TY CSE
2022-23
Design and Analysis of Algorithm

Lect. 1 13/08/2022

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10. **Module 1: Introduction**
    a. Introduction to Algorithm Analysis Time and Space Complexity, Elementary operations and Computation of Time Complexity-Best, worst and Average Case Complexities- Complexity Calculation of simple algorithms. Recurrence Equations: Solution of Recurrence Equations –Iteration Method and Recursion Tree Methods. Master's theorem for complexity computation.
11. **Definition**:

Finite set of Steps to solve a problem.

Steps means Fundamental Instructions (i.e. Instructions which contains the operatn +, −, *, ÷, %, =, etc. are called fundamental Instructions.

12.Characteristics of fundamentals operators:

1. Definiteness : Every fundamental operator should be defined without any ambiguity.

e.g. $i = i + 1$ → Invalid

$i = i + 1$ → Valid

2. Finiteness : Every fundamental instructions should be terminated within finite amount of time.

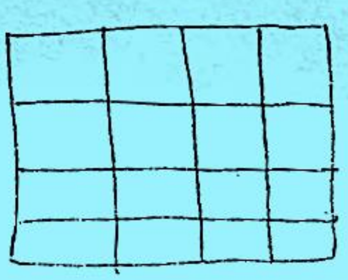
```
i = 1
while (1)
{
    i = i+1 ;      // This type of instn are not allowed .
}
```

3. Every fundamental expressn accepts at least 0 i/p. & provides at most one o/p.

13. Steps to solve problem:

1. Identifying problem Statement :

ex: Arrange 4 queens Q_1, Q_2, Q_3, Q_4 into 4x4 Chess board



② Constraints : No two pins on same row and No two pins in same column & diagonal.

~~Des~~

③ Design Logic : Depending upon the characteristics of problem we can choose any one of the following design strategy for designing logic.

a) Divide & conquer.
b) Greedy method
c) Dynamic programming
d) Backtracking
e) Branch & Bound
f) Bruteforce

etc.

④ Validation : Most of the algorithms validated by using mathematical induction.

⑤ Analysis : The process of comparing two algorithms with respect to time, space, rate of growth, network Bandwidth, number of registers, etc. is called Analysis.

Priory Analysis	Posterior Analysis
① Analysis done before executing. ex. $x = x+1$ Principle: freq. count of fundamental instructions since $x = x+1$ being carried out only one time so its time complexity is $O(1)$.	① Analysis done executing. ex: $x = x+1$ System1 0.1 ns, Sys. 2 0.2 ns, Syst. 3 0.4 ns
② It is independent of O.S., syst. Architecture, Processor speed.	② Dependent
③ It provides estimated values.	③ It provides exact values.
④ All these are uniform values	④ These are non-uniform values.

⑥ Implementation.

⑦ Testing and Debugging

14.Asymptotic notation:

<u>Asymptotic Notation</u>

To compare two algorithm's growth rate, we need notations called Asymptotic Notations.

1) Big-Oh (O) :-

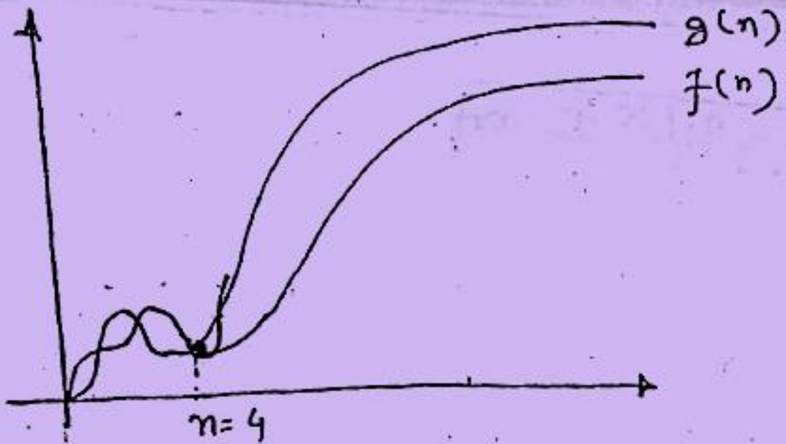
Def: $f(n)$ and $g(n)$ are two functions

$f(n)$ is $O(g(n))$ iff $\exists$ some 'c' and 'k'

$\ni f(n) \leq c \cdot g(n)$; $\forall\ n \geq k$.

ex:

n	$f(n) = n^2$	$g(n) = 2^n$	
1	1	2	$\rightarrow f(n) < g(n)$
2	4	4	$\rightarrow f(n) = g(n)$
3	9	8	$\rightarrow f(n) > g(n)$
4	16	16	$f(n) \leq g(n)$.
5	25	32	
6	36	64	
7	49	128	
8	64	256	
9	81		



(3)

$f(n) = O(g(n)) \Leftrightarrow f(n) \leq g(n).\ ;\ \forall\ n \geq 4$

$n^2 = O(2^n) \quad \forall\ n \geq 4.$

$f(n) = O(g(n)).$

<u>Note</u>:— $f(n) = O(g(n))$ means $g(n)$ is an upper bound to $f(n)$ for <u>large values of n</u>.

