Matrix
- Matrix is a rectangular arrangement of numbers (real number/ complex number) in m rows and n columns is called matrix of order (or size) m by n matrix.
Row (/column) matrix
- A matrix in which there is only one row is called a row matrix.
Null
- A matrix in which every element is zero is called an null matrix.
Equal
- Two matrices are said to be equal if and only if they are of the same order (i.e. they have the same row and column number) and each element of one is equal to the corresponding of the other one.
Square
- An m x n matrix A is said to be a square matrix if m = n, i.e. number of rows is equal to the number of columns.
Diagonal
- A square matrix is called diagonal matrix if each of it's non diagonal element is zero
Scalar
- A diagonal matrix whose diagonal elements are all equal
Unity/ Identity
- A square matrix where diagonal elements are unity (or one) and remaining elements are zero
Sub
Upper (/Lower) triangle
- A square matrix in which all the elements below the principal diagonal are zero
Symmetric
- A square matrix in which aij = aji i.e. (i, j)th element is the same as the (j, i)th element
Skew symmetric
- A square matrix in which aij = -aji i.e. (i, j)th element is the negative (j, i)th element

$$A = \begin{bmatrix} 6 & -2 & 0 \\ -1 & -3 & 1 \\ 5 & 2 & -4 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 & 6 \\ 7 & -2 & 8 \\ 0 & 5 & 9 \end{bmatrix}$$

$$AB = \begin{bmatrix} 6 & -2 & 0 \\ -1 & -3 & 1 \\ 5 & 2 & -4 \end{bmatrix} \begin{bmatrix} 3 & 4 & 6 \\ 7 & -2 & 8 \\ 0 & 5 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 6\times3 + -2\times7 + 0\times0 & 6\times4 + -2\times-2 + 0\times5 & 6\times6 + -2\times8 + 0\times9 \\ -1\times3 + -3\times7 + 1\times0 & -1\times4 + -3\times-2 + 1\times5 & -1\times6 + -3\times8 + 1\times9 \\ 5\times3 + 2\times7 + -4\times0 & 5\times4 + 2\times-2 + -4\times5 & 5\times6 + 2\times8 + -4\times9 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 28 & 20 \\ -24 & 7 & -21 \\ 29 & -4 & 10 \end{bmatrix}$$

Find the inverse of a matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 2 & 9 \end{pmatrix}$

$M_{1,1} = \det\begin{pmatrix} 5 & 6 \\ 2 & 9 \end{pmatrix} = 33$

$M_{1,2} = \det\begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} = -6$

$M_{1,3} = \det\begin{pmatrix} 4 & 5 \\ 7 & 2 \end{pmatrix} = -27$

$M_{2,1} = \det\begin{pmatrix} 2 & 3 \\ 2 & 9 \end{pmatrix} = 12$

$M_{2,2} = \det\begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix} = -12$

$M_{2,3} = \det\begin{pmatrix} 1 & 2 \\ 7 & 2 \end{pmatrix} = -12$

$M_{3,1} = \det\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} = -3$

$M_{3,2} = \det\begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix} = -6$

$M_{3,3} = \det\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}: \quad -3$

cofactors: $\begin{pmatrix} 33 & 6 & -27 \\ -12 & -12 & 12 \\ -3 & 6 & -3 \end{pmatrix}$

$\begin{pmatrix} 33 & 6 & -27 \\ -12 & -12 & 12 \\ -3 & 6 & -3 \end{pmatrix}^T = \begin{pmatrix} 33 & -12 & -3 \\ 6 & -12 & 6 \\ -27 & 12 & -3 \end{pmatrix}$ Let us find the minors of the given matrix as given below:

Now,

$A^{-1} = (1/|A|)$ Adj A

Hence, the inverse of the given matrix is:

$$= \begin{pmatrix} -\frac{11}{12} & \frac{1}{3} & \frac{1}{12} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{3}{4} & -\frac{1}{3} & \frac{1}{12} \end{pmatrix}$$

Solve the following equations by matrix inversion

2x+y+2z=0

2x-y+z=10

x+3y-z=5

The given equation can be written in a matrix form as AX = D, and then by obtaining A^{-1} and multiplying it on both sides, we can solve the given problem.

$$\begin{bmatrix} 2 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix}$$

AX = D where A

$$= \begin{bmatrix} 2 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, D = \begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix}$$

$\Rightarrow\ A^{-1}(AX) = A^{-1}D$

$(A^{-1}A)X = A^{-1}D$

$\Rightarrow IX = A^{-1}D$

$\Rightarrow X = A^{-1}D \ldots\ldots (i)$

Now

$$A^{-1} = \frac{adj\ A}{|A|}; \qquad |A| = \begin{vmatrix} 2 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{vmatrix} = 2(1-3) - 1(-2-1) + 2(6+1) = 13$$

The matrix of co-factors of

$$|A|\ is\ \begin{bmatrix} -2 & 3 & 7 \\ 7 & -4 & -5 \\ 3 & 2 & -4 \end{bmatrix}.\ So,\ adj\ A \quad = \begin{bmatrix} -2 & 7 & 3 \\ 3 & -4 & 2 \\ 7 & -5 & -4 \end{bmatrix}$$

