

THEME _____

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Ex. $(y^4 + 4x^3y + 3x)dx + (x^4 + 4xy^3 + y + 1)dy = 0$

Here $M = y^4 + 4x^3y + 3x$

$$\frac{\partial M}{\partial y} = 4y^3 + 4x^3$$

and $N = x^4 + 4xy^3 + y + 1$

$$\frac{\partial N}{\partial x} = 4x^3 + 4y^3$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The equation is Exact

Soln. becomes,

$$\int (M \text{ keeping } y \text{ as constant}) dx + \int N \text{ terms free from } x dy$$

$$\Rightarrow \int y^4 + 4x^3y + 3x dx + \int y + 1 dy = \text{constant}$$

$$\Rightarrow xy^4 + x^4y + \frac{3}{2}x^2 + \frac{y^2}{2} + y = C.$$

In $x^4 + 4xy^3 + y + 1$, terms free from x are $y+1$ whose integral with respect to y is $\frac{1}{2}y^2 + y$.

Therefore the general solution is

$$y^4x + x^4y + \frac{1}{2}x^2 + \frac{1}{2}y^2 + y = C.$$

Ex. 2. Solve $x(x^2 + y^2 - a^2) dx + y(x^2 - y^2 - b^2) dy = 0$. [Nag. 63; Poona 61]

Solution. Here $M = x^3 + xy^2 - a^2x$, $N = yx^2 - y^3 - b^2y$.

$$\frac{\partial M}{\partial y} = 2xy \text{ and } \frac{\partial N}{\partial x} = 2xy.$$

Since these are equal, the equation is exact,

Integrating M w.r.t. x keeping y as constant, we get

$$\frac{1}{2}x^4 + \frac{1}{2}x^2y^2 - \frac{1}{2}a^2x^2.$$

In N , terms free from x are $-y^3 - b^2y$ whose integral is
 $-\frac{1}{2}y^4 - \frac{1}{2}b^2y^2$.

Hence the general solution is

$$\frac{1}{2}x^4 + \frac{1}{2}x^2y^2 - \frac{1}{2}a^2x^2 - \frac{1}{2}y^4 - \frac{1}{2}b^2y^2 = \text{const.}$$

or $x^4 - y^4 + 2x^2y^2 - 2a^2x^2 - 2b^2y^2 = C$.

Ex. 3. Solve $(x^2 - 2xy + 3y^2) dx + (4y^3 + 6xy - x^2) dy = 0$. [Delhi Hons. 55]

Solution. Here $\frac{\partial M}{\partial y} = -2x + 6y$, $\frac{\partial N}{\partial x} = 6y - 2x$.

Since these are equal the equation is exact.

Integrating M , i.e. $x^2 - 2xy + 3y^2$ w.r.t. x keeping y as constant, we get $\frac{1}{2}x^3 - x^2y + 3y^2x$.

In N , term free from x is $+4y^3$ whose integral is y^4 .

Hence the solution is $\frac{1}{2}x^3 - x^2y + 3y^2x + y^4 = C$.

Ex. 4. Solve $(x - 2e^y) dy + (y + x \sin x) dx = 0$. [Gujrat 61]

Solution. Here $M = y + x \sin x$, $N = x - 2e^y$.

$$\therefore \frac{\partial M}{\partial y} = 1, \frac{\partial N}{\partial x} = 1; \text{ therefore equation is exact.}$$

Integrating $y + x \sin x$ with respect to x keeping y as constant, we get $xy + \int x \sin x dx = xy - x \cos x + \sin x$.

In N , term free from x is $-2e^y$ whose integral with respect to y is $-2e^y$.