* Lower growth rate function = O(higher growth rate function).

pb1 If $f(n) = n^2 + n + 1$ then $f(n) = O(\,)$.

Sol^n:

To prove $n^2 \leq$

$$f(n) = O(g(n)) \iff f(n) \leq c \cdot g(n); \ \forall n \geq k$$

$$n^2 \leq n^2$$

$$n^2 + n \leq n^2 + n^2$$

$$n^2 + n + 1 \leq n^2 + n^2 + n^2$$

$$n^2 + n + 1 \leq 3n^2 . \quad \forall n \geq 1$$

c. k

$$\therefore n^2 + n + 1 = O(n^2) \qquad n \geq 1$$

Shortcut: If $f(n) = a_0 + a_1 n + a_2 n^2 + a_3 n^3 + \dots + a_m n^m$ $(a_m \neq 0)$.

then $f(n) = O(n^m)$.

$n^2 \leq n^3$.

$n^2 + n + 1 \leq n^3$.

$n^2 + n + 1 = O(n^3)$.

14 lack

10.5 lac

$= fn(n) = 10$ lack

<u>Note</u>: Even though n^2, n^3, n^4 are upper bounds to $f(n) = n^2 + n + 1$ take ~~up~~ least upper bounds only.

$\therefore f(n) = O(n^2)$.

(Problem 2): If $f(n) = n!$ then $f(n) = O(\;)$

<u>Sol.</u>: $f(n) = n!$

$= n * (n-1) * (n-2) * n - 2)$

$= n^n \{ 1 \cdot (1 - \frac{1}{n}) (1 - \frac{2}{n}) \cdots \frac{1}{n} \}$

$= O(n^n)$ ($\because$ polynomial of degree n)

③ If $f(n) = \log(n!)$ then $f(n) = O(\ )$.

sol^n: $f(n) = \log(n!)$

$= \log(O(n^n))$ $\quad (\because n! = O(n^n))$

$= O(\log n^n)$ $\quad$ ($\because$ log is constant fun. it satisfies Associative prop).

$= O(n \log n)$.

④ If $f(n) = n^2$ for $n \leq 100$

$= n$ for $n > 100$

and $g(n) = n$ for $n < 1000$

$g(n) = n^3$ for $n \geq 1000$

which of the following is true?

If $f(n) = n^2 \log n$. $g(n) = n \log^{10} n$.

then which of the following is true?

15.Dominance ranking:

(6) $f_1(n) = 2^n$, $f_2(n) = n^{3/2}$, $f_3(n) = n\log_2 n$

$f_4(n) = n^{\log_2 n}$

Arrange f_1, f_2, f_3, f_4 in the increasing order.

(a) $f_3\ f_2\ f_4\ f_1$ (b) $f_2\ f_3\ f_4\ f_1$

(c) $f_3\ f_2\ f_1\ f_4$ (d) $f_2\ f_3\ f_1\ f_4$.

16.

Big-Omega (Ω).

Defn: $f(n)$ is $\Omega(g(n))$ iff $\exists$ some c and 'k' $\ni f(n) \geq c \cdot g(n) : \forall n \geq k$

Problems

Ex 1. If $f(n) = n^2+n+1$ then $f(n) = \Omega(\ \underline{\quad}\ )$.

Soln :-

I. $n^2 \geq n^2$

$n^2 + n \geq n^2$

$\longrightarrow n^2+n+1 \geq n^2 \quad \forall n \geq 0$

$\therefore n^2+n+1 = \Omega(n^2)$.

II. $n^2 \geq n$.

$n^2 + n \geq n$.

$n^2+n+1 \underset{\uparrow}{\geq} \underset{\substack{\uparrow \\ c}}{1} \cdot n \quad \forall n \geq \underset{\substack{\uparrow \\ k}}{0}$

$\therefore n^2+n+1 = \Omega(n)$.

Note: Even though n^2, n are lower boundary to $f(n)$ we have to greatest lower boundary

$\therefore n^2+n+1 = \Omega(n^2)$.

