Belal Sir Part :

20-21

4(a)

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{--- (i)}$$

This eqn known as the general eqn of 2nd degree.

If we transfer this with,

$$x = x' + \alpha, \quad y = y' + \beta$$

then, the eqn will be,

$$a(x'+\alpha)^2 + 2h(x'+\alpha)(y'+\beta) + b(y'+\beta)^2 + 2g(\alpha+x')$$
$$+2f(y'+\beta)+c=0$$

$$\Rightarrow ax'^2 + 2hx'y' + by'^2 + 2(a\alpha + h\beta + g)x' +$$
$$2(h\alpha + b\beta + f)y' + a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c \quad \text{--- (ii)}$$

From eqn (i) & (ii)

If we write,

$$F(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$$F(\alpha,\beta) = a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c$$

$$\therefore \frac{\Delta F(\alpha,\beta)}{\Delta F(\alpha)} = 2a\alpha + 2h\beta + 2g \quad \text{--- (iii)}$$

$$\frac{\Delta F(\alpha,\beta)}{\Delta F(\beta)} = 2h\alpha + 2b\beta + 2f = 0 \quad \text{(IV)}$$

Substituting those value in eq^n (II)

$$ax'^2 + 2hx'y' + by'^2 + \frac{\Delta F(\alpha,\beta)}{\Delta F(\alpha)}x' + \frac{\Delta F(\alpha,\beta)}{\Delta F(\beta)}y' + \underbrace{F(\alpha,\beta)}_{=0} \quad \text{(V)}$$

Expanding this eq^n — (IV) with,

$$a'x'^2 + 2h'x'y' + b'y'^2 + 2g'x' + 2f'y' + c' = 0 \quad \text{(VI)}$$

hence, $a' = a$, $b' = b$, $h' = h$

Then we can write,

(i) $a' + b' = a + b$

(ii) $a'b' - h'^2 = ab - h^2$

From the eq^n $F(\alpha, \beta)$ —

(iii) $\Delta' = \Delta$

4(b) $ax^2+2hxy+by^2+2gx+2fy+c=0$

Invariant : The quantities like $a+b$, $ab-h^2$ and Δ which remain unchange during all posible transformations are called invariants.

→ given eqn,

$$x^2+y^2-2ax+2by+c^2=0 \quad \text{——— (1)}$$

	x'	y'
x	$\cos\theta$	$-\sin\theta$
y	$\sin\theta$	$\cos\theta$

The transformed eqn obtained by if we replacing x by $(x'\cos\theta - y'\sin\theta)$ and y by $(x'\sin\theta + y'\cos\theta)$. The transformed eqn is the,

Here $\theta=\frac{\pi}{4}$ $\qquad y=\frac{x'+y'}{\sqrt{2}}$

$$x = x'\cos\theta - y'\sin\theta$$
$$= \frac{x'-y'}{\sqrt{2}}$$

$$\left(\frac{x'-y'}{\sqrt{2}}\right)^2 - \left(\frac{x'+y'}{\sqrt{2}}\right)^2 - 2a\left(\frac{x'-y'}{\sqrt{2}}\right) + 2b\left(\frac{x'+y'}{\sqrt{2}}\right) + c^2 = 0$$

$$\Rightarrow \frac{x'^2 - 2x'y' + y'^2}{2} - \frac{x'^2 + 2x'y' + y'^2}{2} - 2a\,\frac{x'-y'}{\sqrt{2}} + 2b\,\frac{x'+y'}{\sqrt{2}} + c^2 = 0$$

$$\Rightarrow \frac{\cancel{x'^2} - 2x'y' + \cancel{y'^2} - \cancel{x'^2} - 2x'y' - \cancel{y'^2} - 2\sqrt{2}a x' + 2\sqrt{2}a y'}{2}$$

$$\frac{2\sqrt{2}b x' + 2\sqrt{2} b y' + 2c^2}{2} = 0$$

$$\Rightarrow -4x'y' - 2\sqrt{2}ax' + 2\sqrt{2}ay' + 2\sqrt{2}bx' + 2\sqrt{2}by' + 2c^2 = 0$$

$$\Rightarrow 2x'y' + \sqrt{2}$$

The transforming point is:

