Solution. The equation is of the above type and can be written as $(3x+4y-1)\,dy+(2x+3y+1)\,dx=0$,

i.e., $3\,(x\,dy+y\,dx)+(4y-1)\,dy+(2x+1)\,dx=0$.

Integrating. $3xy+2y^2-y+x^2+x=C$ is the solution.

2.7. Linear Differential Equations

[Poona 63, 61; Nagpur 62, 61; Guj. 61]

A differential equation of the form

$$\frac{dy}{dx}+Py=Q,$$

where P, Q are functions of x or constants, is called the *linear differential equation of the first order*.

To solve this equation, multiply both the sides by $e^{\int P\,dx}$

Then it becomes $e^{\int P\,dx}\dfrac{dy}{dx}+Py\,e^{\int P\,dx}=Qe^{\int P dx}$.

or $$\frac{d}{dx}\left[ye^{\int P\,dx}\right]=Q\,e^{\int P\,dx}.$$

Integrating both the sides, w.r.t. x, we get

$$ye^{\int P\,dx}=\int \left[Qe^{\int Pdx}\right]dx+C,$$

which is the required solution.

Integrating factor (I.F.). It will be noticed that for solving (1), we multiplied it by a factor $e^{\int P\,dx}$ and the equation became readily (directly) integrable. Such a factor is called the integrating factor.

Note. Sometimes a differential equation takes linear form if we *regard x as dependent variable and y as independent variable*. The equation can then be put as $\dfrac{dx}{dy}+Px=Q$, where P, Q are functions of y or constants.

The integrating factor in this case is $e^{\int P\,dy}$ and solution is

$$xe^{\int P\,dy}=\int \left[Qe^{\int P\,dy}\right]dy+C.$$

(See Ex. 1 to 4 pages 21 and 22).

Ex. 1. *Solve* $(1-x^2)\dfrac{dy}{dx}-xy=1$.

[Delhi 68 : Nag. 61]

Solution. The equation can be written as

$$\frac{dy}{dx}-\frac{x}{1-x^2}\,y=\frac{1}{1-x^2}$$

This is now expressed in the linear form

$$P=-\frac{x}{1-x^2},\quad \text{I.F.}=e^{\int P\,dx}=e^{\int\frac{-x}{1-x^2}dx}=e^{\frac{1}{2}\log(1-x^2)}$$
$$=\sqrt{(1-x^2)}.$$

Hence the solution is

$$y.\sqrt{(1-x^2)}=\int\frac{1}{1-x^2}\sqrt{(1-x^2)}\,dx+C.$$

Ex. 2. (a) *Solve* $x\dfrac{dy}{dx}+2y=x^2\log x.$ **[Lucknow 52]**

Solution. The equation is $\dfrac{dy}{dx}+\dfrac{2}{x}y=x\log x.$

$\text{I.F.}=e^{\int(2/x)\,dx}=e^{2\log x}=x^2.$

Hence the solution is

$$y.x^2=C+\int x^2.x\log x\,dx=C+\int x^3\log x\,dx$$
$$=C+\log x.\frac{x^4}{4}-\int\frac{1}{x}.\frac{x^4}{4}dx$$
$$=C+\tfrac{1}{4}x^4\log x-\tfrac{1}{16}x^4$$

or $$y=Cx^{-2}+\tfrac{1}{4}x^2(\log x-\tfrac{1}{4}).$$

Ex. 2. (b) *Solve* $x\dfrac{dy}{dx}+2y=x^4.$

[Bombay B.Sc. 61]

Solution. Equation is $\dfrac{dy}{dx}+\dfrac{2}{x}y=x^3.$ $\text{I.F.}=x^2$ as above.

