -\dfrac{1}{3}, B = \dfrac{1}{3}, C = \dfrac{1}{3}.$

$\therefore (1) \Rightarrow \dfrac{t}{(t+1)(t^2-t+1)} = \dfrac{-1/3}{t+1} + \dfrac{t+1}{3(t^2-t+1)}$

$\therefore I = \dfrac{3}{2}\displaystyle\int_0^1 \left[\frac{-1}{3(t+1)} + \frac{1}{3}\cdot\frac{(t+1)}{t^2-t+1}\right]dt$

$= -\dfrac{1}{2}\displaystyle\int_0^1 \frac{dt}{t+1} + \frac{1}{2}\int_0^1 \frac{\frac{1}{2}(2t-1)+\frac{3}{2}}{t^2-t+1}dt$

$= -\dfrac{1}{2}\displaystyle\int_0^1 \frac{dt}{t+1} + \frac{1}{4}\int_0^1 \frac{(2t-1)dt}{t^2-t+1} + \frac{3}{4}\int_0^1 \frac{dt}{(t-1/2)^2+(\sqrt{3}/2)^2}$

$= -\dfrac{1}{2}[\ln(t+1)]_0^1 + \dfrac{1}{4}[\ln(t^2+t+1)]_0^1 + \dfrac{3}{4}\cdot\dfrac{2}{\sqrt{3}}\left[\tan^{-1}\dfrac{t-1/2}{\sqrt{3}/2}\right]_0^1$

বা $I = -\dfrac{1}{2}[\ln 2 - 0] + \dfrac{1}{4}[0] + \dfrac{3}{2\sqrt{3}}\left[\tan^{-1}\dfrac{2t-1}{\sqrt{3}}\right]_0^1$

$= -\dfrac{1}{2}\ln 2 + \dfrac{3}{2\sqrt{3}}\left[\tan^{-1}\dfrac{1}{\sqrt{3}} - \tan^{-1}\dfrac{(-1)}{\sqrt{3}}\right]$

$= \dfrac{1}{2}\ln\dfrac{1}{2} + \dfrac{3}{2\sqrt{3}}2\tan^{-1}\dfrac{1}{\sqrt{3}} = \dfrac{1}{2}\ln\dfrac{1}{2} + \sqrt{3}\cdot\dfrac{\pi}{6} = \dfrac{1}{2}\ln\dfrac{1}{2} + \dfrac{\pi\sqrt{3}}{6}.$

(x). ধরি $I = \displaystyle\int_0^{\pi/4}\sqrt{\tan x}\,dx$

ধরি $\tan x = z^2$ তবে $\sec^2 x\,dx = 2z\,dz$

$\therefore dx = \dfrac{2z\,dz}{\sec^2 x} = \dfrac{2zdz}{1+\tan^2 x} = \dfrac{2z\,dz}{1+z^4}.$

সীমা : যদি $x = 0$ হয়, তবে $\tan 0 = z^2 \Rightarrow z = 0$

যদি $x = \dfrac{\pi}{4}$ হয়, তবে $\tan\dfrac{\pi}{4} = z^2 \Rightarrow z = 1$

$\therefore I = 2\displaystyle\int_0^1 \frac{z\,z\,dz}{1+z^4} = \int_0^1 \frac{(z^2+1)+(z^2-1)}{z^4+1}dz$

$= \displaystyle\int_0^1 \frac{(z^2+1)\,dz}{z^4+1} + \int_0^1 \frac{(z^2-1)\,dz}{z^4+1}$

$= \displaystyle\int_0^1 \frac{z^2(1+1/z^2)dz}{z^2(z^2+1/z^2)} + \int_0^1 \frac{z^2(1-1/z^2)\,dz}{z^2(z^2+1/z^2)}$

$$= \int_0^1 \frac{(1 + 1/z^2)dz}{(z - 1/z)^2 + 2} + \int_0^1 \frac{(1 - 1/z^2)\,dz}{(z + 1/z)^2 - 2}$$

$$= \int_0^1 \frac{d(z - 1/z)}{(z - 1/z)^2 + (\sqrt{2})^2} + \int_0^1 \frac{d\,(z + 1/z)}{(z + 1/z)^2 - (\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{z - 1/z}{\sqrt{2}}\right]_0^1 + \frac{1}{2\sqrt{2}}\left[ln\ \frac{z + 1/z - \sqrt{2}}{z + 1/z + \sqrt{2}}\right]_0^1$$

বা $I = \frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{z^2 - 1}{z\sqrt{2}}\right]_0^1 + \frac{1}{2\sqrt{2}}\left[ln\ \frac{z^2 + 1 - z\sqrt{2}}{z^2 + 1 + z\sqrt{2}}\right]_0^1$

$$= \frac{1}{\sqrt{2}}\,[0 - \tan^{-1}(-\infty)] + \frac{1}{2\sqrt{2}}\left[ln\ \frac{2 - \sqrt{2}}{2 + \sqrt{2}} - ln\ 1\right]$$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\pi}{2} + \frac{1}{2\sqrt{2}}\,ln\ \frac{\sqrt{2} - 1}{\sqrt{2} + 1}.$$

10(i). ধরি $a + b - x = z$ তবে $- dx = dz$

সীমা ঃ যদি $x = a$ হয়, তবে $z = b$

যদি $x = b$ হয়, তবে $z = a$

$$\therefore \int_a^b f(a + b - x)\,dx = -\int_b^a f(z)\,dz = \int_a^b f(z)\,dz$$

$$= \int_a^b f(x)\,dx$$

(ii). ধরি $x + c = z$ তবে $dx = dz$

সীমা ঃ যদি $x = a - c$ হয়, তবে $z = a$

যদি $x = b - c$ হয়, তবে $z = b$

$$\int_{a-c}^{b-c} f(x + c)\,dx = \int_a^b f(z)\,dz = \int_a^b f(x)\,dx$$

(iii). ধরি $I = \int_0^\pi x\, f(\sin x)\,dx \cdots (1)$

বা $I = \int_0^\pi (\pi - x)\, f\{\sin(\pi - x)\}\,dx = \int_0^\pi (\pi - x)\, f(\sin x)\,dx \cdots (2)$

এখন $(1) + (2) \Rightarrow 2I = \pi \int_0^\pi f(\sin x)\,dx$

$$\therefore I = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx.$$

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