Logarithms

We shall use the following notations:

$$
\begin{aligned}
\lg n &= \log_2 n && \text{(binary logarithm)} ,\\
\ln n &= \log_e n && \text{(natural logarithm)} ,\\
\lg^k n &= (\lg n)^k && \text{(exponentiation)} ,\\
\lg \lg n &= \lg(\lg n) && \text{(composition)} .
\end{aligned}
$$

An important notational convention we shall adopt is that *logarithm functions will apply only to the next term in the formula*, so that $\lg n + k$ will mean $(\lg n) + k$ and not $\lg(n + k)$. If we hold $b > 1$ constant, then for $n > 0$, the function $\log_b n$ is strictly increasing.

For all real $a > 0$, $b > 0$, $c > 0$, and n,

$$
\begin{aligned}
a &= b^{\log_b a} ,\\
\log_c(ab) &= \log_c a + \log_c b ,\\
\log_b a^n &= n \log_b a ,\\
\log_b a &= \frac{\log_c a}{\log_c b} ,\\
\log_b(1/a) &= -\log_b a ,\\
\log_b a &= \frac{1}{\log_a b} ,\\
a^{\log_b c} &= c^{\log_b a} ,
\end{aligned}
$$

where, in each equation above, logarithm bases are not 1.

The approach was first presented by Jon Bentley, Dorothea Haken, and James B. Saxe in 1980

$$T(n) = a\,T\left(\frac{n}{b}\right) + \theta(n^k \log^p n)$$

$a \geq 1, b > 1$ and $k \geq 0$ and $'p'$ is any real number

Case 1: If $a > b^k$, $\quad T(n) = \theta(n^{\log_b a})$

Case 2: If $a = b^k$

(a) If $p > -1 \quad \theta(n^{\log_b a} \log^{p+1} n)$

(b) If $p = -1 \quad \theta(n^{\log_b a} \log_2 \log_2 n)$

(c) If $p < -1 \quad \theta(n^{\log_b a})$

Case 3: If $a < b^k$

(a) If $p \geq 0 \quad \theta(n^k \log^p n)$

(b) If $p < 0 \quad O(n^k)$