Hence the complete solution is

$$xy - x \cos x + \sin x - 2e^y = C.$$

***Ex. 5. (a)** Solve $x dx + y dy = \frac{a^2 (x dy - y dx)}{x^2 + y^2}$.

[Delhi Hons. 62]

Solution. The equation can be put as

$$\left(x + \frac{a^2 y}{x^2 + y^2} \right) dx + \left(y - \frac{a^2 x}{x^2 + y^2} \right) dy = 0.$$

Exact Equations

$$(ii) (x^3 - 4xy - 2y^2) dx + (y^2 - 4xy - 2x^2) dy.$$

Ans. $x^3 + y^3 - 6xy(x+y) = C.$

$$(iii) \cos x (\cos x - \sin a \sin y) dx + \cos y (\cos y - \sin a \sin x) dy = 0.$$

Ans. $2(x+y) \sin 2x + \sin 2y - 4 \sin a \sin x \sin y = C.$

$$(iv) (2xy + y - \tan y) dx + (x^2 - x \tan^2 y + \sec^2 y) dy = 0.$$

[Poona 1964]

Ans. $x^2y + xy - x \tan y + \tan y = C.$

$$(v) (2x^2y + 4x^3 - 12xy^2 + 3y^2 - xe^y + e^{2x}) dy$$

$$+ (12x^2y + 2yx^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y) dx = 0. \quad [\text{Poona 64}]$$

Ans. $4x^3y + x^2y^2 + x^4 - 4y^3x + ye^{2x} - xe^y + y^3 = C.$

3.4. Integrating factors.

If an equation becomes exact after it has been multiplied by a function of x and y , then such a function is called an integrating factor

[Karnatak 61]

3.5. Number of integrating factors.

To show that there is an infinite number of integrating factors for an equation,

$$M dx + N dy = 0.$$

[Karnatak 61]

To prove this let μ be an integrating factor; then

$$\mu(M dx + N dy) = du.$$

Integrating, $u = c$ is a solution.

Now multiplying both the sides by $f(u)$, a function of u , we get $\mu f(u) [M dx + N dy] = f(u) du$.

Expression on the right is directly integrable and therefore so must be the left hand side.

Hence $\mu f(u)$ is also an integrating factor. Since $f(u)$ is an arbitrary function of u , the number of integrating factors is infinite.

3.6. Integrating factor by inspection.

Sometimes an integrating factor can be found by inspection. For this the reader should study the following results :-

Group of terms	I.F.	Exact Differential
$x dy - y dx$	$\frac{1}{x^2}$	$\frac{x dy - y dx}{x^2} = d\left(\frac{y}{x}\right)$
$x dy - y dx$	$\frac{1}{y^2}$	$\frac{y dx - x dy}{-y^2} = d\left(-\frac{x}{y}\right)$
$x dy - y dx$	$\frac{1}{xy}$	$\frac{dy}{y} - \frac{dx}{x} = d\left(\log \frac{y}{x}\right)$
$x dy - y dx$	$\frac{1}{x^2 + y^2}$	$\frac{x dy - y dx}{x^2 + y^2} = \frac{d\left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^2}$ $= d\left[\tan^{-1} \frac{y}{x}\right]$

3.7. Rules for finding the integrating factor.

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$$

Rule I. If $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = f(x)$, a function of x only, then $e^{\int f(x) dx}$ is an integrating factor. [Delhi Hons. 64]

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$$

Rule II. If $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = g(y)$ is a function of y alone, then $e^{\int -g(y) dy}$ is an integrating factor.

We give below some examples to illustrate these rules.

Ex. 1. Solve $(x^2 + y^2 + x) dx + xy dy = 0$.

Solution. $M = x^2 + y^2 + x, N = xy$.

$$\frac{\partial M}{\partial y} = 2y, \frac{\partial N}{\partial x} = y, \text{ equation is not exact.}$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$$

However, $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{2y - y}{xy} = \frac{1}{x}$, a function of x alone.

$$\text{Hence I.F.} = e^{\int \frac{1}{x} dx} = e^{\log x} = x.$$

Multiplying by I.F., the equation becomes

$$(x^2 + xy^2 + x^2) dx + x^2y dy = 0, \text{ exact now (check up).}$$

Integrating, $x^3 + xy^2 + x^2$ with regard to x , keeping y as constant, we get $\frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + \frac{1}{3}x^3$

and in x^2y^2 there is no term free from x . Therefore the solution is

$$\frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + \frac{1}{3}x^3 = C' \quad \text{or} \quad 3x^4 + 4x^3 + 6x^2y^2 = C.$$

Ex. 2. Solve $(x^2 + y^2 + 1) dx - 2xy dy = 0$.