$$x = a + \frac{x'-y'}{\sqrt{2}}$$

$$y = b + \frac{x'+y'}{\sqrt{2}}$$

$$\Rightarrow \left(a+\frac{x'-y'}{\sqrt{2}}\right)^2-\left(b+\frac{x'+y'}{\sqrt{2}}\right)^2-2a\left(a+\frac{x'-y'}{\sqrt{2}}\right)$$
$$+2b\left(b+\frac{x'+y'}{\sqrt{2}}\right)+c^2=0$$

$$\Rightarrow \cancel{a^2}+\cancel{2a\frac{x'-y'}{\sqrt{2}}}+\frac{x^2-2x'y'+y^2}{2}-\left(b^2+\cancel{2\cdot b\frac{x'+y'}{\sqrt{2}}}\right.$$
$$\left.+\frac{x^2+2xy+y^2}{2}\right)-\cancel{2a^2}-\cancel{\frac{2a(x'-2ay')}{\sqrt{2}}}+2b^2+\cancel{\frac{2bx'}{\sqrt{2}}}$$
$$+\cancel{\frac{2by'}{\sqrt{2}}}+c^2=0$$

$$\Rightarrow -a^2+b^2+c^2+\frac{\cancel{x^2}-2xy+\cancel{y^2}-\cancel{x^2}-2xy-\cancel{y^2}}{2}=0$$

$$\Rightarrow -a^2+b^2+c^2-2xy=0$$

$$\Rightarrow \cancel{2xy=a^2-b^2-c^2}$$

$$\Rightarrow 2xy=b^2+c^2-a^2$$

5(a) we know that,

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{—— (i)}$$

Which is the required general eq^n of 2nd degree

~~Whe~~ We have to prove;

eq^n (i) represent a straight line,

Let's the eq^n

$$ax^2+2hxy+by^2=0 \quad \text{—— (ii)}$$

Dividing both side with x^2b ~~in~~ eq^n (ii)

eq^n (i) has homogeneous form is—

$$ax^2+2hxy+by^2=0 \quad \text{—— (ii)}$$

We have to prove eq^n (ii) is a straight line,

Now, Dividing both side with x^2b in eq^n (ii)

$$\frac{ax^2}{bx^2}+\frac{2hxy}{bx^2}+\frac{by^2}{bx^2}=0$$

$$\Rightarrow \frac{a}{b} + \frac{2h}{b}\frac{y}{x} + \left(\frac{y}{x}\right)^2 = 0 \quad \text{——— (iii)}$$

Let, m_1 and m_2 be the roots of this quadratic in y/x.

Then eq^n (iii) must be equivalen to,

$$\left(\frac{y}{x} - m_1\right)\left(\frac{y}{x} - m_2\right) = 0$$

Therefore is satiesffed when,

$$\frac{y}{x} - m_1 = 0 \quad \text{and} \quad \frac{y}{x} - m_2 = 0$$

$$\Rightarrow y - m_1 x = 0 \qquad \Rightarrow y - m_2 x = 0$$

Therefore all the points on the locus represented by eq^n (ii) are on or other of the two straight lines

$$\left.\begin{array}{l} y - m_1 x = 0 \\ y - m_2 x = 0 \end{array}\right\} \text{— Hence the thorem.}$$

5(b) Given eq^n is —

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{—(I)}$$

Let, eq^n (I) point of intersection be (α,β) —

So, $x = X+\alpha$ and $y = Y+\beta$

Substituting x, and y in eq^n (I)

$$a(X+\alpha)^2+2h(X+\alpha)(Y+\beta)+b(Y+\beta)^2+2g(X+\alpha)$$
$$+2f(Y+\beta)+c=0$$

$$\Rightarrow aX^2+2hXY+bY^2+2(a\alpha+h\beta+g)X+$$
$$2(h\alpha+b\beta+f)Y+a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c=0$$

$$\text{—(II)}$$

If the eq^n (II) represent a pair of straight line then X & Y term constant

so, $$a\alpha+h\beta+g=0 \quad \text{—(III)}$$
$$h\alpha+b\beta+f=0 \quad \text{—(IV)}$$

and $a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0$ —— (V)

$\Rightarrow \alpha(\underbrace{a\alpha + h\beta + g}_{eq^n \text{ (III)}}) + \beta(\underbrace{h\alpha + b\beta + f}_{eq^n \text{ (IV)}}) + g\alpha + f\beta + c = 0$

$\Rightarrow 0 + 0 + g\alpha + f\beta + c = 0$ —— (V)

Elliminating α, β from eq^n (III), (IV) & (V) —

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & g \end{vmatrix} = 0$$

which is the required condition

and intersecting points are (α, β).