Solution is $y.x^2=C+\int x^3.x^2\,dx=C+\tfrac{1}{6}x^6.$

Ex. 3. *Solve* $(x^3-x)\dfrac{dy}{dx}-(3x^2-1)\,y=x^5-2x^3+x.$

[Gujrat B.Sc. (Sub.) 1961]

Solution. The equation is

$$\frac{dy}{dx}-\frac{3x^2-1}{x^3-x}\,y=(x^2-1).$$

$$\text{I.F.}=e^{-\int\frac{(3x^2-1)}{x^3-x}}=e^{-\log(x^3-x)}=\frac{1}{x^3-x}.$$

$\therefore$ Solution is $$y.\frac{1}{x^3-x}=C+\int\frac{x^2-1}{x^3-1}dx$$
$$=C+\int\frac{1}{x}\,dx=C+\log x.$$

Ex 4. *Solve* $xp+y=ax^2+bx+c,\ p=\dfrac{dy}{dx}.$

[Delhi Hons. 1957]

Solution. Write the equation as

$$\frac{dx}{dy}+\frac{1}{y\log y}x=\frac{1}{y}.$$

$$\text{I.F.}=e^{\int \frac{1}{y}\frac{1}{\log y}dy}=e^{\log(\log y)}=\log y.$$

$$\therefore\quad x\log y=C+\int\frac{1}{y}\log y\,dy$$

$$=C+\tfrac{1}{2}(\log y)^2 \text{ is the solution.}$$

Ex. 2. *Solve* $dx+x\,dy=e^{-y}\log y\,dy$. [Poona 61]

Solution. The equation can be written as

$$\frac{dx}{dy}+x=e^{-y}\log y,\ \text{I.F.}=e^{y}.$$

$$\therefore\quad xe^{y}=C+\int e^{-y}\log y.e^{y}\,dy$$

$$=C+\int\log y\,dy=C+\log y.y-\int y.\frac{1}{y}dy$$

$$=C+y\log y-y.$$

Ex. 3. *Solve* $(1+y^2)\,dx+(x-\tan^{-1}y)\,dy=0$. [Gujrat 65; Delhi Hons. 65; Pb. 62; Cal. Hons. 62; Agra 67, 58]

Solution. The equation can be written as

$$\frac{dx}{dy}+\frac{1}{1+y^2}x=\frac{\tan^{-1}y}{1+y^2}.$$

$$\text{I.F.}=e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}.$$

$$\therefore\quad xe^{\tan^{-1}y}=\int\frac{\tan^{-1}y}{1+y^2}e^{\tan^{-1}y}dy+C$$

$$=\int te^{t}\,dt+C \text{ where } t=\tan^{-1}y$$

$$=e^{t}(t-1)+C=e^{\tan^{-1}y}(\tan^{-1}y-1)+C.$$

Hence $x=(\tan^{-1}y-1)+Ce^{-\tan^{-1}y}$ is the solution.

Ex. 4. *Solve* $(x+2y^3)\dfrac{dy}{dx}=y.$

[Agra B.Sc. 1956; Raj B.Sc. 56]

Hint. The equation can be written as

$\dfrac{dx}{dy}=x+2y^3$ [linear]. **Ans.** $x=y^3+Cy$.

2·8. Equations reducible to linear form

***I. Bernoulli Equation*.** $\dfrac{dy}{dx}+Py=Qy^n$,

*Known after James Bernoulli. The method of solution was discovered by Leibnitz.

where P and Q are functions of x or constants.

[Nag. T.D.C. 1961; Poona T.D.C. 61 ; Gujrat B.Sc. (Prin.) 58; Poona B.A. (Gen.) 60]

Dividing both the sides by y^n we have

$$y^{-n}\frac{dy}{dx}+Py^{-n+1}=Q. \qquad \text{...(1)}$$

Now put $y^{-n+1}=v$ so that $(1-n)\,y^{-n}\dfrac{dy}{dx}=\dfrac{dv}{dx}$.

Then (1) becomes $\dfrac{1}{1-n}\dfrac{dv}{dx}+Pv=Q$

or $\dfrac{dv}{dx}+P\,(1-n)\,v=(1-n)\,Q$

which is a linear equation in v and x.

II. Equation $f'(y)\dfrac{dy}{dx}+Pf(y)=Q$.

where P and Q are functions of x or constants.

Put $f(y)=v$ so that $f'(y)\dfrac{dy}{dx}=\dfrac{dv}{dx}$.