Solution. $\frac{\partial M}{\partial y} = 2y, \frac{\partial N}{\partial x} = -2y, \text{ not exact.}$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$$

However, $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{2x + 2y}{-2xy} = -\frac{2}{x}$ function of x alone.

$$\therefore \text{I.F.} = e^{-\int \frac{2}{x} dx} = e^{-2 \log x} = \frac{1}{x^2}.$$

Multiplying by $\frac{1}{x^2}$ the equation becomes

$$\left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right) dx - \frac{2y}{x} dy = 0, \text{ exact now.}$$

Integrating, $1 + \frac{y^2}{x^2} + \frac{1}{x^2}$ with regard to x keeping y as constant,

$$\text{we get } x - \frac{y^2}{x} - \frac{1}{x}$$

Exact Equations

as

Therefore the solution is $yx + \frac{2}{y^3}x^3 + y^2 = C$.

Ex. 9. Solve $(3x^2)^4 + 2xy) dx + (2x^3y^3 - x^2) dy = 0$.

[Cal. Hons. 54, 53]

Solution. Here $\frac{\partial M}{\partial y} = 12x^3y^3 + 2x$, $\frac{\partial N}{\partial x} = 6x^2y^3 - 2x$.

$$\text{Now } \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{6x^2y^3 - 4x}{y(3x^2y^3 + 2x)} = \frac{2}{y} \text{ function of } y \text{ alone.}$$

$$\therefore \text{I.F.} = e^{\int -\frac{2}{y} dy} = e^{-2 \log y} = \frac{1}{y^2}.$$

Multiplying by $\frac{1}{y^2}$, the equation becomes

$$\left(3x^2y^2 + \frac{2x}{y}\right) dx + \left(2x^3y - \frac{x^2}{y^2}\right) dy = 0, \text{ exact now.}$$

Integrating $3x^2y^2 + \frac{2x}{y}$ w.r.t. x keeping y as constant, we get

$$x^3y^2 + \frac{x^2}{y}$$

In $2x^3y - \frac{x^2}{y^2}$, there is no term free from x .

Hence the solution is $x^3y^2 + \frac{x^2}{y} = C$

or $x^3y^3 + x^2 = Cy$.

Ex. 10. $(2x^4e^y + 2xy^3 + y) dx + (x^2y^4e^y - x^2y^2 - 3x) dy = 0$.

Solution. We have $\frac{1}{M} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{4}{y}$. \therefore I.F. = $\frac{1}{y^4}$.

Solution is $x^2e^y + \frac{x^2}{y^3} + \frac{x}{y^5} = C$.

3.8. Rule III.

and If $M dx + N dy = 0$ is homogeneous and $Mx + Ny \neq 0$,

then $\frac{1}{Mx + Ny}$ is an integrating factor.

Rule IV.

[Delhi Hons. 61]

If the equation can be written in the form
 $yf(xy) dx + xg(xy) dy = 0$, $f(xy) \neq g(xy)$,

then $\frac{1}{xy[f(xy) - g(xy)]} = \frac{1}{Mx - Ny}$ is an integrating factor.

Ex. 1. Solve $x^2y dx - (x^3 + y^3) dy = 0$.

Solution. The equation is homogeneous and

5

Linear Differential Equations with Constant Coefficients

5.1. Linear Differential Equation

A differential equation of the form

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_n y = X$$

where P_1, P_2, \dots, P_n , and X are functions of x or constants, is called a linear differential equation of n^{th} order.

And if P_1, P_2, \dots, P_n are all constants (not functions of x) and X is some function of x , then the equation is a linear differential equation with constant coefficients.

5.2. The Operator D . It is usual to write

$$D \text{ for } \frac{d}{dx}, D^2 \text{ for } \frac{d^2}{dx^2}, \dots, D^n \text{ for } \frac{d^n}{dx^n}.$$

And in terms of the operator D the differential equation (1) can be written as $[D^n + P_1 D^{n-1} + P_2 D^{n-2} + \dots + P_n] y = X$.

Note. It can be proved that D can be treated as an algebraic quantity in several respects.