5 (c) out of syllabut.

(as far as I know)

5 (d)

$2\lambda xy - y^2 + 4x + 2y + 8 = 0$

$a = 0,\ b = -1,\ h = \lambda,\ g = 2,\ f = 1,\ c = 8$

$\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$= 0\times(-1)\times 8 + \underline{2\times 1\times 2\times \lambda} - 0\times f^2 - (-1)(2)^2 - 8\lambda^2 = 0$

$\Rightarrow 0 + 4\lambda - 0 - 4 - 8\lambda^2 = 0$

$\Rightarrow 2\lambda^2 + \lambda - 1 = 0$

$\Rightarrow 2\lambda^2 + 2\lambda - \lambda - 1 = 0$

$\Rightarrow 2\lambda(\lambda+1) - 1(\lambda+1) = 0$

$\Rightarrow (\lambda+1)(2\lambda-1) = 0$

either, OR $2\lambda - 1 = 0$

$\lambda + 1 = 0$ $\qquad \Rightarrow \lambda = \frac{1}{2}$

$\therefore \lambda = -1$

$\therefore$ The value of $\lambda = -1, \frac{1}{2}$

6 (a)

Parabola $\equiv$ $y^2 = 4ax$

(i) ~~centre~~ ~~0,0~~ vertex $(0,0)$

(ii) Latus rectum $= 4a$

(iii) Focus $(a,0)$

Ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(i) centre $(0,0)$

(ii) foci $(\pm ae, 0)$

(iii) vertices $(\pm a, 0)$

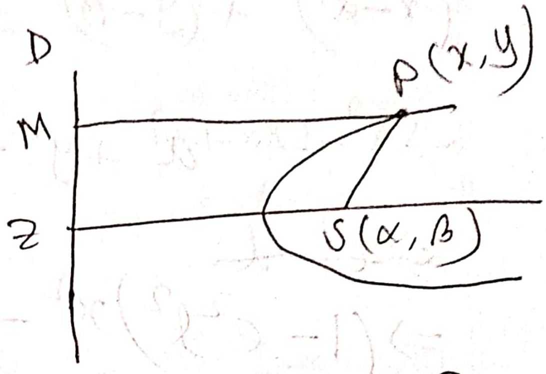
Hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(i) centre $(0,0)$

(ii) foci $(\pm ae, 0)$

(iii) directrix : $x = \pm a/e$

6(b)



Let, $S(\alpha, \beta)$ be the focus and DZ be the directrix.

Let the eq^n of DZ be $Ax + By + C = 0$

and $P(x,y)$ any point on the conic.

Draw PM perpendicular to DZ. Then,

$\frac{SP}{PM} = e$ [e = eccentricity]

$\Rightarrow SP^2 = e^2 PM^2$

$\Rightarrow (x-\alpha)^2 + (y-\beta)^2 = e^2 \left(\frac{Ax + By + C}{\sqrt{A^2+B^2}} \right)$ — (1)

which is the required eq^n of conic.

putting, $l = \frac{A}{\sqrt{A^2+B^2}}$, $m = \frac{B}{\sqrt{A^2+B^2}}$, $n = \frac{C}{\sqrt{A^2+B^2}}$

Now eq^n (i) becomes,

$$(x-\alpha)^2+(y-\beta)^2 = e^2(lx+my+n)^2$$

$$\Rightarrow x^2-2x\alpha+\alpha^2+y^2-2y\beta+\beta^2 = e^2(lx+my+n)^2$$

~~$\Rightarrow e^2 l$~~

$$\Rightarrow (1-e^2l^2)x^2 - 2e^4lmn\,xy + (1-e^2m^2)y^2$$
$$-2(\alpha+e^2ln)x - 2(\beta+e^2mn)y + (\alpha^2+\beta^2-e^2n^2) = 0$$

$$\Rightarrow ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{(iii)}$$

Where $a = 1-e^2l^2$, $b = 1-e^2m^2$ and so on.

The eq^n (iii) represent the general eq^n of 2^{nd} degree.