Shortcut: If $f(n) = a_0 + a_1 n + a_2 n^2 + \cdots a_m n^m$ $(a_m \neq 0)$
then $f(n) = \Omega(n^m)$.

$n^2+n+1 = O(n^2)$ & $n^2+n+1 = \Omega(n^2)$.

$\therefore$ $\underline{n^2+n+1 = \Theta(n^2)}$.

Theta (Θ):
$f(n)$ is $\Theta(g(n))$ iff $f(n)$ is $O(g(n))$
and $f(n)$ is $\Omega\, g(n)$.

Ex:- If $f(n) = n^2+n+1$ then
for $c_1 = 3$, $n^2+n+1 \leq 3n^2$ $\forall n \geq 1$] $f(n) = O(n^2)$
for $c_2 = 1$, $n^2+n+1 \leq 1n^2$ $\forall n \geq 0$] $f(n) = \Omega(n^2)$

i.e. $c_2 \cdot n^2 \leq n^2+n+1 \leq c_1 \cdot n^2$ $\forall n \geq 1$.

$\Rightarrow$ $n^2+n+1 = \Theta(n^2)$.

Shortcut: If $f(n) = a_0 + a_1 n + a_2 n^2 + \cdots a_m n^m$ $(a_m \neq 0)$
then $f(n) = \Theta(n^m)$.

Little-Oh : $f(n)$ is $o(g(n))$ iff

$f(n) < c \cdot g(n) \quad \forall\, n \geq K$
$\forall\, c$.

O	o
1. $\leq$	1. $<$
2. $\exists$ some c	2. $\forall\, c$. for all c

ex Let $f(n) = n^2$ $\Rightarrow f(n) < g(n) \Rightarrow f(n) = o(g(n))$
$g(n) = n^3$, $n^2 = o(n^3)$.

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \Rightarrow f(n) = o(g(n)).$$

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Ex $$\lim_{n \to \infty} \frac{n^2}{n^3} = \lim_{n \to \infty} \frac{1}{n} = 0$$

$\therefore n^2 = o(n^3)$.

Little omega (ω):

$f(n)$ is a $\omega(g(n))$ iff $f(n) > c \cdot g(n)$. $\forall n \geq K$
$\forall c$.

Ω	ω
1. $\geq$	1. $>$
2. $\exists$ some c	2. $\forall c$:

Shortcut: If $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$ then $f(n) = \omega\, g(n)$.

Recap:

1. lower = O (higher)
2. If $f(n) = a_0 + a_1 n + a_2 n^2 + \dots a_m n^m$ $(a_m \neq 0)$
   then $f(n) = O(n^m)$, $\Theta(n^m)$, $\Omega(n^m)$
3. Dominance Ranking
4. $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \begin{cases} 0 & \Rightarrow f(n) = o(g(n)) \\ \infty & \Rightarrow f(n) = \omega(g(n)). \end{cases}$
5. Properties of Asymptotic

Lect. 2 14/08/2022

i) Reflexivity:

$a = a$

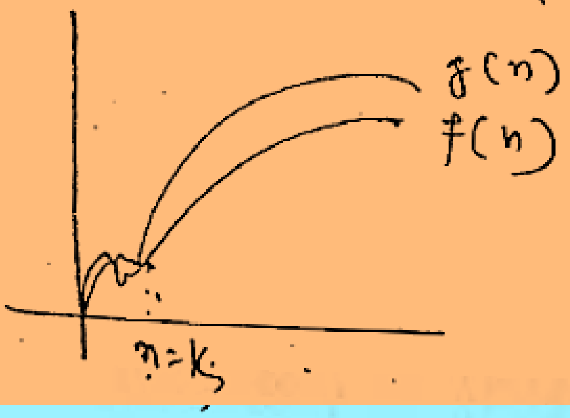
$f(n) \leq 1 \cdot f(n)$. $\forall n \geq k$.

$\Rightarrow f(n) = O(f(n))$

ii) Symmetric:

If $a = b \Rightarrow b = a$.

$\rightarrow f(n) = O(g(n)) \not\Rightarrow g(n) = O f(n)$.



iii) Transitive

If $a = b$, $b = c$, then $a = c$.

If $f(n) = O(g(n))$, $g(n) = O(h(n))$

$\Rightarrow f(n) = O(h(n))$.

iv) Transpose:

$$f(n) = \Theta(g(n))$$
$$\therefore \quad g(n) = \Omega\, f(n).$$

Notation \ prop	Reflexivity	Symmetric	Transitive	Transpose
$O(\leq)$	✓	✗	✓	$O \Rightarrow \Omega$
$\Omega(\geq)$	✓	✗	✓	$\Omega \Rightarrow O$
$\Theta(=)$	✓	✓	✓	$\Theta \Rightarrow \Theta$
$o(<)$	✗	✗	✓	$o \Rightarrow \omega$
$\omega(>)$	✗	✗	✓	$\omega \Rightarrow o$

prob

Consider:

1. $(n+K)^m = \Theta(n^m)$ (where K, m are constants)
2. $2^{n+1} = O(2^n)$
3. $2^{2n+1} = O(2^n)$.