5.3. A Theorem. If $y=y_1, y=y_2, \dots, y=y_n$ are linearly independent solutions of

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n) y = 0, \quad \dots(1)$$

then $y = C_1 y_1 + C_2 y_2 + \dots + C_n y_n$ is the general or complete solution of the differential equation, where C_1, C_2, \dots, C_n are n arbitrary constants.

Let us denote the given equation (1) by $f(D) y = 0$,
where $f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n$.

Since $y=y_1, y=y_2, \dots, y=y_n$ are solutions of the equation,

$$\therefore f(D) y_1 = 0, f(D) y_2 = 0, \dots, f(D) y_n = 0. \quad \dots(2)$$

Now putting $y = C_1 y_1 + C_2 y_2 + \dots + C_n y_n$ in (1), we have

$$D^n (C_1 y_1 + \dots + C_n y_n) + a_1 D^{n-1} (C_1 y_1 + C_2 y_2 + \dots + C_n y_n) + \dots + a_n (C_1 y_1 + C_2 y_2 + \dots + C_n y_n) = 0$$

$$\text{or } C_1 (D^n y_1 + a_1 D^{n-1} y_1 + \dots + a_n) + C_2 (D^n y_2 + a_1 D^{n-1} y_2 + \dots + a_n) + \dots + C_n (D^n y_n + a_1 D^{n-1} y_n + \dots + a_n) = 0$$

$$\text{or } C_1 f(D) y_1 + C_2 f(D) y_2 + \dots + C_n f(D) y_n = 0$$

$$\text{or } C_1 \cdot 0 + C_2 \cdot 0 + \dots + C_n \cdot 0 = 0 \text{ by (2).}$$

Since (1) is satisfied by $y = C_1y_1 + C_2y_2 + \dots + C_ny_n$, it is a solution of (1). Also since it contains n arbitrary constants, it is the general or complete solution of the equation.

5.4. Auxiliary Equation. Consider the differential equation

$$(D^n + a_1D^{n-1} + a_2D^{n-2} + \dots + a_n)y = 0 \quad \dots(1)$$

where a_1, a_2, \dots, a_n are all constants.

Let $y = e^{mx}$ be a solution of this equation. Then putting

$$y = e^{mx}, Dy = me^{mx}, D^2y = m^2e^{mx}, \dots, D^n y = m^n e^{mx},$$

the equation becomes

$$(m^n + a_1m^{n-1} + a_2m^{n-2} + \dots + a_n)e^{mx} = 0.$$

Hence e^{mx} will be a solution of (1) if m is a root of the algebraic equation

$$m^n + a_1m^{n-1} + a_2m^{n-2} + \dots + a_n = 0 \quad \dots(2)$$

This equation in m is called the *Auxiliary equation*.

Note It is observed that the auxiliary equation $f(m) = 0$ gives the same values of m as the equation $f(D) = 0$ gives of D .

Hence $f(D) = 0$, i.e., $D^n + a_1D^{n-1} + \dots + a_n = 0$ can in general be regarded as the auxiliary equation.

Therefore in practice we do not replace D by m to form the auxiliary equation. The equation in D may be regarded as auxiliary equation.

5.5. Solution of equation (1) of the above article.

[Gujrat B.Sc. (Prin.) 58; Gujarat B.Sc. (Subsi.) 65]

Case I. When all the roots of auxiliary equation are real and different.

If m_1, m_2, \dots, m_n be the n different roots of (2), then $y = e^{m_1x}, y = e^{m_2x}, \dots, y = e^{m_nx}$ are all independent solutions of (1). Therefore the general solution of (1) is

$$y = C_1e^{m_1x} + C_2e^{m_2x} + C_3e^{m_3x} + \dots + C_ne^{m_nx}.$$

$$\text{Ex. 1. Solve } \frac{d^3y}{dx^3} - 13\frac{dy}{dx} - 12y = 0.$$

Solution Equation is $(D^3 - 13D - 12)y = 0$.

The auxiliary equation is $(D^3 - 13D - 12) = 0$,

$$\text{i.e., } (D+1)(D+3)(D-4) = 0, D = -1, -3, 4$$

Hence the complete solution is

$$y = C_1e^{-x} + C_2e^{-3x} + C_3e^{4x}.$$

$$\text{Ex. 2. Solve } (D^3 + 6D^2 + 11D + 6)y = 0. \quad [\text{Delhi Pass 67}]$$

Solution A.E. is $(D+1)(D+2)(D+3) = 0, D = -1, -2, -3$.