$\therefore$

$$A^{-1} = \frac{1}{13}\begin{bmatrix} -2 & 7 & 3 \\ 3 & -4 & 2 \\ 7 & -5 & -4 \end{bmatrix}$$

The matrix of co-factors of

$$|A|\ is\ \begin{bmatrix} -2 & 3 & 7 \\ 7 & -4 & -5 \\ 3 & 2 & -4 \end{bmatrix}.\ So,\ adj\ A \quad = \begin{bmatrix} -2 & 7 & 3 \\ 3 & -4 & 2 \\ 7 & -5 & -4 \end{bmatrix}$$

$\therefore$

$$A^{-1} = \frac{1}{13}\begin{bmatrix} -2 & 7 & 3 \\ 3 & -4 & 2 \\ 7 & -5 & -4 \end{bmatrix}$$

$\Rightarrow$ from (i),

$$X = \frac{1}{13}\begin{bmatrix} -2 & 7 & 3 \\ 3 & -4 & 2 \\ 7 & -5 & -4 \end{bmatrix}\begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} 0+70+15 \\ 0-40+10 \\ 0-50-20 \end{bmatrix} = \begin{bmatrix} 85/13 \\ -30/13 \\ -70/13 \end{bmatrix}$$

$\therefore$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 85/13 \\ -30/13 \\ -70/13 \end{bmatrix} \Rightarrow\ x = \frac{85}{13}, y = \frac{-30}{13}, z = \frac{-70}{13}$$

Rank of a Matrix

Example 1: Find the rank of matrix A by using the row echelon form.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$$

Solution:

Given,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$$

Now, we apply elementary transformations.

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

We get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & -6 & -4 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The above matrix is in row echelon form.

Number of non-zero rows = 2

Hence, the rank of matrix A = 2.

Example 2: Find the rank of the given matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

Solution:

Given,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

Now, we transform matrix A to echelon form by using elementary transformation.

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Number of non-zero rows = 2

Hence, the rank of matrix A = 2

Eigenvalue and Eigenvector

Let A be an n×n matrix.

1. An **eigenvector** of A is a *nonzero* vector v in Rn such that Av=λv, for some scalar λ
2. An **eigenvalue** of A is a scalar λ such that the equation Av=λv has a *nontrivial* solution.

If Av=λv for v≠0, we say that λ is the **eigenvalue for** v, and that v is an **eigenvector for** λ.

The **characteristic equation** is the equation which is solved to find a matrix's eigenvalues, also called the characteristic polynomial.

Determine the characteristics roots of the matrix

$$\begin{bmatrix} 4 & -20 & -10 \\ -2 & 10 & 4 \\ 6 & -30 & -13 \end{bmatrix}$$

Solution: Let, A = $\begin{bmatrix} 4 & -20 & -10 \\ -2 & 10 & 4 \\ 6 & -30 & -13 \end{bmatrix}$

The order of A is 3 X 3, so the unit matrix I = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now we have to multiply λ with unit matrix I,

λI = $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

$$A-\lambda I = \begin{bmatrix} 4-\lambda & -20 & -10 \\ -2 & 10-\lambda & 4 \\ 6 & -30 & -13-\lambda \end{bmatrix}$$

= (4-λ)[(10-λ)(-13-λ)+120]+ 20[-2(-13-λ)-24]-10[60-6(10-λ)]

= (4-λ)[-10+3λ +λ^2]+20[2+2λ]-10[6λ]

= 4λ^2+12λ -40-λ^3-3λ^2+10λ +40λ +40-60λ

= λ^3 + 1λ^2 + 2λ

To find roots let |A-I| = 0

λ^3 + 1λ^2 + 2λ = 0

For solving this equation -λ from all the terms

$\lambda^3 + 1\lambda^2 + 2\lambda = 0$

$-\lambda=0$ (or) $\lambda^2-1\lambda -2=0$

$\lambda = 0, -1, 2$

Therefore the characteristic roots (or) Eigen values are x = 0, -1, 2

Determine the characteristics vector of the matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Solution: Let, A = $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

The order of A is 3 X 3, so the unit matrix I = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now we have to multiply λ with unit matrix I,

$\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}$

$= (1-1)\,[(5-\lambda)(1-\lambda) - 1]-1\,[1(1-\lambda) - 3] + 3\,[1-3(5-\lambda)]$

$= (1-\lambda)\,[\lambda^2-6\lambda +4] -1[-2-\lambda] + 3[-14+3\lambda]$

$= -\lambda^3 + 7\lambda^2 - 36$

To find roots let $|A-\Lambda I| = O$

$-\lambda^3 + 7\lambda^2 - 36 = 0$

Therefore the characteristic roots are x = 3, -2, 6

Substitute λ = 3 in the matrix A-λI = $\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix}$

From this matrix we are going to form three linear equations using variables x, y and z

-2x+ 1y + 3z = 0 …………………(1)

