প্রশ্নমালা–2(A)

1(i). ধরি $1 - \cos x = t$ তবে $\sin x\, dx = dt$

$$I = \int \frac{dt}{t^2} = -\frac{1}{t} + c = -\frac{1}{1 - \cos x} + c.$$

(ii). ধরি $1 + \sin x = t$ তবে $\cos x\, dx = dt$

$$\therefore I = \int \frac{dt}{t^2} = -\frac{1}{t} + c = -\frac{1}{1 + \sin x} + c.$$

(iii). ধরি $a\sin x - b\cos x + c = t$ তবে $(a\cos x + b\sin x)\, dx = dt$

$$\therefore I = \int \frac{dt}{t^2} = -\frac{1}{t} + c_1 = \frac{-1}{a\sin x - b\cos x + c} + c_1.$$

(iv). ধরি $a^2\sin^2 x + b^2\cos^2 x = z$

তবে $(2a^2 \sin x \cos x - 2b^2 \cos x \sin x)\, dx = dz$

বা $(a^2 - b^2) \sin 2x\, dx = dz$

$$\therefore I = \frac{1}{a^2 - b^2} \int \frac{dz}{z^2} = \frac{-1}{(a^2 - b^2)} \cdot \frac{1}{z} + c$$

$$= \frac{1}{(b^2 - a^2)(a^2 \sin^2 x + b^2 \cos^2 x)} + c.$$

(v). ধরি $a + b\sin x = t$ তবে $b \cos x\, dx = dt$

$$\therefore I = \frac{1}{b} \int \frac{dt}{t^3} = -\frac{1}{2bt^2} + c = \frac{-1}{2b(a + b\sin x)^2} + c.$$

(vi). ধরি $x + \sin x = z$ তবে $(1 + \cos x)\, dx = dz$

$$I = \int \frac{dz}{z^{1/3}} = \int z^{-1/3}\, dz = \frac{z^{2/3}}{2/3} + c = \frac{3}{2}(x + \sin x)^{2/3} + c.$$

2(i). ধরি $x^2 = t$ তবে $2x\, dx = dt$

$$I = \frac{1}{2} \int 7^t\, dt = \frac{1}{2} \cdot \frac{7^t}{\ln 7} + C = \frac{7^{x^2}}{2\ln 7} + c.$$

(ii). ধরি $x + \frac{1}{x} = t$ তবে $\left(1 - \frac{1}{x^2}\right) dx = dt$

$$\therefore I = \int e^t\, dt = e^t + c = e^{(x + 1/x)} + c.$$

(iii). ধরি $\cos x = t$ তবে $-\sin x\, dx = dt$

$$\therefore I = -\int \frac{dt}{t^3} = \frac{1}{2t^2} + c = \frac{1}{2\cos^2 x} + c.$$

(iv). ধরি $2 + 3\ln x = z$ তবে $\frac{3}{x}dx = dz$

$$\therefore I = \frac{1}{3}\int \sin z\, dz = -\frac{1}{3}\cos z + c = -\frac{1}{3}\cos(2 + 3\ln x) + c.$$

(v). ধরি $\ln(\ln x) = z$ তবে $\frac{1}{\ln x}\cdot\frac{1}{x}dx = dz$

$$\therefore I = \int z\, dz = \frac{1}{2}z^2 + c = \frac{1}{2}[\ln(\ln x)]^2 + c.$$

(vi). ধরি $\ln x = z$ তবে $\frac{1}{x}dx = dz$

$$\therefore I = \int \frac{dz}{\cos^2 z} = \int \sec^2 z\, dz = \tan z + c = \tan(\ln x) + c$$

(vii). ধরি $2 + a\ln x = z$ তবে $\frac{a}{x}dx = dz$

$$I = \frac{1}{a}\int \cos z\, dz = \frac{1}{a}\sin z + c$$

$$= \frac{1}{a}\sin(2 + a\ln x) + c.$$

(viii). ধরি $\ln(\sec x + \tan x) = z$ তবে $\frac{(\sec x\tan x + \sec^2 x)\,dx}{\sec x + \tan x} = dz$

$$\Rightarrow \frac{\sec x(\sec x + \tan x)\,dx}{(\sec x + \tan x)} = dz$$

$$\Rightarrow \sec x\, dx = dz$$

$$I = \int z\, dz = \frac{z^2}{2} + c = \frac{1}{2}[\ln(\sec x + \tan x)]^2 + c.$$

(ix). ধরি $1 - \ln x = z$ তবে $-\frac{1}{x}dx = dz$

$$\therefore I = -\int z^2\, dz = -\frac{1}{3}z^3 + c = -\frac{1}{3}(1 - \ln x)^3 + c$$

(x). ধরি $e^x + \ln x = z$ তবে $(e^x + 1/x)\,dx = dz$

$$I = \int z\, dz = \frac{1}{2}z^2 + c = \frac{1}{2}(e^x + \ln x)^2 + c.$$

(xi). ধরি $e^{\sqrt{x}} = z$ তবে $\frac{e^{\sqrt{x}}}{2\sqrt{x}}dx = dz$

$$\therefore I = 2\int \cos z\, dz = 2\sin z + c = 2\sin e^{\sqrt{x}} + c.$$

(xii). ধরি $x + \ln x = z$ তবে $(1 + 1/x)\,dx = dz$, বা $\dfrac{(x+1)}{x}dx = dz$

$$\therefore I = \frac{1}{2}\int z^2\,dz = \frac{1}{6}z^3 + C = \frac{1}{6}(x + \ln x)^3 + c.$$

3(i). $I = \displaystyle\int \frac{(1+\tan^2 x)\sec^2 x}{\sqrt{\tan x}}dx$; ধরি $\tan x = z^2$ তবে $\sec^2 x\,dx = 2z\,dz$

$$= 2\int\frac{(1+z^4)z\,dz}{z} = 2\int(1+z^4)\,dz = 2\left[z + \frac{z^5}{5}\right] + c.$$

(ii). $I = \displaystyle\int\frac{dx}{(e^x + 1/e^x)^2} = \int\frac{e^{2x}\,dx}{(e^{2x}+1)^2}$

ধরি $e^{2x} + 1 = t$ তবে $2e^{2x}\,dx = dt$

$$I = \frac{1}{2}\int\frac{dt}{t^2} = -\frac{1}{2t} + c = \frac{-1}{2(e^{2x}+1)} + c.$$

(iii). ধরি $1 + x^3 = z^2$ তবে $3x^2\,dx = 2z\,dz$ বা $dx = \dfrac{2zdz}{3x^2}$

$$\Rightarrow \frac{dx}{x} = \frac{2z\,dz}{3x^3} = \frac{2z\,dz}{3(z^2-1)}$$

$$\therefore I = \frac{2}{3}\int\frac{z\,dz}{(z^2-1)z} = \frac{2}{3}\int\frac{dz}{z^2-1} = \frac{1}{3}\ln\frac{z-1}{z+1} + c.$$

(iv). ধরি $1 + e^x = z$ তবে $e^x\,dx = dz$

$$\therefore I = \int z^{1/2}\,dz = \frac{z^{3/2}}{3/2} + c = \frac{2}{3}(1+e^x)^{3/2} + c.$$

(v). ধরি $a + b\cos x = z$ বা $\cos x = (z-a)/b$ তবে $-\sin x\,dx = dz/b$

$$I = \int\frac{2\sin x\cos x}{(a+b\cos x)^2}dx = -\frac{2}{b^2}\int\frac{(z-a)\,dz}{z^2} = -\frac{2}{b^2}\int\left[\frac{1}{z} - \frac{a}{z^2}\right]dz$$

$$= -\frac{2}{b^2}\left[\ln z + \frac{a}{z}\right] + c = -\frac{2}{b^2}\left[\ln(a+b\cos x) + \frac{a}{a+b\cos x}\right] + c.$$

(vi). ধরি $a^x + x^a = z$ তবে $(a^x\ln a + ax^{a-1})\,dx = dz$

$$\Rightarrow a[a^{x-1}\ln a + x^{a-1}]\,dx = dz$$

$$\Rightarrow (a^{x-1}\ln a + x^{a-1})\,dx = dz/a$$

$$I = \frac{1}{a}\int\frac{dz}{z} = \frac{1}{a}\ln z + c$$

$$= \frac{1}{a}\ln[a^x + x^a] + c.$$

(vii). $I = \int\left[\frac{1}{\sqrt{x}} + \frac{\ln x}{x}\right]dx$

$= \int\frac{dx}{\sqrt{x}} + \int \ln x \, d(\ln x) = 2\sqrt{x} + \frac{1}{2}(\ln x)^2 + c.$

4(i). ধরি $\tan^{-1}x = z$ তবে $\frac{dx}{1+x^2} = dz$

$\therefore I = \int\frac{dz}{z^2} = -\frac{1}{z} + c = -\frac{1}{\tan^{-1}x} + c.$

(ii). ধরি $\tan^{-1}x = z$ তবে $\frac{1}{1+x^2}dx = dz$

$\therefore I = \int e^{mz}dz = \frac{e^{mz}}{m} + c = \frac{1}{m}e^{m\tan^{-1}x} + c.$

(iii). ধরি $\tan^{-1}x^4 = z$ তবে $\frac{4x^3}{1+x^8}dx = dz$

$\therefore I = \frac{1}{4}\int z\,dz = \frac{1}{4}\cdot\frac{1}{2}z^2 + c = \frac{1}{8}(\tan^{-1}x^4)^2 + c.$