Now,

$$ab - h^2 = (1 - e^2 l^2)(1 - e^2 m^2) - (e^2 lm)^2$$

$$= 1 - e^2 (\underbrace{l^2 + m^2}_{1})$$

$$\therefore ab - h^2 \ncong 1 - e^2$$

Hence, $ab - h^2 >, =, < 0$

According as the conic is an ellipse ($e<1$), a parabola ($e=1$) or hyperbola ($e>1$)

6 (c) → 6(d) out of syllabus

2015—16

4(a) Effect ≈ 20-21 4(a)

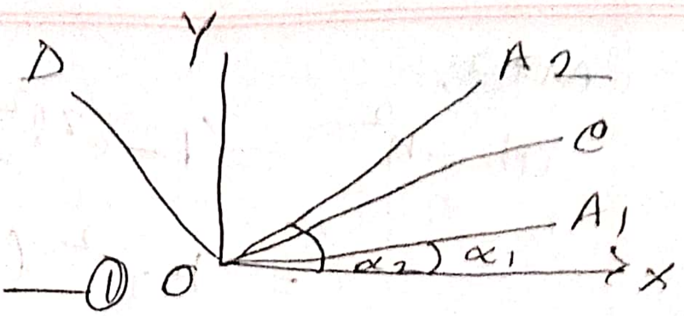
↳ 4 No full set repeat as 20-21 (4 No)

5 No repeat as

6 No out of syllabus

14-15

4 (a)



$ax^2 + 2hxy + by^2 = 0$ —— (1)

Let, OA_1 and OA_2 be the straight line represented by the eq^n (1).

Let their separate eq^n ——

$y - m_1 x = 0$, $y - m_2 x = 0$

Then,

$by^2 + 2hxy + ax^2 = b(y - m_1 x)(y - m_2 x)$

$\therefore m_1 + m_2 = -\frac{2h}{b}$; $m_1 m_2 = \frac{a}{b}$

Let the inclinations of OA_1 and OA_2 be represented by α_1 & α_2 ——

Then $\left.\begin{array}{l}\tan\alpha_1 = m_1 \\ \tan\alpha_2 = m_2\end{array}\right\}$ —— (1a)

Let, OC and OD be respectively the internal & External bisectors of angle A_1OA_2.

Now,

$\angle XOC$

$= \angle XOA_1 + \angle A_1OC$

$= \alpha_1 + \frac{1}{2} \angle A_1OA_2$

$= \alpha_1 + \frac{1}{2} (\angle XOA_2 - \angle XOA_1)$

$= \alpha_1 + \frac{1}{2}(\alpha_2 - \alpha_1)$

$= \frac{1}{2}(\alpha_1 + \alpha_2)$

From the figure, $\angle COD = \pi/2$

$\therefore \angle XOD = \angle XOC + \angle COD$

$= \frac{1}{2}(\alpha_1 + \alpha_2) + \frac{\pi}{2}$

$= \frac{\pi + (\alpha_1 + \alpha_2)}{2}$

If $\angle XOC$ or $\angle XOD = \theta$ then,

$$\tan 2\theta = \frac{\tan\alpha_1 + \tan\alpha_2}{1 - \tan\alpha_1 \tan\alpha_2}$$

$$\Rightarrow \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{m_1 + m_2}{1 - m_1 m_2}$$

$$\Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = \frac{-2h/b}{a/b} = \frac{2h}{a-b}$$

- If (x,y) be any point on either bisector

then $\tan\theta = y/x$

$$\therefore \frac{\cancel{2}\, y/x}{1-\left(\frac{y}{x}\right)^2} = \frac{\cancel{2}h}{a-b}$$

$$\Rightarrow \frac{xy}{x^2-y^2} = \frac{h}{a-b}$$

$$\Rightarrow \frac{x^2-y^2}{xy} = \frac{a-b}{h}$$

4 (b) Given eq^n

$$9x^2+15xy+y^2+12x-11y-15=0$$

Let's first transform the eq^n to parallel axes through the point (α, β) —

$$X = x+\alpha \; ; \; Y = y+\beta$$

$$\Rightarrow 9(x+\alpha)^2 + 15(x+\alpha)(y+\beta) + (y+\beta)^2 + 12(x+\alpha)$$
$$-11(y+\beta) - 15 = 0$$