The complete solution is

$$y = C_1e^{-x} + C_2e^{-2x} + C_3e^{-3x}.$$

5.6. Case II. Auxiliary equation having equal roots.

[Gujrat B.Sc. (Prin.) 59; Poona T.D.C. 61 (S)]

$$\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$$

$$(D^3 + 6D^2 + 11D + 6)y = 0,$$

Let $y = e^{mx}$, then $\frac{dy}{dx} = me^{mx}$, $\frac{d^2y}{dx^2} = m^2e^{mx}$, $\frac{d^3y}{dx^3} = m^3e^{mx}$,

The equation becomes

$$(m^3 + 6m^2 + 11m + 6)e^{mx} = 0$$

Auxiliary equation is

$$(m^3 + 6m^2 + 11m + 6) = 0$$

Roots are $m_1 = -1$, $m_2 = -2$, $m_3 = -3$

The complete solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$$

We have shown in case I § 5.5, that when m_1, m_2, \dots, m_n are all different, the general solution is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x}.$$

But if $m_1 = m_2$ (two roots equal) then this becomes

$$y = (C_1 + C_2)x e^{m_1 x} + C_3 e^{m_2 x} + \dots + C_n e^{m_n x},$$

which clearly contains only $n-1$ arbitrary constants (since $C_1 + C_2$ is equivalent to only one arbitrary constant)

Therefore this is no longer a general solution.

Consider an equation $(D - m_1)^2 y = 0$, a differential equation of second order having both the roots equal.

Put $(D - m_1) v = y$; then (1) becomes

$$(D - m_1) v = 0 \quad \text{or} \quad \frac{dv}{dx} = m_1 v,$$

Separating the variables, $\frac{dv}{v} = m_1 dx$.

Integrating, $\log v = \log C + m_1 x$, or $v = C e^{m_1 x}$ (A.E.)
or $(D - m_1) y = C e^{m_1 x}$ as $v = (D - m_1) y$
or $\frac{dy}{dx} - m_1 y = C e^{m_1 x}$ which is a linear equation of the first order, its L.F. $\equiv e^{-m_1 x}$

$$ye^{-m_1 x} = \int C e^{m_1 x} e^{-m_1 x} dx + C_2$$

$$\text{or } y = (C_1 + C_2) e^{m_1 x}.$$

Therefore the most general solution of

$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n) y = 0$, when two roots of A.E. are equal, is

$$y = (C_1 + C_2 x) e^{m_1 x} + C_3 e^{m_2 x} + \dots + C_n e^{m_n x}.$$

Cor. In case three roots are equal, i.e., $m_1 = m_2 = m_3$, the general solution is

$$y = (C_1 + C_2 x + C_3 x^2) e^{m_1 x} + C_4 e^{m_2 x} + \dots + C_n e^{m_n x}.$$

Ex. 1. Solve $\frac{d^4 y}{dx^4} - \frac{d^3 y}{dx^3} - 9 \frac{d^2 y}{dx^2} - 11 \frac{dy}{dx} - 4y = 0$.

Solution. A.E. is $D^4 - D^3 - 9D^2 - 11D - 4 = 0$, i.e., $(D+1)^3(D-4) = 0$, $D = -1, -1, -1, 4$.

Hence the general solution is

$$y = (C_1 + C_2 x + C_3 x^2) e^{-x} + C_4 e^{4x}.$$

Ex. 2. Solve $(D^2 - 2D^2 - 4D + 8) y = 0$.

(Delhi Pass 1968)

Solution. Auxiliary equation is

$$D^4 - 2D^2 - 4D + 8 = 0 \quad \text{or} \quad (D+2)(D-2)^2 = 0,$$

$$D = -2, 2, 2.$$

$$\therefore y = (C_1 + C_2 x) e^{2x} + C_3 e^{-2x}.$$

$$y = (C_1 + C_2 x) e^{2x} + C_3 e^{-2x}$$

5.7 Case III. Auxiliary equation having imaginary roots.

Let $\alpha \pm i\beta$ be the imaginary roots of an equation of second order (since imaginary roots occur in pairs).