(iv). ধরি $\tan^{-1}x^3 = z$ তবে $\frac{3x^2\,dx}{1+x^6} = dz$

$\therefore I = \frac{1}{3}\int z\,dz = \frac{1}{6}z^2 + c = \frac{1}{6}[\tan^{-1}x^3)^2 + c.$

(v). ধরি $\sin^{-1}x^2 = z$ তবে $\frac{2x\,dx}{\sqrt{1-x^4}} = dz$

$\therefore I = \int z\,dz = \frac{1}{2}z^2 + c = \frac{1}{2}(\sin^{-1}x^2)^2 + c.$

(vi). ধরি $\sin^{-1}x = z$ তবে $\frac{dx}{\sqrt{1-x^2}} = dz$

$\therefore I = \int e^{az}dz = \frac{e^{az}}{a} + c = \frac{1}{a}e^{a\sin^{-1}x} + c.$

(vii). ধরি $\sin^{-1}x = z$ তবে $\frac{dx}{\sqrt{1-x^2}} = dz$

$\therefore I = \int \tan z\,dz = \ln(\sec z) + c = \ln\sec(\sin^{-1}x) + c.$

(viii). ধরি $\sec^{-1}x = z$ তবে $\frac{dx}{x\sqrt{x^2-1}} = dz$

$\therefore I = \int\frac{dz}{z} = \ln z + c = \ln(\sec^{-1}x) + c.$

প্রশ্নমালা–2(B)

1(i). ধরি $4x = z$ তবে $4\,dx = dz$

$$\therefore I = \frac{1}{4}\int \frac{dz}{\sqrt{2^2 - z^2}} = \frac{1}{4}\sin^{-1}\frac{z}{2} + c$$

$$= \frac{1}{4}\sin^{-1}\left(\frac{4x}{2}\right) + c = \frac{1}{4}\sin^{-1}2x + c.$$

(ii). ধরি $\sqrt{2}\,x = z$ তবে $\sqrt{2}\,dx = dz$

$$\therefore I = \frac{1}{\sqrt{2}}\int \sqrt{(\sqrt{5})^2 + z^2}\,dz = \frac{1}{\sqrt{2}}\left[\frac{z\sqrt{5+z^2}}{2} + \frac{5}{2}\sinh^{-1}\frac{z}{\sqrt{5}}\right] + c.$$

(iii). $$I = \int \frac{dx}{(a^2 - x^2)(a^2 + x^2)} = \frac{1}{2a^2}\int\left[\frac{1}{a^2 - x^2} + \frac{1}{a^2 + x^2}\right]dx$$

$$= \frac{1}{2a^2}\left[\frac{1}{2a}\log\frac{a+x}{a-x} + \frac{1}{a}\tan^{-1}\frac{x}{a}\right] + c.$$

2(i). ধরি $x^3 = z$ তবে $3x^2\,dx = dz$

$$I = \int \frac{dz}{1+z^2} = \tan^{-1}z + c = \tan^{-1}x^3 + c.$$

(ii). ধরি $x^2 = z$ তবে $2x\,dx = dz$

$$I = \frac{1}{2}\int \frac{dz}{z^2 + (a^2)^2} = \frac{1}{2a^2}\tan^{-1}\frac{z}{a^2} + c = \frac{1}{2a^2}\tan^{-1}\frac{x^2}{a^2} + c$$

(iii). ধরি $x^2 = z$ তবে $2x\,dx = dz$

$$\therefore I = \frac{1}{2}\int \frac{dz}{z^2 - 1} = \frac{1}{4}\ln\frac{z-1}{z+1} + c$$

$$= \frac{1}{4}\ln\frac{x^2-1}{x^2+1} + c.$$

(iv). ধরি $x^4 = z$ তবে $4x^3\,dx = dz$

$$\therefore I = \frac{1}{4}\int \frac{dz}{\sqrt{(a^4)^2 - z^2}} = \frac{1}{4}\sin^{-1}\frac{z}{a^4} + c = \frac{1}{4}\sin^{-1}\frac{x^4}{a^4} + c.$$

(v). ধরি $x^2 = z$ তবে $2x\,dx = dz$

$$\therefore I = \frac{1}{2}\int \frac{dz}{\sqrt{(a^2)^2 + z^2}} = \frac{1}{2}\sinh^{-1}\frac{z}{a^2} + c = \frac{1}{2}\sinh^{-1}\frac{x^2}{a^2} + c.$$

(vi). ধরি $x^4 = z$ তবে $4x^3\,dx = dz$

$$I = \frac{1}{4}\int \frac{dz}{\sqrt{z^2+1}} = \frac{1}{4}\sinh^{-1}z + c = \frac{1}{4}\sinh^{-1}x^4 + c.$$

(vii). ধরি $x^3 = z$ তবে $3x^2\,dx = dz$

$$I = \frac{1}{3}\int \sqrt{z^2-1}\,dz = \frac{1}{3}\left[\frac{z\sqrt{z^2-1}}{2} - \frac{1}{2}\cosh^{-1}z\right] + c.$$

(viii). ধরি $x^5 = z$ তবে $5x^4\,dx = dz$

$$I = \frac{1}{5}\int \sqrt{a^2-z^2}\,dz = \frac{1}{5}\left[\frac{z\sqrt{a^2-z^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{z}{a}\right] + c.$$

3(i). ধরি $\sin x = z$ তবে $\cos x\,dx = dz$

$$I = \int \frac{dz}{1+z^2} = \tan^{-1}z + c = \tan^{-1}(\sin x) + c.$$

(ii). $I = \int \frac{\cos x\,dx}{5 + 7(1-\sin^2 x)} = \int \frac{\cos x\,dx}{12 - 7\sin^2 x}$

ধরি $\sqrt{7}\sin x = z$ তবে $\sqrt{7}\cos x\,dx = dz$

$$I = \frac{1}{\sqrt{7}}\int \frac{dz}{(2\sqrt{3})^2 - z^2} = \frac{1}{\sqrt{7}}\cdot\frac{1}{2.2\sqrt{3}}\ln\frac{2\sqrt{3}+z}{2\sqrt{3}-z} + c.$$

(iii). ধরি $e^x = z$ তবে $e^x\,dx = dz$

$$I = \int \frac{dz}{\sqrt{1+z^2}} = \sinh^{-1}z + c = \sinh^{-1}(e^x) + c.$$

(iv). ধরি $\cot^{-1}x = z$ তবে $\frac{-1}{1+x^2}\,dx = dz$

$$\therefore I = -\int \frac{dz}{3^2+z^2} = \frac{1}{3}\cot^{-1}\frac{z}{3} + c$$

$$= \frac{1}{3}\cot^{-1}\left[\frac{1}{3}\cot^{-1}x\right] + c.$$

(v). ধরি $\sin^{-1}x = z$ তবে $\dfrac{dx}{\sqrt{1-x^2}} = dz$

$$I = \int \frac{dz}{\sqrt{1+z^2}} = \sinh^{-1}z + c = \sinh^{-1}(\sin^{-1}x) + c.$$

(vi). ধরি $\ln x = z$ তবে $\dfrac{1}{x}dx = dz$.

$$\therefore I = 2\int \frac{dz}{1+z^2} = 2\tan^{-1}z + c = 2\tan^{-1}(\ln x) + c.$$

(vii). ধরি $\tan^3 x = z$ তবে $3\tan^2 x.\ \sec^2 x\, dx = dz$

$$I = \int \frac{dz}{1+z^2} = \tan^{-1}z + c = \tan^{-1}(\tan^3 x) + c.$$

(viii). $I = \displaystyle\int \frac{dx}{e^x + 1/e^x} = \int \frac{e^x\, dx}{(e^x)^2 + 1} = \int \frac{dz}{z^2+1}$ | যখন $e^x = z$

$= \tan^{-1}z + c$ | $\Rightarrow e^x\, dx = dz$

$= \tan^{-1}e^x + c.$

4(i). $I = \displaystyle\int \frac{x^2(1 - 1/x^2)\, dx}{x^2(x^2 + 1/x^2)} = \int \frac{(1 - 1/x^2)\, dx}{(x + 1/x)^2 - 2}$

ধরি $x + 1/x = z$ তবে $(1 - 1/x^2)\, dx = dz$

$$\therefore I = \int \frac{dz}{z^2 - (\sqrt{2})^2} = \frac{1}{2\sqrt{2}} \ln \frac{z - \sqrt{2}}{z + \sqrt{2}} + c = \frac{1}{2\sqrt{2}} \ln \frac{x + 1/x - \sqrt{2}}{x + 1/x + \sqrt{2}}.$$

(ii). $I = \displaystyle\int \frac{1}{x^4+1}dx = \frac{1}{2}\int \frac{(x^2+1) - (x^2-1)}{x^4+1}dx$

$= \dfrac{1}{2}\displaystyle\int \frac{x^2+1}{x^4+1}dx - \frac{1}{2}\int \frac{x^2-1}{x^4+1}dx$; উদা–4 এবং 4(i) এর ন্যায়।

(iii). $I = \dfrac{1}{2}\displaystyle\int \frac{(x^2+a^2) + (x^2-a^2)}{x^4+a^4}dx$