$$\cancel{\Rightarrow 9x^2 + 15xy + y^2 + 2(18\alpha + 24\beta - 6)x}$$
$$\cancel{+(24\alpha + 4\beta + 20)y + 41 =}$$

$$\Rightarrow 9(x^2 + 2x\alpha + \alpha^2) + 15(xy + x\beta + y\alpha + \alpha\beta)$$
$$+(y^2 + 2y\beta + \beta^2) + 12x + 12\alpha - 11y - 11\beta - 15 = 0$$

$$\Rightarrow 9x^2 + 18x\alpha + 9\alpha^2 + 15xy + 15x\beta + 15y\alpha + 15\alpha\beta$$
$$+y^2 + 2y\beta + \beta^2 + 12x + 12\alpha - 11y - 11\beta - 15 = 0$$

$$\Rightarrow 9x^2 + 15xy + y^2 + x(18\alpha + 15\beta + 12) + y(15\alpha + 2\beta$$
$$+9\alpha^2 + 15\alpha\beta + \beta^2 + 12\alpha - 11\beta - 15 = 0 \quad \text{--- (1)}$$

The term x and y in eq^n (1) will be absent if,

$$18\alpha + 15\beta + 12 = 0$$
$$15\alpha + 2\beta - 11 = 0$$

$$\alpha = 1 \; ; \; \beta = -2$$

Now substituting α, β's value in eqn (i)

$$9x^2+15xy+y^2+2=0 \quad \text{——— (iii)}$$

Now, in order to remove the term xy,

Replacing x by $(x\cos\theta - y\sin\theta)$ and y by $(x\sin\theta + y\cos\theta)$

$\longrightarrow$ eqn (iii) will be

$$9(x\cos\theta - y\sin\theta)^2 + 15(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) + (x\sin\theta + y\cos\theta)^2 + 2 = 0$$

$$\Rightarrow 9(x^2\cos^2\theta - 2xy\sin\theta\cos\theta + y^2\sin^2\theta) + 15(x^2\sin\theta\cos\theta + xy\cos^2\theta - xy\sin^2\theta - y^2\sin\theta\cos\theta) + (x^2\sin^2\theta + 2xy\sin\theta\cos\theta + y^2\cos^2\theta) + 2 = 0$$

$$\Rightarrow 9x^2\cos^2\theta - 18\underline{xy\sin\theta}\cos\theta + 9y^2\sin\theta + 15x^2\sin\theta\cos\theta + 15\underline{xy\cos^2\theta} - 15xy\sin^2\theta - 15y^2\sin\theta\cos\theta + x^2\sin^2\theta + \underline{2xy}\sin\theta\cos\theta + y^2\cos^2\theta + 2 = 0 \quad \text{——— (iv)}$$

$\Rightarrow$ By our assumtion

$$-18xy\sin\theta\cos\theta + 15xy\cos^2\theta - 15xy\sin^2\theta + 2xy\sin\theta\cos\theta = 0$$

$\Rightarrow xy(15\cos^2\theta - 16\sin\theta\cos\theta - 15\sin^2\theta) = 0$

$\Rightarrow 15\cos^2\theta - 16\sin\theta\cos\theta - 15\sin^2\theta = 0$

$\Rightarrow -8\sin 2\theta + 15\cos 2\theta = 0$

$\Rightarrow 15\cos 2\theta = 8\sin 2\theta$

$\Rightarrow \frac{15}{8} = \tan 2\theta$

$\therefore 2\theta = \tan^{-1}\left(\frac{15}{8}\right)$

$\theta = \frac{1}{2}\tan^{-1}\left(\frac{15}{8}\right)$

[পারি না — —]

$\tan 2\theta = \frac{2x}{1-x^2}$

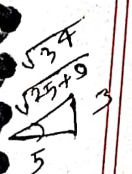
$\Rightarrow \frac{15}{8} = \frac{2x}{1-x^2}$

$\Rightarrow 15 - 15x^2 - 16x = 0$

$\Rightarrow 15x^2 + 16x - 15 = 0$

$\Rightarrow 15x^2 + 15x + x - 15 = 0$

$\Rightarrow 15x(x+1) + 1$



$\theta = \tan^{-1} 3/5$

$\sin\theta = \frac{3}{\sqrt{34}}, \quad \cos\theta = \frac{5}{\sqrt{34}}$

Substituting these in eqⁿ

$9x^2$

পারি না ~