$\therefore$ equation becomes $\dfrac{dv}{dx}+Pv=Q$,

which is a linear equation in v and x.

Note. In each of these equations, single out Q (function of x on the right) and then make suitable substitution to reduce the equation in linear form.

Ex. 1. *Solve* $\dfrac{dy}{dx}=x^3y^3-xy$.

[Karnatak B.Sc. (Prin.) 1960, 62; Agra 61; Bihar 62; Gujrat B.Sc. (Sub.) 61]

Solution. The equation is $\dfrac{dy}{dx}+xy=x^3y^3$.

Dividing by y^3; $\dfrac{1}{y^3}\dfrac{dy}{dx}+x\,.\,\dfrac{1}{y^2}=x^3$.

Put $\dfrac{1}{y^2}=v$, so that $-\dfrac{2}{y^3}\dfrac{dy}{dx}=\dfrac{dv}{dx}$, *i.e.,* $\dfrac{1}{y^3}\dfrac{dy}{dx}=-\frac{1}{2}\dfrac{dv}{dx}$.

$\therefore$ equation becomes $-\frac{1}{2}\dfrac{dv}{dx}+xv=x^3$

or $\dfrac{dv}{dx}-2x\,.\,v=-2x^3$.

Linear, I. F. $=e^{\int -2x\,dx}=e^{-x^2}$.

Hence $ve^{-x^2}=\int -2x^3e^{-x^2}\,dx+C$

$\qquad =\int x^2\,(-2x)\,e^{-x^2}\,dx+C$

3
Equations of First Order and First Degree

Exact Differential Equations and Reduction to Exact Equations

3·1. Exact Differential Equations. [Bombay 61 ; Karnatak 60]

Study the following two differential eqnations :

1. $x\,dy+y\,dx=0$. Solution is $xy=C$.
2. $\sin x \cos y\,dy+\cos x \sin y\,dx=0$.

   Solution is $\sin x \sin y=C$.

We see that these differential equations can be obtained by directly differentiating their solutions. Differential equations of this type are called exact equations and bear the following property :

An exact differential equation can always be obtained from its primitive directly by differentiation, without any subsequent multiplication, elimination etc.

*3·2. Necessarry and Sufficient Condition

To find the necessary and sufficient condition for a differential equation of first degree being exact.

[Poona 63, 61 ; Delhi Hons. 57, 55 ; Nag. 63 : Gujrat 59; Bombay 61]

Let the equation be $M+N\dfrac{dy}{dx}=0$. ...(1)

Let $u=C$ be its primitive. ...(2)

If (1) is exact, it can be obtained by directly differentiating its primitive.

Differentiating (2), we have $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}\cdot\dfrac{dy}{dx}=0$. ...(3)

Comparing (1) and (3) we get $M=\dfrac{\partial u}{\partial x}$ and $N=\dfrac{\partial u}{\partial y}$, so that

$$\frac{\partial M}{\partial y}=\frac{\partial^2 u}{\partial y\partial x},\quad \frac{\partial N}{\partial x}=\frac{\partial^2 u}{\partial x\partial y}.$$

Hence the condition is $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$.

That the condition is necessary has been proved. Now we prove that it is sufficient also, *i.e.* if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then we show that $M+N\frac{dy}{dx}=0$ or $M\,dx+N\,dy=0$ is an exact equation.

Let $\int M\,dx=U$, then $\frac{\partial U}{\partial x}=M$, so that

$$\frac{\partial^2 U}{\partial y\,\partial x}=\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \text{ as } \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}.$$

i.e. $\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right)$

Integrating, $N=\frac{\partial U}{\partial y}+f(y)$, where $f(y)$ is a function of y free from x.

$$\therefore \quad M+N\frac{dy}{dx}=\frac{\partial U}{\partial x}+\left[\frac{\partial U}{\partial y}+f(y)\right]\frac{dy}{dx}$$

$$=\frac{d}{dx}\left[U+\int f(y)\,\frac{dy}{dx}\,dx\right]$$

$$=\frac{d}{dx}\,[U+F(y)].$$

This shows that $M+N\frac{dy}{dx}=0$ is an exact equation.