$= \dfrac{1}{2}\left[\displaystyle\int \frac{x^2+a^2}{x^4+a^4}dx + \int \frac{x^2-a^2}{x^4+a^4}dx\right]$

$= \dfrac{1}{2}[I_1 + I_2]$ ধরি।

$$I_1 = \int \frac{x^2+a^2}{x^4+a^4}dx = \int \frac{x^2(1 + a^2/x^2)}{x^2(x^2 + a^4/x^2)}dx = \int \frac{(1 + a^2/x^2)\, dx}{(x - a^2/x)^2 + 2a^2}$$

ধরি $x - a^2/x = z$ তবে $(1 + a^2/x^2)\, dx = dz$

$$\therefore I_1 = \int \frac{dz}{z^2 + (a\sqrt{2})^2} = \frac{1}{a\sqrt{2}} \tan^{-1} \frac{z}{a\sqrt{2}} + c_1$$

$$= \frac{1}{a\sqrt{2}} \tan^{-1} \left\{\left(\frac{x - a^2/x}{a\sqrt{2}}\right)\right\} + c_1$$

$$I_2 = \int \frac{x^2 - a^2}{x^4 + a^4}\, dx = \int \frac{1 - a^2/x^2}{x^2 + a^4/x^2}\, dx = \int \frac{(1 - a^2/x^2)\, dx}{(x + a^2/x)^2 - 2a^2}$$

ধরি $x + \dfrac{a^2}{x} = z$ তবে $\left(1 - \dfrac{a^2}{x^2}\right) dx = dz$

$$\therefore I_2 = \int \frac{dz}{z^2 - (a\sqrt{2})^2} = \frac{1}{2a\sqrt{2}}\, ln \frac{z - a\sqrt{2}}{z + a\sqrt{2}} + c_2$$

(iv). $$I = \int \frac{(x^2 + 1 - \cos^2 x) \sec^2 x}{1 + x^2}\, dx$$

$$= \int \left[\frac{(x^2 + 1) \sec^2 x}{1 + x^2} - \frac{\cos^2 x \sec^2 x}{1 + x^2}\right] dx$$

$$= \int \left[\sec^2 x - \frac{1}{1 + x^2}\right] dx = \tan x - \tan^{-1} x + c.$$

(v). $$I = \int \sqrt{1 + \frac{1}{\cos x}}\, dx = \int \sqrt{\frac{1 + \cos x}{\cos x}}\, dx$$

$$= \int \sqrt{\frac{2\cos^2 x/2}{1 - 2\sin^2 x/2}}\, dx = \int \frac{\sqrt{2} \cos x/2}{\sqrt{1 - 2\sin^2 x/2}}\, dx$$

ধরি $\sqrt{2} \sin \dfrac{x}{2} = z$ তবে $\sqrt{2}\, \dfrac{1}{2} \cos \dfrac{x}{2}\, dx = dz.$

$$\therefore I = 2\int \frac{dz}{\sqrt{1 - z^2}} = 2\sin^{-1} z + c = 2\sin^{-1}\left(\sqrt{2} \sin \frac{x}{2}\right) + c$$

(vi). $I = \displaystyle\int \frac{\tan x \sec^2 x}{1 + \tan^4 x}\, dx$; [লব ও হরকে $\cos^4 x$ দ্বারা ভাগ করা হয়েছে]

ধরি $\tan^2 x = z \Rightarrow 2\tan x \sec^2 x\, dx = dz$

$$\therefore I = \frac{1}{2} \int \frac{dz}{1 + z^2} = \frac{1}{2} \tan^{-1} z + c$$

$$= \frac{1}{2} \tan^{-1} (\tan^2 x) + c.$$

প্রশ্নমালা–2(C)

1(a):(i). $I = \int \frac{dx}{(x-1/2)^2 + 1 - 1/4} = \int \frac{dx}{(x-1/2)^2 + (\sqrt{3}/2)^2}$

$= \frac{1}{\sqrt{3}/2} \tan^{-1} \frac{x - 1/2}{\sqrt{3}/2} + c$

$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x-1}{\sqrt{3}} + c.$

(ii). $I = \int \frac{dx}{(x-7/2)^2 + 18 - 49/4} = \int \frac{dx}{(x-7/2)^2 + (\sqrt{23}/2)^2}$

$= \frac{1}{\sqrt{23/2}} \tan^{-1} \frac{x - 7/2}{\sqrt{23/2}} + c$

$= \frac{2}{\sqrt{23}} \tan^{-1} \frac{2x-7}{\sqrt{23}} + c.$

(iii). $I = \int \frac{dx}{(x+2)^2 + 1} = \tan^{-1}(x+2) + c.$

(iv). $I = -\int \frac{dx}{x^2 - 2x + 10} = -\int \frac{dx}{(x-1)^2 + 3^2} = -\frac{1}{3} \tan^{-1} \frac{x-1}{3} + c.$

(v). $I = \int \frac{dx}{5(x^2 + 2x/5 + 3/5)} = \frac{1}{5} \int \frac{dx}{(x+1/5) + (\sqrt{14}/5)^2}$

$= \frac{1}{5} \cdot \frac{5}{\sqrt{14}} \tan^{-1} \frac{x + 1/5}{\sqrt{14}/5} + c = \frac{1}{\sqrt{14}} \tan^{-1} \frac{5x+1}{\sqrt{14}} + c.$

(vi). $I = \frac{1}{3} \int \frac{dx}{x^2 - 2x/3 + \frac{4}{3}} = \frac{1}{3} \int \frac{dx}{(x-1/3)^2 + \left(\frac{\sqrt{11}}{3}\right)^2}$

$= \frac{1}{3} \cdot \frac{3}{\sqrt{11}} \tan^{-1} \frac{x - 1/3}{\sqrt{11}/3} + c.$

$= \frac{1}{\sqrt{11}} \tan^{-1} \frac{3x-1}{\sqrt{11}} + c$

b(i). $I = \int \frac{dx}{\sqrt{a^2 - (x-a)^2}} = \sin^{-1} \frac{x-a}{a} + c.$

(ii). $I = \int \frac{dx}{\sqrt{(x+a)^2 - a^2}} = \cosh^{-1} \frac{x+a}{a} + c.$

(iii). $I = \int \frac{dx}{\sqrt{3(x^2 + 4x/3 + 4)}} = \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{(x + 2/3)^2 + (4\sqrt{2}/3)^2}}$

$= \frac{1}{\sqrt{3}} \sinh^{-1} \frac{x + 2/3}{4\sqrt{2}/3} + c$

$= \frac{1}{\sqrt{3}} \sinh^{-1} \frac{3x + 2}{4\sqrt{2}} + c.$

(iv). $I = \int \frac{dx}{\sqrt{(x - 7/2)^2 - (1/2)^2}} = \cosh^{-1} \frac{x - 7/2}{1/2} + c$

$= \cosh^{-1} (2x - 7) + c.$

(v). $I = \int \frac{dx}{(x - 1)^{1/2} (x - 2)^{1/2}} = \int \frac{dx}{\sqrt{(x - 1)(x - 2)}}$

$= \int \frac{dx}{\sqrt{x^2 - 3x + 2}} = \int \frac{dx}{\sqrt{(x - 3/2)^2 - (1/2)^2}}$

$= \cosh^{-1} \frac{x - 3/2}{1/2} + c = \cosh^{-1} (2x - 3) + c.$

(vi). $I = \int \frac{dx}{\sqrt{(x^2 - (\alpha + \beta) x + \alpha\beta)}}$

$= \int \frac{dx}{\sqrt{\left(x - \frac{\alpha + \beta}{2}\right)^2 - \left(\frac{\alpha + \beta}{2}\right)^2 + \alpha\beta}}$

$= \int \frac{dx}{\sqrt{\left(x - \frac{\alpha + \beta}{2}\right)^2 - \left(\frac{\alpha - \beta}{2}\right)^2}} = \cosh^{-1} \frac{x - \frac{\alpha + \beta}{2}}{\frac{\alpha - \beta}{2}} + c$

$= \cosh^{-1} \left(\frac{2x - \alpha - \beta}{\alpha - \beta}\right) + c.$

c(i). $I = \int \sqrt{x^2 - (\alpha + \beta)x + \alpha\beta}\, dx$

$= \int \sqrt{\{x - (\alpha + \beta)/2\}^2 - \{(\alpha - \beta)/2\}^2}\, dx$

$= \frac{\{x - (\alpha + \beta)/2\}}{2} \sqrt{(x - \alpha)(x - \beta)}$

$- \frac{\{(\alpha - \beta)/2\}^2}{2} \cosh^{-1} \frac{x - (\alpha + \beta)/2}{(\alpha - \beta)/2} + c$

(ii). $I = \sqrt{5}\int\sqrt{x^2+\frac{8}{5}x+\frac{4}{5}}\,dx = \sqrt{5}\int\sqrt{\left(x+\frac{4}{5}\right)^2+\left(\frac{2}{5}\right)^2}\,dx$

$= \sqrt{5}\left[\frac{1}{2}\left(x+\frac{4}{5}\right)\sqrt{x^2+\frac{8}{5}x+\frac{4}{5}}\right.$