Then its general solution is $y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$

$$y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$$

$$= e^{\alpha x} [C_1 e^{i\beta x} + C_2 e^{-i\beta x}]$$

$$= e^{\alpha x} [C_1 (\cos \beta x + i \sin \beta x) + C_2 (\cos \beta x - i \sin \beta x)]$$

$$= e^{\alpha x} [(C_1 + C_2) \cos \beta x + (C_1 - C_2) i \sin \beta x]$$

$$= e^{\alpha x} [A \cos \beta x + B \sin \beta x].$$

Note. The above result after suitably adjusting constants may also be written as

$$y = e^{\alpha x} A \cos (\beta x + B) \quad \text{or} \quad y = e^{\alpha x} A \sin (\beta x + B).$$

Imaginary roots repeated. If auxiliary equation has two equal pairs of imaginary roots, i.e., if $\alpha + i\beta$ and $\alpha - i\beta$ occur twice, then general solution is obtained as

$$y = e^{\alpha x} [C_1 + C_2 x] \cos \beta x + (C_3 + C_4 x) \sin \beta x].$$

Cor. If a pair of roots of the auxiliary equation occur in the form of quadratic surd $\alpha \pm \sqrt{\beta}$, where β is +ive, then the corresponding term in the solution may be written as

$$e^{\alpha x} [C_1 \cosh x\sqrt{\beta} + C_2 \sinh x\sqrt{\beta}]$$

$$\text{or } C_1 e^{\alpha x} \cosh (x\sqrt{\beta} + C_2) \quad \text{or} \quad C_1 e^{\alpha x} \sinh (x\sqrt{\beta} + C_2).$$

Ex. 1. Solve $(D^4 + 5D^2 + 6)y = 0$. (Karnatak M. A. 61)

Solution Auxiliary equation is $(D^4 + 5D^2 + 6) = 0$,

$$\text{i.e., } (D^2 + 3)(D^2 + 2) = 0 \quad \therefore D = \pm \sqrt{3}i, \pm \sqrt{2}i.$$

Hence the complete solution is

$$y = C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x + C_3 \cos \sqrt{2}x + C_4 \sin \sqrt{2}x.$$

Ex. 2. Solve $(D^4 - D^3 - D + 1)x = 0$. (Gujrat 58)

Solution. Auxiliary equation is $D^4 - D^3 - D + 1 = 0$

$$\text{or } (D^3 - 1)(D - 1) = 0 \quad \text{or } (D - 1)^2(D^2 + D + 1) = 0$$

$$\text{or } D = 1, 1 - \frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$$

Hence the complete solution is

$$y = (C_1 + C_2 x)e^x + e^{-x/2} \left[C_3 \cos \frac{\sqrt{3}}{2}x + C_4 \sin \frac{\sqrt{3}}{2}x \right].$$

Ex. 3. Solve the differential equation

$$\frac{d^2y}{dx^2} + a \frac{dy}{dx} + by = 0,$$

a, b being constants.

Solution. Proceed yourself.

5.8 Synopsis of the forms of solutions

To solve an equation of the form

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n)y = 0:$$

(Delhi Hous. 66)

1. Find the roots of the auxiliary equation, viz.
 $D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n = 0$.
2. Put the General Solution as follows :

Roots of Auxi. Equation	Complete Solution
Case I All roots $m_1, m_2, m_3, \dots, m_n$ real and different.	$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x}$
Case II $m_1 = m_2$ but other roots real and different.	$y = (C_1 + C_2 x) e^{m_1 x} + C_3 e^{m_3 x} + \dots + C_n e^{m_n x}$
Case III (Imag. Roots)	
1. $\alpha \pm i\beta$, a pair of imaginary roots.	Corresponding part of the general solution is $e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$ or $C_1 e^{\alpha x} \cos (\beta x + C_2)$ or $C_1 e^{\alpha x} \sin (\beta x + C_2)$.
2. $(\alpha \pm i\beta), (\alpha \pm i\beta)$ repeated twice.	Corresponding part of general solution is $y = e^{\alpha x} [(C_1 + C_2 x) \cos \beta x + (C_3 + C_4 x) \sin \beta x]$.