3·3. Working Rule (Remember it).

If the equation $M\,dx+N\,dy=0$ satisfies the condition

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x},$$

then it is exact. To integrate it,

(*i*) integrate M with regard to x regarding y as constant;

(*ii*) find out those terms in N which are free from x and integrate them with regard to y;

(*iii*) add the two expressions so obtained and equate the sum to an arbitrary constant.

This gives the general solution of the given exact equation.

Ex. 1. $(y^4+4x^3y+3x)\,dx+(x^4+4xy^3+y+1)\,dy=0$ **[Karnatak 60]**

Solution Here $M=y^4+4x^3y+3x$ and $N=x^4+4xy^3+y+1$.

$\frac{\partial M}{\partial y}=4y^3+4x^3$ and $\frac{\partial N}{\partial x}=4x^3+4y^3$.

Since these are equal, the equation is exact.

To find solution of the differential equation, integrating M *i.e.* y^4+4x^3y+3x w.r.t. x, keeping y as constant, we get $y^4x+x^4y+\frac{3}{2}x^2$.

$$=\int -te^t\,dt+C \text{ where } t=-x^2$$
$$=-te^t+e^t+C=-e^{-x^2}(x^2-1)+C$$

Hence $\quad v=1-x^2+Ce^{x^2}$ or $\dfrac{1}{y^2}=1-x^2+Ce^{x^2}$

Ex. 2. *Solve* $\dfrac{dy}{dx}+xy=xy^2$. [Karnatak 1960]

Solution. Dividing by y^2, $y^{-2}\dfrac{dy}{dx}+xy^{-1}=x$.

Put $y^{-1}=v$, so that $-y^{-2}\dfrac{dy}{dx}=\dfrac{dv}{dx}$.

$\therefore$ equation is $\dfrac{dv}{dx}-xv=-x$.

I. F. $=e^{\int -x\,dx}=e^{-\frac{1}{2}x^2}$.

$\therefore \quad ve^{-\frac{1}{2}x^2}=C-\int xe^{-\frac{1}{2}x^2}$

$\quad =C+\int e^t\,dt$, where $-\frac{1}{2}x^2=t$, $-x\,dx=dt$

or $\quad y^{-1}e^{-\frac{1}{2}x^2}=C+e^t=C+e^{-\frac{1}{2}x^2}$

or $\quad y^{-1}=Ce^{\frac{1}{2}x^2}+1$ is the solution.

Ex. 3. *Solve* $\dfrac{dy}{dx}+\dfrac{2}{x}y=\dfrac{y^3}{x^3}$. [Nag. 1958]

Solution. Dividing by y^3, $y^{-3}\dfrac{dy}{dx}+\dfrac{2}{x}y^{-2}=\dfrac{1}{x^3}$.

Put $y^{-2}=v$, so that $-2y^{-3}\dfrac{dy}{dx}=\dfrac{dv}{dx}$.

$\therefore$ equation becomes $-\frac{1}{2}\dfrac{dv}{dx}+\dfrac{2}{x}v=\dfrac{1}{x^3}$

or $\dfrac{dv}{dx}-\dfrac{4}{x}v=-\dfrac{2}{x^3}$.

I. F. $=e^{\int(-4/x)\,dx}=e^{-4\log x}=\dfrac{1}{x^4}$.

$\therefore \quad v\dfrac{1}{x^4}=\int -\dfrac{2}{x^3}\cdot\dfrac{1}{x^4}\,dx+C=C+\dfrac{1}{3x^6}$

or $\dfrac{1}{y^2}\cdot\dfrac{1}{x^4}=\dfrac{1}{3x^6}+C$ is the solution.

***Ex. 4.** *Solve* $\dfrac{dy}{dx}(x^2y^3+xy)=1$.

[Sagar 1962; Raj. 63; Cal. Hons. 62; Luck. 63]

Solution. The equation can be written as

$\dfrac{dx}{dy}-xy=x^2y^3$.