$\left. + \{(2/5)^2/2\}\sinh^{-1}\left\{(x+4/5)^2/\frac{2}{5}\right\}+C.\right]$

(iii). $I = \sqrt{2}\int\sqrt{2+\frac{3}{2}x-x^2}\,dx$

$= \sqrt{2}\int\sqrt{(\sqrt{41}/4)^2-(x-3/4)^2}\,dx$

$= \sqrt{2}\left[\frac{1}{2}(x-3/4)^2\ \sqrt{2+3x/2-x^2}\right.$

$\left. + \left\{(\sqrt{41}/4)^2/2\right\}\sin^{-1}\{(x-3/4)/(\sqrt{41}/4)\}\right]+c$

(iv). $I = \sqrt{2}\int\sqrt{2-\frac{3}{2}x-x^2}\,dx$

$= \sqrt{2}\int\sqrt{(\sqrt{41}/4)^2-(x+3/4)^2}$

$= \sqrt{2}\left[\frac{(x+3/4)}{2}\sqrt{2-3x/2-x^2}+\frac{41/16}{2}\sin^{-1}\frac{x+3/4}{\sqrt{41}/4}\right]+c$

$= \frac{1}{8}(4x+3)\sqrt{4-3x-2x^2}+\frac{41\sqrt{2}}{32}\sin^{-1}\frac{4x+3}{\sqrt{41}}+c.$

2(a):(i). $I = \int\frac{\frac{7}{2}(2x-2)-2}{x^2-2x+35}\,dx$

$= \frac{7}{2}\int\frac{(2x-2)dx}{x^2-2x+35}-2\int\frac{dx}{(x-1)^2+(\sqrt{34})^2}$

$= \frac{7}{2}\ln(x^2-2x+35)-(2/\sqrt{34})\tan^{-1}\{(x-1)/\sqrt{34}\}+c$

(ii). $I = \frac{1}{2}\int\frac{2x\,dx}{x^2+a^2} = \frac{1}{2}\ln(x^2+a^2)+c$

(iii). $I = \int\frac{\frac{2}{6}(6x+4)+3-\frac{4}{3}}{3x^2+4x-7}\,dx$

$= \frac{1}{3}\int\frac{(6x+4)\,dx}{3x^2+4x-7}+\frac{5}{3}\int\frac{dx}{3(x^2+4x/3-7/3)}$

$$= \frac{1}{3} \ln (3x^2 + 4x - 7) + \frac{5}{9} \int \frac{dx}{(x + 2/3)^2 - (5/3)^2}$$

$$= \frac{1}{3} \ln (3x^2 + 4x - 7) + \frac{5}{9} \cdot \frac{1}{2.\ 5/3} \ln \frac{x + 2/3 - 5/3}{x + 2/3 + 5/3} + c.$$

(iv). $$I = \int \frac{-\frac{1}{6}(-6 - 18) - 3}{1 - 6x - 9x^2} dx$$

$$= -\frac{1}{6} \int \frac{(-6 - 18x)\, dx}{(1 - 6x - 9x^2)} - 3 \int \frac{dx}{-9\left(x^2 + 2x/3 - 1/9\right)}$$

$$= -\frac{1}{6} \ln (1 - 6x - 9x^2) + \frac{1}{3} \int \frac{dx}{(x + 1/3)^2 - (\sqrt{2}/3)^2}$$

$$= -\frac{1}{6} \ln (1 - 6x - 9x^2) + \frac{1}{3} \cdot \frac{1}{2.\ \sqrt{2}/3} \ln \frac{(x + 1/3) - \sqrt{2}/3}{(x + 1/3) + \sqrt{2}/3} + c$$

$$= \frac{1}{2\sqrt{2}} \ln \left(\frac{3x + 1 - \sqrt{2}}{3x + 1 + \sqrt{2}}\right) - \frac{1}{6} \ln (1 - 6x - 9x^2) + c.$$

(v). $$I = \int \frac{\frac{2}{4}(4x - 2) + 8}{2x^2 - 2x + 1} dx$$

$$= \frac{1}{2} \int \frac{(4x - 2)\, dx}{2x^2 - 2x + 1} + 8 \int \frac{dx}{2(x^2 - x + 1/2)}$$

$$= \frac{1}{2} \ln (2x^2 - 2x + 1) + 4 \int \frac{dx}{(x - 1/2)^2 + (1/2)^2} + c_1.$$

(vi). $$I = \int \frac{\frac{3}{2}(2x - 1) - 1 + \frac{3}{2}}{x^2 - x + 2} dx$$

$$= \frac{3}{2} \int \frac{(2x - 1)\, dx}{x^2 - x + 2} + \frac{1}{2} \int \frac{dx}{(x - 1/2)^2 + (\sqrt{7}/2)^2}$$

$$= \frac{3}{2} \ln (x^2 - x + 2) + \frac{1}{2}\left(\frac{2}{\sqrt{7}}\right) \tan^{-1} \frac{x - 1/2}{\sqrt{7}/2} + c.$$

(vii). $$I = \int \frac{\frac{5}{6}(6x + 2) - 2 - \frac{5}{3}}{3x^2 + 2x + 1} dx$$

$$= \frac{5}{6} \int \frac{(6x + 2)\, dx}{3x^2 + 2x + 1} - \frac{11}{3} \int \frac{dx}{3(x^2 + 2x/3 + 1/3)}$$

$$= \frac{5}{6} \ln (3x^2 + 2x + 1) - \frac{11}{9} \int \frac{dx}{(x + 1/3)^2 + (\sqrt{2}/3)^2}$$

$$= \frac{5}{6} \ln (3x^2 + 2x + 1) - \frac{11}{9} \left(\frac{3}{\sqrt{2}}\right) \tan^{-1} \frac{x + 1/3}{\sqrt{2}/3} + c.$$

b(i). $I = \int \frac{\frac{1}{4}(4x - 8)\, dx}{\sqrt{2x^2 - 8x + 5}} = \frac{1}{4} \cdot 2\sqrt{2x^2 - 8x + 5} + c.$

(ii). $I = \int \frac{(2x - 2) + 7}{\sqrt{x^2 - 2x + 2}} dx = \int \frac{(2x - 2)dx}{\sqrt{x^2 - 2x + 2}} + 7 \int \frac{dx}{\sqrt{(x - 1)^2 + 1}}$

$$= 2\sqrt{x^2 - 2x + 2} + 7 \sinh^{-1} (x - 1) + c.$$

(iii). $I = \int \frac{\frac{1}{2}(2x - 1) + \frac{3}{2}}{\sqrt{x^2 - x + 1}} dx$

$$= \frac{1}{2} \int \frac{(2x - 1)\, dx}{\sqrt{x^2 - x + 1}} + \frac{3}{2} \int \frac{dx}{\sqrt{(x - 1/2)^2 + (\sqrt{3}/2)^2}}$$

$$= \frac{1}{2} \cdot 2\sqrt{x^2 - x + 1} + \frac{3}{2} \cdot \sinh^{-1} \frac{x - 1/2}{\sqrt{3}/2} + c$$

(iv). $I = \int \frac{\frac{1}{2}(2x - 2)\, dx}{\sqrt{4 + x^2 - 2x}} = \frac{1}{2} \cdot 2\sqrt{4 + x^2 - 2x} + c.$

(v). $I = \int \frac{\frac{1}{4}(8x + 4) - 2}{\sqrt{4x^2 + 4x + 2}} dx$

$$= \frac{1}{4} \int \frac{(8x + 4)\, dx}{\sqrt{4x^2 + 4x + 2}} - 2 \int \frac{dx}{\sqrt{4(x^2 + x + 1/2)}}$$

$$= \frac{1}{4} \cdot 2\sqrt{4x^2 + 4x + 2} - \frac{2}{2} \int \frac{dx}{\sqrt{(x + 1/2)^2 + (1/2)^2}}$$

$$= \frac{1}{2}\sqrt{4x^2 + 4x + 2} - \sinh^{-1} \frac{(x + 1/2)}{1/2} + c.$$

c(i). $I = \int \left\{\frac{1}{2}(2x - 1) - \frac{1}{2}\right\} \sqrt{x^2 - x + 1}\, dx$

$$= \frac{1}{2} \int (2x - 1)\sqrt{x^2 - x + 1}\, dx - \frac{1}{2} \int \sqrt{(x - 1/2)^2 + (\sqrt{3}/2)^2}\, dx$$

$$=\frac{1}{2}\cdot\frac{(x^2-x+1)^{3/2}}{3/2}-\frac{1}{2}\left[\frac{(x-1/2)\sqrt{x^2-x+1}}{2}\right.$$

$$\left.+\frac{(\sqrt{3}/2)^2}{2}\sinh^{-1}\frac{x-1/2}{\sqrt{3}/2}\right]+c$$

$$=\frac{1}{3}(x^2-x+1)^{3/2}-\frac{1}{8}(2x-1)\sqrt{x^2-x+1}$$

$$-\frac{3}{16}\sinh^{-1}\frac{(2x-1)}{\sqrt{3}}+c$$

$$=\frac{1}{24}[8(x^2-x+1)-3(2x-1)]\sqrt{x^2-x+1}$$

$$-\frac{3}{16}\sinh^{-1}\frac{(2x-1)}{\sqrt{3}}+c.$$

(ii). $I=\int\left\{\frac{1}{2}(2x+2)+3\right\}\sqrt{x^2+2x+7}\,dx$

$$=\frac{1}{2}\int(2x+2)\sqrt{x^2+2x+7}\,dx+3\int\sqrt{(x+1)^2+(\sqrt{6})^2}\,dx$$