1x + 2y + 1z = 0 …………………(2)

3x + 1y – 2z = 0 …………………(3)

By solving (1) and (2) we get the eigen vector

The eigen vector x = $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Substitute λ = -2 in the matrix A-λI = $\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix}$

From this matrix we are going to form three linear equations using variables x, y and z

3x+ 1y + 3z = 0 …………………..(4)

1x + 7y + 1z = 0 …………………(5)

3x + 1y + 3z = 0 …………………(6)

By solving (4) and (5) we get the eigen vector

The eigen vector y = $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Substitute λ = 6 in the matrix A-λI = $\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix}$

Simillarly, by solving it, we get,

The eigen vector z = $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Cayley-Hamilton Theorem

• "A square matrix satisfies its own characteristic equation".

• Every square matrix satisfied its own characteristic equation

I.e. if A is an n x m matrix whose characteristic equation is,

Suppose the characteristic polynomial of an n × n square matrix, A, is given as
p(λ) = λn+an-1λn-1+...+a1λ+a0
Then p(A) = An+an-1An-1+...+a1A+a0In= 0

Thus, p(A) = 0.

Scalar Quantity

A scalar quantity is defined as the physical quantity with only magnitude and no direction.

Vector Quantity

A vector quantity is defined as the physical quantity that has both directions as well as magnitude.

WORKED OUT EXAMPLES
Example 1
Find the angle between the vectors 2i + j + k -j + 2k and also find a unit vector perpendicular to the above vector [D.U.P. 1980]

Let a = 2i + j + k and b = i - j + 2k
then a.b = (2i + j + k) · (i - j + k)
= 2i i - j j + 2k · k + 0
= 2-1 + 2
= 3.
Also |a| = |2i + j + k| $=\sqrt{2^2+1^2+1^2}$ √6
|b| = |i - j + 2k| = $\sqrt{1^2+(-1)^2+2^2}$ = √6
Let ∅ be the angle between the two vectors a and b then
cos θ= $\frac{a.b}{|a||b|}$= $\frac{3}{\sqrt{6}\sqrt{6}}$= $\frac{3}{6}$= $\frac{1}{2}$= cos60◦

Second portion
Again a × b = (2i + j + k) × (i - j + 2k)
= 2i × i - 2i × j + 4i × k + j ×i-j×j
+2j×k + k × i - k × j + 2k × k
= 0 - 2k - 4j-k – 0 + 2i + j+i+0
=3i-3j-3K
Now the required unit vector e is given by
e=$\frac{a \times b}{|a \times b|}$ = $\frac{|3i-3j-3k|}{\sqrt{3^2+(-3)^2+(-3)^2}}$ =$\frac{3(i-j-k)}{3\sqrt{3}}$

Example 5
What is the unit vector perpendicular to each of the vector's **u = i + 2j+3k** and **v=5i - j + 2k,** Calculate the sine of the angle between these vectors. D
Solution: u * v = (i + 2j+3k) x (5i-j+2k)

$$\begin{vmatrix} I & j & k \\ 1 & 2 & 3 \\ 5 & -1 & 2 \end{vmatrix}$$

=7i+13j-11k
:: |u* v|= $\sqrt{(72+132+(-11)2)}$= $\sqrt{339}$
If n be a unit vector perpendicular to the plane of u and B, then since u, v is also perpendicular to the plane of u and v, we have:

$$\frac{u*v}{|u*v|} = \frac{7i+13j-11k}{\sqrt{339}} = \frac{7}{\sqrt{339}}i + \frac{13}{\sqrt{339}}j - \frac{11}{\sqrt{339}}k$$

Again, |u|=|i+2j+3k|=$\sqrt{1^2+2^2+3^2}=\sqrt{14}$
|v|=|5i-j+2k|=$\sqrt{5^2+(-1)^2+2^2}=\sqrt{30}$
Now let 6 be the angle between the directions of u and v then u *v=|u| |v| $\sin\theta$ n
Or, $\sin\theta$ = $\frac{u*v}{|u||v|n}$
Or, $\sin\theta = \frac{|u*v|}{|u||v|} = \frac{\sqrt{339}}{\sqrt{14}\sqrt{30}}$

$\sin\theta=\sqrt{\frac{339}{420}}$

Example 7. Prove that [c * a a * b b * c] = [a b c]^2
Proof : [c*a a*b b*c] = (c*a) .{(c*a). (a*b). (b*c)}
= {(c*a) * (a*b)}. (b*c)
Let c*a =P then we have
(c*a) * (a*b) = P * (a*b)
= (p.b)a – (p.a)b
Now putting the value of **P** in the above equation, we have
(c*a) * (a*b) = (c*a.b)a – (c*a.a)b
=[cab]a – 0 *since* **c * a.a = 0**
= [cab]a.
[c*a a*b b*c] = {(c*a) * (a*b) * (b*c)}
= [cab]a . (b*c)
=[cab] . [abc] *since* **[cab]=[abc]**

= [abc].[abc] = [abc]^2

Hence **[c*a a*b b*c]= [abc]^2.**