Simple for loop

```
sum = 0
for (i = 1 ; i <= n ; i = i+2)
{
    sum = sum + i ;
}
```

$\rightarrow O\left(\frac{n+1}{2}\right) = O(n+1)$

$= O(n).$

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```
sum = 0
for ( i = 1 ; i <= n ; i = i * 2)
{
    sum = sum + i;
}
```

$\longrightarrow O(1 + \log_2 n)$

$O(\log_2 n).$

```
j = 1 .
while (j <= n)
{
    j = j * 2 ;
}
```

How many ~~compilations~~ comparisons performed.

```
sum = 0
for (i = n; i > 0; i = i/2)
  {  sum = sum + i;
  }
```

Nested For loop Questions

Q.
```
sum = 0
for (i = 1; i <= n; i = i+1)
  {
     for (j = 1; j <= n; j = j+1)
        {
           sum = sum + j;
        }
  }
```

$\underline{\underline{Sol^n}}$

i	1	2	3	....	n
j	n	n	n	...	n
sum	n	n	n	....	n

$= O(n * n)$

$= O(n^2)$.

It is not the value
number of times executed

<u>means</u>:

for i = 1. n times will be executed

for i = 2 n times ——"——

⋮

i = n ——"——

Q. for (i=1; i ≤ n; i = i+1)

{

for (j = 1; j ≤ n; j = j*2)

{ sum = sum+j;

}

}.

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i	1	2	3	--	n
j	$\log_2 n$	$\log_2 n$	$\log_2 n$	---	$\log_2 n$ times executes.
sum	$\log_2 n$ +	$\log_2 n$ +	$\log_2 n$ +	.. -	$\log_2 n$.

$= O(n \log_2 n)$

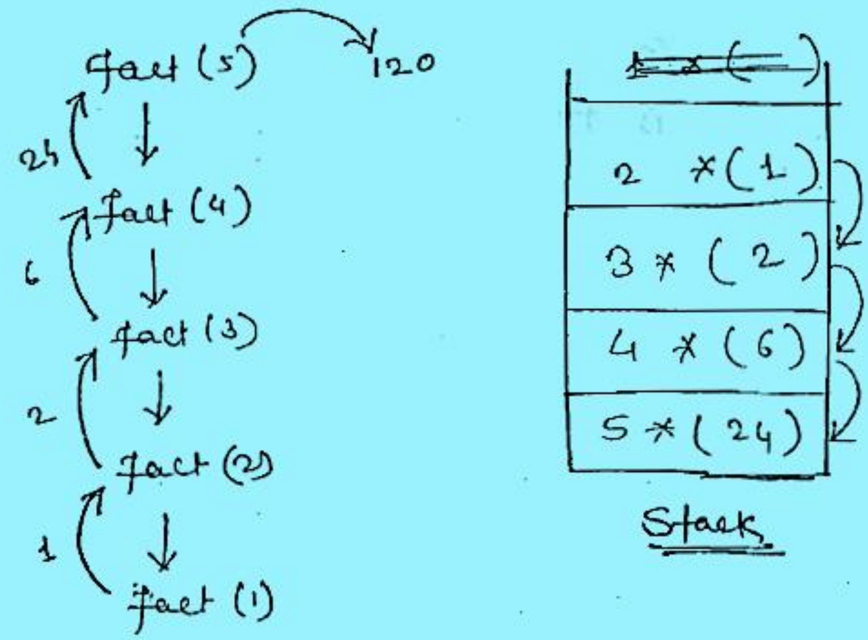
Q.

```
for (i = 1; i ≤ n; i = i+1)
{
for (j = 1; j ≤ i; j = j*2)
{ sum = sum + j;
}
```

Time Complexity of Recursive Algorithm.

```
int fact (int n)
{
    if (n==0 || n==1)  // Base condition
        return 1.
    else
        return n x fact(n-1) ; // induct^n cond^n
}
```

⇒ Execution



Note: 1) Time complexity of recursive algorithm
= Number of function calls

$\therefore$ Time complexity of fact = $O(n)$.

2) Space Complexity of Recursive Algorithm
= Depth of Recursion tree

$\therefore$ Space Complexity of fact (n) = $O(n-1)$
= $O(n)$.

17.Back-Substitution Method