বা $I=\frac{1}{2}\cdot\frac{(x^2+2x+7)^{3/2}}{3/2}+3\left[\frac{(x+1)\sqrt{x^2+2x+7}}{2}\right.$

$$\left.+\frac{6}{2}\sinh^{-1}\frac{x+1}{\sqrt{6}}\right]+c.$$

(iii). $I=\frac{1}{2}\int 2x\sqrt{x^2+a^2}\,dx+a\int\sqrt{x^2+a^2}\,dx$

$$=\frac{1}{2}\frac{(x^2+a^2)^{3/2}}{3/2}+a\left[\frac{x\sqrt{x^2+a^2}}{2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a}\right]+c.$$

(iv). $I=\int\{(2x-4)+1\}\sqrt{x^2-4x+13}\,dx$

$$=\int(2x-4)\sqrt{x^2-4x+13}\,dx+1\int\sqrt{(x-2)^2+3^2}\,dx$$

$$=\frac{(x^2-4x+13)^{3/2}}{3/2}+\left[\frac{(x-2)\sqrt{x^2-4x+13}}{2}\right.$$

$$\left.+\frac{9}{2}\sinh^{-1}\frac{(x-2)}{3}\right]+c.$$

(v). $I=\int\left\{\frac{1}{4}(4x+2)+\frac{3}{2}\right\}\sqrt{2x^2+2x+1}\,dx$

$$=\frac{1}{4}\int(4x+2)\sqrt{2x^2+2x+1}\,dx+\frac{3}{2}\int\sqrt{2\,(x^2+x+1/2)}\,dx$$

$$=\frac{1}{4}\frac{(2x^2+2x+1)^{3/2}}{3/2}+\frac{3\sqrt{2}}{2}\int\sqrt{(x+1/2)^2+(1/2)^2}\,dx$$

$$=\frac{1}{6}(2x^2+2x+1)^{3/2}+\frac{3\sqrt{2}}{2}\left[\frac{1}{2}(x+1/2)\sqrt{x^2+x+1/2}\right.$$

$$\left.\frac{(1/2)^2}{2}\sinh^{-1}\left\{\frac{(x+1/2)}{1/2}\right\}\right]+c.$$

(vi). $I=\int\left\{\frac{1}{4}(8x+8)+1\right\}\sqrt{4x^2+8x+5}\,dx$

$$=\frac{1}{4}\int(8x+8)\sqrt{4x^2+8x+5}\,dx+2\int\sqrt{x^2+2x+5/4}\,dx$$

$$\therefore I=\frac{1}{4}\cdot\frac{(4x^2+8x+5)^{3/2}}{3/2}+2\int\sqrt{(x+1)^2+(1/2)^2}\,dx$$

$$=\frac{1}{6}(4x^2+8x+5)^{3/2}+2\left[\frac{(x+1)\sqrt{x^2+2x+5/4}}{2}\right.$$

$$\left.+\left\{\frac{(1/2)^2}{2}\right\}\sinh^{-1}\left\{\frac{x+1}{1/2}\right\}\right]+c.$$

(vii). $I=\int\left\{\frac{1}{2}(2x)-1\right\}\sqrt{x^2-1}\,dx$

$$=\frac{1}{2}\int 2x(x^2-1)^{1/2}\,dx-\int\sqrt{x^2-1}\,dx$$

$$=\frac{1}{2}\cdot\frac{(x^2-1)^{3/2}}{3/2}-\left\{\frac{x\sqrt{x^2-1}}{2}-\frac{1}{2}\cosh^{-1}x\right\}+c.$$

3(i). $I=\int\frac{\frac{a}{c}(cx-d)+b+ad/c}{cx-d}\,dx=\int\frac{a}{c}\,dx+\frac{bc+ad}{c}\int\frac{dx}{cx-d}$

$$=\frac{ax}{c}+\frac{bc+ad}{c}\cdot\frac{1}{c}\ln(cx-d)+c_1.$$

(ii). $I=\int\frac{(x^2+a^2)-2a^2}{x^2+a^2}\,dx=\int dx-2a^2\int\frac{dx}{x^2+a^2}$

$$=x-2a^2.\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\left(\frac{x}{a}\right)+c.$$

(iii). $I=\int\frac{(x^2-4)+4}{x^2-4}\,dx=\int dx+4\int\frac{dx}{x^2-2^2}$

$$=x+4.\left(\frac{1}{2.2}\right)\ln\frac{x-2}{x+2}+c.$$

(ii). ধরি $x^2 = z$ তবে $2xdx = dz$

$$\therefore I = \int \frac{dz}{z^2 + 2z + 2} = \int \frac{dz}{(z+1)^2 + 1} = \tan^{-1}(z+1) + c.$$

(iii). ধরি $\tan x = z$ তবে $\sec^2 x\, dx = dz$

$$\therefore I = \int \frac{dz}{z^2 + 4z + 3} = \int \frac{dz}{(z+2)^2 - 1^2} = \frac{1}{2} \ln \frac{z+2-1}{z+2+1} + c.$$

(iv). ধরি $e^x = z$ তবে $e^x dx = dz$

$$\therefore I = \int \frac{dz}{z^2 + 2z + 10} = \int \frac{dz}{(z+1)^2 + 3^2} = \frac{1}{3} \tan^{-1} \frac{z+1}{3} + c.$$

(v). ধরি $x^3 = z$ তবে $3x^2\, dx = dz$

$$\therefore I = \frac{1}{3} \int \frac{dz}{z^2 - 6z + 5} = \frac{1}{3} \int \frac{dz}{(z-3)^2 - 2^2} = \frac{1}{3} \cdot \frac{1}{2.2} \ln \frac{z-3-2}{z-3+2} + c.$$

(vi). ধরি $\ln x = z$ তবে $\frac{1}{x} dx = dz.$

$$\therefore I = \int \frac{dz}{z^2 + 7z + 10} = \int \frac{dz}{(z + 7/2)^2 - (3/2)^2}$$

$$= \frac{1}{2.3/2} \ln \frac{x + 7/2 - 3/2}{x + 7/2 + 3/2} + c.$$

(vii). ধরি $\sin x = z$ তবে $\cos x\, dx = dz$

$$\therefore I = \int \frac{dz}{\sqrt{5z^2 - 12z + 4}} = \int \frac{dz}{\sqrt{5\,(z^2 - 12z/5 + 4/5)}}$$

$$= \frac{1}{\sqrt{5}} \int \frac{dz}{\sqrt{(z - 6/5)^2 - (4/5)^2}} = \frac{1}{\sqrt{5}} \cosh^{-1} \frac{z - 6/5}{4/5} + c$$

$$= \frac{1}{\sqrt{5}} \cosh^{-1} \frac{5z - 6}{4} + c.$$

প্রশ্নমালা–2(D)

1(i). ধরি $x+3=z^2$, বা $x=z^2-3$ তবে $dx=2z\,dz$

$$\therefore I=\int\frac{2z\,dz}{(z^2-3+1)z}=2\int\frac{dz}{z^2-(\sqrt{2})^2}=\frac{2}{2\sqrt{2}}\ln\frac{z-\sqrt{2}}{z+\sqrt{2}}+c$$

(ii). ধরি $x-4=z^2$, বা $x=z^2+4$ তবে $dx=2z\,dz$

$$\therefore I=\int\frac{2z\,dz}{(z^2+4+3)z}=2\int\frac{dz}{z^2+(\sqrt{7})^2}=\frac{2}{\sqrt{7}}\tan^{-1}\frac{z}{\sqrt{7}}+c$$

(iii). ধরি $x+8=z^2$, বা $x=z^2-8$ তবে $dx=2z\,dz$

$$\therefore I\int\frac{2z\,dz}{(1-z^2+8)z}=2\int\frac{dz}{3^2-z^2}=\frac{2}{2.3}\ln\frac{3+z}{3-z}+c$$

(iv). ধরি $x+9=z^2$ বা $x=z^2-9$ তবে $dx=2z\,dz$

$$\therefore I=\int\frac{2z\,dz}{(z^2-9)z}=2\int\frac{dz}{z^2-3^2}=\frac{2}{2.3}\ln\frac{z-3}{z+3}+c$$

(v). ধরি $x=z^2$ তবে $dx=2z\,dz$

$$\therefore I=\int\frac{2z\,dz}{(1+z^2)z}=2\int\frac{dz}{1+z^2}=2\tan^{-1}z+c=2\tan^{-1}\sqrt{x}+c$$

(vi). ধরি $x-2=z^2$, বা $x=z^2+2$ তবে $dx=2z\,dz$

$$\therefore I=\int\frac{2z\,dz}{\{3(z^2+2)+7\}z}=2\int\frac{dz}{3z^2+13}=\frac{2}{3}\int\frac{dz}{z^2+(\sqrt{13/3})^2}$$

$$=\frac{2}{3}\cdot\frac{1}{\sqrt{13}/\sqrt{3}}\tan^{-1}\frac{z}{\sqrt{13}/\sqrt{3}}+c=\frac{2}{\sqrt{39}}\tan^{-1}\frac{\sqrt{3}z}{\sqrt{13}}+c$$