Ex. 1. Solve $\frac{d^4 y}{dx^4} - a^4 y = 0$.

Solution. The auxiliary equation is $(D^4 - a^4) = 0$

or $(D^2 - a^2)(D^2 + a^2) = 0, D = \pm a, \pm ai$.

\therefore solution is $y = C_1 e^{ax} + C_2 e^{-ax} + (C_3 \cos ax + C_4 \sin ax)$.

Ex. 2. Solve $\frac{d^4 y}{dx^4} + m^4 y = 0$.

[Agra B. Sc. 55]

Solution. Auxiliary equation is $D^4 + m^4 = 0$

or $(D^2 + m^2)^2 - 2m^2 D^2 = 0$

or $(D^2 - \sqrt{2m}D + m^2)(D^2 + \sqrt{2m}D + m^2) = 0$.

When $D^2 - \sqrt{2m}D + m^2 = 0, D = \frac{m \pm mi}{\sqrt{2}}$.

When $D^2 + \sqrt{2m}D + m^2 = 0, D = \frac{-m \pm mi}{\sqrt{2}}$.

i.e., roots of auxiliary equation are $\frac{m}{\sqrt{2}} \pm \frac{m}{\sqrt{2}} i, -\frac{m}{\sqrt{2}} \pm \frac{m}{\sqrt{2}} i$.

Hence the general solution is

$$y = e^{(m/\sqrt{2})x} C_1 \cos\left(\frac{m}{\sqrt{2}}x + C_2\right) + e^{(-m/\sqrt{2})x} C_3 \cos\left(\frac{m}{\sqrt{2}}x + C_4\right).$$

5.9. General solution of $(D^n + a_1 D^{n-1} + \dots + a_n) y = X$ (1)

[Bombay 61 : Gujarat 52]

To show that if $y = Y$ is a complete solution of

$$(D^n + a_1 D^{n-1} + \dots + a_n) y = 0 \quad \dots (2)$$

and $y = u$ is a particular solution of (1); then $y = Y + u$ is a general solution of (1). [Nagpur B.Sc. 55 (S)]

Since $y = Y$ is a solution of (2), we have

$$(D^n + a_1 D^{n-1} + \dots + a_n)(Y) = 0. \quad \dots (3)$$

Also since $y = u$ is a solution of (1), we have

$$(D^n + a_1 D^{n-1} + \dots + a_n) u = X. \quad \dots (4)$$

Adding (3) and (4), we have

$$(D^n + a_1 D^{n-1} + \dots + a_n)(Y + u) = X.$$

This shows that $y = Y + u$ is a solution of (1). Now Y being a general solution of (2) contains n arbitrary constants and as such $Y + u$ also contains n arbitrary constants. Therefore $y = Y + u$ is a general solution of (1).

Note 1. In the general solution $y = Y + u$ of the equation (1), Y is called the Complementary Function (C.F.) and u is called the Particular Integral (P. I.) and thus

The General Solution = C.F. + P.I.

2. The solution Y of (2) can be determined by the methods discussed above. The problem is now to find the particular integral u of (1). We give below certain methods of finding u .

Ex. Define the Complementary Function and Particular Integral for the linear differential equation with constant coefficients $f(D)y = X$. [Karnatak 62]

5.10. Meaning of the symbol $\frac{1}{f(D)}$.

Def. $\frac{1}{f(D)} X$ is that function of x , free from arbitrary constants, which when operated by $f(D)$ gives X .

Thus $f(D) \cdot \frac{1}{f(D)} X = X$.

Therefore $f(D)$ and $\frac{1}{f(D)}$ are inverse operators (i.e. they cancel each other's effect on the function on which they operate)

Thus the symbol $\frac{1}{D}$ stands for integration.

5.11. $\frac{1}{f(D)} X$ is the particular integral of $f(D) y = X$.

Clearly $\frac{1}{f(D)} X$ will be solution of (1) if it satisfies (1).

So putting $\frac{1}{f(D)} X$ for y in (1), we get

$f(D) \frac{1}{f(D)} X = X$ i.e., $X = X$, which is true.