(vii). ধরি $x+5=z^2$, বা $x=z^2-5$ তবে $dx=2z\,dz$

$$\therefore I=\int\frac{2z\,dz}{\{2(z^2-5)+3\}z}=2\int\frac{dz}{2z^2-7}=\frac{2}{2}\int\frac{dz}{z^2-(\sqrt{7/2})^2}$$

$$=\frac{1}{2\sqrt{7}/\sqrt{2}}\ln\frac{z-\sqrt{7}/\sqrt{2}}{z+\sqrt{7}/\sqrt{2}}+c=\frac{1}{\sqrt{14}}\ln\frac{\sqrt{2}z-\sqrt{7}}{\sqrt{2}z+\sqrt{7}}+c$$

$$=\frac{1}{\sqrt{14}}\ln\frac{\sqrt{2x+10}-\sqrt{7}}{\sqrt{2x+10}+\sqrt{7}}+c.$$

(iv). ধরি $x = \frac{1}{z}$ তবে $dx = \frac{-dz}{z^2}$

$$\therefore I = -\int \frac{dz}{z^2 (1/z) \sqrt{2 + 1/z - 1/z^2}} = -\int \frac{dz}{z\sqrt{2 + 1/z - 1/z^2}}$$

$$= -\int \frac{dz}{\sqrt{2z^2 + z - 1}} = -\frac{1}{\sqrt{2}} \int \frac{dz}{\sqrt{z^2 + z/2 - 1/2}}$$

$$= -\frac{1}{\sqrt{2}} \int \frac{dz}{\sqrt{(z + 1/4)^2 - (3/4)^2}} = -\frac{1}{\sqrt{2}} \cosh^{-1} \frac{z + 1/4}{3/4} + c$$

(v). ধরি $x - 1 = 1/z$ বা $x = 1 + 1/z$ তবে $dx = \frac{-dz}{z^2}$

$$\therefore I = -\int \frac{dz}{z^2 . 1/z \sqrt{(1 + 1/z)^2 + 2(1 + 1/z) + 2}}$$

$$= -\int \frac{dz}{z \sqrt{5 + 4/z + 1/z^2}}$$

$$= -\int \frac{dz}{\sqrt{5z^2 + 4z + 1}} = -\frac{1}{\sqrt{5}} \int \frac{dz}{\sqrt{z^2 + 4z/5 + 1/5}}$$

$$= -\frac{1}{\sqrt{5}} \int \frac{dz}{\sqrt{(z + 2/5)^2 + (1/5)^2}}$$

$$= -\frac{1}{\sqrt{5}} \sinh^{-1} \frac{z + 2/5}{1/5} + c$$

(vi). ধরি $x = \frac{1}{z}$ তবে $dx = \frac{-dz}{z^2}$

$$\therefore I = -\int \frac{dz}{z^2 (1/z) \sqrt{1 + 1/z^2}} = -\int \frac{dz}{z \sqrt{1 + 1/z^2}} = -\int \frac{dz}{\sqrt{z^2 + 1}}$$

$$= -\sinh^{-1} z + c. \quad = -\sinh^{-1} \left(\frac{1}{x}\right) + c$$

(vii). ধরি $x - 2 = \frac{1}{z}$, বা $x = 2 + \frac{1}{z}$ তবে $dz = -\frac{dz}{z^2}$.

$$\therefore I = -\int \frac{dz}{z^2 (1/z) \sqrt{\{4(2 + 1/z)^2 - 5(2 + 1/z) + 3\}}}$$

$$= -\int \frac{dz}{z\sqrt{9 + 11/z + 4/z^2}}$$

$$= -\int \frac{dz}{\sqrt{9z^2 + 11z + 4}} = -\frac{1}{3} \int \frac{dz}{\sqrt{z^2 + 11z/9 + 4/9}}$$

বা $I = -\frac{1}{3}\int \frac{dz}{\sqrt{\{(z+11/18)^2 + (\sqrt{23}/18)^2\}}}$

$$= -\frac{1}{3}\sinh^{-1}\frac{z+11/18}{\sqrt{23}/18} + c = -\frac{1}{3}\sinh^{-1}\frac{18z+11}{\sqrt{23}} + c$$

(viii). ধরি $x = \frac{1}{z}$ তবে $dx = \frac{-dz}{z^2}$

$$\therefore I = -\int\frac{dz}{z^2(1/z)\sqrt{1/z^2 + a^2}} = -\int\frac{dz}{z\sqrt{1/z^2+a^2}} = -\int\frac{dz}{\sqrt{1+a^2z^2}}$$

$$= -\frac{1}{a}\int\frac{dz}{\sqrt{(1/a)^2 + z^2}} = -\frac{1}{a}\sinh^{-1} az + c = -\frac{1}{a}\sinh^{-1}\left(\frac{a}{x}\right) + c$$

(ix). ধরি $3x + 1 = \frac{1}{z}$ বা $3x = \frac{1}{z} - 1$ তবে $3dx = \frac{-dz}{z^2}$

$$\therefore I = -\frac{1}{3}\int\frac{dz}{z^2(1/z)\sqrt{(1/z-1)^2 + 1/z - 1 - 1}}$$

$$= -\frac{1}{3}\int\frac{dz}{z\sqrt{1/z^2 - 1/z - 1}}$$

$$= -\frac{1}{3}\int\frac{dz}{\sqrt{1-z-z^2}} = -\frac{1}{3}\int\frac{dz}{\sqrt{(\sqrt{5}/2)^2 - (z+1/2)^2}}$$

$$= -\frac{1}{3}\sin^{-1}\frac{z+1/2}{\sqrt{5}/2} + c = -\frac{1}{3}\sin^{-1}\frac{2z+1}{\sqrt{5}} + c$$

(x). ধরি $x + 3 = \frac{1}{z}$ বা $x = \frac{1}{z} - 3$ তবে $dx = \frac{-dz}{z^2}$

$$\therefore I = -\int\frac{dz}{z^2(1/z)\sqrt{(1/z-3)^2+1}} = -\int\frac{dz}{z\sqrt{10 - 6/z + 1/z^2}}$$

$$= -\int\frac{dz}{\sqrt{10z^2 - 6z + 1}} = -\frac{1}{\sqrt{10}}\int\frac{dz}{\sqrt{z^2 - 3z/5 + 1/10}}$$

$$= -\frac{1}{\sqrt{10}}\int\frac{dz}{\sqrt{(z-3/10)^2 + (1/10)^2}}$$

$$= -\frac{1}{\sqrt{10}}\sinh^{-1}\frac{z-3/10}{1/10} + c$$

$$= -\frac{1}{\sqrt{10}}\sinh^{-1}(10z-3) + c.$$

$$\therefore\ I_2 = -2\int\frac{dz}{z^2(1/z)\sqrt{1/z^2-2}} = -2\int\frac{dz}{\sqrt{1-2z^2}}$$

$$= -\frac{2}{\sqrt{2}}\int\frac{dz}{\sqrt{(1/\sqrt{2})^2-z^2}} = -\sqrt{2}\sin^{-1}\frac{z}{1/\sqrt{2}}+c_2$$

$$= -\sqrt{2}\sin^{-1}z\sqrt{2}+c_2 = -\sqrt{2}\sin^{-1}(\sqrt{2}/x)+c_2$$

(iv). ধরি $x^2 = z$ তবে $2x\,dx = dz$

$$\therefore I = \int\frac{dz}{(1-z)\sqrt{z^2-1}}$$

আবার ধরি $1-z = 1/u$, বা $z = 1-1/u$ তবে $dz = \frac{du}{u^2}$

$$\therefore I = \int\frac{du}{u^2(1/u)\sqrt{(1-1/u)^2-1}} = \int\frac{du}{u\sqrt{1/u^2-2/u}} = \int\frac{du}{\sqrt{1-2u}}$$

$$= -\frac{1}{2}\int\frac{-2\,du}{\sqrt{1-2u}} = -\frac{1}{2}.2\sqrt{1-2u}+c$$

$$= \sqrt{1-\frac{2}{1-z}}+c = -\sqrt{\frac{-1-z}{1-z}}+c = -\sqrt{\frac{z+1}{z-1}}+c$$

(v). $I = \int\frac{\{(x+1)-1\}\,dx}{(x+1)\sqrt{x^2+1}} = \int\frac{dx}{\sqrt{x^2+1}} - \int\frac{dx}{(x+1)\sqrt{x^2+1}}$

$$= \sinh^{-1}x - I_1;\ \text{যখন}\ I_1 = \int\frac{dx}{(x+1)\sqrt{x^2+1}}$$

ধরি $x+1 = 1/z$, বা $x = 1/z-1$ তবে $dx = \frac{-dz}{z^2}$

$$\therefore I_1 = -\int\frac{dz}{z^2(1/z)\sqrt{(1/z-1)^2+1}} = -\int\frac{dz}{z\sqrt{2-2/z+1/z^2}}$$

$$= -\int\frac{dz}{\sqrt{2z^2-2z+1}} = -\frac{1}{\sqrt{2}}\int\frac{z}{\sqrt{z^2-z+1/2}}$$

$$= -\frac{1}{\sqrt{2}}\int\frac{dz}{\sqrt{(z-1/2)^2+(1/2)^2}} = -\frac{1}{\sqrt{2}}\sinh^{-1}\frac{z-1/2}{1/2}+c$$

$$= -\frac{1}{\sqrt{2}}\sinh^{-1}(2z-1)+c = -\frac{1}{\sqrt{2}}\sinh^{-1}\frac{1-x}{1+x}+c$$

4(i). ধরি $x + 1 = z^2$, বা $x = z^2 - 1$ তবে $dx = 2z\,dz$

$$\therefore I = \int \frac{2z\,dz}{(z^2-1)^2 z} = 2\int \frac{dz}{(z^2-1)^2} = 2\int \frac{dz}{(z-1)^2(z+1)^2}$$

এখন $\dfrac{1}{(z-1)^2(z+1)^2} = \dfrac{A}{z-1} + \dfrac{B}{(z-1)^2} + \dfrac{C}{z+1} + \dfrac{D}{(z+1)^2}$

$\Rightarrow A = -\dfrac{1}{4}, B = \dfrac{1}{4}, C = \dfrac{1}{4}$ এবং $D = \dfrac{1}{4}$

$$\therefore \frac{1}{(z-1)^2(z+1)^2} = \frac{1}{4}\left[-\frac{1}{z-1} + \frac{1}{(z-1)^2} + \frac{1}{z+1} + \frac{1}{(z+1)^2}\right] \cdots (1)$$

কিন্তু আমাদের আছে $I = 2\displaystyle\int \frac{1}{(z-1)^2(z+1)^2}dz$

বা $I = \dfrac{2}{4}\displaystyle\int\left[-\frac{1}{z-1} + \frac{1}{(z-1)^2} + \frac{1}{(z+1)} + \frac{1}{(z+1)^2}\right]dz$ [(1) নং দ্বারা]

$$= \frac{1}{2}\left[-\ln(z-1) - \frac{1}{z-1} + \ln(z+1) - \frac{1}{z+1}\right] + c$$

$$= \frac{1}{2}\left[\ln\frac{z+1}{z-1} - \left\{\frac{1}{z-1} + \frac{1}{z+1}\right\}\right]$$

$$= \frac{1}{2}\left[\ln\frac{z+1}{z-1} - \frac{2z}{z^2-1}\right]$$

(ii). ধরি $x + 1 = z^2$, বা $x = z^2 - 1$ তবে $dx = 2z\,dz$

$$\therefore I = \int \frac{2z\,dz}{\{(z^2-1)^2 - 4\}z} = 2\int \frac{dz}{(z^2+1)(z^2-3)}$$

$$= \frac{1}{2}\int\left[\frac{1}{z^2-3} - \frac{1}{z^2+1}\right]dz$$

$$= \frac{1}{2}\int \frac{dz}{z^2 - (\sqrt{3})^2} - \frac{1}{2}\int \frac{dz}{z^2+1}$$

$$= \frac{1}{2}\cdot\frac{1}{2\sqrt{3}}\ln\frac{z-\sqrt{3}}{z+\sqrt{3}} - \frac{1}{2}\tan^{-1}z + c$$

(iii). ধরি $x + 1 = z^2$ বা $x = z^2 - 1$ তবে $dx = 2z\,dz$

$$I = \int \frac{x\,dx}{\{(x+1)^2+1\}\sqrt{x+1}} = \int \frac{(z^2-1)\,2z\,dz}{\{(z^2)^2+1\}z} = 2\int \frac{(z^2-1)\,dz}{z^4+1}$$

$$= 2\int \frac{z^2(1 - 1/z^2)\,dz}{z^2(z^2 + 1/z^2)} = 2\int \frac{(1 - 1/z^2)\,dz}{(z + 1/z)^2 - 2}$$

যখন $I_1 = \dfrac{1}{6\sqrt{6}} \displaystyle\int \frac{dx}{\sqrt{(x+7/12)^2-(1/12)^2}}$

$$= \frac{1}{6\sqrt{6}} \cosh^{-1} \frac{x+7/12}{1/12} + c$$

(vii). $I = \displaystyle\int \frac{(x-3)\,dx}{\sqrt{(3x-4).(x-3)}} = \int \frac{(x-3)\,dx}{\sqrt{3x^2-13x+12}}$

$$= \int \frac{(1/6)(6x-13)-14/6}{\sqrt{3x^2-13x+12}} dx$$

$$= \frac{1}{6}\int \frac{(6x-13)\,dx}{\sqrt{3x^2-13x+12}} - \frac{5}{6}\int \frac{dx}{\sqrt{3(x^2-13x/3+4)}}$$

$$= \frac{1}{6}.2\sqrt{3x^2-13x+12} - I_1$$

যখন $I_1 = \dfrac{5}{6\sqrt{3}} \displaystyle\int \frac{dx}{\sqrt{(x-13/6)^2-(5/6)^2}}$

$$= \frac{5}{6\sqrt{3}} \cosh^{-1} \frac{x-13/6}{5/6}.$$

(viii). $I = \displaystyle\int \frac{(a+x)\,dx}{\sqrt{x(a+x)}} = \int \frac{(a+x)\,dx}{\sqrt{x^2+ax}} = \int \frac{(1/2)(2x+a)+a/2}{\sqrt{x^2+ax}} dx$

$$= \frac{1}{2}\int \frac{(2x+a)\,dx}{\sqrt{x^2+ax}} + \frac{a}{2}\int \frac{dx}{\sqrt{(x+a/2)^2-(a/2)^2}}$$

$$= \frac{1}{2}.2\sqrt{x^2+ax} + \frac{a}{2}\cosh^{-1}\frac{x+a/2}{a/2} + c$$

(ix). $I = \displaystyle\int \frac{(a+x)\,dx}{\sqrt{(a-x)(a+x)}} = \int \frac{(a+x)\,dx}{\sqrt{a^2-x^2}} = a\int \frac{dx}{\sqrt{a^2-x^2}} + \int \frac{x\,dx}{\sqrt{a^2-x^2}}$

$$= a\int \frac{dx}{\sqrt{a^2-x^2}} - \frac{1}{2}\int \frac{-2x\,dx}{\sqrt{a^2-x^2}}$$

$$= a\sin^{-1}\frac{x}{a} - \frac{1}{2}.2\sqrt{a^2-x^2} + c$$

(x). $I = \displaystyle\int \frac{x(1+x)\,dx}{\sqrt{(1-x)(1+x)}} = \int \frac{(x+x^2)\,dx}{\sqrt{1-x^2}} = \int \frac{x\,dx}{\sqrt{1-x^2}} + \int \frac{x^2\,dx}{\sqrt{1-x^2}}$

$$= \int \frac{x\,dx}{\sqrt{1-x^2}} + \int \frac{-1(1-x^2)+1}{\sqrt{1-x^2}} dx$$

$$= -\frac{1}{2}\int \frac{-2x\,dx}{\sqrt{1-x^2}} - \int \sqrt{1-x^2}\,dx + \int \frac{dx}{\sqrt{1-x^2}}$$

বা $I = -\frac{1}{2}.2\sqrt{1-x^2} - \left[\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right] + \sin^{-1}x + c$

$= -\frac{1}{2}(x+2)\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x + c$

(xi). $I = \int\frac{(x-1)\,dx}{x\sqrt{(x+1)(x-1)}} = \int\frac{(x-1)\,dx}{x\sqrt{x^2-1}}$

$= \int\frac{dx}{\sqrt{x^2-1}} - \int\frac{dx}{x\sqrt{x^2-1}} = \cosh^{-1}x - \sec^{-1}x + c$

6(i). ধরি $x = 1/z$ তবে $dx = \frac{-dz}{z^2}$

$\therefore I = -\int\frac{dz}{z^2(1+1/z^2)\sqrt{1-1/z^2}} = -\int\frac{z\,dz}{(z^2+1)\sqrt{z^2-1}}$

আবার ধরি $z^2 - 1 = u^2$, বা $z^2 = u^2 + 1$ তবে $2z\,dz = 2u\,du$

$\therefore I = -\int\frac{u\,du}{(u^2+1+1)u} = -\int\frac{du}{u^2+(\sqrt{2})^2} = -\frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}}$

$= -\frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{z^2-1}}{\sqrt{2}} + c = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\sqrt{\frac{1}{x^2}-1}\right) + c$

(ii). ধরি $x = 1/z$ তবে $dx = \frac{-dz}{z^2}$

$\therefore I = -\int\frac{dz}{z^2(1/z^2)\sqrt{1-1/z^2}} = -\int\frac{z\,dz}{\sqrt{z^2-1}} = -\frac{1}{2}\int\frac{2z\,dz}{\sqrt{z^2-1}}$

$= -\frac{1}{2}.2\sqrt{z^2-1} + c = -\sqrt{1/x^2-1} + c = \frac{-\sqrt{1-x^2}}{x} + c$

(iii). ধরি $x = \frac{1}{z}$ তবে $dx = \frac{-dx}{z^2}$

$I = -\int\frac{dz}{z^2(4+1/z^2)\sqrt{1/z^2+2}} = -\int\frac{z\,dz}{(4z^2+1)\sqrt{1+2z^2}}$

আবার ধরি $1 + 2z^2 = u^2$, বা $2z^2 = u^2 - 1$ তবে $4z\,dz = 2u\,du$

$\therefore I = -\frac{1}{2}\int\frac{u\,du}{\{2(u^2-1)+1\}u} = -\frac{1}{2}\int\frac{du}{2u^2-1} = -\frac{1}{4}\int\frac{du}{u^2-(1/\sqrt{2})^2}$

বা $I = -\frac{1}{4}.\frac{1}{2.1/\sqrt{2}}\, ln\frac{u-1/\sqrt{2}}{u+1/\sqrt{2}} + c = -\frac{1}{4\sqrt{2}}\, ln\frac{u\sqrt{2}-1}{u\sqrt{2}+1} + c$