It means that $\frac{1}{f(D)} X$ is a particular solution of (1).

Therefore to find the particular solution of $f(D) y = X$, we should find the value of $\frac{1}{f(D)} X$.

Note. We know that in solving $f(D) y = 0$, $f(D) = 0$ forms the auxiliary equation, which can be resolved into linear factors (real or imaginary). Therefore $\frac{1}{f(D)}$ can be resolved into partial

fractions. The partial fractions will be of the form $\frac{1}{D-\alpha}$ where α is real or imaginary.

5.12. To show that $\frac{1}{D-\alpha} X = e^{\alpha x} \cdot \frac{1}{D} (e^{-\alpha x} X)$.

Suppose $y = \frac{1}{D-\alpha} X$; then $(D-\alpha) y = X$.

or $\frac{dy}{dx} - \alpha y = X$; this is linear in y , as $D \equiv \frac{d}{dx}$.

\therefore Integrating factor $= e^{\int P dx} = e^{\int -\alpha dx} = e^{-\alpha x}$

and the solution is $ye^{-\alpha x} = \int e^{-\alpha x} X dx$.

(constant is not added as it is the particular solution)

or $y = e^{\alpha x} \int e^{-\alpha x} X dx$

$= e^{\alpha x} \frac{1}{D} (Xe^{-\alpha x})$ as $\frac{1}{D} \equiv \text{integration}$.

5.13. Working rule for finding the Particular integral of $f(D) y = X$.

Let $f(D) = (D-\alpha_1)(D+\alpha_2)\dots(D-\alpha_n)$.

Then resolving into partial fraction, we get

$$\frac{1}{f(D)} = \frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \dots + \frac{A_n}{D-\alpha_n} \text{ say.}$$

Now particular integral

$$= \frac{1}{f(D)} X = \left\{ \frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \dots + \frac{A_n}{D-\alpha_n} \right\} X$$

$$= A_1 \frac{1}{D-\alpha_1} X + A_2 \frac{1}{D-\alpha_2} X + \dots + A_n \frac{1}{D-\alpha_n} X$$

$$= A_1 e^{\alpha_1 x} \int e^{-\alpha_1 x} X dx + A_2 e^{\alpha_2 x} \int e^{-\alpha_2 x} X dx + \dots$$

$$+ A_n e^{\alpha_n x} \int e^{-\alpha_n x} X dx.$$

which can in general be evaluated and thus the particular integral can be found.

Particular Integral in some special cases.

5.14. Particular Integral when $X = e^{ax}$

[Nagpur 61; Poona 61; Karnatak 61;
Gujrat 59; Bombay 61]

By successive differentiation, we find that

$$e^{\alpha x} = e^{\alpha x}, \quad \dots(1)$$

$$De^{\alpha x} = ae^{\alpha x} \quad \dots(2)$$

$$D^2 e^{ax} = a^2 e^{ax}, \quad \dots (3)$$

.....
.....

$$D^n e^{ax} = a^n e^{ax}, \quad \dots (n)$$

If $f(D) = (D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n)$, then multiplying (1), (2), (3).....(n) by $a_n, a_{n-1}, \dots, 1$ respectively and adding, we obtain

$$f(D) e^{ax} = f(a) e^{ax}$$

Now operating on both the sides by $\frac{1}{f(D)}$,

$$\frac{1}{f(D)} f(D) e^{ax} = \frac{1}{f(D)} f(a) e^{ax}$$

$$\text{or } e^{ax} = f(a) \frac{1}{f'(D)} e^{-ax} \text{ or } \frac{1}{f(a)} e^{ax} = \frac{1}{f'(D)} e^{-ax},$$

dividing by $f(a) \neq 0$

Therefore $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$, provided that $f(a) \neq 0$.

Ex. 1. Solve $\frac{d^2y}{dx^2} - 2k \frac{dy}{dx} + k^2y = 0$.

[Nagpur 1957]

Solution. Auxiliary equation is $D^3 - 2kD + k^2 = 0$,

$$\text{i.e., } (D-k)^2=0 \quad \text{or} \quad D=k, k.$$

$$\therefore \text{C.F.} = (C_1 + C_2 x) e^{kx},$$