$= \frac{-1}{4\sqrt{2}}\, ln\frac{\sqrt{2+4z^2}-1}{\sqrt{2+4z^2}+1} = \frac{-1}{4\sqrt{2}}\, ln\frac{\sqrt{2+4/x^2}-1}{\sqrt{2+4/x^2}+1} + c$

(iv). ধরি $x = \frac{1}{z}$ তবে $dx = \frac{-dz}{z^2}$

$$I = -\int \frac{dz}{z^2(2/z^2 + 5)\sqrt{1/z^2 - 4}} = -\int \frac{z\,dz}{(2 + 5z^2)\sqrt{1 - 4z^2}}$$

ধরি $1 - 4z^2 = u^2$, বা $4z^2 = 1 - u^2$ তবে $8z\,dz = -2u\,du$

$$I = \frac{1}{4}\int \frac{u\,du}{\left\{2 + \frac{5}{4}(1 - u^2)\right\}u} = \frac{1}{4}\int \frac{4\,du}{13 - 5u^2} = \frac{1}{5}\int \frac{du}{(\sqrt{13/5})^2 - u^2}$$

$$= \frac{1}{5}\cdot\frac{1}{2\sqrt{13}/\sqrt{5}}\, ln \frac{\sqrt{13}/\sqrt{5} - u}{\sqrt{13}/\sqrt{5} + u} + c$$

$$= \frac{1}{2\sqrt{65}}\, ln \frac{\sqrt{13} - u\sqrt{5}}{\sqrt{13} + u\sqrt{5}} + c$$

(v). ধরি $x = \frac{1}{z}$ তবে $dx = \frac{-dz}{z^2}$

$$I = -\int \frac{dz}{z^2(1/z^2 + 1)\sqrt{2/z^2 - 1}} = -\int \frac{z\,dz}{(1 + z^2)\sqrt{2 - z^2}}$$

ধরি $2 - z^2 = u^2$, বা $z^2 = 2 - u^2$ তবে $2z\,dz = -2u\,du$

$$I = \int \frac{u\,du}{(1 + 2 - u^2)u} = \int \frac{du}{(\sqrt{3})^2 - u^2} = \frac{1}{2\sqrt{3}}\, ln \frac{\sqrt{3} + u}{\sqrt{3} - u} + c$$

7(i). ধরি $x^2 = t$ তবে $2x\,dx = dt$

$$\therefore I = \int \frac{dt}{(1 - t)\sqrt{t - 1}}$$

আবার ধরি $t - 1 = z^2$, বা $t = 1 + z^2$ তবে $dt = 2z\,dz$

$$\text{বা } I = \int \frac{2z\,dz}{(1 - 1 - z^2)z} = 2\int \frac{dz}{-z^2} = \frac{2}{z} + c = \frac{2}{\sqrt{t - 1}} + c$$

(ii). $I = \int \frac{x\,dx}{(x^2 + 4)\sqrt{x^2 + 9}} + \int \frac{dx}{(x^2 + 4)\sqrt{x^2 + 9}} = I_1 + I_2$ ধরি

এখন $I_1 = \int \frac{x\,dx}{(x^2 + 4)\sqrt{x^2 + 9}}$; ধরি $x^2 = t$ তবে $2x\,dx = dt$

$$= \frac{1}{2}\int \frac{dt}{(t + 4)\sqrt{t + 9}}; \text{ ধরি } t + 9 = v^2 \text{ তবে } dt = 2v\,dv$$

$$= \frac{1}{2}\int \frac{2v\,dv}{(v^2 - 9 + 4)v} = \int \frac{dv}{v^2 - (\sqrt{5})^2} = \frac{1}{2\sqrt{5}}\, ln \frac{v - \sqrt{5}}{v + \sqrt{5}} + c_1$$

এবং $I_2 = \int \frac{dx}{(x^2+4)\sqrt{x^2+9}}$; ধরি $x = \frac{1}{z}$ তবে $dx = \frac{-dz}{z^2}$

$$= -\int \frac{dz}{z^2(1/z^2+4)\sqrt{1/z^2+9}} = -\int \frac{z\,dz}{(1+4z^2)\sqrt{1+9z^2}}$$

ধরি $1+9z^2 = u^2$, বা $z^2 = \frac{1}{9}(u^2-1)$ তবে $2z\,dz = \frac{1}{9}2u\,du$

$$I_2 = -\frac{1}{9}\int \frac{u\,du}{\left\{1+\frac{4}{9}(u^2-1)\right\}u} = -\frac{1}{9}\int \frac{9du}{5+4u^2} = -\frac{1}{4}\int \frac{du}{(\sqrt{5}/2)^2+u^2}$$

$$= -\frac{1}{4}\cdot\frac{2}{\sqrt{5}}\tan^{-1}\frac{u}{\sqrt{5}/2} + c_2 = -\frac{1}{2\sqrt{5}}\tan^{-1}2\frac{\sqrt{1+9z^2}}{\sqrt{5}} + c_2$$

$$= -\frac{1}{2\sqrt{5}}\tan^{-1}\frac{2\sqrt{1+9/x^2}}{\sqrt{5}} + c_2$$

$$= -\frac{1}{2\sqrt{5}}\tan^{-1}\frac{2\sqrt{x^2+9}}{x\sqrt{5}} + c_2$$

(iii). $I = \int \frac{(1+x^2)\,dx}{(1-x^2)\sqrt{1+x^2}} = \int \frac{-1(1-x^2)+2}{(1-x^2)\sqrt{1+x^2}}dx$

$$= -\int \frac{dx}{\sqrt{1+x^2}} + 2\int \frac{dx}{(1-x^2)\sqrt{1+x^2}} = -\sinh^{-1}x + I_1$$

যখন $I_1 = 2\int \frac{dx}{(1-x^2)\sqrt{1+x^2}}$; ধরি $x = \frac{1}{z}$ তবে $dx = \frac{-dz}{z^2}$

$$= -\int \frac{dz}{z^2(1-1/z^2)\sqrt{1+1/z^2}} = -\int \frac{z\,dz}{(z^2-1)\sqrt{z^2+1}}$$

আবার ধরি $z^2+1 = u^2$ বা $z^2 = u^2-1$ তবে $2z\,dz = 2u\,du$

$$I_1 = -\int \frac{u\,du}{(u^2-1-1)u} = -\int \frac{du}{u^2-(\sqrt{2})^2} = -\frac{1}{2\sqrt{2}}\ln\frac{u-\sqrt{2}}{u+\sqrt{2}} + c_1$$

$$= -\frac{1}{2\sqrt{2}}\ln\frac{\sqrt{z^2+1}-\sqrt{2}}{\sqrt{z^2+1}+\sqrt{2}} + c_1 = -\frac{1}{2\sqrt{2}}\ln\frac{\sqrt{1/x^2+1}-\sqrt{2}}{\sqrt{1/x^2+1}+\sqrt{2}}$$

(iv). $I = \int \frac{x(x+2)-(x-1)}{(x+2)\sqrt{x-1}}dx = \int \frac{x\,dx}{\sqrt{x-1}} - \int \frac{\sqrt{x-1}}{x+2}dx = I_1 - I_2$

এখন $I_1 = \int \frac{x\,dx}{\sqrt{x-1}} = \int \frac{(x-1)+1}{\sqrt{x-1}}\,dx = \int (x-1)^{1/2}\,dx + \int \frac{dx}{\sqrt{x-1}}$

$= \frac{2}{3}(x-1)^{3/2}) + 2\sqrt{x-1} + c_1$

এবং $I_2 = \int \frac{(x-1)\,dx}{(x+2)\sqrt{x-1}} = \int \frac{(x+2)-3}{(x+2)\sqrt{x-1}}\,dx$

$= \int \frac{dx}{\sqrt{x-1}} - 3\int \frac{dx}{(x+2)\sqrt{x-1}}$

$= 2\sqrt{x-1} - 3\int \frac{2z\,dz}{(z^2+1+2)z}$ যখন $x-1 = z^2 \Rightarrow dx = 2z\,dz$

$= 2\sqrt{x-1} - 6\int \frac{dz}{z^2+(\sqrt{3})^2} = 2\sqrt{x-1} - \frac{6}{\sqrt{3}}\tan^{-1}\frac{z}{\sqrt{3}} + c_2$

(v). $I = \int \left[\frac{x^2(x+1) - x(x+1) + (x+1) - 1}{(x+1)}\right]\frac{dx}{\sqrt{x-2}}$

$= \int \left[x^2 - x + 1 - \frac{1}{x+1}\right]\frac{dx}{\sqrt{x-2}}$

ধরি $x-2 = z^2$ বা $x = z^2+2$ তবে $dx = 2z\,dz$

$\therefore I = \int \left[(z^2+2)^2 - (z^2+2) + 1 - \frac{1}{z^2+2+1}\right]\frac{2z\,dz}{z}$

$= 2\int \left[z^4 + 4z^2 + 4 - z^2 - 2 + 1 - \frac{1}{z^2+3}\right]dz$

$= 2\int \left[z^4 + 3z^2 + 3 - \frac{1}{z^2+(\sqrt{3})^2}\right]dz$

$= 2\left[\frac{z^5}{5} + \frac{3z^3}{3} + 3z - \frac{1}{\sqrt{3}}\tan^{-1}\frac{z}{\sqrt{3}}\right] + c.$

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